Proving that the Boolean Algebra forms a vector space

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1 Defining the Boolean algebra

We define a Boolean algebra as a set of B elements $a, b, \ldots$ with the following properties:

1. B has two binary operators $\wedge$ or $\cdot$ (logical AND) and $\vee$ or $+$ (logical OR)
2. Idempotence
   - $a \wedge a = a \vee a = a$
3. Commutative law
   - $a \wedge b = b \wedge a$
   - $a \vee b = b \vee a$
4. Associative law
   - $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   - $a \vee (b \vee c) = (a \vee b) \vee c$
5. Absorption law
   - $a \wedge (a \vee b) = a \vee (a \wedge b) = a$
6. Mutual distributiveness
   - $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
   - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
7. B contains universal bounds $\emptyset$ (empty set) and I (universal set)
   - $\emptyset \wedge a = \emptyset$
   - $\emptyset \vee a = a$
   - $I \wedge a = a$
   - $I \vee a = I$
8. B has a unary operator $a \to a'$ such that
   - $a \wedge a' = \emptyset$
   - $a \vee a' = I$

If the truth values a, b are interpreted as integers $0, 1$ our operators can be expressed with ordinary arithmetic, or by minimum/maximum functions:

1. $a \wedge b = a \times b = \min(a, b)$
2. $a \vee b = a + b - (a \times b) = \max(a, b)$
3. $\neg a$ or $\bar{a} = 1 - a$

We may also express $a \wedge b$, $a \vee b$, and $\neg a$ with a truth table

a	b	$a \wedge b$	$a \vee b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Table 1: Truth table for binary operators

a	$\neg a$
0	1
1	0

Table 2: Truth table for unary operator

2 Defining a field

bar

3 Boolean algebra as a field

foobar

4 Fields as vector spaces

barfoo

5 Boolean algebra as a vector space

foobar