1 Induction Proofs

Ex. 1 — Show that for any $n \in \mathbb{N}_1$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Ex. 2 — Consider the sequence (a_n) defined by

$$\begin{cases} a_1 = 6, \\ a_{n+1} = \frac{a_n}{3}(1 - e^{-a_n^2}), \quad \text{if } n \geq 1 \end{cases}$$

(a) Show that $a_n \in (0, 6]$ for any $n \in \mathbb{N}_1$

(b) Show that the sequence is decreasing

(c) Prove that the sequence is convergent and determine it's limit

Ex. 3 — Consider a sequence (b_n) defined by

$$\begin{cases} b_1 = \frac{1}{2}, \\ b_{n+1} = b_n(b_n - 1) + 1, \quad \text{if } n \geq 1 \end{cases}$$

(a) Show that $b_n \in (0, 1)$ for any $n \geq 1$

(b) Show that the sequence is increasing

(c) Prove that the sequence is convergent and determine it's limit

Ex. 4 — Consider the sequence (u_n) defined by

$$\begin{cases} u_1 = 1, \\ u_{n+1} = 1 + \frac{u_n}{2}, \quad \text{if } n \geq 1 \end{cases}$$

(a) Show that $u_n \leq 2$ for any $n \in \mathbb{N}_1$

(b) Show that the sequence is increasing

(c) Prove that the sequence is convergent and determine it's limit

Ex. 5 — Let $\alpha \in [0,1]$, consider the sequence (b_n) given by

$$\begin{cases} b_1 = \alpha, \\ b_{n+1} = b_n - b_n^4, & \text{if } n \geq 1 \end{cases}$$

(a) Show that b_n is monotonically decreasing

(b) Show that $b_n \in [0,1] \forall n \in \mathbb{N}_1$

(c) Prove that b_n is convergent and calculate $\lim b_n$

Ex. 6 — Consider the sequence (a_n) defined by

$$\begin{cases} a_1 = 1, \\ a_n = \frac{3a_{n-1}}{n}, & \text{if } n \geq 2 \end{cases}$$

Show that

$$a_n = \frac{3^{n-1}}{n!} \qquad \forall n \geq 1$$

2 Limits I

Ex. 7 — Calculate or show that it does not exist in $\overline{\mathbb{R}}$

(a) $\lim \frac{(n+1)!-n!}{n!(n+2)}$

(b) $\lim(-1)^n \frac{10^n}{n!}$

(c) $\lim \frac{5n!+5n}{n^n+2}$

(d) $\lim \sqrt{\frac{e^n+2}{n!}}$

(e) $\lim \frac{(-1)^n n}{n!+4}$

(f) $\lim \frac{\sqrt[3]{n+4}}{\sqrt[3]{n}+4}$

(g) $\lim \sqrt[n]{\frac{n+2^n}{2+5^n}}$

(h) $\lim \frac{3n^4-2n}{(n^2+3)(1+5n^2)}$

(i) $\lim(\frac{2}{3} + \cos(3n\pi))^n$

(j) $\lim \frac{3+\cos(e^{-n})}{n+\sqrt{n!}}$

(k) $\lim \frac{(-1)^n n}{n!+5}$

(l) $\lim \frac{\cos(n\pi)}{\sin(1/n\pi)+1}$

(m) $\lim \frac{e^{3n}+1}{2^n+n^2}$

(n) $\lim \frac{\sqrt{n}+n^3}{(n+\sqrt{n})(n^2+n^{3/2})}$

(o) $\lim(\cos(\frac{\pi}{4}+n\pi)+1)^n$

(p) $\lim \frac{(3n)!}{n!(2n)!}$

(q) $\lim \frac{\arccos(1/n)}{\cos(\pi/n)}$

(r) $\lim \frac{2n!+3^n}{n^{50}+n!}$

(s) $\lim \sqrt[n]{\frac{\arctan n}{1+e^n}}$

3 Limits II

Ex. 8 — Calculate or show that it does not exist in $\overline{\mathbb{R}}$

(a) $\lim_{x\to 0} \frac{e^x-1}{x-e^{3x}+1}$

(b) $\lim_{x\to+\infty}(3x^2+1)^{1/x}$

(c) $\lim_{x\to 1} \frac{x-1}{\tan(x-1)}$

(d) $\lim_{x\to 2}(x-1)^{\frac{1}{2-x}}$

(e) $\lim_{x\to-\infty} xe^{x^2}$

(f) $\lim_{x\to+\infty}(\log(2x))^{1/x}$

(g) $\lim_{x\to 1} \frac{1}{x-1}\int_0^{\log x} e^{\sin t}dt$

(h) $\lim_{x\to 0}(x^2+1)^{\frac{1}{x^2}}$

(i) $\lim_{x\to 1+} \frac{\arctan(x-1)}{x^3-3x+2}$

(j) $\lim_{x\to 0} \frac{\arctan 2x}{\tan x}$

(k) $\lim_{x\to+\infty}(2e^x+1)^{\frac{1}{x}}$

(l) $\lim_{x\to 0} \frac{1-\cos x}{\arcsin x}$

(m) $\lim_{x\to 0} x(\log x)^2$

(n) $\lim_{x\to 0} \frac{2\int_0^x(1-e^{t^2})dt}{x^2}$

4 Functions

Ex. 9 — Consider the function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -xe^x, & \text{if } x < 0 \\ \alpha \arctan(x^2 - 2x), & \text{if } x \geq 0 \end{cases}$$

Where α is a real constant.

(a) Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$

(b) Justify whether f is continuous

(c) Let $f'_+(0) = -2\alpha$, determine α such that f is differentiable at $x = 0$. Justify that f is differentiable in $\mathbb{R}$ and calculate it's derivative

(d) Determine the local extrema and the monotonous intervals of f (with $\alpha = 1/2$)

(e) Indicate the co-domain of f (with $\alpha = 1/2$)

Ex. 10 — The function $h\colon \mathbb{R} \to \mathbb{R}$

$$h(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous at only one point. Is it differentiable at that point?

Ex. 11 — Consider the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{e^{-x}}{x+1}, & \text{if } x > 0 \\ 2 + \log(1 - x), & \text{if } x < 0 \end{cases}$$

(a) Show that f is continuous. Will f be extensible by continuity on the point $x = 0$?

(b) Calculate in $\overline{\mathbb{R}}$

$$\lim_{x\to-\infty} f(x), \qquad \lim_{x\to-\infty} f(x)$$

(c) Show that f is differentiable, and calculate $f'(x) \quad \forall x \in \mathbb{R} \setminus \{0\}$. Use the result to determine the monotonous intervals of f

(d) Determine the co-domain of f

Ex. 12 — Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} -x^2 e^x, & \text{if } x \leq 0 \\ (1 - \cos x)e^{-x} & \text{if } x > 0 \end{cases}$$

(a) Calculate, if it exists in $\overline{\mathbb{R}}$, $\lim_{x \to +\infty} f(x)$

(b) Show that f is continuous at the point $x = 0$ and calculate,if it exists, $f'(0)$

(c) Determine the differentiability domain of f, and calculate it's derivative.

(d) Show that f has one and only one local extremum on the interval $(-\infty, 0)$. Prove that it's an absolute minimum of f in $\mathbb{R}$.

(e) Show that the co-domain of f is a closed and limited interval.

Ex. 13 — Given a function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} \alpha + \log \frac{1}{1+x^2} & \text{if } x < 0 \\ \arctan \frac{1}{x} & \text{if } x > 0 \end{cases}$$

(a) Calculate, if it exists, $\lim_{x \to -\infty}$ and $\lim_{x \to +\infty}$

(b) Determine $\alpha \in \mathbb{R}$ such that f is extensible by continuity at $x = 0$

(c) Determine the domain of differentiability of f, and calculate it's derivative.

(d) Determine the monotony intervals of f, as well as it's extrema, if they exist.

(e) Assuming $\alpha = 0$, what will be the co-domain of f?

Ex. 14 — Consider the function $f\colon \mathbb{R} \setminus \{-\pi/2, \pi/2\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{\pi}{2(\sin x + 1)}, & \text{if } |x| < \pi/2 \\ \arctan(e^{x - \pi/2}), & \text{if } |x| > \pi/2 \end{cases}$$

(a) Study f with regards to continuity.

(b) Calculate $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$

(c) Decide whether f extendable by continuity at $-\pi/2$ and $\pi/2$

(d) Calculate the derivative f', and determine it's monotonous intervals.

Ex. 15 — Consider the function $f\colon(-1,+\infty)\to\mathbb{R}$:

$$f(x)=\begin{cases}\log\sqrt{1-x^2}, & \text{if } -1<x\le 0\\ x^2e^{1-x^2}, & \text{if } x>0\end{cases}$$

(a) Study f with regards to continuity

(b) Calculate $\lim_{x\to-1^+}f(x)$ and $\lim_{x\to+\infty}f(x)$

(c) Determine the derivative f'

(d) Determine the monotony intervals of f, as well as the local maxima and minima.

5 Primitives

Ex. 16 — Calculate the primitive of the function

$$\frac{2x+3}{x^2+2}$$

that vanishes at $x=0$

Ex. 17 — Determine a primitive for the following functions:

(a) $\frac{1}{\sqrt{e^x-1}}$

(b) $\frac{3x}{1+x^4}$

(c) $\frac{3x}{\sqrt{4+x^2}}$

(d) $\frac{1}{x^2+4}$

(e) $\frac{\sin\sqrt[3]{x}}{\sqrt[3]{x}}$

(f) $\frac{\cos(\log x)}{x}$

(g) $\log(1+x)^2$

Ex. 18 — Determine the function $g\colon\mathbb{R}\to\mathbb{R}$ that satisfies

$$\begin{cases}g'(x)=\frac{3x^2+1}{x^2+2} & \forall x\in\mathbb{R}\\ g(0)=2\end{cases}$$

Ex. 19 — Calculate the function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies

$$\forall x \in \mathbb{R} \quad f'(x) = \frac{1+x}{9+x^2} \quad \text{and} \quad f(0) = \log(3)$$

Ex. 20 — Write the general expression for the primitives in $(0, +\infty)$ of

$$\frac{1}{x\sqrt{x+1}}$$

6 Integration

Ex. 21 — Calculate the area of region delimited by:

(a) $\{(x,y) \in \mathbb{R}^2 \colon x \le y \le -x^2 + 2\}$
(b) $\{(x,y) \in \mathbb{R}^2 \colon x^2 - \pi x \le y \le -\sin x\}$
(c) $\{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le 1, \frac{\pi}{4}x \le y \le \arctan x\}$
(d) The triangle described by the lines $y = x, \;\; y = 2x, \;\; y = 3x - 2$

Ex. 22 — Compute the following integrals

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{2+\sin^2 x} dx$
(b) $\int_0^1 \frac{4x}{4+x^2} dx$
(c) $\int_{\log 2}^{\log 3} \frac{e^x}{(e^x-1)^2} dx$
(d) $\int_1^2 \frac{1}{x(4+log^2(x))} dx$
(e) $\int_{-1}^0 \frac{\log(x+2)}{(x+2)^2} dx$
(f) $\int_0^1 (\frac{x}{2} + x^3) \arctan x \, dx$
(g) $\int_1^{\log 2} \frac{e^x}{\sqrt{e^x-1}} dx$
(h) $\int_2^7 \frac{1}{(x+1)\sqrt{x+2}} dx$

Ex. 23 — Determine the value of the constant $c \in \mathbb{R}$ such that $f'(0) = 0$, with

$$f(x) = \int_x^{cx+2} e^{-t^2} dt$$

Ex. 24 — Let $g \in C(\mathbb{R})$ be an odd function, and f the function given by

$$f(x) = \int_1^{x^2-1} xg(t)dt$$

Calculate $f'(x)$ for any $x \in \mathbb{R}$, and show that $f'(0) = 0$

Ex. 25 — Determine a function $f : \mathbb{R} \to \mathbb{R}$, differentiable and non-null, such that

$$f^3(x) = \int_{\pi/2}^{x} \frac{\cos t}{2 - \sin t} dt$$

7 Taylor Polynomial

Ex. 26 — Let $f \in C^2(\mathbb{R})$ and $g(x) = f(e^x) \forall x \in \mathbb{R}$. Let $3 - x + 2(x-1)^2$ be the second order Taylor polynomial of f relative to point 1, determine the second order MacLaurin polynomial of g.

Ex. 27 — Let $f \in C^4(\mathbb{R})$ be such that its third degree Taylor polynomial at point 2 is constant. Given that $f^{(4)}(2) = 1$, justify that $f(2)$ is an extremum of f and classify it.

Ex. 28 — Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function , and consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \int_x^{x^2} h(t)dt$$

(a) Show that $f''(1) - 2f'(1) = 3h'(1)$

(b) If $p_1(x) = x - 1$ is the first order Taylor polynomial of h at point 1, show that f has a local minimum at $x = 1$

8 Series

Ex. 29 — Analyse the following series and determine whether they are absolutely convergent, conditionally convergent, or divergent

(a) $\sum_{n=1}^{+\infty} \frac{1}{3+2n}$

(b) $\sum_{n=1}^{+\infty} \frac{\cos(2n)}{n^3}$

(c) $\sum_{n=1}^{+\infty} \frac{n^3}{e^{2n}}$

(d) $\sum_{n=1}^{\infty} \frac{2+n\sqrt{n}}{1+3n^4}$

(e) $\sum_{n=2}^{\infty} \frac{3^n(1+(-1)^n)}{(2\pi)^{n+1}}$

(f) $\sum_{n=2}^{+\infty} \log(\arctan(n+1)) - \log(\arctan n)$

(g) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt[4]{n^3+1}}$

(h) $\sum_{n=1}^{\infty}(\frac{n}{n+1} - \frac{n+1}{n+2})$

(i) $\sum_{n=0}^{\infty} 2^{-3n}$

(j) $\sum_{n=1}^{+\infty} \frac{\sqrt[3]{n^3+2}}{n^2\sqrt{n+9}}$

(k) $\sum_{n=1}^{+\infty} \frac{\arctan n}{e^n}$

(l) $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\sin(2n)}{n^3}$

Ex. 30 — Show that if the series $\sum_{n=1}^{+\infty} a_n$ converges, then $\sum_{n=1}^{+\infty} \frac{2n-1}{2n}$ also converges.

Ex. 31 — Determine the values of $x \in \mathbb{R}$ for which the following series is absolutely convergent, conditionally convergent, or divergent

(a) $\sum_{n=1}^{+\infty} \frac{n^n(x-1)^n}{(n+3)^n}$

(b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{\sqrt{n}+1}$

(c) $\sum_{n=1}^{\infty} \frac{\log n}{2^n}(x-2)^n$

9 Proofs

Ex. 32 — Show that for any $x > 0$

$$\frac{x}{1+x} < \log(1+x) < x$$

Hint: Mean Value Theorem

Ex. 33 — Let $h \in C(\mathbb{R})$ such that $h(x) = h(x+2)$ for all $x \in \mathbb{R}$, and

$$\phi(x) = \int_0^x h(t)dt - \int_0^{x+2} h(t)dt$$

Prove that ϕ is identically 0 if and only if $\int_0^2 h(t)dt = 0$

Ex. 34 — Let (a_n) be a limited sequence with terms in $(1, +\infty)$, and (b_n) another sequence such that

$$b_n = \frac{na_n}{n + a_n} \qquad n \in \mathbb{N}_1$$

Show that (b_n) has convergent subsequences.

Ex. 35 — Let f and g be real functions, defined and continuous on the interval $[a, b]$, such that

$$\int_a^b f(t)dt = 2\int_a^b g(t)dt$$

Show that there is $c \in [a, b]$ such that $f(c) = 2g(c)$

Ex. 36 — Let $h\colon [0, +\infty) \to \mathbb{R}$ be a continuous function for which $\lim_{x\to+\infty} h(x) = c \in \mathbb{R}$.

(a) Show that there is at least one solution for $h(x) = \frac{x^2-1}{x}$ in the interval $(0, +\infty)$

(b) Suppose h is differentiable in $(0, +\infty)$ and

$$\forall x \in (0, +\infty) \quad h'(x < 1)$$

Show that $h(x) = \frac{x^2-1}{x}$ has one and only one solution

Ex. 37 — Let $f\colon [0, 1] \to \mathbb{R}$ be a continuous function. Show that for all $x \in [0, 1]$

$$\int_0^x \int_0^u f(t)dtdu = \int_0^x (x - t)f(t)dt$$

Ex. 38 — Let f be a function integrable in $[0, 1]$. Show that

$$\lim_{n\to\infty} \int_0^1 x^{n+1} f(x)dx = 0$$

Ex. 39 — Let $G\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there is $m \in \mathbb{R}$ such that $\{x \in \mathbb{R}\colon G(x) \leq m\}$ is limited and not empty. Show that G has an absolute minimum.

Ex. 40 — Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x \to -\infty} g(x) = \alpha > 0$$

Show that

$$\lim_{x \to -\infty} \int_x^0 e^{x-t} g(t) dt = \alpha$$

Hint: Show that $\lim_{x \to -\infty} \int_x^0 e^{-t} g(t) dt = +\infty$

Ex. 41 — Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} \psi(x) = +\infty$ and $\lim_{x \to -\infty} \psi(x) = -\infty$. Decide if $G\colon \mathbb{R} \to \mathbb{R}$ defined as

$$G(x) = \frac{\psi(x)}{1 + \psi^2(x)}$$

Has a maximum and a minimum.

Ex. 42 — Show that if $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are convergent series with positive terms, then $\sum_{n=1}^{+\infty} a_n b_n$ is a convergent series. Will this hold true if $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are series with an oscillating sign?

Ex. 43 — Let I be an open interval, $a \in I$ and $\rho\colon I \to \mathbb{R}$ a function 2-times differentiable, such that $\rho''(x) > 0$ for any $x \in I$. Also let

$$g(x) = \rho'(a)(x - a) + \rho(a)$$

Show that $\rho(x) > g(x)$ for all $x \in I \setminus \{a\}$

10 Notation

1. $[a, b]$ for a closed interval, (a, b) for an open one.
2. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$