About comprehensive quantity classes of which the value is nor algebraic nor reducible to algebraic irrational numbers

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1. A long time ago, I gave a presentation about this subject at the Science Academy. I also had two Notes inserted in volume XVIII of Comptes rendus (pages 883 and 910; sessions of the 13th and 20th of May 1844). I believe these two Notes need to be reproduced and completed here. The first Note is formulated as the following:

> To give examples of continued fractions of which we can prove that their value is not at the root of any algebraic equation
>
> $$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$
>
> $a, b, \ldots, g$ being integers, it is sufficient to recollect that $\frac{p_0}{q_0}$ and $\frac{p}{q}$, being two successive reductions of the continued fraction that expresses the development of an immeasurable x of this equation, the incomplete quotient μ, that comes after the reduction $\frac{p}{q}$, serves to form the next reduction, ends up (that results from a Lagrange formula, referencing the Memoires of Berlin, 1768) being, for very big q values, constantly inferior to
>
> $$\pm\frac{df(p,q)}{qf(p,q)dp}$$
>
> essentially a positive expression where we assume
>
> $$f(p,q) = q^n f(\frac{p}{q}) = ap^n + bp^{n-1}q + \cdots + hq^n$$
>
> Given that the abstraction is made of signs, we have now, with more certainty,
>
> $$\mu < \frac{df(p,q)}{qdp}$$
>
> since $f(p,q)$ is an equal integer at least to the unity if we admit (what is allowed) that the equation
>
> $$f(x) = 0$$
>
> has been stripped of any measurable factor; $f(p,q) = 0$ would give, indeed,
>
> $$f(\frac{p}{q}) = 0$$
>
> Now represented by $f'(x)$ the differential of $f(x)$, the inequality above will become
>
> $$\mu < q^{n-2} f'(\frac{p}{q})$$
>
> However, $f'(\frac{p}{q})$ is a finite quantity going towards the limit $f'(x)$, like $\frac{p}{q}$ to the limit x. Designating with A a certain fixed number,which is superior to this limit, we will be sure to have
>
> $$\mu < Aq^{n-2}$$
>
> Thereby, the incomplete quotients of a continued fraction representing the root x of an algebraic equation of the degree n, in rational coefficients, are subject to never pass the product of a certain constant number by the force $(n-2)^{\text{nd}}$ of the denominator of the previous reduction.
>
> It is sufficient to give the incomplete quotients μ a mode of formation that makes them grow beyond a predetermined term, to obtain continued fractions of which the value will not be able to satisfy any algebraic equation itself; it will happen, for example, that if, starting with any first incomplete quotient, we form every following μ with the help of the previous reduced $\frac{p}{q}$, according to the law $\mu = q^q$, or according to the law $\mu = q^m$, m being the index of the μ row.
>
> Moreover, the previous method, which offered the first, is not the only nor is it the simplest that we can use. Let's add that there are also analogue theorems for ordinary series. In particular, we cite the series
>
> $$\frac{1}{l} + \frac{1}{l^{1\cdot 2}} + \frac{1}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{1}{l^{1\cdot 2\cdot 3\cdots m}} + \cdots$$
>
> l being a whole number.

2. The second Note contains a new and more simple demonstration of the theorem to which I was led by the Lagrange formula. The real power of our method, as we will see, is independent of this formula.

If $x, x_1, x_2, \ldots, x_{n-1}$ are the n roots (the first real, the others real or imaginary) of the algebraic equation

$$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$

which we can assume is irreducible, and where $a, b, \ldots, g, h$ are integers that are either positive, negative or zero, as we wish. Let's designate by $\frac{p_0}{q_0}$, $\frac{p}{q}$ two consecutive reductions of the continued fraction in which x develops; and by z the complete quotient that comes after, so we have

$$\frac{p}{q} - x = \pm \frac{1}{q(qz + q_0)}$$

Finally, stating

$$f(p, q) = q^n f(\frac{p}{q}) = ap^n + bp^{n-1}q + \cdots + hq^n$$

By the decomposition of $f(\frac{p}{q})$ in factors, with the help of the roots $x, x_1, \ldots, x_{n-1}$, we find

$$\frac{p}{q} - x = \pm \frac{1}{q(qz + q_0)} = \frac{f(p, q)}{q^n \cdot a\left(\frac{p}{q} - x\right) \cdots \left(\frac{p}{q} - x_{n-1}\right)}$$

However, in order to converge the reduction $\frac{p}{q}$ towards x, the quantity

$$a\left(\frac{p}{q} - x_1\right) \cdots \left(\frac{p}{q} - x_{n-1}\right)$$

also converges towards a finite limit,

$$a(x - x_1) \cdots (x - x_{n-1})$$

there is thus a certain maximum A below which the limit will always remain. On the other hand, $f(p, q)$ is a whole number, at least equal to the unity, abstraction made of the sign. We have thus

$$\frac{1}{q(qz + q_0)} > \frac{1}{Aq^n}$$

of which

$$z < Aq^{n-2} - \frac{q_0}{q} < Aq^{n-2}$$

inequality subsists, even more so, when we substitute the complete z quotient of the integer part that it contains, namely the incomplete quotient μ. The theorem we had in mind is hereby proved simply, without having to fall back on the Lagrange formula that we used earlier. We can, incidentally, apply a similar method to diverse development genres of which the irrational quantities are susceptible, and obtain that way interesting results.

3. Let's add some developments to what preceded. Still regarding a real root x of the equation, irreducible and with whole coefficients,

$$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$

which, if $n > 1$, will also have these other roots $x_1, \ldots, x_{n-1}$, essentially irrational or imaginary and different to x. But let's stop using, to get closer and closer to x, reductions of continued fractions, and let's use any fraction $\frac{p}{q}$. In doing so, as here above,

$$f(p, q) = ap^n + bp^{n-1}q + \cdots + hq^n$$

we will be sure, if $n > 1$, that the absolute value of the integer $f(p, q)$ is at least equal to the unity, and we can again use the equation

$$\frac{p}{q} - x = \frac{f(p,q)}{q^n \cdot a\left(\frac{p}{q} - x_1\right) \cdots \left(\frac{p}{q} - x_{n-1}\right)}$$

this consequence, by designating by A a certain fixed number, we must have (abstraction made of the sign) for all fractions $\frac{p}{q}$ which we now use,

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

But the case of $n = 1$ has to be, in turn, examined closely. This case could not be presented right away; because if we assume the continued fraction, in which we developed x, made up of an infinite number of terms, we had irrational x and $n > 1$. But here, still assuming infinite numbers of successive fractions $\frac{p}{q}$ of which x is the limit, we need to consider the case $n = 1$ as possible.

To deal with this case, whether

$$f(x) = ax + b = 0$$

or

$$\frac{p}{q} - x = \frac{ap + bq}{aq}$$

$$ap + bq = 0 \tag{1}$$

$$x = \frac{p}{q} \tag{2}$$

$$\frac{p}{q} - x > \frac{1}{aq} \tag{3}$$

$$\frac{p}{q} - x > \frac{1}{Aq} \tag{4}$$

$$\frac{p}{q} - x > \frac{1}{Aq^n} \tag{5}$$

$$\frac{p}{q} - x > \frac{1}{Aq^n} \tag{6}$$

$$\frac{p}{q} - x > \frac{1}{Aq^n} \tag{7}$$

$$\frac{p}{q} - x > \frac{1}{Aq^i} \tag{8}$$

$$\frac{p}{q} - x > \frac{1}{Aq} \tag{9}$$

$$\frac{p}{q} - x > \frac{1}{Aq^2} \tag{10}$$

$$\frac{p}{q} - x > \frac{1}{Aq^n} \tag{11}$$

$$x = \frac{1}{l} + \frac{1}{l^{1\cdot 2}} + \frac{1}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{1}{l^{1\cdot 2\cdot 3\cdots m}} + \cdots \tag{12}$$

$$q = l^{1\cdot 2\cdots m} \tag{13}$$

$$x - \frac{p}{q} = \frac{1}{l^{1\cdot 2\cdots m(m+1)}} + \cdots < \frac{2}{q^{m+1}} \tag{14}$$

$$\frac{p}{q} - x > \frac{1}{Aq^n} \tag{15}$$

$$x = \frac{k_1}{l} + \frac{k_2}{l^{1\cdot 2}} + \frac{k_3}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{k_m}{l^{1\cdot 2\cdots m}} + \cdots \tag{16}$$

$$x = \frac{1}{l} + \frac{1}{l^4} + \frac{1}{l^9} + \cdots + \frac{1}{l^{m^2}} + \cdots \tag{17}$$

$$q = l^{m^2} \tag{18}$$

$$x - \frac{p}{q} = \frac{1}{l(m+1)^2} + \cdots < \frac{2}{l^{2m+1}\cdot q} \tag{19}$$

$$x = \frac{1}{l} + \frac{1}{l_1} + \frac{1}{l_2} + \cdots + \frac{1}{l_{m-1}} + \cdots \tag{20}$$

$$l_m = l_{m-1}^{n+1} \tag{21}$$

$$q = l_{m-1} \tag{22}$$

$$x - \frac{p}{q} = \frac{1}{l_m} + \cdots < \frac{2}{q^{n+1}} \tag{23}$$

$$x = \frac{1}{l} + \frac{1}{u_1} + \frac{1}{u_1 l_2} + \cdots + \frac{1}{u_1 l_2 \cdots l_{m-1}} + \cdots \tag{24}$$

$$ax^n + bx^{n-1} + \cdots + gx + h = 0 \tag{25}$$

$$\frac{p}{q} - x = \frac{f(p,q)}{q^n \cdot a\left(\frac{p}{q} - x_1\right) \cdots \left(\frac{p}{q} - x_{n-1}\right)} \tag{26}$$