On very large sets of numbers that are neither algebraic nor reducible to irrational algebraic

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1. A long time ago, I gave a presentation about this subject at the Science Academy. I also had two Notes inserted in volume XVIII of Comptes rendus (pages 883 and 910; sessions of the 13th and 20th of May 1844). I believe these two Notes need to be reproduced and completed here. The first Note is formulated as the following:

> To give examples of continued fractions of which we can prove that their value is not at the root of any algebraic equation
>
> $$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$
>
> $a, b, \ldots, g$ being integers, it is sufficient to recollect that $\frac{p_0}{q_0}$ and $\frac{p}{q}$, being two successive reductions of the continued fraction that expresses the development of an immeasurable x of this equation, the incomplete quotient μ, that comes after the reduction $\frac{p}{q}$, serves to form the next reduction, ends up (that results from a Lagrange formula, referencing the Memoires of Berlin, 1768) being, for very big q values, constantly inferior to
>
> $$\pm\frac{df(p,q)}{qf(p,q)dp}$$
>
> essentially a positive expression where we assume
>
> $$f(p,q) = q^n f(\frac{p}{q}) = ap^n + bp^{n-1}q + \cdots + hq^n$$
>
> Given that the abstraction is made of signs, we have now, with more certainty,
>
> $$\mu < \frac{df(p,q)}{qdp}$$
>
> since $f(p,q)$ is an equal integer at least to the unity if we admit (what is allowed) that the equation
>
> $$f(x) = 0$$
>
> has been stripped of any measurable factor; $f(p,q) = 0$ would give, indeed,
>
> $$f(\frac{p}{q}) = 0$$
>
> Now represented by $f'(x)$ the differential of $f(x)$, the inequality above will become
>
> $$\mu < q^{n-2} f'(\frac{p}{q})$$
>
> However, $f'(\frac{p}{q})$ is a finite quantity going towards the limit $f'(x)$, like $\frac{p}{q}$ to the limit x. Designating with A a certain fixed number,which is superior to this limit, we will be sure to have
>
> $$\mu < Aq^{n-2}$$
>
> Thereby, the incomplete quotients of a continued fraction representing the root x of an algebraic equation of the degree n, in rational coefficients, are subject to never pass the product of a certain constant number by the force $(n-2)^{\text{nd}}$ of the denominator of the previous reduction.
>
> It is sufficient to give the incomplete quotients μ a mode of formation that makes them grow beyond a predetermined term, to obtain continued fractions of which the value will not be able to satisfy any algebraic equation itself; it will happen, for example, that if, starting with any first incomplete quotient, we form every following μ with the help of the previous reduced $\frac{p}{q}$, according to the law $\mu = q^q$, or according to the law $\mu = q^m$, m being the index of the μ row.
>
> Moreover, the previous method, which offered the first, is not the only nor is it the simplest that we can use. Let's add that there are also analogue theorems for ordinary series. In particular, we cite the series
>
> $$\frac{1}{l} + \frac{1}{l^{1\cdot 2}} + \frac{1}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{1}{l^{1\cdot 2\cdot 3\cdots m}} + \cdots$$
>
> l being a whole number.

2. The second Note contains a new and more simple demonstration of the theorem to which I was led by the Lagrange formula. The real power of our method, as we will see, is independent of this formula.

If $x, x_1, x_2, \ldots, x_{n-1}$ are the n roots (the first real, the others real or imaginary) of the algebraic equation

$$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$

which we can assume is irreducible, and where $a, b, \ldots, g, h$ are integers that are either positive, negative or zero, as we wish. Let's designate by $\frac{p_0}{q_0}$, $\frac{p}{q}$ two consecutive reductions of the continued fraction in which x develops; and by z the complete quotient that comes after, so we have

$$\frac{p}{q} - x = \pm\frac{1}{q(qz + q_0)}$$

Finally, stating

$$f(p, q) = q^n f(\frac{p}{q}) = ap^n + bp^{n-1}q + \cdots + hq^n$$

By the decomposition of $f(\frac{p}{q})$ in factors, with the help of the roots $x, x_1, \ldots, x_{n-1}$, we find

$$\frac{p}{q} - x = \pm\frac{1}{q(qz + q_0)} = \frac{f(p, q)}{q^n \cdot a\left(\frac{p}{q} - x\right)\cdots\left(\frac{p}{q} - x_{n-1}\right)}$$

However, in order to converge the reduction $\frac{p}{q}$ towards x, the quantity

$$a\left(\frac{p}{q} - x_1\right)\cdots\left(\frac{p}{q} - x_{n-1}\right)$$

also converges towards a finite limit,

$$a(x - x_1)\cdots(x - x_{n-1})$$

there is thus a certain maximum A below which the limit will always remain. On the other hand, $f(p, q)$ is a whole number, at least equal to the unity, abstraction made of the sign. We have thus

$$\frac{1}{q(qz + q_0)} > \frac{1}{Aq^n}$$

of which

$$z < Aq^{n-2} - \frac{q_0}{q} < Aq^{n-2}$$

inequality subsists, even more so, when we substitute the complete z quotient of the integer part that it contains, namely the incomplete quotient μ. The theorem we had in mind is hereby proved simply, without having to fall back on the Lagrange formula that we used earlier. We can, incidentally, apply a similar method to diverse development genres of which the irrational quantities are susceptible, and obtain that way interesting results.

3. Let's add some developments to what preceded. Still regarding a real root x of the equation, irreducible and with whole coefficients,

$$f(x) = ax^n + bx^{n-1} + \cdots + gx + h = 0$$

which, if $n > 1$, will also have these other roots $x_1, \ldots, x_{n-1}$, essentially irrational or imaginary and different to x. But let's stop using, to get closer and closer to x, reductions of continued fractions, and let's use any fraction $\frac{p}{q}$. In doing so, as here above,

$$f(p, q) = ap^n + bp^{n-1}q + \cdots + hq^n$$

we will be sure, if $n > 1$, that the absolute value of the integer $f(p,q)$ is at least equal to the unity, and we can again use the equation

$$\frac{p}{q} - x = \frac{f(p,q)}{q^n \cdot a\left(\frac{p}{q} - x_1\right)\cdots\left(\frac{p}{q} - x_{n-1}\right)}$$

this consequence, by designating by A a certain fixed number, we must have (abstraction made of the sign) for all fractions $\frac{p}{q}$ which we now use,

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

But the case of $n = 1$ has to be, in turn, examined closely. This case could not be presented right away; because if we assume the continued fraction, in which we developed x, made up of an infinite number of terms, we had irrational x and $n > 1$. But here, still assuming infinite numbers of successive fractions $\frac{p}{q}$ of which x is the limit, we need to consider the case $n = 1$ as possible.

To deal with this case, whether

$$f(x) = ax + b = 0$$

or

$$\frac{p}{q} - x = \frac{ap + bq}{aq}$$

Should it be possible that the numerator $ap + bq$ were zero, we would not be able to draw any conclusion. But if we have ensured it by any way that we never have

$$ap + bq = 0$$

namely

$$x = \frac{p}{q}$$

we could then argue that we have

$$\frac{p}{q} - x > \frac{1}{aq}$$

or even

$$\frac{p}{q} - x > \frac{1}{Aq}$$

writing A instead of a. The general formula

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

will thus subsist even in the case of $n = 1$.

That being said, granting that the quantity x as such, forming infinite numbers of fractions $\frac{p}{q}$ going towards x, but of which none are exactly equal to x, we end up recognizing that the inequality

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

does not always occur. It has to be concluded that x cannot be the root of an equation of the degree n. Let's also add that x will not be root of any equation of inferior degree i; because with

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

we will have, a fortiori,

$$\frac{p}{q} - x > \frac{1}{Aq^i}$$

for every exponent $i < n$. Therefore, having determined that the inequality

$$\frac{p}{q} - x > \frac{1}{Aq}$$

is in default, we will conclude that the value x is not rational. If the higher inequality

$$\frac{p}{q} - x > \frac{1}{Aq^2}$$

is inadmissible, we will conclude that x is not rational, nor even the root of a second-degree equation; and so on. Finally, should it happen that the inequality

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

generally takes place in default, any finite number we select for n, we will be able to claim that x is not even an algebraic irrational.

4. If, for example,

$$x = \frac{1}{l} + \frac{1}{l^{1\cdot 2}} + \frac{1}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{1}{l^{1\cdot 2\cdot 3\cdots m}} + \cdots$$

l being a whole number. Contributing to $\frac{p}{q}$ the rough value, but essentially too small of x, that give the m's the first terms of the series, we will have

$$q = l^{1\cdot 2\cdots m}$$

and

$$x - \frac{p}{q} = \frac{1}{l^{1\cdot 2\cdots m(m+1)}} + \cdots < \frac{2}{q^{m+1}}$$

If the exponent m grows indefinitely, this last quantity decreases faster than any fraction having a constant numerator and a proportionate denominator of a given power of q, so that the inequality

$$\frac{p}{q} - x > \frac{1}{Aq^n}$$

always ends up being in default. From this, I conclude that x is not rational, nor even expressible by algebraic irrationals.

We will easily get to a similar conclusion for the much more general series

$$x = \frac{k_1}{l} + \frac{k_2}{l^{1\cdot 2}} + \frac{k_3}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{k_m}{l^{1\cdot 2\cdots m}} + \cdots$$

$k_1, k_2, k_3, \ldots, k_m, \ldots$ designating whole numbers, positive or negative, of which the absolute value does not exceed a certain maximum k. So if we take $l = 10$, and $k_1, k_2, \ldots$, freely, from 0 until 9, we will form undefined decimal fractions of which the value will never be able to expressed algebraically. I believe I remember that there is such a theorem, in a letter from Goldbach to Euler; but I do not know whether its proof has been provided.

Now let's say

$$x = \frac{1}{l} + \frac{1}{l^4} + \frac{1}{l^9} + \cdots + \frac{1}{l^{m^2}} + \cdots$$

Still assuming for the rough value of x the sum of the first m terms of the series, we will have

$$q = l^{m^2}$$

and

$$x - \frac{p}{q} = \frac{1}{l(m+1)^2} + \cdots < \frac{2}{l^{2m+1} \cdot q}$$

This last quantity, the denominator containing the product of q by l^{2m+1} that grows with m, will end up decreasing faster than $\frac{1}{Aq}$. But the only thing we can conclude from that is that x is not rational.

As a new example, if

$$x = \frac{1}{l} + \frac{1}{l_1} + \frac{1}{l_2} + \cdots + \frac{1}{l_{m-1}} + \cdots$$

l being an integer, and every term has as a denominator the $(n+1)^{\text{th}}$ power of the previous denominator, so that

$$l_m = l_{m-1}^{n+1}$$

It is obvious that l_{m-1} will be a power of l, consequently, we will have

$$q = l_{m-1}$$

and

$$x - \frac{p}{q} = \frac{1}{l_m} + \cdots < \frac{2}{q^{n+1}}$$

x will thus not be the root of any algebraic equation of a degree equal or inferior to n.

We would also have other examples, considering the series

$$x = \frac{1}{l} + \frac{1}{u_1} + \frac{1}{u_1 l_2} + \cdots + \frac{1}{u_1 l_2 \cdots l_{m-1}} + \cdots$$

where $l, l_1, l_2, \ldots, l_{m-1}, \ldots$ designate increasingly bigger whole numbers. This circumstance where l_m grows beyond any limit means that x could no longer be rational. And if l_m grows fast enough with the m index, we will be sure that x is not even an algebraic irrational.

5. Finally, let us observe that, if we assume $a, b, \ldots, g, h$ as imaginary and complex integers, then move towards a imaginary root x of the equation

$$ax^n + bx^{n-1} + \cdots + gx + h = 0$$

with the help of fractions $\frac{p}{q}$ of which the two terms would also be complex integers, we would again find the equation

$$\frac{p}{q} - x = \frac{f(p,q)}{q^n \cdot a\left(\frac{p}{q} - x_1\right) \cdots \left(\frac{p}{q} - x_{n-1}\right)}$$

and substituting the imaginary's modules, of which we would easily deduce

$$\text{mod } \left(\frac{p}{q} - x\right) > \frac{1}{A \ (\text{mod } q)^n}$$

which allows to extend the results of the imaginaries that we just developed for real quantities. This way, we recognize, for example, that, whichever be the complex integer l, the sum of the series

$$\frac{1}{l} + \frac{1}{l^{1\cdot 2}} + \frac{1}{l^{1\cdot 2\cdot 3}} + \cdots + \frac{1}{l^{1\cdot 2\cdot 3\cdots m}} + \cdots$$

is never algebraically expressible.