Lecture 2
Applied Cryptography (Part 2)

ENGG5105/CSC5470 Computer and Network Security

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Roadmap

- Number theory
- Public key cryptography
  - RSA
  - Diffie-Hellman
  - DSA
  - Certificates

Number Theory

- Number theory deals with the arithmetic operations of cryptographic algorithms
- Public key cryptography is based on number theory.

- We'll describe the number theory concepts necessary to understand the public key algorithms, but only in sufficient detail for an intuitive understanding.

Modular Arithmetic

- If m and n are two integers and n > 0, the **remainder** of m divided by n is the smallest non-negative integer that differs from m by a multiple of n
  - For 3, 13 and -7 divided by 10, remainder = 3
- In modular arithmetic, two integers are **equivalent** if they differ by a multiple of n

Modular Arithmetic

- General rules:
  - $a + b \bmod n = [(a \bmod n) + (b \bmod n)] \bmod n$
  - $a - b \bmod n = [(a \bmod n) - (b \bmod n)] \bmod n$
  - $a * b \bmod n = [(a \bmod n) * (b \bmod n)] \bmod n$
- E.g., $12 * 25 \bmod 7 = (12 \bmod 7) * (25 \bmod 7) \bmod 7 = 5 * 4 \bmod 7 = 6$.
- Note that
  - $a^b \bmod n = (a \bmod n)^b \bmod n$
  - But $a^b \bmod n \neq a^{b \bmod n} \bmod n$
- We sometimes drop “mod n” for brevity

Primes

- A positive integer p is **prime** iff it is evenly divisible by exactly two positive integers (itself and 1)
- Question: is the set of primes infinite?
- Many crypto algorithms work on very large primes, which can be found efficiently using randomized algorithms
- Fact: a random hundred-digit number has a 1 in 230 chance of being prime

Greatest Common Divisor (GCD)

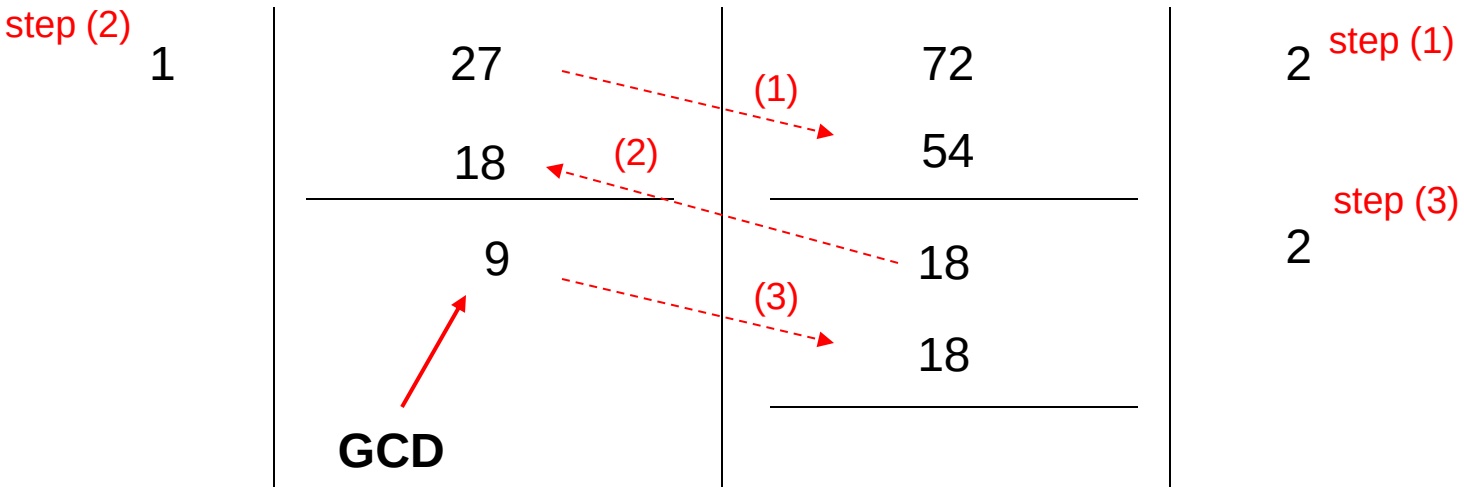
- The GCD of two integers is the largest integer that evenly divides them
  - gcd(0, x) = x
- Two integers are relatively prime if their GCD = 1.
- For two integers a and b, a is the multiplicative inverse mod n of b (and vice versa) if a * b mod n = 1 (or a^{-1} = b mod n)
- Both GCD and multiplicative inverse can be found efficiently via Extended Euclidean algorithm

Facts about Multiplicative Inverse

- $a^{-1} = x \pmod{n}$ has a unique solution x if a and n are relatively prime
- $a^{-1} \pmod{n}$ doesn't exist if a and n are not relatively prime
- If n is prime, all integers 1, 2, …, n-1 are relatively prime to n

Euclidean Algorithm

- Euclidean algorithm is used to find GCD.
- Example: find GCD of 27 and 72



- GCD of 27 and 72 = 9.

Extended Euclidean Algorithm

- Given p = 5, q = 11. Then n = 55 and (p-1)(q-1) = 40. Also, e = 7, which is relatively prime to 40. Find d such that de = 1 mod (p-1)(q-1), i.e., 7d = 1 mod 40
- Solve for x, y such that 7x + 40y = 1.

	z = 7x + 40y			z = 7x + 40y			
	7	1	0	40	0	1	
step (2) 1	5	-5	1	35	5	0	5 step (1)
	2	6	-1	5	-5	1	step (3) 2
				4	12	-2	
				1	-17	3	

- d = x = -17 = 23 (mod 40).

Chinese Remainder Theorem

- Let n_1, n_2, ..., n_k be positive integers that are pairwise relatively prime. Then for any given integers a_1, a_2, …, a_k, there exists an integer x, where $0 < x < n_1n_2…n_k$ such that $x = a_1 \pmod{n_1}$, $x = a_2 \pmod{n_2}$, …, $x = a_k \pmod{n_k}$.
- Used to prove correctness of RSA

Z_n^*

- $\mathbf{Z_n}$ is the set of all integers modulo n.
- $\mathbf{Z_n^*}$ is the set of mod n integers that are relatively prime to n.
  - E.g., If n is prime, then $\mathbf{Z_n^*} = \{1, 2, \ldots, n-1\}$
  - Note that 0 is not in Z_n^*
- $\mathbf{Z_n^*}$ is closed under multiplication (i.e., if a, b are in $\mathbf{Z_n^*}$, then ab is in $\mathbf{Z_n^*}$)

Euler Totient Function Φ(n)

- **Euler Totient function Φ(n)** is defined as the number of elements in $\mathbf{Z_n^*}$.
  - E.g., if n is prime, then Φ(n) = n-1
  - If n = pq for primes p and q, what is Φ(n)?
- **Euler Theorem**: For all a in $\mathbf{Z_n^*}$, $a^{\Phi(n)} = 1 \bmod n$.
- **Fermat's Little Theorem**: If n is prime and for any integer a that is relative prime to n, $a^{n-1} = 1 \bmod n$

Roadmap

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- Public key cryptography
  - RSA
  - Diffie-Hellman
  - DSA
  - Certificates

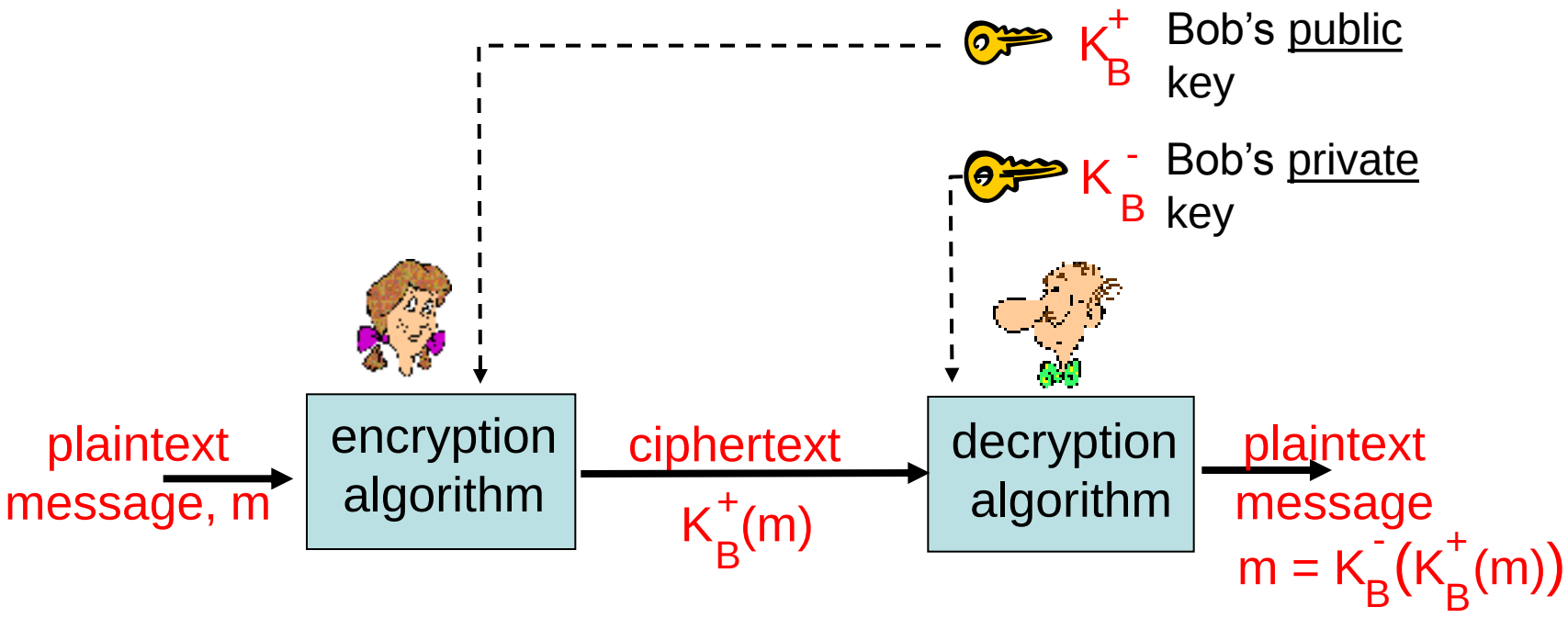
Threat Model

- RSA encryption:
  - Provide confidentiality against eavesdropping (by "honest-but-curious" attackers)
- Diffie-Hellman
  - Provide secure key agreement over an insecure channel
- DSA:
  - Provide authentication for messages
- Certificates:
  - Provide authentication for public keys

Public Key Cryptography

- Symmetric key cryptography
  - Assumption: both sender and receiver share the same secret key
  - But how to agree on the same key first?
- **Public key cryptography**
  - Use two keys – public and private keys
    - Public key for encryption – known to everyone
    - Private key for decryption – known to receiver only
  - Complement symmetric key crypto

Public Key Cryptography



- Requirements:
  - $m = K^-_B\ K^+_B\ (m)$
  - Given K^+_B, computationally infeasible to tell K^-_B

RSA

- Developed by Rivest, Shamir & Adleman of MIT in 1977
- Best known & widely used public-key scheme
- Based on exponentiation in $\mathbf{Z_n^*}$over integers
  - Exponentiation takes $O((\log n)^3)$ operations (easy)
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - Factorization takes $O(e^{\log n \log \log n})$ operations (hard)
  - Exponential with respect to the number of bits of keys
  - Naïve approach (quadratic sieve): Let m = sqrt(n). See if the first m values are divisible by n.
  - More advanced algorithm (number field sieve)
  - Reference: http://en.wikipedia.org/wiki/Integer_factorization

RSA: Create Public/Private Key Pair

- Choose two large prime numbers p and q (e.g., 1024 bits each)
- Compute n = pq. Hence $\Phi(n) = (p-1)(q-1)$.
- Choose e ($e < n$) relatively prime to $\Phi(n)$
- Compute d s.t. $ed = 1 \bmod \Phi(n)$
- Public key is (n, e). Private key is (n, d).
- Note that p and q must be kept secret.

RSA: Encryption/Decryption

- Given message $m < n$,
  - encryption: **$c = m^e \bmod n$**
  - decryption: **$m = c^d \bmod n$**
- Why it works?
  - $c^d = m^{ed} \bmod n$
    $= m^{\Phi(n)k + 1} \bmod n$ (for some k)
    $= m$ (by Euler's theorem)

RSA Example: Key Setup

- Select primes: $p=17$ & $q=11$
- Calculate $n = pq = 17 \times 11 = 187$
- Calculate $\phi(n) = (p-1)(q-1) = 16 \times 10 = 160$
- Select e: $\gcd(e,160)=1$; choose $e=7$
- Determine d: $de = 1 \bmod 160$ and $d < 160$
  Value is $d=23$ since $23 \times 7 = 161 = 10 \times 160 + 1$
- Publish public key $PU=\{7,187\}$
- Keep secret private key $PR=\{23,187\}$

RSA Example: En/Decryption

- sample RSA encryption/decryption is:
- given message $M = 88$ ($88<187$)
- encryption:

$$C = 88^7 \bmod 187 = 11$$

- decryption:

$$M = 11^{23} \bmod 187 = 88$$

Security of RSA

- Security of RSA is based on the factorization assumption:
  - factoring a large number n = pq is hard.
- Fact: The problem of computing d from the public key (n, e), and the problem of factoring n, are computationally equivalent

Efficiency of RSA

- Fact: RSA is no less secure (by far) if e is always chosen to be the same number
- Typical values of e: 3 and 65537.
- If e = 3, some practical constraints need to be imposed
  - If m is small, then $c = m^3$ is small. Attackers can readily determine m by computing the ordinary cube root.

Attacks on RSA

- If e = 3, and the same message m is sent encrypted to three different parties with different moduli (e.g., $c_i = m^3 \bmod n_i$ for i=1, 2, 3), then an attacker can determine m from c_i.
- How?
  - Attacker can efficiently find x s.t. $x = c_i \bmod n_1 n_2 n_3$.
  - Since $m < n_i$ for each i, $m^3 < n_1 n_2 n_3$, by Chinese Remainder Theorem, $x = m^3$.
  - Also, $m^3 \bmod n_1 n_2 n_3$ will just be m^3. Then m can be obtained by the cube root of m^3.

Attacks on RSA

- Scenario: a central authority selects a single RSA modulus n, and distributes a distinct encryption/decryption exponent pair (e_i, d_i) to each entity in a network.
- Common modulus attack:
  - if a single message m were encrypted and sent to two or more entities in the network, then there is a technique by which an eavesdropper could recover m
- Design rule: each n must be associated with a unique (e, d).

Public-Key Cryptography Standard (PKCS)

- Useful to have some standard for encoding of information that is signed or encrypted via RSA, so that different implementations can interwork, and pitfalls are avoided.
- PKCS defines the encoding of RSA keys, signatures, encrypted messages.
- You'll know more when using OpenSSL.

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  - Diffie-Hellman
  - DSA
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Diffie-Hellman Key Exchange

- First public-key type scheme proposed
  - Earlier than RSA
- By Diffie & Hellman in 1976 along with the exposition of public key concepts
  - note: now know that Williamson (UK CESG) secretly proposed the concept in 1970
- A practical method for public exchange of a secret key

Diffie-Hellman Key Exchange

- A public-key distribution scheme
  - cannot be used to exchange an arbitrary message
  - rather it can establish a common key
  - known only to the two participants
    - Can be extended to more than two parties
- value of key depends on the participants (and their private and public key information)
- based on exponentiation in $\mathbf{Z_n^*}$ (modulo a prime or a polynomial) - easy
- security relies on the difficulty of computing discrete logarithms (similar to factoring) – hard

Diffie-Hellman Setup

- Two users A and B agree on global (public) parameters:
  - large prime integer p
  - g being a generator (primitive root) mod p
- Each user (eg. A) generates their key
  - chooses a secret key (number): $x_A < p$
  - compute their **public key**: $y_A = g^{xA} \bmod p$
- each user makes public that key y_A

Diffie-Hellman Setup

- An integer x, where $1 \leq x \leq p-1$, is said to be the primitive root of prime p if the set {x, x^2, x^3, …, x^{p-1} mod p} is a collection of all distinct integers between 1 and p-1 inclusively.
- You may assume g is a primitive root of p
- How to find the primitive root / generator is beyond the scope here.

Diffie-Hellman Key Exchange

- Key exchange process:
  - A sends to B: $y_A = g^{xA}$
  - B sends to A: $y_B = g^{xB}$
- shared session key for users A & B is K_{AB}:

  $K_{AB} = g^{x_A.x_B} \bmod p$

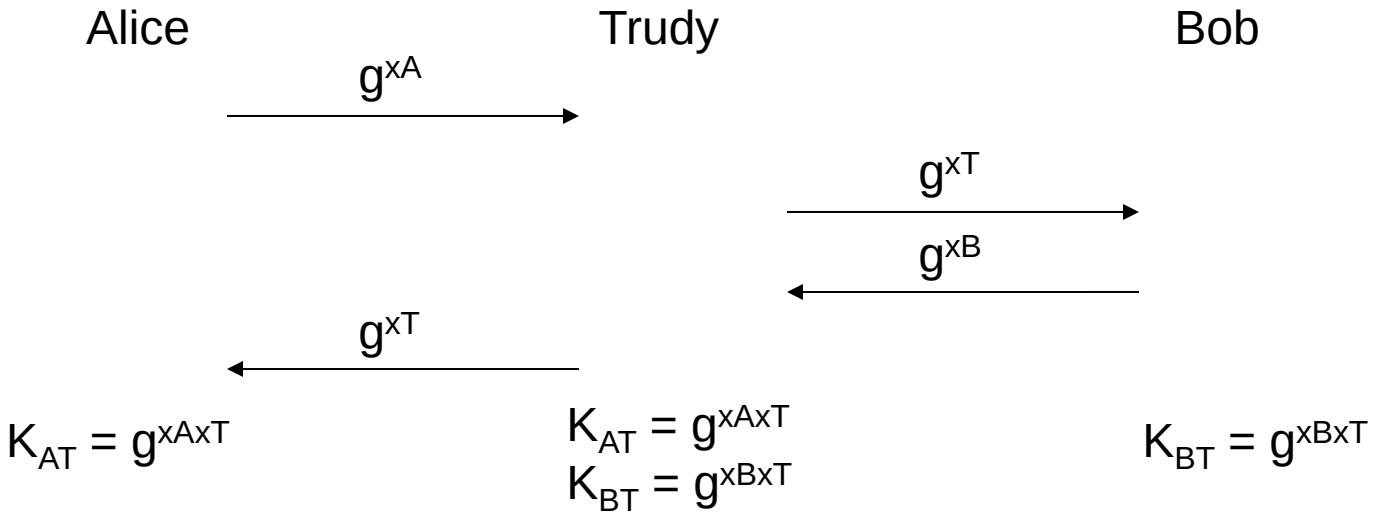
  $= y_A^{x_B} \bmod p$ (which **B** can compute)

  $= y_B^{x_A} \bmod p$ (which **A** can compute)
- K_{AB} is used as session key in private-key encryption scheme between Alice and Bob
- attacker needs an x, must solve discrete log

Diffie-Hellman Example

- users Alice & Bob who wish to swap keys:
- agree on prime $p=353$ and $g=3$
- select random secret keys:
  - A chooses $x_A=97$, B chooses $x_B=233$
- compute respective public keys:
  - $y_A=3^{97} \bmod 353 = 40$ (Alice)
  - $y_B=3^{233} \bmod 353 = 248$ (Bob)
- compute shared session key as:
  - $K_{AB}= y_B^{x_A} \bmod 353 = 248^{97} = 160$ (Alice)
  - $K_{AB}= y_A^{x_B} \bmod 353 = 40^{233} = 160$ (Bob)

Man-in-the-middle Attack on DH



- Trudy can intercept messages between Alice and Bob
- Why MITM attack work?
  - Alice and Bob can't verify if the public keys are actually from Alice and Bob, respectively

Security of DH

- Security of DH is based on discrete log assumption
  - Given g, p, g^x mod p, it is computationally infeasible to find x.
- More precisely, it's based on Diffie-Hellman assumption:
  - Given g, p, g^x mod p, g^y mod p, it is computationally infeasible to find g^{xy} mod p.

Defenses Against MITM

- Authenticated Key Exchange
- Examples:
  - Encrypt public keys with pre-shared secret
  - Encrypt public keys with other side's public keys
  - **Sign DH values with private keys**
    - will show how this approach works in a moment

Digital Signatures

- Digital signatures provide the ability to:
  - verify author, date & time of signature
  - authenticate message contents
  - be verified by third parties to resolve disputes
- Analogous to hand-written signatures
- Goal is similar to Message Authentication Code (MAC), but using public key crypto.

Roadmap

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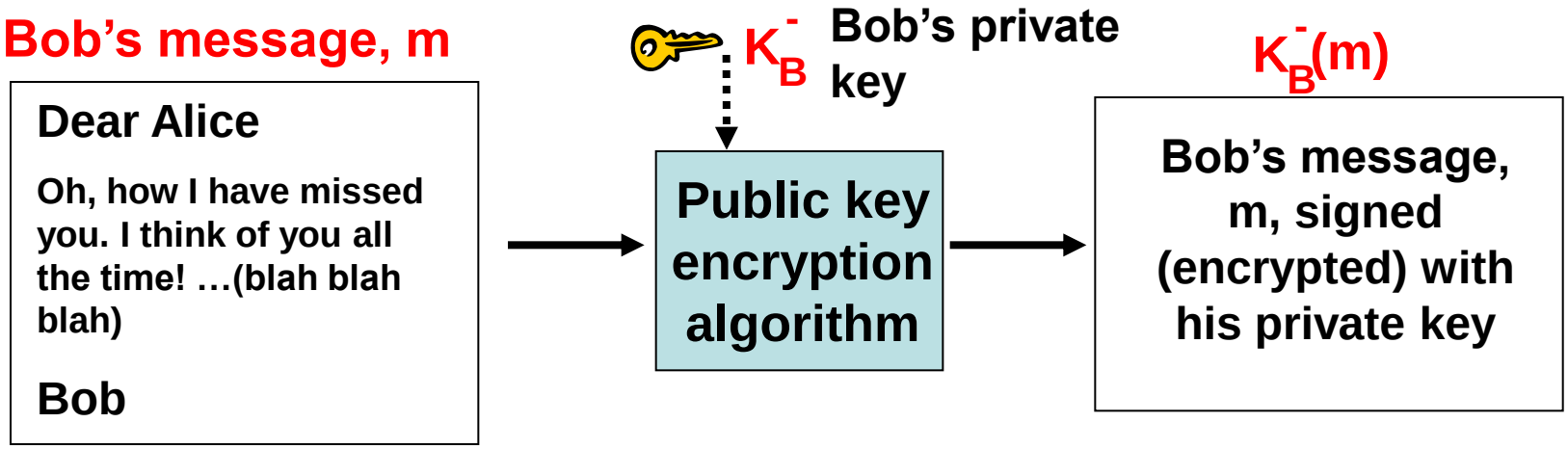
Digital Signatures: Idea

- Encryption/decryption
  - Sender encrypts with public key
  - Receiver decrypts with private key
- **Signature/verification** – two main operations
  - Sender signs with private key
  - Receiver verifies with public key
- Can be directly based on RSA, but we will show more efficient approaches.

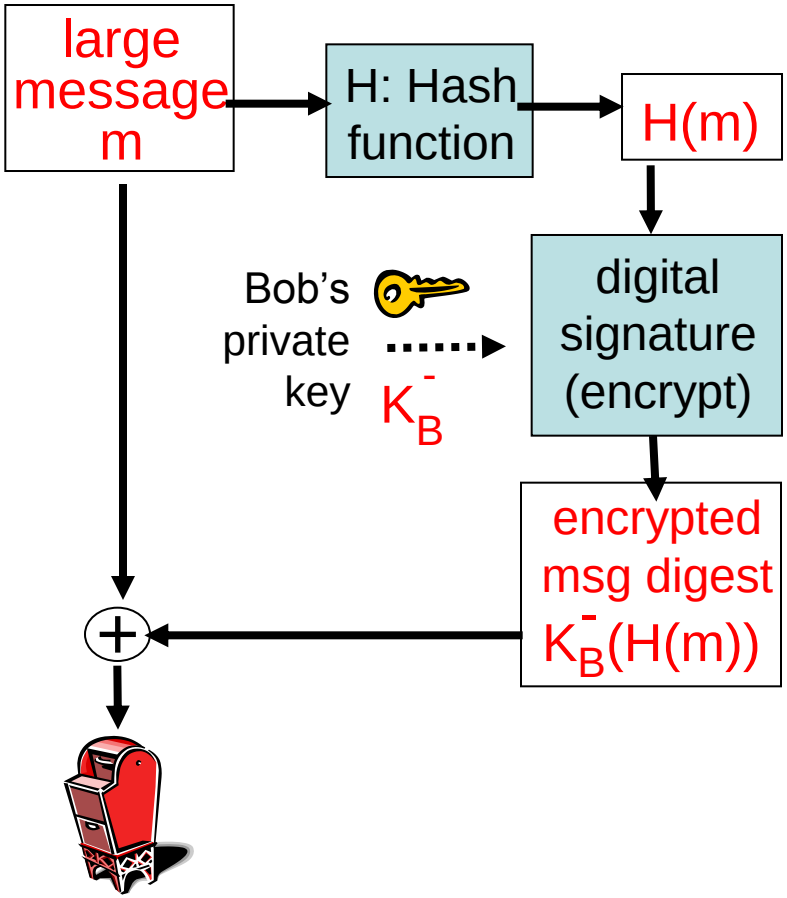
Digital Signatures: Naive Approach

- Naive approach: sign message m directly

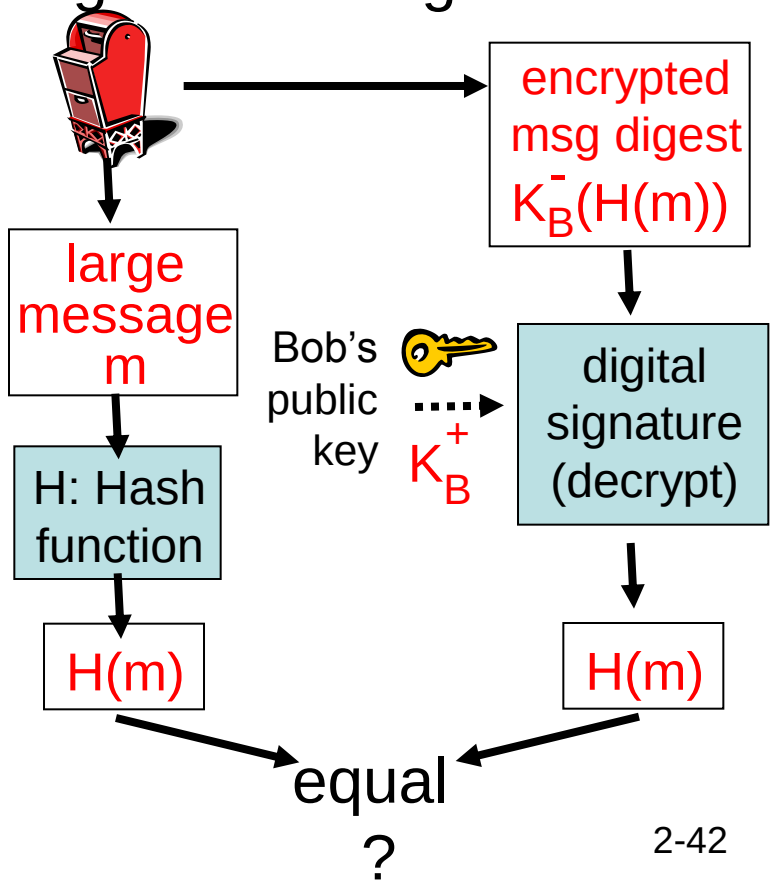


Digital Signatures: Signed Message Digest

Bob sends digitally signed message:



Alice verifies signature and integrity of digitally signed message:



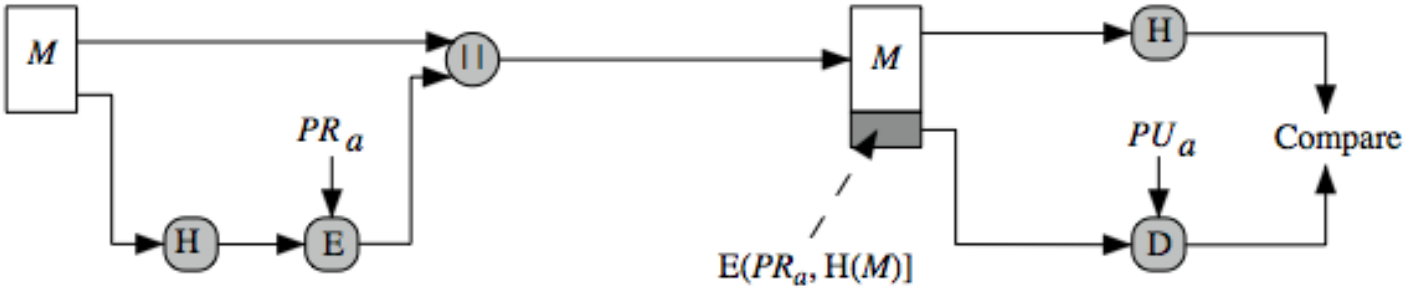
Digital Signatures with Encryption

- Signing before encrypting:
  - Alice signs the message with her private key
    - $S_A(M)$
  - Alice encrypts the message **and signature** with Bob's public key
    - $E_B(S_A(M))$
  - Bob decrypts message with his private key
    - $D_B(E_B(S_A(M))) = S_A(M)$
  - Bob verifies with Alice's public key
    - $V_A(S_A(M)) = M$
- Why "sign before encrypt", but not "sign after encrypt"?

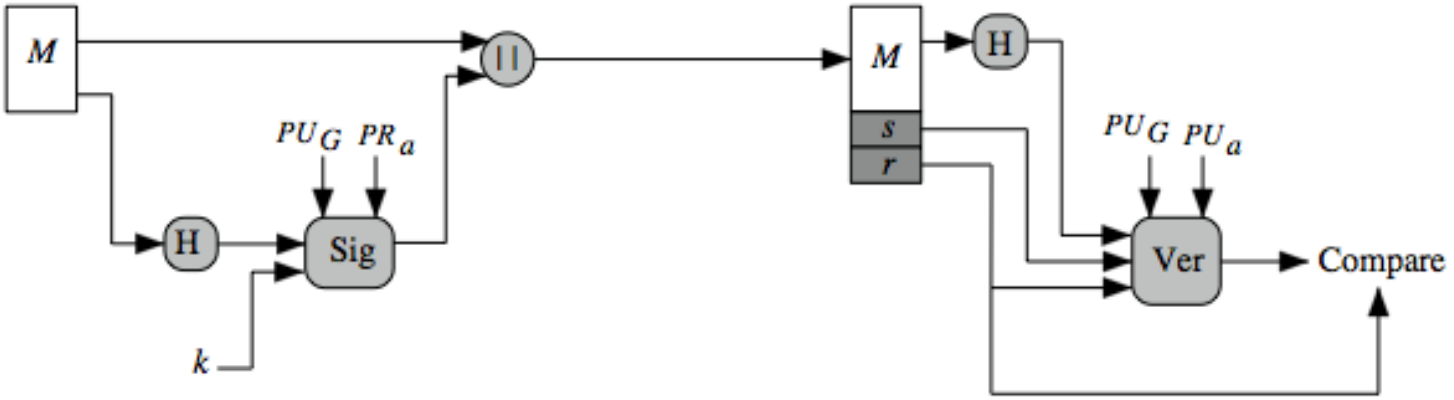
Digital Signature Standard (DSS)

- DSS
  - a US Govt approved signature standard designed by NIST & NSA in early 90's
  - based on the SHA hash algorithm
- DSS is the standard, DSA is the algorithm
- DSA
  - digital signature only
  - unlike RSA, cannot be used for encryption and key exchange
  - It is a public-key technique

DSA vs. RSA



(a) RSA Approach



(b) DSS Approach

Digital Signature Algorithm (DSA)

- Creates a 320 bit signature with 512-1024 bit security
- Smaller and faster than RSA
- A digital signature scheme only, not for encryption or key exchange
- Security depends on difficulty of computing discrete logarithms
- Details not covered here.

Digital Signature with Key Exchange

- Let's revisit Diffie-Hellman Key Exchange
- A sends to B: g^{xA}, $sign_A(g^{XA})$
- B sends to A: g^{xB}, $sign_B(g^{XB})$
- A and B verify with B's and A's public keys, resp.
- This defends against MITM attack (why?)
- But, how A and B obtain the public keys for signatures in the first place?

Challenge/Response

- A simple way of authentication
  - one party presents a question ("challenge") and another party must provide a valid answer ("response") to be authenticated
- One possible implementation:
  - A sends to B: random number *R*
  - B sends to A: $sign_B(R)$
  - A verifies if $sign_B(R)$ matches *R* using B's public key

Roadmap

- Number theory
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  - RSA
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  - Certificates

Public Key Infrastructure (PKI)

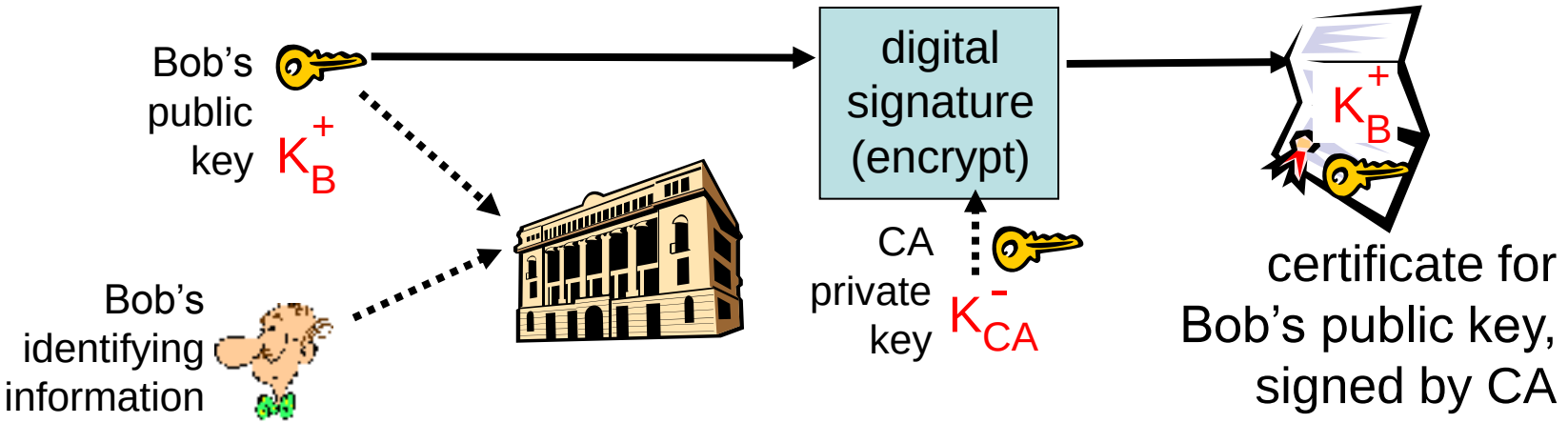
- We need some **trusted public key authority** to authenticate keys
- **Public key infrastructure (PKI)** consists of the components necessary to securely distribute public/private keys.

Public Key Certificates

- **Certificates** allow key exchange without real-time access to public-key authority
- A certificate binds **identity** to **public key**
  - usually with other info such as period of validity, rights of use etc
- with all contents **signed** by a trusted Public-Key or Certificate Authority (CA)
- can be verified by anyone who knows the public-key authorities public-key
  - usually pre-installed in applications

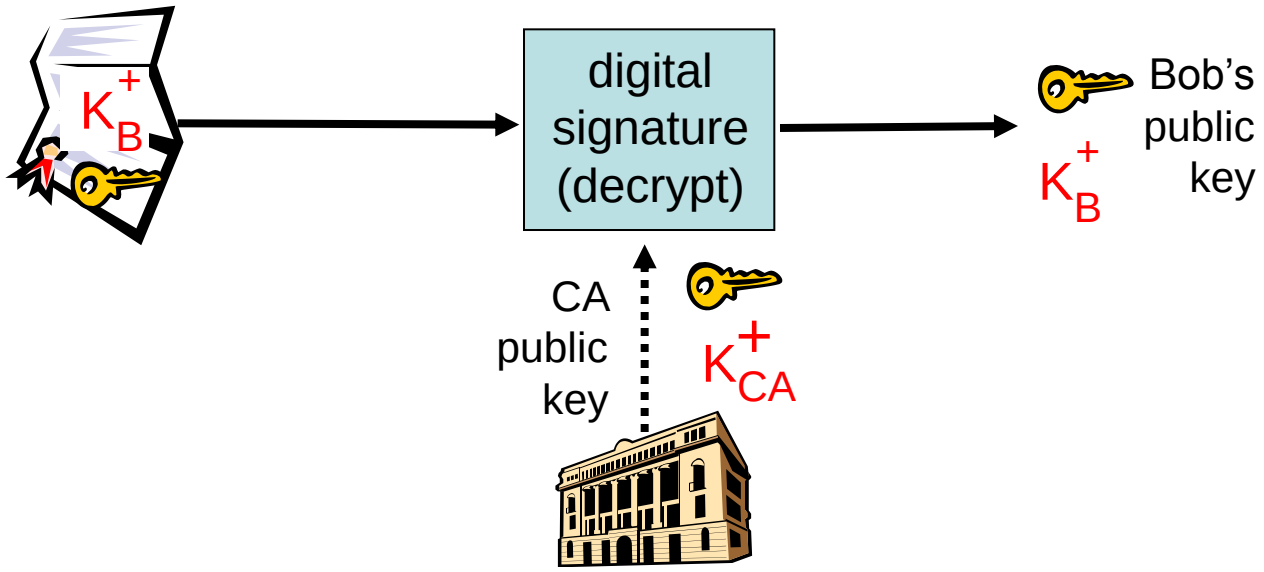
Operations on Certificate

- **Certification authority (CA):** binds public key to particular entity (e.g., Bob) .
- Bob registers his public key with CA.
  - Bob provides "proof of identity" to CA.
  - CA creates certificate binding Bob to his public key.
  - certificate containing Bob's public key digitally signed by CA – CA says "this is Bob's public key"



Operations on Certificates

- When Alice wants Bob's public key:
  - gets Bob's certificate (Bob or elsewhere).
  - apply CA's public key to Bob's certificate, get Bob's public key



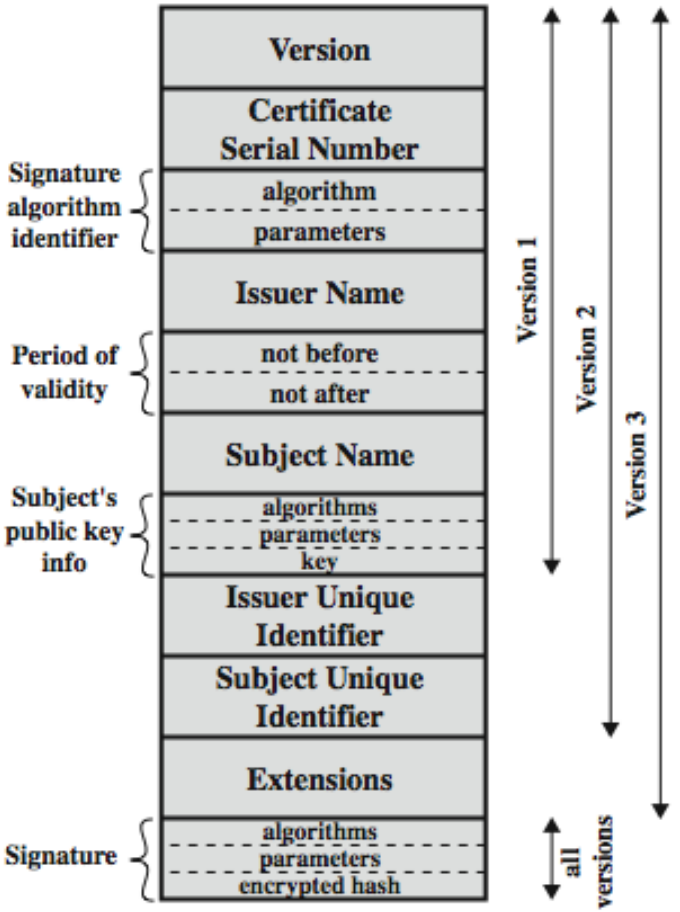
Long-term/Short-term Keys

- Certificates aim to bind keys. But how are they different with the key exchange algorithms (e.g., Diffie-Hellman)?
  - Certificates deal with **long-term public/private keys**, which remain the same for a very long time
  - Key exchange algorithms deal with **short-term public/private keys**, which change in every session

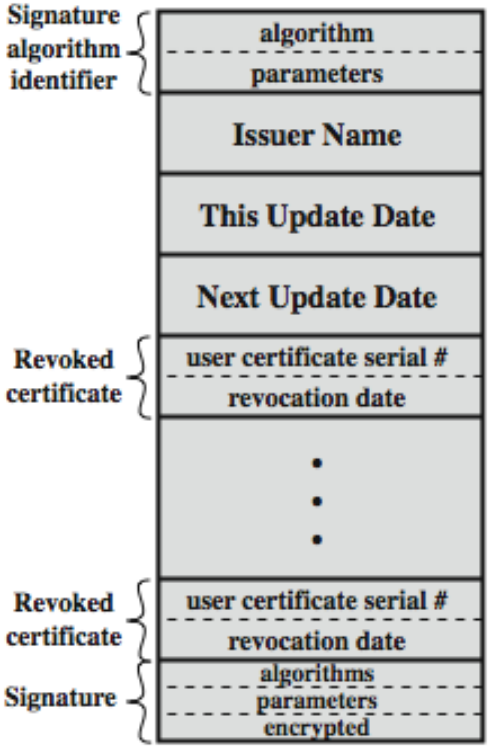
X.509 Authentication Service

- X.509 defines framework for authentication services
  - directory may store public-key certificates
  - with public key of user signed by certification authority
- It also defines authentication protocols
- It uses public-key crypto & digital signatures
  - algorithms not standardised, but RSA recommended
- X.509 certificates are widely used
  - have 3 versions

X.509 Certificate



(a) X.509 Certificate



(b) Certificate Revocation List

CRL discussed later

Obtaining a Certificate

- Any user with access to CA can get any certificate from it
- Only the CA can modify a certificate
- Because certificates cannot be forged, they can be placed in a public directory

CA Hierarchy

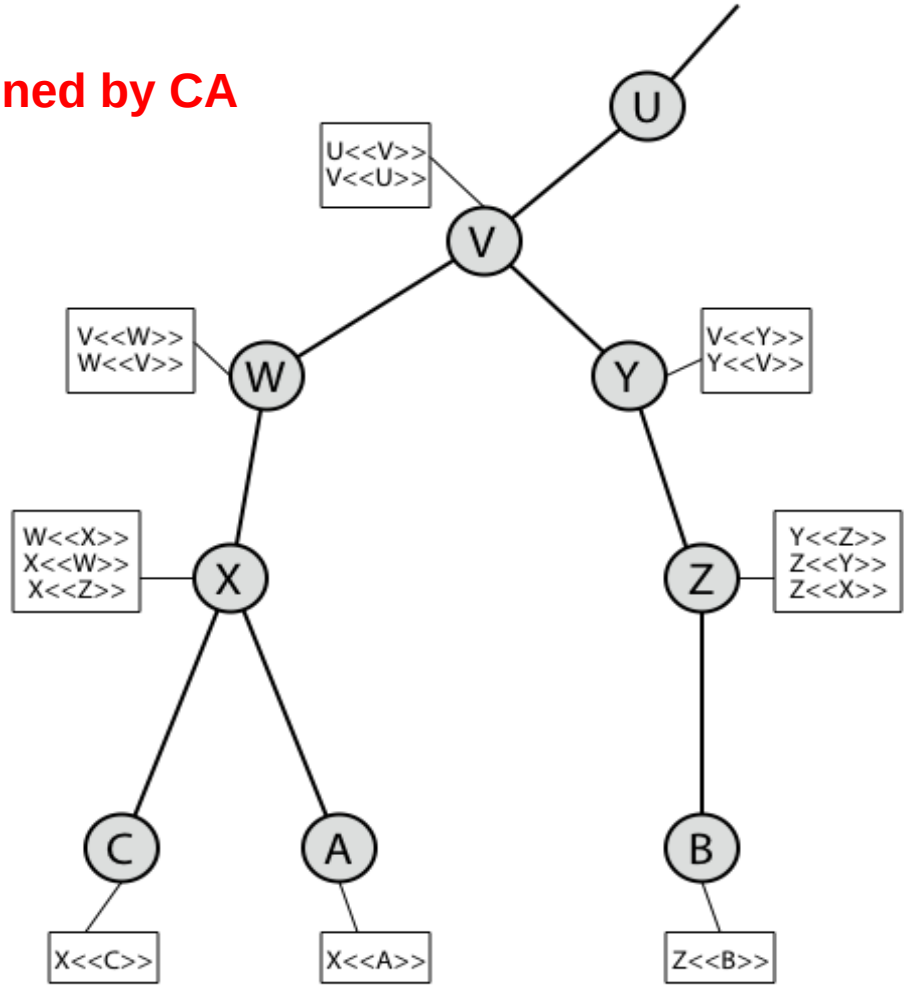
- If both users share a common CA then they are assumed to know its public key
- Otherwise CA's must form a hierarchy
- Use certificates linking members of hierarchy to validate other CA's
  - each CA has certificates for clients (forward) and parent (backward)

CA Hierarchy Use

Denote
CA<<A>> certificate for A signed by CA

X and W can mutually verify each other

Cert of X signed by W
Cert of W signed by X
Cert of Z signed by X



CA Hierarchy Use

- Chains of certificates:
  - A acquires B certificate using chain: X<<W>>W<<V>>V<<Y>>Y<<Z>>Z<<B>>
  - B acquires A certificate using chain: Z<<Y>>Y<<V>>V<<W>>W<<X>>X<<A>>
- Each user only needs to contact one CA (A contacts X, B contacts Z), and unwraps the certificates signed by other CAs through the chain of trusts

Certificate Revocation

- certificates have a period of validity
- may need to revoke before expiry, eg:
  1. user's private key is compromised
  2. user is no longer certified by this CA
  3. CA's certificate is compromised
- CA’s maintain list of revoked certificates
  - the Certificate Revocation List (CRL)
- users should check certificates with CA’s CRL

References

- Kaufman et al., Ch. 5, 6, 7
- Stallings, Ch. 8 – 14
- Kurose & Ross, Ch. 8