Applying the Axioms of Additive Conjoint Measurement to a Hierarchy of Latent Variable Models

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Score scales and latent structure

- We are concerned with our ability to adequately characterize psychological and educational variables.
  - Based on responses to a set of questions, how can we determine the correct generating model?
  - Should we model respondent's differences as qualitative differences? as an order? as distances?
- This issue is not only of theoretical importance, considering that its results provide the foundations of many debates in policy and substantive research.
- In other words, the inferences that we make based on this decisions matter.

A tale of two methods

- There have been a number of theoretical challenges regarding the posibility, in principle, of making this distinctions in psychometric (references).
- In this presentation we present our joint work based on two previous lines of research on this topic.
  - Torres Irribarra and Diakow – Can we determine the underlying structure (qualitative, ordinal, interval) by fit comparisons between models that place progressively more stringent monotonicity and scale assumptions on the latent variable?
  - Domingue – Do data possess sufficient structure, as judged by the axioms of conjoint measurement, to yield interval scales?

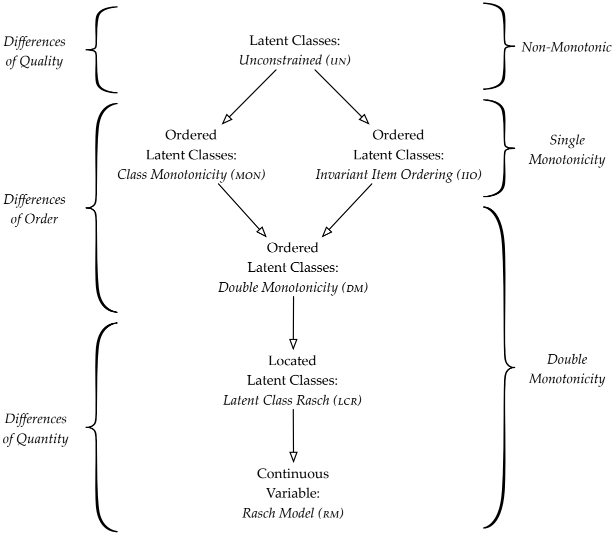
Organizing Models According to their Latent Structure

Our proposed framework organizes these basic latent variable models according to two kinds of restrictions placed on the latent variable:

Monotonicity, which implies that the probability a correct response is a non-decreasing function of increasing proficiency and/or is a non-increasing function of increasing difficulty.

Scale, which implies that the differences between persons (and items) are quantitative in nature.

The Tenable Assessment Framework of Latent Variable Models



Six Latent Variable Models

1. Unconstrained Latent Class Model (UN):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic}$$

2. Ordered Latent Class Model with Class Monotonicty (MON):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i$$

3. Ordered Latent Class Model with Invariant Item Ordering (IIO):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c$$

Six Latent Variable Models (cont.)

4. Ordered Latent Class Model with Double Monotonicity (DM):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i,$$
$$\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c$$

5. Located Latent Class Model or Latent Class Rasch Model (LCR):

$$logit[Pr(x_{ic} = 1|\theta_c, \delta_i)] = \theta_c - \delta_i$$

6. Rasch Model (RSH):

$$logit[Pr(x_{ip} = 1|\theta_p, \delta_i)] = \theta_p - \delta_i$$

Additive Conjoint Measurement (ACM)

- The axioms of additive conjoint measurement provide a means of testing the hypothesis that testing data can be placed onto a scale with equal-interval properties.
  - Solvability,
  - the Archimedean Condition and,
  - Cancellation axioms.
- However, the axioms are difficult to verify given that item responses may be subject to measurement error.

Cancellation axioms

Single cancellation: Rows or columns can be consistently ordered. Easiest to see in a 3×3 matrix formed by the selection of 3 items and 3 abilities:

$$\begin{array}{ccc} \cdot & x_{12} & x_{13} \\ x_{21} & \cdot & x_{23} \\ x_{31} & x_{32} & \cdot \end{array}.$$

Two forms:

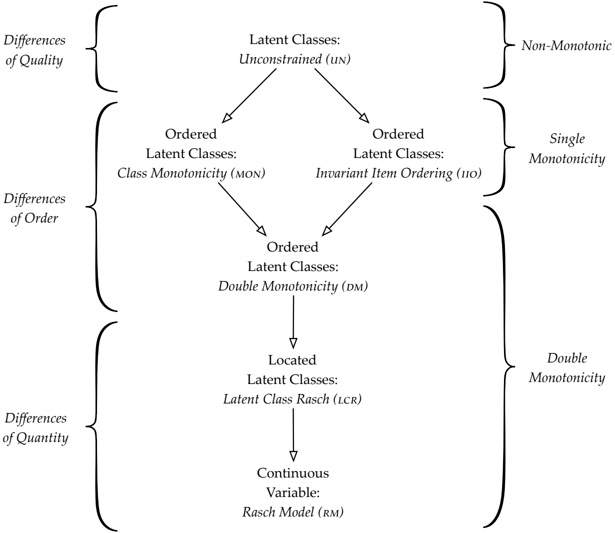
1. If $x_{21} < x_{12} \& x_{32} < x_{23} \text{then} x_{31} < x_{13}$.
2. If $x_{21} > x_{12} \& x_{32} > x_{23} \text{then} x_{31} > x_{13}$.

Key is that double cancellation imposes some order on the very messy minor diagonal.

Applying the axioms of ACM

- Domingue expanded and improved a Bayesian method for imposing order restrictions from additive conjoint measurement while estimating the probability of a correct response.
- This method allows the examination of the single and double cancelation axioms by examining the relevant cancellation constraints and estimating credible intervals to check if observed data laid within.
- Domingue (In Press) applied the methodology to confirm if the use of the Rasch Model was supported by a reading assessment intentionally designed to support an equal-interval scaling.

The Framework and the Axioms



Hypotheses/Questions

1. Data generated under an UN model should yield the most violations.
2. The ammount of single cancelation violations under the MON and IIO models should be similar.
3. The DM model should comply with both single cancelations, but not with double cancelation.
4. The LCR and RM should have the smalles number of violations.
5. In summ, the expected order is:
   - UN > (MON $\sim$ IIO) > DM > (LCR $\sim$ RM)

If these hypotheses hold, then we potentially have a fairly straightforward criteria to recover the generating latent structure.

Simulation design

generate data under each of six models
1000 people in 6 latent classes and 50 items in 6 latent groups
50 replications for each model

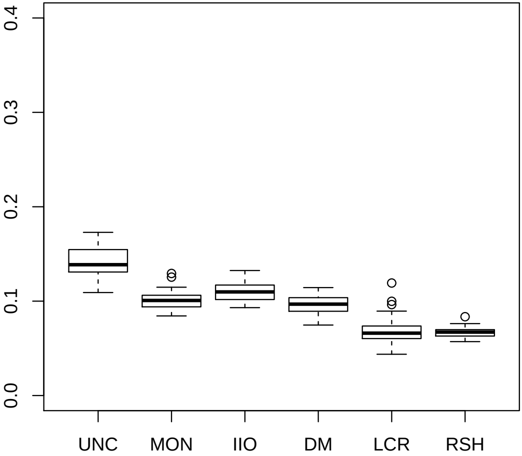
Analysis

ConjointChecks, each single cancellation and double cancellation
% weighted violations

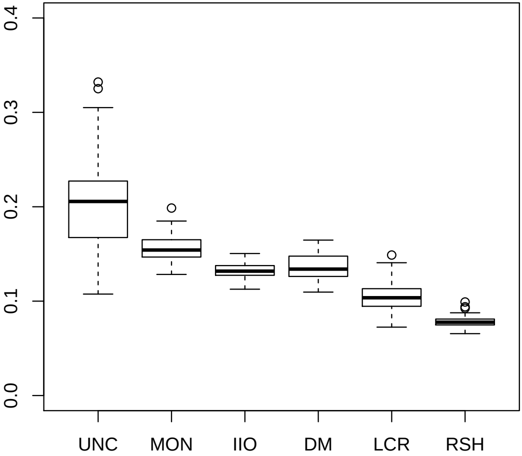
Double cancellation

[include figure]

Single cancellation – Person ordering



Single cancellation – Item ordering



Person monotonicity versus item ordering

precision / aggregation
how this simulation is different from how we usually treat persons and items

Reconsidering double cancellation

Further thoughts

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Appendix Slides