Applying the Axioms of Additive Conjoint Measurement to a Hierarchy of Latent Variable Models

Ronli Diakow[1]* Benjamin Domingue[2]*
David Torres Irribarra[3]*

[1]New York University

[2]University of Colorado at Boulder

[3]University of California, Berkeley

International Meeting of the Psychometric Society
July 25, 2013

*Authors are listed in alphabetical order.

Score scales and latent structure

- We want to appropriately characterize psychological and educational variables.
  - Are differences between respondents categorical? ordered? distances?
  - How can we determine the ("correct") generating model?
- This issue has both theoretical and practical importance.
  - Foundation of many debates in policy and substantive research
  - Any decision matters for subsequent inferences

A tale of two methods

- Joint work based on two previous lines of research:
  - Torres Irribarra and Diakow – Can we determine the structure (qualitative, ordinal, interval) of a latent variable by model fit comparisons within a hierarchical model framework that places progressively more stringent monotonicity and scale assumptions on the latent variable?
  - Domingue – Do data possess sufficient structure, as judged by the axioms of conjoint measurement, to yield interval scales?

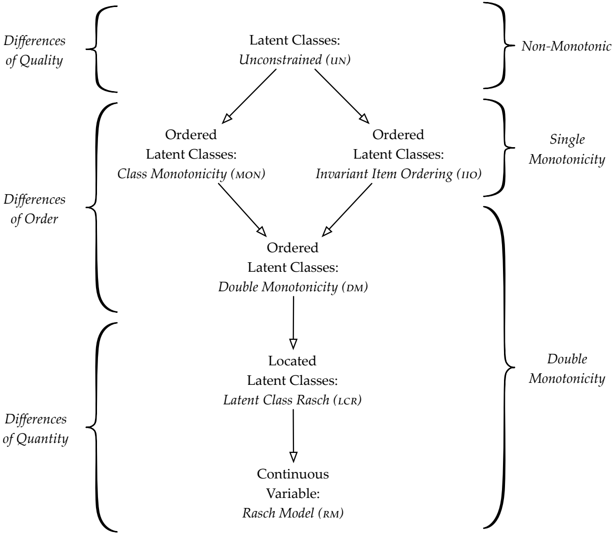
Organizing models by their latent structure

Two kinds of restrictions that a model places on the latent variable:

Monotonicity, which implies that the probability of a correct response is a non-decreasing function of increasing proficiency and/or is a non-increasing function of increasing difficulty.

Scale, which implies that the differences between persons (and items) are quantitative (i.e. interval) in nature.

A hierarchy of latent variable models



Six latent variable models

1. Unconstrained Latent Class Model (UN):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic}$$

2. Ordered Latent Class Model with Class Monotonicty (MON):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i$$

3. Ordered Latent Class Model with Invariant Item Ordering (IIO):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c$$

Six latent variable models (cont.)

4. Ordered Latent Class Model with Double Monotonicity (DM):

$$\begin{aligned}
\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic}, \\
\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i, \\
\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c
\end{aligned}$$

5. Located Latent Class Model or Latent Class Rasch Model (LCR):

$$\text{logit}[Pr(x_{ic} = 1|\theta_c, \delta_i)] = \theta_c - \delta_i$$

6. Rasch Model (RM):

$$\text{logit}[Pr(x_{ip} = 1|\theta_p, \delta_i)] = \theta_p - \delta_i$$

Additive Conjoint Measurement (ACM)

- Axioms of additive conjoint measurement:
  - Describe conditions that need to be satisfied for an interval scale to exist for an attribute
  - Provide a means of testing the hypothesis that data is consistent with an interval scale
  - Are difficult to verify given (likely) measurement error
- Focus on the cancelation axioms

Cancelation axioms

Consider a 3×3 probability matrix formed by the selection of 3 person abilities and 3 item difficulties:

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix}$$

- Single cancelation: Rows (people) or columns (items) can be consistently ordered.
  - Implies that the major (left-leaning) diagonal is ordered.
- Examples:
  - Rows/People: If $x_{11} < x_{21}$, then $x_{12} < x_{22}$ & $x_{13} < x_{23}$.
  - Columns/Items: If $x_{11} < x_{12}$, then $x_{21} < x_{22}$ & $x_{31} < x_{32}$.

Cancelation axioms

Consider a 3×3 probability matrix formed by the selection of 3 person abilities and 3 item difficulties:

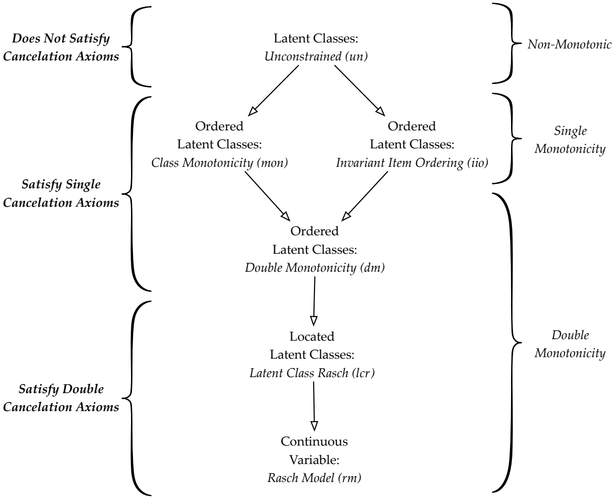
$$\begin{array}{ccc} \cdot & x_{12} & x_{13} \\ x_{21} & \cdot & x_{23} \\ x_{31} & x_{32} & \cdot \end{array}$$

- Double cancelation: Additive constraints
  - Imposes some order on the (very messy) minor (right-leaning) diagonal.
- Two forms:
  - If $x_{21} < x_{12}$ & $x_{32} < x_{23}$ then $x_{31} < x_{13}$.
  - If $x_{21} > x_{12}$ & $x_{32} > x_{23}$ then $x_{31} > x_{13}$.

Applying the axioms of ACM

- Domingue expanded and improved a Bayesian method for checking the single and double cancelation constraints
  - Accounts for measurement error in the observed proportions
- The method implemented in the R package ConjointChecks:
  - Estimate the posterior for the probability of a correct response within each cell using relevant cancelation constraints to form the jumping distribution
  - Check if the observed proportions correct fit within the estimated 95% credible interval

The framework and the axioms



Hypotheses

Model	Satisfies cancelation axiom		
	Single		Double
	Rows	Columns	
UN			
MON	✓		
IIO		✓	
DM	✓	✓	
LCR	✓	✓	✓
RM	✓	✓	✓

Expected order for number of violations of cancelation axioms:

$$\text{UN} > (\text{MON} \overset{?}{\sim} \text{IIO}) > \text{DM} > (\text{LCR} \overset{?}{\sim} \text{RM})$$

This should lead to fairly straightforward criteria to recover the generating latent structure. **However...**

Hypotheses

Model	Satisfies cancelation axiom		
	Single		Double
	Rows	Columns	
UN			
MON	✓		
IIO		✓	
DM	✓	✓	
LCR	✓	✓	✓
RM	✓	✓	✓

Expected order for number of violations of cancelation axioms:

$$\text{UN} > (\text{ MON} \overset{?}{\sim} \text{IIO }) > \text{DM} > (\text{ LCR} \overset{?}{\sim} \text{RM })$$

This should lead to fairly straightforward criteria to recover the generating latent structure. **However...**

Hypotheses

Model	Satisfies cancelation axiom		
	Single		Double
	Rows	Columns	
UN			
MON	✓		
IIO		✓	
DM	✓	✓	
LCR	✓	✓	✓
RM	✓	✓	✓

Expected order for number of violations of cancelation axioms:

$$\text{UN} > (\text{MON} \overset{?}{\sim} \text{IIO}) > \text{DM} > (\text{LCR} \overset{?}{\sim} \text{RM})$$

This should lead to fairly straightforward criteria to recover the generating latent structure. However...

Hypotheses

Model	Satisfies cancelation axiom		
	Single		Double
	Rows	Columns	
UN			
MON	✓		
IIO		✓	
DM	✓	✓	
LCR	✓	✓	✓
RM	✓	✓	✓

Expected order for number of violations of cancelation axioms:

$$\text{UN} > (\text{ MON} \overset{?}{\sim} \text{IIO }) > \text{DM} > (\text{ LCR} \overset{?}{\sim} \text{RM })$$

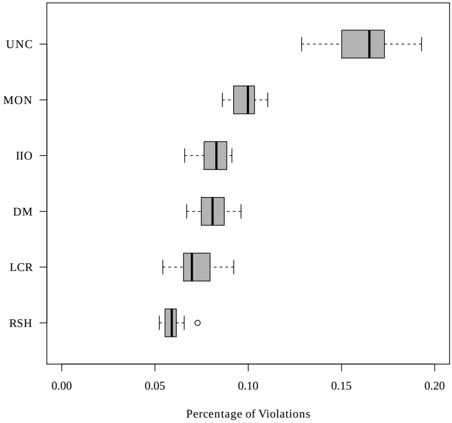
This should lead to fairly straightforward criteria to recover the generating latent structure. **However...**

Simulation design and analysis

- Generate data under each of six models
  1. Use original data from Torres Irribarra and Diakow
     - 5000 people in 2-6 classes
     - 10 items
     - 30 replications per model
  2. Simulate new data
     - 1000 people in 6 classes
     - 50 items in 6 groups
     - 50 replications per model
- Check for violations of the cancelation axioms using ConjointChecks
  - Double cancelation and each single cancelation
- Record the % of (weighted) violations for each model

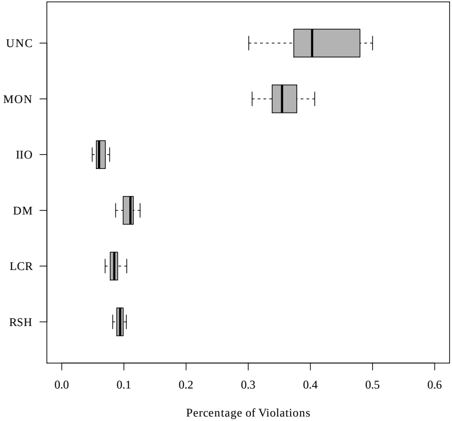
Simulation 1: Results for single cancelation

Person ordering

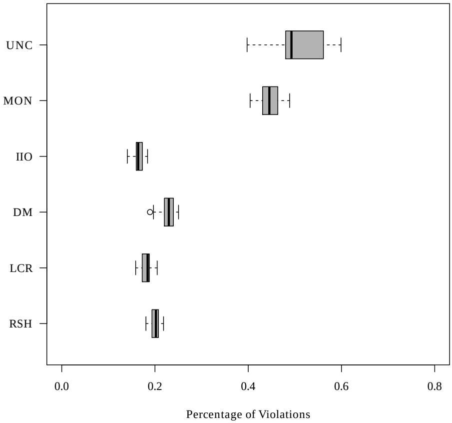


Simulation 1: Results for single cancelation

Item ordering



Simulation 1: Results for double cancelation

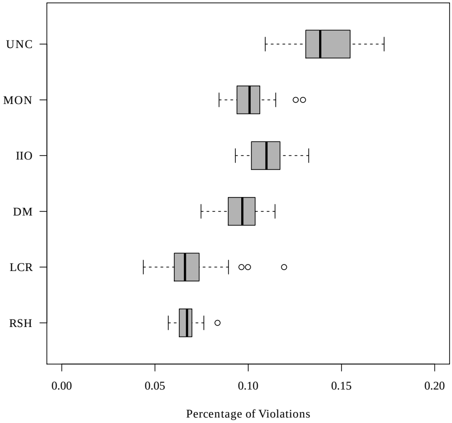


Simulation 1: Results summary

Model	Average percentage of violations		
	Single		Double
	Rows	Columns	
UN	0.16	0.41	0.51
MON	0.10	0.36	0.45
IIO	0.08	0.06	0.16
DM	0.08	0.11	0.23
LCR	0.07	0.08	0.18
RM	0.06	0.09	0.20

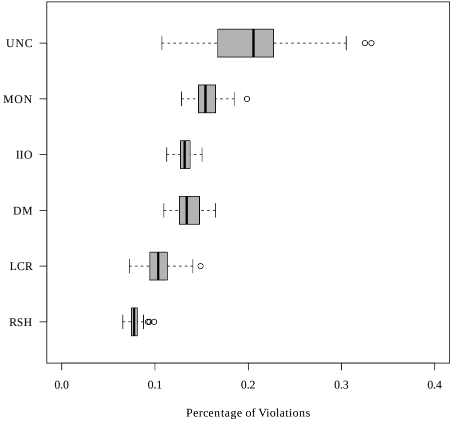
Simulation 2: Single cancelation

Person ordering

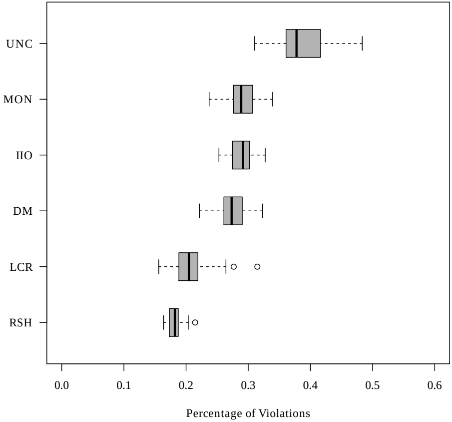


Simulation 2: Single cancelation

Item ordering



Simulation 2: Double cancelation



Simulation 2: Results summary

Model	Average percentage of violations		
	Single		Double
	Rows	Columns	
UN	0.21	0.14	0.38
MON	0.16	0.10	0.29
IIO	0.13	0.11	0.29
DM	0.14	0.10	0.27
LCR	0.10	0.07	0.21
RM	0.08	0.07	0.18

Discussion

- Person monotonicity versus item ordering
  - Precision and/or aggregation
  - Formally, people and items are symmetric; in the real world, we rarely treat them symmetrically.
- Reconsidering double cancelation
  - Checking for double cancelation did not add stringency
  - Observed data is very noisy relative to differences in true probabilities
  - Casts doubt on the utility of this method for checking double cancelation

Next steps

- Can we manipulate data generation such that we can accurately predict changes in the proportion of violations?
- Are the double cancelation results idiosyncratic, or do they apply more generally?

Applying the Axioms of Additive Conjoint Measurement to a Hierarchy of Latent Variable Models

ben.domingue@gmail.com	Benjamin Domingue
dti@berkeley.edu	David Torres Irribarra
rdiakow@berkeley.edu	Ronli Diakow

Appendix Slides