Applying the Axioms of Additive Conjoint Measurement to a Hierarchy of Latent Variable Models

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Score scales and latent structure

- We want to adequately characterize psychological and educational variables.
  - Are differences between respondents categorical? ordered? distances?
  - How can we determine the ("correct") generating model?
- This issue has both theoretical and practical importance.
  - Foundation of many debates in policy and substantive research
  - Any decision matters for subsequent inferences

A tale of two methods

- Joint work based on two previous lines of research:
    - Torres Irribarra and Diakow – Can we determine the structure (qualitative, ordinal, interval) of a latent variable by model fit comparisons within a hierarchical model framework that places progressively more stringent monotonicity and scale assumptions on the latent variable?
    - Domingue – Do data possess sufficient structure, as judged by the axioms of conjoint measurement, to yield interval scales?

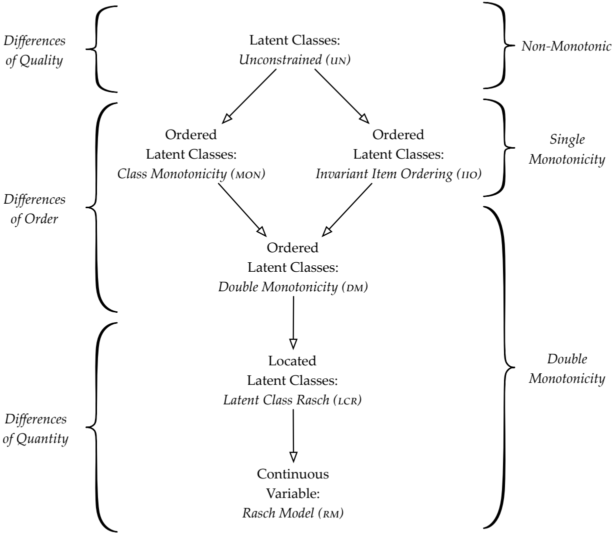
Organizing models by their latent structure

Two kinds of restrictions that a model places on the latent variable:

Monotonicity, which implies that the probability a correct response is a non-decreasing function of increasing proficiency and/or is a non-increasing function of increasing difficulty.

Scale, which implies that the differences between persons (and items) are quantitative (i.e. interval) in nature.

A hierarchy of latent variable models



Six latent variable models

1. Unconstrained Latent Class Model (UN):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic}$$

2. Ordered Latent Class Model with Class Monotonicty (MON):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i$$

3. Ordered Latent Class Model with Invariant Item Ordering (IIO):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c$$

Six latent variable models (cont.)

4. Ordered Latent Class Model with Double Monotonicity (DM):

$$\text{logit}[Pr(x_{ic} = 1|c)] = \text{logit}[\pi_{i|c}] = \beta_{ic},$$
$$\beta_{ic} \leq \beta_{ic'} \text{ for all } c < c' \text{ and for all } i,$$
$$\beta_{ic} \leq \beta_{i'c} \text{ for all } i < i' \text{ and for all } c$$

5. Located Latent Class Model or Latent Class Rasch Model (LCR):

$$logit[Pr(x_{ic} = 1|\theta_c, \delta_i)] = \theta_c - \delta_i$$

6. Rasch Model (RM):

$$logit[Pr(x_{ip} = 1|\theta_p, \delta_i)] = \theta_p - \delta_i$$

Additive Conjoint Measurement (ACM)

- Axioms of additive conjoint measurement:
  - Describe conditions that need to be satisfied for an interval scale to exist for an attribute
  - Provide a means of testing the hypothesis that data is consistent with an interval scale
  - Are difficult to verify given (likely) measurement error
- Focus on the cancelation axioms

Cancelation axioms

Consider a 3×3 probability matrix formed by the selection of 3 item difficulties and 3 person abilities:

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix}$$

- Single cancelation: Rows or columns can be consistently ordered.
  - Implies that the major (left-leaning) diagonal is ordered.
- Examples:
  - Rows: If $x_{11} < x_{21}$, then $x_{12} < x_{22}$ & $x_{13} < x_{23}$.
  - Columns: If $x_{11} < x_{12}$, then $x_{21} < x_{22}$ & $x_{31} < x_{32}$.

Cancelation axioms

Consider a 3×3 probability matrix formed by the selection of 3 item difficulties and 3 person abilities:

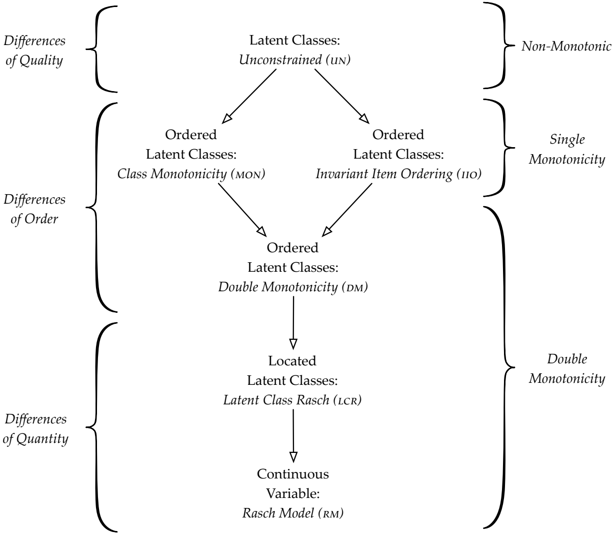
$$\begin{array}{ccc} \cdot & x_{12} & x_{13} \\ x_{21} & \cdot & x_{23} \\ x_{31} & x_{32} & \cdot \end{array}$$

- Double cancelation: Additive constraints
  - Imposes some order on the (very messy) minor (right-leaning) diagonal.
- Two forms:
  - If $x_{21} < x_{12}$ & $x_{32} < x_{23}$ then $x_{31} < x_{13}$.
  - If $x_{21} > x_{12}$ & $x_{32} > x_{23}$ then $x_{31} > x_{13}$.

Applying the axioms of ACM

- Domingue expanded and improved a Bayesian method for checking the single and double cancelation constraints
  - Accounts for measurement error in the observed proportions
- The method implemented in the R package ConjointChecks:
  - Estimate the posterior for the probability of a correct response within each cell using relevant cancelation constraints to form the jumping distribution
  - Check if the observed proportions correct fit within the estimated 95% credible interval

The Framework and the Axioms



Hypotheses

The expected order for the number of violations of the cancelation axioms is UN > (MON $\sim$ IIO) > DM > (LCR $\sim$ RM)

Model	Violates cancelation axiom		
	Single		Double
	Rows	Columns	
UN	✓	✓	✓
MON		✓	✓
IIO	✓		✓
DM			✓
LCR			
RM			

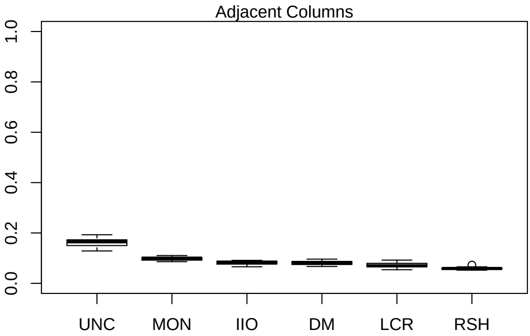
If these hypotheses hold, then we potentially have a fairly straightforward criteria to recover the generating latent structure.

Simulation design and analysis

- Generate data under each of six models
  1. Use original data from Diakow and Torres Irribarra
     - 5000 people in 2-6 classes
     - 10 items
     - 30 replications per model
  2. Simulate new data
     - 1000 people in 6 classes
     - 50 items in 6 groups
     - 50 replications per model
- Check for violations of the cancelation axioms using ConjointChecks
  - Double cancelation and each single cancelation separately
- Record the % of (weighted) violations for each model

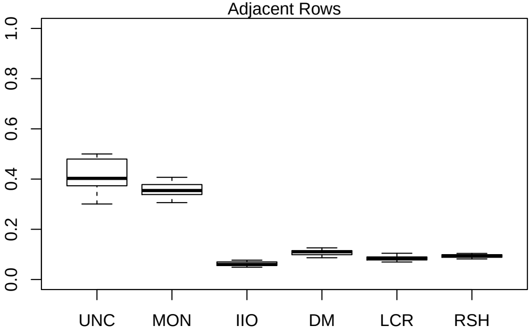
Simulation 1: Results for single cancelation

Person ordering

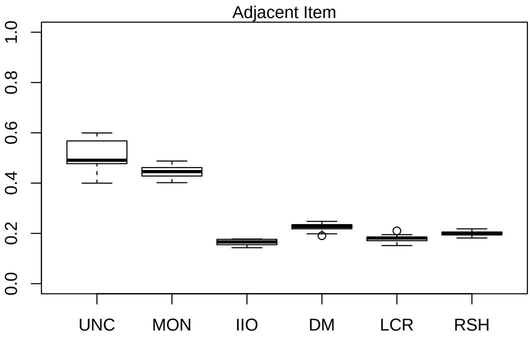


Simulation 1: Results for single cancelation

Item ordering

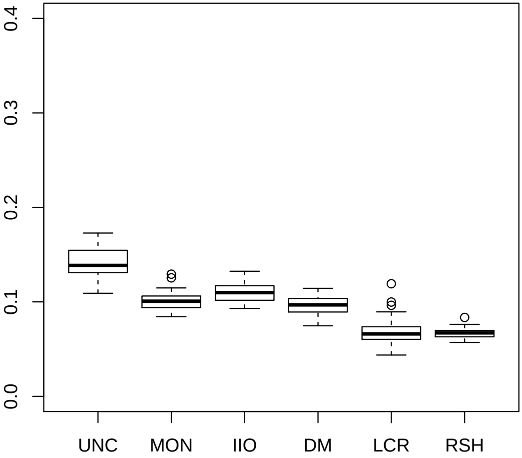


Simulation 1: Results for double cancelation



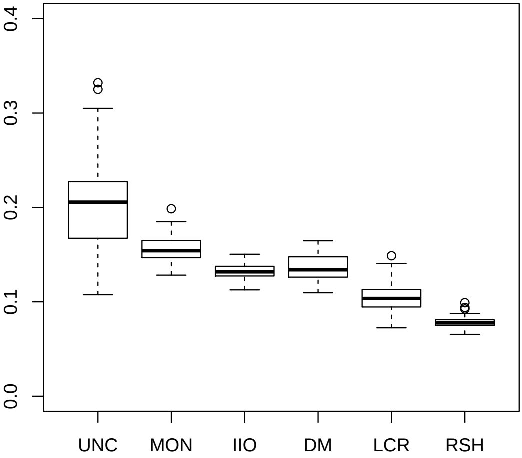
Simulation 2: Single cancelation

Person ordering



Simulation 2: Single cancelation

Item ordering



Simulation 2: Double cancelation

[include figure]

Discussion

- Person monotonicity versus item ordering
  - precision / aggregation
  - we don't usually treat persons and items as symmetric, though the models/checks formally do
- Reconsidering double cancelation
  - standard testing data too noisy to check this?

Summary

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Appendix Slides