THE BERNOULLI DISTRIBUTION

The Bernoulli Distribution.

Learning outcomes:

- To know the definition of a Bernoulli random variable

- To be able to calculate the mean and variance of a Bernoulli random variable.

The **Bernoulli distribution** arises from considering the result of a **single trial** of an experiment.

A **Bernoulli random variable** X can take only two possible values: 1 to represent "success", and 0 to represent "failure".

We define p as the probability of success, so $P(X = 1) = p$, and $P(X = 0) = 1 - p$, where $0 \leqslant p \leqslant 1$.

The probability distribution for a Bernoulli random variable X with probability of success p is shown in the table.

x_i	0	1
p_i	$1-p$	p

Now
$$\begin{aligned} E(X) &= \sum x_i p_i \\ &= 0(1-p) + 1(p) \\ &= p \end{aligned}$$

and
$$\begin{aligned} \mathrm{Var}(X) &= \sum x_i^{\,2} p_i - \mu^2 \\ &= 0^2(1-p) + 1^2(p) - p^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

A Bernoulli random variable X with probability of success p has:

- mean $\mathrm{E}(X) = p$
- variance $\mathrm{Var}(X) = \sigma^2 = p(1-p)$
- standard deviation $\sigma = \sqrt{p(1-p)}$

A Bernoulli sequence has the following properties:

- Each trial results in one of two outcomes
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent (the outcome of any trial is not affected by the outcome of any previous trial)

A pen is randomly selected from a box containing 7 red pens and 3 blue pens. Let $X = 1$ if the selected pen is red, and $X = 0$ if the selected pen is blue.

a Explain why X is a Bernoulli random variable.

b Find the mean and standard deviation of X.

a X can only take two possible values: 1 or 0.

$\mathrm{P}(X = 1) = \frac{7}{10} = p$

$\mathrm{P}(X = 0) = \frac{3}{10} = 1 - p$

$\therefore$ X is a Bernoulli random variable with probability of success $p = \frac{7}{10}$.

b The mean $\mathrm{E}(X) = p = \frac{7}{10}$

The standard deviation $\sigma = \sqrt{p(1-p)}$

$$= \sqrt{\frac{7}{10} \times \frac{3}{10}}$$

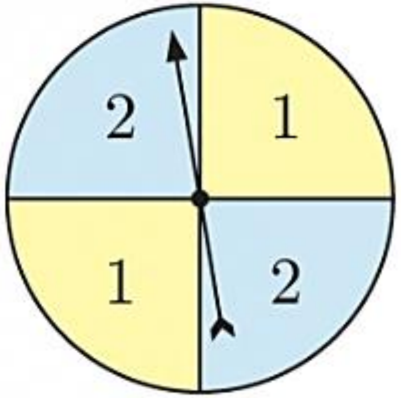
$$\approx 0.458$$

1 Decide whether X is a Bernoulli random variable:

a A coin is tossed. $X = 1$ if the result is heads, $X = 0$ if the result is tails.

b Two coins are tossed. $X =$ the number of heads tossed.

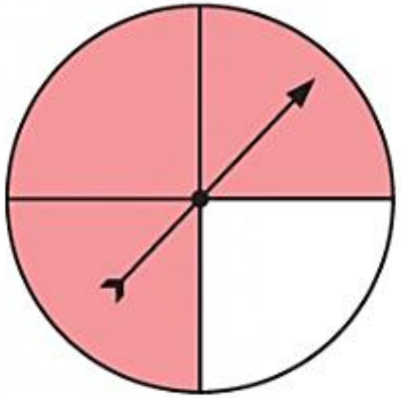
c $X =$ the result when the spinner alongside is spun once.



d A person enters a store. $X = 1$ if the person purchases something, $X = 0$ otherwise.

e A letter of the alphabet is chosen at random. $X =$ the number of vowels chosen.

2 Consider the spinner alongside. Let $X = 1$ if the outcome is red, and $X = 0$ if the outcome is white.



a Explain why X is a Bernoulli random variable.

b Find:

i the mean of X **ii** the standard deviation of X.

1 **a** yes **b** no **c** no **d** yes **e** yes

2 **a** $P(X = 1) = \frac{3}{4} = p$, $P(X = 0) = \frac{1}{4} = 1 - p$

b **i** $\frac{3}{4}$ **ii** $\frac{\sqrt{3}}{4} \approx 0.433$

3 A regular six-sided die is rolled once. Let X represent the number of sixes rolled.

a Explain why X is a Bernoulli random variable.

b Find:

i the mean of X **ii** the variance of X.

4 Let $X = 1$ if there is rain tomorrow, and $X = 0$ if there is no rain tomorrow. The probability of rain tomorrow is $p = 0.4$.

a Construct the probability distribution for X.

b Find the mode of the distribution.

c Find:

i $\mathrm{E}(X)$ **ii** $\mathrm{Var}(X)$.

5 Tokens numbered 1 to 20 are placed in a bag, and one is selected at random. Let $X = 1$ if a composite number is selected, and $X = 0$ otherwise. Let $Y = 1$ if a number greater than 5 is selected, and $Y = 0$ otherwise.

a Find the probability of success for each distribution.

b Which random variable has the higher:

i mean **ii** standard deviation?

3 **a** $\mathrm{P}(X = 1) = \frac{1}{6} = p$, $\mathrm{P}(X = 0) = \frac{5}{6} = 1 - p$

b **i** $\frac{1}{6}$ **ii** $\frac{5}{36}$

4 **a**

x_i	0	1
p_i	0.6	0.4

b 0

c **i** $\mathrm{E}(X) = 0.4$ **ii** $\mathrm{Var}(X) = 0.24$

5 **a** X: $p = \frac{11}{20}$, Y: $p = \frac{3}{4}$ **b** **i** Y **ii** X

6 Suppose X is a Bernoulli random variable with probability of success p. Find the value of p such that $\mathrm{Var}(X)$ is maximised.

6 $p = \frac{1}{2}$

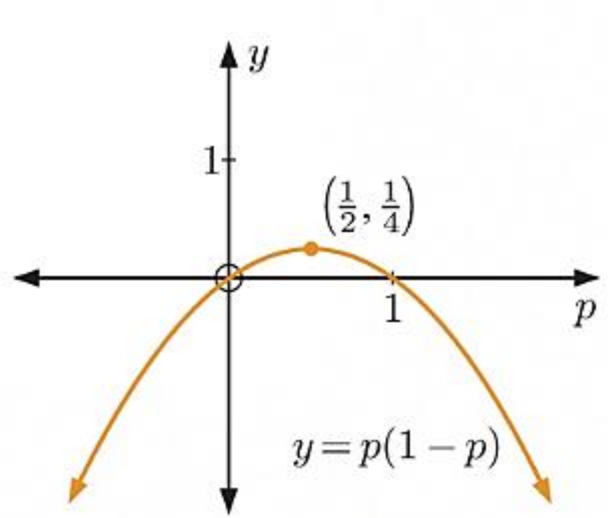
$$Var(X) = p(1-p)$$
$$= p - p^2$$

$$\frac{dX}{dp} = 1 - 2p$$

Let $\frac{dX}{dp} = 0$

$$p = \frac{1}{2}$$

Var(X) is maximised when $p = \frac{1}{2}$



Combinations (revision)

A **combination** is a selection of objects *without* regard to order or arrangement.

For example, the possible teams of 3 people selected from A, B, C, D, and E are:

ABC	ABD	ABE	ACD	ACE	ADE
BCD	BCE	BDE			
CDE					

We only list ABC and not ACB, BAC, BCA, CAB, and CBA. These are different *permutations* of the players, but they are all the same *combination*.

There are a total of 10 combinations of the 5 letters taken 3 at a time.

This is written as $C_3^5 = 10$ or $\binom{5}{3} = 10$

$$C_3^5 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = \frac{5!}{3! \times 2!}.$$

The number of combinations on n distinct symbols taken r at a time is

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

On the calculator $math \rightarrow prob \rightarrow 3$

How many different teams of 3 can be selected from a squad of 8 if:

a there are no restrictions **b** the team must include the captain?

a There are 8 players for selection and we can choose any 3 of them.
This can be done in $\binom{8}{3} = 56$ different ways.

b The captain must be included *and* we need any 2 of the other 7.
This can be done in $\binom{1}{1} \times \binom{7}{2} = 21$ different ways.

Recall: $\Pr(E) = \frac{n(E)}{n(s)}$ The probability of an event is the number of ways the event can occur over the number in the sample space (total number of outcomes)

So in the problem above the probability of selecting a team of 3, which contains the captain, from a squad of 8 is:

$$\frac{\binom{1}{1}\binom{7}{2}}{\binom{8}{3}} = 0.375$$

An equestrian team of 4 is to be chosen from 7 male and 6 female riders. How many different teams can be chosen if:

a there are no restrictions

b there must be two riders of each sex

c at least one rider of each sex is required?

a There are $7+6=13$ riders up for selection, and we need to choose any 4 of them. So there are $\binom{13}{4}=715$ different teams.

b The males can be chosen in $\binom{7}{2}$ ways and the females in $\binom{6}{2}$ ways.

$\therefore$ $\binom{7}{2}\times\binom{6}{2}=315$ different teams can be chosen.

c We could have 3 males and 1 female **or** 2 males and 2 females **or** 1 male and 3 females.

$\therefore$ $\binom{7}{3}\times\binom{6}{1}+\binom{7}{2}\times\binom{6}{2}+\binom{7}{1}\times\binom{6}{3}=665$ different teams can be chosen.

Note: Since the only other alternatives are 4M *and* 0F or 0M *and* 4F, we could have used $\binom{13}{4}-\binom{7}{4}\times\binom{6}{0}-\binom{7}{0}\times\binom{6}{4}$

7 How many different teams of 7 can be chosen from a training squad of 15?

8 **a** How many different committees of 5 can be selected from a club of 27 members?

b How many of these committees consist of the president and 4 others?

9 A physics exam contains 7 questions. Students must answer both questions 1 and 2, and any 3 of the remaining questions. How many different selections are possible?

10 A team of 8 is selected from a squad of 14. The captain and vice-captain must both be included. How many different teams can be selected?

7 6435 different teams

8 **a** 80 730 different committees

b 14 950 different committees

9 10 different selections

10 924 different teams

11 The committee of a table tennis club consists of 3 people chosen from 6 male and 8 female members.

a If there are no restrictions, how many different committees can be chosen?

b How many of the different possible committees consist of:

i 3 males **ii** 2 males and 1 female **iii** 1 male and 2 females

iv 3 females?

12 A committee of 4 singers is chosen from 7 tenors and 9 sopranos.
Find the number of ways of selecting the committee if:

a there are no restrictions

b it must contain 2 tenors and 2 sopranos

c it must contain at least 2 sopranos.

11	**a**	364	**b**	**i**	20	**ii**	120	**iii**	168	**iv**	56		
12	**a**	1820	**b**	756		**c**	1470						

EXERCISE 5.01 (PAGE 202)

- Question 1, 4, 6, 7, 8, 10