EXPECTED VALUE

Expected Value

Learning outcomes:

- To understand and be able to calculate the expected value or mean of a discrete random variable.

- To understand and be able to calculate the variance of a discrete random variable

If a die is rolled 120 times, on how many occasions will we *expect* the result to be a "six"?

The possible outcomes from each roll are 1, 2, 3, 4, 5, and 6, and these outcomes are equally likely to occur. We would therefore expect $\frac{1}{6}$ of them to be a "six".

$\frac{1}{6}$ of 120 is 20, so we *expect* 20 of the 120 rolls of the die to yield a "six".

However, this does not mean that we *will* get 20 sixes when we roll a die 120 times.

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we **expect** the event to occur is np.

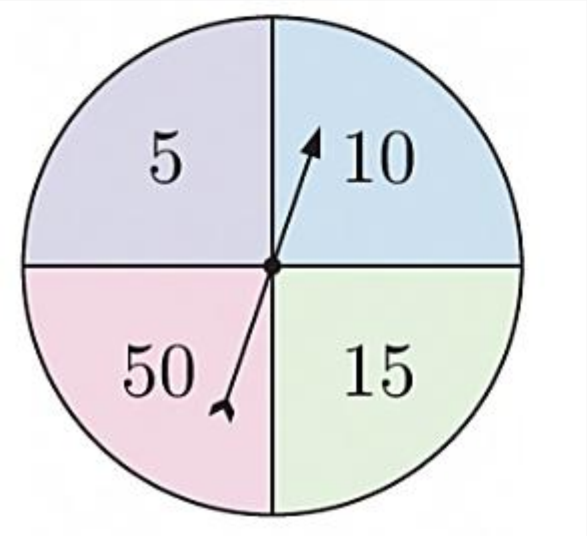
Each time a footballer kicks for goal, he has a $\frac{3}{4}$ chance of being successful.

During a particular game he has 12 kicks for goal. How many goals would you expect him to score?

$p = \mathrm{P}(\text{goal}) = \frac{3}{4}$

$\therefore$ we expect the footballer to score $np = 12 \times \frac{3}{4} = 9$ goals.

When the spinner alongside is spun, players are awarded the resulting number of points. On average, how many points can we *expect* to be awarded per spin?



For every 4 spins, we would expect that on average, each score will be spun once. The total score in this case would be $50+15+10+5=80$, which is an average of $\frac{80}{4}=20$ points per spin.

Alternatively, we can write the average score as

$$\begin{aligned}
&\tfrac{1}{4}(50+15+10+5) \\
&= \tfrac{1}{4}\times 50+\tfrac{1}{4}\times 15+\tfrac{1}{4}\times 10+\tfrac{1}{4}\times 5 \\
&= 20 \text{ points.}
\end{aligned}$$

It is impossible to score 20 points on any given spin, but over many spins we *expect* an average score of 20 points per spin.

For a random variable X with possible values $x_1, x_2, x_3, ..., x_n$ and associated probabilities $p_1, p_2, ..., p_n$, the **expected value** of X is

$$\begin{aligned} \mathrm{E}(X) = \mu &= \sum_{i=1}^{n} x_i p_i \\ &= x_1 p_1 + x_2 p_2 + ... + x_n p_n \end{aligned}$$

A magazine store recorded the number of magazines purchased by its customers in one week. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four, and 5% purchased five. Let X be the number of magazines sold to a randomly selected customer.

Find the expected number of magazines bought by each customer. Explain what this means.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05) \\ &= 2.39\end{aligned}$$

In the long run, the average number of magazines purchased per customer is 2.39.

1 Find $\mathrm{E}(X)$ for the following probability distributions:

a

x_i	1	2	3
p_i	0.4	0.5	0.1

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

1 **a**
$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.4) + 2(0.5) + 3(0.1) \\ &= 1.7\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 0(0.1) + 1(0.2) + 2(0.15) + 3(0.2) + 4(0.35) \\ &= 2.5\end{aligned}$$

c
$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 0(0.2) + 2(0.35) + 5(0.27) + 10(0.18) \\ &= 3.85\end{aligned}$$

d
$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 10\left(\tfrac{1}{4}\right) + 15\left(\tfrac{1}{3}\right) + 30\left(\tfrac{1}{12}\right) + 60\left(\tfrac{1}{3}\right) \\ &= 30\end{aligned}$$

6 Lachlan randomly selects a ball from a bag containing 5 red balls, 2 green balls, and 1 white ball.

He is then allowed to take a number of lollies from a lolly jar. The number of lollies is determined by the colour of the ball, as shown in the table.

Find the average number of lollies that Lachlan would expect to receive.

Colour	*Number of lollies*
Red	4
Green	6
White	10

6 There are a total of $5 + 2 + 1 = 8$ balls.

Number of lollies	4	6	10
Probability	$\frac{5}{8} = 0.625$	$\frac{2}{8} = 0.25$	$\frac{1}{8} = 0.125$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\
&= (4 \times 0.625) + (6 \times 0.25) + (10 \times 0.125) \\
&= 2.5 + 1.5 + 1.25 \\
&= 5.25 \text{ lollies}
\end{aligned}
$$

On average, Lachlan would expect to receive 5.25 lollies.

10 Every Thursday, Zoe meets her friends in the city for dinner. There are two car parks nearby, the costs for which are shown below:

Car park A

Time	*Cost*
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	*Cost*
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

Zoe's dinner takes 1 - 2 hours 20% of the time, 2 - 3 hours 70% of the time, and 3 - 4 hours 10% of the time.

a Which car park is cheapest for Zoe if she stays:

i 1 - 2 hours **ii** 2 - 3 hours **iii** 3 - 4 hours?

b When Zoe parks her car, she does not know how long she will stay. Which car park do you recommend for her? Explain your answer.

10 **a** **i** car park B **ii** car park A **iii** car park B

b Let \$$X$ be the amount Zoe pays for parking. When Zoe parks her car at car park A:

$$\begin{aligned} E(X) &= (7 \times 0) + (12 \times 0.2) + (15 \times 0.7) + (19 \times 0.1) \\ &= 2.4 + 10.5 + 1.9 \\ &= \$14.80 \end{aligned}$$

When Zoe parks her car at car park B:

$$\begin{aligned} E(X) &= (6.5 \times 0) + (11 \times 0.2) + (16 \times 0.7) + (18.5 \times 0.1) \\ &= 2.2 + 11.2 + 1.85 \\ &= \$15.25 \end{aligned}$$

$\therefore$ Zoe should choose car park A as it has the lower expected cost.

Fair Games

In gambling, the **expected gain** of the player from each game is the expected return or payout from the game, less the amount it cost them to play.

The game will be **fair** if the expected gain is zero.

Suppose X represents the gain of a player from each game.
The game is **fair** if $\mathrm{E}(X) = 0$.

In a game of chance, a player spins a square spinner labelled 1, 2, 3, 4. The player wins an amount of money according to the table alongside. Determine:

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a the expected return for one spin of the spinner

b the expected *gain* of the player if it costs \$5 to play each game

c whether you would recommend playing this game.

a Let Y denote the return or payout from each spin.

Each outcome is equally likely, so the probability for each outcome is $\frac{1}{4}$

$\therefore$ the expected return $= \mathrm{E}(Y) = \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4$.

b Let X denote the *gain* of the player from each game.

Since it costs \$5 to play the game, the expected gain $= \mathrm{E}(X) = \mathrm{E}(Y) - \$5$

$$= \$4 - \$5$$
$$= -\$1$$

c Since $\mathrm{E}(X) \neq 0$, the game is not fair. In particular, since $\mathrm{E}(X) = -\$1$, we expect the player to lose \$1 on average with each spin. We would not recommend that a person play the game.

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- Question 1, 2, 3, 6, 14