Stable Finite Difference Approximations

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The purpose of this analysis is to explore stable recursive finite difference equations for application to compartment models.

The logistic growth law is commonly used in many fish models and is a useful starting point. The logistic is often written as its derivative:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Starting with the usual finite difference approximation

$$\frac{dN}{dt} \approx \frac{N_t - N_{t-\Delta t}}{\Delta t} = f(N)_t,$$

the question becomes which time levels, N_t and $N_{t-\Delta t}$, should be used in $f(N)_t$?

There are 4 obvious possibilities:

The first possibility is to require the right hand side to depend entirely on the previous time step:

$$\frac{dN}{dt} \approx \frac{N_t - N_{t-\Delta t}}{\Delta t} = rN_{t-\Delta t}\left(1 - \frac{N_{t-\Delta t}}{K}\right)$$

This approximation leads to

$$N_t = N_{t-\Delta t} + \Delta t r N_{t-\Delta t}\left(1 - \frac{N_{t-\Delta t}}{K}\right) \tag{1}$$

which is sometimes known as "explicit" time stepping.

There are two possibilities where the right hand side depends on different combinations of N_t and $N_{t-\Delta t}$, that is "semi-implicit" time stepping.

$$\frac{dN}{dt} \approx \frac{N_t - N_{t-\Delta t}}{\Delta t} = rN_t\left(1 - \frac{N_{t-\Delta t}}{K}\right)$$

so that

$$N_t = \frac{N_{t-\Delta t}}{1 - \Delta t r - \Delta t \frac{r}{K} N_{t-\Delta t}}. \tag{2}$$

Alternatively,

$$\frac{dN}{dt} \approx \frac{N_t - N_{t-\Delta t}}{\Delta t} = rN_{t-\Delta t}\left(1 - \frac{N_t}{K}\right)$$

so that

$$N_t = \frac{N_{t-\Delta t}(1 + \Delta t r)}{1 + \frac{\Delta t r N_{t-\Delta t}}{K}}. \tag{3}$$

The final possibility is to use fully implicit time stepping where the right hand side depends only on the current time step:

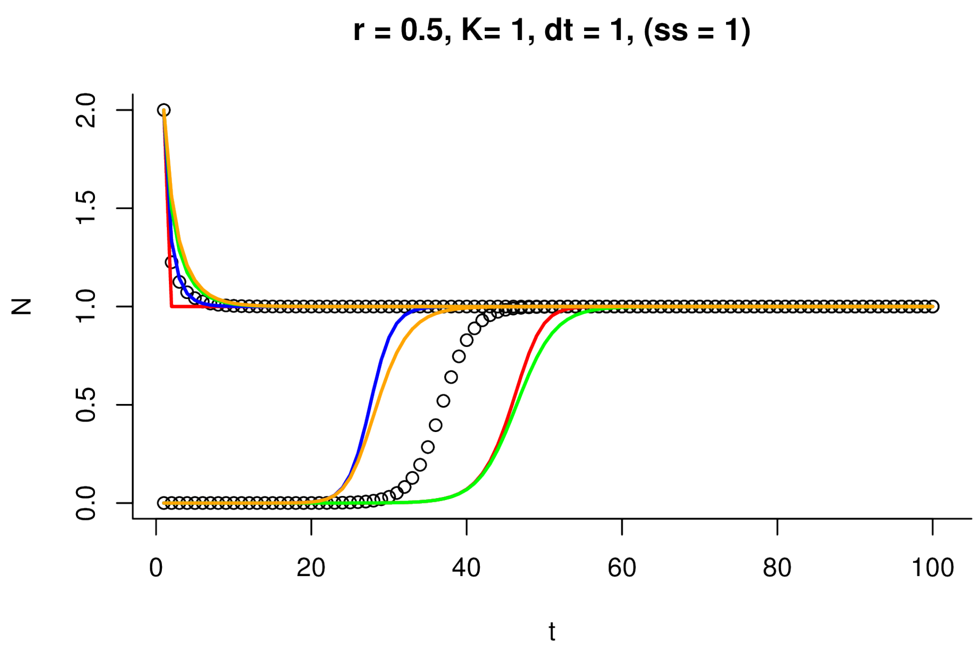
$$\frac{dN}{dt} \approx \frac{N_t - N_{t-\Delta t}}{\Delta t} = rN_t\left(1 - \frac{N_t}{K}\right)$$

This expression can be rearranged into standard quadratic form and solved using the quadratic formula,

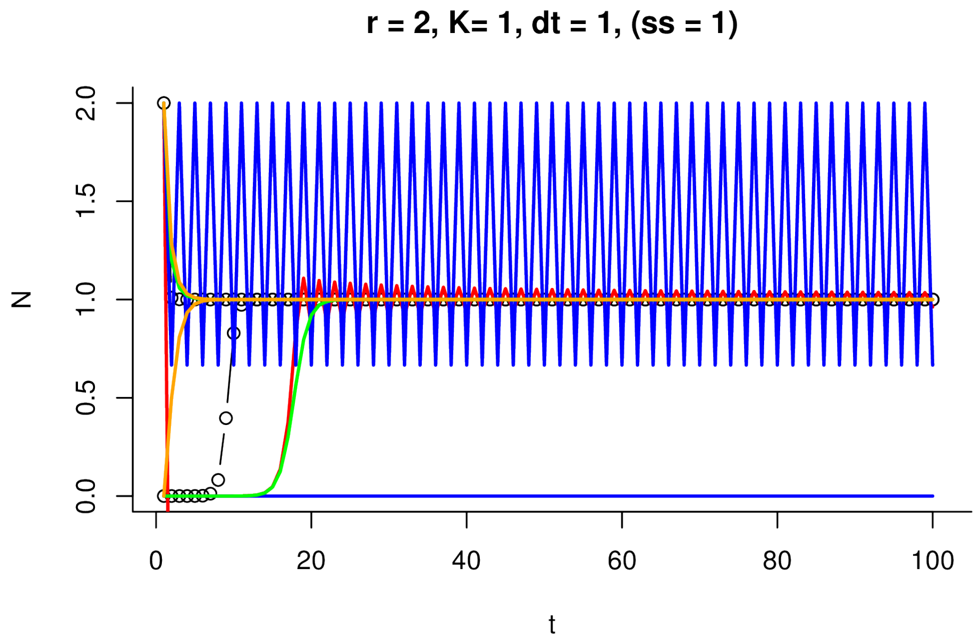
$$\frac{\Delta t r}{K} {N_t}^2 + (1 - \Delta t r)N_t - N_{t-\Delta t} = 0 \tag{4}$$

There are of course two roots; it turns out that one is positive and one is negative. This equation of course be solved by completing the square which might lead to a more instructive formula.

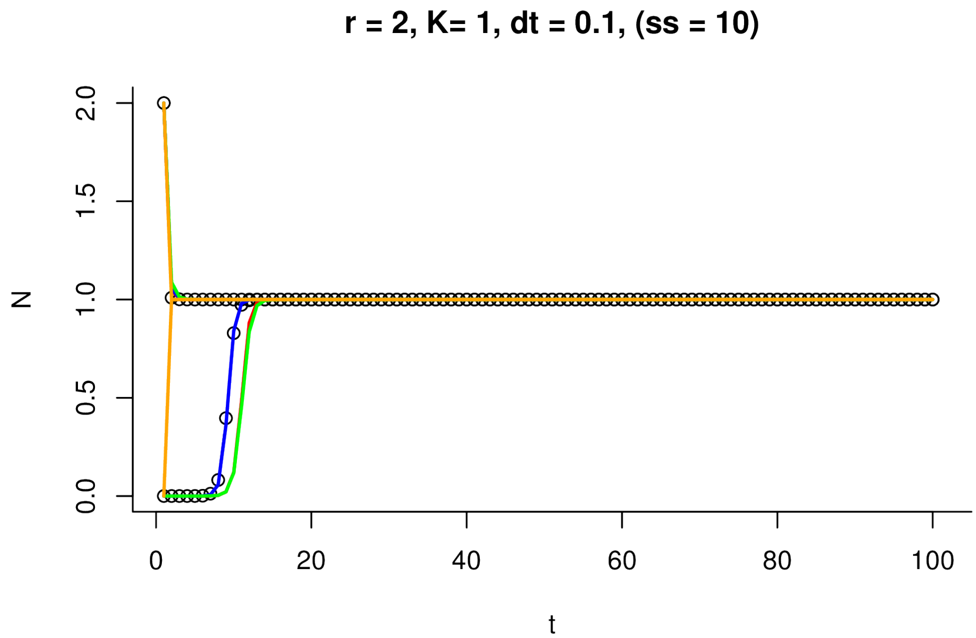
The following figures show the effects of r and Δt on stability and accuracy. The open black circles represent the analytic integral of the logistic equation as given by Quinn and Deriso (1999); the red line is explicit time stepping, equation 1; the blue line is semi-implicit time stepping, equation 2; the green line is semi-implicit time stepping, equation 3; the orange line is fully implicit time stepping, equation 4;



For reasonably small rate of increase r, all 4 approximations are stable, but quite inaccurate near the inflection point.



At high values of r, explicit approximation (1) and the semi-implicit approximation (2) are unstable, but the semi-implicit approximation (3) and fully implicit approximation (4) are stable. Again all of the approximations are innacurate near the inflection point.



As might be expected, decreasing the time step, Δt, at high values of r increases both stability and accuracy, especially for the semi-implicit time stepping, equation 2.