Integrating the Schaefer Model Differential Equation

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1 Integrating Schaefer Models

The widely used Schaefer (1954) fisheries stock assessment model is a simple extension of the logistic population model with a term added to represent removals from the population due to fishing.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - FN = N(r - F - \frac{r}{K}N) \tag{1}$$

where N is the population size, r is the instantaneous growth rate (t^{-1}), K is the asymptotic population size in the same units as N, and F, is the instantaneous rate of removal due to fishing (t^{-1}). Equation (1) reduces to the logistic model if F is assumed to be zero. These models are usually integrated numerically with "explict" finite difference methods to compute an approximation of the value of N at some time. Such approximations are often unstable for values of r large relative to the time step used in the finite difference solution. The performance of estimation methods in which logistic

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models are embedded in a statistical procedure depending on numerical function minimizers, i.e, models build using ADMB, TMB and BUGS, is greatly improved in accuracy, speed and use of computing resources if analytical solutions are used in preference to finite difference approximations.

1.1 Single population

Quinn and Deriso (1999) show the integral of the logistic ODE and point out that it can be obtained using integration by partial fractions. Numerous mathematics tutorials can be found on the World Wide Web use integration of the logistic ODE to illustrate the technique of integration by partial fractions. The same procedure can be used to integrate the Schaefer ODE. Equation (1) is rearranged slightly and variables separated to become

$$\frac{K}{N(K(r-F)-rN)}dN = dt. \tag{2}$$

The fraction in the left hand side can be factored into two parts,

$$\frac{K}{N(K(r-F)-rN)} = \frac{A}{N} + \frac{B}{(K(r-F)-rN)}. \tag{3}$$

A and B are constants that can be found by solving $K = A(K(r-f) - rn)) + BN$ setting $N = K$ and $N = 0$; $A = \frac{1}{r-F}$ and $B = 1 + \frac{F}{r-F}$. The desired integral becomes

$$\int \frac{K}{N(K(r-F)-rN)}dN = \int dt$$

$$\int \frac{A}{N}dN + \int \frac{B}{K(r-F)-rN}dN = \int dt$$

$$\frac{1}{r-F}\int \frac{1}{N}dN + (1+\frac{F}{r-F})\int \frac{1}{K(r-F)-rN}dN = \int dt$$

$$\frac{1}{r-F}\log|N| + \frac{1}{r}(1+\frac{F}{r-F})\log|K(r-f)-rN| + logC = t$$

$$\log|N| - \log|K(r-F)-rN| + logC = t(r-F)$$

$$\frac{|N|}{|K(r-F)-rN|}\cdot C = e^{t(r-F)}$$

$$\frac{|K(r-F)-rN|}{C|N|} = e^{-t(r-F)}$$

where C is the constant of integration. Setting $|N| = N_t$, the population size at time t, yields

$$N_t = \frac{K(r-F)}{Ce^{-t(r-F)} + r} \tag{4}$$

A formula suitable for computing population size at successive time steps can be found by setting $N_t = N_{t-\Delta t}$ at time $t = t - \Delta t$ in equation (4). The integration constant, C, becomes

$$C = \left(\frac{K(r-f)}{N_{t-\Delta t}} - r\right)e^{(t-\Delta t)(r-F)}, \tag{5}$$

and finally

$$N_t = \frac{K(r-F)}{\frac{K(r-F)}{N_{t-\Delta t}}e^{-\Delta t(r-F)} - re^{-\Delta t(r-F)} - r} \tag{6}$$

Further simplification of this equation may be possible, but I have not found it. In any case, equation (6) is the only general solution of the Schaefer ODE that I have seen, and it appears to work well in numerical applications.

1.2 Two populations with exchange

The motivation for the two popultion Schaefer model with exchange is developed fully in Appendix **??**.

The basic equations can be written

$$\frac{dN_1}{dt} = N_1\Big(r - F - T_{12} - 2(1-q)\frac{r}{K}N_2 - \frac{r}{K}N_1\Big) \tag{7}$$

$$\frac{dN_2}{dt} = N_2\Big(r - F - T_{12} - 2q\frac{r}{K}N_1 - \frac{r}{K}N_2\Big) + T_{21} \tag{8}$$

where N_1 the biomass of fish originating in region 1 and residing in region 1, and N_2 is the biomass of fish originating in region 2 but residing in region 1. The parameters r, K and F are unchanged from equation (1), and T_{12} is the emigration rate from region 1 (t^{-1}), T_{21} is the rate of immigration of biomass from region 2 to region 1 in units of biomass per time, and q; $(0 <$

$q < 1$) partitions the mortality caused by "competition" between the two subpopulations. Substitute

$$Z_1 = F + T_{12} + 2(1-q)\frac{r}{K}N_2 \tag{9}$$

$$Z_2 = F + T_{12} + 2q\frac{r}{K}N_1 \tag{10}$$

into equations (7) and (8) respectively to produce model equations in a similar form to equation (1)

$$\frac{dN_1}{dt} = N_1(r - Z_1 - \frac{r}{K}N_1) \tag{11}$$

$$\frac{dN_2}{dt} = N_2(r - Z_2 - \frac{r}{K}N_2) + T_{21}. \tag{12}$$

Equation (11) can be integrated in the same manner as equation (1) to yield

$$N_{1t} = \frac{K(r - Z_1)}{C_1 e^{-t(r-Z_1)} + r}. \tag{13}$$

All that remains is to find an equivalent integral for equation (12) and a means to simultaneously solve for C_1 and C_2.

$$N_2 = -\frac{\sqrt{Z^2-2\,r\,Z+r^2-4\,a\,c}\,\left(e^{\frac{t\,\sqrt{Z^2-2\,r\,Z+r^2-4\,a\,c}}{2}-\frac{C_2\,\sqrt{Z^2-2\,r\,Z+r^2-4\,a\,c}}{2}}+1\right)+r\,\left(e^{\frac{t\,\sqrt{Z^2-2\,r\,Z+r^2-4\,a\,c}}{2}-\frac{C_2\,\sqrt{Z}}{}}\right.}{a\,\left(2\,e^{\frac{t\,\sqrt{Z^2-2\,r\,Z+r^2-4\,a\,c}}{2}-\frac{C_2\,\sqrt{Z^2-2\,r\,Z+r^2-4\,a}}{2}}\right.}$$

$$C_2 = \frac{t\,\sqrt{\left(r-Z\right)^2-4\,a\,c}-2\,\log\left(-\frac{Z}{-Z+\sqrt{\left(r-Z\right)^2-4\,a\,c}+2\,a\,N_2+r}-\frac{\sqrt{\left(r-Z\right)^2-4\,a\,c}}{-Z+\sqrt{\left(r-Z\right)^2-4\,a\,c}+2\,a\,N_2+r}+\frac{2\,a\,N}{-Z+\sqrt{\left(r-Z\right)^2-}}\right.}{\sqrt{\left(r-Z\right)^2-4\,a\,c}}$$