University of Toronto
Department of Electrical and Computer Engineering
ECE311 Dynamic Systems and Control
Homework 2

Transfer Functions and Solutions of Linear Differential Equations

1. For the system described by the differential equation

$$\dddot{y} + 3\ddot{y} - 2\dot{y} + y = \ddot{u} - 3\dot{u} + 2u,$$

find the transfer function from u to y. Next, find a state space representation of the system.

2. For the system with state space representation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = [1 \;\; 0 \;\; 0]\mathbf{x},$$

find the transfer function from u to y.

3. **Part 1.** You are given a linear system with input u and output y. When[1] $u(t) = \sin t \cdot \mathbf{1}(t)$ and all initial conditions are set to zero, it is found experimentally that $y(t) = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right]\mathbf{1}(t)$. Find the transfer function of the system from u to y.

   Part 2. Consider the system with state space representation

$$\dot{x}_1 = x_2 + u$$
$$\dot{x}_2 = u$$
$$y = x_1 + x_2 + u.$$

   Find the transfer function from u to y.

4. For the following transfer function, write the corresponding differential equation.

$$\frac{X(s)}{F(s)} = \frac{s+2}{s^3 + 8s^2 + 9s + 15}.$$

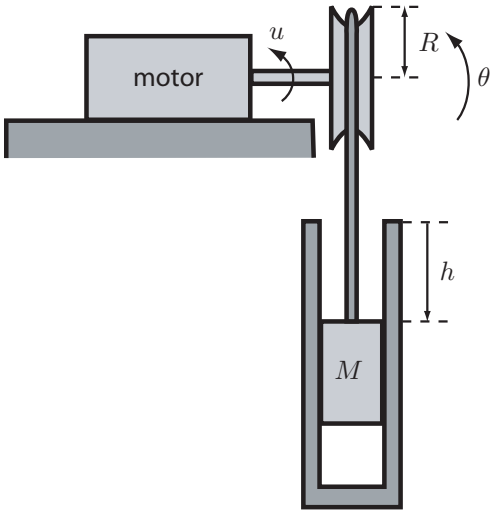
5. Find the transfer function, $G(s) = Y(s)/R(s)$, for the following system represented in state space.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r$$
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

6. A pulley of radius R and moment of inertia I is connected to the shaft of a DC motor. The pulley is connected via a cable to a piston of mass M. The rotation of the pulley in the clockwise or

[1] $\mathbf{1}(t)$ denotes the unit step function, i.e., $\mathbf{1}(t) = 1$ for $t \geq 0$ and $\mathbf{1}(t) = 0$ for $t < 0$

counterclockwise direction makes the piston slide vertically upward or downward in a cylinder. The contact between piston and cylinder is subject to viscous friction with coefficient K_v.



Assume that

- The torque (moment) u imparted by the DC motor is the control input (in other words, neglect the dynamics of the DC motor)
- The cable is *rigid* and it doesn't slip.

Do the following:

(a) Find the equations of motion of the system.

Hint. Note that the displacement h is related to the angle θ by the relation $h = -R\theta + h_0$, where h_0 is some constant. Pick $h_0 = 0$ in this problem. Due to this relation, if we know the evolution of θ, we also know the evolution of h. Therefore, the entire system can be modeled by the rotation of the pulley, giving rise to an equation of the form $I\ddot{\theta}$ =total torque. The pulley is subject to two torques: u in the positive direction of rotation, and a torque in the negative direction due to the cable tension, T. The mass is subject to three forces: Mg pointing downward, T pointing upward, and viscous friction pointing upward. Impose the constraint $\ddot{h} = -R\ddot{\theta}$ and solve for the tension T. You need to write a model that depends only on θ and $\dot{\theta}$.

(a) Suppose that the position of the cylinder is measured by means of an optical encoder. Let $\tilde{u} := u - RMg$. Find the transfer function from $\tilde{u}$ to h.

(a) Suppose now that the angular velocity of the shaft, $\dot{\theta}$, is measured by means of a tachometer. Find the transfer function from $\tilde{u}$ to $\dot{\theta}$.