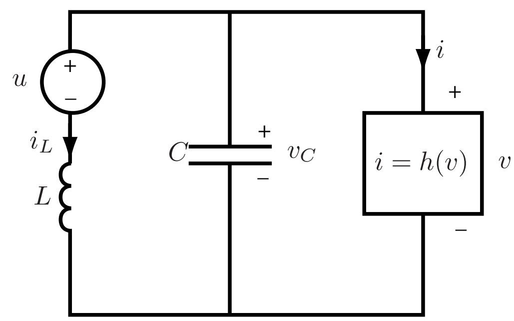
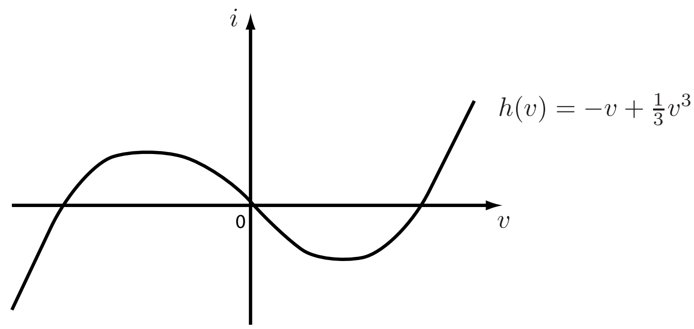
University of Toronto
Department of Electrical and Computer Engineering
ECE311 Dynamic Systems and Control
Homework 1

1 Models of Control Systems

1. The controlled Van der Pol oscillator has the circuit diagram shown below.



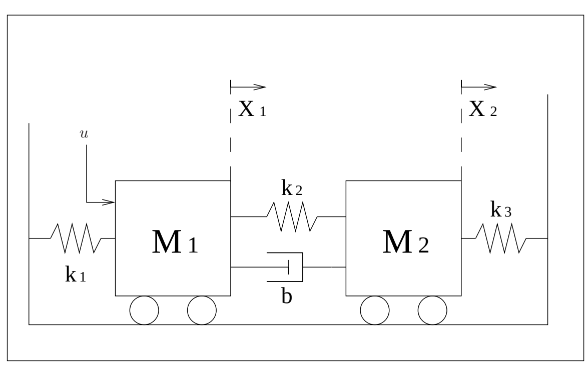
The output of the system is the voltage across the capacitor, v_C. The nonlinear element of the circuit has a characteristic $i = h(v)$, depicted below, with $h(v) = -v + \frac{1}{3}v^3$.



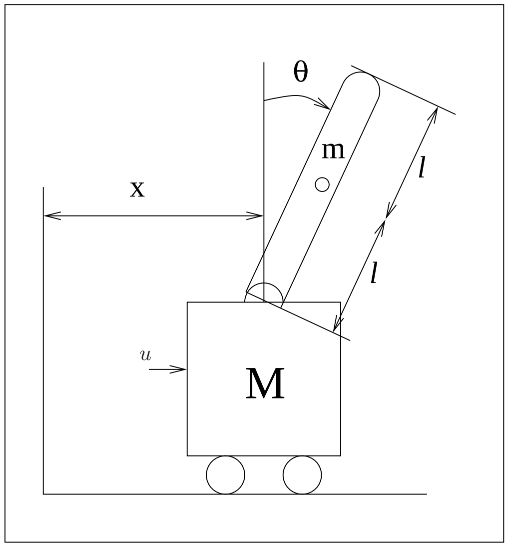
(i) Write the mathematical model of the system in terms of v_C, i_L, and u.

(ii) Letting $x_1 = i_L$ and $x_2 = v_C$, write the state-space model of the system.

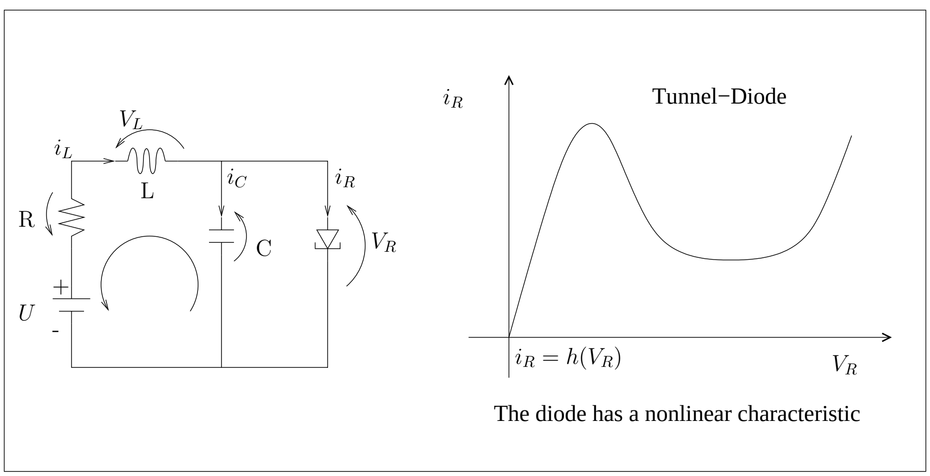
2. Write the equations of motion of the system below, where x_1 and x_2 represent deviations from the equilibrium position.



3. Write the equations of motion of the system below.
    - Suppose the center of gravity of the pendulum rod is at its geometric center.
    - Neglect the moment of inertia of the wheels.



4. (a) Write the equations of the circuit below.
   (b) Write the state-variable model of the system.
   (c) Find the equilibria holding u constant.



5. You are given a nonlinear system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + u \\
\dot{x}_2 &= -2x_2 + x_3 \\
\dot{x}_3 &= e^{x_1} x_2 + u \\
y &= x_1 + x_2.
\end{aligned}$$

   (i) Setting $u = 1$, find the equilibrium point of the system.
   (ii) Find a linearized model of the system about the equilibrium point you found in part (i).

2 Laplace transforms and the solutions of linear differential equations

Main Properties of the Laplace Transform

Below is a table of Laplace transform pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$\mathbf{1}(t)$	$1/s$
$t\mathbf{1}(t)$	$1/s^2$
$t^k\mathbf{1}(t)(k \text{ integer})$	$k!/s^{k+1}$
$e^{at}\mathbf{1}(t)$	$\dfrac{1}{s-a}$
$t^k e^{at}\mathbf{1}(t)$	$\dfrac{k!}{(s-a)^{k+1}}$
$\sin(at)\mathbf{1}(t)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)\mathbf{1}(t)$	$\dfrac{s}{s^2+a^2}$

where $\mathbf{1}(t)$ denotes the unit step, i.e.

$$\mathbf{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The following is a list of the main properties of Laplace transforms. Using these properties you'll be able to solve the next two problems without computing any integral. In what follows, we denote $F(s) := \mathcal{L}\{f(t)\}$, $G(s) := \mathcal{L}\{g(t)\}$.

P1: Linearity	$(\forall c_1, c_2 \in \mathbb{R})\ \ \mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s).$
P2: Differentiation	$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-).$
P3: Integration	$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s).$
P4: Convolution	$\mathcal{L}\{f(t) \star g(t)\} = F(s)G(s).$
P5: Time shift	Let $T > 0$. Then, $\mathcal{L}\{f(t-T)\mathbf{1}(t-T)\} = e^{-Ts}F(s).$
P6: Shift in s	$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$
P7: Multiplication by t	$\mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}.$

1. Find the Laplace transform of each of the following time-signals.

    1. $f(t) = 3t^2 e^{-t} \cdot \mathbf{1}(t)$
    2. $f(t) = \sin t \cos t \cdot \mathbf{1}(t)$
    3. $f(t) = \sin(t-3) \cdot \mathbf{1}(t)$
    4. $f(t) = \sin(t-3) \cdot \mathbf{1}(t-3)$
    5. $f(t) = t\sin(t-3)\mathbf{1}(t)$
    6. $f(t) = (t-3)e^{t-3}\mathbf{1}(t)$
    7. $f(t) = (t-3)e^{t-3}\mathbf{1}(t-3)$
    8. $f(t) = te^{-at} + 2t\cos t$
    9. $f(t) = t^2 + e^{-at}\sin bt$

2. Find the inverse Laplace transform of each of the following functions.

$$1. \qquad F(s) = \frac{s^2+s+1}{(s+1)(s+2)(s+3)} \tag{1}$$

$$2. \qquad F(s) = \left(\frac{s^2+s+1}{(s+1)(s+2)(s+3)}\right) e^{-s} \tag{2}$$

$$3. \qquad F(s) = \frac{e^{-2s}}{s-3} \tag{3}$$

$$4. \qquad F(s) = \frac{e^{1-s}}{s} \tag{4}$$

$$5. \qquad F(s) = \frac{1}{s^6} \tag{5}$$

$$6. \qquad F(s) = \frac{10}{s(s+1)(s+10)} \tag{6}$$

$$7. \qquad F(s) = \frac{1}{s(s+2)^2} \tag{7}$$

$$8. \qquad F(s) = \frac{2(s+2)}{(s+1)(s^2+4)} \tag{8}$$

3. Given $F(s) = \frac{2-s}{(s-1)(s+2)}$, find $\lim_{t\to+\infty} f(t)$.

4. Find the solution $y(t)$ to each of the following differential equations

   1. $\ddot{y} + 2\dot{y} + 5y = \exp(-t)\,\mathbf{1}(t)$ $\qquad y(0) = \dot{y}(0) = 0$
   2. $\ddot{y} - y = \mathbf{1}(t)$ $\qquad y(0) = 1, \dot{y}(0) = 1$
   3. $y^{(4)} + 4y^{(3)} + 8y^{(2)} + 8y^{(1)} + 4y = 3\dot{u} + 4u$ $\qquad y(0) = \dot{y}(0) = \cdots = y^{(3)}(0) = 0$, $u = \mathbf{1}(t)$
   4. $\ddot{y} + \dot{y} + 3y(t) = 0;$ $\qquad y(0) = \alpha, \dot{y}(0) = \beta$