Initialization of Multistep

29/05/18

Given the ODE system:

$$M u_t = -L(t)u + g(t), \quad \boxed{u \in \mathbb{R}^m}$$

Now. Consider the ST linear systems for each ODE discretization, with the s starting values for each method appended to the RHS vector.

s-step method.

Let $u_n \approx u(t_n)$.

$\boxed{\text{AB: (Adams-Bashforth)}}$

AB1: Let $A_n \equiv \frac{1}{\delta t_n} M_n$; $B \equiv -\frac{1}{\delta t_n}(M_n - L_n)$,

Then

$$\begin{bmatrix} A_1 & & & \\ B_1 & A_2 & & \\ & \ddots & \ddots & \\ & & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_0 u_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{n_t-1} \end{bmatrix},$$

$$Au = b, \quad u, b \in \mathbb{R}^{mn_t}, \quad A \in \mathbb{R}^{mn_t \times n_t m}$$

AB2: Let $A_n \equiv \frac{1}{\delta t_n} M_n$; $B_n \equiv -\left(\frac{M_n}{\delta t_n} - \frac{3}{2} L_n\right)$;

$$C_n \equiv -\frac{1}{2} L_{n-2}.$$

Have,

$$\begin{bmatrix} A_2 & & & & \\ B_2 & A_3 & & & \\ C_2 & B_3 & A_4 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n_t-2} & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_1 u_1 - C_0 u_0 \\ -C_1 u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3g_1 - g_0 \\ 3g_2 - g_1 \\ 3g_3 - g_2 \\ \vdots \\ 3g_{n_t} - g_{n_t-2} \end{bmatrix}$$

$Au = b, \quad u, b \in \mathbb{R}^{m(n_t-1)}, \quad A \in \mathbb{R}^{m(n_t-1)\times(n_t-1)m}$ 2.

AB3: Let $A_n \equiv \frac{1}{\delta t_n} M_n$; $B_n \equiv -\left(\frac{1}{\delta t_n} M_n - \frac{23}{12} L_{n-1}\right)$;

$$C_n \equiv -\frac{4}{3} L_n \; ; \; D_n \equiv \frac{5}{12} L_{n-3} ,$$

Then have

$$\begin{bmatrix} A_3 & & & & & & & \\ B_3 & A_4 & & & & & & \\ C_3 & B_4 & A_5 & & & & & \\ D_3 & C_4 & B_5 & A_6 & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & D_{n_t-3} & C_{n_t-2} & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_2 u_2 - C_1 u_1 - D_0 u_0 \\ -C_2 u_2 - D_1 u_1 \\ -D_2 u_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$+ \frac{1}{12} \begin{bmatrix} 23 g_2 - 16 g_1 + 5 g_0 \\ 23 g_3 - 16 g_2 + 5 g_1 \\ 23 g_4 - 16 g_3 + 5 g_2 \\ \vdots \\ 23 g_{n_t-1} - 16 g_{n_t-2} + 5 g_{n_t-3} \end{bmatrix}$$

$Au = b$ with $u, b \in \mathbb{R}^{m(n_t-2)}$ & $A \in \mathbb{R}^{m(n_t-2)\times(n_t-2)m}$.

Adams-Moulton:

AM1: Let $A_n \equiv \frac{1}{\delta t_n} M_n + L_n$; $B_n \equiv \frac{-1}{\delta t_n} M_n$

Then, we have

$$\begin{bmatrix} A_1 & & & \\ B_1 & A_2 & & \\ & & \ddots & \\ & & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_0 u_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n_t} \end{bmatrix}$$

$$Au = b, \text{ with } u, b \in \mathbb{R}^{n_t}, \; A \in \mathbb{R}^{n_t \times n_t}$$

AM2: Let $A_n \equiv \left(\frac{1}{\delta t_n} M_n + \frac{1}{2} L_n\right)$; $B_n \equiv -\left(\frac{1}{\delta t} M_n - \frac{1}{2} L_{n-1}\right)$;

Have

$$\begin{bmatrix} A_1 & & & \\ B_1 & A_2 & & \\ & \ddots & \ddots & \\ & & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_0 u_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 + g_0 \\ g_2 + g_1 \\ \vdots \\ g_{n_t} + g_{n_t-1} \end{bmatrix} \times \frac{1}{2}$$

$$Au = b \text{ with } u, b \in \mathbb{R}^{m n_t}, \; A \in \mathbb{R}^{m n_t \times n_t m}$$

AM3: Let $A_n \equiv \frac{1}{\delta t} M_n + \frac{5}{12} L_n$;

$$B_n \equiv -\left(\frac{1}{\delta t} M_n - \frac{2}{3} L_n\right);$$

$$C_n \equiv -\frac{1}{12} L_n .$$

Will have

$$\begin{bmatrix} A_2 & & & & & \\ B_2 & A_3 & & & & \\ C_2 & B_3 & A_4 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & C_{n_t-2} & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_1 u_1 - C_0 u_0 \\ -C_1 u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} +$$

$$\frac{1}{12} \begin{bmatrix} 5g_2 + 8g_1 - g_0 \\ 5g_3 + 8g_2 - g_1 \\ 5g_4 + 8g_3 - g_2 \\ \vdots \\ 5g_{n_t} + 8g_{n_t-1} - g_{n_t-2} \end{bmatrix}$$

Have linear system $Au = b$ with $u, b \in \mathbb{R}^{m(n_t-2)}$,

$A \in \mathbb{R}^{m(n_t-2) \times (n_t-2)m}$.

$\underline{\text{Backward Differentiation Formulas (BDF)}}$

BDF1: (same as AM1)

BDF2: Let $A_n \equiv \frac{1}{\delta t_n} M_n + \frac{2}{3} L_n$; $B_n \equiv -\frac{4}{3\delta t_n} M_n$;

$$C_n \equiv \frac{1}{3\delta t_n} M_n$$

Will have

$$\begin{bmatrix} A_2 & & & & \\ B_2 & A_3 & & & \\ C_2 & B_3 & A_4 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n_t-2} & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_1 u_1 - C_0 u_0 \\ -C_1 u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_{n_t} \end{bmatrix}$$

Have

$$Au = b, \quad u, b \in \mathbb{R}^{m(n_t-1)}, \quad A \in \mathbb{R}^{m(n_t-1) \times (n_t-1)m}$$

I think there's a typo on Ben's notes here. 6.

BDF3 : Let $A_n \equiv \frac{1}{\delta t_n} M_n + \frac{6}{11} L_n$;

$$B_n \equiv -\frac{18}{11\,\delta t_n} M_n; \quad ; \quad C_n \equiv \frac{9}{11\,\delta t_n} M_n;$$

$$D_n \equiv \frac{-2}{11\,\delta t_n} M_n$$

Then we'll have

$$\begin{bmatrix} A_3 & & & & & & & \\ B_3 & A_4 & & & & & & \\ C_3 & B_4 & A_5 & & & & & \\ D_3 & C_4 & B_5 & A_6 & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & D_{n_t-3} & C_{n_t-2} & B_{n_t-1} & A_{n_t} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ \vdots \\ \vdots \\ \vdots \\ u_{n_t} \end{bmatrix} = \begin{bmatrix} -B_2u_2 - C_1u_1 - D_0u_0 \\ -C_2u_2 - D_1u_1 \\ -D_2u_2 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$+ \frac{6}{11} \begin{bmatrix} g_3 \\ g_4 \\ g_5 \\ g_6 \\ \vdots \\ \vdots \\ \vdots \\ g_{n_t} \end{bmatrix}$$

Have linear system $Au = b$, with $u, b \in \mathbb{R}^{m(n_t-3)}$

and $A \in \mathbb{R}^{m(n_t-3)\times(n_t-3)m}$

And now a summary table indicating what has to be generated for each method.

Multistep	starting values required	starting values solver
AB1	u_0	N/A
AB2	u_0, u_1	ERK2
AB3	u_0, u_1, u_2	ERK3
AM1	u_0	N/A
AM2	u_0	N/A
AM3	u_0, u_1	DIRK3
BDF1	u_0	N/A
BDF2	u_0, u_1	DIRK2
BDF3	u_0, u_1, u_2	DIRK3