In this article we endeavor to test the validity of Euler's method against analytical means. As a subject, a system of decaying nuclei were used with given Ordinary Differential Equations. Eventually, it was found that Euler's method is a useful algorithm in certain situations, but not perfect.

# Section I - Introduction

Consider a system consisting of two species, called A and B, which have the interesting tendency to turn into something new after some passage of time. Indeed, these two species are actually nuclei, and their "transformation" is actually a decay process with a specific quality. A decays and turns into B, while B decays into some undisclosed form.

$$A \rightarrow B \qquad \textit{Figure 1}$$

$$B \rightarrow ? \qquad \textit{Figure 2}$$

As one might assume from Figures 1 and 2, the population of B is affected by the decay rate of A. After all, as A nuclei decay, they increase the amount of B nuclei in the system. Despite this fact, the B nuclei still decay with time and *decrease* the amount of B nuclei. Thus, the system can be described by competing rates.

$$\frac{dN_A}{dt} = \frac{-N_A(t)}{\tau_A} \qquad \textit{Equation 1}$$

$$\frac{dN_B}{dt} = \frac{N_A(t)}{\tau_A} - \frac{N_B(t)}{\tau_B} \qquad \textit{Equation 2}$$

Where $N_A$ , $N_B$ represent the population of nuclei A and B, respectively, and $\tau_A$, $\tau_B$ represent the decay constant of their nuclei, in units of time.

Equations 1 and 2 represent the ordinary differential equations that describe the system. Note that the ODEs describe exactly what was expected with regard to the B nuclei. In Equation 2 the influx of new B nuclei is represented by the positive term, while the loss of B nuclei is represented by the negative term. Still, how can we better describe the system? A model of the populations without derivatives would grant the best picture of the system's state at a given time. To gain that, however, more information is needed.

To accurately integrate with any method, consider something more specific than before. A system, frozen in time at t = 0. Whether we are operating on the scope of seconds, hours, years or more will not affect the math since an initial state is 0 for all scales. Later, Equations 1 and 2 will be normalized, but for now measurements need to be taken.

The new system still possesses two nuclei A and B, with the same properties outlined above, and the added bonus that their initial counts are also known: $N_A(0) = 200$ and $N_B(0) = 5$. In terms of visualizing what our models may end up looking like, these initial counts help a lot.

For example, when looking at the population of A nuclei, it is clear that whatever form $N_A(t)$ takes, it will involve a decline from 200 nuclei. That initial value is the peak of the A model. Similarly, we can attempt to imagine the structure of $N_B(t)$. Noting that it starts with a count of 5, will the model fall to zero before climbing, or climb before declining? This depends on the form of $N_A$ and $N_B$ so it's time to move on to the next step.

# Section II - Method

To find models of the two populations, there are two paths afforded to us. One involves the application of analytical means to solve the coupled ODE system. Specifically, application of Mathematica or "on paper" calculations. The other path utilizes Euler's method to plot a numerical model. Our interest here is in comparing the two methods. Exact results versus an approximation.

Before anything, however, the equations need to be normalized. Once they are, we can use a general timescale to describe their decay. In this scenario, let $t_0 = \tau_A = 1$. For the A nuclei, the timescale and decay constant leave the expression. For the B nuclei, $\tau_B$ still remains and its value has a notable affect on the model.

$$\bar{t} = \frac{t}{t_0} \Rightarrow t = \bar{t}\,t_0 \qquad \textit{Equation 3}$$

$$\frac{dN_i}{dt} = \frac{dN_i}{\bar{t}\,t_0} = \frac{1}{t_0}\frac{dN_i}{d\bar{t}} = \frac{1}{t_0}\dot{N}_i \qquad \textit{Equation 4}$$

Now, with normalized time, what path should be taken? As a check, the exact results will be compared to the numerical ones, meaning the Euler method should be explained. It exists as an extension of the Taylor Series, which is a method used to approximate a function as a series of its derivatives. The idea is sound, using the instantaneous rate of change at a single point to trace a path along a series of sequential ones.

$$F(x) \approx f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

*Equation 5*

Now, expand that idea. Instead let $a$ be a small change $\Delta t$. From Figure Three, only take the terms that are relevant to your situation. If we apply the algorithm to our population model, for instance…

$$N_i(t + \Delta t) \approx N_i(t) + \dot{N}_i \Delta t \qquad \textit{Equation 6}$$

Equation 6 is the esteemed Euler Algorithm for this problem. It progresses the data one "step" from the previous point, using the derivative of that previous point. Interestingly, we actually know the first derivative of our $N_i$ models and also have a starting point. As a result, we can readily apply this algorithm to our cases.

# Section III - Results

A look at the coupled ODE shows that Equation 1, dealing with the A nuclei, is the simpler of the two. First I solved the ODE by hand. It was homogenous and rather straightforward once the characteristic equation was solved. The final model of A's population is given in Equation 8, below.

$$\dot{N}_A + \frac{N_A}{1} = 0 \qquad \textit{Equation 7}$$

$$N_A(t) = N_{A,0}e^{-t} = 200e^{-t} \quad \textit{Equation 8}$$

So, the population of A is a function of exponential decay. This falls in line with what was predicted in Section I, with A having a negative trend from its peak at 200. From an analytical standpoint Equation 8 holds well. It's derivative is itself, but negative. At t = 0, the exponential factor goes to 1, leaving a count of 200. This is consistent with the initial state of the system and the first order ODE.

Next, Euler's method was applied. Equation 6 only works if a starting point is known. Thankfully, we know that our starting value for A nuclei is 200. The hurdle after that was finding a suitable step, $dt$. Trial and error found me the best fit.

At first I considered a $dt$ of 1/2, but found that the plot given didn't really match the exponential curve I expected. From there, I overcompensated and tried $dt = 0.01$. The effect wasn't wholly surprising. On a small interval the slope should hardly change between points, so when I saw what appeared to be a linear line I understood that I'd gone too low. Finally, I settled on $dt = 0.1$ and got the desired curve. The approximation plot, and the comparison with the exact curve, are shown in the figures below.

Note that the interval of time plotted is related to the amount of steps taken and dt size. In my code the first quantity was called "lastStep" and capped at 50 in most cases. In other words, 50 points were plotted.

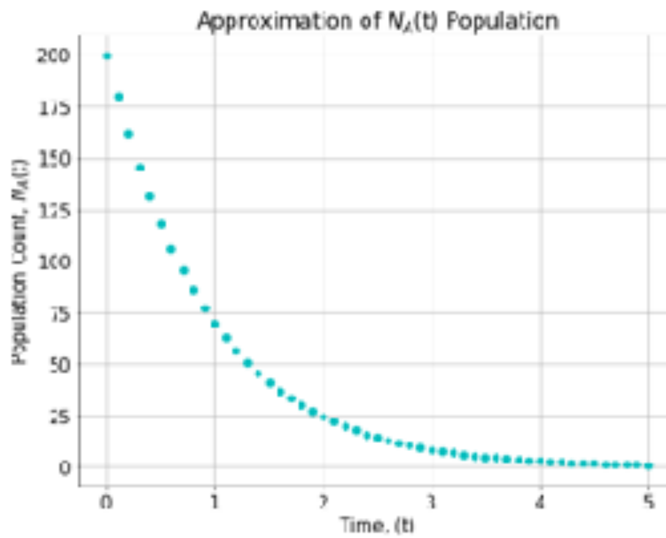


*Figure 3*. Plot of Population of A nuclei against time. Follows the expected trend of decay toward zero and away from 200, but these numbers were found through Euler's Method (Equation 8)

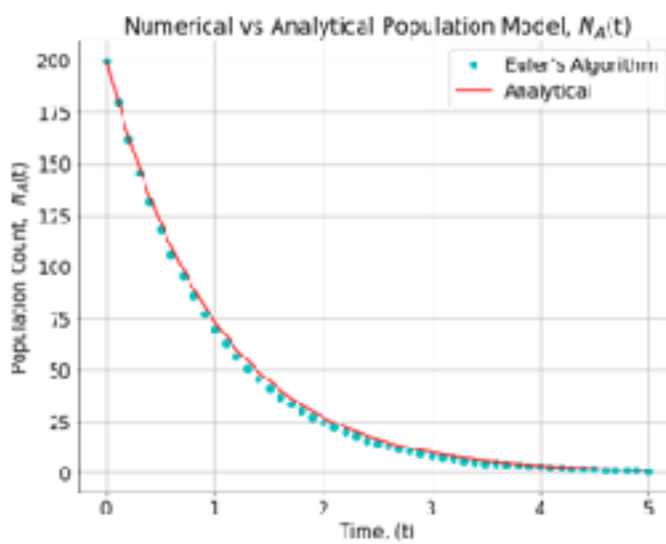


*Figure 4*. Comparison of Numerical data against Analytical functions (Equations 6 and 8) for the same time values. Very little deviation is seen here, though in the range of $.5 \leq t \leq 2.5$ there is noticeable error. This is actually seen in later plots as well for a similar interval.

Figures 3 and 4 describe an agreement between analytical and numerical methods. At least, they do so for the simple ODE that described A nuclei's decay rate. Unfortunately, there isn't much to be said about the plots beyond that due to the nature of the system. Looking at B generates more thought. Recall the fact that Equation 2 had a dependency on both $N_A$ and $N_B$ functions. We already discussed why this would be, but back then there was no expression for the A population. Now there is, and we can substitute this expression in to change Equation 2 into a solvable ODE.

$$\dot{N}_B + \frac{N_B}{\tau_B} = 200e^{-t} \qquad \textit{Equation 9}$$

Equation 9 has the form of an ODE constrained by a forcing function. Thus, the final solution of $N_B$ will be a linear combination of the homogenous solution and particular solution. Both solutions, however, are very dependent on the value of $\tau_B$ chosen. Still, I believe I should clear something up. It relates to the normalized time variable, seen in Equations 3 and 4. The latter displayed a generalized version of the derivatives. When plugged into Equation 2, and after a bit of algebra, the following is found.

$$\frac{dN_B}{d\bar{t}} = \frac{200e^{-\bar{t}}}{\frac{\tau_A}{t_0}} - \frac{N_B}{\frac{\tau_B}{t_0}} \qquad \textit{Equation 10}$$

But recall that $t_0 = \tau_A = 1$, and so the while the ratio underneath $N_A$ vanishes, the ratio under $N_B$ becomes

$$\frac{\tau_B}{\tau_A} \qquad \textit{Equation 11}$$

And while it may not be immediately clear, the value of this new, normalized, decay constant will influence the shape of $N_B$ greatly. So commit Equation 11 to memory as we look at general cases. For example, let's consider a case in which $\tau_B = \tau_A = 1$, causing the ratio to also equal 1. What would our population function look like?

On paper, the solution to the ODE can be found using Laplacian Transforms. The process is tedious and not worth outlining. In fact, for the other cases Mathematica or a similar program were used to save time. Still, the following equation describes our B population for this scenario.

$$\dot{N}_B + \frac{N_B}{1} = 200e^{-t} \qquad \textit{Equation 12}$$

$$N_B(t) = e^{-t}(5 + 200t) \qquad \textit{Equation 13}$$

To clarify, for the ODE seen in Equation 12, Equation 13 is a valid solution. It is a combination of the homogenous and particular solutions and has the properties that were predicted in the introduction. As a test, for $t = 0$ we see that the 200 term goes to 0 and the exponential term goes to 1, leaving a count of 5 B nuclei, as expected for the initial state of the system. How did the numerical method fare?

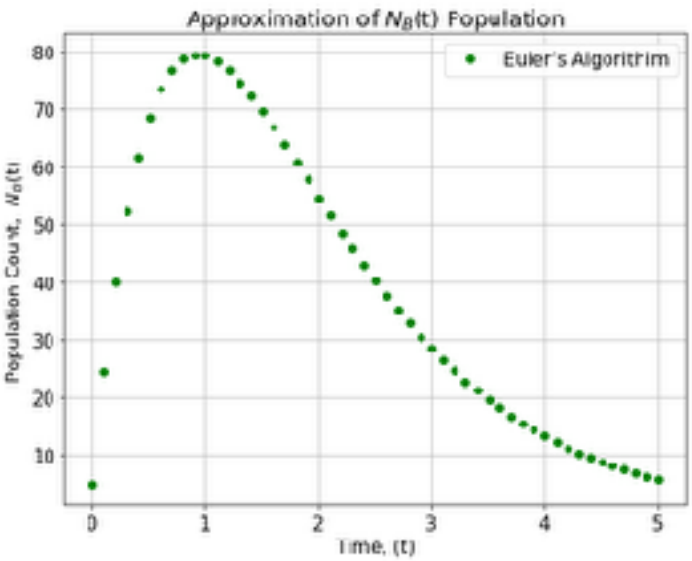


*Figure 5.* Plot of Population of B nuclei against time. Follows the expected climb from 5 toward some peak before descending. These numbers were found through Euler's Method (Equation 8) and with the constraint that $\tau_B = \tau_A$

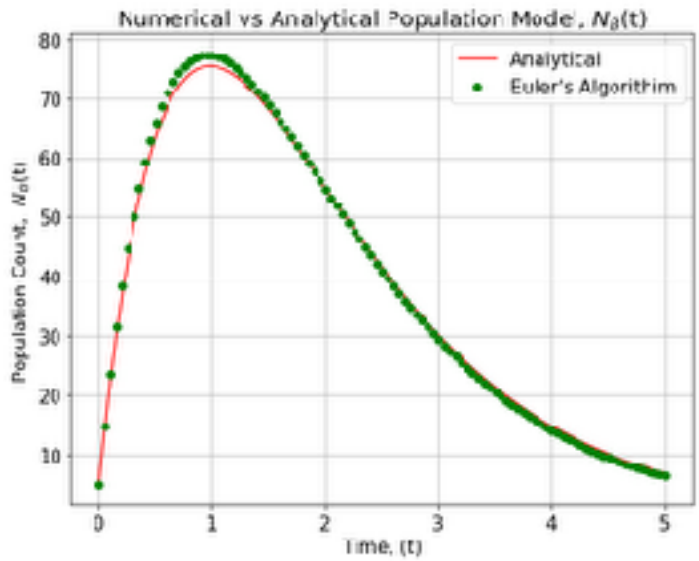


*Figure 7.* Adjusted plot with $dt\ = 0.05$ and maximum steps of 100. Clearly error size has been minimized.

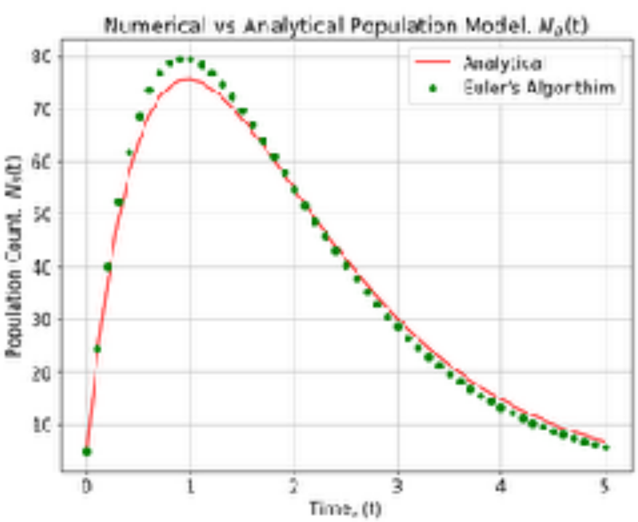


*Figure 6.* Comparison of Numerical data against Analytical function (Equations 6 and 13) for the same time values. The deviation here is more noticeable here than in Figure 4, but the fit is satisfactory. Most concerning is the error at the peak. Like Figure 5, this is strictly for the case where $\tau_B = \tau_A$

Once again we see an agreement between numerical and analytical result, though less so with this case. The fact that the general shapes are similar implies that we're on the right track, however. In an attempt to minimize the deviation, however, the step size was decreased and amount of steps was increased. Specifically, $dt\ = 0.05$ and lastStep = 100.

Is it possible to explain the behavior of $N_B(t)$? It's shape is very different from the A model, after all. A look at the original ODE, Equation 2, keeping in mind the final form of the B model, can offer some help. It appears that the positive quantity representing new B nuclei steadily increases the population of B in the system. This explains the initial rise.

Why the peak then, in Figure 7? What stops the addition of B? Mathematically, it's valid to assume that the exponential decay term overtakes the linear growth term at some $t = t_{max}$. As a result, the curve passes its inflection point and assumes a negative trend. Mathematically, this is reasonable and accurate.

Physically, however, it may be as simple as taking a look at Figure 3 and recognizing that at some $t = t_{min}$, the population of A nuclei approaches 0. Without A nuclei to decay into B, the growth in B slows down and, eventually, stops. "Stop" is used abstractly here, since the exponential never reaches 0. Regardless, the steadying of the decay curve in Figure 7 can likely be attributed to the decrease in available A nuclei.

Moving on, it's time to return to Equation 11. A new case for the ratio must be considered. Our options now have us considering $\tau_B < \tau_A$ and $\tau_B > \tau_A$. Of the two, we'll first consider the latter. A decay constant greater than 1 would decrease the impact of the $N_B$ term in Equation 2, implying a shallower descent after the peak is reached. Analytically, the function takes on the following form, found through Mathematica. In this case, to actually generate a plot, $\tau_B = 10$ was chosen for our decay constant.

$$\dot{N}_B + \frac{N_B}{10} = 200e^{-t} \quad \textit{Equation 14}$$

$$N_B(t) = \frac{5}{9}e^{-t}(409e^{9t/10} - 400)$$

*Equation 15*

The usual checks show that the analytical result, Equation 15, satisfies the ODE seen in Equation 14 and accurately describes the initial state of the system. How does this compare to the numerical results?

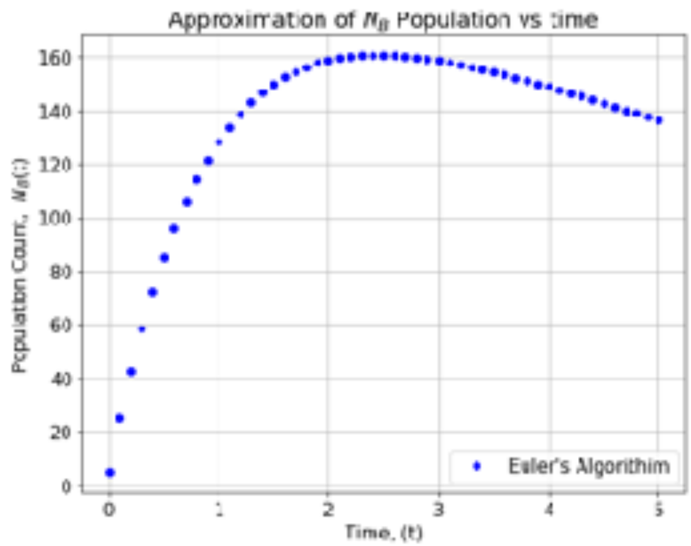


*Figure 8.* Plot of Population of B nuclei against time. Follows the expected climb from 5 toward some peak before slowly descending. These numbers were found through Euler's Method (Equation 8) and with the constraint that $\tau_B > \tau_A$

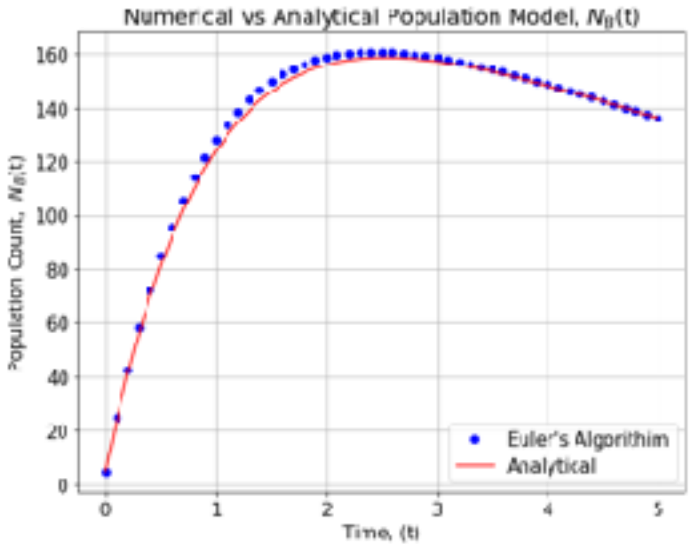


*Figure 9.* Comparison of Numerical data against Analytical function (Equations 6 and 15) for the same time values. The deviation here is noticeable around the peak, but this is likely due to the step of $dt = 0.1$. Note that this is strictly for the case where $\tau_B > \tau_A$

Figures 8 and 9 confirm a general agreement between numerical and analytical methods. They also support the prediction on the trend of the curve. Indeed, on the same time interval as Figures 7 and 4, the negative slope is shallower (smaller in magnitude) than the curves seen previously. In other words, having a larger decay constant lessens the decay of the B nuclei.

Our choice of $\tau_B$ also affected the peak of the B nuclei. Comparing Figure 9 to Figure 7, the former has a maximum of ~160 nuclei, while the latter peaked at ~75 nuclei. Evidently, the larger decay constant decreases the loss rate and, as a consequence, allows more B nuclei to remain in the system.

It's time to inspect the final case, in which $\tau_B < \tau_A$. For this test, I chose to let $\tau_B = \dfrac{1}{10}$ and decided to see if 9it was possible to predict the curve here as well.

Since we know the shape of the ODE, see Equation 9, we understand that for whatever value of $\tau_B$ is chosen, its effect will be inverted due to the fact that it is a denominator. This is how I guessed at the shallow descent seen in Figure 9. It is also how I can guess at a steeper decay slope and smaller peak for this plot. After all, a small value in the denominator has a large reciprocal. Thus, the small constant will cause B's decay term to have a larger impact on the rate.

Once again, Mathematica was used to solve this ODE. Shown below are the system and solution.

$$\dot{N}_B + 10N_B = 200e^{-t} \qquad \textit{Equation 16}$$

$$N_B(t) = \frac{5}{9}e^{-10t}(40e^{9t} - 31) \qquad \textit{Equation 17}$$

The exponential decay term is larger than it has been in either of the cases, which encourages my expectations of a steeper slope. Plugging $t = 0$ into the model gives a count of 5 B nuclei, as expected. So, at first glance, this solution is satisfactory.

Below are the relevant plots for this case, though some alterations had to be made to the time step and lastStep quantities in the code. It should be noted that for all of the previous plots, ignoring Figure 7, I used the same dt and lastStep values listed earlier in this section.

This was not the case for Figures 10 and 11, as the deviation between the numerical and analytical models was too large to ignore.

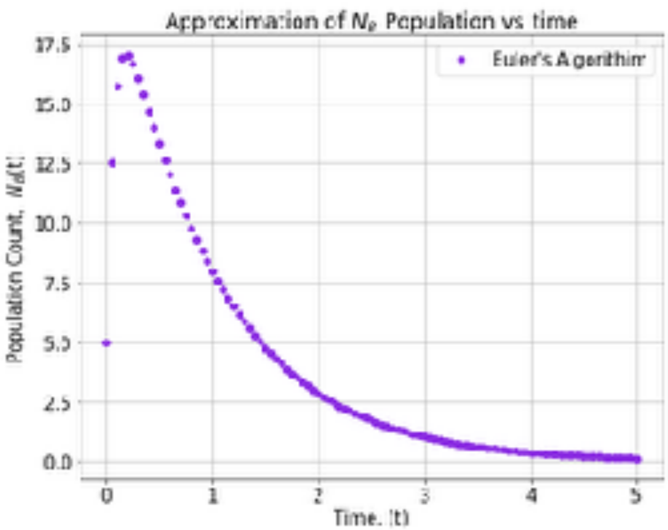


*Figure 10.* Plot of Population of B nuclei against time. Follows the expected climb from 5 toward some peak before steeply descending. These numbers were found through Euler's Method (Equation 8) and with the constraint that $\tau_B < \tau_A$ and adjustment of $dt = 0.05$ , lastStep = 200

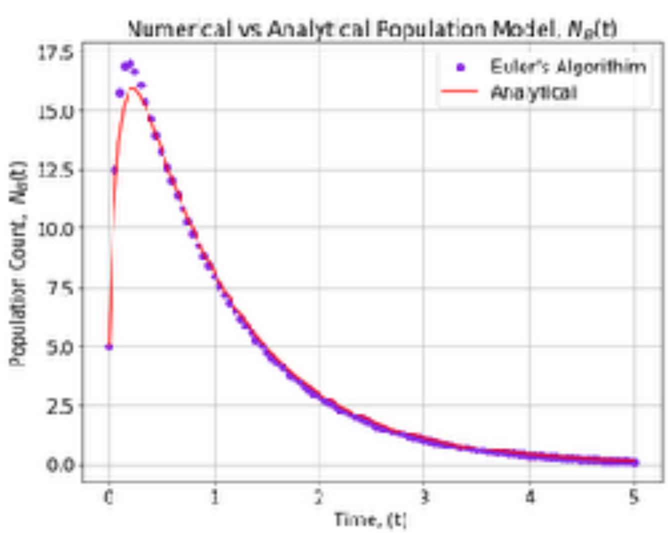


*Figure 11.* Comparison of Numerical results against Analytical function (Equations 6 and 15) for the same time values. Only holds for $\tau_B < \tau_A$

So, prediction wise, we were right. There is a steeper slow, and a lower peak as a result of that. It would seem, then, that the methods garnered useful results. We have models for the A and B nuclei in the system for various scenarios, and we did so with two very different ways.

# Section III - Conclusion

Inspection of the results of Section II show that, when comparing analytical and numerical models, our results matched very well. That was the goal of all of this, so confirmation is great. Still, some thoughts:

It became a trend as the models were found that, the more complicated the function, the more deviation we saw in the final comparison plot. It seems that Euler's method's is great at approximating shapes and trends, but can easily fall apart when the ODE in question gets to be a bit difficult. The errors seen were corrected because I could see the correct function and force a fit by adjusting dt and lastStep. What would happen if I simply wasn't able to use calculus or Mathematica to solve a a differential equation? The only way to check the numerical model, then, would be to use experimental results. That would be an interesting addition to this experiment; the comparison of algorithms, analytical models, and reality.