The Shape of Monotone and Skew Monotone Pattern Avoiding Permutations

Benjamin Fineman

Department of Mathematics
University of California, Davis

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Definition of Permutation Patterns

Definition: A π-pattern in σ is a subsequence of σ that has the same relative order as π.
Example: The permutation $1432 \in S_4$ contains three 132-patterns:

- 143
- 142
- 132.

If σ contains no copies of π, we say that σ avoids π.

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Origins of Pattern Avoiding Permutations

Connections to Sorting

- 1915 (MacMahon): first appearance (enumerated permutations avoiding 123)
- 1968 (Knuth): stack-sorting

Theorem

(Knuth) A permutation σ can be stack-sorted if and only if it avoids the pattern 231.

Counting Pattern Avoiding Permutations

Definition: $S_n(\pi)$=set of permutations in S_n which avoid π.
Examples: $S_4(12) = \{4321\}$, so $|S_4(12)| = 1$.
$S_4(132) =$
$\{1234, 2134, 2341, 2314, 3124, 3214, 3241$
$3412, 3421, 4123, 4213, 4231, 4312, 4321\}$
Thus: $|S_4(132)| = 14$.

How big is $|S_n(\pi)|$?

- If π is any pattern of length 3, $|S_n(\pi)| = C_n \approx 4^n/P(n)$. (MacMahon 1915, Knuth '68)
- π is pattern of length 4, there are three pattern classes:
  - $|S_n(1234)| \approx 9^n/P(n)$ (Gessel '90)
  - $|S_n(1423)| \approx 8^n$ (Bóna '97)
  - $|S_n(1324)|$ between 9.8^n and 13.8^n (Beven '14, Bóna '15)
- No known enumeration for any pattern of length > 4, however, some asymptotics known.
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Stanley-Wilf Conjecture '81

Upper Bound for the number of pattern avoiding permutations

Theorem

(Marcus and Tardos '04) For any pattern π there exists a constant c such that $|S_n(\pi)| < c^n$.

Pattern avoiding permutations are sparse: $|S_n| = n! \approx e^{n \log n - n}$

Current Research Directions for Pattern Avoiding Permutations

- Generalize either concept of pattern or avoidance:
  - generalized patterns (Babson and Steingrímsson '00)
  - allow a small number (low density) of patterns
  - avoid more than one pattern simultaneously (Atkinson '99, West '96, and others)
- Better understand structure
  - shape (Pak and Miner '13, Atapour and Madras '13, Hoffman, Rizzolo, and Slivken '14 and others)
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Results

Low density of patterns

Definition: Let $T_n(\pi)$ be the set of permutations containing a "small number" of π-patterns.
"small number" means fewer than $\delta^k n^k$, where k is the length of the pattern π.

Theorem

(F.) For any pattern π there are constants $a \leq b < 1$, such that

$$(a^n)n! \leq |T_n(\pi)| \leq (b^n)n!$$

Permutation Diagram

Diagram for 352987461 in S_9

			X					
				X				
					X			
							X	
	X							
						X		
X								
		X						
								X

Sketch of Upper Bound Proof

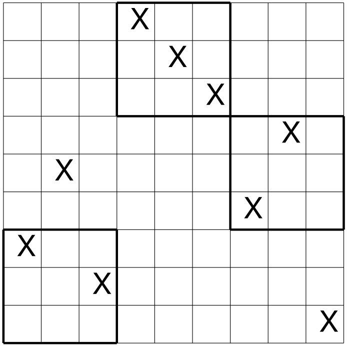
Special Case: Consider permutations in S_9 containing fewer than eight 132-patterns.

Idea: Find a condition that all such permutations must satisfy.

Sketch of Upper Bound Proof

Permutation Diagrams and Block Partitions

Diagram for 352987461 in S_9



Sketch of Lower Bound Proof

Idea: Find a relatively large number of permutations containing a small number of π-patterns.

Strategy: Split the problem into 2 cases depending on π.

- π starts with a decreasing sequence (Ex: $\pi = 4213$.)
- π starts with an increasing sequence (Ex: $\pi = 1324$.)

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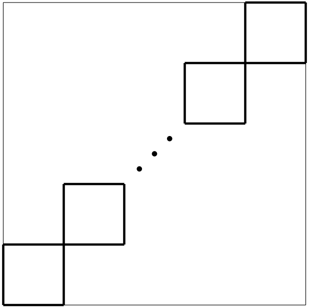

- Question: Select a permutation uniformly at random from $S_n(\pi)$. What does the permutation diagram look like?
- π is a pattern of length 3 (well studied) (Hoffman, Rizzolo, and Slivken '14, Pak and Miner '14)
- π pattern of length k and starts with a k, then with high probability, some neighborhood of the upper left hand corner contains no X. (Atapour and Madras '13)

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Shape Results for Monotone Pattern-Avoiding Permutations

Definition

For a permutation $\sigma \in S_n$, let $\mathcal{D}_t(\sigma)$ denote the set of indices $i \in [n]$ such that $|i - \sigma(i)| > t$ and let $D_t = |\mathcal{D}_t(\sigma)|$.

Theorem

(F., Slivken) Let $\epsilon > 0$, and let $\mathbb{P}_n^{\searrow k}$ be the uniform probability measure on $S_n[k+1, k, \ldots 1]$. There exists $c > 0$ such that for n sufficiently large, we have

$$\mathbb{P}_n^{\searrow k}\left[D_{n^{1/2+\epsilon}} > 0\right] = O(\exp(-cn^{\epsilon})).$$

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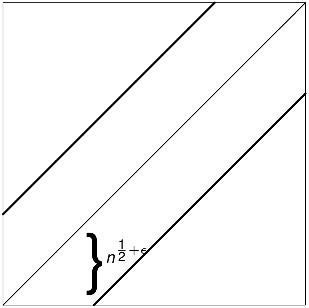
Shape Results for Pattern-Avoiding Permutations

In the permutation diagram of σ, with high probability, all X's are close to the diagonal.



Sketch of Proof

- Associate each permutation in $S_n(k+1, \ldots, 1)$ with a unique pair (A, B) of sequences
- Work with these sequences.

Rank of σ_i

Definition

Given σ, let the **rank** of σ_i=length of the longest decreasing sequence in σ that terminates at σ_i.

Example: if $\sigma = 4213$ the ranks of $4, 2, 1, 3$ are $1, 2, 3, 2$ respectively.

1. σ avoids $[k + 1 \dots 1]$, all ranks between 1 and k.
2. elements of same rank form increasing sequence.
3. σ is the union of k increasing sequences.

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$\sigma \in S_n(k+1,\ldots,1) \to (A,B)$

Theorem

(Bóna) $|S_n(k+1,\ldots,1)| < k^{2n}$.

1. At most k^n ways to assign a rank to each number $1,2\ldots n$ (B sequence)
2. At most k^n ways to assign a rank to each position. (A sequence)

$\sigma \in S_n(k+1, \ldots, 1) \to (A, B)$

Example: $352187469 \in S_9$ avoids 4321.

- $A = (1, 1, 2, 3, 1, 2, 3, 3, 1)$ (Ranks of positions)
- $B = (3, 2, 1, 3, 1, 3, 2, 1, 1)$ (Ranks of numbers)

$A_i = rank(\sigma_i)$

$B_i = rank(i)$.

Recovering the permutation from the pair (A, B)

Example: $352187469 \in S_9$

1									
1									
2									
3									
1									
3									
1									
2									
3									
	1	1	2	3	1	2	3	3	1

Recovering the permutation from the pair (A, B)

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1									
1									
2									
3									
1									
3									
1	X								
2									
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1					X				
2						X			
3								X	
1		X							
3							X		
1	X								
2			X						
3				X					
	1	1	2	3	1	2	3	3	1

Chernoff/Hoeffding's Inequality ('52, '63)

Theorem

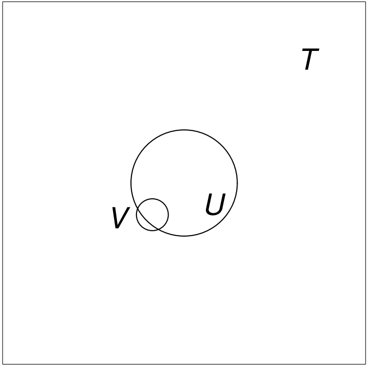
Let $Y_1, \ldots, Y_n$ be independent 0-1 *random variables with* $\mathbb{E}[Y_i] = p$. *Let* $Y = \sum_{i=1}^{n} Y_i$, $\mu = \mathbb{E}[Y] = np$. *Then*

$$\mathbb{P}[|Y - \mu| \geq \lambda] \leq \exp\left(-\frac{2\lambda^2}{n}\right).$$

- Y_i indicator function for A_i (or $(B_i) = r$

Sketch of the Proof

$$\mathbb{P}(V|U) = \frac{|V \cap U|}{|U|} \leq \frac{|V|}{|U|}$$



$T =$ all pairs of sequences

$U =$pairs $\leftrightarrow$ perm. avoiding $k, k-1\dots 1$

$V =$"bad pairs" (far from diagonal)

$|T| = k^{2n}$

$|U| \approx k^{2n}/P(n)$

$|V|/|U| \approx \exp(-cn^{\epsilon})$

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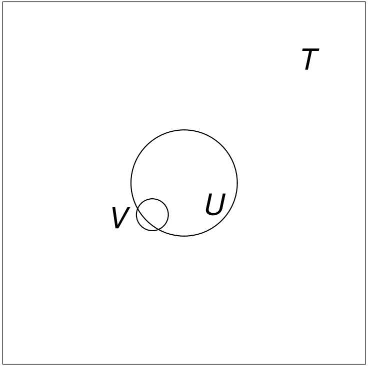

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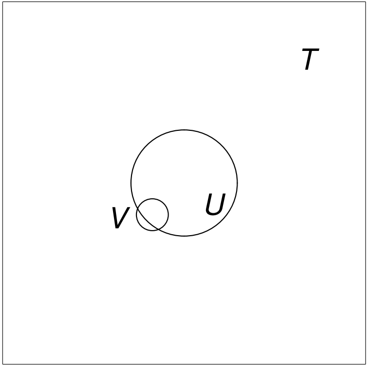

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Bijection of Backelin, West, and Xin

Special Case: Bijection from $S_n([k+1, k, \ldots, 1])$ to $S_n([k+1, k, \ldots m+1, 1, 2, \ldots m])$

- Bijection preserves Left-to-Right Maxima
- Bijection works iteratively
- Example: $S_n(4321) \to S_n(4312) \to S_n(4123)$

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$S_n(4321) \to S_n(4312)$

Search for instances of $[4, 3, 1, 2]$, and switch lowest 1 and 2.

							X	
	X							
				X				
								X
						X		
					X			
X								
			X					
		X						

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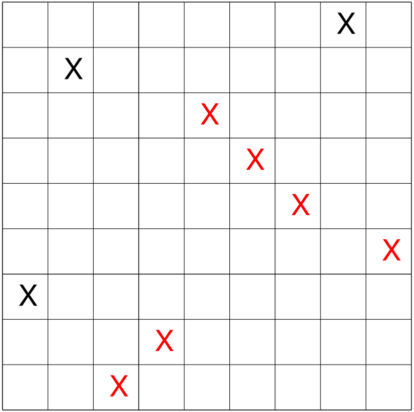
								X
X								
			X					
		X						

$S_n(4312) \to S_n(4123)$

Search for instances of $[4, 1, 2, 3]$, and switch $123 \to 312$.



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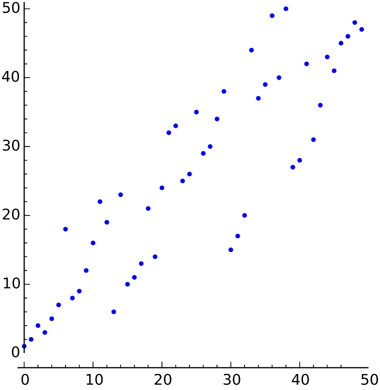
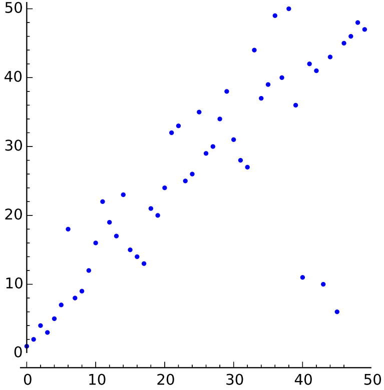
							X	
	X							
				X				
					X			
						X		
		X						
X								
								X
			X					

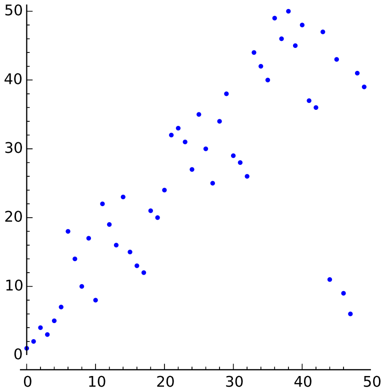
BWX Bijection - 4321 avoiding



BWX Bijection - 4312 avoiding



BWX Bijection - 4123 avoiding



Shape Results for Pattern-Avoiding Permutations

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For a permutation $\sigma \in S_n$, let $\mathcal{D}_t(\sigma)$ denote the set of indices $i \in [n]$ such that $|i - \sigma(i)| > t$ and let $D_t = |\mathcal{D}_t(\sigma)|$.

Theorem

Let $\delta > \epsilon > 0$, and let $\mathbb{P}_n^{k \vee m}$ be the uniform probability measure on $S_n^{k \vee m} = S_n(k+1, \cdots, m+1, 1 \cdots, m)$. Then there exists a $c > 0$ such that for n sufficiently large we have

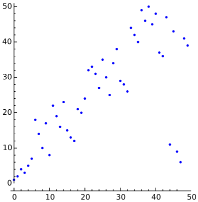
$$\mathbb{P}_n^{k \vee m}\left[D_{n^{1/2+\delta}} > n^{1+\epsilon-\delta}\right] = O(\exp(-cn^{2\epsilon})).$$

Shape Results for Pattern-Avoiding Permutations

Definition

For a permutation $\sigma \in S_n$, let $\mathcal{D}_t(\sigma)$ denote the set of indices $i \in [n]$ such that $|i - \sigma(i)| > t$. Also, let $A^- = \{i : \sigma(i) - i < 0\}$ and let $A^+ = \{i : \sigma(i) - i \geq 0\}$

- if $\sigma \in S_n^{k \vee m}$, with high probability, we have:
  1. $\mathcal{D}_{n^{1/2+\delta}} \in A^-$
  2. $\sigma(i) - i < n^{1/2+\epsilon}$ for $i \in A^+$



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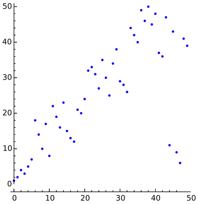

Sketch of Proof

- $A^- = \{i : \sigma(i) - i < 0\}$
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1. $\sum_{i \in [n]} \sigma(i) - i = 0.$
2. $[n] = A^+ \cup A^-$
3. $\sum_{i \in A^+} \sigma(i) - i = \sum_{i \in A^-} i - \sigma(i)$

$$|\mathcal{D}_{n^{1/2+\delta}}| n^{1/2+\delta} \leq \sum_{\mathcal{D}_{n^{1/2+\delta}}} i - \sigma(i)$$

$$\leq \sum_{A^-} i - \sigma(i) = \sum_{A^+} \sigma(i) - i \leq (n) n^{1/2+\epsilon}.$$

Thus, with high probability, we have:

$$|\mathcal{D}_{n^{1/2+\delta}}| \leq n^{1+\epsilon-\delta}.$$

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$$|\mathcal{D}_{n^{1/2+\delta}}| \leq n^{1+\epsilon-\delta}.$$

Sketch of Proof

- $A^- = \{i : \sigma(i) - i < 0\}$
- $\mathcal{D}_{n^{1/2+\delta}} \in A^-$
- $\sigma(i) - i < n^{1/2+\epsilon}$ for $i \in A^+$

1. $\sum_{i \in [n]} \sigma(i) - i = 0.$
2. $[n] = A^+ \cup A^-$
3. $\sum_{i \in A^+} \sigma(i) - i = \sum_{i \in A^-} i - \sigma(i)$

$$|\mathcal{D}_{n^{1/2+\delta}}| n^{1/2+\delta} \leq \sum_{\mathcal{D}_{n^{1/2+\delta}}} i - \sigma(i)$$

$$\leq \sum_{A^-} i - \sigma(i) = \sum_{A^+} \sigma(i) - i \leq (n) n^{1/2+\epsilon}.$$

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Extensions and Future Work

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- shape results for other classes of patterns, e.g. 4213
- shape results for permutations with a low density of a given pattern

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$S_n(4213)$

Theorem

(Bóna/Simion and Schmidt '85) Given $\sigma \in S_n(4213)$, there is a unique permutation $\sigma \in S_n(312)$ that has the same left-to-right maxima (values and positions are the same.)

- 312 is a skew-monotone pattern
- unsure of multiplicities, i.e. how many 4213 avoiding permutations correspond to "bad" 312 avoiding permutations.

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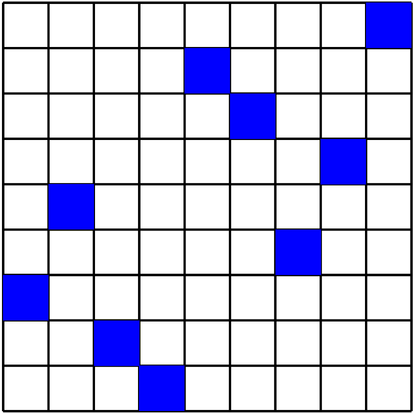
Low Density of 4321 patterns

Example: $352187469 \in S_9$

								X
				X				
					X			
							X	
	X							
						X		
X								
		X						
			X					

Low Density of 4321 patterns

Example: $\sigma \in S_n$, n large, $\leq (\delta n)^4$ 4321-patterns



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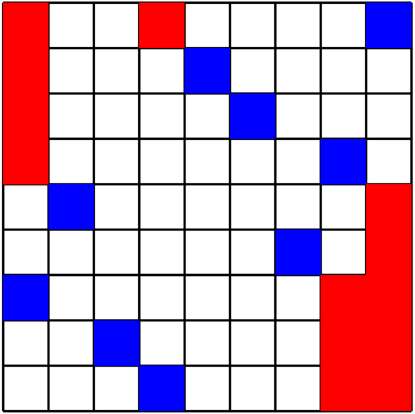

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