2.7 Radiation Level Calculation

In a nuclear power production plant, the daily radiation level is modeled as a normally distributed random variable with mean $\mu = 100\,\mu\mathrm{Sv/day}$ and variance $\sigma^2 = 50\,\mu\mathrm{Sv}^2/\mathrm{day}^2$. An alarm system is configured to activate when radiation levels exceed a critical threshold of $120\,\mu\mathrm{Sv/day}$. This document calculates the expected radiation level per hour in scenarios where the alarm has been activated.

Given that the radiation level exceeds $120\,\mu\mathrm{Sv/day}$, we seek to calculate the conditional expectation $E[X|X > 120]$, where X represents the daily radiation level. This expectation will then be converted to an hourly rate by dividing by the total number of hours in a day.

Daily Conditional Expectation

The conditional expectation of X given $X > 120$ can be found using the properties of the normal distribution. This calculation involves integrating the tail of the normal distribution beyond $120\,\mu\mathrm{Sv/day}$, weighted by the radiation level, and normalized by the probability of exceeding this threshold. Mathematically, this is expressed as:

$$E[X|X > 120] = \mu + \sigma \frac{\phi\left(\frac{120-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{120-\mu}{\sigma}\right)},$$

where ϕ and Φ represent the standard normal probability density function and cumulative distribution function, respectively.

Conversion to Hourly Rate

To find the expected radiation level per hour, we divide the conditional daily expectation by 24:

$$E[X_{\text{hour}}|X > 120] = \frac{E[X|X > 120]}{24}.$$

Conclusion

This calculation provides the expected radiation level per hour under the condition that the daily radiation level exceeds the alarm threshold of $120\,\mu\mathrm{Sv/day}$. It leverages the properties of the normal distribution and the concept of conditional expectation.

3) (Bonus) Introduction to $Y = (X, X)$

Consider X as a continuous real-valued random variable and define $Y = (X, X)$, which maps X into a two-dimensional space, $\mathbb{R}^2$. We aim to show that Y is not a continuous random variable in $\mathbb{R}^2$ and to describe the probability distribution of Y.

Continuity of Y

A continuous random variable in $\mathbb{R}^2$ would have a probability density function (pdf) that assigns probabilities to regions in the plane. For Y to be considered continuous, it would need a pdf $f_Y(y_1, y_2)$ such that the probability $P((Y_1, Y_2) \in A)$ for any region A in $\mathbb{R}^2$ is given by the integral of f_Y over A.

However, the transformation $Y = (X, X)$ implies that all probability mass of Y is concentrated on the line $y_1 = y_2$ in $\mathbb{R}^2$, with zero probability mass elsewhere. This characteristic precludes Y from having a conventional two-dimensional pdf, indicating that Y is not a continuous random variable in the typical sense used in $\mathbb{R}^2$.

Distribution of Y

Although Y does not have a standard two-dimensional pdf, its distribution can still be described. Given that X has a pdf $f_X(x)$, the distribution of Y is such that all probability is concentrated along the line $y_1 = y_2$. This means that for any function of Y, say $g(Y)$, the expectation $E[g(Y)]$ would depend on the distribution of X and can be computed considering Y's unique concentration of probability.

Conclusion

The random variable $Y = (X, X)$ illustrates a case where, despite originating from a continuous random variable X, the resultant Y does not conform to the definition of a continuous random variable in $\mathbb{R}^2$. Its distribution is uniquely defined by the line $y_1 = y_2$ and is directly related to the probability distribution of X.