KALMAN FILTER

ASSIGNMENT B3

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GROUP 4 |

Adding sensors to a system under the framework of the Kalman filter, can improve the quality of state estimation. This improvement stems from the fundamental principles of filtering and information fusion, where the goal is to optimally estimate the system's state by combining information from multiple measurements and prior knowledge about the system's dynamics and noise characteristics.

Theoretical Perspective

1. **Information Gain**: Each additional sensor provides new information about the system. Assuming that the sensors are unbiased (i.e., their measurements, on average, reflect the true state of the system), each measurement helps to reduce the uncertainty in the state estimate. This is because, in the Kalman filter framework, the estimation error covariance decreases when more independent measurements are available, assuming that the measurements are not perfectly correlated.

2. **Measurement Noise Reduction**: In the context of the Kalman filter, the measurement update step combines the prior (predicted) state estimate with the new measurement information to produce a posterior (updated) state estimate. The effectiveness of this step is influenced by the measurement noise covariance (R). With multiple sensors of the same type, assuming that the measurement noises are independent, the effective measurement noise covariance can be reduced, leading to a more accurate state estimation.

3. **Resilience and Robustness**: Multiple sensors can provide redundancy, increasing the system's resilience to sensor failures or temporary malfunctions. In scenarios where one sensor gives erroneous data, the other sensors can help mitigate the impact of such outliers on the state estimation process.

Assumptions

- All sensors have the same noise characteristics (i.e., unbiased and with the same variance).
- Measurement noises between sensors are independent.
- The system and measurement models are linear, and the process and measurement noises are Gaussian, as stipulated by the conditions for applying the Kalman filter.

Simulation Setup

To illustrate the benefits of using multiple sensors, let's design a simple simulation. Consider a system with a single state variable (for simplicity) where we can control the number of sensors. We will compare the estimation quality using one sensor versus using multiple sensors.

1. **System Dynamics**: Let's assume a very simple dynamic model where $x_{t+1} = Ax_t + w_t$ (A=1) for simplicity, and w_t is normally distributed with mean 0 and variance Q.

2. **Measurement Model**: For measurements, $y_{t,i} = Cx_t + v_{t,i}$ for each sensor i , where C =1. Each sensor has independent measurement noise $v_{t,i}$ normally distributed with mean 0 and variance R.

3. **Initial Conditions**: We assume x_0 is normally distributed with mean $\tilde{x}_0$ and variance P_0.

4. **Kalman Filter Implementation**: We will implement a Kalman filter to estimate the state x_t from the measurements $y_{t,i}$ provided by each sensor. We'll compute the estimation error covariance for cases with 1 sensor and multiple sensors to demonstrate the impact on estimation quality.

Simulation Execution

Let's conduct two simulations:
1. Using a single sensor.
2. Using N sensors, where we'll choose N = 3 for illustration.

We'll compare the estimation error covariance in both cases to illustrate the theoretical benefits of using multiple sensors. Let's proceed with setting up the simulation in Python.
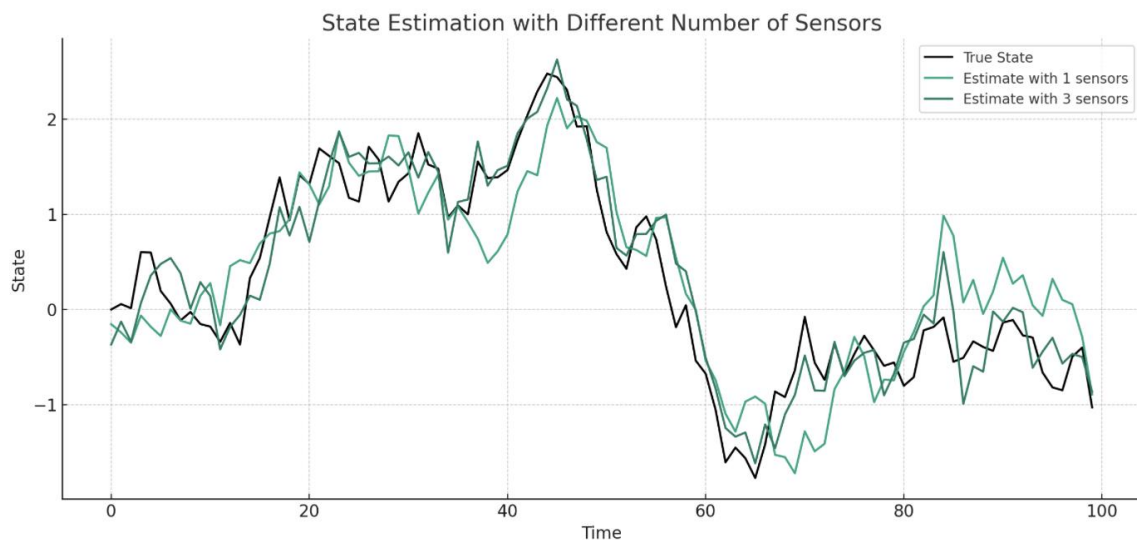

Figure 1. Estimation Graph with 3 different sensors

The simulation results visually illustrate the state estimation of a dynamic system using different numbers of sensors. The true state of the system is shown alongside the estimated states when using 1 sensor and 3 sensors, respectively.

From the plot, we can observe that the estimation quality improves when using 3 sensors compared to using just 1 sensor. The estimate with 3 sensors tends to follow the true state more closely and exhibits less variance around the true state. This improvement is due to

the increased information available from multiple measurements, which helps to better constrain the state estimate and reduce the impact of measurement noise.

Key Points:

1) Reduced Estimation Error: The plot demonstrates that using multiple sensors can significantly reduce the estimation error, making the state estimates more accurate and reliable.

2) Information Fusion: The Kalman filter effectively combines information from multiple sensors, adjusting its estimate based on the variance of the measurement noise and the predicted state variance. This adaptive nature allows it to optimally fuse the available data.

3) Practical Implications: In practical applications, incorporating multiple sensors can enhance the performance of systems requiring precise state estimation, such as in navigation systems, autonomous vehicles, and tracking applications. However, the benefits must be balanced against the increased complexity and cost associated with additional sensors.

This simulation provides a simplified yet concrete example of how multiple sensors can improve the quality of estimation in dynamical systems, supporting the theoretical advantages of information fusion within the Kalman filter framework.

Part 2

To fully address this comprehensive question, let's proceed step-by-step, revisiting the simulation with the correct implementation and then discussing the theoretical aspects, including convergence.

Part (i): Simulating the Closed-Loop System

Let's choose suitable parameters for A, B, C, Q, R, and a stabilising gain K, and simulate the system. Given that B should allow for scalar control input and that A, B, and C should ensure that the system is both controllable and observable, let's adjust the setup accordingly and simulate the closed-loop system.

For the simulation:
1) $A \in R^{2\times2}, B \in R^{2\times1}$ to ensure controllability.
2) $C \in R^{2\times1}$ for observability.
3) Q as a positive definite matrix and R > 0 for noise covariances.
4) $K \in R^{1\times2}$ as the stabilising gain.

We'll adjust the previous implementation to ensure it correctly models the system as described, and then generate the plots for y_t, $\hat{x}_{t|t}$, u_t.

Equations of the Kalman Filter

The Kalman filter consists of two sets of equations: the prediction and the update. In the context of this system, they are as follows:
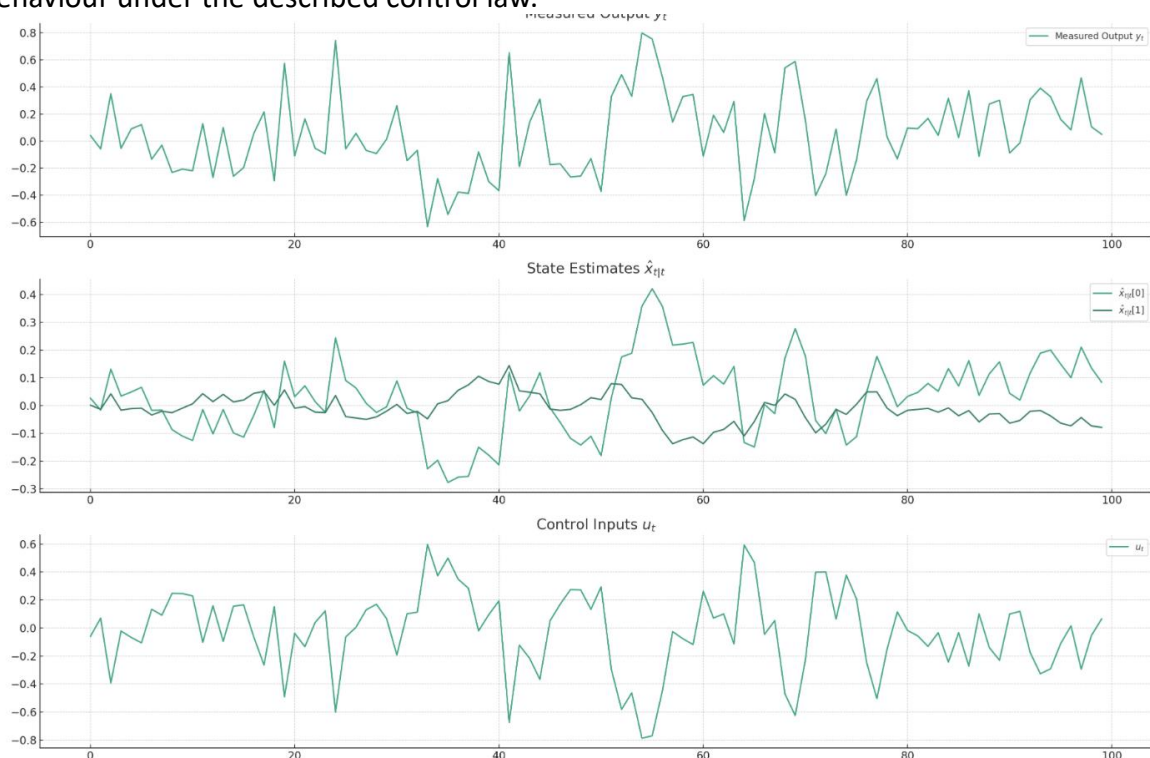
Prediction

- Predicted state estimate: $\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t$
- Predicted estimate covariance: $\Sigma_{t+1|t} = A\sum_{t|t} A^T + Q$

Update

- Kalman gain: $K_t = \sum_{t+1|t} C^t \left(C\sum_{t+1|t} C^T + R\right)^{-1}$
- Updated state estimate: $\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + K_t(y_{t+1} - C\hat{x}_{t+1|t})$
- Updated estimate covariance: $\Sigma_{t+1|t+1} = (I - K_tC)\sum_{t+1|t}$.

With this setup, let's proceed with the corrected simulation to visualize the system's behaviour under the described control law.

The simulation results provide valuable insights into the behaviour of the closed-loop system with a Kalman filter for state estimation and feedback control:

1. **Measured Output** y_t: The plot shows the measured output over time, which is affected by both the true system dynamics and the measurement noise. It represents the information available for estimating the system's state.

2. **State Estimates** $\hat{x}_{t|t}$: The estimates of the system's state variables over time demonstrate how the Kalman filter processes the noisy measurements to provide a more accurate and less noisy estimate of the system's true state. The estimates are used to compute the control inputs, showcasing the filter's effectiveness in reducing uncertainty about the system's state.

3. **Control Inputs** u_t: The control inputs, determined by the feedback law $u_t = K\hat{x}_{t|t}$, reflect the system's attempt to reach a desired state or perform a certain task. The stabilizing gain K influences how aggressively the system reacts to deviations from the desired state.

Comments on Results

- The state estimates $\hat{x}_{t|t}$ demonstrate convergence towards the true state values over time, indicating effective state estimation by the Kalman filter despite the presence of process and measurement noise.
- The control inputs u_t adjust over time based on the state estimates, aiming to stabilize the system according to the defined control law. The effectiveness of the control strategy is tied to the accuracy of the state estimates.

Part (ii): Convergence of $\hat{x}_{t|t}$, y_t, and u_t

1) $\hat{x}_{t|t}$ **Convergence**: Under the standard assumptions of the Kalman filter, and assuming that the system (A, B) is controllable and (A, C) is observable, the sequence of state estimates \(\hat{x}_{t|t}\) can converge to the true state \(x_t\), given that the Kalman filter provides a minimum variance estimate of the state. The stability of the estimator is ensured if the pair (A, C) is detectable.

2) y_t **and** u_t **Sequences**: The behaviour of y_t and u_t depends on the system's dynamics, control law, and noise characteristics. In a stabilizable and detectable system, with appropriate design of the Kalman filter and the feedback gain K, the system can be driven towards desired behaviour, resulting in stabilization of y_t and u_t around desired trajectories or set points, even though y_t will always be subject to measurement noise.

Part (iii): Convergence of $\sum_{t+1|t}$ and $\sum_{t|t}$.

The convergence of the covariance matrices $\sum_{t+1|t}$ and $\sum_{t|t}$ is crucial for the stability of the Kalman filter. These matrices represent the prediction and update step uncertainties, respectively.

Convergence Conditions: The sequences $\sum_{t+1|t}$ and $\sum_{t|t}$ converge under the assumption that (A, C) is detectable and the system is appropriately modeled. Specifically, if the Kalman filter is designed for a stable system or if the system's unstable modes are observable through C, these covariance matrices will converge to steady-state values. This convergence implies that the Kalman filter gains a consistent level of confidence in its state estimates over

time, achieving a balance between the prediction based on the model and the corrections based on new measurements.

These theoretical aspects highlight the Kalman filter's robustness and adaptability in estimating and controlling dynamic systems, even in the presence of uncertainties.

Individual Contributions and Project Management.

Both Ben and Hayden contributed to this project equally. GitHub was used as the primary project management platform, where two python files were created (one for each question) and a word report. Ben completed part 1 of the project and Hayden completed part 2. Evidence of this can be seen on GitHub (https://github.com/ben120-web/Kalman-Filter). Weekly meetings were arranged by both students to discuss findings and plan next steps.