1. For a beta distribution $P(p|a,b)$ which represents prior distribution and a binomial distribution $P(m|N,p)$ which represents likelihood distribution, the posterior distribution will be a beta distribution

proof: posterior $= \dfrac{P(p|a,b) \cdot P(m|N,p)}{P(m)}$

$$= \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot C_m^n \, p^m (1-p)^{N-m} \cdot p^{a-1} (1-p)^{b-1}}{\int_0^1 C_m^n \, p^m (1-p)^{N-m} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1} (1-p)^{b-1} \, dp}$$

where $\Gamma(\cdot)$ is gamma function;

we use $\frac{1}{B(a,b)}$ to represent $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$

$$\Rightarrow \frac{\frac{1}{B(a,b)} \cdot C_m^n \cdot p^{m+a-1} \cdot (1-p)^{N-m+b-1}}{\frac{C_m^n}{B(a,b)} \cdot \int_0^1 p^{m+a-1} \cdot (1-p)^{N-m+b-1} \, dp}$$

$$= \frac{1}{B(m+a, N-m+b)} p^{m+a-1} \cdot (1-p)^{N-m+b-1}$$

$$= \mathrm{Beta}(p|m+a, N-m+b) \quad \#$$

2. Prior distribution: $P(\lambda) = \mathrm{Gamma}(\lambda; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \lambda^{\alpha-1} \cdot e^{-\beta\lambda}$

likelihood distribution: $P(x|\lambda) = \mathrm{Possion}(x;\lambda) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$

Proof posterior distribution is a Gamma distribution

proof: $\dfrac{P(x|\lambda) \cdot P(\lambda)}{P(x)}$

$$= \frac{\dfrac{\lambda^{x} \cdot e^{-\lambda}}{x!} \cdot \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \lambda^{\alpha-1} \cdot e^{-\beta\lambda}}{\int_0^\infty p(x|\lambda) \cdot p(\lambda)\, d\lambda}$$

$$= \frac{\dfrac{\beta^{\alpha}}{x!\,\Gamma(\alpha)} \cdot \lambda^{x+\alpha-1} \cdot e^{-(1+\beta)\lambda}}{\dfrac{\beta^{\alpha}}{x!\,\Gamma(\alpha)} \cdot \int_0^\infty \lambda^{x+\alpha-1} \cdot e^{-(1+\beta)\lambda}\, d\lambda}$$

$$\because \int_0^\infty \lambda^{\alpha-1} \cdot e^{-\beta\lambda}\, d\lambda = \left(1 \Big/ \frac{\beta^{\alpha}}{\Gamma(\alpha)}\right)$$

$$\Rightarrow \frac{(1+\beta)^{x+\alpha}}{\Gamma(x+\alpha)} \cdot \lambda^{x+\alpha-1} \cdot e^{-(1+\beta)\lambda}$$

$$= \text{Gamma}(\lambda ; x+\alpha, 1+\beta)$$

#