# A beginner's introduction to Bayesian inference

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# Who am I?

- Statistician working on data science, machine learning and statistical inference across the university.
- User of Bayesian statistics for the past X years.
- Born in the same town as Thomas Bayes (Tunbridge Wells).

# Course timetable

- 2-3pm: (Interactive) lecture.
- 3-4pm: Problems and (hopefully) answers.

# Course outcomes

By the end of this course you should:

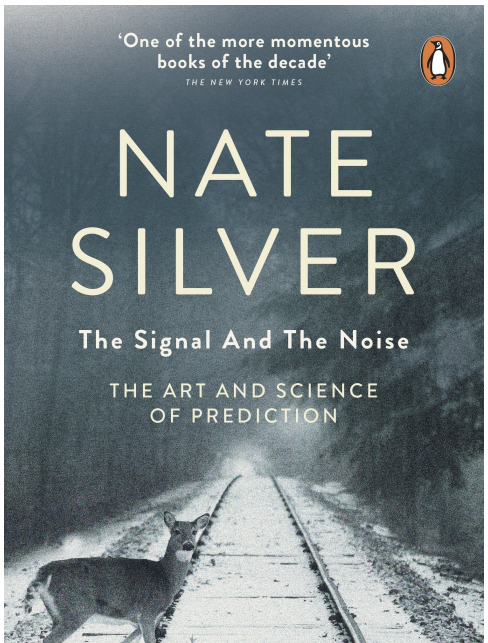
- Know what probability distributions are and why they are used in modelling.
- Understand the goal of statistical inference.
- Appreciate how Bayesian and frequentist approaches to inference achieve this goal.
- Know the elements required to do Bayesian inference and appreciate how they affect inferences.
- Know why exact Bayesian inference is *hard*.
- See how conjugate priors provide a slight remedy.

# Why don't more people use Bayesian inference?

- Most existing texts put a strong emphasis on its (seemingly) complex mathematical basis.
- Poor explanation of why computational sampling (usually MCMC) is needed.
- The view that Bayesian inference is more wishy-washy than frequentist inference.

# Tangible benefits of Bayesian inference

- Simple and intuitive model building (unlike frequentist statistics there is no need to remember lots of specific formulae).
- Exhaustive and creative model testing.
- The best predictions; for example, Nate Silver.
- Allows estimation of models that would be impossible in frequentist statistics: especially true in epidemiology!



# Outline

# Example: how to estimate disease prevalence?

- Suppose we take a sample of $N$ study participants from the population.
- We take their blood and use a clinical test to determine presence / absence of disease: finding $X$ are disease-positive.

Question: How do we use these data to estimate disease prevalence (with uncertainty)?

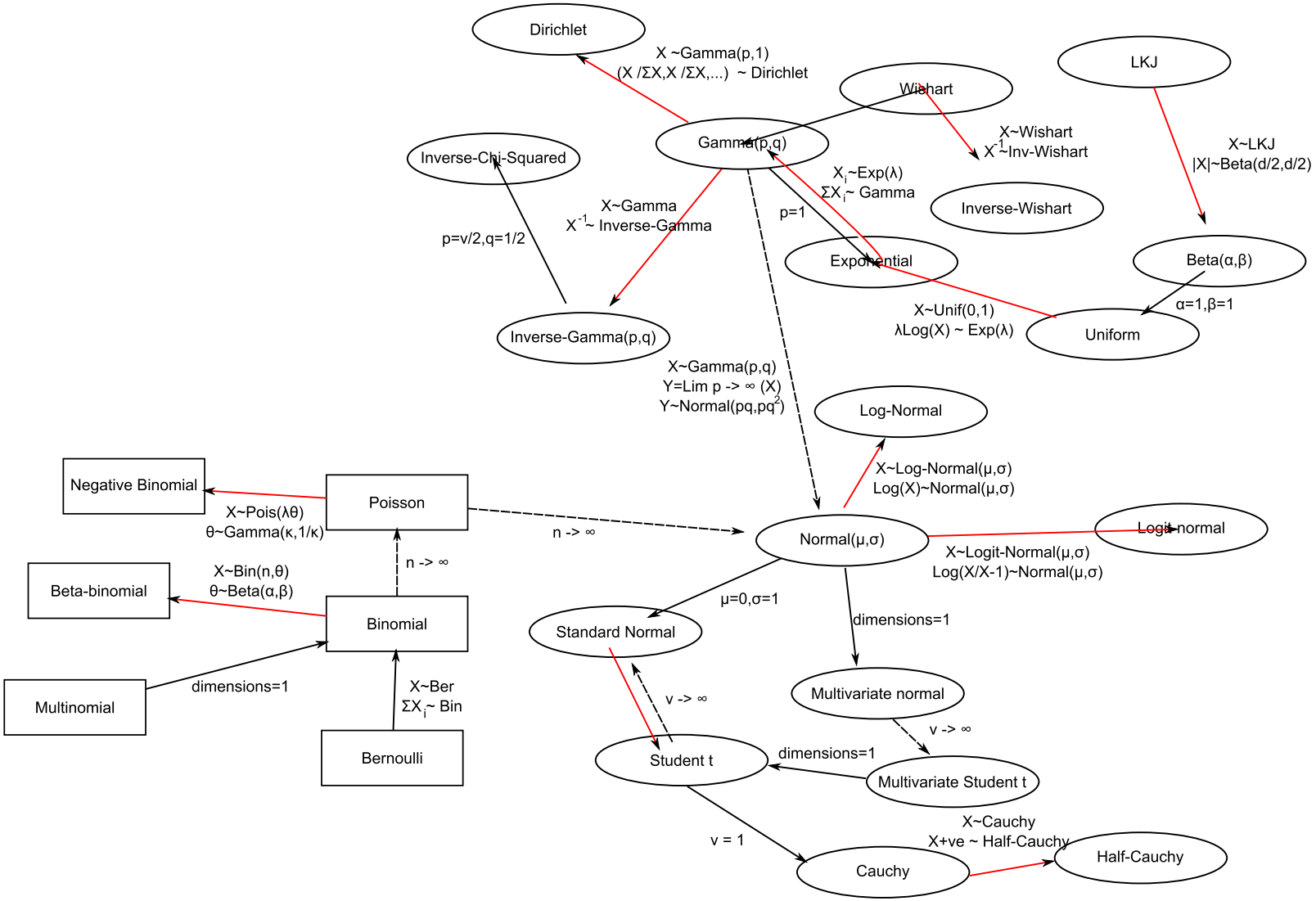
# Building a model to explain these data

We don't know a lot about how our data were produced:

- How exactly participants were picked.
- How the disease is distributed in the population.
- How the clinical test works.

Due to uncertainty $\Longrightarrow$ use a model that encompasses uncertainty: i.e. one that uses probability distributions.

# Which probability distribution?



# How to choose a probability model?

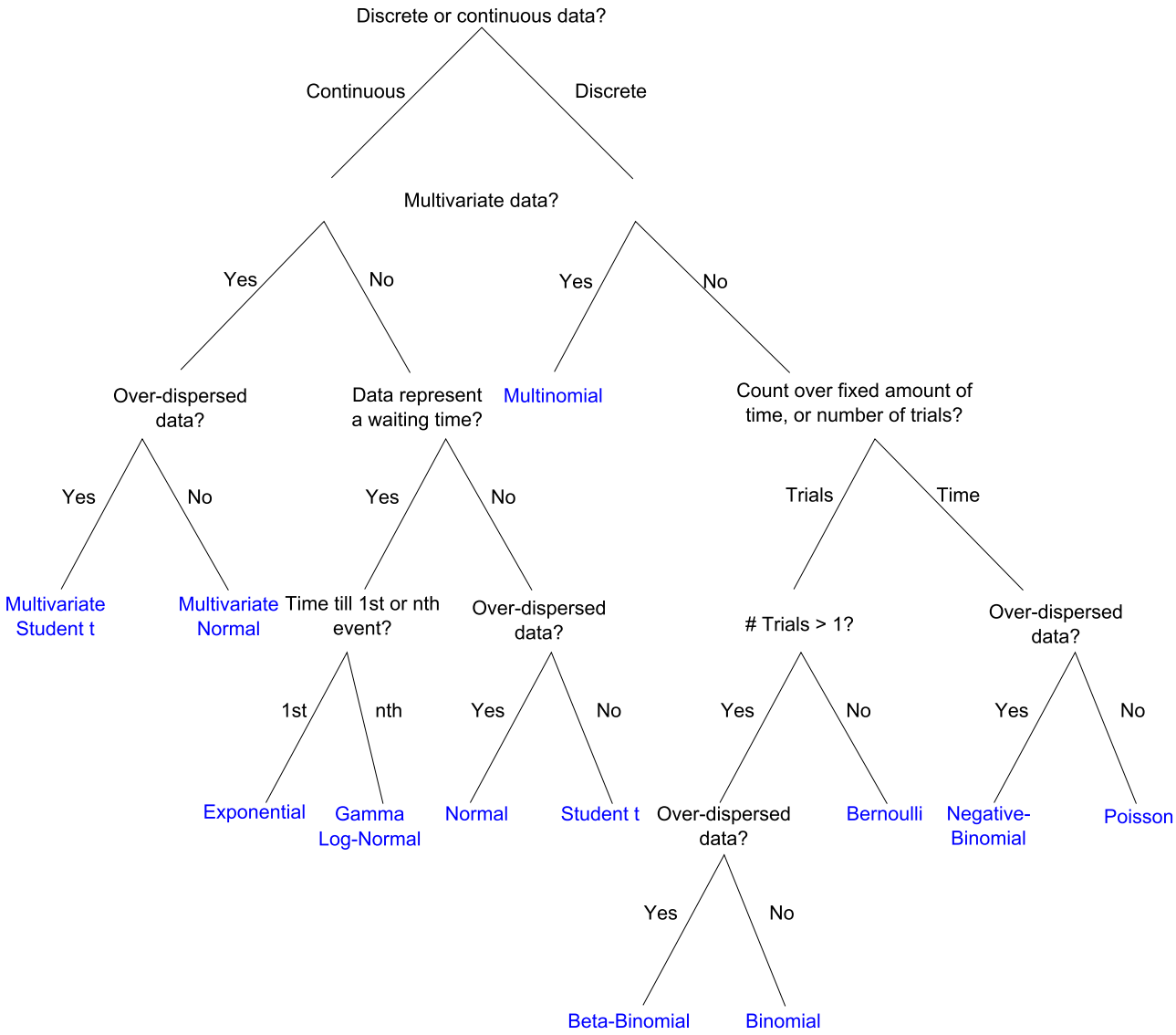
Characteristics of our data:

1. Our sample size $N$ is fixed.
2. Our data $X$ are discrete and can take values $0, 1, 2, ..., N-1, N$.

Assumptions:

1. Individuals represent independent samples from a population.
2. Those individuals are drawn from the same population.

# Which probability model satisfies these conditions?



# Binomial model: introduction

Analogy: count of disease-positive cases in a sample of size $N$ $\sim$ count of a coin landing heads up in $N$ flips of it.

If we assume the clinical test is perfect:

- $\Pr(+) = \theta$ is the proportion of disease-positive individuals in the population.
- Analogous to $\Pr(H) = \theta$, the probability the coin lands heads up: $0 \leq \theta \leq 1$.

# Binomial model probability

The probability of a given number of heads $X$ depends on:

- $\Pr(H) = \theta$.
- The number of possible ways to obtain result. E.g. if $N = 2$, there are two ways to obtain $X = 1$: $(1, 0)$ or $(0, 1)$.

# Binomial model probability

The probability for a given $X$ is:

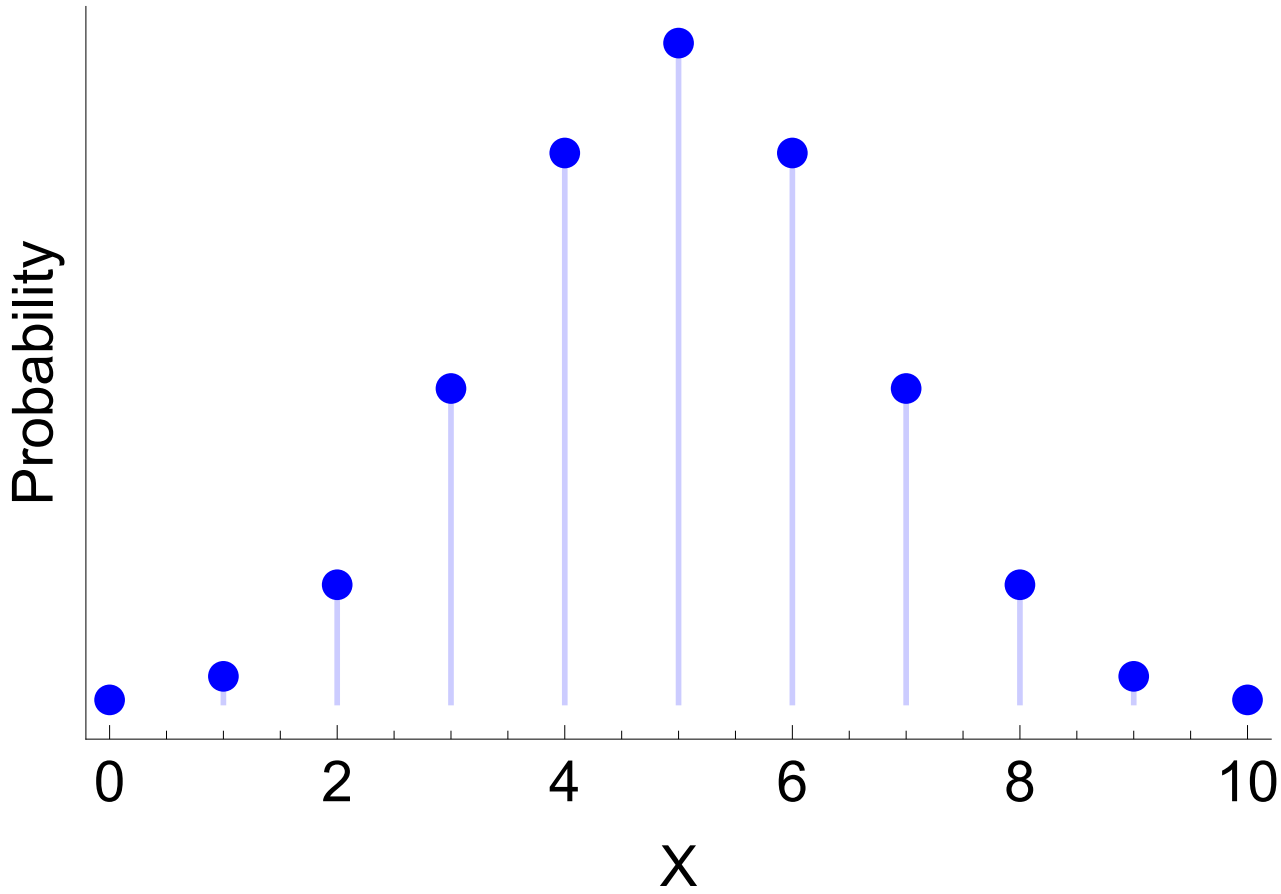
$$\Pr(X|\theta) = \binom{N}{X}\theta^X(1-\theta)^{N-X}. \tag{1}$$

We often use the following notation as shorthand:

$$X \sim \mathcal{B}(N, \theta). \tag{2}$$

# Binomial model probabilities: visualised

Suppose $\theta = 0.5$ and $N = 10$.



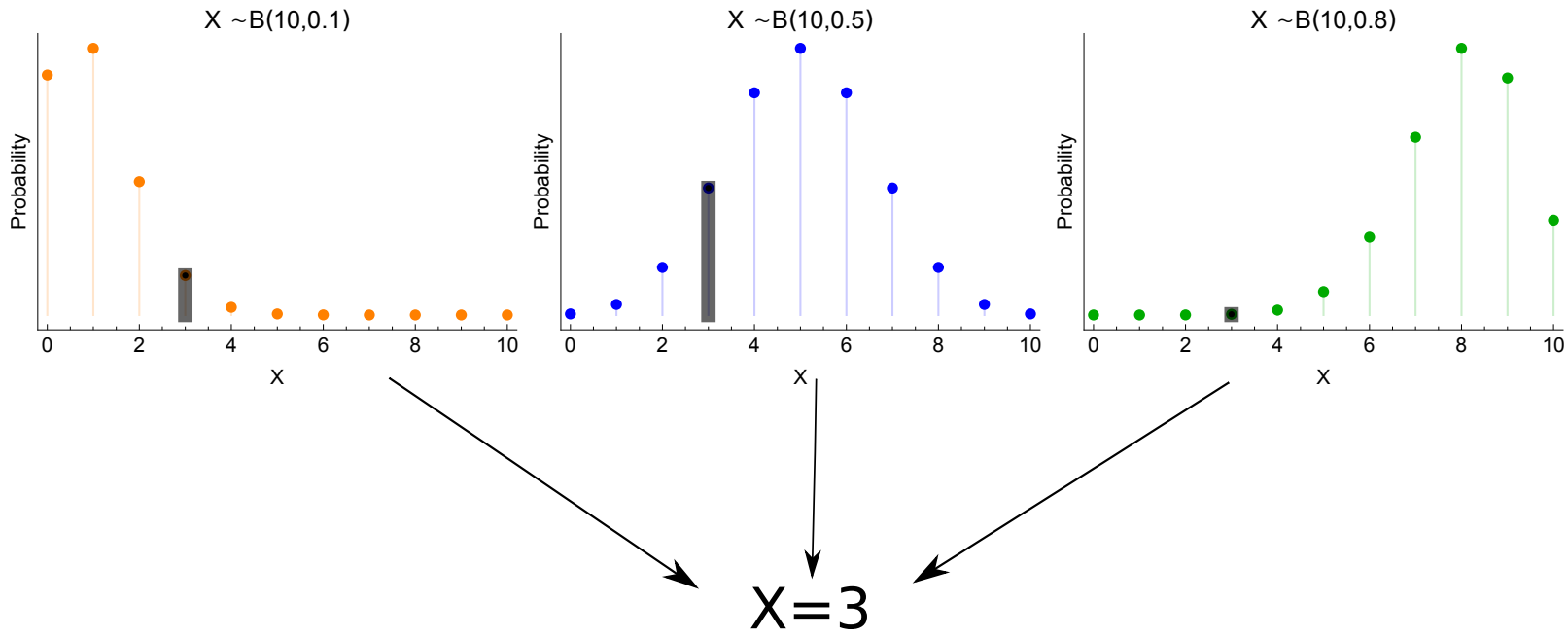
# Many ways of generating data

Suppose:

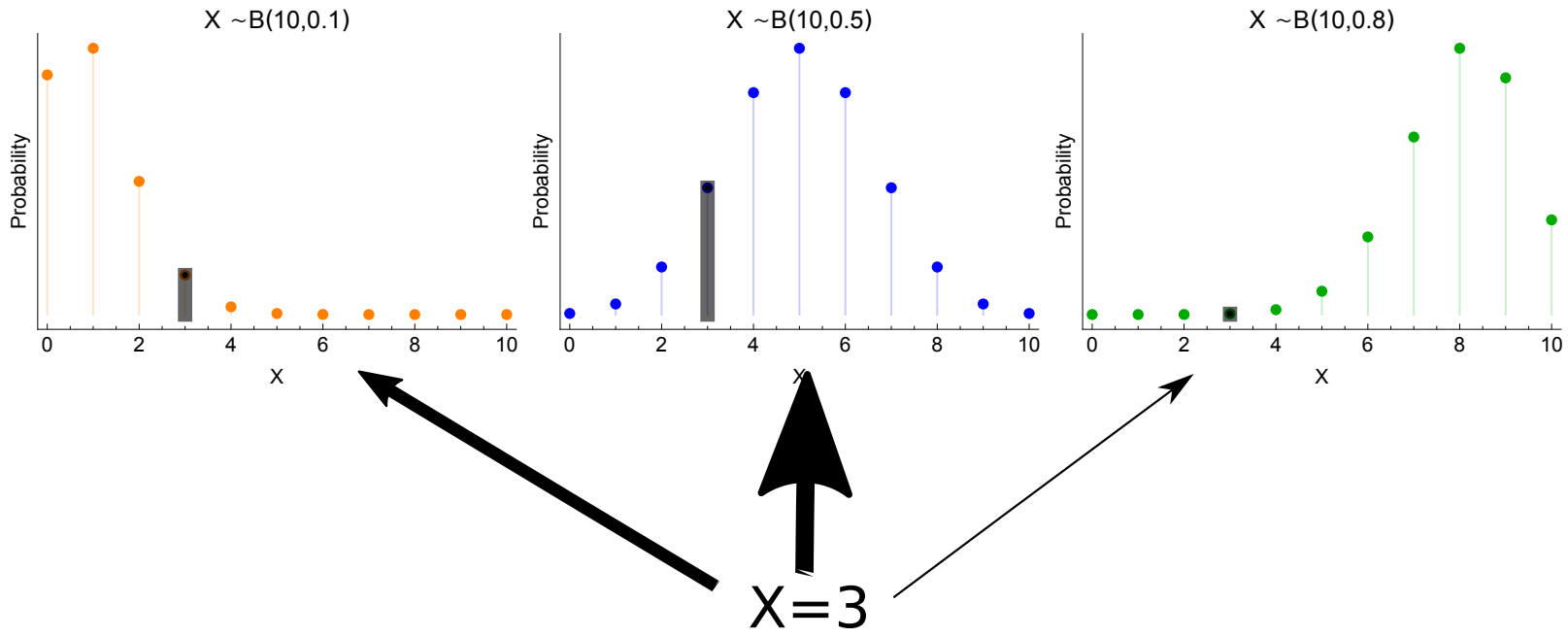
- We take blood from $N = 10$ patients.
- And find that $X = 3$ individuals are disease-positive.

How could this have happened?

# Many worlds are consistent with data



# Aim of inference: determine which worlds are most likely



# Inference is effectively inverting our model

- Forward model: our probability model $X \sim \mathcal{B}(10, \theta)$ gives us an (infinite) number of ways to *generate* data: one for each value of $\theta$.
- Inverse model: in inference, instead start with $X$ and want to run process in reverse to determine which values of $\theta$ could have generated it.

Inference amounts to going from an effect – the data – back to its cause – the parameter values.

# Likelihoods versus probability distributions

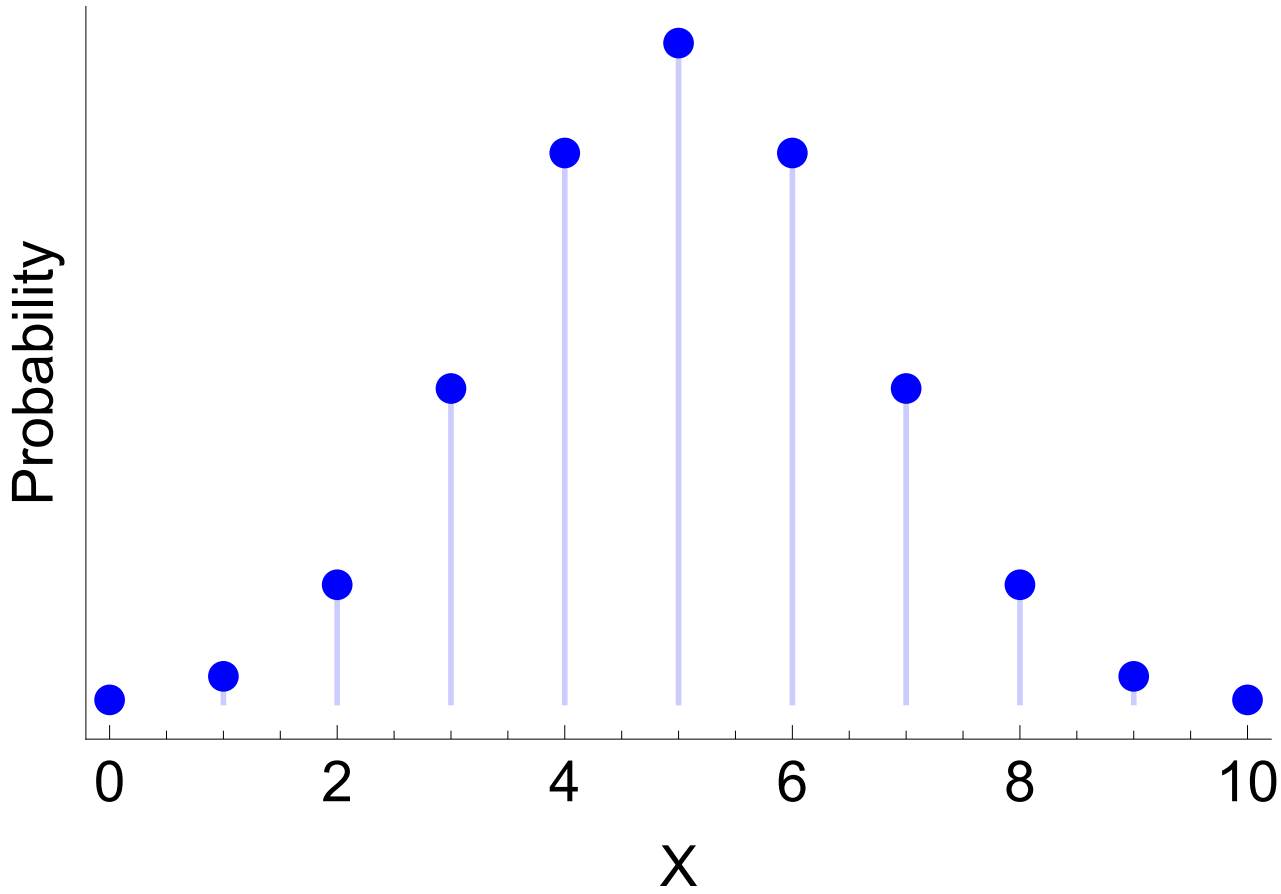
The binomial probability model:

$$\Pr(X|\theta) = \binom{N}{X}\theta^X(1-\theta)^{N-X}. \tag{3}$$

can be used to calculate the probability of different values of $X$ for a fixed $\theta$. This amounts to using the *forward* or *generative* model.

# A probability distribution

For $\theta = 0.5$, we can calculate probabilities:



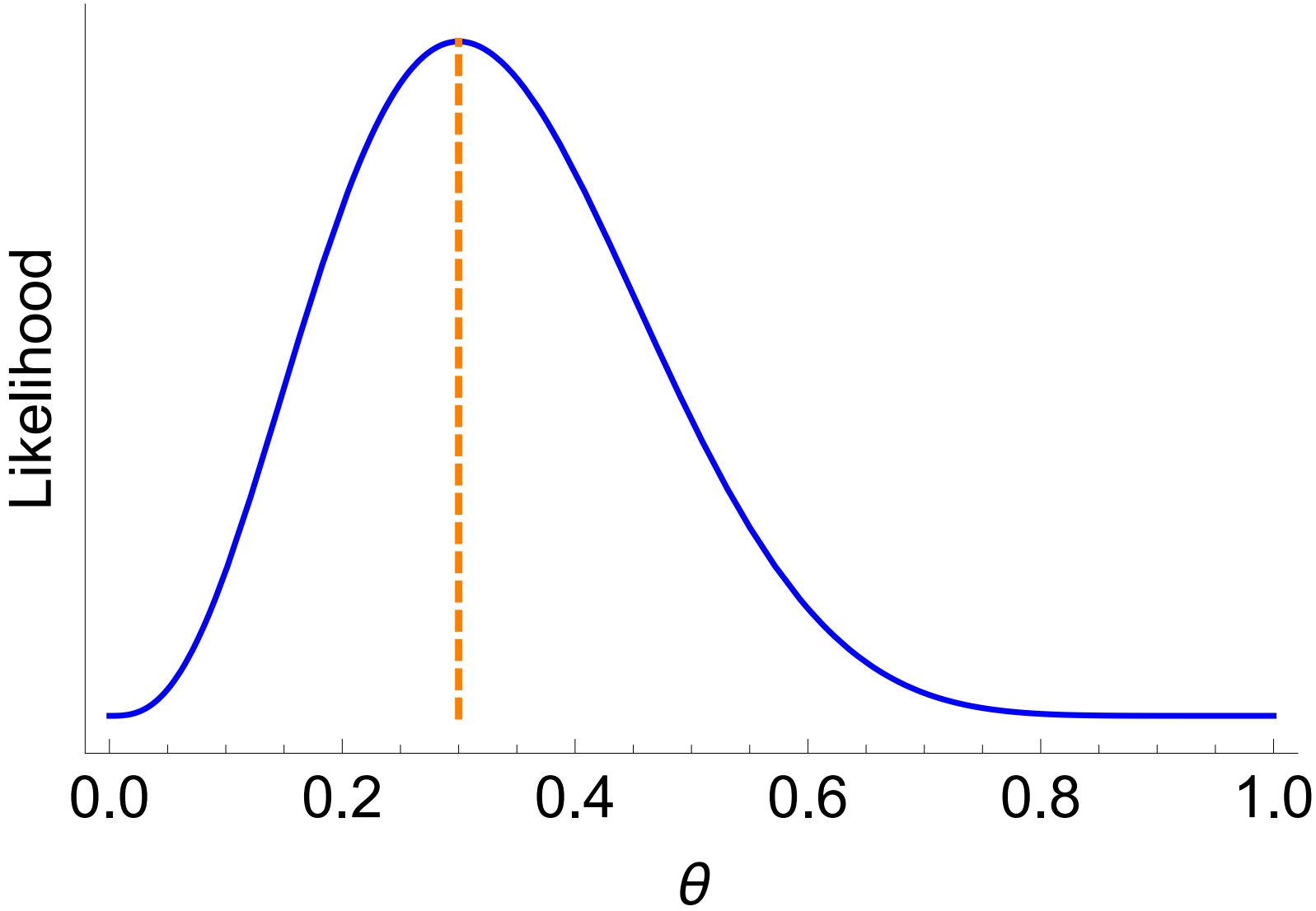
# Likelihoods versus probability distributions

In inference, we have fixed $X = 3$ – our observed data. Now we can vary $\theta$ and use:

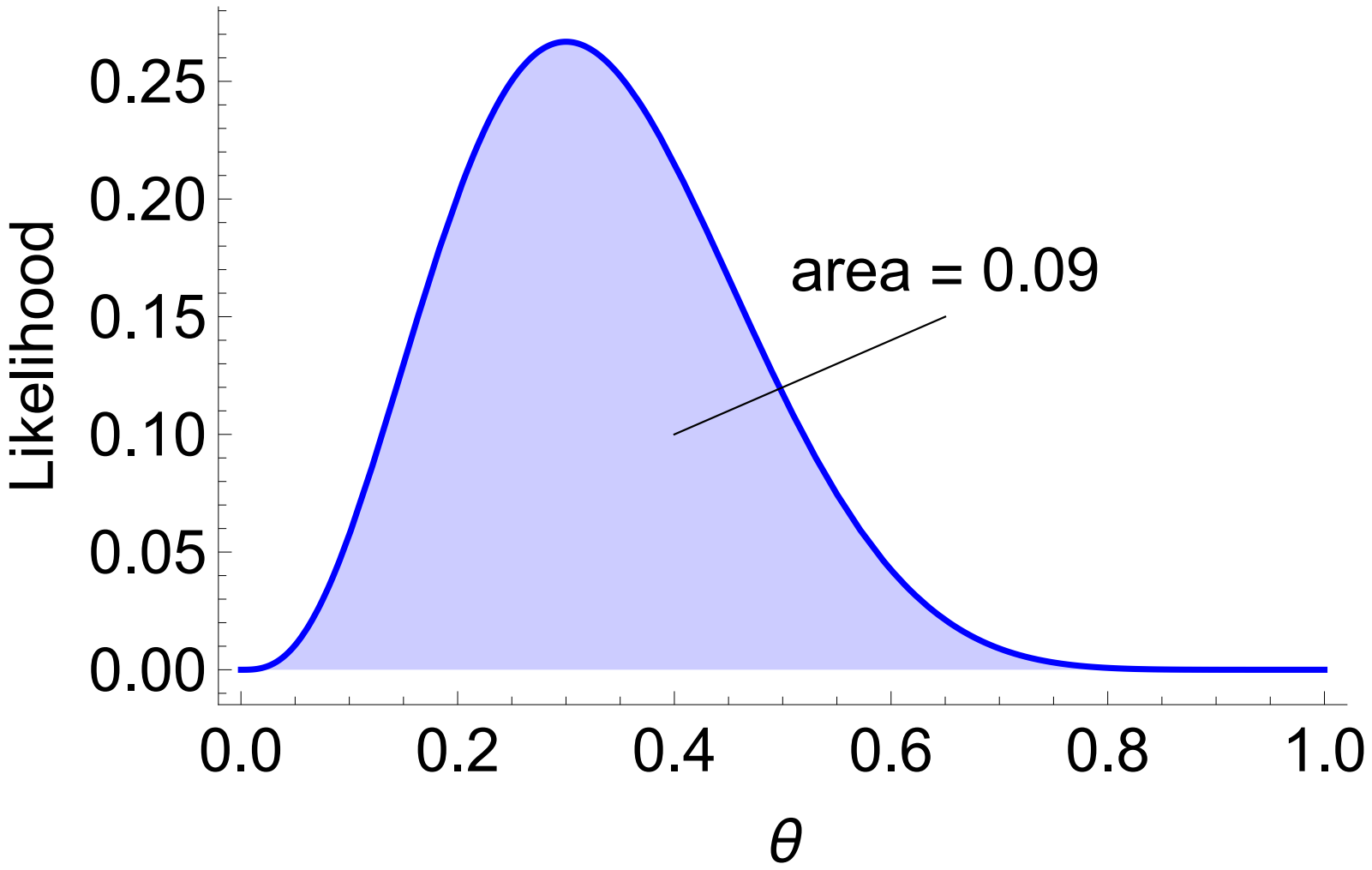
$$\Pr(X|\theta) = \binom{N}{X}\theta^X(1-\theta)^{N-X} = 120\theta^3(1-\theta)^7 \qquad (4)$$

to calculate what are known as likelihoods of each value of $\theta$.

# Likelihood function



# Why is a likelihood function not a valid probability distribution?



Questions?

# Why do we care about likelihoods?

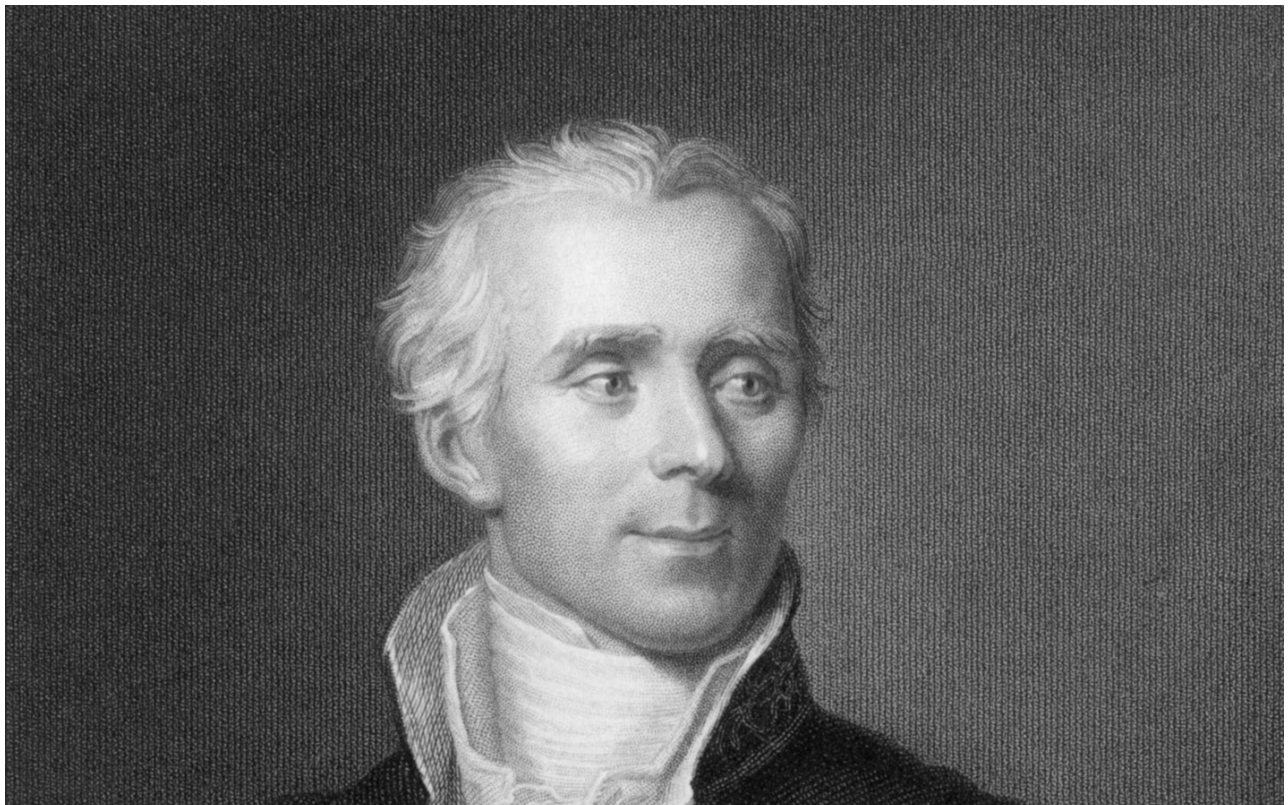
Two predominant approaches to inference:

- Frequentist inference.
- Bayesian inference.

Both use likelihoods as a basis of inference.

# The aim of inference: inverting the likelihood

- Both frequentists and Bayesians essentially invert: $p(X|\theta) \rightarrow p(\theta|X)$.
- Both attempts to convert the likelihood into a probability distribution.
- Their methods of inversion are *different*.



# Frequentist inversion: null hypothesis testing

Frequentist inference considers a single hypothesis $\theta$ about data generating process at a time.

$$H_0 : \text{A hypothesis } \theta \text{ is true} \tag{5}$$

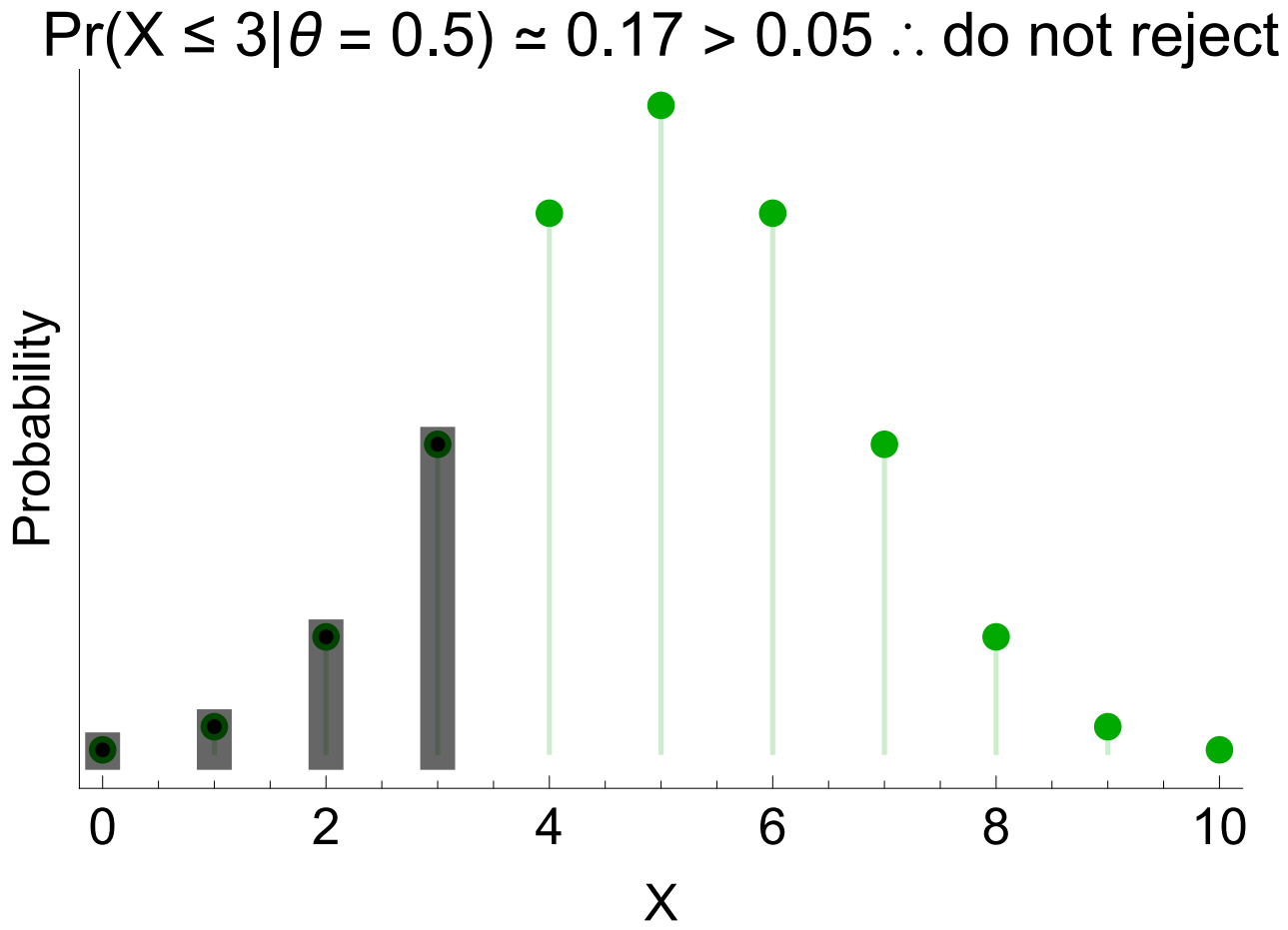
$$H_1 : \text{A hypothesis } \theta \text{ is false} \tag{6}$$

Frequentists use a rule of thumb:

- If $Pr(\text{data as or more extreme than } X|\theta) < 0.05$, then $\theta$ is false, $\implies p(\theta|X) = 0$
- If $Pr(\text{data as or more extreme than } X|\theta) \geq 0.05$, then $\theta$ *could* be true, $\implies p(\theta|X) =?$

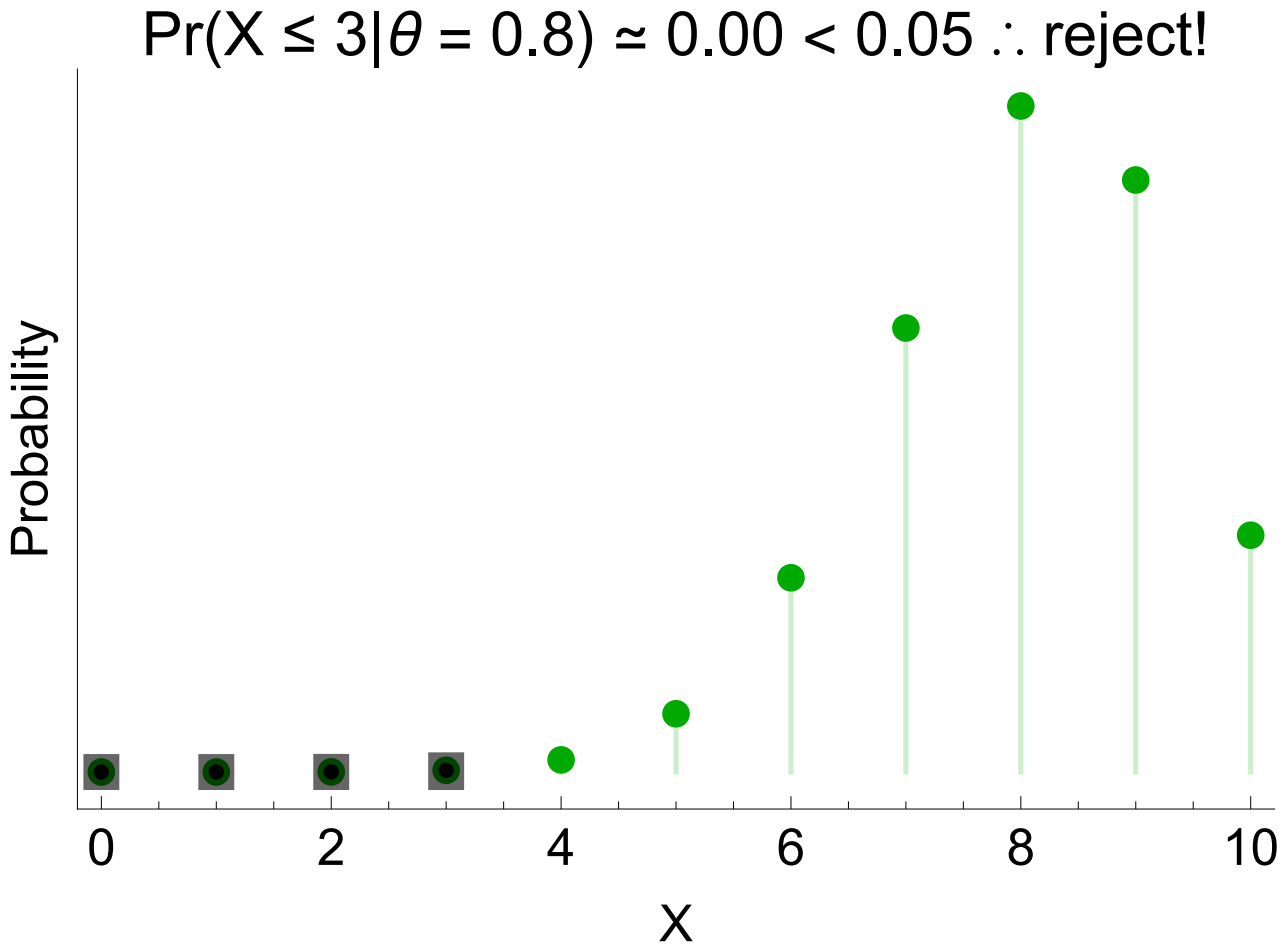
# Frequentist inversion: null hypothesis testing

- For $X = 3$ we can carry out a series of these hypothesis tests across a range of $\theta$.
- For example, assume $H_0 : \theta = 0.5$:



# Frequentist inversion: null hypothesis testing

- Now, assume $H_0 : \theta = 0.8$:



# Frequentist inversion: null hypothesis testing

If we carry out a series of similar hypothesis tests over the range of $\theta$ we find the 90% confidence intervals (90% because we have used two one sided 5% test sizes):

$$0.09 \leq \theta \leq 0.61 \tag{7}$$

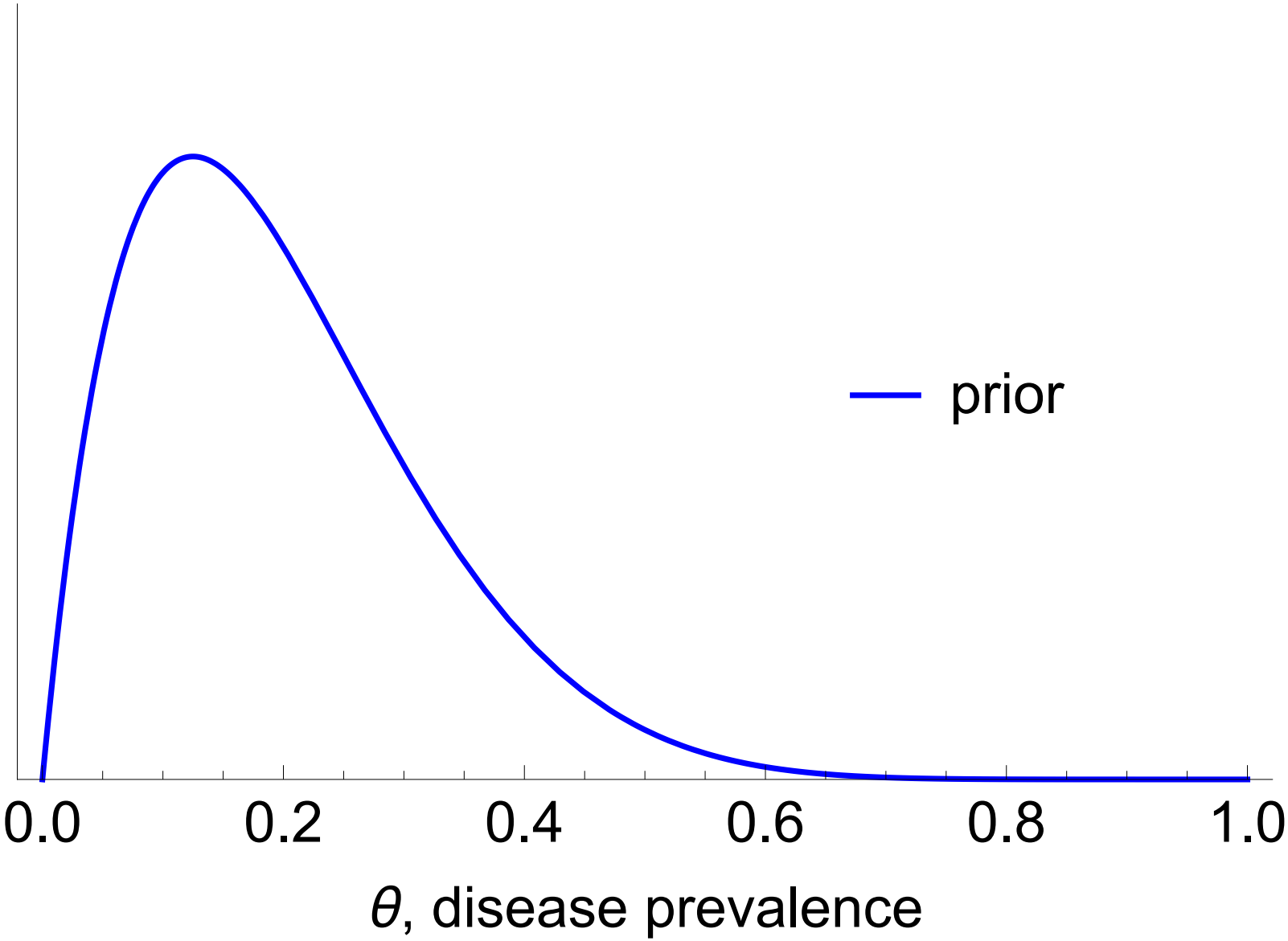
# Bayesian inversion

Bayesians instead use a rule consistent with the rules of probability known as *Bayes' rule*:

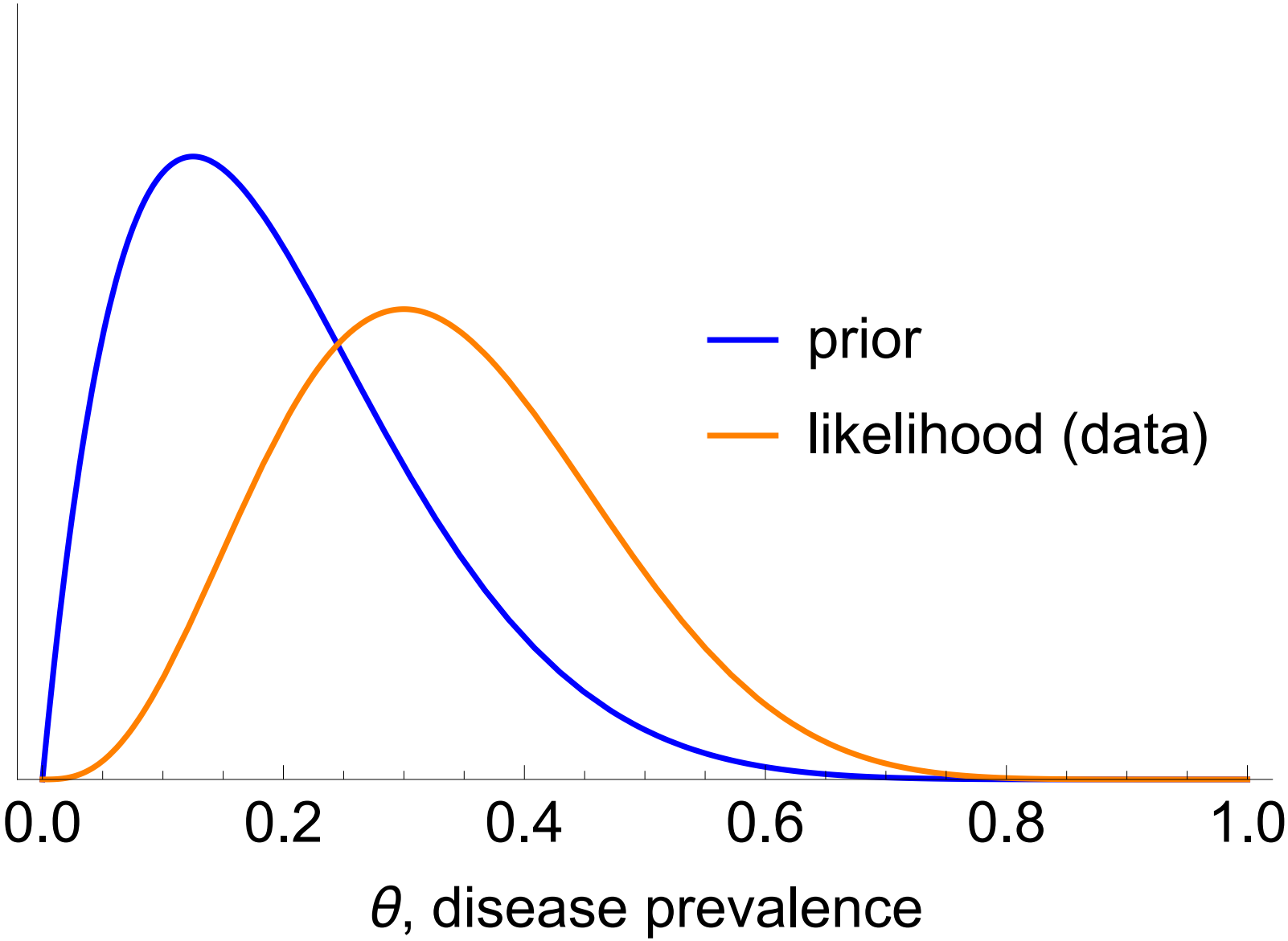
$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{8}$$

Resulting in an accumulation of evidence (not binary decision) across *all* potential hypotheses $\theta$.

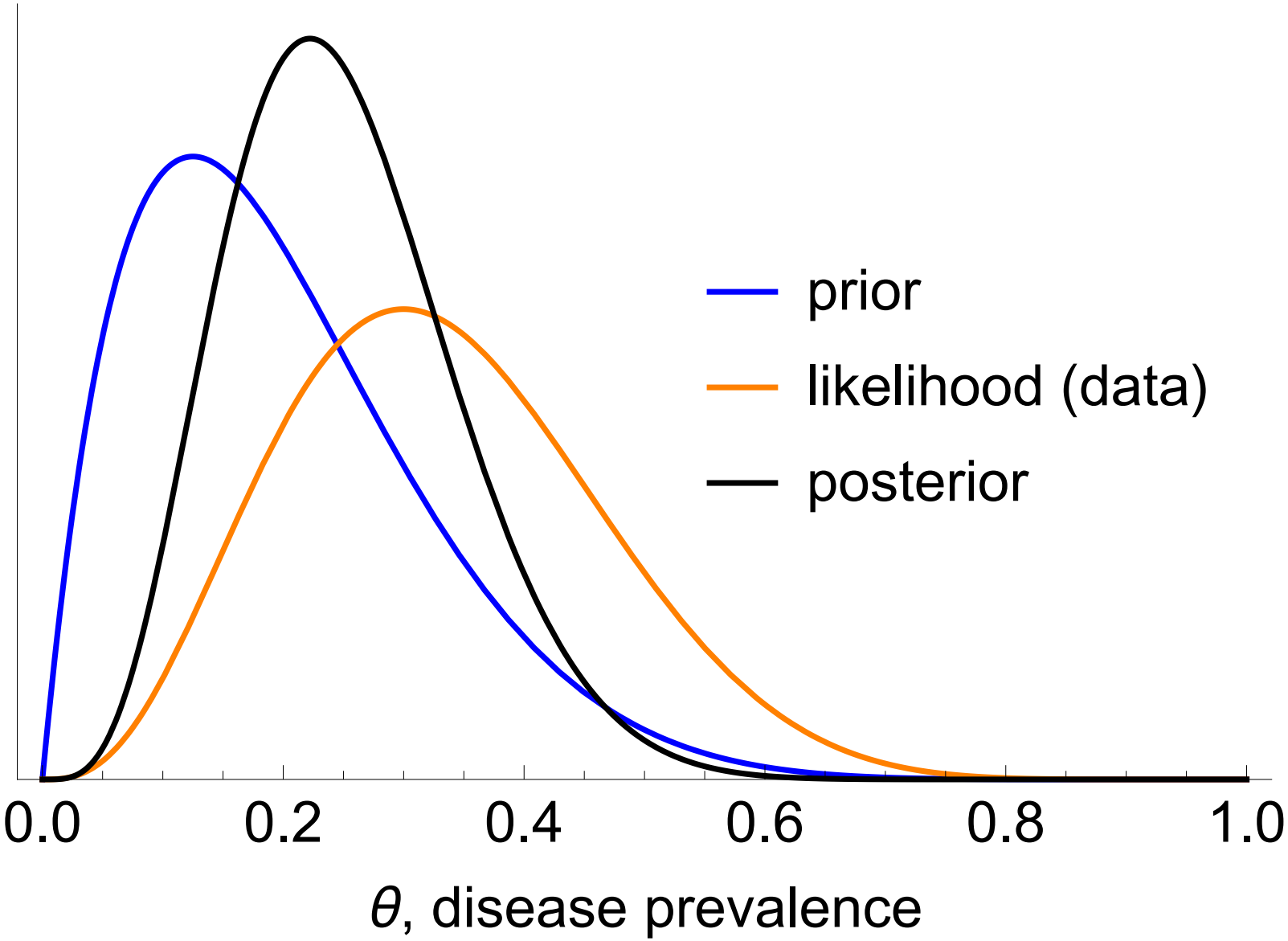
# Bayesian inversion



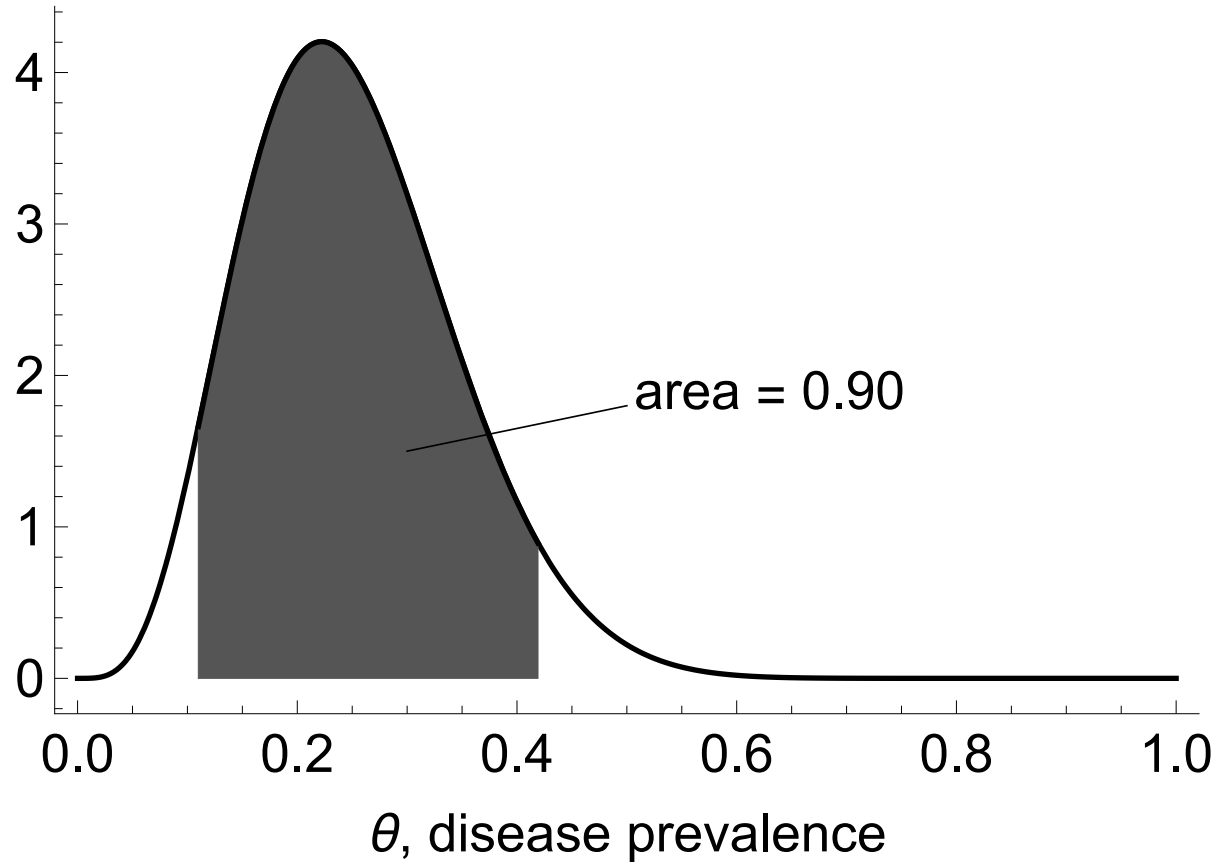
# Bayesian inversion



# Bayesian inversion



# Bayesian credible intervals




$\Longrightarrow$ find a 90% central posterior interval of $0.11 \leq \theta \leq 0.41$.

Questions?

# Bayes' rule for inference

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{9}$$

But what do these terms mean?

# Likelihood summary

$$p(\theta|X) = \frac{\textcircled{p(X|\theta)} \times p(\theta)}{p(X)} \tag{10}$$

- In our example, $\theta$ is the disease prevalence.
- $X$ is the data.
- $p(X|\theta)$ represents the *likelihood*.
- Remember *not* a probability distribution because $\theta$ varies.
- Encapsulates many **subjective** judgements about analysis.

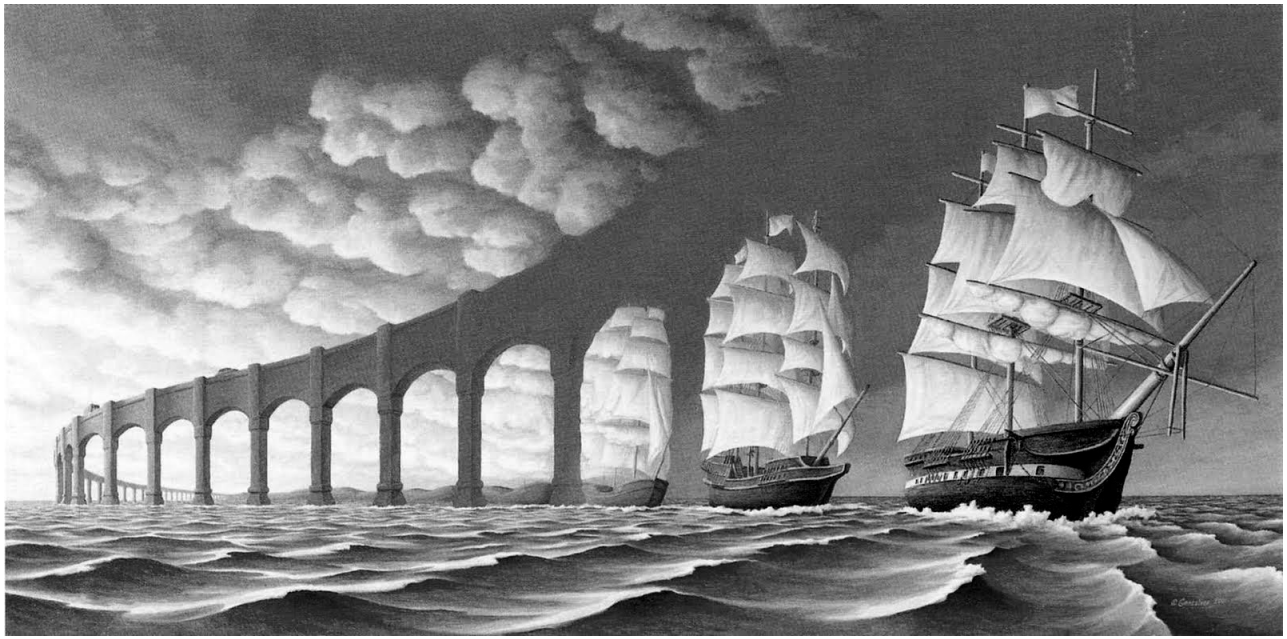
# Priors summary

$$p(\theta|X) = \frac{p(X|\theta) \times \boxed{p(\theta)}}{p(X)} \tag{11}$$

- $p(\theta)$ represents the *prior*.
- A valid probability distribution.
- Similar to the likelihood; it is also subjective.

# No "objective" rule for priors

- Embody subjective assumptions about state of the world.
- Essentially measure $Pr(\text{cause}|\text{pre-data knowledge})$.
  - Since knowledge differs between subjects $\Longrightarrow$ different priors.
- Can be informed by pre-experimental data (for example, previous studies or from a collection of previous studies).



# Denominator summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{12}$$

- $p(X)$ represents the *denominator*.
- Two different interpretations:
  - Before we collect $X$ it is the **prior predictive distribution**.
  - When we have data $X = 3$ it is simply a number (that normalises the posterior) known as the **evidence** or **marginal likelihood**.
- Calculated from the numerator.
- Source of some difficulty of **exact** Bayesian inference (return to this later).

# Posteriors summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{13}$$

- $p(\theta|X)$ represents the *posterior*.
- A valid probability distribution.
- Starting point for all further analysis in Bayesian inference.

# Intuition behind Bayesian analyses

Bayes' rule:

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{14}$$

Tells us that:

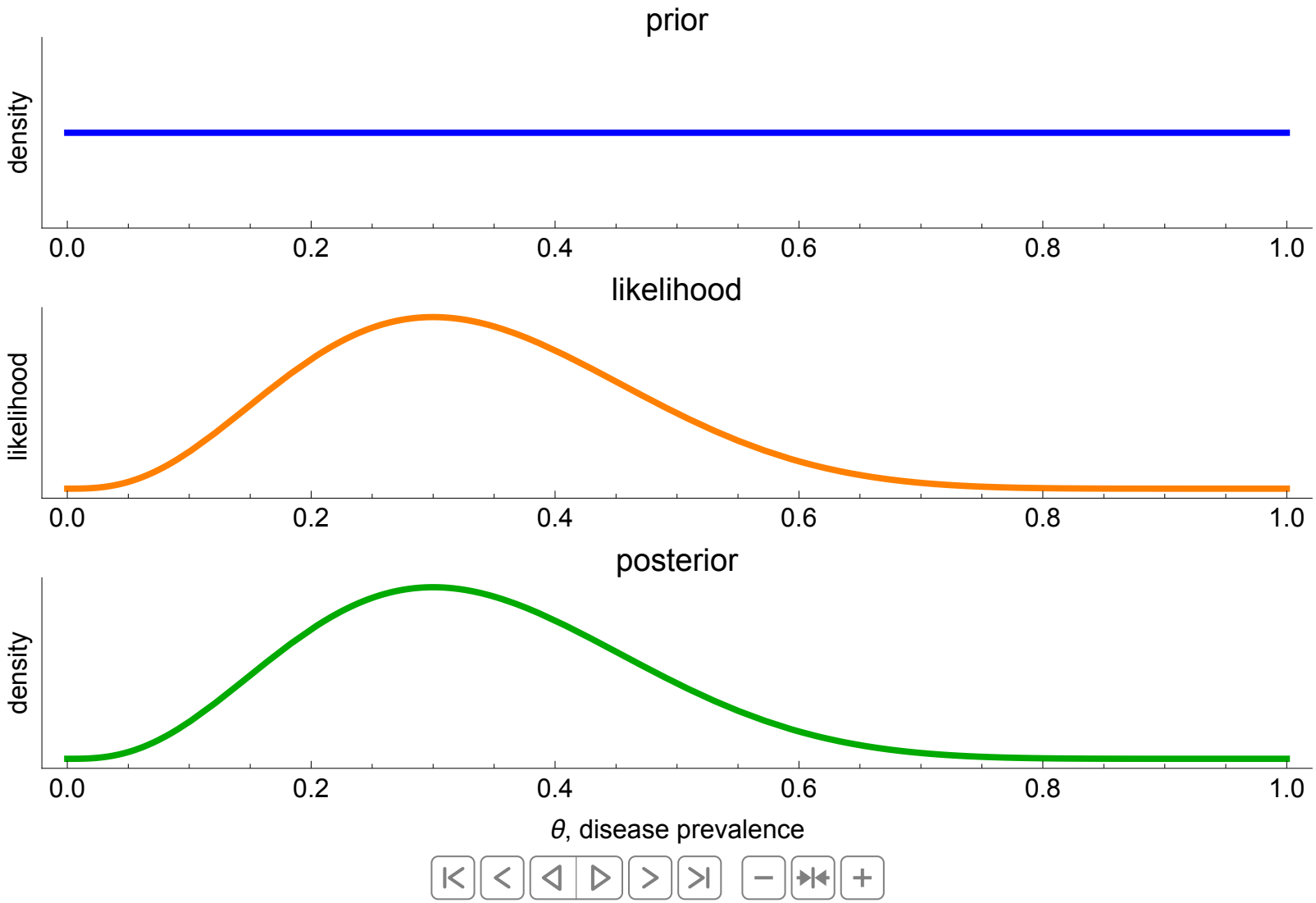
$$p(\theta|X) \propto p(X|\theta) \times p(\theta) \tag{15}$$

Because $p(X)$ is independent of $\theta$

$\Longrightarrow$ the posterior is essentially a weighted (geometric) mean of the prior and likelihood.

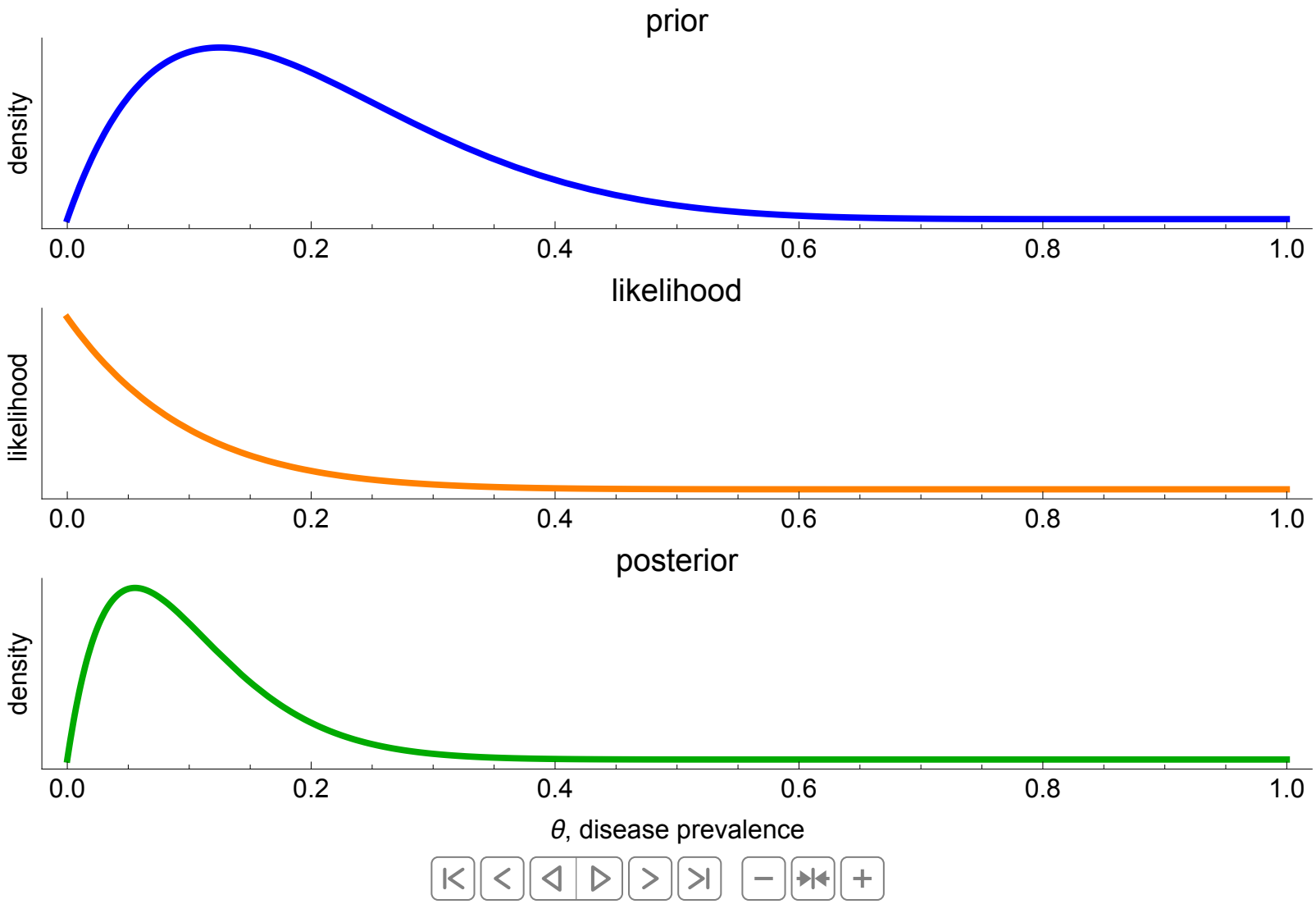
# Intuition behind Bayesian analyses: prior

Consider $N = 10$ where $X = 3$.



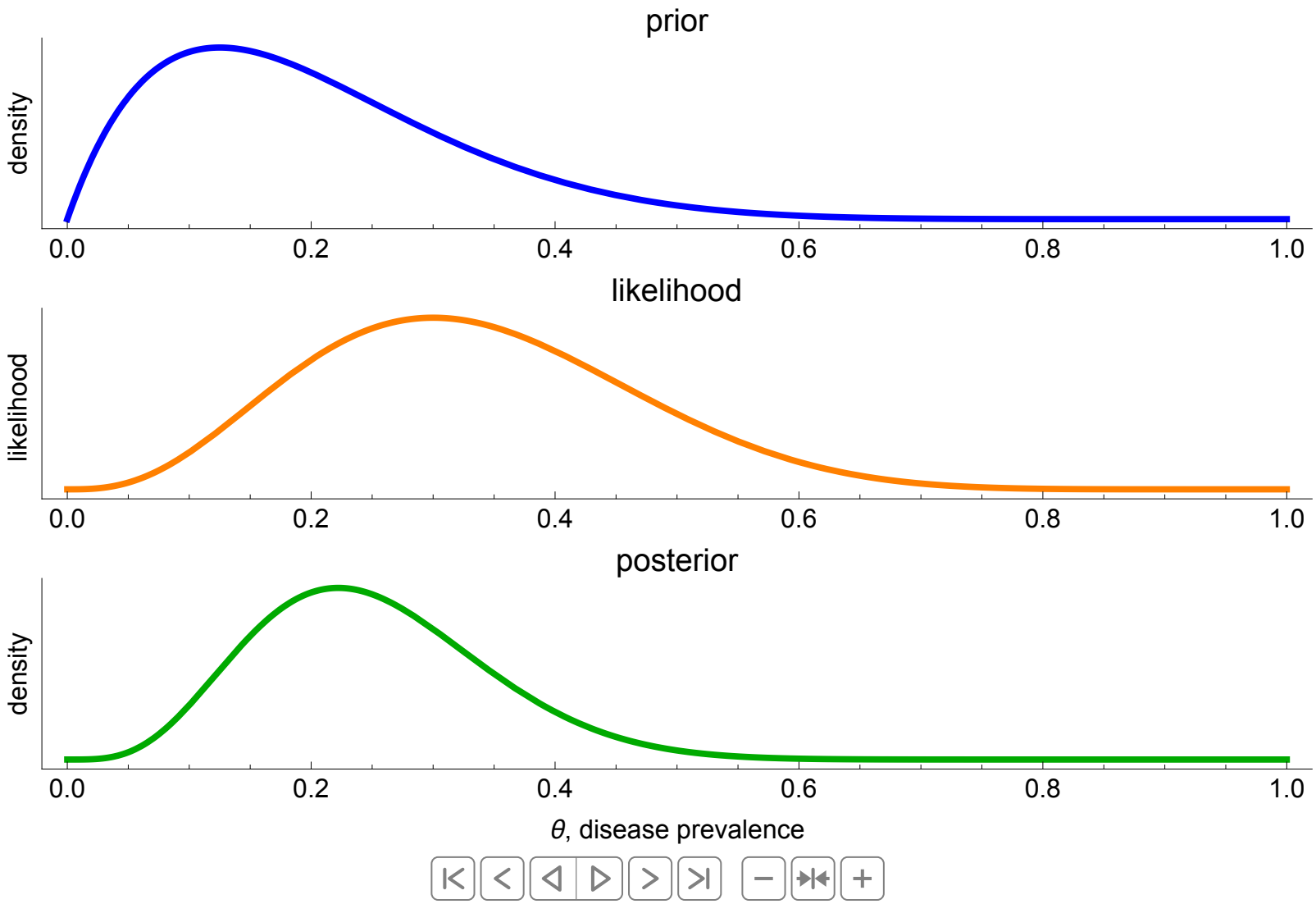
# Intuition behind Bayesian analyses: likelihood

Now holding prior constant and varying $X$.



# Intuition behind Bayesian analyses: sample size

Constant prior and proportion with disease; sample size↑.



Questions?

# The denominator revisited

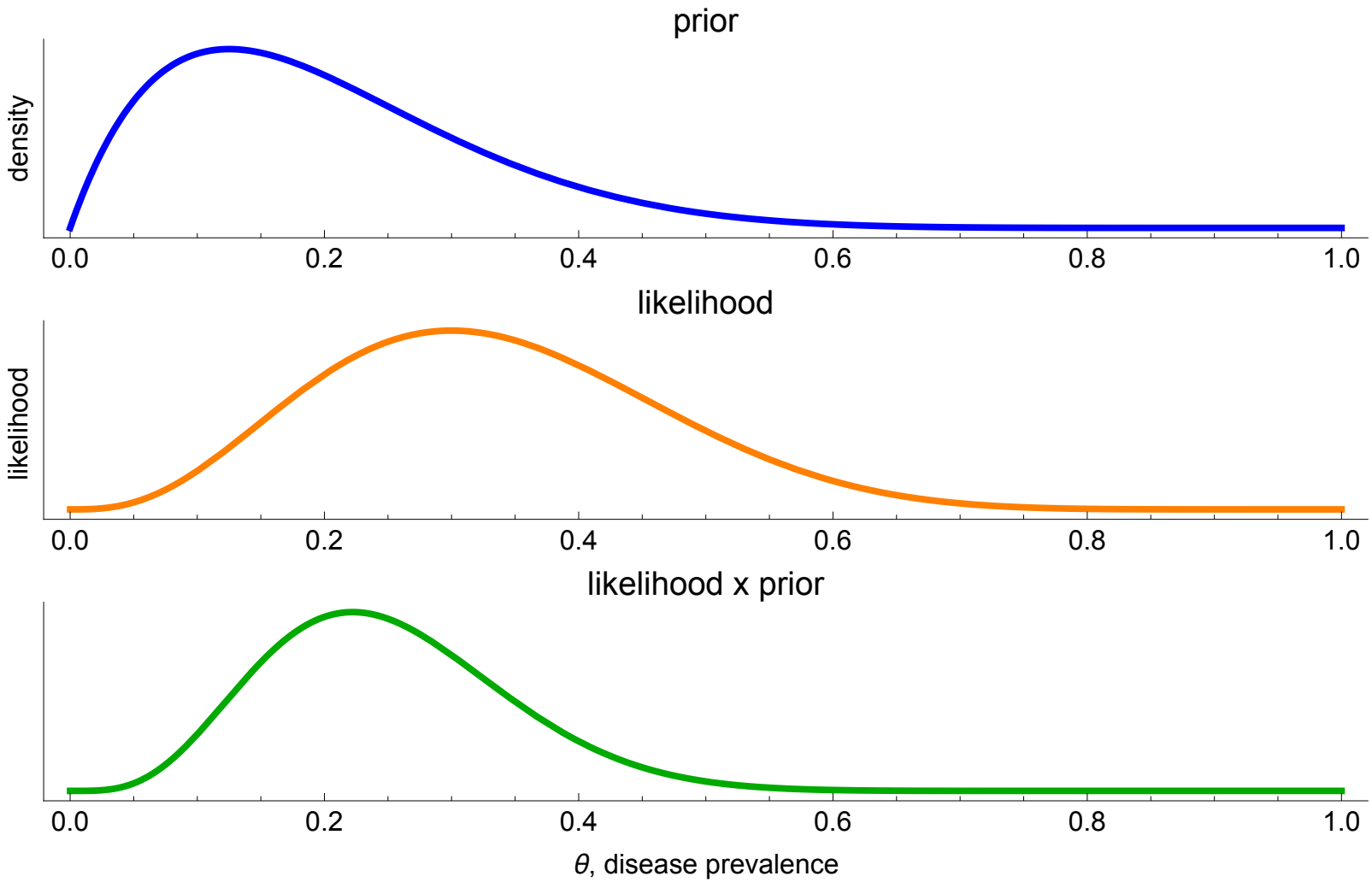
$$p(\theta|X=3) = \frac{p(X=3|\theta) \times p(\theta)}{p(X=3)} \tag{16}$$

Where we suppose we have $X = 3$ disease-positive out of a sample of 10 in our example. We obtain the denominator by averaging out all $\theta$ dependence. This is equivalent to integrating across all $\theta$:

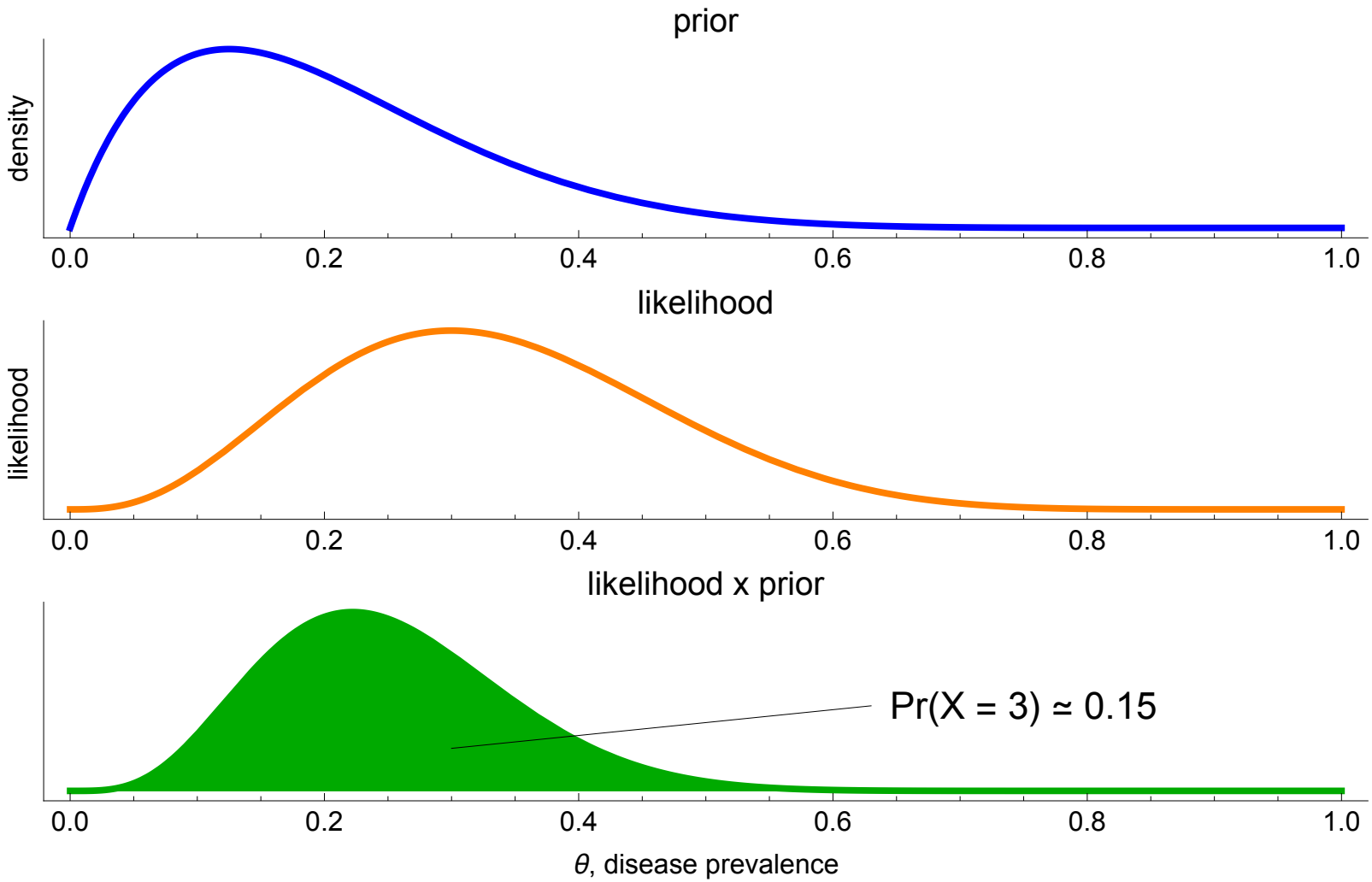
$$p(X=3) = \int_0^1 p(X=3|\theta) \times p(\theta) \mathrm{d}\theta \tag{17}$$

This is equivalent to working out an **area** under a curve.

# The denominator as an area



# The denominator as an area



# Calculating the denominator in 2 dimensions

If we considered a different model where there were two parameters $\theta_1 \in (0,1),\ \theta_2 \in (0,1) \implies$ :

$$p(X=3) = \int_0^1 \int_0^1 p(X=3|\theta_1,\theta_2) \times p(\theta_1,\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \qquad (18)$$

This is equivalent to working out a **volume** contained within a surface.

# Calculating the denominator in $d$ dimensions

If we considered a different model where there were $d$ parameters $(\theta_1, ..., \theta_d)$ all defined to lie between 0 and 1 $\implies$ :

$$p(X = 3) = \int_0^1 ... \int_0^1 p(X = 3|\theta_1, ..., \theta_d) \times p(\theta_1, ..., \theta_d) \mathrm{d}\theta_1 ... \mathrm{d}\theta_d \tag{19}$$

This is equivalent to working out a $(d + 1)$-dimensional **volume** contained within a $d$-dimensional (hyper-surface)!

# The difficult denominator

- Calculating the denominator possible for $d <\sim 3$ using computers.
- Numerical quadrature and many other approximate schemes struggle for larger $d$.
- Many models have **thousands** of parameters.

Arrrghhh!

# What are conjugate priors?

Judicious choice of prior and likelihood can make posterior calculation trivial.

- Choose a likelihood $L$.
- Choose a prior $\theta \sim f \in F$, where:
  - $F$ is a family of distributions.
  - $f$ is a member of that **same** family.
- If posterior, $\theta|X \sim f' \in F \implies$ conjugate!
- In other words both the **prior** and **posterior** are members of the same distribution!

# Conjugate priors

- For likelihood (if independent and identically-distributed):

$$X \sim \mathcal{B}(10, \theta) \implies p(X|\theta) \propto \theta^X (1-\theta)^{10-X} \tag{20}$$

- For prior assume a beta distribution (a reasonable choice if $\theta \in (0, 1)$):

$$\theta \sim \text{beta}(a, b) \implies p(\theta) \propto \theta^{a-1} (1-\theta)^{b-1} \tag{21}$$

- Numerator of Bayes' rule for inference:

$$p(X|\theta) \times p(\theta) \propto \theta^X (1-\theta)^{10-X} \times \theta^{a-1} (1-\theta)^{b-1} \tag{22}$$

# Conjugate priors

- Numerator of Bayes' rule for inference:

$$p(X|\theta) \times p(\theta) \propto \theta^X (1-\theta)^{10-X} \times \theta^{a-1}(1-\theta)^{b-1}$$
$$= \theta^{X+a-1}(1-\theta)^{10-X+b-1}$$

- This has same $\theta$-dependence as a beta$(X+a, 10-X+b)$ density $\implies$ must be this distribution!
- $\therefore$ a beta prior is *conjugate* to a binomial likelihood.

# Table of common conjugate pairs of likelihoods and priors

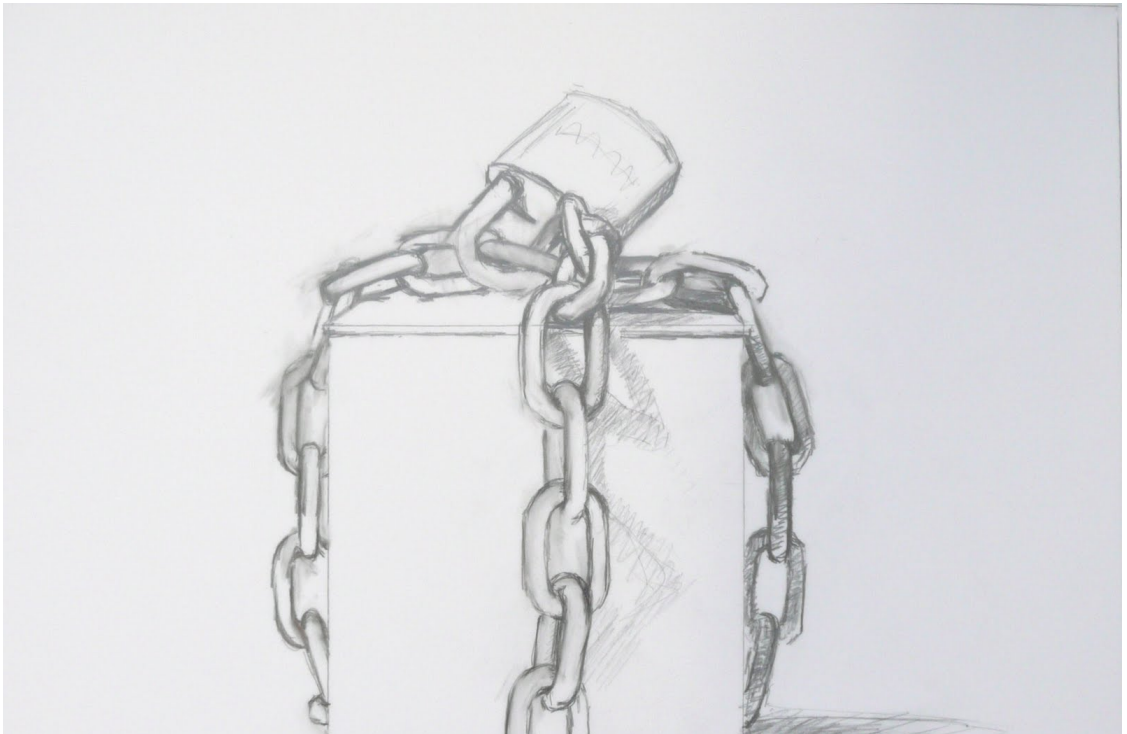
No need to do any integrals! Just lookup rules:

| **Likelihood** | **Prior** | **Posterior** |
|---|---|---|
| Bernoulli | beta$(\alpha, \beta)$ | beta$(\alpha + \sum_{i=1}^{n} X_i, \beta + n - \sum_{i=1}^{n} X_i)$ |
| Binomial | beta$(\alpha, \beta)$ | beta$(\alpha + \sum_{i=1}^{n} X_i, \beta + \sum_{i=1}^{n} N_i - \sum_{i=1}^{n} X_i)$ |
| Poisson | Gamma$(\alpha, \beta)$ | Gamma$(\alpha + \sum_{i=1}^{n} X_i, \beta + n)$ |
| Multinomial | Dirichlet$(\boldsymbol{\alpha})$ | Dirichlet$(\boldsymbol{\alpha} + \sum_{i=1}^{n} \boldsymbol{X}_i)$ |
| Normal | Normal-inv-$\Gamma$ | Normal-inv-$\Gamma$ |

# Limits of conjugate modelling

Using conjugate priors is limiting because:

- Often restricted to univariate problems.
  - $\Longrightarrow$ we could just use numerical quadrature instead.
- Required to use relevant conjugate prior for a given likelihood $\Longleftarrow$ may not be sufficient to capture pre-data beliefs of analyst.

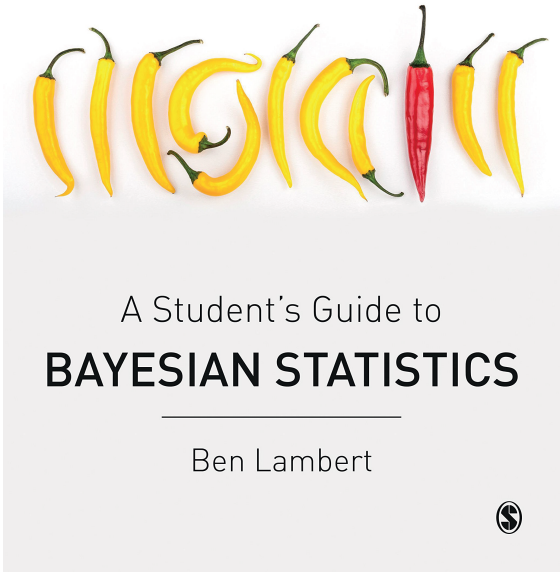


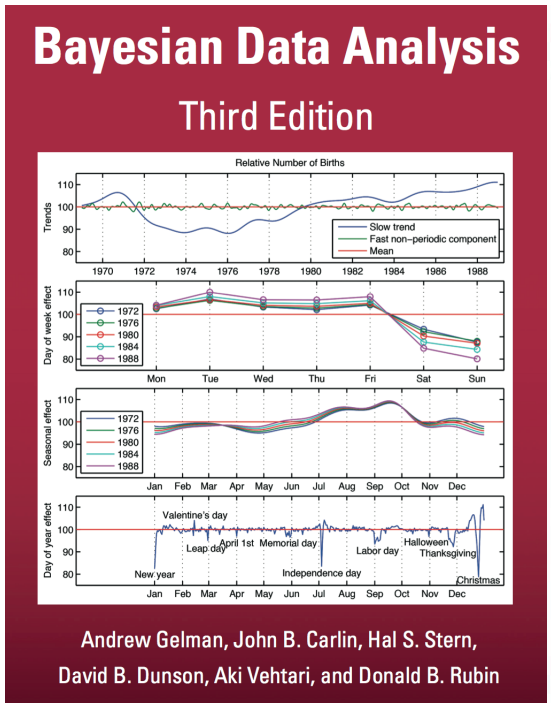
## Longer-term solution

Sampling (usually MCMC)!
But that's another story.

Questions?

# Books


A Student's Guide to
BAYESIAN STATISTICS
Ben Lambert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Third Edition
Relative Number of Births
Trends
Slow trend
Fast non−periodic component
Mean
Day of week effect
Seasonal effect
Day of year effect
Valentine's day
New year
Leap day
April 1st
Memorial day
Independence day
Labor day
Halloween
Thanksgiving
Christmas
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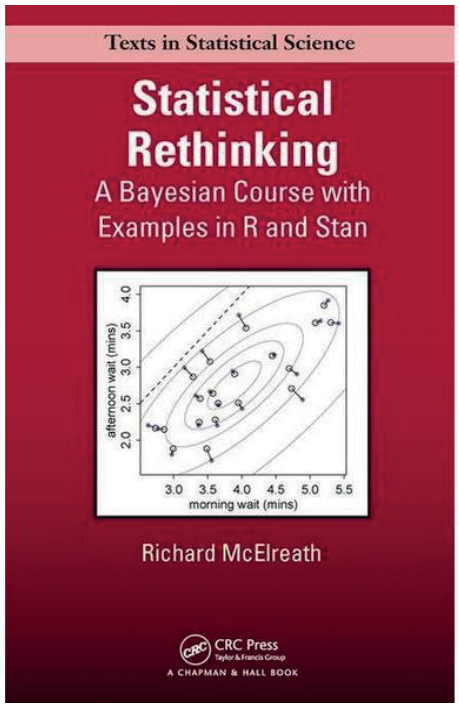
David B. Dunson, Aki Vehtari, and Donald B. Rubin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A Bayesian Course with
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morning wait (mins)
Richard McElreath
CRC Press
A CHAPMAN & HALL BOOK

# Free lectures

- Richard McElreath's has a great YouTube lecture series.
- I have a series on YouTube called "A Student's Guide to Bayesian Statistics".

# Not sure I understand?

Bayesian statistics:

$$p(\theta|\boldsymbol{D}) = \frac{p(\boldsymbol{D}|\theta) \times p(\theta)}{p(\boldsymbol{D})} \tag{23}$$

Beigeian statistics:

$$p(\theta|\boldsymbol{D}) = \frac{p(\boldsymbol{D}|\theta) \times p(\theta)}{p(\boldsymbol{D})} \tag{24}$$