A beginner's introduction to Bayesian inference

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Who am I?

- Statistician working on data science, machine learning and statistical inference across the university.
- User of Bayesian statistics for the past X years.
- Born in the same town as Thomas Bayes (Tunbridge Wells).

Lecture outcomes

By the end of this lecture you should:

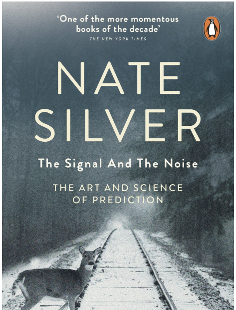
- Know what probability distributions are and why they are used in modelling.
- Understand the goal of statistical inference.
- Appreciate how Bayesian and frequentist approaches to inference achieve this goal.
- Know the elements required to do Bayesian inference and appreciate how they affect inferences.
- Know why exact Bayesian inference is *hard*.
- See how conjugate priors provide a slight remedy.

Why don't more people use Bayesian inference?

- Most existing texts put a strong emphasis on its (seemingly) complex mathematical basis.
- Poor explanation of why computational sampling (usually MCMC) is needed.
- The view that Bayesian inference is more wishy-washy than frequentist inference.

Tangible benefits of Bayesian inference

- Simple and intuitive model building (unlike frequentist statistics there is no need to remember lots of specific formulae).
- Exhaustive and creative model testing.
- The best predictions; for example, Nate Silver.
- Allows estimation of models that would be impossible in frequentist statistics: especially true in epidemiology!



Outline

1. An introduction to statistical modelling
2. The goal of statistical inference
3. Frequentist and Bayesian world views
4. Elements of Bayes' rule for inference
5. The difficulty with exact Bayesian inference
6. Conjugate priors

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Example: how to estimate disease prevalence?

- Suppose we take a sample of N study participants from the population.
- We take their blood and use a clinical test to determine presence / absence of disease: finding X are disease-positive.

Question: How do we use these data to estimate disease prevalence (with uncertainty)?

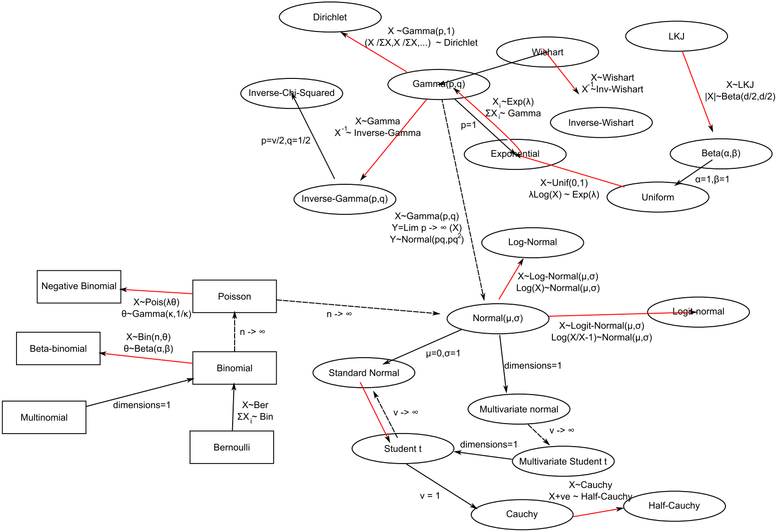
Building a model to explain these data

We don't know a lot about how our data were produced:

- How exactly participants were picked.
- How the disease is distributed in the population.
- How the clinical test works.

Due to uncertainty $\implies$ use a model that encompasses uncertainty: i.e. one that uses probability distributions.

Which probability distribution?



How to choose a probability model?

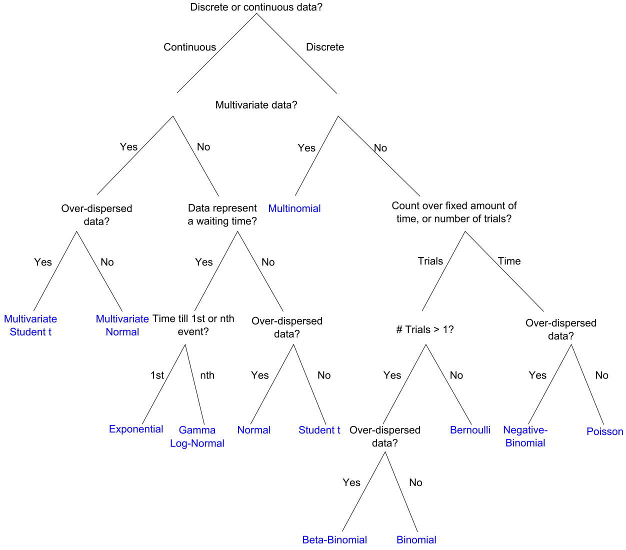
Characteristics of our data:

1. Our sample size N is fixed.
2. Our data X are discrete and can take values $0, 1, 2, ..., N-1, N$.

Assumptions:

1. Individuals represent independent samples from a population.
2. Those individuals are drawn from the same population.

Which probability model satisfies these conditions?



Binomial model: introduction

Analogy: count of disease-positive cases in a sample of size N $\sim$ count of a coin landing heads up in N flips of it.

If we assume the clinical test is perfect:

- $\Pr(+) = \theta$ is the proportion of disease-positive individuals in the population.
- Analogous to $\Pr(H) = \theta$, the probability the coin lands heads up: $0 \leq \theta \leq 1$.

Binomial model probability

The probability of a given number of heads X depends on:

- $\Pr(H) = \theta$.
- The number of possible ways to obtain result. E.g. if $N = 2$, there are two ways to obtain $X = 1$: $(1, 0)$ or $(0, 1)$.

Binomial model probability

The probability for a given X is:

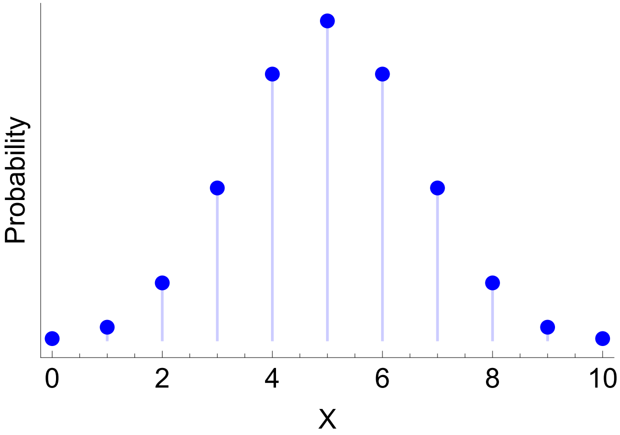
$$\Pr(X|\theta) = \binom{N}{X}\theta^X(1-\theta)^{N-X}. \tag{1}$$

We often use the following notation as shorthand:

$$X \sim \mathcal{B}(N, \theta). \tag{2}$$

Binomial model probabilities: visualised

Suppose $\theta = 0.5$ and $N = 10$.



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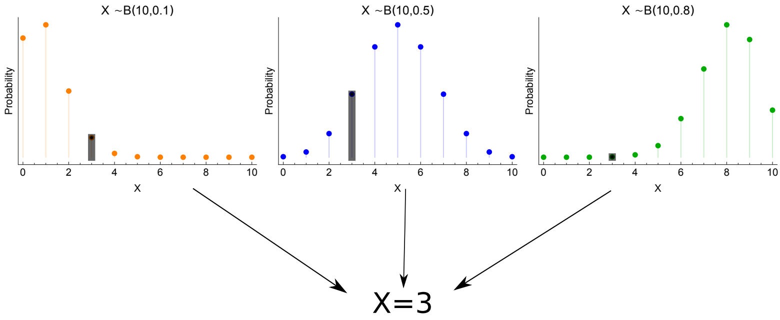
Many ways of generating data

Suppose:

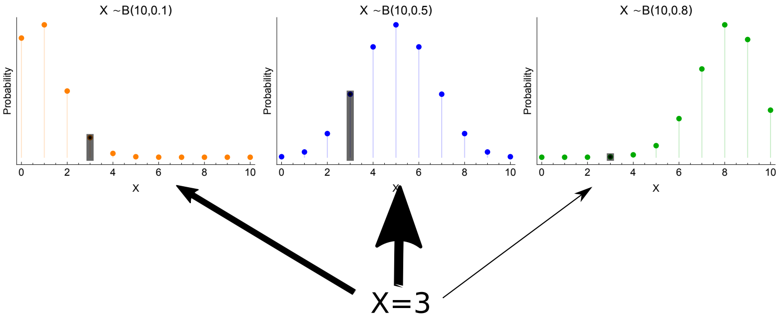
- We take blood from $N = 10$ patients.
- And find that $X = 3$ individuals are disease-positive.

How could this have happened?

Many worlds are consistent with data



Aim of inference: determine which worlds are most likely



Inference is effectively inverting our model

- Forward model: our probability model $X \sim \mathcal{B}(10, \theta)$ gives us an (infinite) number of ways to *generate* data: one for each value of θ.
- Inverse model: in inference, instead start with X and want to run process in reverse to determine which values of θ could have generated it.

Inference amounts to going from an effect – the data – back to its cause – the parameter values.

Likelihoods versus probability distributions

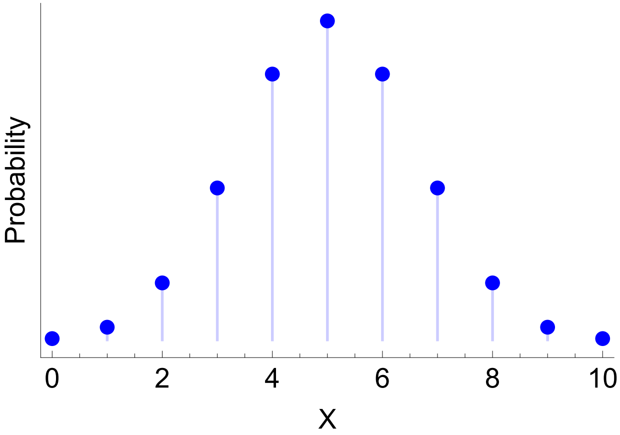
The binomial probability model:

$$\Pr(X|\theta) = \binom{N}{X}\theta^{X}(1-\theta)^{N-X}. \tag{3}$$

can be used to calculate the probability of different values of X for a fixed θ. This amounts to using the *forward* or *generative* model.

A probability distribution

For $\theta = 0.5$, we can calculate probabilities:



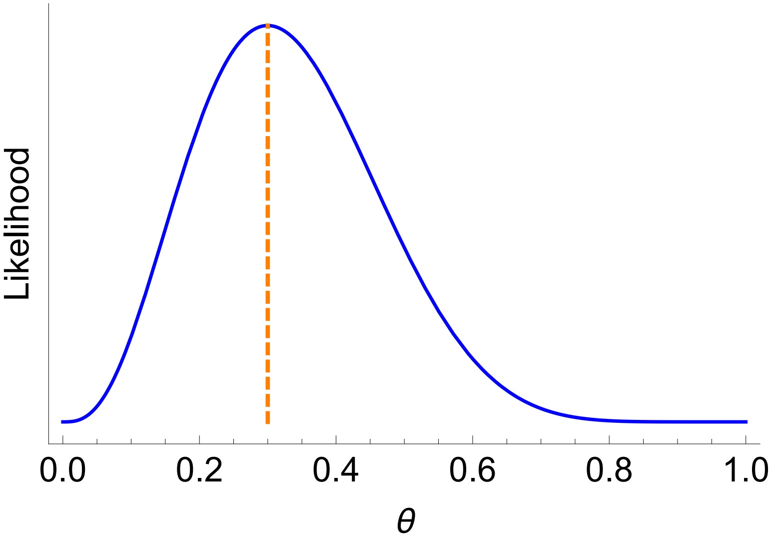
Likelihoods versus probability distributions

In inference, we have fixed $X = 3$ – our observed data. Now we can vary θ and use:

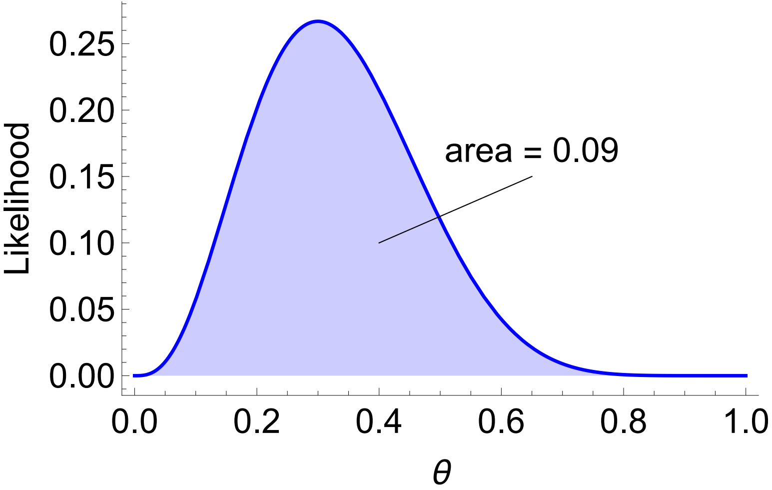
$$\Pr(X|\theta) = \binom{N}{X}\theta^X(1-\theta)^{N-X} = 120\theta^3(1-\theta)^7 \qquad (4)$$

to calculate what are known as likelihoods of each value of θ.

Likelihood function



Why is a likelihood function not a valid probability distribution?



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Why do we care about likelihoods?

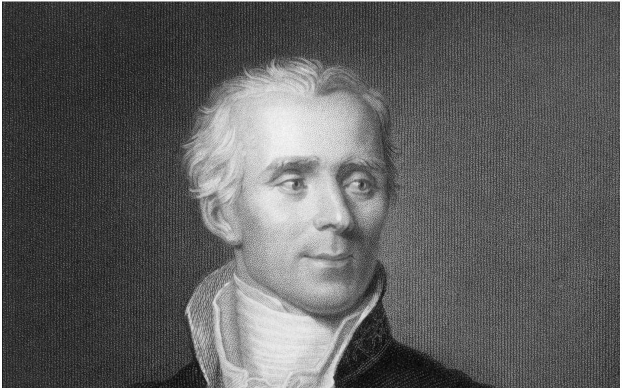
Two predominant approaches to inference:

- Frequentist inference.
- Bayesian inference.

Both use likelihoods as a basis of inference.

The aim of inference: inverting the likelihood

- Both frequentists and Bayesians essentially invert: $p(X|\theta) \rightarrow p(\theta|X)$.
- Both attempts to convert the likelihood into a probability distribution.
- Their methods of inversion are *different*.



Frequentist inversion: null hypothesis testing

Frequentist inference considers a single hypothesis θ about data generating process at a time.

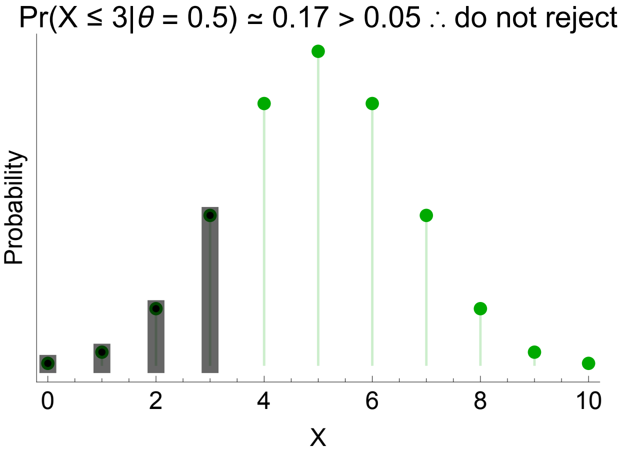
$$H_0 : \text{A hypothesis } \theta \text{ is true} \tag{5}$$
$$H_1 : \text{A hypothesis } \theta \text{ is false} \tag{6}$$

Frequentists use a rule of thumb:

- If $Pr(\text{data as or more extreme than } X|\theta) < 0.05$, then θ is false, $\implies p(\theta|X) = 0$
- If $Pr(\text{data as or more extreme than } X|\theta) \geq 0.05$, then θ *could* be true, $\implies p(\theta|X) =?$

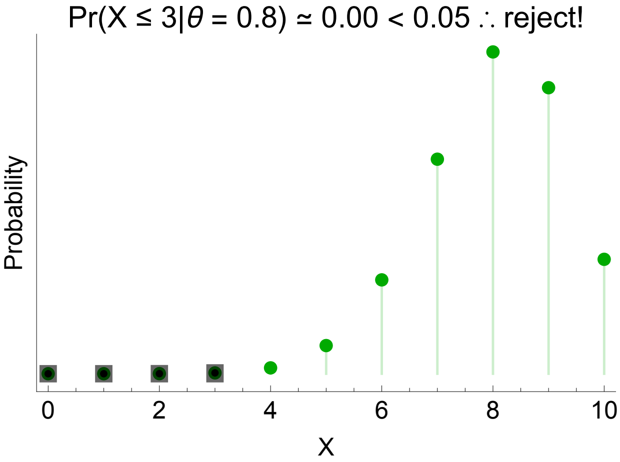
Frequentist inversion: null hypothesis testing

- For $X = 3$ we can carry out a series of these hypothesis tests across a range of θ.
- For example, assume $H_0 : \theta = 0.5$:



Frequentist inversion: null hypothesis testing

- Now, assume $H_0 : \theta = 0.8$:



Frequentist inversion: null hypothesis testing

If we carry out a series of similar hypothesis tests over the range of θ we find the 90% confidence intervals (90% because we have used two one sided 5% test sizes):

$$0.09 \leq \theta \leq 0.61 \tag{7}$$

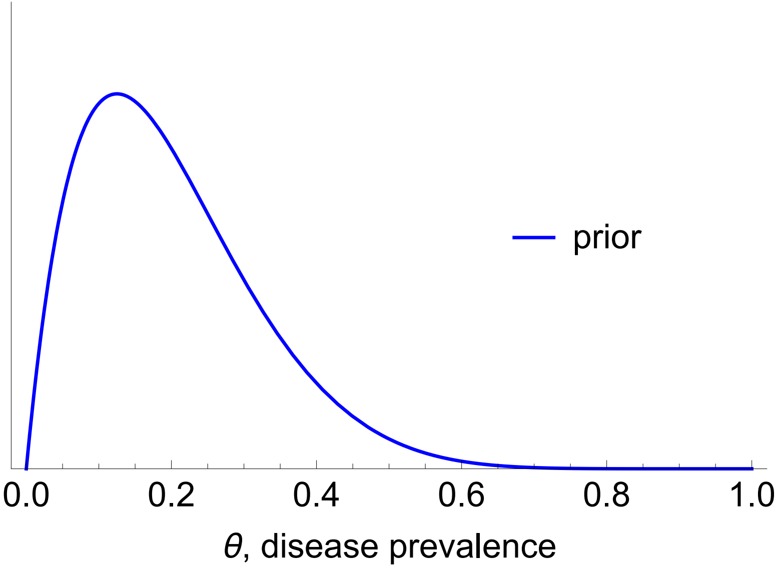
Bayesian inversion

Bayesians instead use a rule consistent with the rules of probability known as *Bayes' rule*:

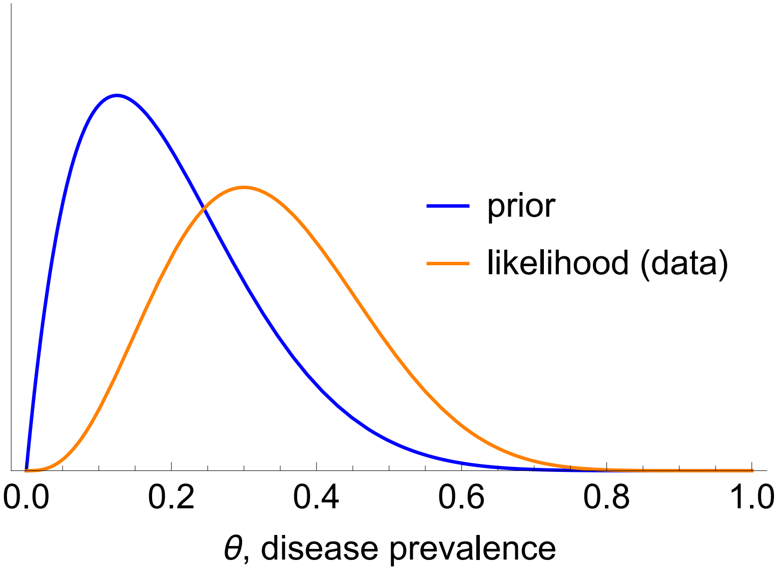
$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{8}$$

Resulting in an accumulation of evidence (not binary decision) across *all* potential hypotheses θ.

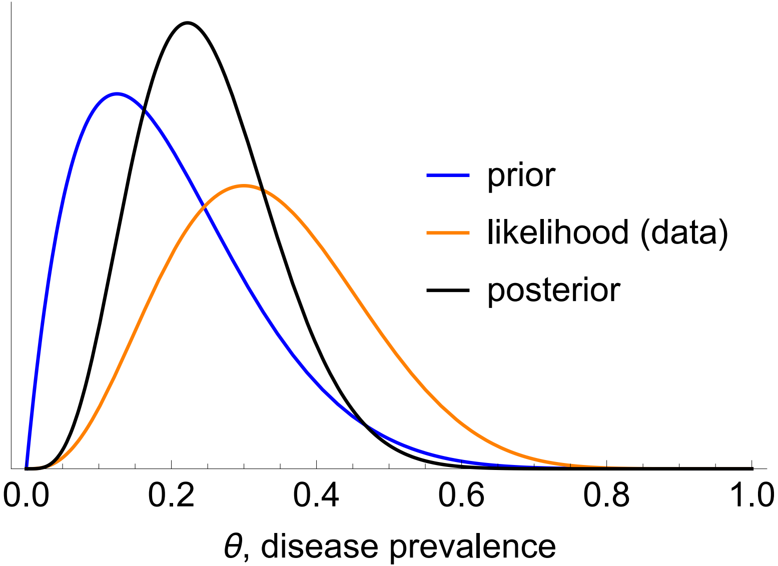
Bayesian inversion



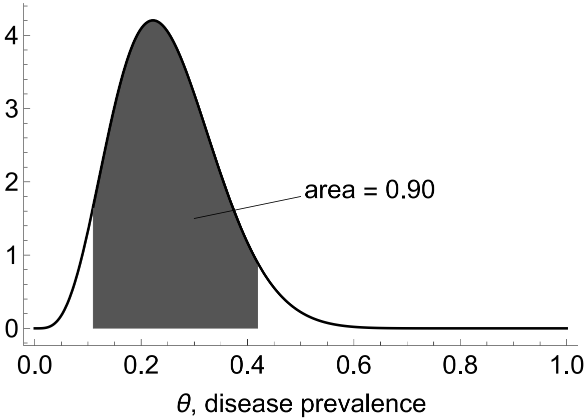
Bayesian inversion



Bayesian inversion



Bayesian credible intervals



$\implies$ find a 90% central posterior interval of $0.11 \leq \theta \leq 0.41$.

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Bayes' rule for inference

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{9}$$

But what do these terms mean?

Likelihood summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{10}$$

- In our example, θ is the disease prevalence.
- X is the data.
- $p(X|\theta)$ represents the *likelihood*.
- Remember *not* a probability distribution because θ varies.
- Encapsulates many **subjective** judgements about analysis.

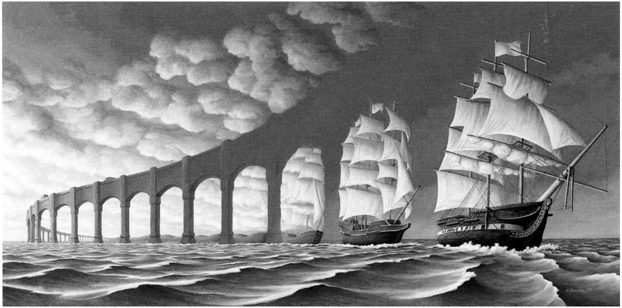
Priors summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{11}$$

- $p(\theta)$ represents the *prior*.
- A valid probability distribution.
- Similar to the likelihood; it is also subjective.

No "objective" rule for priors

- Embody subjective assumptions about state of the world.
- Essentially measure $Pr(\text{cause}|\text{pre-data knowledge})$.
  - Since knowledge differs between subjects $\implies$ different priors.
- Can be informed by pre-experimental data (for example, previous studies or from a collection of previous studies).



Denominator summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{12}$$

- $p(X)$ represents the *denominator*.
- Two different interpretations:
  - Before we collect X it is the **prior predictive distribution**.
  - When we have data $X = 3$ it is simply a number (that normalises the posterior) known as the **evidence** or **marginal likelihood**.
- Calculated from the numerator.
- Source of some difficulty of **exact** Bayesian inference (return to this later).

Posteriors summary

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{13}$$

- $p(\theta|X)$ represents the *posterior*.
- A valid probability distribution.
- Starting point for all further analysis in Bayesian inference.

Intuition behind Bayesian analyses

Bayes' rule:

$$p(\theta|X) = \frac{p(X|\theta) \times p(\theta)}{p(X)} \tag{14}$$

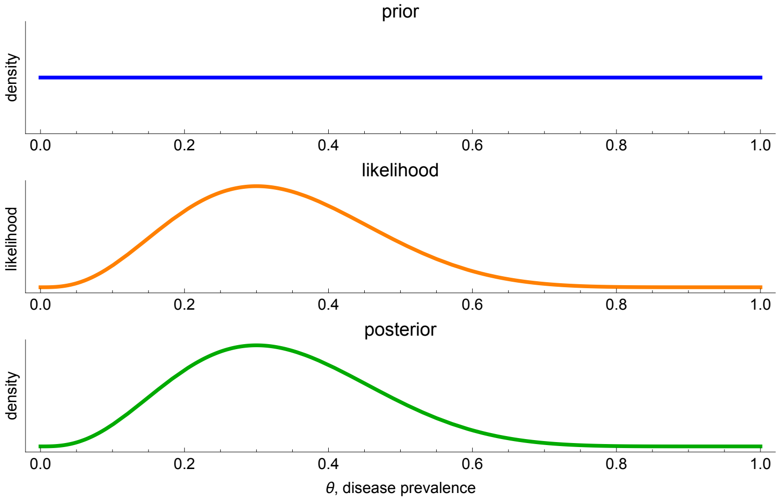
Tells us that:

$$p(\theta|X) \propto p(X|\theta) \times p(\theta) \tag{15}$$

Because $p(X)$ is independent of θ
$\implies$ the posterior is essentially a weighted (geometric) mean of the prior and likelihood.

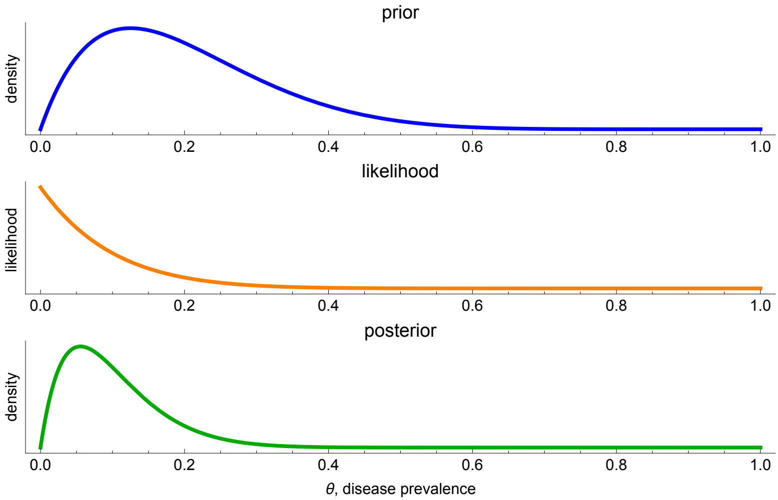
Intuition behind Bayesian analyses: prior

Consider $N = 10$ where $X = 3$.



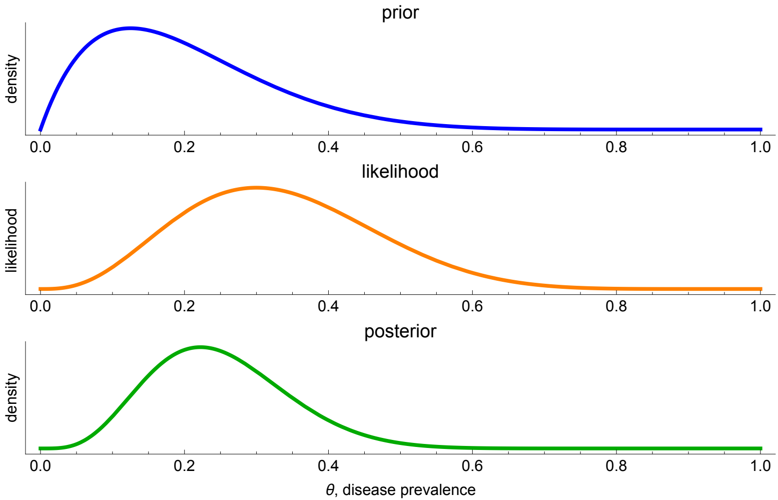
Intuition behind Bayesian analyses: likelihood

Now holding prior constant and varying X.



Intuition behind Bayesian analyses: sample size

Constant prior and proportion with disease; sample size$\uparrow$.



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The denominator revisited

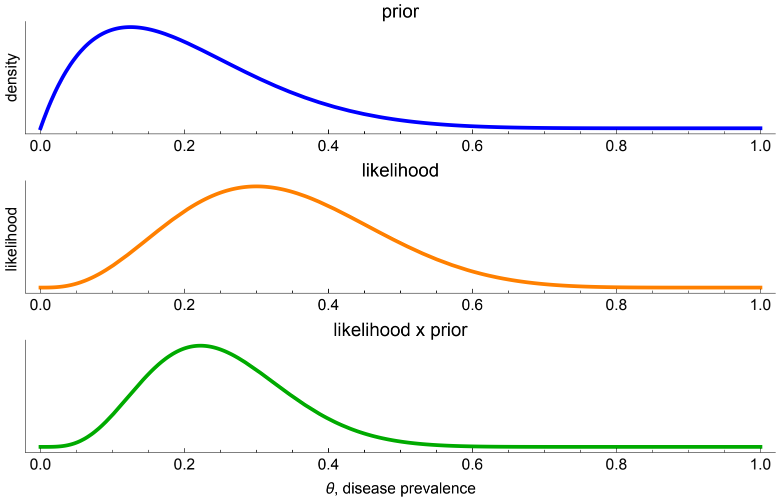
$$p(\theta|X=3) = \frac{p(X=3|\theta) \times p(\theta)}{p(X=3)} \tag{16}$$

Where we suppose we have $X = 3$ disease-positive out of a sample of 10 in our example. We obtain the denominator by averaging out all θ dependence. This is equivalent to integrating across all θ:

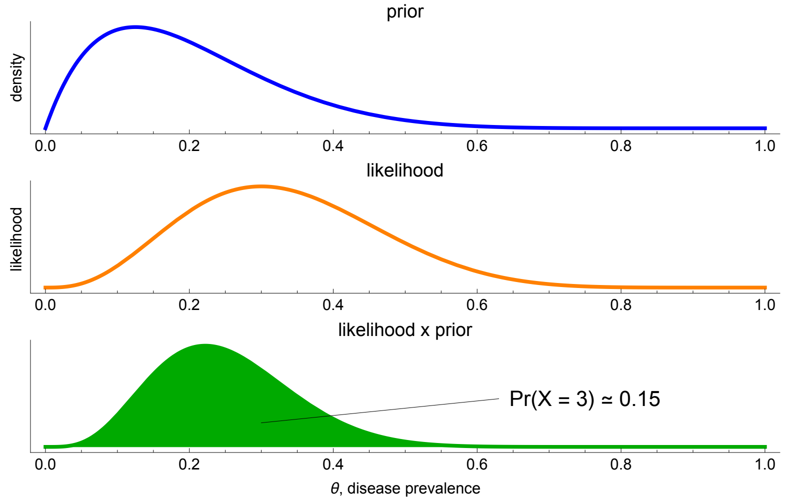
$$p(X=3) = \int_0^1 p(X=3|\theta) \times p(\theta)\mathrm{d}\theta \tag{17}$$

This is equivalent to working out an **area** under a curve.

The denominator as an area



The denominator as an area



Calculating the denominator in 2 dimensions

If we considered a different model where there were two parameters $\theta_1 \in (0,1),\ \theta_2 \in (0,1) \implies$:

$$p(X=3) = \int_0^1 \int_0^1 p(X=3|\theta_1,\theta_2) \times p(\theta_1,\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \tag{18}$$

This is equivalent to working out a **volume** contained within a surface.

Calculating the denominator in d dimensions

If we considered a different model where there were d parameters $(\theta_1, ..., \theta_d)$ all defined to lie between 0 and 1 $\implies$:

$$p(X=3) = \int_0^1 ... \int_0^1 p(X=3|\theta_1, ..., \theta_d) \times p(\theta_1, ..., \theta_d) \mathrm{d}\theta_1 ... \mathrm{d}\theta_d \tag{19}$$

This is equivalent to working out a $(d+1)$-dimensional **volume** contained within a d-dimensional (hyper-surface)!

The difficult denominator

- Calculating the denominator possible for $d <\sim 3$ using computers.
- Numerical quadrature and many other approximate schemes struggle for larger d.
- Many models have **thousands** of parameters.

Arrrghhh!

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What are conjugate priors?

Judicious choice of prior and likelihood can make posterior calculation trivial.

- Choose a likelihood L.
- Choose a prior $\theta \sim f \in F$, where:
  - F is a family of distributions.
  - f is a member of that **same** family.
- If posterior, $\theta | X \sim f' \in F \implies$ conjugate!
- In other words both the **prior** and **posterior** are members of the same distribution!

Conjugate priors

- For likelihood (if independent and identically-distributed):

$$X \sim \mathcal{B}(10, \theta) \implies p(X|\theta) \propto \theta^X (1-\theta)^{10-X} \tag{20}$$

- For prior assume a beta distribution (a reasonable choice if $\theta \in (0, 1)$):

$$\theta \sim \text{beta}(a, b) \implies p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1} \tag{21}$$

- Numerator of Bayes' rule for inference:

$$p(X|\theta) \times p(\theta) \propto \theta^X (1-\theta)^{10-X} \times \theta^{a-1}(1-\theta)^{b-1} \tag{22}$$

Conjugate priors

- Numerator of Bayes' rule for inference:

$$\begin{aligned} p(X|\theta) \times p(\theta) &\propto \theta^X (1-\theta)^{10-X} \times \theta^{a-1}(1-\theta)^{b-1} \\ &= \theta^{X+a-1}(1-\theta)^{10-X+b-1} \end{aligned}$$

- This has same θ-dependence as a beta$(X+a, 10-X+b)$ density $\implies$ must be this distribution!
- $\therefore$ a beta prior is *conjugate* to a binomial likelihood.

Table of common conjugate pairs of likelihoods and priors

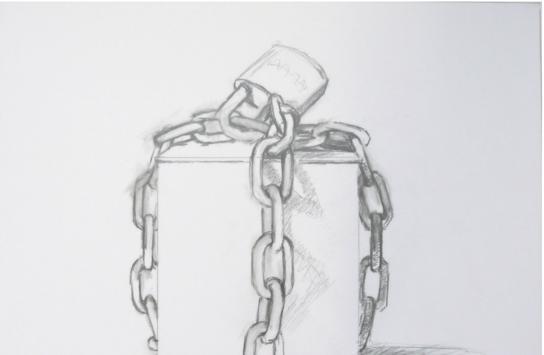
No need to do any integrals! Just lookup rules:

Likelihood	**Prior**	**Posterior**
Bernoulli	beta(α, β)	beta$(\alpha + \sum_{i=1}^{n} X_i, \beta + n - \sum_{i=1}^{n} X_i)$
Binomial	beta(α, β)	beta$(\alpha + \sum_{i=1}^{n} X_i, \beta + \sum_{i=1}^{n} N_i - \sum_{i=1}^{n} X_i)$
Poisson	Gamma(α, β)	Gamma$(\alpha + \sum_{i=1}^{n} X_i, \beta + n)$
Multinomial	Dirichlet$(\boldsymbol{\alpha})$	Dirichlet$(\boldsymbol{\alpha} + \sum_{i=1}^{n} \boldsymbol{X}_i)$
Normal	Normal-inv-Γ	Normal-inv-Γ

Limits of conjugate modelling

Using conjugate priors is limiting because:

- Often restricted to univariate problems.
  - $\Longrightarrow$ we could just use numerical quadrature instead.
- Required to use relevant conjugate prior for a given likelihood $\Longleftarrow$ may not be sufficient to capture pre-data beliefs of analyst.

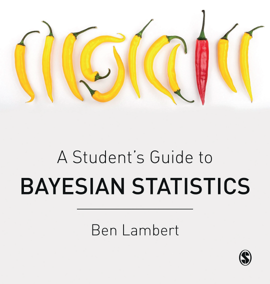


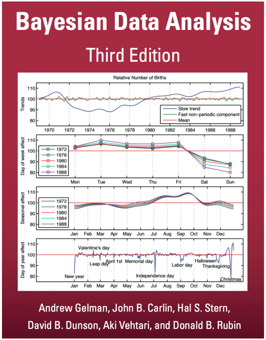
Longer-term solution

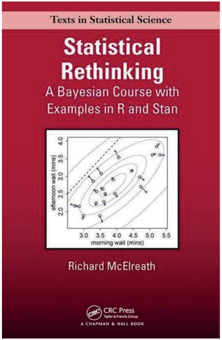
Sampling (usually MCMC)!
But that's another story.

Questions?

Books


A Student's Guide to
BAYESIAN STATISTICS
Ben Lambert


Bayesian Data Analysis
Third Edition
Relative Number of Births
Andrew Gelman, John B. Carlin, Hal S. Stern,
David B. Dunson, Aki Vehtari, and Donald B. Rubin


Texts in Statistical Science
Statistical Rethinking
A Bayesian Course with Examples in R and Stan
afternoon wait (mins)
morning wait (mins)
Richard McElreath
CRC Press
A CHAPMAN & HALL BOOK

Free lectures

- Richard McElreath's has a great YouTube lecture series.
- I have a series on YouTube called "A Student's Guide to Bayesian Statistics".

Not sure I understand?

Bayesian statistics:

$$p(\theta|\boldsymbol{D}) = \frac{p(\boldsymbol{D}|\theta) \times p(\theta)}{p(\boldsymbol{D})} \tag{23}$$

Beigeian statistics:

$$p(\theta|\boldsymbol{D}) = \frac{p(\boldsymbol{D}|\theta) \times p(\theta)}{p(\boldsymbol{D})} \tag{24}$$