An introduction to probability

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Who am I?

- Statistician working mainly in epidemiology.
- Claim to probability fame: born in the same town where Thomas Bayes lived (Tunbridge Wells, UK).

Course outline

- 9am-10.30am: follow along lecture and problems
- 10.30am-10.45am: refreshments break
- 10.45am-midday: follow along lecture and problems

Note: problems will involve both pen and paper type questions *and* those using R.
Levels: each problem set will have at least one *advanced* question.
Time: if you don't finish the questions in time, don't worry.

Resources

- Introduction to probability, Blitzstein and Hwang. Open source book available here:
  `https://projects.iq.harvard.edu/stat110/home`
- Seeing theory, Kunin et al. A beautiful online resource that has lots of creative ways to think about probability.
  `https://seeing-theory.brown.edu/`

Outline

1. What is probability and why do we need it?
2. Probability and counting
3. Conditional probability
4. Random variables and probability distributions
5. Joint distributions
6. Marginal and conditional probability
7. Bayes' rule
8. Continuous probability distributions

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What is probability?

Mathematics is the logic of certainty; probability [theory] is the [mathematical] logic of uncertainty

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Bitzstein and Hwang, 2019

Why do we need probability theory?

Life and science are full of unknown things. We say these things are *uncertain*.
Faced with these, we can give up; *Probability theory* gives us a way to make assumptions about uncertain phenomena which allows us to make progress with understanding without having to know everything.

> There are known knowns. These are things we know that we know. There are known unknowns. That is to say, there are things that we know we don't know. But there are also unknown unknowns. There are things we don't know we don't know.
>
> Donald Rumsfeld, 2002

Who uses probability?

It is used in:

- Statistics: probability is the foundational language of it
- Biology: e.g. inheritance of genes
- Meteorology: e.g. weather forecasts are generated using probabilities
- Epidemiology: e.g. analysing randomised clinical trials and fitting models to epidemiological data
- Physics: our current best explanation of the universe at small scales (quantum theory) is based on probability

The difficulties of probability

If we rely on our intuitions, we can easily get things wrong when dealing with uncertainty.

So, we need careful mathematical analysis.

Fortunately, *simulation* using R / Python / etc. can also really help us to understand.

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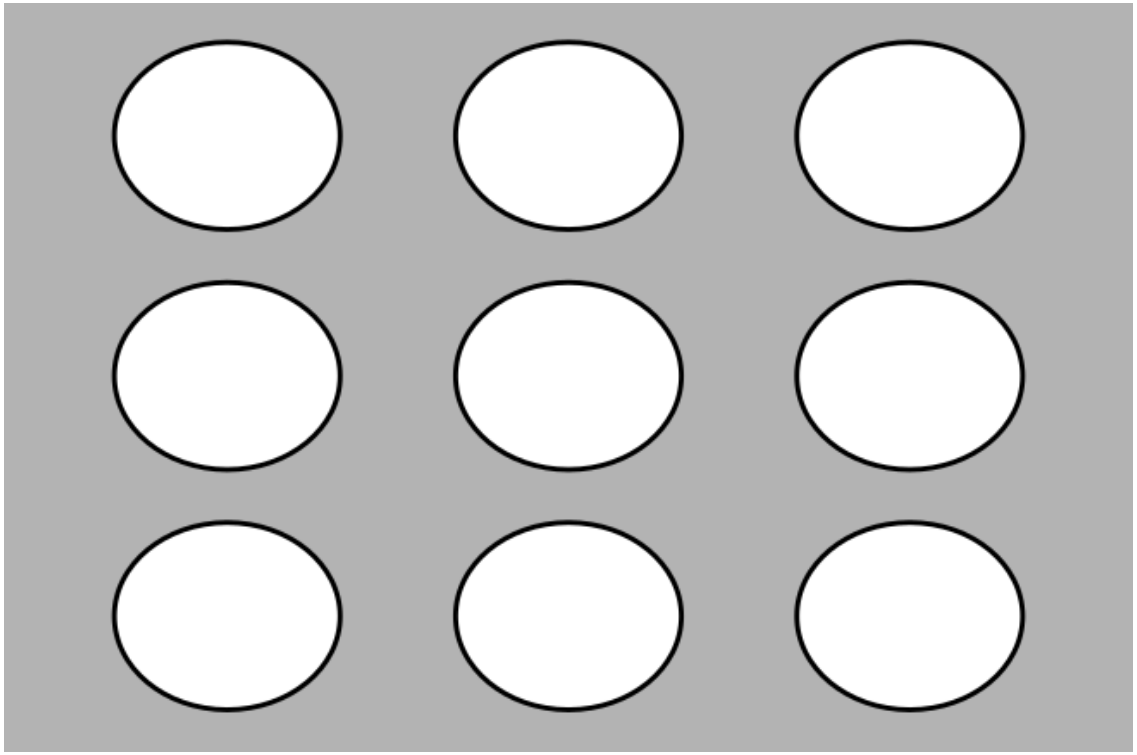
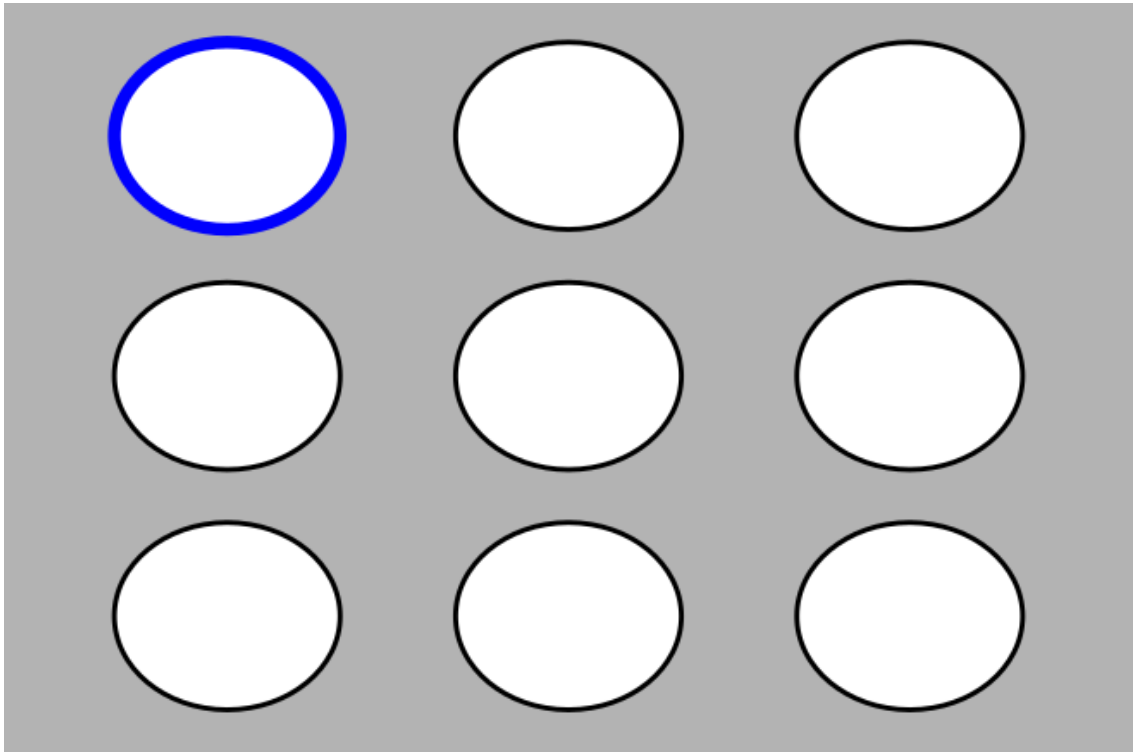
Blitzstein and Hwang's Pebble World

As an example, consider reaching into a bag to pull out one of nine pebbles: we call this *pebble world*.



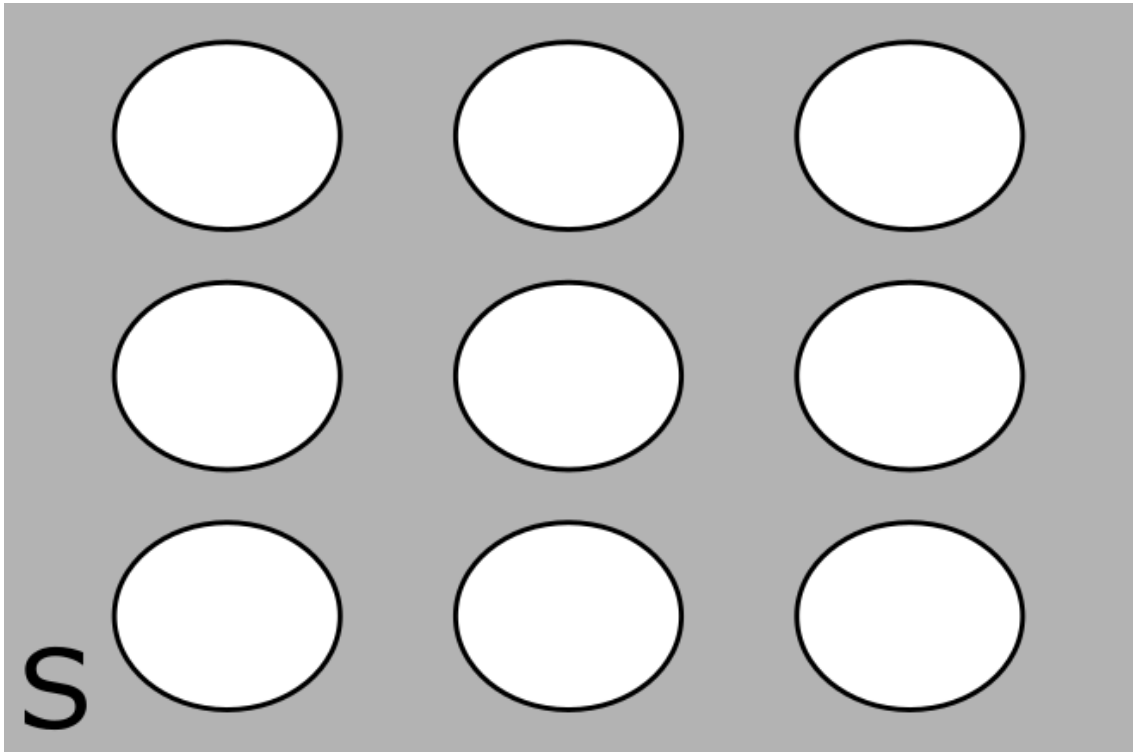
Outcomes

An *outcome* is a possible result of some activity. Here pulling one particular pebble out of the bag would be an outcome.



Sample spaces

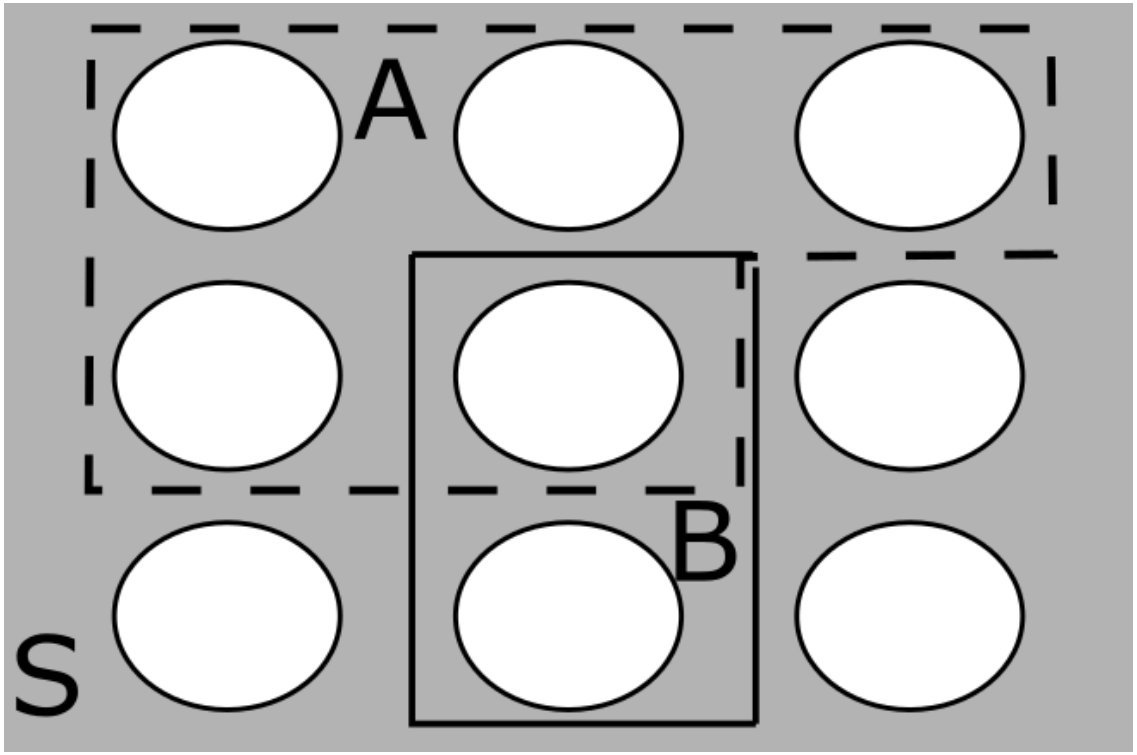
The *sample space*, S, is the *set* of all possible outcomes of an experiment. Here, it is the set of all pebbles.



Events

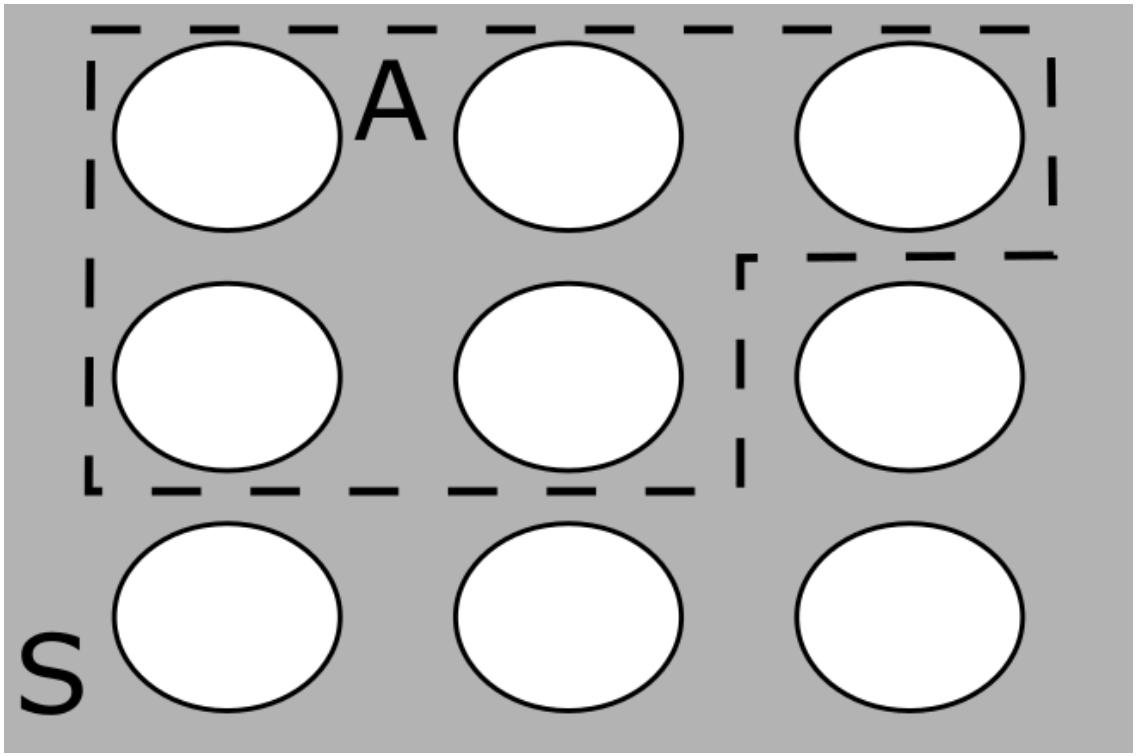
An *event* is a *set* of possible outcomes. For example, below event A corresponds to selecting one of five pebbles; event B to selecting one of two.

As we can see, two or more events can happen at once.



Probabilities of events

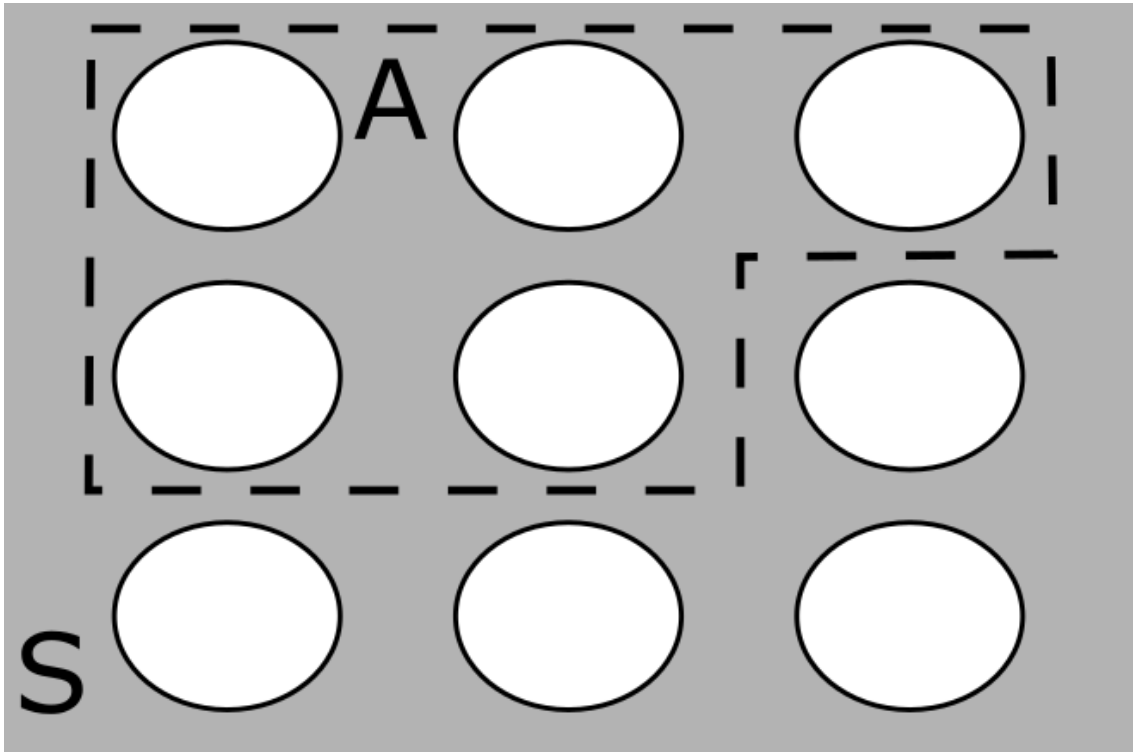
What do we mean by the probability of A occurring? We write this as: $\mathbb{P}(A)$.



Probabilities from counting

If all pebbles equally likely to be drawn:

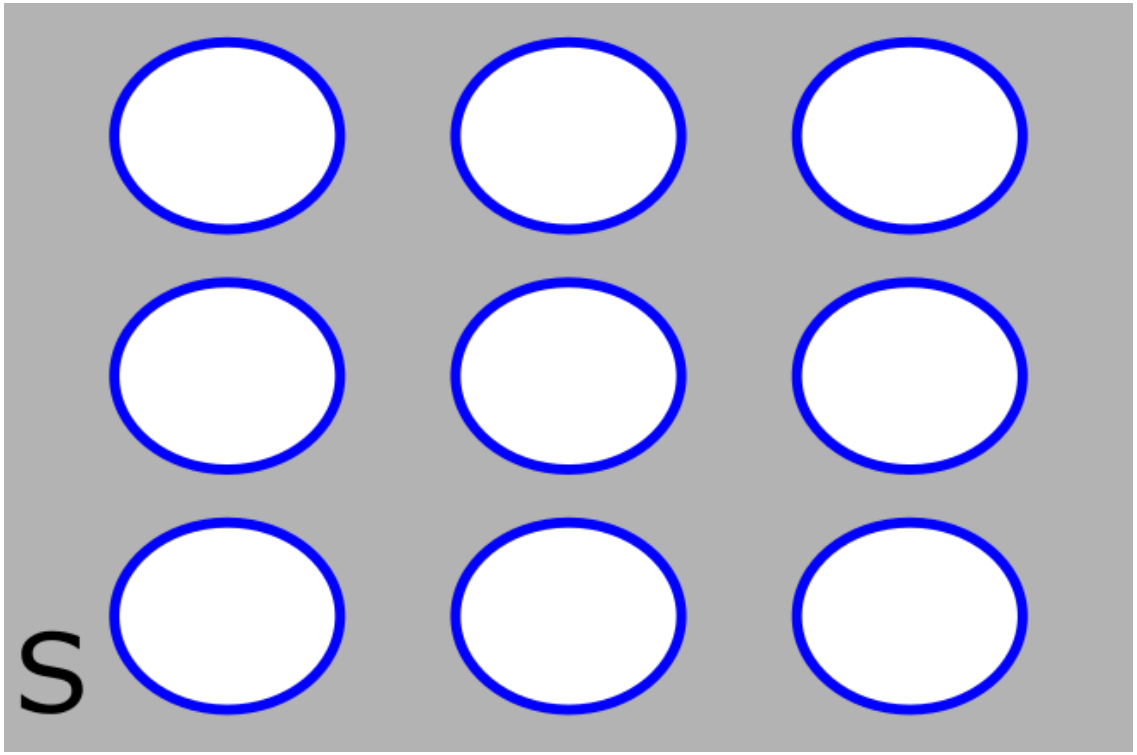
$$\mathbb{P}(A) = \frac{\text{number of pebbles in } A}{\text{number of pebbles in } S} = \frac{5}{9} \tag{1}$$



Probability of an event in S

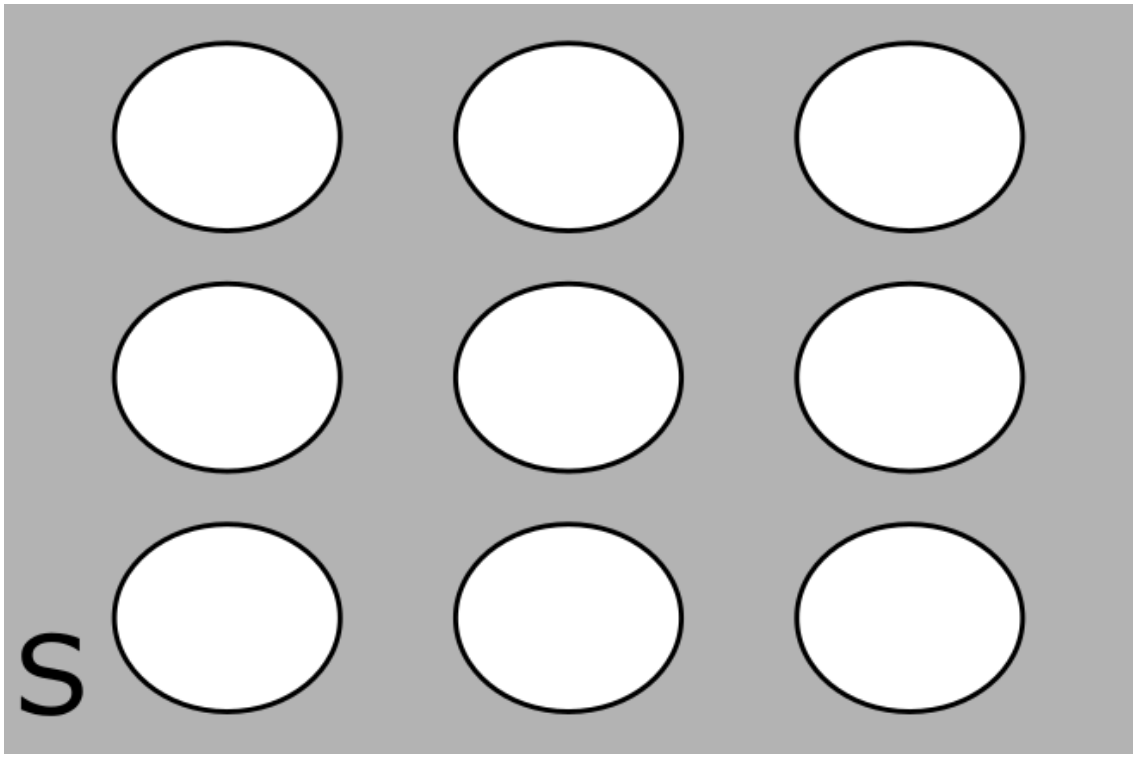
Consider the probability that some event in S occurs:

$$\mathbb{P}(S) = \frac{\text{number of pebbles in } S}{\text{number of pebbles in } S} = \frac{9}{9} = 1 \tag{2}$$



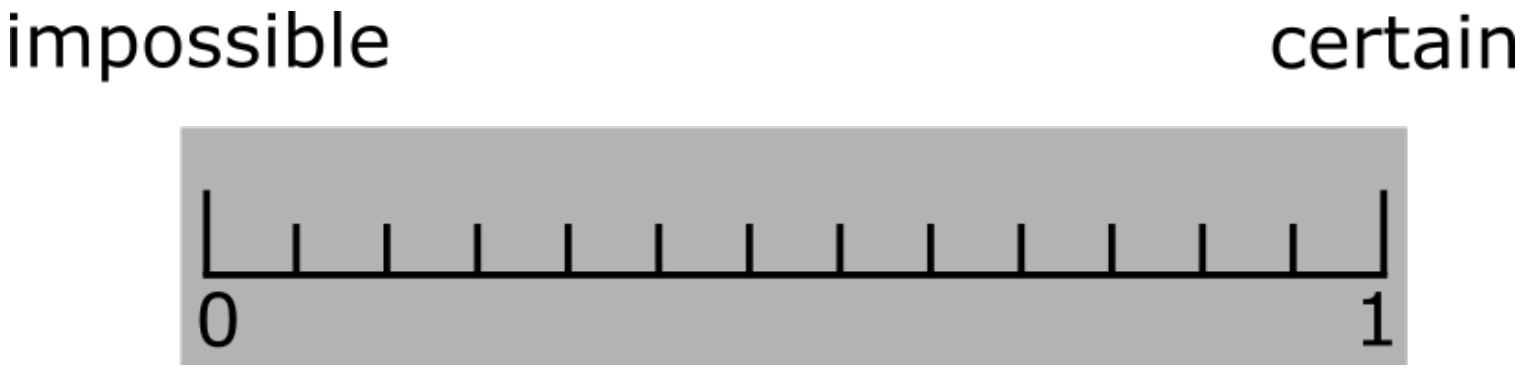
What about the probability that no event in S occurs?

$$\mathbb{P}(\text{not } S) = \frac{0}{9} = 0 \tag{3}$$



Defining probability

A probability of an event must be bounded between 0 and 1[1].



[1]Note: "impossible" isn't 100% accurate here but you can start out by thinking of probabilities this way.

Interpreting probabilities

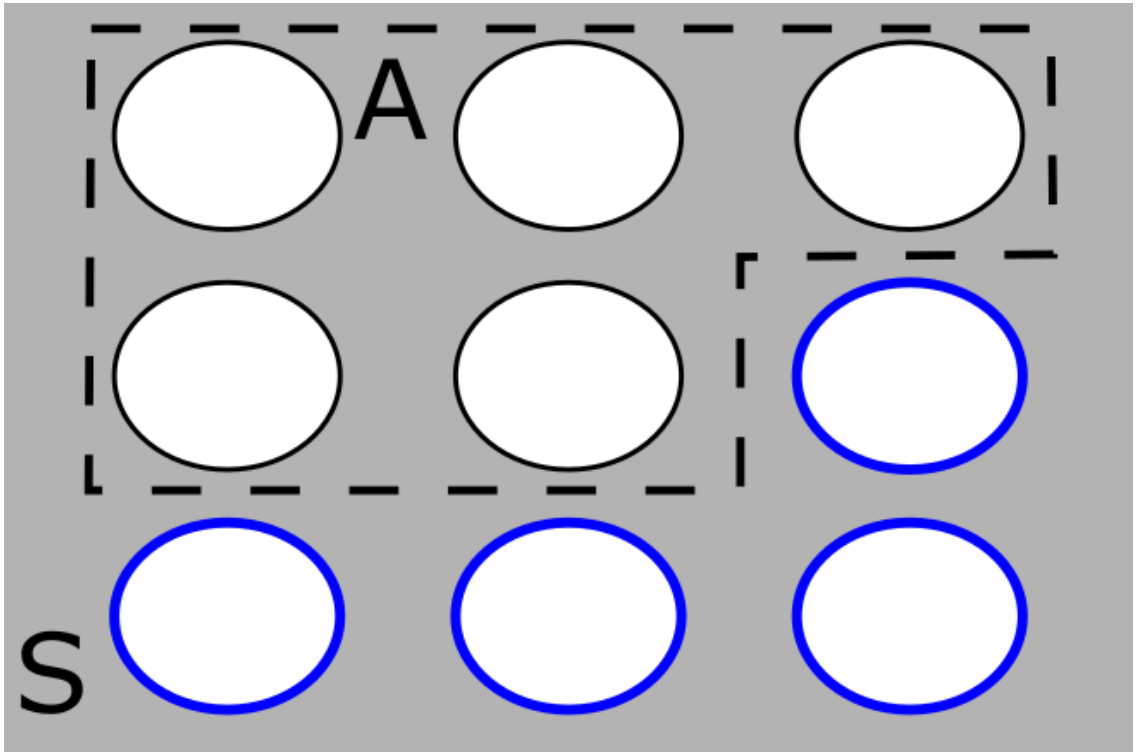
There are many schools of thought for what probabilities mean. Two common ones are:

- *Frequentist*. Think of probabilities as frequencies that would be obtained under many (actually infinite) repetitions of an experiment. E.g. flipping a coin a large number of times and using fraction of heads as $\mathbb{P}(H)$.
- *Bayesian*. Suppose probabilities reflect an underlying subjective belief about the chance of events occurring.

An event not occurring

We can also determine the probability that A does not occur:

$$\mathbb{P}(\text{not } A) = \frac{\text{number of pebbles not in } A}{\text{number of pebbles in } S} = \frac{4}{9} \tag{4}$$



Question

Can anyone think of an alternative way of determining the probability that A does not occur?

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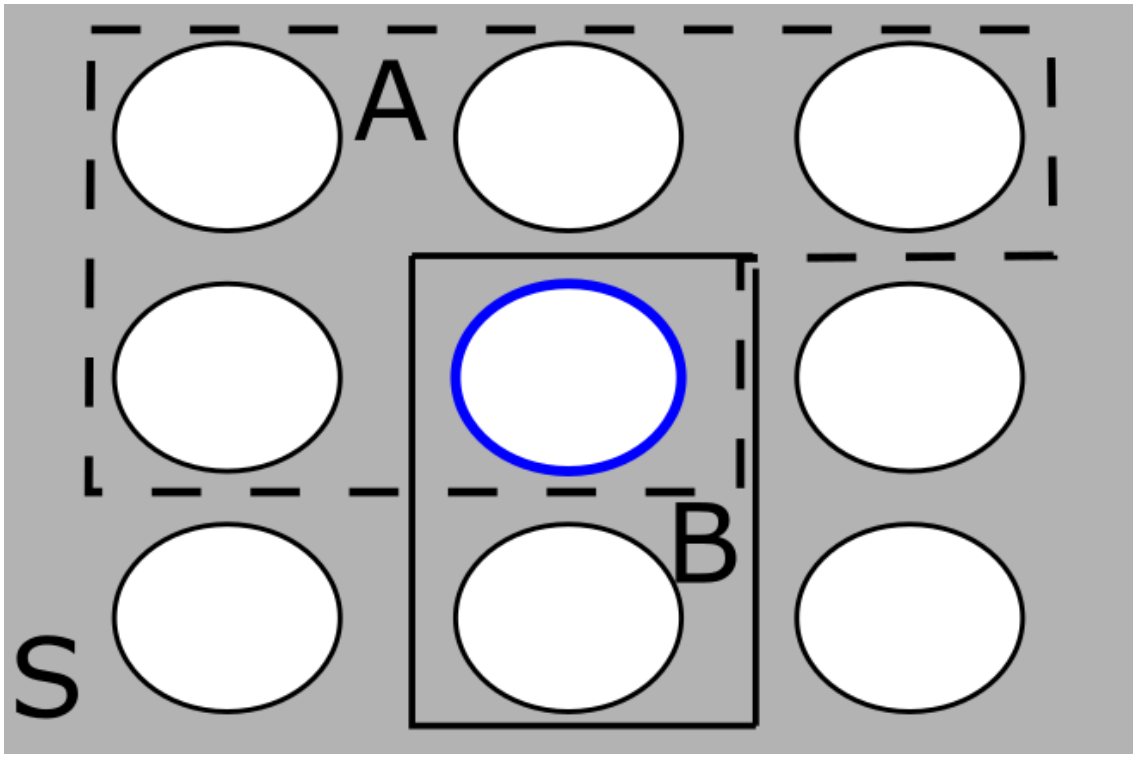
$$\mathbb{P}(\text{not } A) = \mathbb{P}(S) - \mathbb{P}(A) = 1 - \mathbb{P}(A) = 1 - \frac{5}{9} \qquad (5)$$

Combinations of events: intersection

We can determine the probability of A *and* B occurring by determining the overlap between these two events:

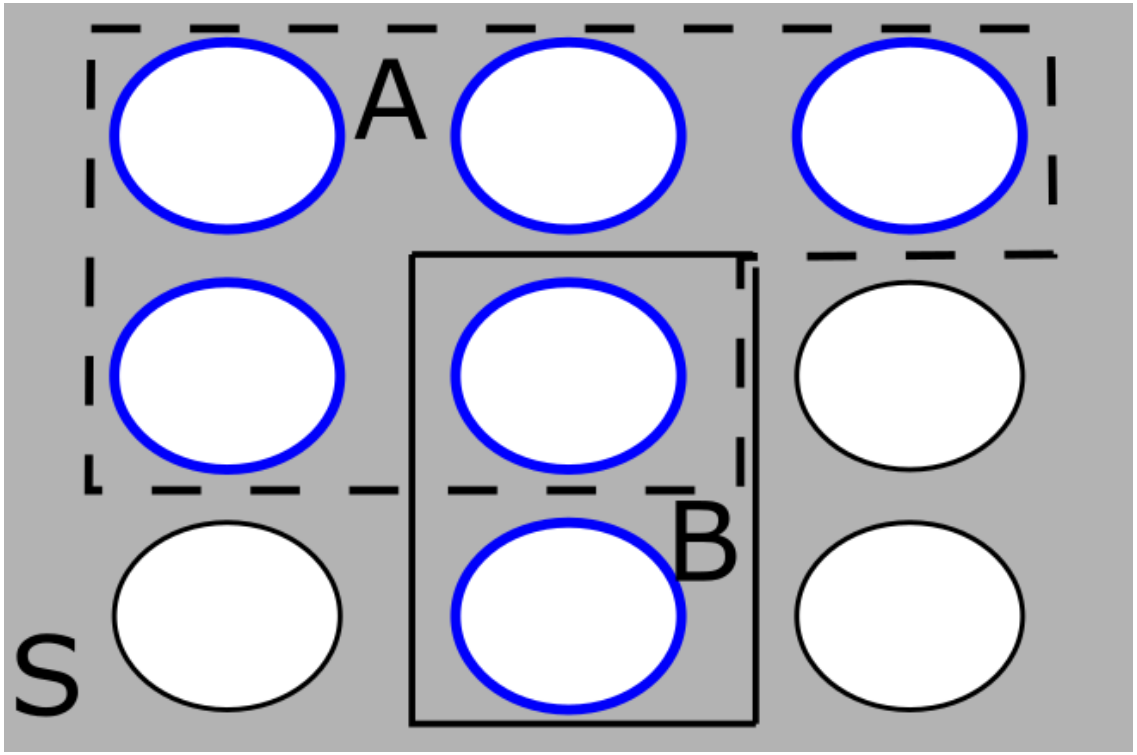
$$\mathbb{P}(A \cap B) = \mathbb{P}(A, B) = \frac{\text{number of pebbles in both } A \text{ and } B}{\text{number of pebbles in } S} = \frac{1}{9} \tag{6}$$



Combinations of events: union

We can determine the probability of A *and/or* B occurring by:

$$\mathbb{P}(A \cup B) = \frac{\text{number of pebbles in either } A \text{ or } B \text{ or both}}{\text{number of pebbles in } S} = \frac{6}{9} \tag{7}$$



Question

Can anyone think of an alternative way of determining the probability of the union of A and B?

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$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \tag{8}$$

$$= \frac{5}{9} + \frac{2}{9} - \frac{1}{9} = \frac{6}{9} \tag{9}$$

Questions?

Problems

Consider a six-sided die with numbers 1-6 on each face that is thrown once.

1. What's the probability that a six is thrown?
2. What's the probability that an even number is thrown?
3. Suppose two dice are thrown, what's the probability that the their sum adds up to 11 or less?
4. *Advanced*: if six dice are thrown, what's the probability that one of each of the numbers is obtained?

Example: a random sweet

Suppose you visit a sweet shop. The shop produces both ice cream and cake. It also offers three sauces: strawberry, vanilla and chocolate.

Unlike most shops, you don't get a choice. The way the shopkeeper allocates food is they randomly choose either a ice cream or cake (selecting either with equal probability). They then randomly select a sauce from the three available (again with equal probability of each).

Suppose you like anything with chocolate sauce. What's the probability that you obtain this?

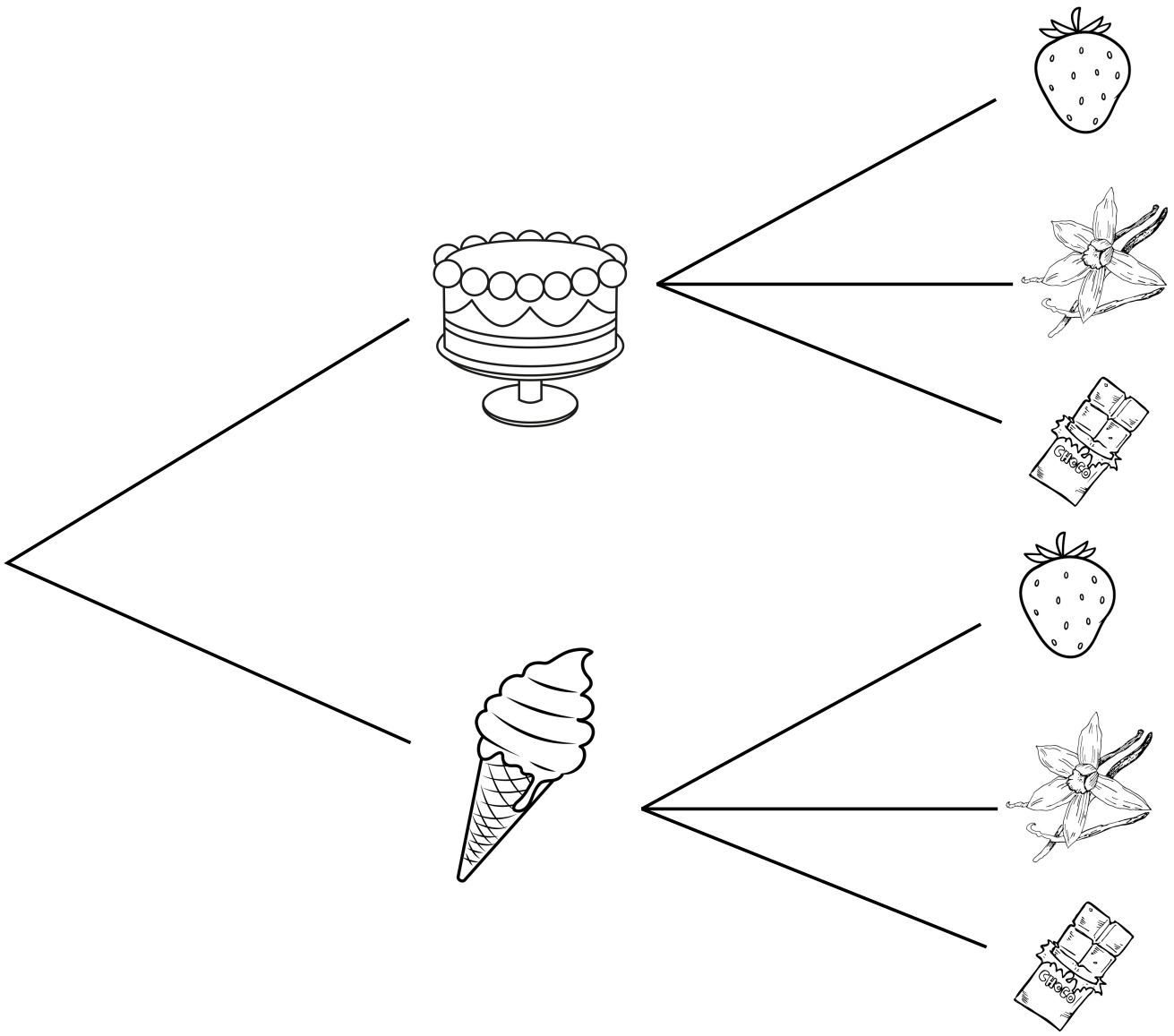
A naive way to count

How many outcomes are there? Cake with strawberry, cake with vanilla, cake with chocolate, ice cream with strawberry, ice cream with vanilla, ice cream with chocolate. So there are *six* outcomes.

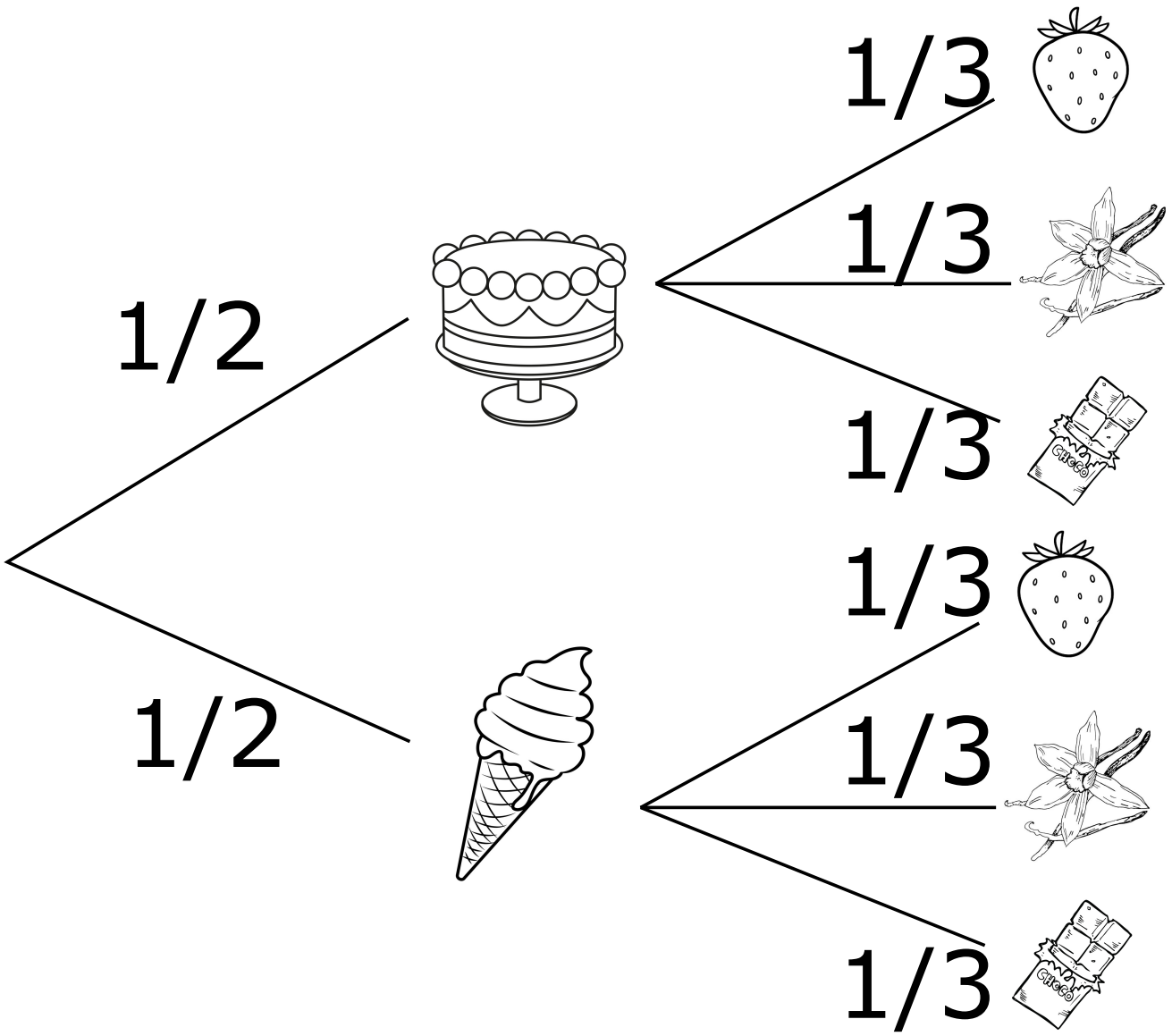
How many of these have chocolate sauce: two.

$$\mathbb{P}(\text{chocolate}) = \frac{2}{6} \tag{10}$$

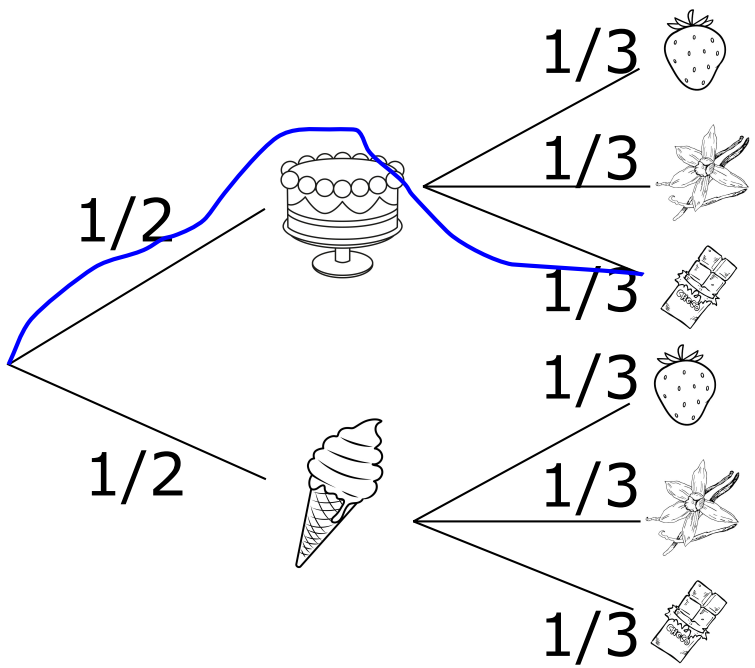
A sweet tree: making counting easier



Using trees to determine probabilities



What's the probability of cake with chocolate sauce?

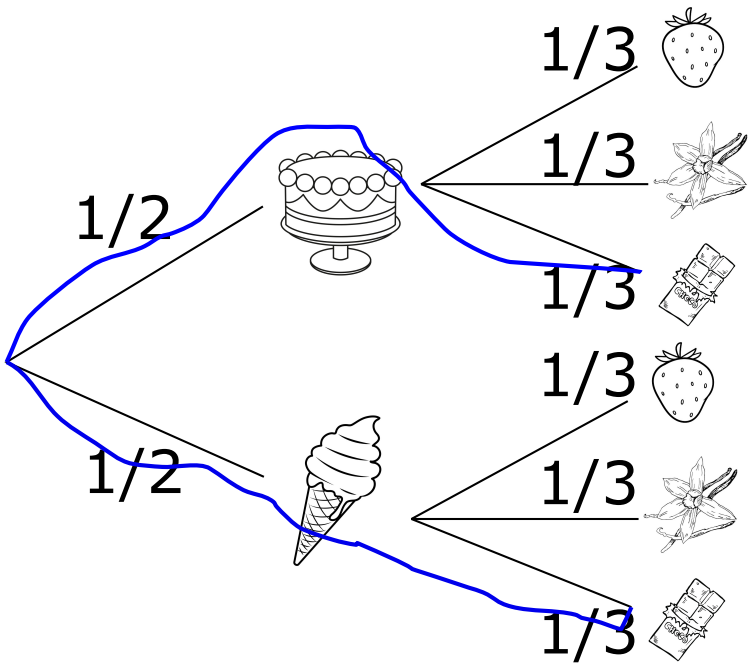


To obtain this probability, we take the probabilities obtained along the path and multiply:

$$\mathbb{P}(\text{chocolate cake}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \tag{11}$$

This is equivalent to counting possibilities (if all choices equally likely).

Chocolate probability



Sum up probabilities:

$$\mathbb{P}(\text{chocolate}) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{2}{6} \tag{12}$$

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What is conditioning?

When we receive new information, we want to take it into account to make better predictions.

Effectively, learning something about the world (typically) helps us to reduce our own uncertainty.

Conditioning is how this is handled in statistics.

Independence

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Use breast cancer example.

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