

# 1 Nondimensionalization

## 1.1 Initial value problem

$$\begin{aligned}\frac{dC}{d\tau} &= \frac{k_1}{k_2 + Z^2} + \frac{k_3}{k_4 + K^2} - d_1 C \\ \frac{dZ}{d\tau} &= \frac{ak_5 K^2}{k_6 + K^2} - d_2 Z \\ \frac{dK}{d\tau} &= \frac{rk_7}{k_8 + C^2} - d_3 K\end{aligned}\tag{1}$$

Initial conditions

$$C(0) = 1, \quad Z(0) = 0, \quad K(0) = 0\tag{2}$$

## 1.2 Units of dimensional parameters

$C, K$  and  $Z$  are concentrations with units  $L^{-3}$  and therefore all the LHS terms e.g.,  $\frac{dC}{d\tau}$  have units  $L^{-3}T^{-1}$

- $d_1, d_2, d_3$  have units  $T^{-1}$
- $k_2, k_4, k_6, k_8$  have units of  $L^{-6}$
- $k_1, k_3$  have units  $L^{-9}T^{-1}$
- $k_7$  has units  $L^{-9}T^{-1}$  if  $r$  is dimensionless
- $k_5$  has units  $L^{-3}T^{-1}$  if  $a$  is dimensionless

## 1.3 Nondimensional variables

The nondimensional variables are ...

$$\begin{aligned}y_1 &= \frac{k_2 d_1}{k_1} C \\ y_2 &= \frac{k_6 d_1}{k_4 k_5} Z \\ y_3 &= \frac{1}{\sqrt{k_4}} K\end{aligned}\tag{3}$$

- $\frac{k_2 d_1}{k_1}$  has units  $L^3$  therefore  $y_1$  is dimensionless
- $\frac{k_6 d_1}{k_4 k_5}$  has units  $L^3$  therefore  $y_2$  is dimensionless
- $\frac{1}{\sqrt{k_4}}$  has units  $L^3$  therefore  $y_3$  is dimensionless

## 1.4 Nondimensional groups (parameters)

The nondimensional groups (parameters) are ...

$$\begin{aligned}
 p_1 &= \frac{k_4^2 k_5^2}{k_2 k_6^2 d_1} \\
 p_2 &= \frac{k_2 k_3}{k_1 k_4} \\
 p_3 &= \frac{k_4}{k_6} \\
 p_4 &= \frac{d_2}{d_1} \\
 p_5 &= \frac{k_2^2 k_7 d_1}{k_1^2 \sqrt{k_4}} \\
 p_6 &= \frac{k_2^2 k_8 d_1^2}{k_1^2} \\
 p_7 &= \frac{d_3}{d_1}
 \end{aligned} \tag{4}$$

The following parameters are dimensionless trivially since they are ratios of dimensional parameter with equal units

- $p_3$  is dimensionless
- $p_4$  is dimensionless
- $p_7$  is dimensionless

The remaining parameters are not so obviously dimensionless

- $p_1$  has units  $\frac{L^{-12} \cdot L^{-6} T^{-2}}{L^{-6} \cdot L^{-12} \cdot T^{-2}}$
- $p_2$  has units  $\frac{L^{-6} \cdot L^{-9} T^{-1}}{L^{-9} T^{-1} \cdot L^{-6}}$
- $p_5$  has units  $\frac{L^{-12} \cdot L^{-9} T^{-1} \cdot T^{-1}}{L^{-18} T^{-2} \cdot L^{-3}}$
- $p_6$  has units  $\frac{L^{-12} \cdot L^{-6} \cdot T^{-2}}{L^{-18} \cdot T^{-2}}$

## 1.5 Nondimensionalizing governing equations

The terms to multiply through by are most easily determined from the decay term.

- Equation One – multiply by  $\frac{k_2}{k_1}$
- Equation Two – multiply by  $\frac{k_6}{k_4 k_5}$
- Equation Three – multiply by  $\frac{1}{d_1 \sqrt{k_4}}$

### 1.5.1 Equation One

$$\begin{aligned}
LHS &= \frac{1}{d_1} \frac{k_2 d_1}{k_1} \frac{dD}{d\tau} = \frac{1}{d_1} \frac{dy_1}{d\tau} \\
RHS(1) &= \frac{1}{1 + Z^2/k_2} = \frac{1}{1 + (k_4 k_5 y_2 / k_6 d_1)^2 / k_2} = \frac{1}{1 + p_1 y_2^2} \\
RHS(2) &= \frac{k_2 k_3}{k_1 k_4} \frac{1}{1 + k_4 y_3^2 / k_4} = \frac{p_2}{1 + y_3^2} \\
RHS(3) &= \frac{k_2 d_1}{k_1} C = y_1
\end{aligned} \tag{5}$$

### 1.5.2 Equation Two

$$\begin{aligned}
LHS &= \frac{1}{d_1} \frac{k_6 d_1}{k_4 k_5} \frac{dZ}{d\tau} = \frac{1}{d_1} \frac{dy_2}{d\tau} \\
RHS(1) &= \frac{k_6}{k_4} \frac{K^2}{k_6(1 + K^2/k_6)} = \frac{K^2/k_4}{(1 + k_4/k_6 * K^2/k_4)} = \frac{y_3^2}{1 + p_3 y_3^2} \\
RHS(2) &= \frac{d_2}{d_1} * \frac{k_6 d_1}{k_4 k_5} Z = p_4 y_2
\end{aligned} \tag{6}$$

### 1.5.3 Equation Three

$$\begin{aligned}
LHS &= \frac{1}{d_1 \sqrt{k_4}} \frac{dK}{d\tau} = \frac{1}{d_1} \frac{dy_3}{d\tau} \\
RHS(1) &= \frac{k_2^2 k_7 d_1}{d_1^2 \sqrt{k_4}} * \left( \frac{k_1^2}{k_2^2 d_1^2} \frac{1}{k_8 + C^2} \right) = p_5 \left( \frac{1}{k_2^2 d_1^2 / k_1^2 (k_8 + C^2)} \right) = p_5 \frac{1}{p_6 + y_1^2} \\
RHS(2) &= \frac{d_3}{d_1 \sqrt{k_4}} K = \frac{d_3}{d_1} y_3 = p_7 y_2
\end{aligned} \tag{7}$$