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1 Nondimensionalization

1.1 Initial value problem

$$\frac{dC}{d\tau} = \frac{k_1}{k_2 + Z^2} + \frac{k_3}{k_4 + K^2} - d_1 C \tag{1}$$
$$\frac{dZ}{d\tau} = \frac{a k_5 K^2}{k_6 + K^2} - d_2 Z \tag{2}$$
$$\frac{dK}{d\tau} = \frac{r k_7}{k_8 + C^2} - d_3 K \tag{3}$$

Initial conditions

$$C(0) = 1, \quad Z(0) = 0, \quad K(0) = 0 \tag{4}$$

1.2 Units of dimensional parameters

C, K and Z are concentrations with units L^{-3} and therefore all the LHS terms e.g., $\frac{dC}{d\tau}$ have units $L^{-3}T^{-1}$

- d_1, d_2, d_3 have units T^{-1}
- k_2, k_4, k_6, k_8 have units of L^{-6}
- k_1, k_3 have units $L^{-9}T^{-1}$
- k_7 has units $L^{-9}T^{-1}$ if r is dimensionless
- k_5 has units $L^{-3}T^{-1}$ if a is dimensionless

1.3 Nondimensional variables

The nondimensional variables are ...

$$y_1 = \frac{k_2 d_1}{k_1} C \qquad y_2 = \frac{k_6 d_1}{k_4 k_5} Z \qquad y_3 = \frac{1}{\sqrt{k_4}} K \tag{5}$$

- $\dfrac{k_2 d_1}{k_1}$ has units L^3 therefore y_1 is dimensionless
- $\dfrac{k_6 d_1}{k_4 k_5}$ has units L^3 therefore y_2 is dimensionless
- $\dfrac{1}{\sqrt{k_4}}$ has units L^3 therefore y_3 is dimensionless

1.4 Nondimensional groups (parameters)

The nondimensional groups (parameters) are ...

$$\left.\begin{aligned}
p_1 &= \frac{k_4^2 k_5^2}{k_2 k_6^2 d_1^2} & p_2 &= \frac{k_2 k_3}{k_1 k_4} \\
p_3 &= \frac{k_4}{k_6} & p_4 &= \frac{d_2}{d_1} \\
p_5 &= \frac{k_2^2 k_7 d_1}{k_1^2 \sqrt{k_4}} & p_6 &= \frac{k_2^2 k_8 d_1^2}{k_1^2} \\
p_7 &= \frac{d_3}{d_1}
\end{aligned}\right\} \tag{6}$$

Parameters p_3, p_4 and p_7 are trivially dimensionless since they are ratios of dimensional parameter with equal units. The remaining parameters are not so obviously dimensionless.

- p_1 has units $\dfrac{L^{-12} \cdot L^{-6} T^{-2}}{L^{-6} \cdot L^{-12} \cdot T^{-2}}$
- p_2 has units $\dfrac{L^{-6} \cdot L^{-9} T^{-1}}{L^{-9} T^{-1} \cdot L^{-6}}$
- p_5 has units $\dfrac{L^{-12} \cdot L^{-9} T^{-1} \cdot T^{-1}}{L^{-18} T^{-2} \cdot L^{-3}}$
- p_6 has units $\dfrac{L^{-12} \cdot L^{-6} \cdot T^{-2}}{L^{-18} \cdot T^{-2}}$

1.5 Nondimensionalizing governing equations

The terms to multiply through by are most easily determined from the decay term. The time scale is $\dfrac{1}{d_1}$.

1.5.1 Equation (1)

Multiply (1) by $\dfrac{k_2}{k_1}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{d_1}\frac{k_2 d_1}{k_1}\frac{dC}{d\tau} = \frac{1}{d_1}\frac{dy_1}{d\tau} = \frac{dy_1}{dt} \\
\text{RHS}(1) &= \frac{1}{1 + Z^2/k_2} = \frac{1}{1 + (k_4 k_5 y_2 / k_6 d_1)^2 / k_2} = \frac{1}{1 + p_1 y_2^2} \\
\text{RHS}(2) &= \frac{k_2 k_3}{k_1 k_4}\left(\frac{1}{1 + k_4 y_3^2 / k_4}\right) = \frac{p_2}{1 + y_3^2} \\
\text{RHS}(3) &= \frac{k_2 d_1}{k_1} C = y_1
\end{aligned} \tag{7}$$

1.5.2 Equation (2)

Multiply (2) by $\dfrac{k_6}{k_4 k_5}$

$$
\begin{aligned}
\mathrm{LHS} &= \frac{1}{d_1}\frac{k_6 d_1}{k_4 k_5}\frac{dZ}{d\tau} = \frac{1}{d_1}\frac{dy_2}{d\tau} = \frac{dy_2}{dt}\\
\mathrm{RHS}(1) &= \frac{k_6}{k_4}\left(\frac{K^2}{k_6(1+K^2/k_6)}\right) = \frac{K^2/k_4}{(1+(k_4/k_6)(K^2/k_4))} = \frac{y_3^2}{1+p_3 y_3^2}\\
\mathrm{RHS}(2) &= \frac{d_2}{d_1}\left(\frac{k_6 d_1}{k_4 k_5}\right) Z = p_4 y_2
\end{aligned}
\tag{8}
$$

1.5.3 Equation (3)

Multiply (3) multiply by $\dfrac{1}{d_1\sqrt{k_4}}$

$$
\begin{aligned}
\mathrm{LHS} &= \frac{1}{d_1\sqrt{k_4}}\frac{dK}{d\tau} = \frac{1}{d_1}\frac{dy_3}{d\tau} = \frac{dy_3}{dt}\\
\mathrm{RHS}(1) &= \left(\frac{k_2^2 k_7 d_1}{d_1^2\sqrt{k_4}}\right)\left(\frac{k_1^2}{k_2^2 d_1^2}\frac{1}{k_8+C^2}\right) = p_5\left(\frac{1}{k_2^2 d_1^2/k_1^2(k_8+C^2)}\right) = \frac{p_5}{p_6+y_1^2}\\
\mathrm{RHS}(2) &= \left(\frac{d_3}{d_1\sqrt{k_4}}\right) K = \frac{d_3}{d_1} y_3 = p_7 y_2
\end{aligned}
\tag{9}
$$