



PERGAMON

Studies in History and Philosophy of  
Modern Physics 34 (2003) 161–187

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Studies in History  
and Philosophy  
of Modern Physics

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# The non-relativistic limits of the Maxwell and Dirac equations: the role of Galilean and gauge invariance

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Received 18 November 2001; received in revised form 12 February 2002; accepted 26 February 2002

Dedicated to Rob Clifton

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## Abstract

The aim of this paper is to illustrate four properties of the non-relativistic limits of relativistic theories: (a) that a massless relativistic field may have a meaningful non-relativistic limit, (b) that a relativistic field may have more than one non-relativistic limit, (c) that coupled relativistic systems may be “more relativistic” than their uncoupled counterparts, and (d) that the properties of the non-relativistic limit of a dynamical equation may differ from those obtained when the limiting equation is based directly on exact Galilean kinematics. These properties are demonstrated through an examination of the non-relativistic limit of the familiar equations of first-quantized QED, i.e., the Dirac and Maxwell equations. The conditions under which each set of equations admits non-relativistic limits are given, particular attention being given to a gauge-invariant formulation of the limiting process especially as it applies to the electromagnetic potentials. The difference between the properties of a limiting theory and an exactly Galilean covariant theory based on the same dynamical equation is demonstrated by examination of the Pauli equation.

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**Keywords:** Maxwell equations; Dirac equations; Galilean invariance; Gauge invariance; Non-relativistic limits of relativistic theories

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## 1. Introduction

Considerable sophistication and insight has been added to the philosophical treatment of theory reduction in science since Nagel's seminal 1961 treatment of the issue (Nagel, 1961). A recent book by Batterman (2002) provides a useful review of many of these developments, as well as providing a detailed analysis of the nature of asymptotic reasoning in the delicate case of singular limits in physical theories. Batterman is right to claim that "most of the literature on [intertheoretic] reduction suffers ... from a failure to pay sufficient attention to detailed features of the respective theories and their interrelations" (2002, p. 5). This arguably applies even to the relatively humdrum case of one theory smoothly approaching another in a relevant correspondence limit, or so we shall argue.

The examples we shall use in this paper concern non-relativistic theories that can be seen as limiting cases of relativistic theories. In our view, these examples illustrate widely overlooked subtleties of the correspondence process that deserve more emphasis. Specifically, we shall illustrate four properties of these limits:

- (a) that a relativistic theory of a massless field may have a meaningful non-relativistic limit,
- (b) that a relativistic field theory may have more than one non-relativistic limit,
- (c) that coupled relativistic systems may be "more relativistic" than their uncoupled counterparts, and
- (d) that the properties of the non-relativistic limit of a dynamical equation may differ from those obtained when the limiting equation is based directly on Galilean kinematics.

All of these properties are features of usual first-quantized QED based on Maxwell's and Dirac's equations, to which we shall confine our analysis.

It is well known that in classical mechanics the behaviour of a charged particle in an external electromagnetic field can be derived from a Lagrangian which contains, apart from the free Lagrangian, terms which depend on the electromagnetic scalar and vector potentials (see, for example, Corben & Stehle, 1960). The resulting equations of motion will only be strictly Galilean covariant if the concomitant transformations of the electromagnetic potentials take a certain form, which differs from that associated with boosts in (relativistic) Maxwell theory. A similar state of affairs holds in non-relativistic single-particle quantum mechanics, as we have recently shown (Brown & Holland, 1999).<sup>1</sup>

It has been known at least since the 1973 work of Le Bellac and Lévy-Leblond (1973) that two sets of Maxwell-type "field" equations exist which are fully Galilean covariant, and which can be regarded as different non-relativistic limits of standard Maxwell theory. It is the so-called "magnetic limit" (see below) which involves transformations of the potentials under boosts that appear in the Galilean covariant

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<sup>1</sup>Since the publication of Brown and Holland (1999), we have become aware of further articles that address this question. See: Wong (1971), Steinwedel (1976), Celeghini, Lusanna, and Sorace (1976), Kraus (1980). We are grateful to Prof. G. Stedman for bringing these papers to our attention.

version of both classical and quantum mechanics. Both limits correspond to static theories, in which, analogously to the Newtonian gravitational field, the electromagnetic field no longer has dynamical degrees of freedom. In the context of the other (electric) limit, it has been remarked that the field acts merely as a “convenient tool to describe the instantaneous interaction between the charges and currents” (Jammer & Stachel, 1980, p. 6). It is remarkable that Galilean covariant versions of electromagnetism may be obtained by removing the displacement current (the magnetic limit) or the Faraday induction term (the electric limit) in Maxwell’s equations, but as we shall see below, further conditions must hold if the theories so obtained are to be viewed as limiting forms of Maxwell theory.

At first sight it may appear odd that a massless relativistic field should have a meaningful non-relativistic limit, the electromagnetic field in particular apparently being an intrinsically relativistic object. After all, comparing with the analogous problem in relativistic matter theories (see Section 4), the absence of a characteristic mass relative to which other energies may be deemed small would appear to mitigate against the possibility of defining a meaningful limiting process. In fact, this is not a problem. Although the literature on the limiting forms of electromagnetic theory is sparse, and to our knowledge there are no textbook discussions, the analogous problem in general relativity has been widely analysed. The masslessness of the gravitational field poses no problem there, the limit being defined in terms of the magnitudes of metrical components and low-velocity conditions on sources and test particles. Indeed, the non-relativistic limit plays a pivotal correspondence role in establishing the validity of Einstein’s equations (Misner, Thorne, & Wheeler, 1973). Of course, there is no necessity for an analogous correspondence argument in electromagnetic theory, but if there had been it is clear that the absence of a calibrating (photon) mass would not have been an impediment to achieving a consistent non-relativistic limit (this is obtained using conditions closely analogous to those employed for gravitation).

It is important to note that the usual cases in which electromagnetic forces are successfully treated in both classical and quantum charged-particle mechanics do *not* involve conditions that correspond to the Le Bellac–Lévy-Leblond magnetic limit. In the typical case, the treatment is neither Galilean nor Lorentz covariant. (A similar situation would occur if a Galilean covariant version of the field equations were combined with a relativistic treatment of fast-moving charges.) This does not represent a significant anomaly, because one can view the treatment as an approximation (under the usual conditions in which Lorentzian kinematics can be replaced by low-velocity ones) to the fundamental theory defined entirely in Minkowski spacetime and consistent with its symmetries. Nevertheless, it is significant for our purposes that there is a regime, namely that described by Le Bellac and Lévy-Leblond, in which the particle physics obeys the low-velocity symmetry.

In Section 2 we discuss the meaning we attach to the term “non-relativistic limit.” In Section 3 we study the limiting forms of Maxwell’s equations for a preassigned external source (e.g., the Dirac current), and in Section 4 treat the analogous problem for the Dirac equation with a given external electromagnetic field. In both

cases the external systems must obey constraints in order that the limiting equations are covariant. In Section 5 we study the case where the Maxwell and Dirac fields are coupled and find that even stronger constraints must be imposed to achieve a non-relativistic limit. An examination of the Pauli equation in Section 6 shows that the results obtained from the low-velocity limit of a theory based on Lorentzian kinematics generally differ from those obtained when the limiting theory is based directly on Galilean kinematics. Greenberger has also made this point recently in a related context (Greenberger, 2001).

## 2. Defining the non-relativistic limit

### 2.1. *The two limits*

According to [Le Bellac and Lévy-Leblond \(1973, p. 218\)](#), the term “non-relativistic” means “in agreement with the principle of Galilean relativity,” taking this principle to mean Galilean covariance.<sup>2</sup> This is how the term is construed in the relativity literature (see, for example, the treatment of the “Newtonian limit” in [Dixon, 1978](#)). Within the somewhat different culture of quantum texts, discussion of the non-relativistic limit of relativistic wave equations is generally confined to the formalities of the limiting process, and the sense in which the results may be deemed “non-relativistic” is not considered further. What criteria should one apply to judge whether one has indeed entered the non-relativistic domain? Typical treatments concentrate on the passage to relatively small velocities and energies, and a key assumption seems to be that the speed of light should cease to play any physical role in the limiting wave equation so that, if it appears at all, it is merely as, say, a parameter associated with a certain choice of units—this is the case, for example, with the two limits (22) and (29) (of the Maxwell theory studied below). It is not obvious, however, that criteria such as small velocities and energies, and the non-appearance of the speed of light, are in themselves sufficient to guarantee entry to the non-relativistic regime (although they may be a necessary means to this end). One must also consider whether the limiting theory is covariant with respect to limiting forms of the Lorentz transformation.

In our approach we define the operation of “taking the non-relativistic limit of a relativistic theory” to comprise the following three components:

- (a) The imposition of constraints on the relevant kinematical and dynamical quantities within a given inertial frame. This will result, in particular, in restricted

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<sup>2</sup>It may be useful here to avoid a possible confusion. Galileo’s original 1632 relativity principle was defined independently of the form of the coordinate transformations, and was effectively re-adopted by Einstein as a postulate in his special theory of relativity ([Brown & Sypel, 1995](#)). Thus, the principle of Galilean covariance is but one way of implementing the relativity principle. Whether a non-relativistic limit of a given relativistic theory exists or not, it follows from the relativistic theory that Galileo’s original relativity principle will hold in all circumstances.

forms of the associated dynamical equations. (Note that it may be undesirable to impose constraints on all the relevant quantities—see Section 2.2).

- (b) The application of these constraints to establish rules for the transformation of the kinematical and dynamical quantities into other inertial frames for small relative velocities.
- (c) The demonstration of the consistency of the first step by showing that the relations of constraint and the limiting dynamical equations are covariant with respect to the limiting transformation rules.

A theory obeying this last requirement will be called a “non-relativistic limit” of the relativistic theory. As noted, a relativistic theory may admit more than one distinct non-relativistic limiting theory. It is important to appreciate also that the question of whether the limiting equations are covariant with respect to the *exact* Galilean group is a separate issue. In the examples studied in this paper, it happens that the non-relativistic limits we obtain, where they exist, are also exactly covariant. Whether this is a necessary requirement of any such limiting process will not be discussed here. However, an important point about our approach does need to be stressed at this preliminary stage.

In a penetrating treatment of the reduction of Einstein’s (general relativistic) theory of gravity to the Newtonian theory (“reduction” here being used in the physicists’, not philosophers’ sense!<sup>3</sup>) Rohrlich (1989) compares what he calls the “dimensionless” process of reduction to the “dimensional” one. The former generally involves taking suitable dimensionless quantities—the ratio of two physical quantities of the same dimensions—to be negligibly small. The latter involves taking limits of dimensional parameters such as the light speed  $c$  or Planck’s constant  $\hbar$ . Rohrlich emphasizes that the dimensionless process represents a case of “factual” approximation and that the dimensional approximation is “counterfactual” (because, for example, it is simply a fact that  $c$  is finite), and although he highlights a number of problems associated with the latter, he does not reject it. Indeed, he uses its application in the case of gravitational theory to buttress an argument against the radical Kuhn–Feyerabend thesis of meaning change across paradigms. For our part, we are more suspicious of the counterfactual dimensional process of reduction, and we emphasize that the conditions referred to in component (a) above are consistent with the more “physical” dimensionless approach.

We now describe how the above procedure is to be implemented. In developing their limiting theory, Le Bellac and Lévy-Leblond (1973) noted that there are two distinct low-velocity limits of the Lorentz transformation of a relativistic 4-vector, and connected the two different non-relativistic limits of Maxwell’s equations with the possible low-velocity limits of physically relevant relativistic 4-vectors. In particular, they characterized the limits in terms of two limiting forms for the transformations of the electromagnetic potentials, a procedure we adopted previously (Brown & Holland, 1999) (but see Section 2.2).

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<sup>3</sup>Frequently, physicists say general relativity reduces to Newtonian gravity in the appropriate limit, for example, and philosophers say the reverse; see Nickles (1973) and Batterman (2002, p. 5).

To see how these limits emerge, let  $\mathbf{v}$  be the relative velocity of two inertial frames. With respect to a Lorentz transformation, an arbitrary 4-vector  $(u_0, \mathbf{u})$  transforms as follows:

$$u'_0 = \gamma \left( u_0 - \frac{\mathbf{v} \cdot \mathbf{u}}{c} \right), \quad (1)$$

$$\mathbf{u}' = \frac{(\gamma - 1)}{v^2} (\mathbf{u} \cdot \mathbf{v}) \mathbf{v} + \mathbf{u} - \frac{\gamma \mathbf{v} u_0}{c}, \quad (2)$$

where  $\gamma = (1 - v^2/c^2)^{-1/2}$ . In the limit  $|\mathbf{v}|/c \ll 1$  (as noted above, in the limiting procedures applied here  $c$  is a fixed number in the units used) we obtain  $\gamma \rightarrow 1$  and

$$u'_0 = u_0 - \frac{\mathbf{v} \cdot \mathbf{u}}{c}, \quad (3)$$

$$\mathbf{u}' = \mathbf{u} - \frac{\mathbf{v} u_0}{c}. \quad (4)$$

Regarded as exact, this transformation does not generate a group. It does generate a (approximate) group, however, for a set of boosts that obey the condition  $|\mathbf{v}|/c \ll 1$ . For example, for two successive boosts  $\mathbf{v}, \mathbf{v}'$  we have for the zeroth component (3), in the third inertial frame,

$$u''_0 = u_0 \left( 1 + \frac{\mathbf{v} \cdot \mathbf{v}'}{c^2} \right) - \frac{(\mathbf{v} + \mathbf{v}') \cdot \mathbf{u}}{c} \quad (4a)$$

and the coefficient of  $u_0$  is unity in this approximation. We can go further, and in particular obtain an exact group, if the relative magnitudes of the vector components are constrained. Consider first a timelike vector. Then when  $|\mathbf{u}| \ll |u_0|$  we obtain

$$u'_0 = u_0, \quad \mathbf{u}' = \mathbf{u} - \frac{\mathbf{v} u_0}{c}. \quad (5)$$

If instead we start with a spacelike vector, then when  $|\mathbf{u}| \gg |u_0|$  we have

$$u'_0 = u_0 - \frac{\mathbf{v} \cdot \mathbf{u}}{c}, \quad \mathbf{u}' = \mathbf{u}. \quad (6)$$

We thus obtain the two limiting cases of Le Bellac and Lévy-Leblond. It will be noted that each of the sets of transformations (5) and (6) generates an exact group, that is, the group property holds for all values of the relative velocity. This explains the importance of these transformations as limiting cases—we expect that these are the possible transformations that would be used in a free-standing Galilean covariant theory. Note that the ultra-timelike or ultra-spacelike character of the 4-vector is preserved by the respective transformation only for boosts that obey the condition  $|\mathbf{v}|/c \ll 1$ . As we shall see below, one can apply analogous procedures to other geometrical objects (tensors and spinors), where again two limits are also generally obtained for each object.

Applying this procedure to the Minkowski spacetime coordinates  $(ct, \mathbf{x})$  we have after the first limiting stage

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \quad t' = t - \frac{\mathbf{v} \cdot \mathbf{x}}{c^2}. \quad (7)$$

A similar result is obtained for the limit of the 4-gradient operator:

$$\nabla' = \nabla + \frac{\mathbf{v}}{c^2} \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \quad (8)$$

As noted, the transformations (7) and (8) generate groups within the approximation considered. Then, under the additional assumption of ultra-timelike separations,  $|\mathbf{x}| \ll ct$ , we get from (7) the usual Galilean transformation equations (an exact group):

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \quad t' = t. \quad (9)$$

From (9) we expect that the limiting form of the 4-gradient operator will be

$$\nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \quad (10)$$

However, this transformation follows from (8) only if the derivatives of relevant functions are such that expressions involving the (second) time-derivative term appearing in the first equation in (8) are negligible in this approximation. *The feasibility of implementing this condition depends on the context and requires careful examination.* As we shall see in Section 3.4 below, the condition need not be interpreted to mean that, when applied to a given function, the first derivative must dwarf the second. Indeed, we shall find that it can be the case that we must retain the time-derivative term in the limit in order to obtain consistent equations. Moreover, in passing to the limit we have to bear in mind that the operations of differentiation and of taking the limit  $|\mathbf{v}|/c \ll 1$  do not generally commute. We must therefore use the formulas (8) and complete the limiting process at the end of the calculation. We shall denote the condition on the derivatives as “ $\nabla \gg (\mathbf{v}/c^2)\partial/\partial t$ ”, with the proviso that it must be applied with care.

In the opposite (ultra-spacelike) limit  $|\mathbf{x}| \gg ct$  we obtain

$$\mathbf{x}' = \mathbf{x}, \quad t' = t - \mathbf{v} \cdot \mathbf{x}/c^2 \quad (11)$$

(also an exact group). But whereas (9) is a recognized symmetry transformation, the status of (11) is different: it represents merely a re-synchronization of clocks, i.e., a redefinition of the spacelike hyperplanes of simultaneity. We shall not discuss this second transformation further (apart from noting that the equations of motion in non-relativistic classical and quantum mechanics are not covariant under the transformation).

The conditions imposed on the Lorentzian kinematics so far do not yet suffice to regain the full Galilean kinematics because the relativistic velocity transformation rule still differs from the Galilean one if the velocity of the observed system—not to be confused with the relative velocity  $\mathbf{v}$  of the frames—is relativistic. (The Fresnel drag coefficient which follows from this rule is measurable even when  $|\mathbf{v}|/c \ll 1$  because the object system is light itself.) Thus, we shall suppose throughout that the charges involved in the source terms,<sup>4</sup> as well as any test charges, move at non-relativistic speeds, and therefore that the Galilean rule for transforming velocities is

<sup>4</sup>This does not mean that currents are always ultra-timelike—see the magnetic limit in Section 3.2.

valid

$$\dot{\mathbf{x}}' = \dot{\mathbf{x}} - \mathbf{v}. \quad (12)$$

Eqs. (9), (10) and (12) are the defining equations of Galilean kinematics. It is useful to collect together the conditions under which they are valid as limiting cases of the full Lorentzian kinematic relations:

$$|\mathbf{v}|/c \ll 1, \quad |x| \ll ct, \quad |\dot{\mathbf{x}}| \ll c, \quad \nabla \gg (\mathbf{v}/c^2)\partial/\partial t. \quad (13)$$

The last three constraints are independent of one another.<sup>5</sup> They are all Galilean invariant relations in the sense that their validity in one frame implies (from (9), (12) and (10), respectively) their validity in a slowly relatively moving frame.

One of the points we wish to illustrate in this paper is that, for a given physical system, there is a difference between a free-standing theory of the system that is consistent with exact Galilean kinematics (i.e. (9) and hence (10) and (12)), and the non-relativistic limit of a theory consistent with Lorentzian kinematics, even though each theory employs the same dynamical equations. Each theory is generally supplemented by constraints or exhibits properties that are extraneous or unnatural in the other. Indeed, the two theories may be incompatible. As an example of unnaturalness, the Dirac current defines flow lines of probability that are bounded by the speed of light. This constraint remains intact when we pass to the non-relativistic domain but there is no such restriction on the current associated with the Pauli equation when the latter is regarded as based directly on Galilean kinematics. We shall see an example of incompatibility in Section 6.

## 2.2. *The electromagnetic potentials*

The method of characterizing the low-velocity limit just outlined that culminates in an exact group is acceptable when applied to 4-vectors whose absolute values have physical significance (such as the electric current). In the case of the electromagnetic potentials, however, it is not a satisfactory technique because the absolute values of the 4-vector components are devoid of physical significance—they depend on the choice of gauge. One of our main aims in what follows is to formulate the limiting process, in particular as it is applied to the potentials, in a fully gauge invariant way (thereby improving the treatments of [Brown & Holland, 1999](#), [Le Bellac & Lévy-Leblond, 1973](#)). (We have to develop a consistent limiting theory for the potentials as they mediate the field–matter interaction.) The way to avoid introducing gauge-dependent assumptions is to employ the transformation (3) and (4) without resorting to the second stage equations. As noted, this generates an approximate group. Going to the second stage gives an exact group but, as stated, this breaks gauge invariance. Hence, if we insist on not restricting the gauge, there is no limiting form of the potentials that would coincide with the transformation we would expect if we started

<sup>5</sup>It is noteworthy how often the first and third of these conditions are conflated, and the importance of the remaining conditions overlooked. See, for example, [Batterman \(2002, pp. 78–79\)](#).



from a free-standing Galilean covariant theory, based on exact groups of transformations (for the latter transformation see Section 6.1).

### 3. The limiting forms of Maxwell's equations

#### 3.1. The electric limit

We have, for a particle of unit charge,

$$\begin{aligned}\partial_\nu F^{\nu\mu} &= J^\mu, \\ \partial_{(\mu} F_{\nu\sigma)} &= 0\end{aligned}\tag{14a}$$

or in 3 + 1 terms,

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\epsilon_0, \quad \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t, \quad \nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t + \mu_0 \mathbf{J}.\end{aligned}\tag{14b}$$

Here  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , where  $A_\mu$  is the electromagnetic 4-potential, and  $J^\mu = (c\rho, \mathbf{J})$  is the current 4-vector. These equations are covariant with respect to the Lorentz transformation

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{(1 - \gamma)(\mathbf{v} \cdot \mathbf{E})\mathbf{v}}{v^2},\tag{15}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - (1/c^2)\mathbf{v} \times \mathbf{E}) + \frac{(1 - \gamma)(\mathbf{v} \cdot \mathbf{B})\mathbf{v}}{v^2}\tag{16}$$

and the transformation (1) and (2) for  $J^\mu$ . We need to consider the relative magnitude of the field components to determine the low-velocity limits ( $|\mathbf{v}|/c \ll 1$ ) of these relations. As with the case of 4-vectors, it turns out that there are two limits that are relevant to the Galilean regime.

In the “electric limit” we assume there is an inertial frame in which the fields obey the following two conditions. First, the electric field is dominant,

$$|\mathbf{E}| \gg c|\mathbf{B}|.\tag{17}$$

From (15) and (16) this implies that the fields transform as follows when we pass to a slowly relatively moving inertial frame:

$$\mathbf{E}' = \mathbf{E},\tag{18}$$

$$\mathbf{B}' = \mathbf{B} - (1/c^2)\mathbf{v} \times \mathbf{E}.\tag{19}$$

These relations imply that the same dominant condition will obtain in the primed frame:

$$|\mathbf{E}'| \gg c|\mathbf{B}'|.\tag{20}$$

The second condition is that the time-variation of the magnetic field is negligible:<sup>6</sup>

$$\partial \mathbf{B}/\partial t \approx 0.\tag{21}$$

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<sup>6</sup>Despite being a dimensional restriction this is not a counterfactual one (cf. Section 2.1).

Under this condition, the limiting form of Maxwell's Equations (14b) is the usual set but without the Faraday induction term:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho/\varepsilon_0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= 0, & \nabla \times \mathbf{B} &= (1/c^2)\partial\mathbf{E}/\partial t + \mu_0\mathbf{J}.\end{aligned}\quad (22)$$

In these equations we have to consider the transformation properties of the 3-current and density. We shall now demonstrate that the truncated Maxwell equations (22), subject to conditions (17) and (21), are covariant with respect to the limiting boost transformation Eqs. (8), (18) and (19) if

$$\begin{aligned}\rho'(\mathbf{x}', t') &= \rho(\mathbf{x}, t), \\ \mathbf{J}'(\mathbf{x}', t') &= \mathbf{J}(\mathbf{x}, t) - \mathbf{v}\rho(\mathbf{x}, t).\end{aligned}\quad (23)$$

This is the limit of the transformation of a timelike 4-vector, when

$$|c\rho| \gg |\mathbf{J}|. \quad (24)$$

(which also holds in the primed frame). We shall assume Eq. (22) are valid in the original frame and derive their validity in the primed frame. We have, for three of Maxwell's equations,

$$\begin{aligned}\nabla' \cdot \mathbf{B}' &= \nabla \cdot \mathbf{B} - \frac{1}{c^2} \nabla \cdot (\mathbf{v} \times \mathbf{E}) \quad \text{using (21) and } \mathbf{v} \cdot \left( \mathbf{v} \times \frac{\partial \mathbf{E}}{\partial t} \right) = 0 \\ &= \nabla \cdot \mathbf{B} - \frac{1}{c^2} \mathbf{v} \cdot (\nabla \times \mathbf{E}) \\ &= 0 \quad \text{from (22),}\end{aligned}$$

$$\begin{aligned}\nabla' \cdot \mathbf{E}' - \rho'/\varepsilon_0 &= \nabla \cdot \mathbf{E} + \frac{\mathbf{v}}{c^2} \cdot \frac{\partial \mathbf{E}}{\partial t} - \rho/\varepsilon_0 \\ &= \nabla \cdot \mathbf{E} - \rho/\varepsilon_0 + \mathbf{v} \cdot (\nabla \times \mathbf{B} - \mu_0\mathbf{J}) \quad \text{from (22)} \\ &= \nabla \cdot (\mathbf{E} - \mathbf{v} \times \mathbf{B}) - \rho/\varepsilon_0 \quad \text{from (24)} \\ &= \nabla \cdot \mathbf{E} - \rho/\varepsilon_0 \quad \text{from (17)} \\ &= 0 \quad \text{from (22),}\end{aligned}$$

$$\begin{aligned}\nabla' \times \mathbf{B}' - (1/c^2)\partial\mathbf{E}'/\partial t' - \mu_0\mathbf{J}' &= \nabla \times \mathbf{B} - (1/c^2)\partial\mathbf{E}/\partial t - \mu_0\mathbf{J} - (1/c^2)\nabla \times (\mathbf{v} \times \mathbf{E}) - (1/c^4)\mathbf{v} \\ &\quad \times (\mathbf{v} \times \partial\mathbf{E}/\partial t) - (1/c^2)(\mathbf{v} \cdot \nabla)\mathbf{E} + \mu_0\mathbf{v}\rho \\ &= \nabla \times \mathbf{B} - (1/c^2)\partial\mathbf{E}/\partial t - \mu_0\mathbf{J} \\ &= 0 \quad \text{from (22),}\end{aligned}$$

where in the second line the fifth term ( $\propto (1/c^2)(\mathbf{v}|\partial\mathbf{E}/\partial t||\mathbf{v}|\cos(\mathbf{v}, \partial\mathbf{E}/\partial t) - |\mathbf{v}|^2\partial\mathbf{E}/\partial t)$ ) is negligible with respect to the second ( $\propto \partial\mathbf{E}/\partial t$ ) and, substituting  $\varepsilon_0\nabla \cdot \mathbf{E}$  for  $\rho$ , the fourth, sixth and seventh terms cancel. To demonstrate the covariance of

the final Maxwell equation we must compute

$$\nabla' \times \mathbf{E}' = \nabla \times \mathbf{E} + \frac{\mathbf{v}}{c^2} \times \frac{\partial \mathbf{E}}{\partial t}. \quad (25)$$

To do this we first show that the last (time-derivative) term in (25) has the same magnitude as a particular expression involving the space derivatives. Consider the following expression:

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{E} &= c^2 \nabla \times \mathbf{B} - c^2 \mu_0 \mathbf{J} - \nabla \times (\mathbf{v} \times \mathbf{E}) + \mathbf{v} \nabla \cdot \mathbf{E} \quad \text{from (22)} \\ &= \nabla \times (c^2 \mathbf{B} - \mathbf{v} \times \mathbf{E}) + \frac{1}{\epsilon_0} (\mathbf{v} \rho - \mathbf{J}) \quad \text{since } c^2 = 1/\epsilon_0 \mu_0. \end{aligned} \quad (26)$$

In the last line, the terms in the first bracket are comparable due to (19), and the ultra-timelikeness of the current (24), implies that the terms in the final bracket are comparable. Hence, going back to the first line of (26), the first pair of terms taken together is comparable to the second pair of terms taken together, and thus the two terms on the left-hand side have the same magnitude:

$$\left| \frac{\partial \mathbf{E}}{\partial t} \right| \approx |(\mathbf{v} \cdot \nabla) \mathbf{E}|. \quad (27)$$

The magnitude of the right-hand side of (25) is therefore equal to

$$\begin{aligned} |\text{rhs}| &= \left| \nabla \times \mathbf{E} + (1/c^2) \mathbf{v} \times (\mathbf{v} \cdot \nabla) \mathbf{E} \right| \\ &= \left| \nabla \times \mathbf{E} + (1/c^2) \mathbf{v} \times (\nabla \times (\mathbf{v} \times \mathbf{E})) \right| \\ &= \left| \nabla \times [\mathbf{E} + (1/c^2) \mathbf{v} \times (\mathbf{v} \times \mathbf{E})] - (1/c^2) \mathbf{v} \mathbf{v} \cdot (\nabla \times \mathbf{E}) \right| \\ &= \left| \nabla \times [\mathbf{E} + (1/c^2) (\mathbf{v} |\mathbf{E}| |\mathbf{v}| \cos(\mathbf{v}, \mathbf{E}) - |\mathbf{v}|^2 \mathbf{E})] - (1/c^2) \mathbf{v} \mathbf{v} \cdot (\nabla \times \mathbf{E}) \right| \\ &= |\nabla \times \mathbf{E}|. \end{aligned} \quad (28)$$

Eq. (25) therefore reduces to

$$\nabla' \times \mathbf{E}' = \nabla \times \mathbf{E} = 0 \quad \text{from (22)} \quad (29)$$

which establishes the covariance of the fourth (truncated) Maxwell equation.

The final relation whose covariance needs to be checked in this approximation is the restriction (21). This is easy to prove starting from the full Maxwell equations in the original frame and in a relatively slowly moving frame. Assuming (21) is valid in the first frame and using the result just proved we have

$$0 = \partial \mathbf{B} / \partial t = -\nabla \times \mathbf{E} = -\nabla' \times \mathbf{E}' = \partial \mathbf{B}' / \partial t'. \quad (30)$$

Thus, the field Eq. (22) and the attendant constraints (17), (21) and (24) that define the electric limit are all preserved under the limiting transformations.

It is obvious from our treatment that it is also true that the truncated Maxwell equations (22) are covariant with respect to the *exact* Galilean group (that is, with respect to the transformations used above for the fields and current but with  $\nabla' = \nabla$  and no further constraints on the fields or the relative velocity). Thus, it is legitimate to claim that there is both a meaningful non-relativistic limit of Maxwell's equations, and a meaningful Galilean covariant theory of electromagnetism. We have shown

here that the dynamical equations of one version of the latter emerge as limiting cases of the full Lorentzian Maxwell equations (rather than belonging to, say, an entirely different theory). As noted, the free-standing Galilean covariant theory is not subject to the constraints we employed above to derive its dynamical equations from the full relativistic theory (and these constraints are not preserved with respect to the exact Galilean group). Hence, the free-standing equations will admit solutions that are not limits of solutions to Maxwell's equations (namely, those that do not obey the relevant constraints).

As an example, we may consider the following solution of (22) (in a region free of sources) that [Le Bellac and Lévy-Leblond \(1973\)](#) proposed is the limiting case in this theory of an electromagnetic wave:

$$\mathbf{E} = 0, \quad \mathbf{B} = \mathbf{B}_0 e^{i\omega t}, \quad \mathbf{B}_0 = \text{constant}. \quad (31)$$

In fact, this solution cannot be the limit of a genuine Maxwell field as it does not obey conditions (17) and (21) characterizing the approximation.

Later we shall examine the analogous relation between the Dirac equation and its non-relativistic limit, the Pauli equation. In that case we find a significant difference between the properties of the Pauli equation regarded as a limit or as a free-standing theory at the level of the transformation rules for the fields (specifically, for the electromagnetic potentials).

As a final observation about the electric limit, we note that Eq. (22) imply the following “wave equations” obeyed by the fields:

$$\nabla^2 \mathbf{E} = \nabla \rho / \epsilon_0, \quad \nabla^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{J}. \quad (32)$$

As expected in this approximation, there is no retarded propagation (cf. the quasi-stationary approximation; [Portis, 1978](#)). Eq. (32) are the limiting forms of the full relativistic wave equations (note that for the electric field this follows not by assuming that the second time-derivative of the field is negligible but through application of constraint (21); the latter obviously guarantees that the second time-derivative of the magnetic field vanishes).

### 3.2. *The magnetic limit*

The second non-relativistic limiting case of Maxwell's equations, the “magnetic limit,”<sup>7</sup> is obtained when

$$|\mathbf{E}| \ll c|\mathbf{B}| \quad (33)$$

so that we get from (15) and (16)

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad (34)$$

$$\mathbf{B}' = \mathbf{B}. \quad (35)$$

<sup>7</sup>Le Bellac and Lévy-Leblond have constructed other Galilean covariant theories of electromagnetism, but these are not to be considered as limits of the usual Maxwell theory (1973, Section 3). Further evidence that no limits exist other than the magnetic and electric ones is provided in (1973, Appendix A).

Once again, the dominant condition holds in the primed frame,

$$|\mathbf{E}'| \ll c|\mathbf{B}'|. \quad (36)$$

This time we introduce the restriction that the time variation of the electric field is negligible:

$$\partial \mathbf{E} / \partial t = 0. \quad (37)$$

The appropriate limiting form of Maxwell's equations (14b) in this case is therefore:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho / \epsilon_0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B} / \partial t, & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \end{aligned} \quad (38)$$

that is, the usual set but without the displacement current. We can repeat the analysis of Section 3.1 to show that this set of equations is covariant with respect to the limiting boost transformations (8), (34) and (35) if the source terms obey the following transformation properties:

$$\begin{aligned} \rho'(\mathbf{x}', t') &= \rho(\mathbf{x}, t) - (1/c^2) \mathbf{v} \cdot \mathbf{J}(\mathbf{x}, t), \\ \mathbf{J}'(\mathbf{x}', t') &= \mathbf{J}(\mathbf{x}, t). \end{aligned} \quad (39)$$

This is the limiting behaviour of a spacelike 4-vector, when

$$|c\rho| \ll |\mathbf{J}|. \quad (40)$$

Such a rule of transformation is possible for a finite 4-current only if it describes a mixture of positive and negative charges.

Repeating the proof given for the electric limit, it is also easily shown that condition (37) is true in the entire class of frames we are considering. Thus, all aspects of the magnetic-limit theory obey the non-relativistic symmetry. And once again, Eq. (38) are covariant with respect to the exact Galilean group.

In both limits it must be the case that the continuity equation derived from the full Maxwell equations remains valid. In the electric case the continuity equation is derived in the usual way. In the magnetic case we see from the last equation in (38) that  $\nabla \cdot \mathbf{J} = 0$ . For the continuity equation to be valid it must therefore be the case that  $\partial \rho / \partial t = 0$  in the magnetic limit. This is indeed confirmed by combining (37) with the first equation in (38).

### 3.3. Dirac current as source

The limiting process applied to Maxwell's equations is compatible with either timelike or spacelike currents. We now specialize to the case of the Dirac current  $J^\mu = j^\mu = c\bar{\psi}\gamma^\mu\psi$  where  $\psi$  is a 4-spinor. The latter, as is well known, is timelike,

$$j^\mu j_\mu \geq 0, \quad (41)$$

with a positive zeroth component in all Lorentz frames:

$$j^0 = c\psi^+\psi \geq 0. \quad (42)$$

Hence, in the ultra-timelike limit the 4-current has the transformation law (cf. (23)):

$$\begin{aligned} j'^0 &= j^0, \\ \mathbf{j}' &= \mathbf{j} - (\mathbf{v}/c)j^0. \end{aligned} \quad (43)$$

It follows that Maxwell's equations (14) with a Dirac source reduce uniquely in the low-velocity limit to the electric set (22). In particular, we see that the exclusion of time-varying densities in the magnetic limit is incompatible with the general time-dependence of the Dirac current.

### 3.4. The potentials

The existence, definition, and properties of the potentials are consequences of Maxwell's equations. In order to introduce a set of potentials in each of the limiting theories in a consistent way, they should be limiting cases of the full set of relativistic potentials. In particular, the relation between the fields and the potentials should be a limiting case of the usual relativistic relation, and indeed we expect that the limiting relations will simply be the usual formulas of the exact theory,

$$\mathbf{E}(\mathbf{x}, t) = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V, \quad \mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A} \quad (44)$$

and similarly in the primed frame. It is easy to see that these formulas may indeed be derived from both sets of truncated Maxwell equations (22) and (38). In the magnetic case (38) this is obvious. In the electric case, we have  $\mathbf{B} = \nabla \times \mathbf{A}$ , and  $\partial \mathbf{B}/\partial t = 0$  implies that  $\partial \mathbf{A}/\partial t = \nabla \phi$  for some function  $\phi$ . In addition,  $\nabla \times \mathbf{E} = 0$  implies that  $\mathbf{E} = \nabla \theta$ . Defining the function  $V = -\phi - \theta$  and substituting for  $\theta$ , we then obtain (44). It will be noted that we have invoked here the condition of time-independent magnetic fields. If we had not used this condition we could still have deduced from (22) the existence of a set of potentials but these would not be related to the fields in the usual way. Finally, the covariance of each set of truncated Maxwell equations implies that we can infer relations (44) in each low-velocity-related frame.

The components of the potentials defined in this way will be the limit of 4-vector components.<sup>8</sup> As noted in Section 2.2, to preserve gauge invariance, the form of the limiting transformation of the potentials is defined by the first stage (3) and (4) of the limiting process (note that  $A_0 = V/c$ ):

$$\begin{aligned} V' &= V - \mathbf{v} \cdot \mathbf{A}, \\ \mathbf{A}' &= \mathbf{A} - (\mathbf{v}/c)(V/c). \end{aligned} \quad (45)$$

These expressions are the transformation rules for the components of the potentials in the non-relativistic domain, and will be used to characterize both limiting cases (electric and magnetic)—in particular, we will not attempt to

<sup>8</sup>We are assuming here, as is usual, that the vector potential is a covariant 4-vector. In fact, the Lorentz covariance of Maxwell's equations implies only that the vector potential is a 4-vector up to gauge transformations, i.e., it transforms under a gauge-dependent Lorentz transformation.

distinguish the limits through the relative magnitudes of the components since, as emphasized, this will introduce a gauge dependence into the theory. Instead, we shall characterize the two limits in terms of the magnitudes of gauge-invariant functions of the potentials (i.e., the field strengths).

We aim to show that the limiting forms of the transformations of the fields are compatible with (45). Using (45) and (8) we find using simple vector algebra:

$$\begin{aligned}
 \mathbf{B}' &= \nabla' \times \mathbf{A}' \\
 &= \left( \nabla + \frac{\mathbf{v}}{c^2} \frac{\partial}{\partial t} \right) \times (\mathbf{A} - (\mathbf{v}/c)(V/c)) \\
 &= \mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E} \text{ since } \nabla \times (\mathbf{v}V) = -\mathbf{v} \times \nabla V \text{ and } \mathbf{v} \times \mathbf{v} = 0
 \end{aligned} \tag{46}$$

and

$$\begin{aligned}
 \mathbf{E}' &= -\frac{\partial \mathbf{A}'}{\partial t'} - \nabla' V' \\
 &= \mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\mathbf{v}}{c^2} \frac{\partial}{\partial t} \mathbf{v} \cdot \mathbf{A} + \frac{\mathbf{v}}{c^2} (\mathbf{v} \cdot \nabla) V \text{ since } \nabla(\mathbf{v} \cdot \mathbf{A}) = \mathbf{v} \times \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{A} \\
 &= \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{\mathbf{v}}{c^2} \mathbf{v} \cdot \mathbf{E}.
 \end{aligned} \tag{47}$$

The required field transformation laws (18) and (19), and (34) and (35), now follow in a gauge invariant way if we pass to the second stage of the limiting process and impose in turn the restrictions  $|\mathbf{E}| \gg c|\mathbf{B}|$  and  $|\mathbf{E}| \ll c|\mathbf{B}|$  (the latter being interpreted as conditions on the potentials). Note that we could only carry through this procedure in a gauge invariant way by using expression (8) for  $\nabla'$ . As anticipated in Section 2.1, the effective neglect of the time-derivative term appearing in the gradient operator is implemented indirectly through restrictions on the relative magnitudes of the field strengths. As also mentioned in Section 2.1, an important exception to this procedure is that in the case of the electric limit we obtain transformation (19) only by retaining the time-derivative term in (46) in the limit.

Conversely, starting from the limiting field transformation Eqs. (18) and (19), and (34) and (35), we can derive expressions (45) up to a gauge. This we do by adding to the limiting transformation equations terms that will be negligible in the corresponding limit, and that result in (46) and (47) in both cases. Assuming (44), we can then work backwards to obtain (45) up to a gauge transformation.

We have thus demonstrated that the low-velocity limiting process is compatible with the transformation law (45) for the potentials and can be carried through without the need to impose further restrictions on the relative values of the components, which as we have emphasized would break gauge invariance.

#### 4. The limiting forms of Dirac's equation

##### 4.1. Derivation of the Pauli equation

The Dirac equation for a particle of mass  $m$ , unit charge, and external electromagnetic field  $A_\mu$ ,

$$\gamma^\mu (i\hbar\partial_\mu - A_\mu)\psi = mc\psi, \quad (48)$$

is well known to yield the Pauli wave equation in the low-energy limit (e.g., [Messiah, 1961](#)). However, guided by our experience with Maxwell's equations, we might expect that there will be two limiting forms of the Dirac equation, corresponding to different choices of the relative magnitudes of the spinor components. In the electromagnetic case this was possible because electric charge has two signs (so that the limiting current 4-vector may be ultra-timelike or ultra-spacelike). And indeed, although the details of the Dirac limiting process differ from the electromagnetic case (in the latter, the potentials obey a second order equation, there is no mass term to provide a calibration, and the coupling with external fields differs), formally there are two limiting forms of Dirac's equation corresponding to different signs of the mass. Hence, only one of these limits will be of physical interest.

We shall follow the “method of eliminating small components” at the level it appears in typical textbook presentations. More detailed treatments of this, as well as other methods, are available (see, for example, [Kutzelnigg, 1989, 1990](#)). We start by writing down the transformation of a Dirac spinor with respect to a pure Lorentz boost  $\mathbf{v}$  ([Messiah, 1961](#)):<sup>9</sup>

$$\psi'(\mathbf{x}', t') = S(\mathbf{v})\psi(\mathbf{x}, t), \quad S(\mathbf{v}) = \frac{1}{\sqrt{2(1+\gamma)}} \left[ 1 + \gamma - \frac{(\boldsymbol{\alpha} \cdot \mathbf{v})}{c} \gamma \right], \quad (49)$$

where  $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ . We split the 4-spinor  $\psi$  into two 2-component spinors:

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (50)$$

Using the representation

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (51)$$

where  $\sigma_i$ ,  $i = 1, 2, 3$ , are the Pauli matrices and  $\mathbf{I}$  is the  $2 \times 2$  unit matrix, the transformation (49) becomes in the limit  $|\mathbf{v}|/c \ll 1$ ,

$$\begin{aligned} \phi' &= \phi - \frac{\boldsymbol{\sigma} \cdot \mathbf{v}}{2c} \chi, \\ \chi' &= -\frac{\boldsymbol{\sigma} \cdot \mathbf{v}}{2c} \phi + \chi. \end{aligned} \quad (52)$$

<sup>9</sup>This transformation is valid in the presence of external fields. Note that the matrix  $S$  is not unitary as  $\psi^\dagger \psi$  is the zeroth component of a 4-vector, not a scalar.



Now we consider the two possibilities for the relative magnitudes of the 2-spinors (usually called “large” and “small”). When  $|\phi| \gg |\chi|$  we get:

$$\begin{aligned}\phi' &= \phi, \\ \chi' &= -\frac{\boldsymbol{\sigma} \cdot \mathbf{v}}{2c}\phi + \chi\end{aligned}\quad (53)$$

and hence  $|\phi'| \gg |\chi'|$  also. When  $|\phi| \ll |\chi|$  we obtain:

$$\begin{aligned}\phi' &= \phi - \frac{\boldsymbol{\sigma} \cdot \mathbf{v}}{2c}\chi, \\ \chi' &= \chi\end{aligned}\quad (54)$$

and  $|\phi'| \ll |\chi'|$ . These two sets of relations constitute the possible low-velocity limits of the transformations of a Dirac 4-spinor. Since the conditions are independent of the phase, they are gauge invariant. We shall now associate these limits with two distinct special cases of Dirac’s equation. It is restriction (53) that is implemented in the usual limiting process applied to the Dirac equation.

In representation (51), (48) becomes

$$\begin{aligned}i\hbar\frac{\partial\phi}{\partial ct} - A_0\phi - \boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\chi &= mc\phi, \\ i\hbar\frac{\partial\chi}{\partial ct} - A_0\chi - \boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\phi &= -mc\chi.\end{aligned}\quad (55)$$

As is usual, we factor out a rest energy term:

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \Psi \\ \mathbf{X} \end{pmatrix} \exp(-imc^2t/\hbar).\quad (56)$$

Then the spinors  $\Psi$  and  $\mathbf{X}$  satisfy the equations

$$\begin{aligned}i\hbar\frac{\partial\Psi}{\partial t} - V\Psi - c\boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\mathbf{X} &= 0, \\ i\hbar\frac{\partial\mathbf{X}}{\partial t} - V\mathbf{X} - c\boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\Psi &= -2mc^2\mathbf{X}.\end{aligned}\quad (57)$$

These equations are exact. We next suppose that the rest energy  $mc^2$  dominates the energy corresponding to the first two terms in the second equation:

$$\left| i\hbar\frac{\partial\mathbf{X}}{\partial t} - V\mathbf{X} \right| \ll |2mc^2\mathbf{X}|.\quad (58)$$

This is a gauge invariant condition. Then from (57) we have

$$\mathbf{X} = \frac{1}{2mc}\boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\Psi.\quad (59)$$

If we suppose that the momentum term is of the order  $mu$ , (59) implies that  $|\mathbf{X}| = O(u/c)|\Psi|$ , and this case therefore corresponds to the first limiting condition above (53). This order of magnitude estimate is in accord with (58). (For a more refined discussion and numerical estimates of the various terms in an atomic context, see [Moss, 1973](#)).

To check that we have indeed entered the non-relativistic regime as a consequence of the assumption (58), we examine in turn the transformation properties of (59), (57) and (58). To this end, we need to know the transformation laws of the 2-spinors. To obtain these we write:

$$\begin{pmatrix} \phi' \\ \chi' \end{pmatrix} = \begin{pmatrix} \Psi' \\ X' \end{pmatrix} \exp(-imc^2 t' / \hbar). \quad (60)$$

Combining this with (53) and (56) and expanding  $t'$  to order  $\mathbf{v}^2/c^2$  gives<sup>10</sup>

$$\begin{pmatrix} \Psi' \\ X' \end{pmatrix}(\mathbf{x}', t') = f(\mathbf{x}, t) \begin{pmatrix} \mathbf{I} & 0 \\ -\frac{\boldsymbol{\sigma} \cdot \mathbf{v}}{2c} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \Psi \\ X \end{pmatrix}(\mathbf{x}, t), \quad (61)$$

where

$$f = \exp[(i/\hbar)(-m \mathbf{v} \cdot \mathbf{x} + \tfrac{1}{2}mv^2 t)]. \quad (62)$$

This is the well-known law for the transformation of a pair of Pauli spinors under a Galilean boost (Lévy-Leblond, 1967). Note that the unitary factor  $f$  is the same as that found for the Schrödinger wavefunction (Brown & Holland, 1999).

To prove the low-velocity covariance of (59), assume it is true in the primed frame. Bearing in mind the invariance of the matrices  $\gamma^\mu$  under a Lorentz transformation, we expect  $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ . Substituting for  $\nabla'$ ,  $\mathbf{A}'$  and the spinors from (8), (45) and (61), respectively, we get

$$X = \frac{1}{2mc} \boldsymbol{\sigma} \cdot (-i\hbar \nabla - \mathbf{A})\Psi - \frac{\mathbf{v} \cdot \boldsymbol{\sigma}}{2mc^3} \left( i\hbar \frac{\partial \Psi}{\partial t} - V\Psi - \tfrac{1}{2}m\mathbf{v}^2\Psi \right). \quad (63)$$

In the limit, the second (gauge invariant) term on the right-hand side is  $O(\mathbf{v}^2/c^2)$  (assuming  $u \approx |\mathbf{v}|$ ) times the first term and may be neglected to yield (59).

Next, we examine the covariance of the first equation in (57). Assuming this is valid in the primed frame we obtain, after some tedious algebra,

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} - V\Psi - c\boldsymbol{\sigma} \cdot (-i\hbar \nabla - \mathbf{A})X &= \frac{\mathbf{v}^2}{2c^2} \left( i\hbar \frac{\partial \Psi}{\partial t} - V\Psi - \tfrac{1}{2}m\mathbf{v}^2\Psi \right) \\ &\quad - \frac{\mathbf{v} \cdot \boldsymbol{\sigma}}{c} \left( i\hbar \frac{\partial X}{\partial t} - VX - \tfrac{1}{2}m\mathbf{v}^2X \right). \end{aligned} \quad (64)$$

To our order of approximation, the right-hand side is negligible. Notice that the negligible terms in (63) and (64) are gauge invariant only because we employed expression (8) for the spatial gradient.

Finally, it is easy to check that restriction (58) is covariant, using the full Dirac equation in the original frame and in a relatively slowly moving frame. From the

<sup>10</sup>Note that this is the only place in the Dirac limiting procedure that we need to directly use the transformation properties of the spacetime coordinates (these were not needed at all in the analogous procedure for Maxwell's equations).

second Eq. (57) and using the covariance of (59) we have

$$\begin{aligned} i\hbar \frac{\partial X'}{\partial t'} - V'X' &= c\boldsymbol{\sigma}' \cdot (-i\hbar\nabla' - \mathbf{A}')\Psi' - 2mc^2X' \\ &= c\boldsymbol{\sigma} \cdot (-i\hbar\nabla - \mathbf{A})\Psi - 2mc^2X \\ &= i\hbar \frac{\partial X}{\partial t} - VX. \end{aligned} \quad (64a)$$

Then, since  $|X'| \approx |X|$ , we deduce that (58) is valid in the primed frame.

We conclude that all relevant aspects of the limiting theory are covariant. Substituting (59) into the first equation in (57) we recover finally the Pauli equation for the two-component spinor  $\Psi$ :

$$i\hbar \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} = \left[ \frac{1}{2m}(-i\hbar\nabla - \mathbf{A}(\mathbf{x}, t))^2 + V(\mathbf{x}, t) + \mu \mathbf{B}(\mathbf{x}, t) \cdot \boldsymbol{\sigma} \right] \Psi(\mathbf{x}, t), \quad (65)$$

where  $\mu = -\hbar/2m$ . We have demonstrated that, within the conditions of approximation we have specified, this is covariant with respect to the limiting boost.

This is the first equation obtainable from Dirac's equation in the low-energy limit. To obtain the second limit, corresponding to the spinor transformation (54), we use the decomposition (56) but with  $m$  replaced by  $-m$ . Repeating the above derivation the result is that the dominant 2-spinor, now  $X$ , obeys Pauli's Equation (65) with a negative mass. Unlike the electromagnetic case there is, therefore, only one physical limiting form of the matter wave equation.

#### 4.2. Limit of the Dirac current

In this approximation the components of the Dirac current  $j^\mu = c\bar{\psi}\gamma^\mu\psi$  become

$$\rho(\equiv j^0/c) = \Psi^+\Psi, \quad (66)$$

$$\begin{aligned} \mathbf{j} &= c(\Psi^+\boldsymbol{\sigma}X + X^+\boldsymbol{\sigma}\Psi) \\ &= (\hbar/2mi)(\Psi^+\nabla\Psi - (\nabla\Psi^+)\Psi) - (\rho/m)\mathbf{A} + (\hbar/2m)\nabla \times (\Psi^+\boldsymbol{\sigma}\Psi). \end{aligned} \quad (67)$$

The latter is the well-known Pauli current which includes a spin-dependent contribution (the last term). As expected (Section 3.3), the limiting current components obey the ultra-timelike condition,  $|j^0| \gg |\mathbf{j}|$ , and applying (61) to the first line of (67) implies the transformation rule (16).

#### 4.3. Restriction on the electromagnetic field

It is not obvious from the limiting treatment of the Dirac equation we have given whether the external electromagnetic field must obey certain conditions in order that the Pauli theory is covariant, and if so what these conditions are. This is because the limiting process involves only the potentials whose transformation laws carry no signature of the conditions obeyed by the fields. In fact, just as in the case of Maxwell's equations where the external source had to be either ultra-timelike or ultra-spacelike for the limiting field equations to be covariant, so here the

assumptions we have made on the Dirac field are consistent only if the external electromagnetic field is restricted, and indeed only magnetic-limit and electric-limit fields are admissible.

To show this, we need to look at the properties of those equations implied by Dirac's equation in which the field strengths appear explicitly. An example is Ehrenfest's theorem applied to the Pauli equation. Using the physical momentum operator  $\mathbf{p} = -i\hbar\nabla - \mathbf{A}$  and taking averages in the state  $\Psi$ , we obtain from (65)  $d\langle\mathbf{p}\rangle/dt = \mathbf{F}$ , where

$$\mathbf{F} = \langle \mathbf{E} + \frac{1}{2m}(\mathbf{p} \times \mathbf{B} - \mathbf{B} \times \mathbf{p}) - \mu\nabla(\mathbf{B} \cdot \boldsymbol{\sigma}) \rangle. \quad (68)$$

In order to examine the transformation properties of the limiting mean force (68), it is easier, and clearer physically, if we express it in terms of the components of the Pauli current. Using partial integration, we obtain

$$\mathbf{F} = \int (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}) d^3x. \quad (69)$$

This has the form of the classical Lorentz force acting on a continuous charge distribution, but where the density and current are given by the limiting quantum mechanical expressions (66) and (67).

In the non-relativistic domain we expect that force is an invariant quantity with respect to boosts. Suppose  $\mathbf{E}' = \mathbf{E} + \mathbf{G}$ ,  $\mathbf{B}' = \mathbf{B} + \mathbf{H}$ . Then, using (43), we get from (69)

$$\mathbf{F}' = \mathbf{F} + \int \rho(\mathbf{G} + (\mathbf{j}/\rho) \times \mathbf{H} - \mathbf{v} \times (\mathbf{B} + \mathbf{H})) d^3x. \quad (70)$$

There are two cases for which  $\mathbf{F}' = \mathbf{F}$ . The first arises from the observation that in the integral term in (70)  $\mathbf{G}$ ,  $\mathbf{B}$  and  $\mathbf{H}$  are independent of  $\mathbf{j}$ . Hence, if this term vanishes for all quantum states we must have  $\mathbf{H} = 0$ , and the fields transform according to the magnetic limit, (34) and (35). In this case, since  $c\rho \gg |\mathbf{j}|$  and  $|\mathbf{E}| \ll c|\mathbf{B}|$ , the two terms in (69) are of comparable magnitude. The other case occurs if the second term in (69) is negligible with respect to the first so that

$$\mathbf{F} = \int \rho\mathbf{E} d^3x. \quad (71)$$

This will happen when  $|\mathbf{E}| \gg c|\mathbf{B}|$ , i.e., in the electric limit (18) and (19), and  $\mathbf{F}$  is clearly invariant since  $\rho' = \rho$  and  $\mathbf{E}' = \mathbf{E}$ .

Hence, the mean force is Galilean invariant only when the fields transform according to the magnetic or electric limits.

## 5. Coupled Maxwell–Dirac equations

We have examined the limiting forms of the Maxwell and Dirac equations when the external fields in each are pre-assigned. Consider now the coupled Maxwell–

Dirac equations:

$$\begin{aligned}\gamma^\mu(i\hbar\partial_\mu - A_\mu)\psi &= mc\psi, \\ \partial_\nu F^{\nu\mu} &= j^\mu.\end{aligned}\tag{72}$$

When the fields are quantized these are of course the basic equations of QED, but here we shall consider them in their first quantized form (for further details see, for example, Barut, 1980). The fields codetermine one another and we wish to examine the nature of the non-relativistic limit of Eq. (72).

A way of expressing the coupling is that the equation obeyed by each field is now non-linear. Consider Dirac's equation. Substituting  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  into Maxwell's equations gives

$$\partial_\nu \partial^\nu A^\mu - \partial_\nu \partial^\mu A^\nu = j^\mu.\tag{73}$$

In the Lorenz gauge ( $\partial_\nu A^\nu = 0$ ) (73) reduces to the inhomogeneous wave equation and has the solution (Barut, 1980):

$$A^\mu(x) = \tilde{A}^\mu(x) + \int G(x - x') j^\mu(x') d^4x',\tag{74}$$

where  $G(x - x')$  is the Green function and  $\tilde{A}^\mu(x)$  obeys the homogeneous wave equation. The  $A_\mu$  appearing in Dirac's equation is now shorthand for the non-linear function of  $\psi$  (74).

We may obtain the non-relativistic limit of the non-linear Dirac equation by repeating the steps in Section 4.1 which led to the Pauli Equation (65), for clearly nothing in the process is altered if  $A_\mu$  is a function of  $\psi$ . However, the issue of what conditions must be obeyed by the electric and magnetic fields in order that the Pauli equation is covariant, discussed in Section 4.3, needs to be reexamined. We saw in Section 3.3 that, with the Dirac current as source, Maxwell's equations have a non-relativistic limit only for electric-limit fields. In Section 4.3 we saw that the Dirac equation has a non-relativistic limit for electric-limit and magnetic-limit fields. In the latter case these constraints on the electromagnetic field are evidently incompatible when the matter and electromagnetic fields are coupled. In the former (electric limit) case the constraints are apparently compatible in the coupled case. However, they are of a different nature to the uncoupled case. Now the constraints (17) and (21) on the  $\mathbf{E}$  and  $\mathbf{B}$  fields represent, via (16) and (74), additional constraints on the  $\psi$  field (and vice versa). Assuming these additional conditions are self-consistent, we obtain in this way a meaningful non-relativistic limit for the coupled equations. Note though that the coupling has made the system "more relativistic" than the individual uncoupled cases, in that there are less special cases for which the field equations are covariant with respect to low-velocity transformations.

We conclude that, again assuming that the constraints on the system are self-consistent, the interacting matter–electromagnetic system has a meaningful non-relativistic limit. This comprises the coupled Pauli–Maxwell (electric-limit) equations. This result is the quantum mechanical analogue of a result stated previously for the coupled Maxwell–Lorentz equations in classical electrodynamics (Le Bellac & Lévy-Leblond, 1973).

## 6. Galilean covariance of the Pauli equation

### 6.1. Transformation rules for the wavefunction and the potentials

In Section 4 we showed how certain constraints on the wavefunction and the electromagnetic field in the Lorentz covariant theory result in the Pauli equation, and this was demonstrated to be covariant with respect to the limiting transformation equations. Consider the converse question: if we start from the Pauli equation in the presence of external potentials and assert its covariance, what can we deduce about the behaviour of the wavefunction and the potentials under the limiting transformations?

As we shall show here, the answer depends on the kinematical framework within which the wave equation is set. In the limiting domain of Lorentzian kinematics studied in Section 4, we expect to recover rules (61) and (45). If instead we start from the Pauli equation within the realm of pure Galilean kinematics, that is, where the defining relations (9) and (10) are valid exactly, we anticipate that there may be different laws of transformation for the wavefunction and/or the potentials. Indeed, it is open to investigation whether the latter can be treated as limiting cases of genuine electromagnetic potentials. Here we derive these laws through a direct examination of the transformation of the Pauli equation when subject to (9) and (10). The mathematical origin of the difference in the transformations that we do indeed find here and those used in Section 4 is the behaviour of the spatial gradient operator under boosts.

The Pauli equation in the boosted frame will read

$$i\hbar \frac{\partial \Psi'(\mathbf{x}', t')}{\partial t'} = \left[ \frac{1}{2m} (-i\hbar \nabla' - \mathbf{A}'(\mathbf{x}', t'))^2 + V'(\mathbf{x}', t') + \mu \mathbf{B}'(\mathbf{x}', t') \cdot \boldsymbol{\sigma}' \right] \Psi'(\mathbf{x}', t'). \quad (75)$$

Writing  $\Psi' = U\Psi$ , where  $U$  is a  $2 \times 2$  matrix, we aim to find the necessary and sufficient conditions on  $U$  and the external fields so that both (65) and (75) are true. The method is a generalization of that used in the Schrödinger case (Brown & Holland, 1999).

In Section 4 we argued that  $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ . This result will now be derived as a component of our demonstration. To this end, we assume only that  $\sigma'_i$ ,  $i = 1, 2, 3$ , are constant matrices obeying the usual anti-commutation relations obeyed by  $\sigma_i$ . Since  $\sigma'_i$  is a  $2 \times 2$  matrix we may write

$$\sigma'_i = \sum_{j=1}^3 c_{ij} \sigma_j + \lambda_i \mathbf{I}. \quad (76)$$

The anti-commutation relations imply that  $\lambda_i = 0$ ,  $i = 1, 2, 3$ .

Substituting  $\Psi' = U\Psi$  into (75) and using (10) and (65) gives a relation of the form

$$a\Psi + \mathbf{b} \cdot \nabla \Psi = 0, \quad (77)$$

where  $a, \mathbf{b}$  are spacetime-dependent matrices. Since  $\Psi$  and  $\nabla \Psi$  are independent functions,  $a = \mathbf{b} = 0$ . These conditions imply that

$$\nabla U = (i/\hbar)(-m\mathbf{v} + \mathbf{A}' - \mathbf{A})U \quad (78)$$

and

$$\begin{aligned} i\hbar\left(\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U\right) &= (V' - V)U + \mu(\mathbf{B}' \cdot \mathbf{s}' - \mathbf{B} \cdot \mathbf{s})U \\ &\quad + \frac{1}{2m}[\mathrm{i}\hbar U \nabla \cdot (\mathbf{A}' - \mathbf{A}) + U(A'^2 - A^2) \\ &\quad - \hbar^2 \nabla^2 U + 2\mathrm{i}\hbar \mathbf{A}' \cdot \nabla U]. \end{aligned} \quad (79)$$

Taking the curl of  $(1/U)$  times (78) implies that  $\mathbf{B}' = \mathbf{B}$ . Inserting this and (78) in (79) gives

$$\frac{\partial}{\partial t}U = (\mathrm{i}/\hbar)(\tfrac{1}{2}m\mathbf{v}^2 + \mu\mathbf{B} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}') + V - V' - \mathbf{v} \cdot \mathbf{A}')U. \quad (80)$$

The gradient of (80) gives, using (78) and (44),

$$\mathbf{E}' - \mathbf{E} - \mathbf{v} \times \mathbf{B} = \mu \nabla [\mathbf{B} \cdot (\boldsymbol{\sigma}' - \boldsymbol{\sigma})]. \quad (81)$$

Now the left-hand side of (81) is a multiple of the unit matrix and the right-hand side is, by (76), a sum of Pauli matrices. Hence the left-hand side is zero and we find the relation  $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ . The right-hand side gives

$$\sum_i B_i(\mathbf{x}, t)(\boldsymbol{\sigma}'_i - \boldsymbol{\sigma}_i) = \text{function of } t. \quad (82)$$

But  $B_i$ ,  $i = 1, 2, 3$ , is arbitrary ( $\mathbf{B}$  is constrained in relation to  $\mathbf{B}'$ , not in itself). Therefore  $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ .

We see that the field strengths have the transformation laws (34) and (35), corresponding to the magnetic limit of Maxwell's equations. But the transformation relations for the potentials, specifically the vector potential, that are implied by the above equations do not correspond to those obtained in the low-velocity limit (45). Rather, we have from (78) and (80) that  $U$  is proportional to the unit matrix, and these relations may therefore be written

$$V' = V - \frac{\partial \theta}{\partial t} - \mathbf{v} \cdot (\mathbf{A} + \nabla \theta), \quad (83)$$

$$\mathbf{A}' = \mathbf{A} + \nabla \theta, \quad (84)$$

where

$$\theta = m\mathbf{v} \cdot \mathbf{x} - \tfrac{1}{2}m\mathbf{v}^2 t - \mathrm{i}\hbar \log U. \quad (85)$$

Eqs. (83) and (84) contain a gauge transformation  $\theta$ . Since we are interested only in pure boosts, we may set this to zero and deduce from (85) that  $U = f\mathbf{I}$ , as in (61). It will be observed that the scalar potential transforms as in the limiting case (45) but the vector potential does not—the scalar potential contribution is missing.

Gathering these results, we conclude that for a pure Galilean boost applied to the Pauli equation, the wavefunction and potentials transform as follows (the same as in

the Schrödinger case (Brown & Holland, 1999)):

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= f(\mathbf{x}, t)\Psi(\mathbf{x}, t), \quad f = \exp[(i/\hbar)(-m\mathbf{v} \cdot \mathbf{x} + \tfrac{1}{2}mv^2t)], \\ V'(\mathbf{x}', t') &= V(\mathbf{x}, t) - \mathbf{v} \cdot \mathbf{A}(\mathbf{x}, t), \\ \mathbf{A}'(\mathbf{x}', t') &= \mathbf{A}(\mathbf{x}, t).\end{aligned}\tag{86}$$

Comparing with the limiting case studied in Section 4, the wavefunction and scalar potential transform in the same way, and the field strengths correspond to the magnetic limit of an electromagnetic field. On the other hand, these results differ from those of Section 4 in that the vector potential does not transform in the way expected of a limiting electromagnetic vector potential, and fields corresponding to the electric limit of an electromagnetic field are not admissible. The significance of the first discrepancy will be addressed below. As regards the second discrepancy, there is nothing in the derivation of (86) that requires us to impose the magnetic-limit condition  $|\mathbf{E}| \ll c|\mathbf{B}|$ . That is, we can certainly insert potentials in the Pauli equation that correspond to a pure electric field (using (44); e.g., Coulomb potential) while maintaining exact Galilean covariance (a possible special case of (86) is  $\mathbf{A} = 0$ ). However, such a field cannot be regarded as a limit of a genuine Maxwell field because the magnetic condition is violated.

## 6.2. Gauge covariance and Galilean kinematics

The gauge-independence of the treatment starting in Section 3.4 above was achieved because transformations (45) for the potentials were combined with transformations (8) for the differential operators. Each of these transformations follows from their standard relativistic counterparts when  $|\mathbf{v}|/c \ll 1$ , but they do not strictly conform to the Galilean regime. What does this mean?

The fact that the first—and only the first—of transformations (8) differs from its Galilean counterpart follows from the fact that in (7) the temporal coordinate of an event is not invariant. This non-invariance does not mean that clocks still dilate; if the Einstein synchrony convention is retained in the  $|\mathbf{v}|/c \ll 1$  regime, then only a residual relativity of simultaneity obtains. This discrepancy with standard Galilean kinematics can be ignored, as we have seen, if  $|\mathbf{x}| \ll ct$ . This condition is not in itself Lorentz invariant, but when  $|\mathbf{v}|/c \ll 1$ , it is coordinate independent. (The same remark holds for conditions (17) and (33). Note that an alternative way to rescue Galilean kinematics that does not limit spatial distances is by way of implementing a suitable resynchronization of clocks in the primed frame, one which depends of course on  $\mathbf{v}$ .) But from the point of view of the effective validity and covariance of the Pauli equation, and the magnetic limit of Maxwell's equations, we have seen that full Galilean kinematics does not seem necessary. Indeed the common notion that the Galilean transformations (9) and (10) represent the natural non-relativistic limit of the Lorentz transformations is surely open to question. (It might be objected here that the limiting kinematics must satisfy the exact group requirement, in which case (7) is ruled out. But as mentioned in footnote 2, the relativity principle holds in all



circumstances, even when coordinate systems are related by transformations which do not form an exact group when  $|\mathbf{v}|/c \ll 1$ ).

The independent imposition of Galilean kinematics comes moreover with a price. If the Pauli equation is required to be exactly Galilean covariant, the gauge independence of the treatment breaks down, at least in so far as this theory has any connection with the relativistic one. The transformations for the potentials in (86) can be derived from (45) only under the assumption

$$|\mathbf{A}| \gg |V/c|. \quad (87)$$

But consider an arbitrary gauge transformation:

$$\begin{aligned} \bar{\mathbf{A}} &= \mathbf{A} + \nabla\theta, \\ \bar{V} &= V - \frac{\partial\theta}{\partial t}, \end{aligned} \quad (88)$$

where  $\theta$  is assumed to be a Galilean scalar. The new gauge will be consistent with (86) only if  $\theta$  is constrained by

$$|\nabla\theta| \gg \left| \frac{\partial\theta}{c\partial t} \right|. \quad (89)$$

As a free-standing theory, the Pauli equation in Galilean spacetime is wholly gauge invariant, but coordinate covariance of the Pauli equation is compatible with the full gauge symmetry only when this wave equation is regarded as an appropriate approximation to the relativistic Dirac theory.

Finally, as noted previously (Section 2.1), there are other ways in which the freestanding and limiting theories differ. Thus, not all solutions to the former will be limits of solutions to the latter. One signature of this is that the freestanding Pauli current can imply superluminal flow lines (for a comparison between the predictions of the Dirac and Schrödinger equations in a context relevant to this point, see [Leavens & Sala Mayato, 1998](#)).

## 7. Conclusion

It is an old result that, at the level of kinematics, the Galilean group is a limiting case of the Lorentz group ([Inonu & Wigner, 1953](#)). This in itself does not, however, constitute a general demonstration of the emergence of the expected Galilean relations in the non-relativistic limit of low velocities. To examine this, one must investigate the relation between the two symmetries in the context of specific dynamical theories.

To this end, we have examined the limiting forms of the Maxwell and Dirac equations when the spacetime transformations approach their Galilean form. Both sets of equations have well-defined limits, and the limiting process can be carried through in a gauge-invariant way. We have found that, in this domain, Maxwell's equations have two limiting forms which, if the electric-limit or magnetic-limit constraints are satisfied, form low-velocity covariant systems. Likewise, in this

domain, Dirac's equation in the presence of an electromagnetic field has a unique limiting form (the Pauli equation) which, if the electric-limit or magnetic-limit constraints are obeyed, is low-velocity covariant. Both the Maxwell and Dirac equations in the presence of "external fields" have, therefore, non-relativistic limits in the sense we defined this term in Section 2.1. The coupled Maxwell–Dirac equations also apparently possess a meaningful non-relativistic limit, insofar as the additional constraints implied by the coupling are consistent. We have further shown by examination of the Pauli equation that, for a given physical system, the properties of a theory of the system obtained as the non-relativistic limit of a relativistic theory are not generally the same as those flowing from the direct imposition of Galilean kinematics.

### Acknowledgements

We wish to thank the referees for a careful reading of an earlier version of this manuscript and very useful suggestions for improvements, and Roman Jackiw, Claus Laemmerzahl and Eugen Merzbacher for valuable correspondence.

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