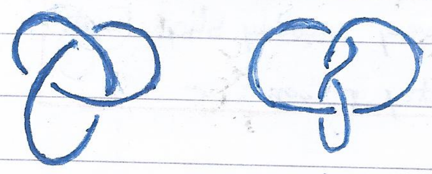
The trefoil and figure-eight are not slice – An introduction to knot concordance

31.10.18

IMPRS Seminar



3_1 $\quad$ 4_1

① What are slice knots – and why do we care about them?

② Seifert surfaces and the Alexander polynomial

③ Obstructing sliceness

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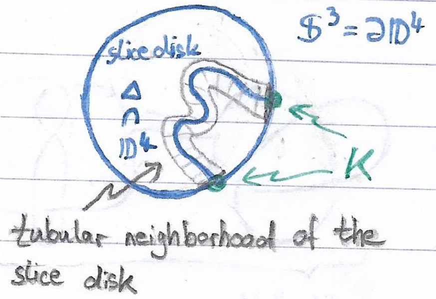
①

Def: $K : S^1 \hookrightarrow S^3$ is topologically slice, if it bounds a (smoothly) locally flat (smoothly) embedded disk $D^2 \subset D^4$.

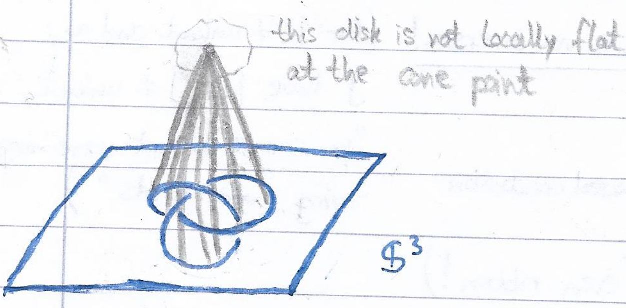
half-dim. picture:



Locally flat: The slice disk $\Delta^2 \subset D^4$ is required to have a tubular neighborhood $\Delta^2 \times D^2$ s.th. $(\Delta^2 \times D^2) \cap S^3$ is a tubular neighborhood $K \times D^2$ for K.

(Comment: In this case locally flat $\Rightarrow$ flat)

Rem: $D^4 = \mathrm{Cone}(S^3)$ contains the cone on K



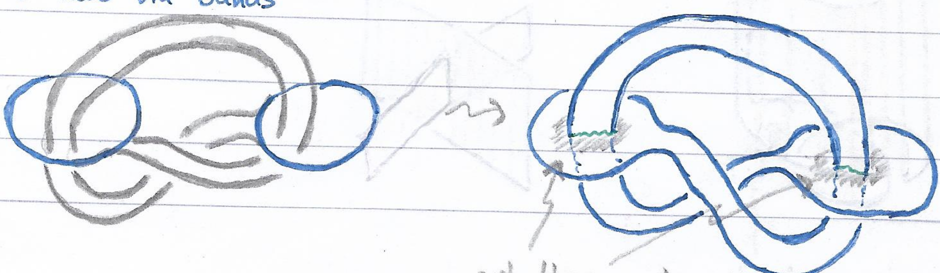
$\rightsquigarrow$ existence of the tubular neighborhood is the essential part of the definition

Important class of examples:

Start with disjoint circles and connect via bands

bounds singular disk in S^3 with two arcs of self-intersection "ribbon singularities"

"stevedore's knot" 6_1



push these parts off S^3 slightly into D^4 $\rightsquigarrow$ get a (smooth) slice disk

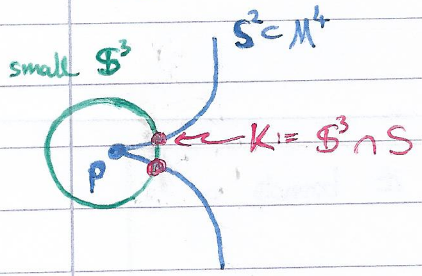
Rmk: Knots which are build in this way are called ribbon knots.

$\boxed{\text{Slice-Ribbon-Conjecture [Fox, 1960s]: Is every smoothly slice knot secretly ribbon?}}$

Applications of slice knots:

•) Creating smooth surfaces in 4-manifolds:

$p \in \underset{\text{surface}}{S^2} \hookrightarrow M^4$ Locally knotted at a point p.

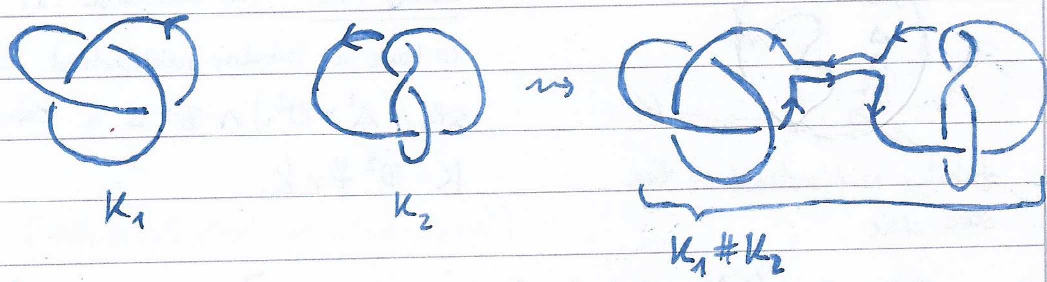


- Locally around p, the surface is a cone on the knot K
- if K is slice knot, the cone can be replaced by a slice disk

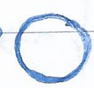
⇝ can remove a singularity

•) Knot Concordance group:

Def: Connected sum of (oriented) knots



(isotopy cl. of Knots, #) commutative monoid with neutral element the unknot. ⇝

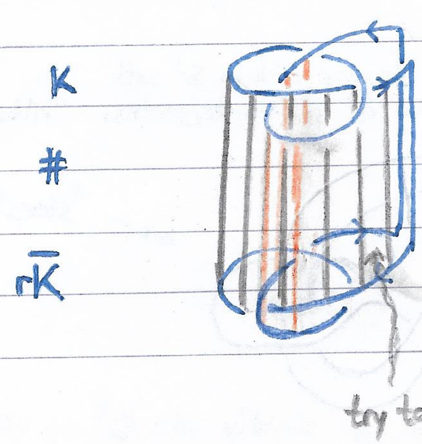


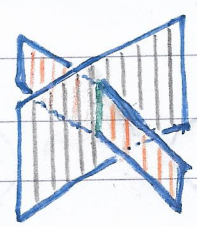
But there are no inverses! (For K ≠ unknot and any J have K#J ≠ unknot, "you cannot unknot something by tying more knots")

$r: S^3 \to S^3$ is a reflection, $\bar{\ }$ denotes reversed orientation

Observation: $K \# r\bar{K}$ is slice (even ribbon!)

Observe that the singularities are only of the ribbon type





CAT = (Top), (Smooth)

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Def.: $\mathcal{C}_1^{CAT} := \frac{\{\text{oriented knots } S^1 \hookrightarrow S^3\}}{\{\text{Concordance}\}}$ ← i.e. $[K]=[J]$ iff. $K \# r\overline{J}$ $\overset{CAT}{\text{slice}}$

abelian group under

$[K] + [J] := [K \# J]$, $0 = [\text{unknot}] = [\text{any slice knot}]$

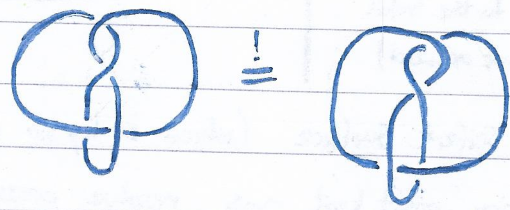
From observation: $-[K] = [\underbrace{r\overline{K}}_{\text{concordance inverse of } K}]$

Algebraic structure of $\mathcal{C}_1$ is complicated, for example

$\mathcal{C}_1 \twoheadrightarrow \mathbb{Z}^\infty$ [Tristram 1969, Milnor 1968]

⇒ not finitely generated

Ex: Torsion elements $4_1 \overset{!}{=}_{\text{or.}} r\overline{4_1}$ "figure eight is amphichiral."



$2 \cdot [4_1] = [4_1] + [r\overline{4_1}] = [4_1 \# r\overline{4_1}] = 0$, so the figure-eight knot represents an element of order 2 in $\mathcal{C}_1$. [↗ later will see that $[4_1] \neq 0$ in $\mathcal{C}_1$]

Open problem: Apparently nobody has been able to find torsion of any other order, nor has ruled out their existence.

② Slice knots are rare! The rest of the talk will develop methods to show that there are non-slice knots

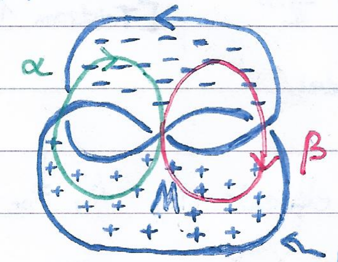
Def.: A Seifert surface for a knot $S^1 \overset{K}{\subset} S^3$ is a
- connected
- bicollared (so in particular oriented), i.e. embedding of M can be thickened to $M \times [-1,1] \hookrightarrow S^3$, $M = M \times \{0\}$
- compact

surface $M^2 \subset S^3$ with $\partial M = K$.

Ex.:

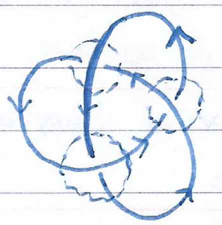
not a Seifert surface for the trefoil (Möbius strip, and thus not oriented)

Draw trefoil like this:

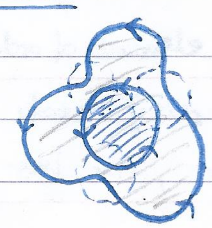


Claim: Every knot has a Seifert surface (which is by no means unique!)

Seifert's algorithm: Start with any projection, orient knot ⇝ resolve crossings



Top view:

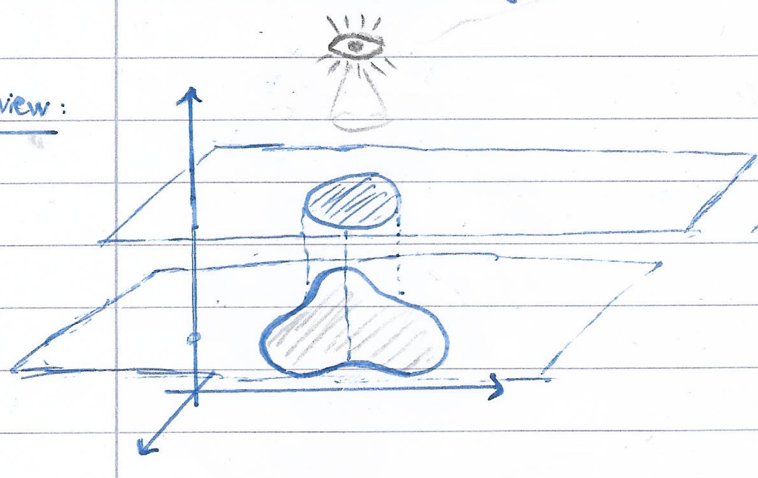


obtain (in this case two) Seifert circles bounding disks

⇝ stack disks at different heights

Side view:

⇝

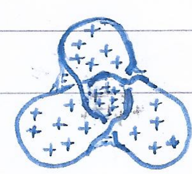


⇝ insert (twisted) bands for each crossing

⇝

side view:

top view:



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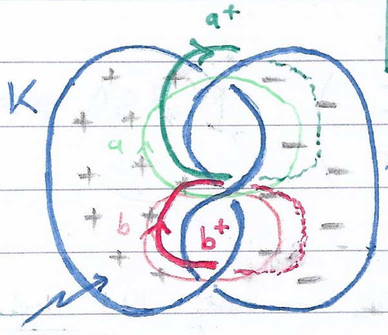
Def.: (Seifert form for K) $\quad f: H_1(\mathring{M}) \times H_1(\mathring{M}) \longrightarrow \mathbb{Z}$

$$f(x,y) := \mathrm{Lk}(x, y^+) \quad \left(= \mathrm{Lk}(x^-, y)\right)$$

here: right-handed trefoil



Seifert surface M

Explaination: $y \in H_1(\mathring{M})$ represented by 1-cycle in $\mathring{M}$

$\rightsquigarrow$ y^+ / y^- denotes homology cycle carried by $y \times \{1\}$ / $y \times \{-1\}$ in bicollar

& 1-cycles in S^3 with disjoint carriers have well-defined linking numbers

Depends on: ·) choice of Seifert surface $M^2 \subset S^3$ for K

·) choice of bicollar $\mathring{M} \times [-1,1] \subset S^3 \setminus K$

·) After choice of basis $e_1, \dots, e_{2g} \in H_1(\mathring{M})$ as $\mathbb{Z}$-module

$\rightsquigarrow$ Seifert matrix

$$V = (v_{ij}), \quad v_{ij} := \mathrm{Lk}(e_i, e_j^+)$$

⚠ Seifert matrix V is not a knot invariant

(but we will soon derive knot invariants from it)

Ex. for the choice of $\mathring{M}$, bicollar and basis above:

$$V = \begin{array}{c c} & \begin{array}{cc} a^+ & b^+ \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \end{array}$$

Def.: (Alexander Polynomial)

$$\Delta_K(t) \underset{\xi}{\doteq} \det(V^T - t \cdot V)$$

only defined up to multiplication by units $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$

Rmk.: ·) From this definition, it is absolutely not clear why this should not depend on our choices of Seifert surface, bicollar, bases, ...

·) There is a more conceptual approach to the Alexander polynomial:

$V^T - t \cdot V$ is a presentation matrix for the first homology of the infinite cyclic cover of the knot complement, $H_1(\widetilde{S^3 \setminus K})$ as $\mathbb{Z}[t, t^{-1}]$-module

where

$$\begin{array}{c} \widetilde{S^3 \setminus K} \\ \downarrow \\ S^3 \setminus K \end{array}$$

covering corresponds to subgroup $[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)] < \underbrace{\pi_1(S^3 \setminus K)}_{=: G}$

Always for fundamental groups of knot complements $G/[G,G] \cong \mathbb{Z}$

Now for our goal to obstruct sliceness:

3 **Key Prop.:** $K \subset S^3$ topologically slice, M any Seifert surface for K
$\Rightarrow$ $\exists$ basis for $H_1(\mathring{M})$ such that the associated Seifert matrix has the block form

$$\underbrace{\left(\begin{array}{c|c} B & C \\ \hline D & 0 \end{array}\right)}_{\text{matrices of this form are called algebraically slice}}$$

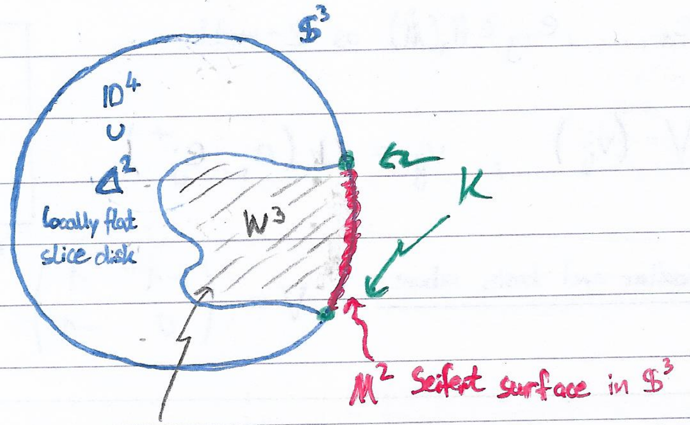
B, C, D square integer matrices

matrices of this form are called <u>algebraically</u> slice

K slice knot $\underset{\not\Leftarrow}{\Rightarrow}$ K alg. slice

in the classical case of e_1, there is a diff. between alg. and top. slice

Pf. Sketch:



Claim: $\exists$ bicollared 3-mfld W^3 in D^4 s.th. $W \cap S^3 = M$ and $\partial W = M \cup \Delta$

Inclusion hom. $H_1(\mathring{M}) \xrightarrow[\text{iso.}]{\cong} H_1(M \cup \Delta) = H_1(\partial W)$

Claim: $\exists$ basis for $H_1(\partial W)$ represented by 1-cycles, half of which bound rational 2-chains in W

$\Rightarrow$ Say basis $a_1, \dots, a_g, \underbrace{a_{g+1}, \dots, a_{2g}}_{\text{s.th. bound 2-chains in } W}$ for $H_1(\mathring{M})$

$\Rightarrow$ For $g+1 \leq i,j \leq 2g$ have that a_i and a_j^+ bound disjoint 2-chains in D^4

push out into the bicollar the one bounded by a_j

$\Rightarrow$ $\mathrm{lk}(a_i, a_j^+) = 0$ $\Rightarrow$ Seifert matrix has required form $\square$

Fox-Milnor-Condition: The Alexander polynomial of a slice knot in S^3 has the form

$$\Delta(t) = p(t) \cdot p(t^{-1})$$

with $p(t) \in \mathbb{Z}[t]$

Pf.: $\Delta(t) = \det(V^T - t\cdot V)$

$$= \det\left(\begin{array}{c|c} B^T - tB & D^T - tC \\ \hline C^T - tD & 0 \end{array}\right)$$

$$\doteq \dots$$

$$\doteq (-t)^g \det(C^T - t\cdot D) \det(C^T - t^{-1} D) \qquad \square$$

In particular: The so-called determinant $\underbrace{|\Delta(-1)|}$ of a slice knot is a square integer.

$H_1(\Sigma_2)$ is finite group, its order is given by this determinant

Ex.: •) $\Delta_{3_1}(t) = t^2 - t + 1 \rightsquigarrow |\Delta_{3_1}(-1)| = 3$ not a square

$\Rightarrow 3_1$ is not slice

•) $\Delta_{4_1}(t) = t^2 - 3t + 1 \rightsquigarrow |\Delta_{4_1}(-1)| = 5$ not a square

$\Rightarrow 4_1$ is not slice $\}$ signature alone could not detect this

Excursion: Signatures

Def: $K \subset S^3$ knot with Seifert matrix V

Signature of K:

$$\sigma(K) := \operatorname{sign}(\underbrace{V + V^T}_{\text{symmetrized Seifert form}})$$

twofold branched cyclic cover of K

presentation matrix for $H_1(\Sigma_2)$

Properties: ·) Depends only on K (up to orientation-preserving homeomorphism of S^3) and not on the choice of Seifert surface; $\sigma(\text{knot})$ always even number

·) $\sigma(rK) = -\sigma(K)$ ⇝ in some cases can distinguish knots from their mirror image

·) Additive: $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$

·) $\sigma(\text{slice knot}) = 0$

Pf: K $\overset{\text{top.}}{\text{slice}}$ $\Rightarrow$ K algebraically slice

Recall: $V \sim \begin{array}{c} \left(\begin{array}{c|c} B & C \\ \hline D & 0 \end{array}\right)\begin{array}{l} \}g \\ \}g \end{array} \\ \begin{array}{cc} \underbrace{\ }_{g} & \underbrace{\ }_{g} \end{array} \end{array}$

Fact from algebra: Non-singular symmetric bilinear form that vanishes on a half-dim. subspace $\Rightarrow$ signature $= 0$ □

Upshot: σ descends to a homomorphism

$$\text{knots}/\text{concordance} = \mathcal{C}_1 \xrightarrow{\sigma} \mathbb{Z}$$

Ex: $\sigma(\text{left-handed trefoil}) = 2$ ⇝ alternative proof that trefoil is not slice & we see that right-handed and left-handed trefoil are two different knots!

$\dots, [r3_1], [3_1], [3_1 \# 3_1], [3_1 \# 3_1 \# 3_1], \dots$ are all distinct in the knot concordance group, $[3_1] \in \mathcal{C}_1$ has infinite order

$\mathcal{C}_1 \xrightarrow{\sigma} 2\mathbb{Z}$