§5.1 Kirby Calculus

<u>Plan:</u>
- Handle cancellation/sliding in 2 and 3-dim.
- How does this look in Kirby diagrams?
- Different view on 1-handles
- <u>Examples</u> along the way

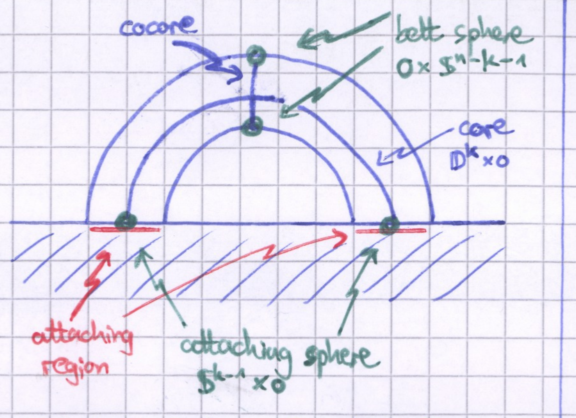
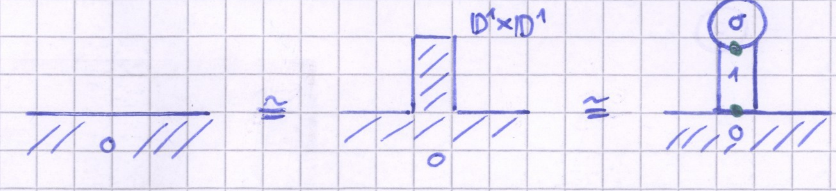


Fig. 4.1: Anatomy of a k-handle

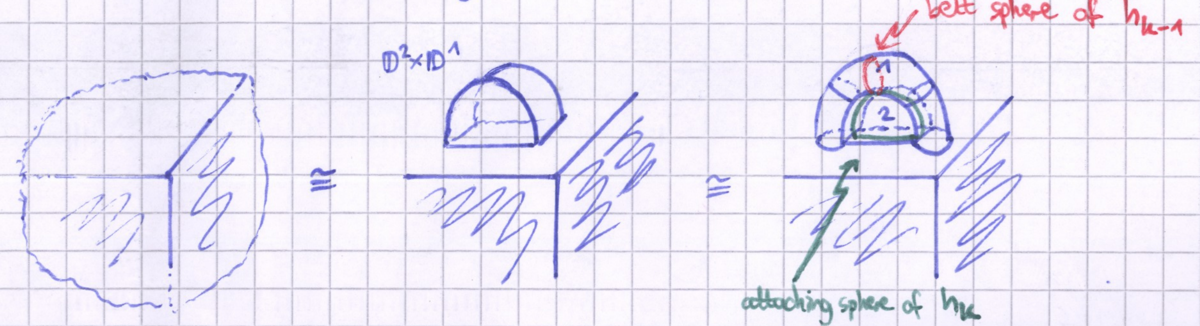
Concrete pictures in 2-/3-dim., abstract description later ↗

<u>Cancellation:</u>

<u>2-dim:</u>

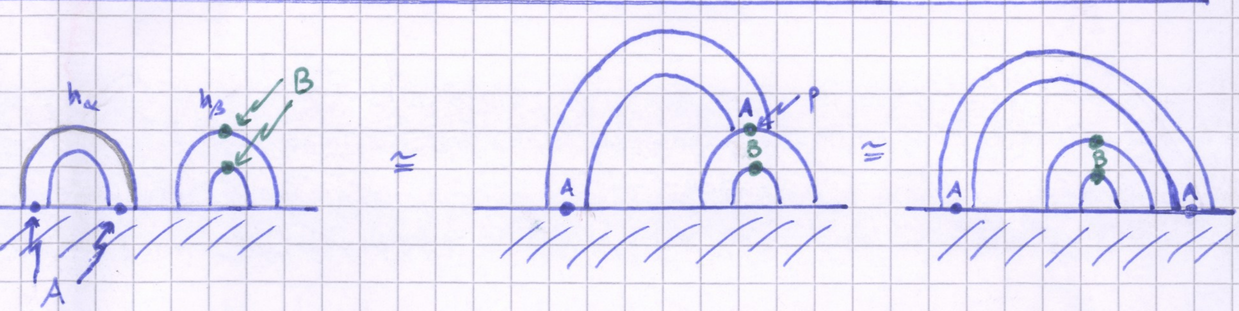


<u>3-dim:</u>



<u>Prop. 4.2.9:</u> A $(k-1)$-handle $h_\beta^{(k-1)}$ and a k-handle $h_\alpha^{(k)}$ $(1 \leq k \leq n)$ can be <u>cancelled</u>, provided that the <u>attaching sphere of $h_\alpha^{(k)}$</u> intersects the <u>belt sphere of $h_\beta^{(k-1)}$</u> transversely in a single (unique) point. (regardless of framing)

<u>Handle slides:</u>



Careful with framings! ↗later

<u>Def 4.2.10:</u> Given two k-handles h_α and h_β $(0 < k < n)$ attached to ∂X, $= D^k \times D^{n-k}$

a handle slide of h_1 over h_2 is given as follows: Isotope the attaching sphere A of h_1 in $\partial(X \cup h_2)$, pushing it through the belt sphere B of h_2.

slide it along a disk $D^k \times \text{pt.} \subset \partial h_2$

Intermediate stage:
- spheres will intersect in one point p
- $$\underbrace{T_pA}_{\dim = k-1} \oplus \underbrace{T_pB}_{\dim = n-k-1} \subseteq \underbrace{T_p(\partial(X \cup h_2))}_{\dim = n-1} \text{ of codim } 1$$

Prop. 4.2.7: handles can be attached in order of increasing index

$\underbrace{\text{dimension-counting and transversality argument}}$, "Miss the belt sphere, miss it all"

$\Rightarrow$ can push A off of B in two possible directions

one direction: original picture

other direction: result of the handle slide

This is a <u>complete set of moves</u>:

Thm. 4.2.12: Given any two (relative) handle decompositions (ordered by increasing index) for a compact X, it is possible to get from one to the other by a sequence of

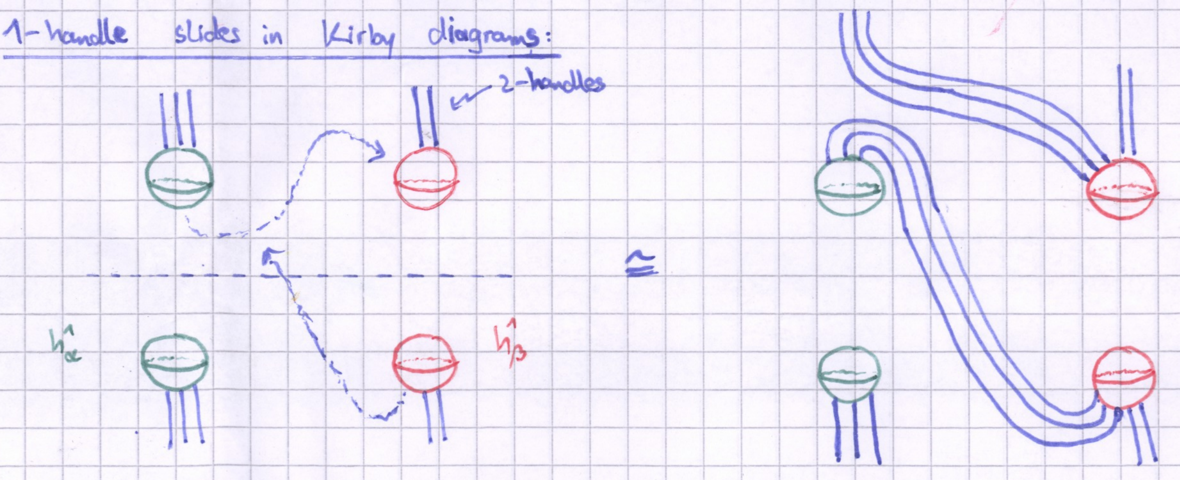
- handle slides
- creating/annihilating canceling handle pairs ("handle birth")
- isotopies within levels

Back to Handle slides: <u>Ex. of 1-handle slide in 2-dim:</u>

Handle slide changed: •) the attaching sphere of h^1_α (unlinked it from that of h^1_β)

•) the framing (here element of $\pi_0(O(1)) \cong \mathbb{Z}/(2)$)

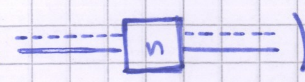
<u>1-handle slides in Kirby diagrams:</u>



i.e. take attaching ball of h^1_α

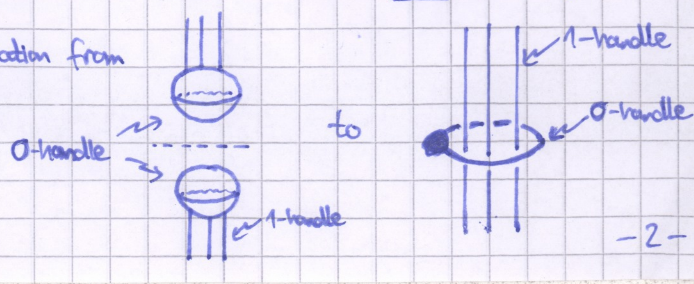
push it through the 1-handle h^1_β

keep track of framings (e.g. by using double-strand notation



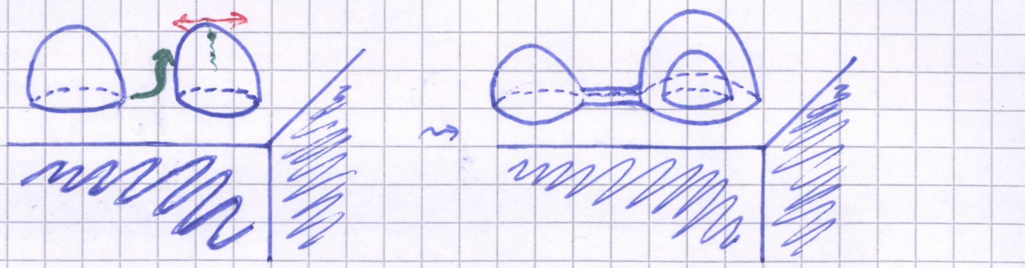
)

↗ ~~easier~~ later when we change notation from

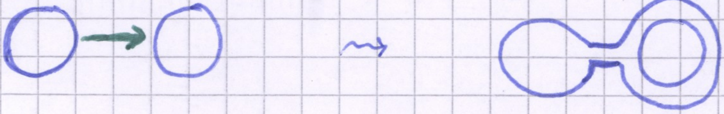


Ex. of 2-handle slide in 3-dim.:

Core disks:



Attaching circles:



2-handle slides in Kirby diagrams:

2-handles h_α^2, h_β^2 attached along

framed knots K_α, K_β

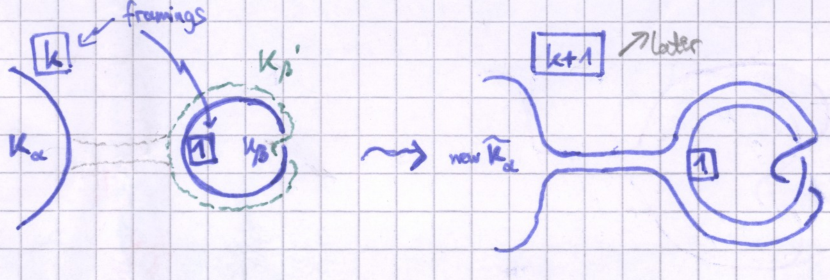
parallel curve K_β' determining the framing on K_β bounds a disk $\underbrace{D^2 \times \{pt.\}}_{\text{parallel copy of the core of } h_\beta^2,} \subset \partial(Y \cup h_\beta)$ lives in the upper boundary of the "ribbon" handle h_β^2

slide h_α by isotoping K_α over one such disk

Form a $\underbrace{\text{band-sum}}_{\text{connected sum along some band}}$ of K_α and K_β'

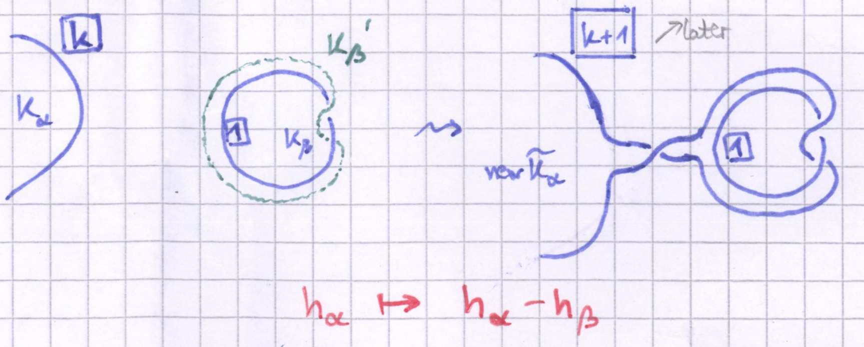
(may precede the slide by any isotopy, so we are allowed to use any band disjoint from the rest of the link, the choice might affect the resulting link)

Ex. I:



Change of basis in H_2: $h_\alpha \mapsto h_\alpha + h_\beta$ ("handle addition")

Ex. II: (one possible alternative):



$h_\alpha \mapsto h_\alpha - h_\beta$ ("handle subtraction")

Aside: Homology from handles

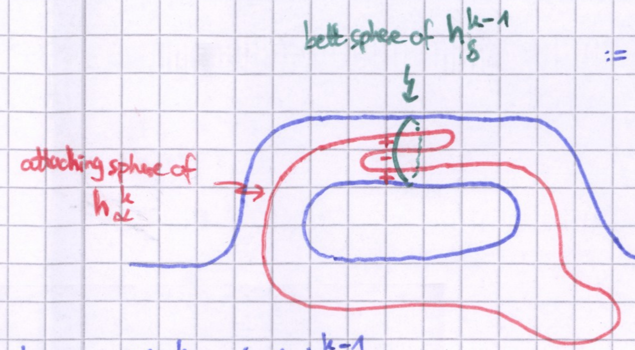
Chain complex: $C_k = \mathbb{Z}\{k\text{-handles } h_\alpha^k\}$

boundary maps $\partial_k : C_k \to C_{k-1}$ given by

$$\partial_k(h_\alpha^k) := \sum \underbrace{\langle h_\alpha^k \mid h_\delta^{k-1} \rangle}\, h_\delta^{k-1}$$

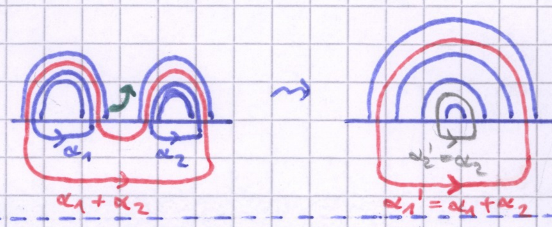
:= incidence number of h_α^k with h_δ^{k-1}

:= intersection number of the attaching sphere of h_α^k with the belt sphere of h_δ^{k-1}



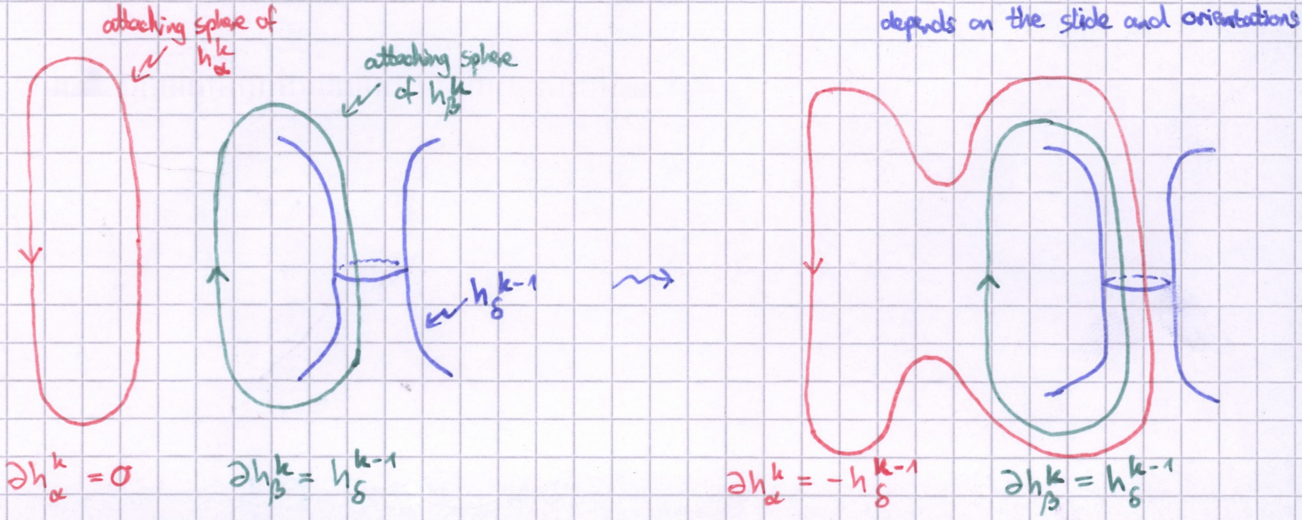
here: $\partial h_\alpha^k = (+1)\cdot h_\delta^{k-1}$

Alternative picture (Fig. 5.7):



Algebraic effect of sliding: Sliding h_α^k over h_β^k modifies ∂_k the same way as would changing the basis of C_k by replacing h_α^k by $h_\alpha^k \underset{\uparrow}{\pm} h_\beta^k$

depends on the slide and orientations



$\partial h_\alpha^k = 0$ $\quad$ $\partial h_\beta^k = h_\delta^{k-1}$ $\qquad\rightsquigarrow\qquad$ $\partial h_\alpha^k = -h_\delta^{k-1}$ $\quad$ $\partial h_\beta^k = h_\delta^{k-1}$

For determining the framing on the slid handle:

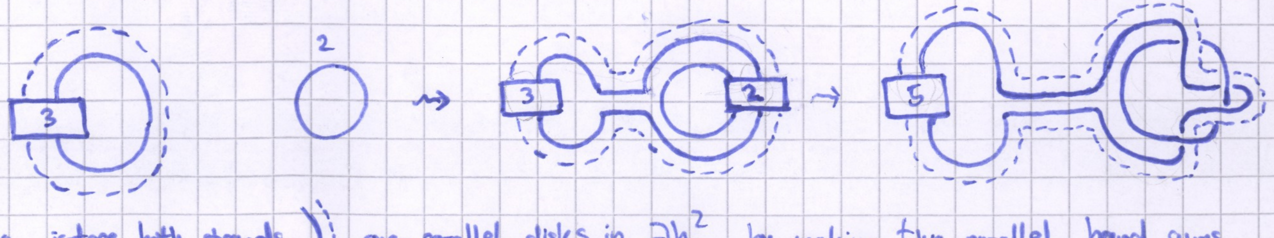
If 2-handles are attached to D^4:

$$\text{framing}(\text{new } \widetilde{h}_\alpha) = (\alpha \pm \beta)\cdot(\alpha \pm \beta) = \alpha\cdot\alpha + \beta\cdot\beta \pm 2\cdot\alpha\cdot\beta$$

$$= \text{framing}(h_\alpha) + \text{framing}(h_\beta) \pm 2\cdot \text{Linking}(\text{knot } h_\alpha, \text{knot } h_\beta)$$

in our pictures this was zero

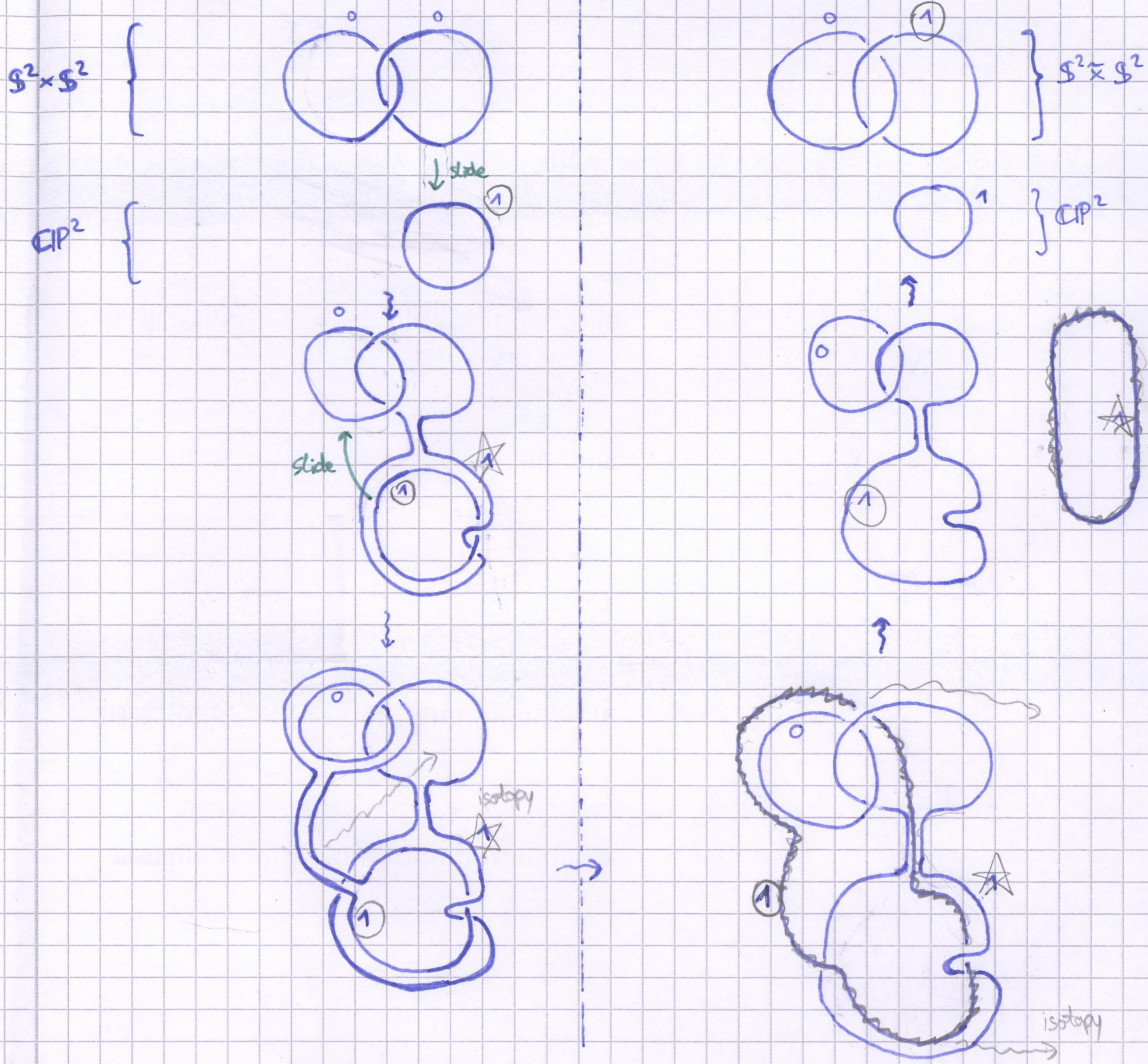
Alternative: double strand notation:



i.e. isotope both strands)) over parallel disks in ∂h_β^2 by making two parallel band sums

Fun example: $\mathbb{CP}^2 \# S^2\times S^2 \cong \mathbb{CP}^2 \# S^2 \tilde{\times} S^2$

Pf:

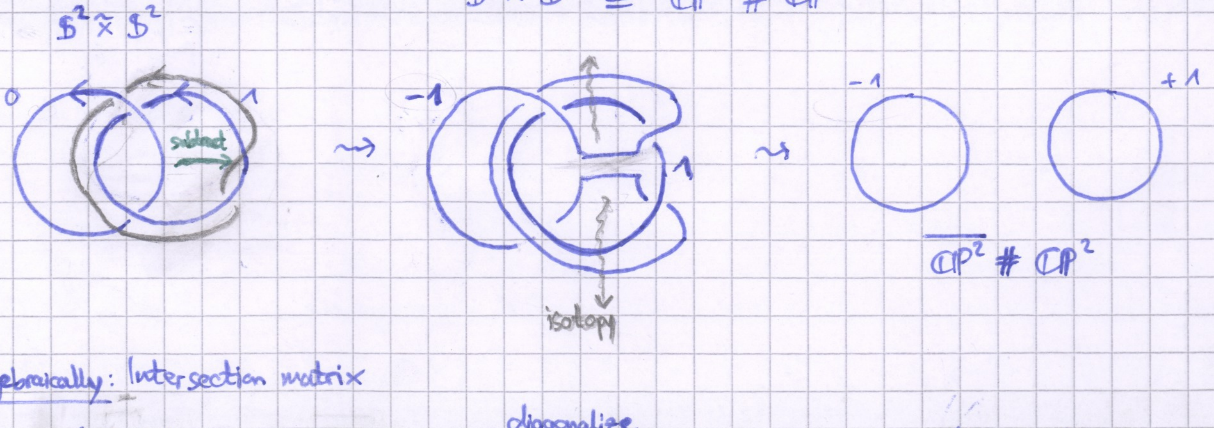


In general **Lemma:** If M^4 has odd intersection form, then there is a diffeomorphism
closed, simply connected, oriented, smooth

$$M \# S^2\times S^2 \cong M \# S^2 \tilde{\times} S^2$$

Example: Analogous to the 2-dim. case $S^1 \tilde{\times} S^1 \cong \mathbb{RP}^2 \# \mathbb{RP}^2$ we have

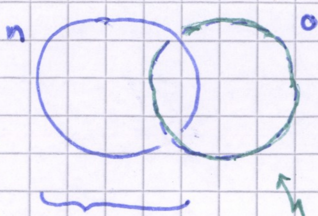
$$S^2 \tilde{\times} S^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$$



Algebraically: Intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{diagonalize}} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

Example: S^2-bundle over S^2

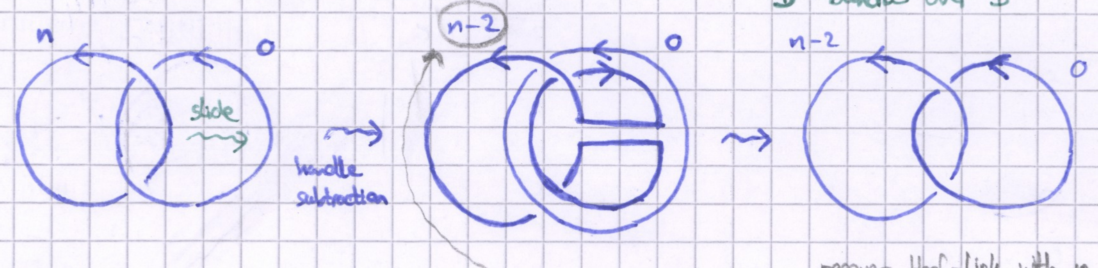


$\cup$ 4-handle

$X = \mathbb{D}^2$-bundle over S^2 with Euler-number n

adding 0-framed meridian gives $DX = X \cup_{id_{\partial X}} \overline{X}$ (double)

S^2-bundle over S^2



new framing

recover Hopf-Link with n reduced by 2

Intersection matrix:

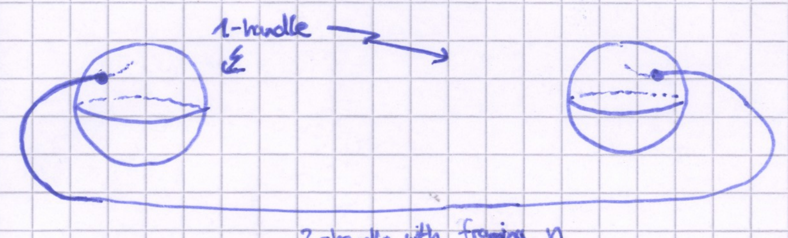
$$\ominus \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} n-1 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} n-2 & 1 \\ 1 & 0 \end{pmatrix}$$

Corollary: Diffeomorphism type of $D\left(\begin{matrix} S^2\text{-bundle with Euler number } n \\ \text{over } S^2 \end{matrix}\right)$ depends only on n modulo 2

Handle cancellation:

Diagrams for a pair of cancelling 1- and 2-handles:

$k=2$:

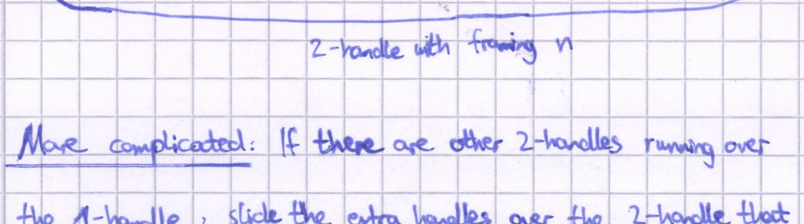


Formally: A $(k-1)$-handle h^{k-1} and a k-handle h^k can be cancelled if the attaching sphere of h^k intersects the belt sphere of h^{k-1} transversely in a unique point (regardless of framing)

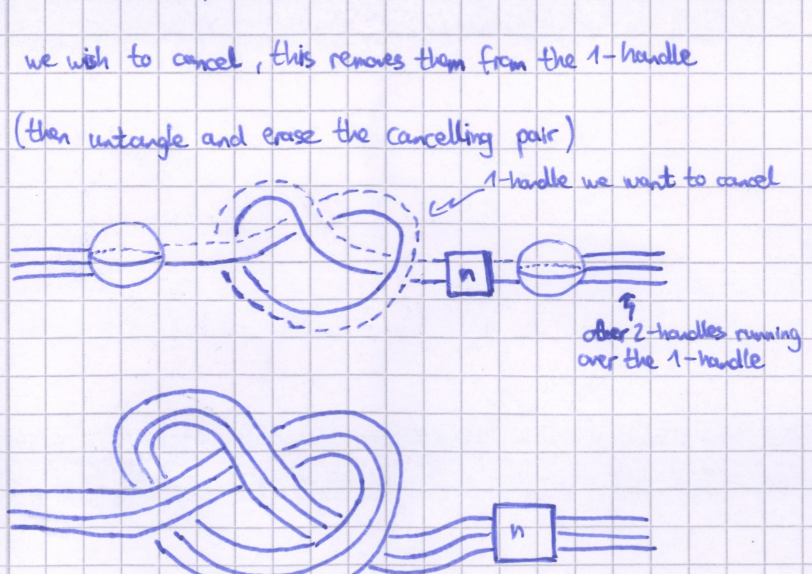
Note: Unique isotopy class of framed emb. of an intervall in any 3-mfld. $\rightsquigarrow$ essentially a unique way to draw cancelling 1-handle/2-handle pair:

- unknot attaching circle
- slide it off any other 1-handles by an isotopy in ∂X_1

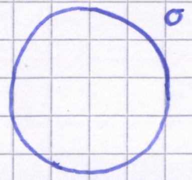
More complicated: If there are other 2-handles running over the 1-handle; slide the extra handles over the 2-handle that we wish to cancel, this removes them from the 1-handle (then untangle and erase the cancelling pair)



$\rightsquigarrow$



Model for a cancelling 2-handle/3-handle pair:



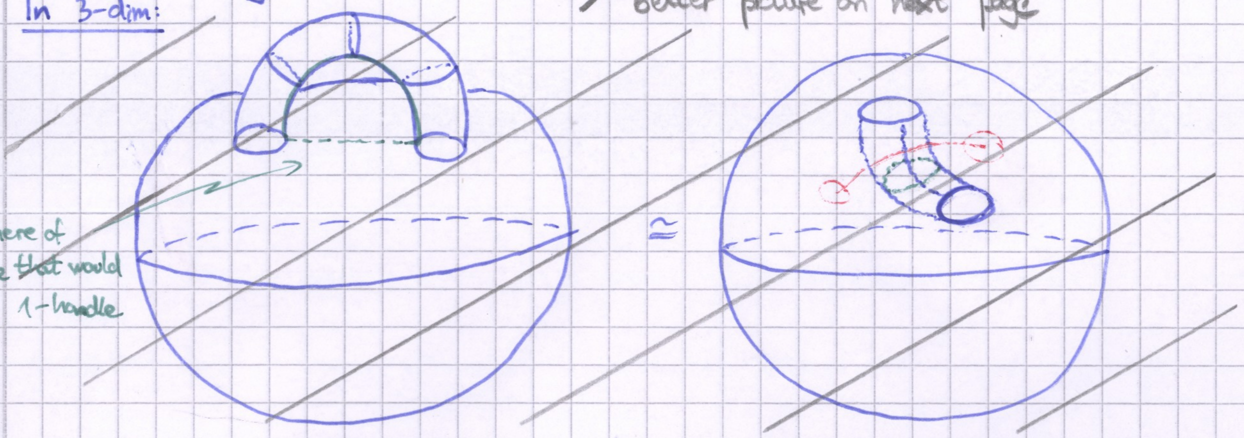
Fact: Any cancelling 2-3 pair can be made to look like this.

Alternative way of attaching a 1-handle:

In 2-dim:

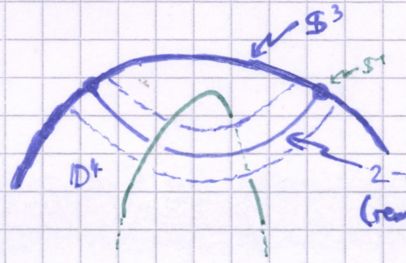
In 3-dim:

better picture on next page

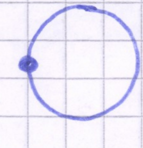


"Adding a 1-handle is the same as removing the 2-handle that cancels it"

In 4-dim: Schematic:

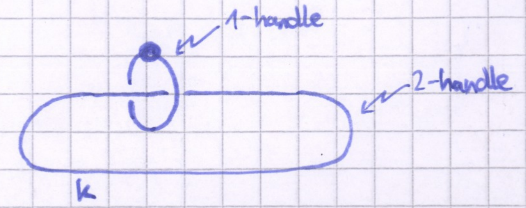


Alternative diagram for attaching an (orientation-preserving) 1-handle:

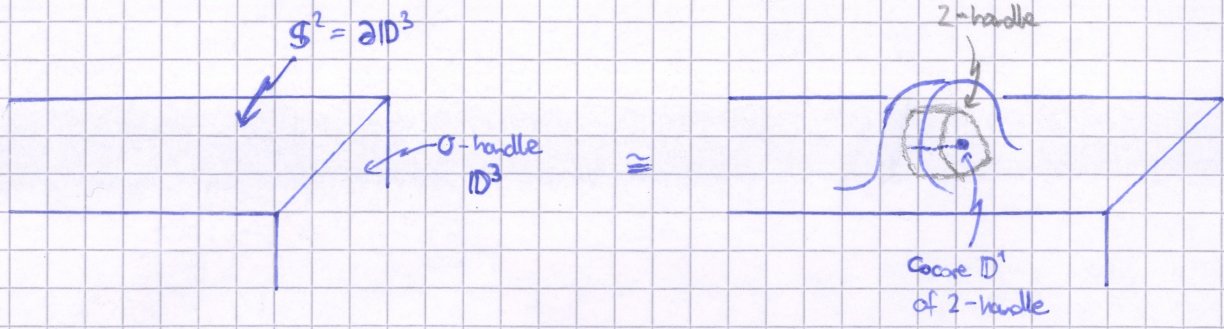


"Dotted circle notation", introduced by [Akbulut]

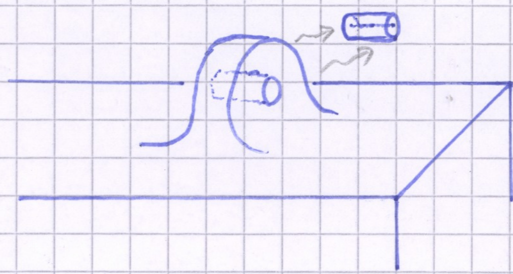
New diagram for a pair of cancelling 1- and 2-handles:

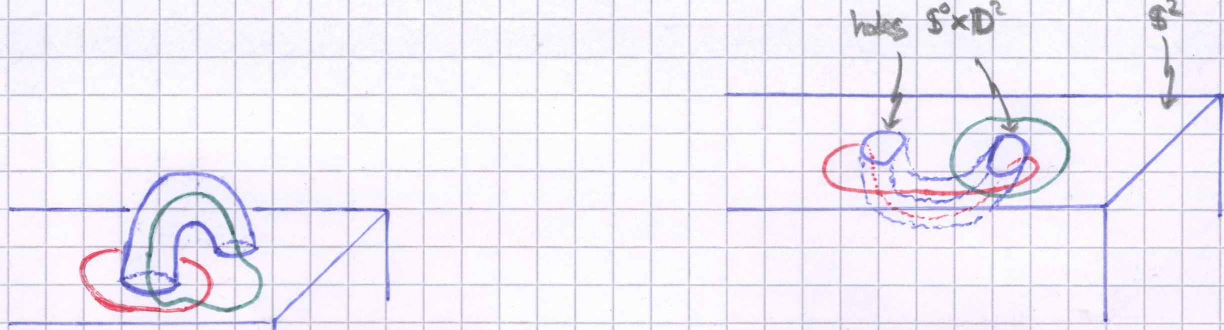


Attaching a 1-handle in 3-dim:



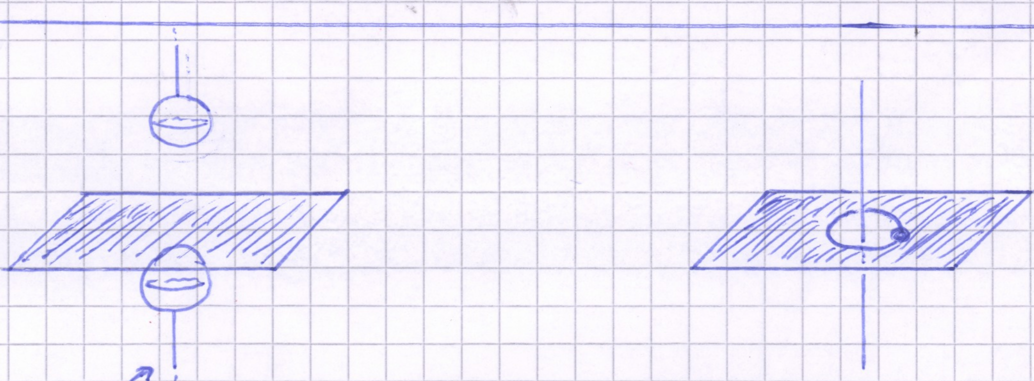
Remove tubular neighborhood $\mathbb{D}^1 \times \mathbb{D}^2$ of cocore:





circles running over the 1-handle $\rightsquigarrow$ now contained in $\mathbb{S}^2$

circles going under the 1-handle $\rightsquigarrow$ now split in $\mathbb{S}^2 \setminus$ attaching region $\mathbb{S}^0 \times \mathbb{D}^2$



2 handle running over 1-handle $\rightsquigarrow$ comes fully into $\mathbb{S}^3$

part of surface running between spheres (under the 1-handle) $\rightsquigarrow$ moves into $\mathrm{int}(\mathbb{D}^4)$