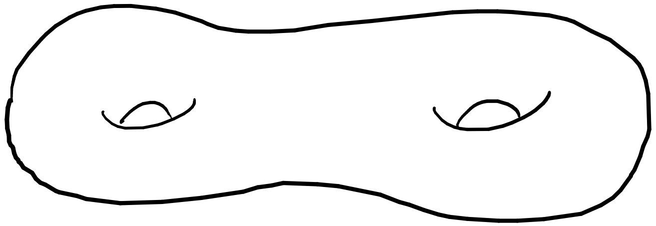
K-OS, 2021-09-30, 60 min talk

Handlebodies, Trivial tangles and Group Trisections for Knotted Surfaces

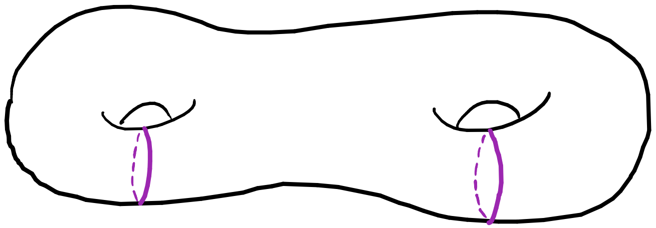
with Sarah Blackwell, Rob Kirby, Michael Klug and Vincent Longo

Benjamin Matthias Ruppik, 3rd year PhD student at the Max-Planck-Institute for Mathematics, Bonn

Handlebodies:

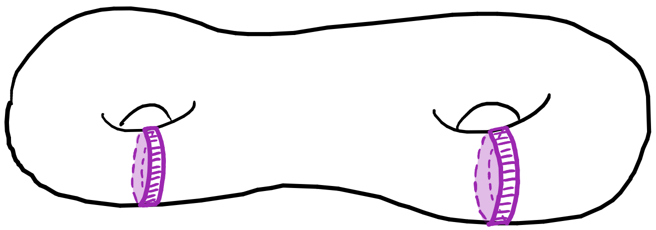


surface Σ_g



cut system of a handlebody:

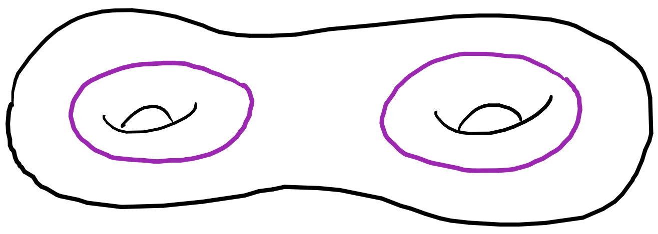
curves on Σ_g

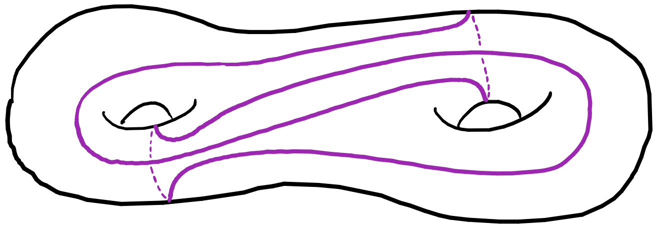


attach 2-handles along the curves

fill 2-sphere boundaries with 3-balls

Can you see the handlebodies?





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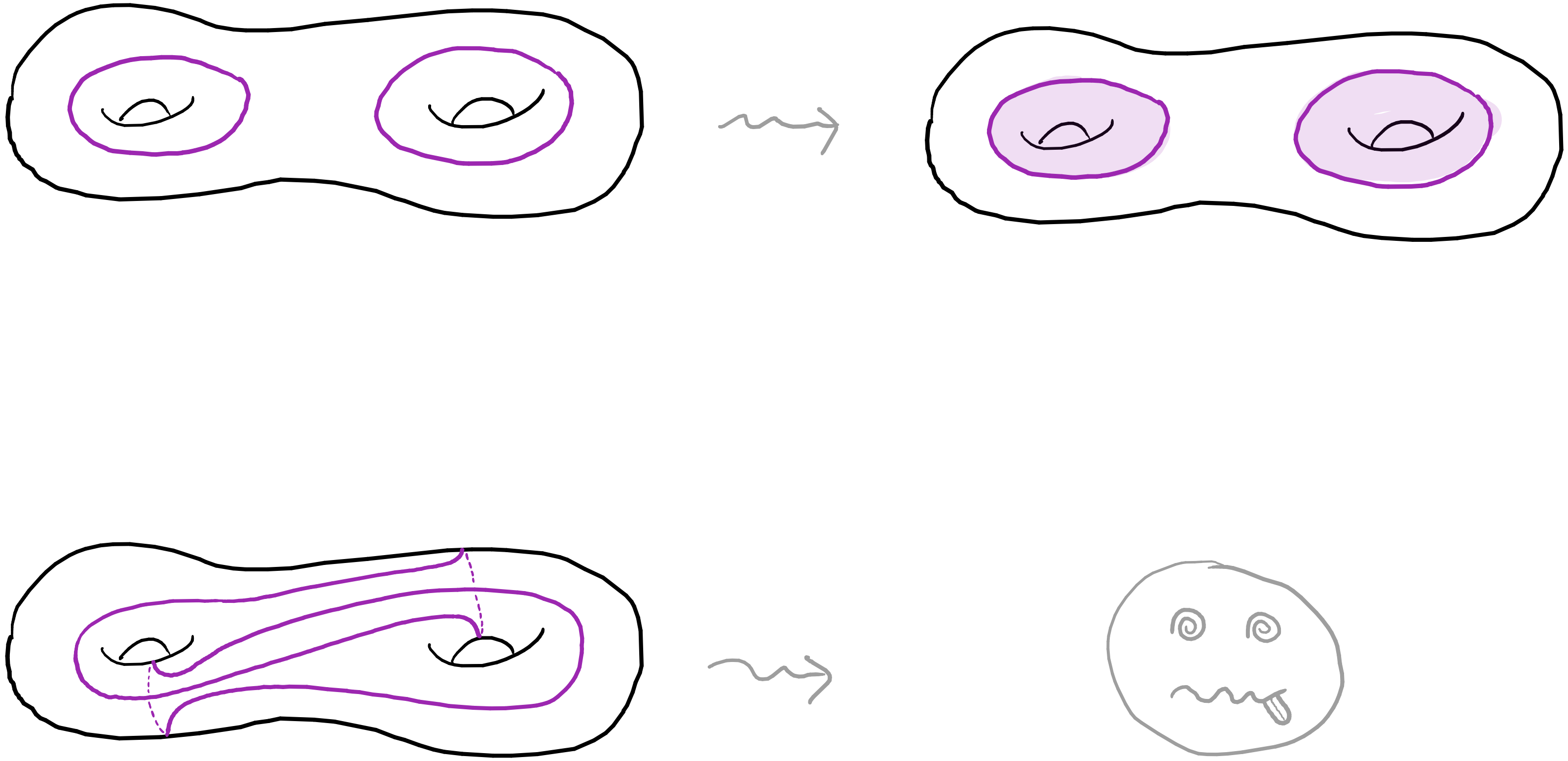
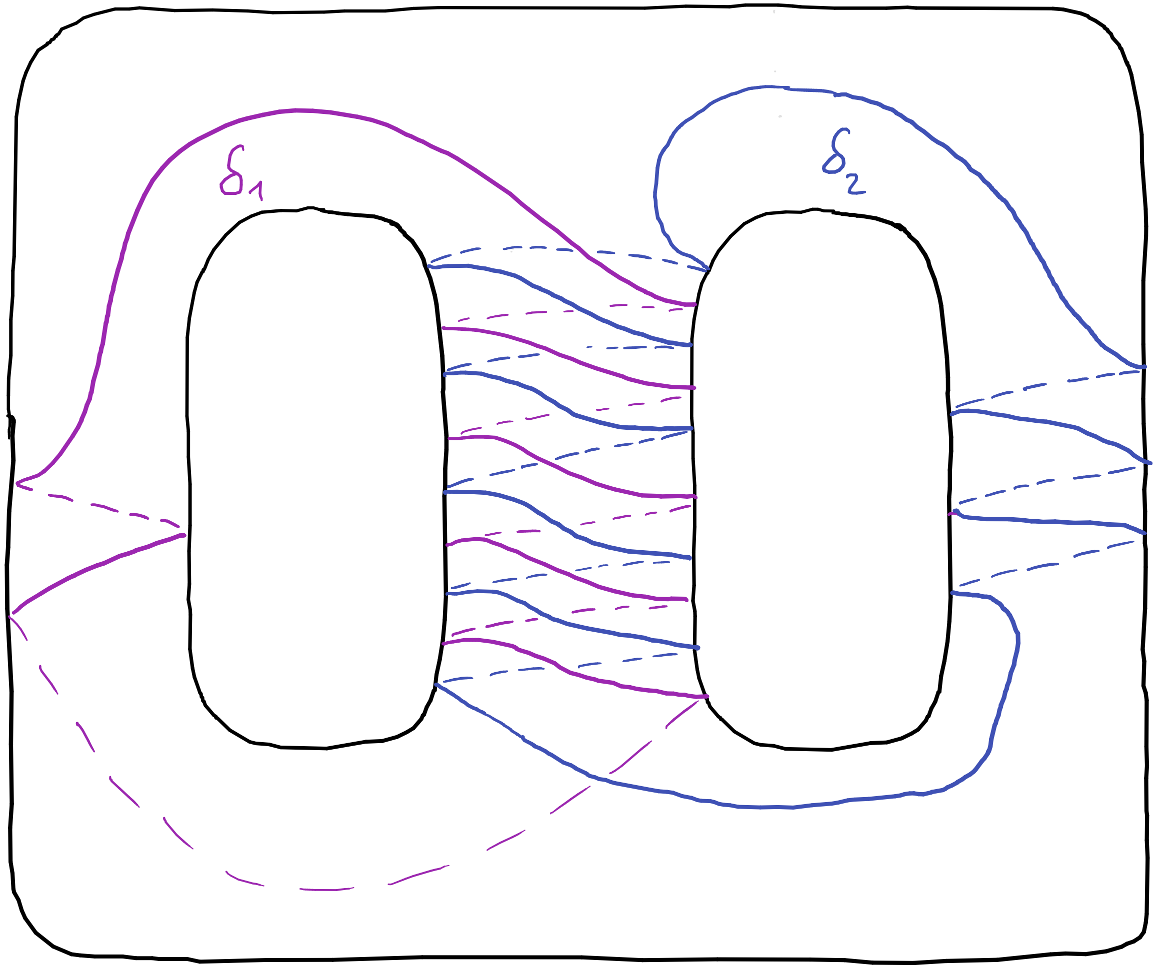
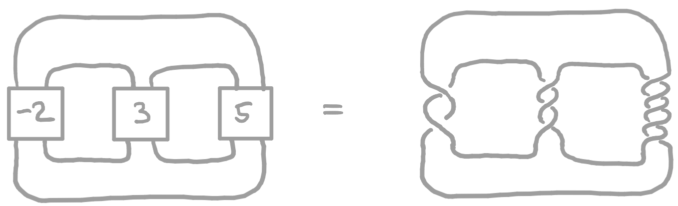


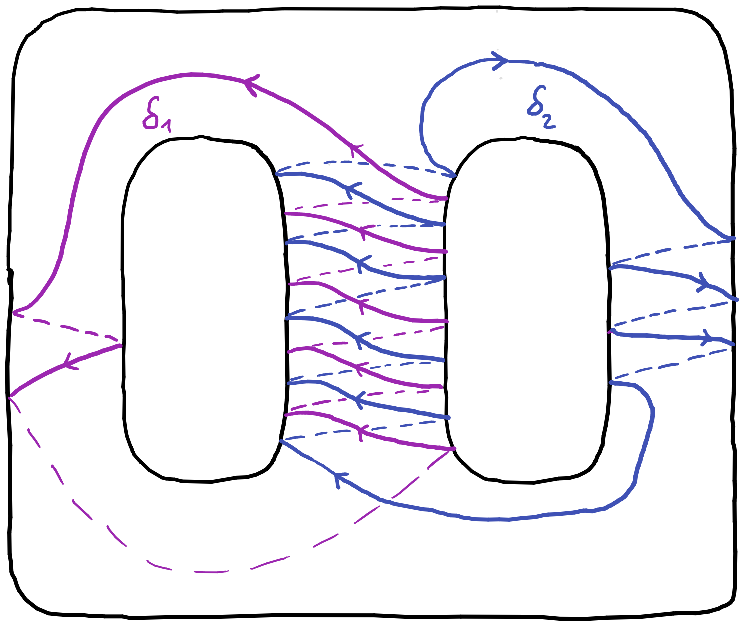


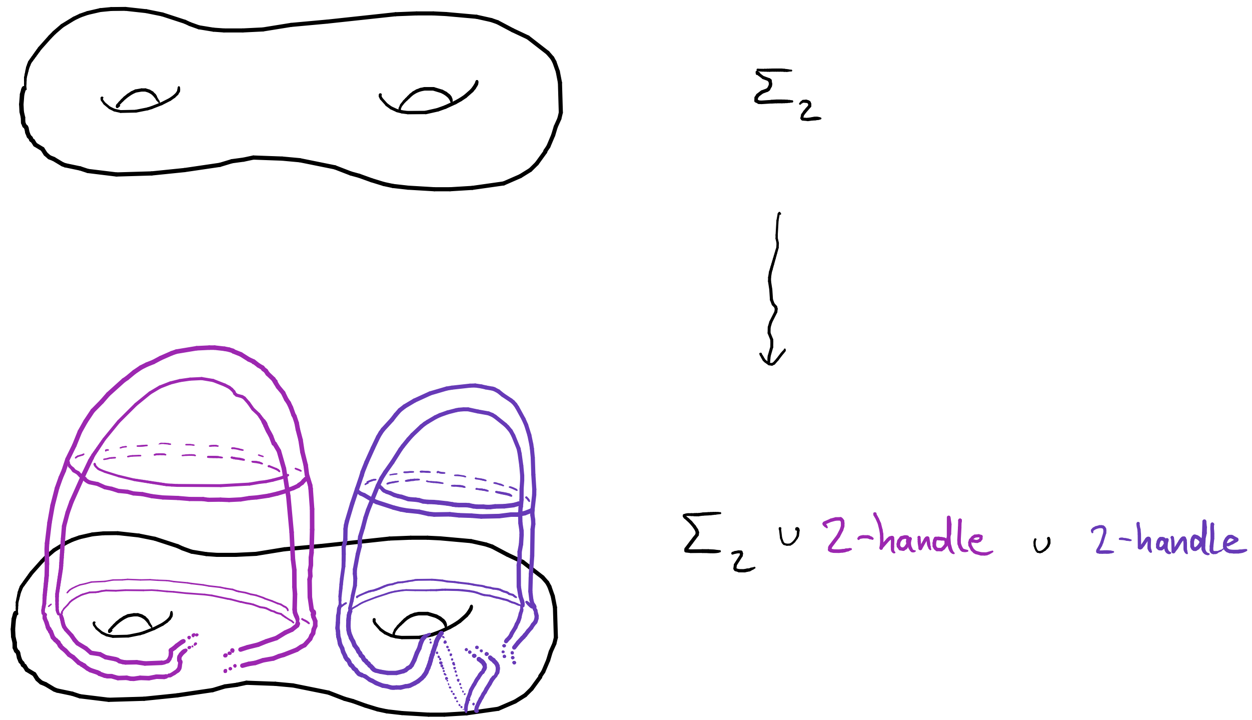
<u>Side remark:</u> This is one of the handlebodies in a genus 2 Heegaard diagram for the 3-mfld. $P = $ Poincaré homology sphere

$P =$ double branched cover $\Sigma_2(K)$ of S^3 branched over

$K = (-2,3,5)$ Pretzel knot

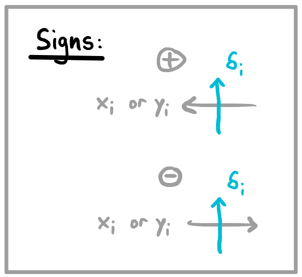


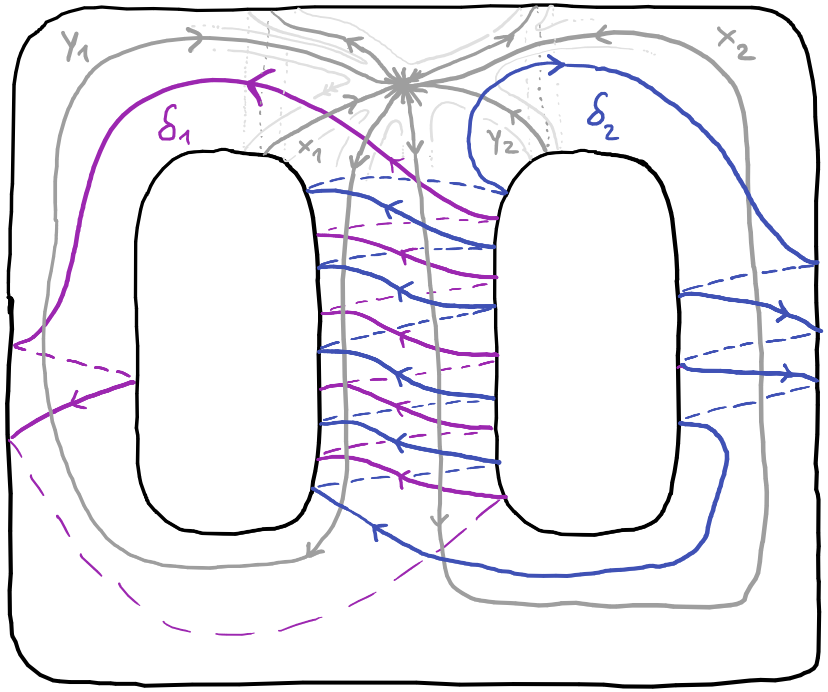

δ_1
δ_2


Σ_2
$\Sigma_2 \cup$ 2-handle $\cup$ 2-handle

Topology

Algebra





$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

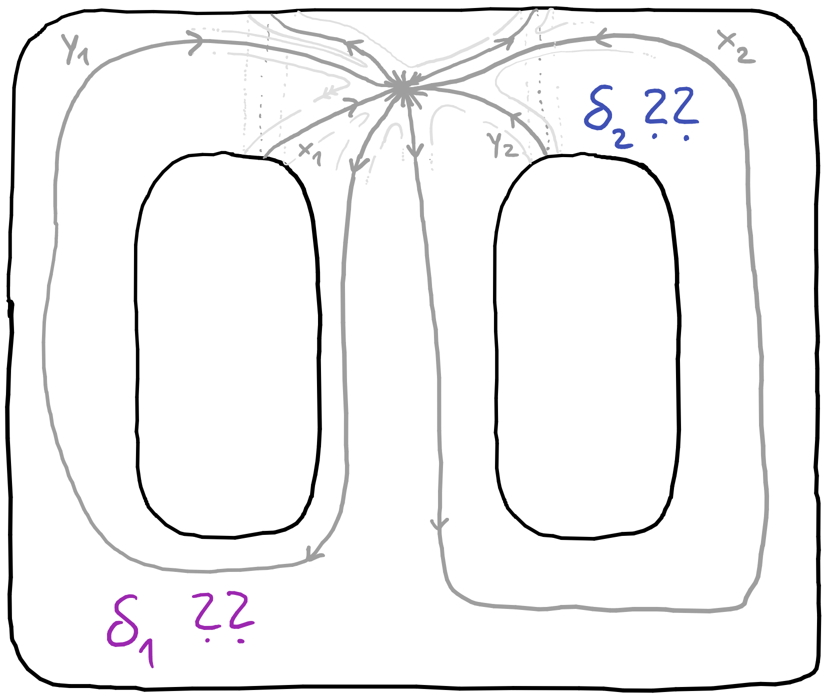
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^{3}$$

$$y_2 \longmapsto d_2$$



Topology

$\uparrow$

Algebra

$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

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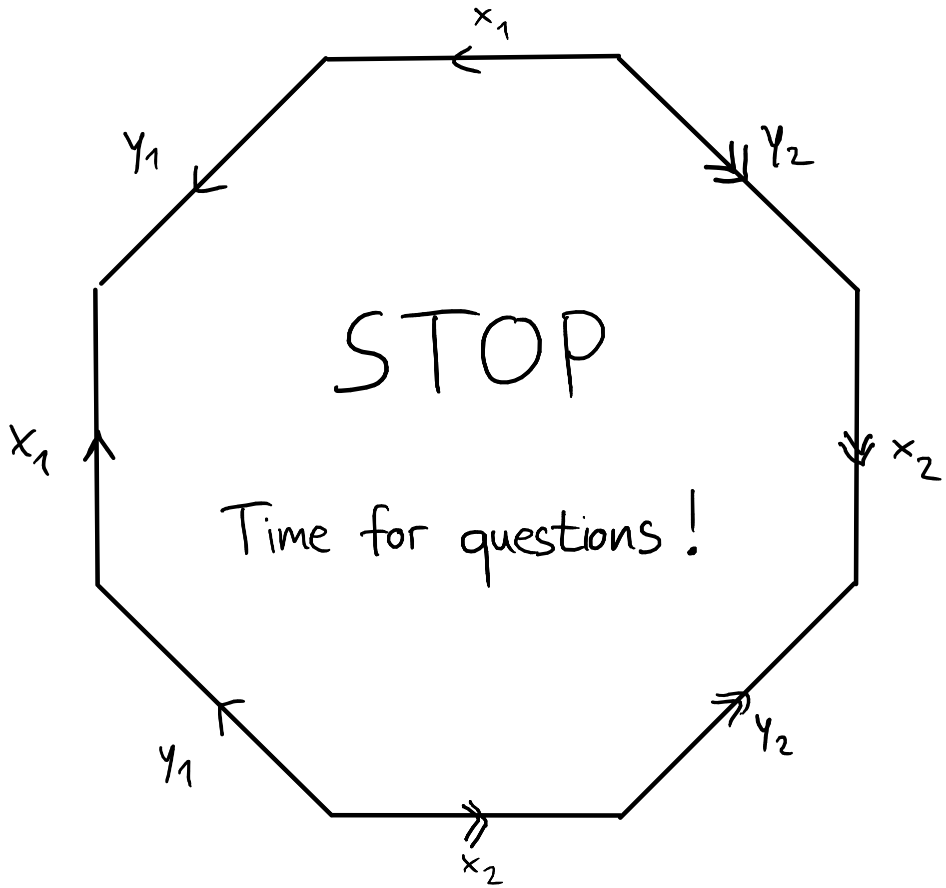


$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

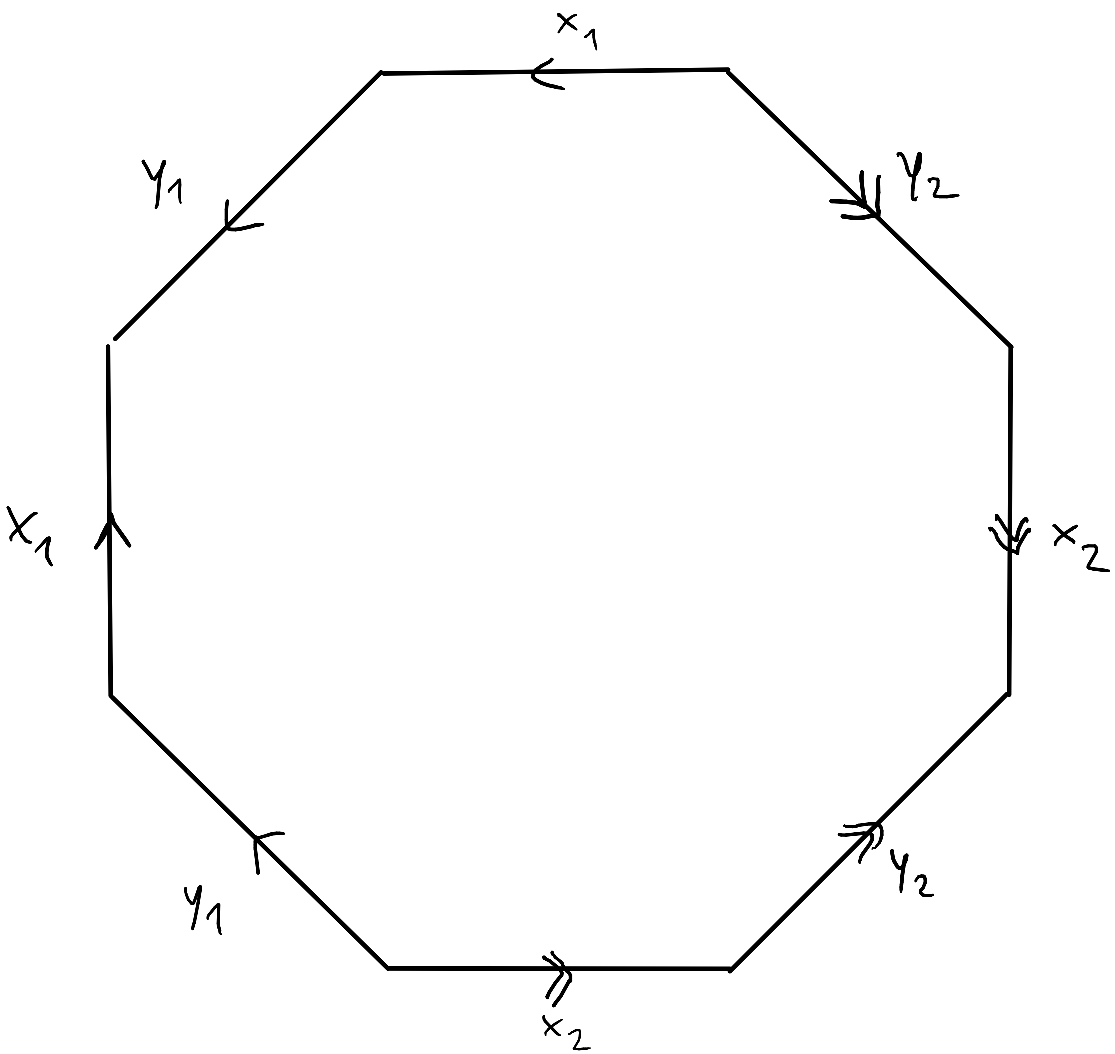
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \, x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$\begin{aligned} x_1 &\longmapsto d_1^{-1} \\ y_1 &\longmapsto (d_1 d_2)^5 \cdot d_1^{-2} \\ x_2 &\longmapsto (d_1 d_2)^5 \cdot d_2^{3} \\ y_2 &\longmapsto d_2 \end{aligned}$$

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

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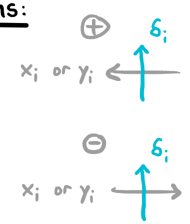
Signs:



Colour coding: d_1 d_2

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

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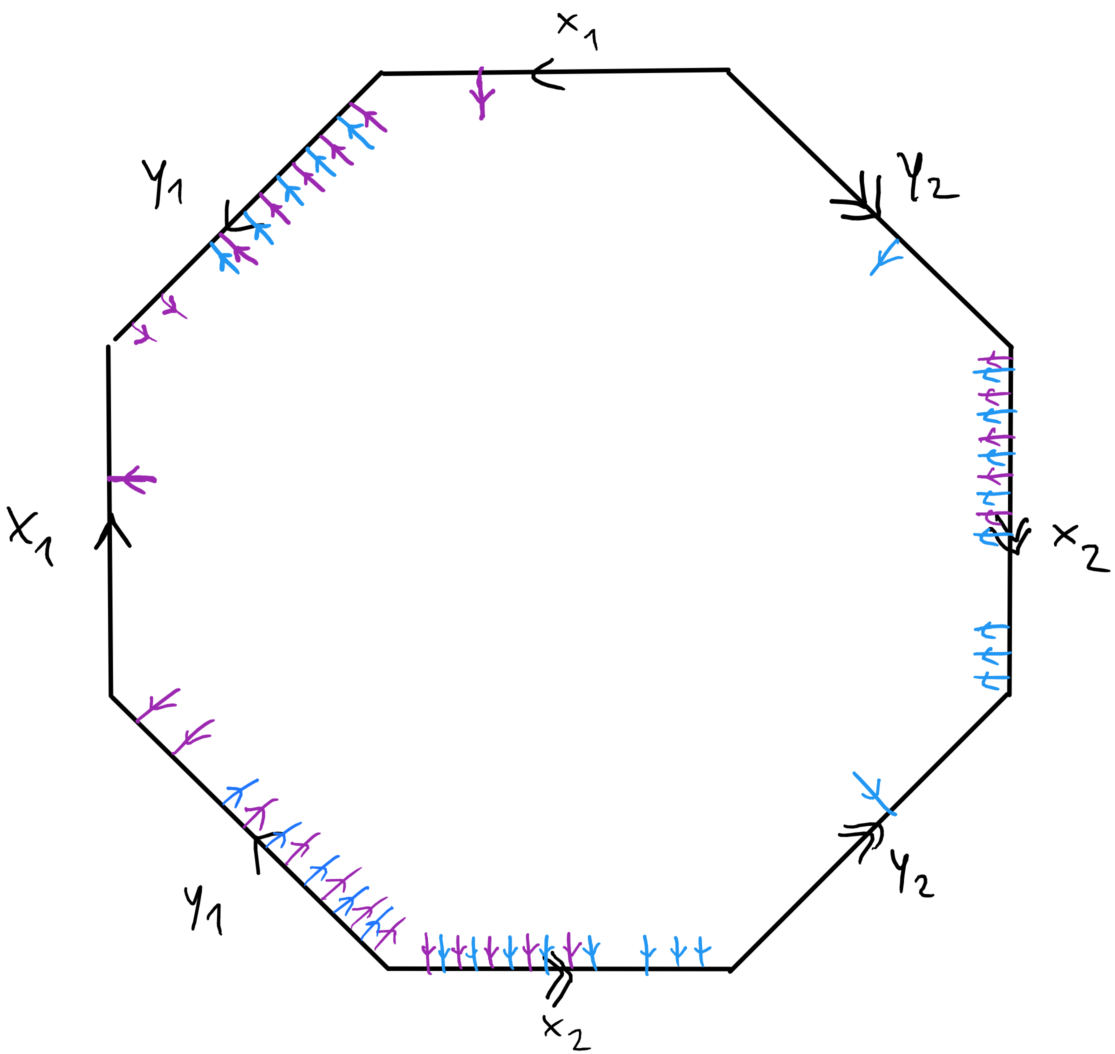
$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

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Signs:

$\oplus$ $\quad \delta_i$ $\quad x_i$ or y_i

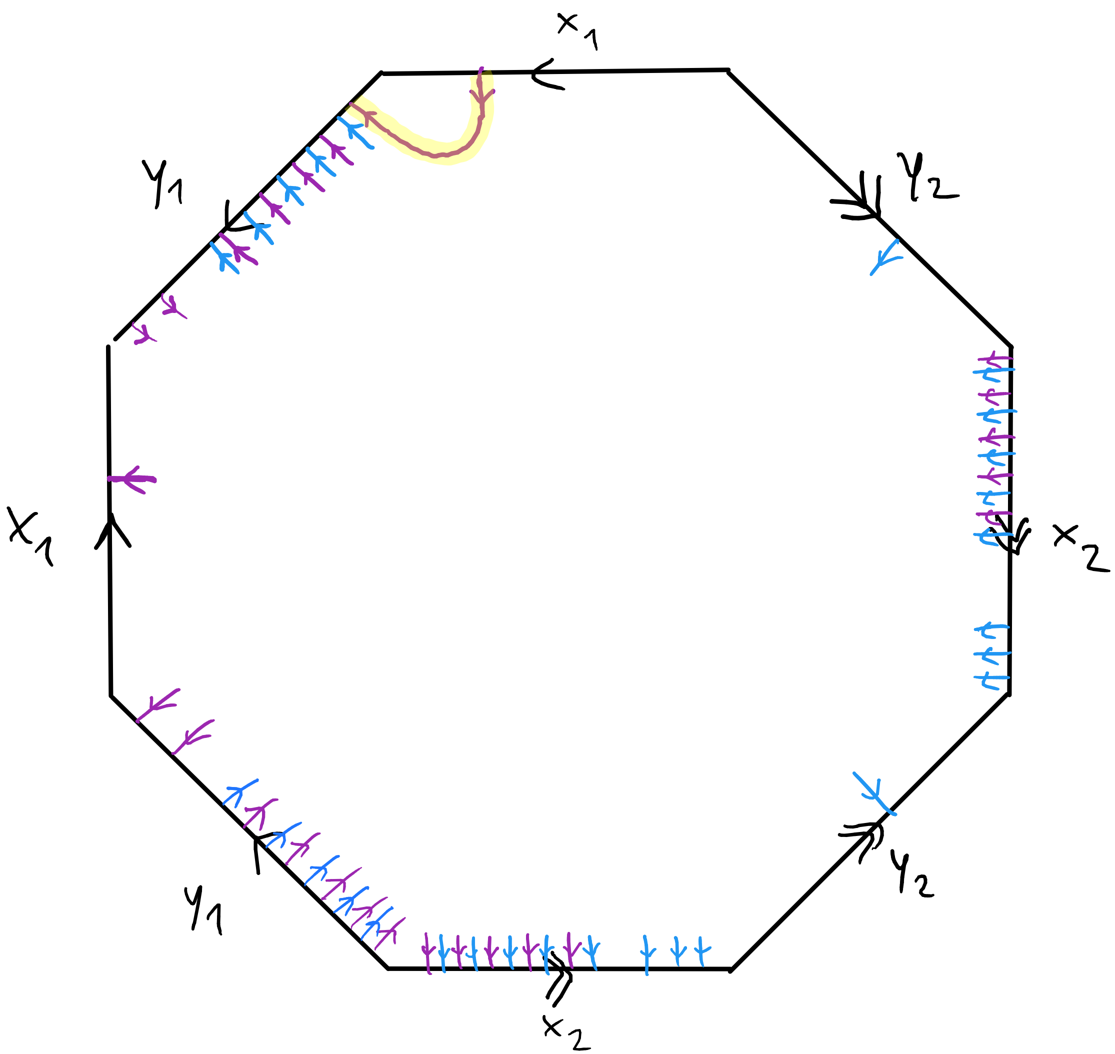
$\ominus$ $\quad \delta_i$ $\quad x_i$ or y_i

Colour coding: d_1 $\quad d_2$

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[\cancel{d_1^{-1}}][(\cancel{d_1}d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1d_2)^{-5}][(d_1d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1d_2)^{-5}][d_2^{-1}]$$



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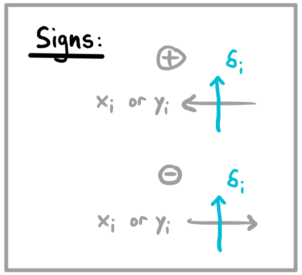
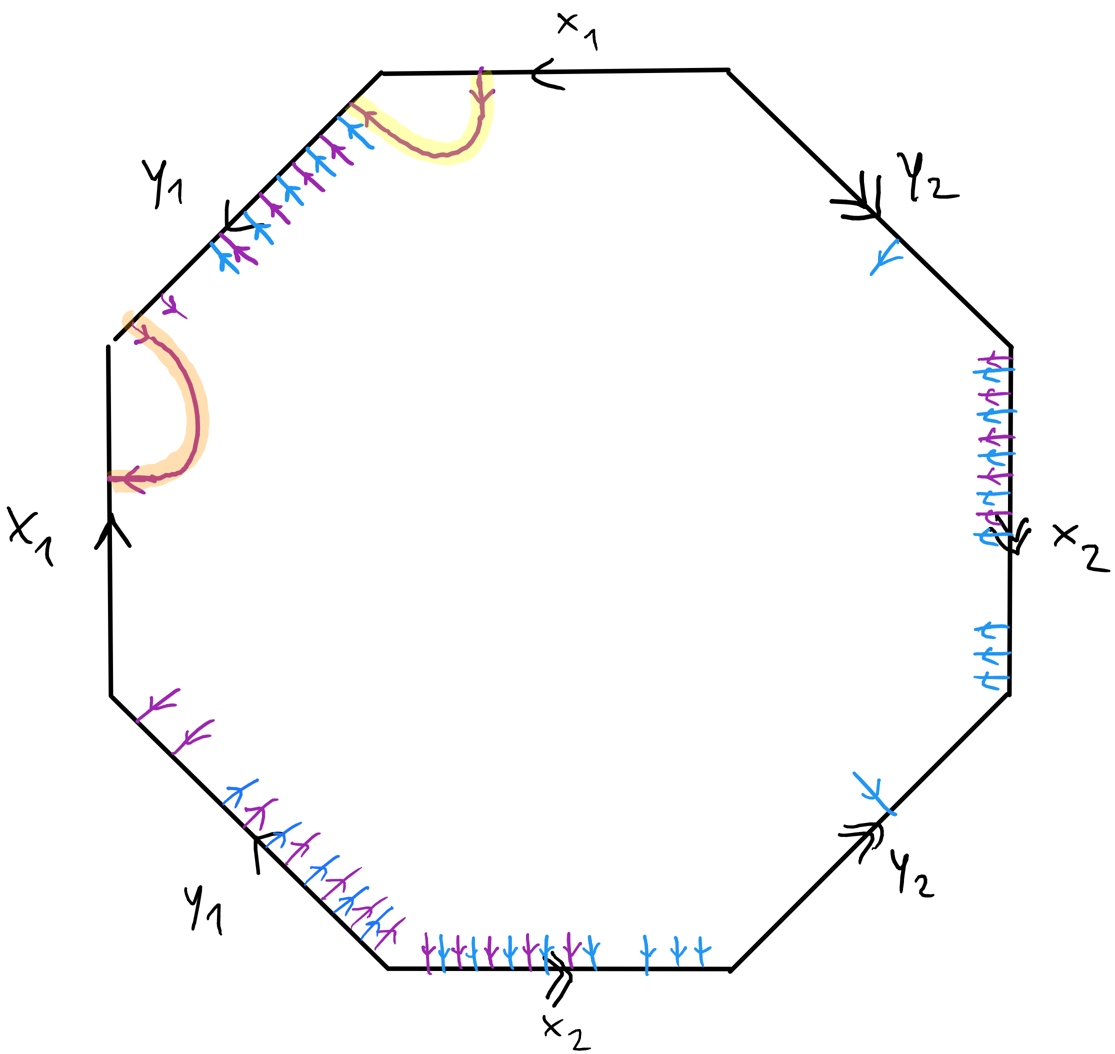


Colour coding: d_1 d_2

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[\cancel{d_1^{-1}}][(\cancel{d_1} d_2)^5 \cancel{d_1^{-2}}][\cancel{d_1}][d_1^{2} (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3} (d_1 d_2)^{-5}][d_2^{-1}]$$



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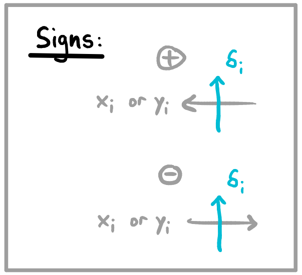
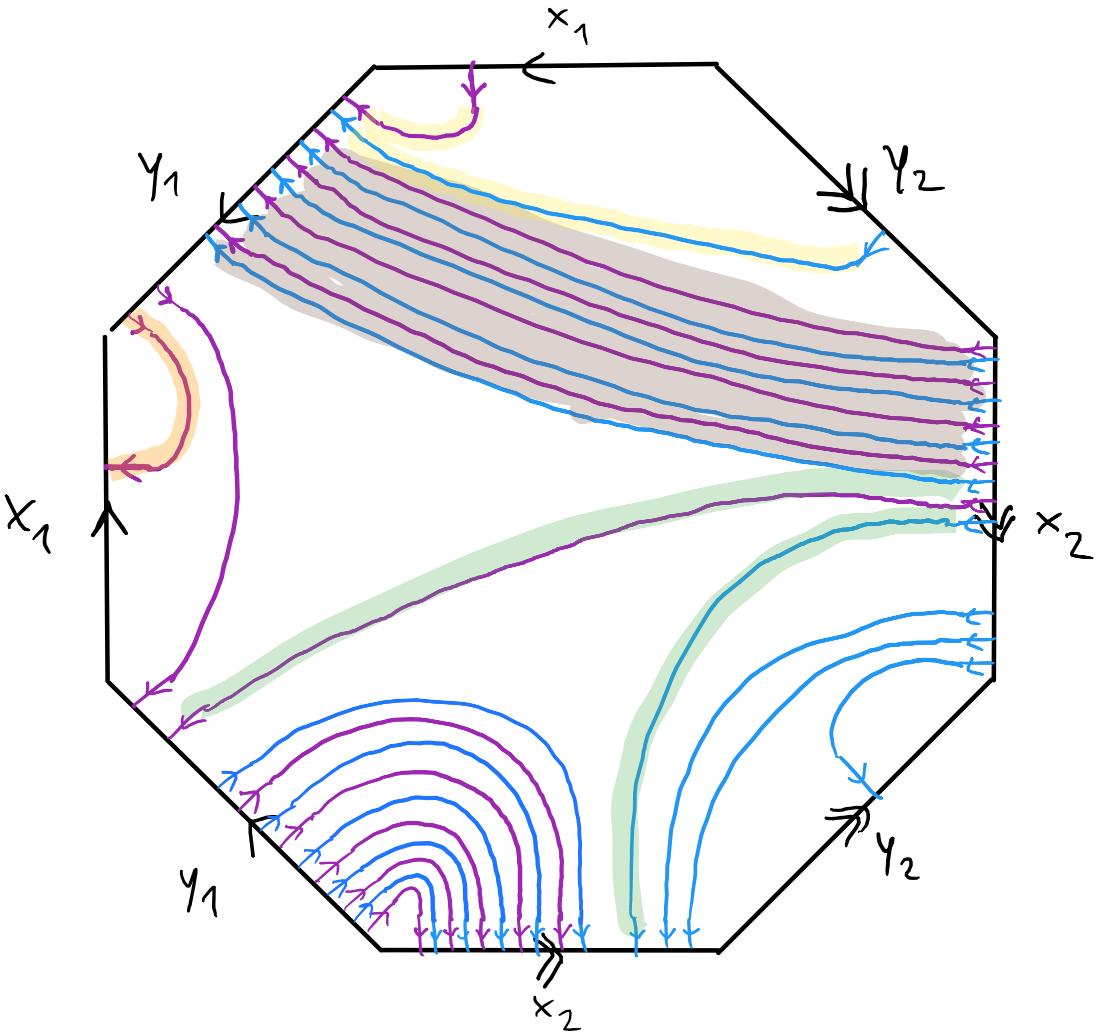


Colour coding: d_1, d_2

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$\underset{x_1}{[d_1^{-1}]}\underset{y_1}{[(d_1d_2)^5 d_1^{-2}]}\underset{x_1^{-1}}{[d_1]}\underset{y_1^{-1}}{[d_1^{2}(d_1d_2)^{-5}]}\underset{x_2}{[(d_1d_2)^5 d_2^{3}]}\underset{y_2}{[d_2]}\underset{x_2^{-1}}{[d_2^{-3}(d_1d_2)^{-5}]}\underset{y_2^{-1}}{[d_2^{-1}]}$$



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Signs:

$\oplus$ δ_i x_i or y_i

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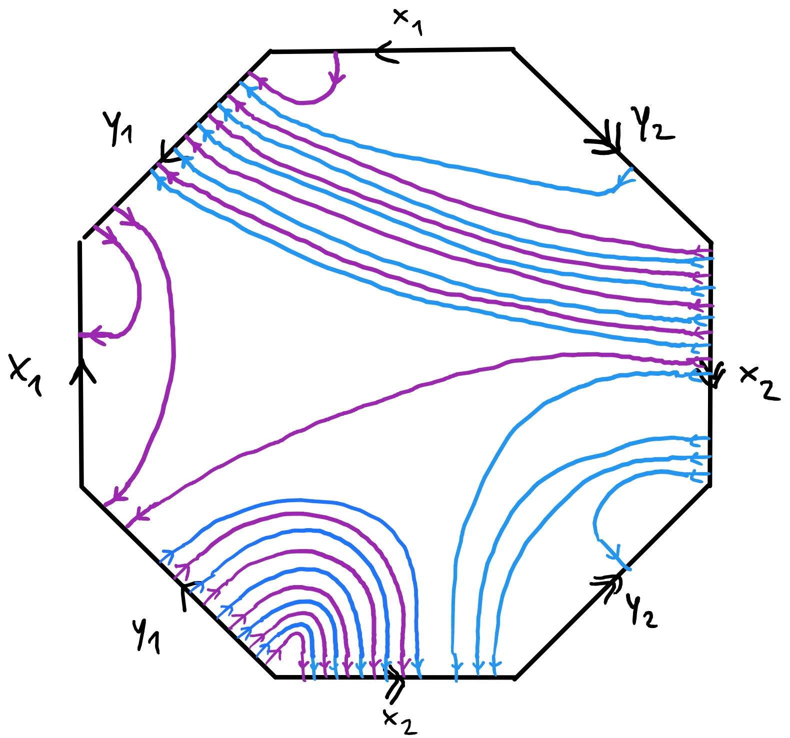
Colour coding: d_1 d_2

Topology



Algebra

$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

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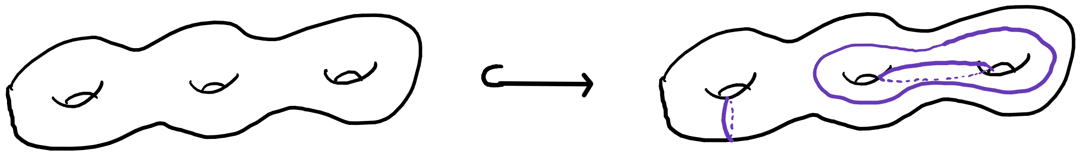
From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} F_g$ (surface group $\twoheadrightarrow$ free group)

is uniquely realized geometrically by a handlebody.

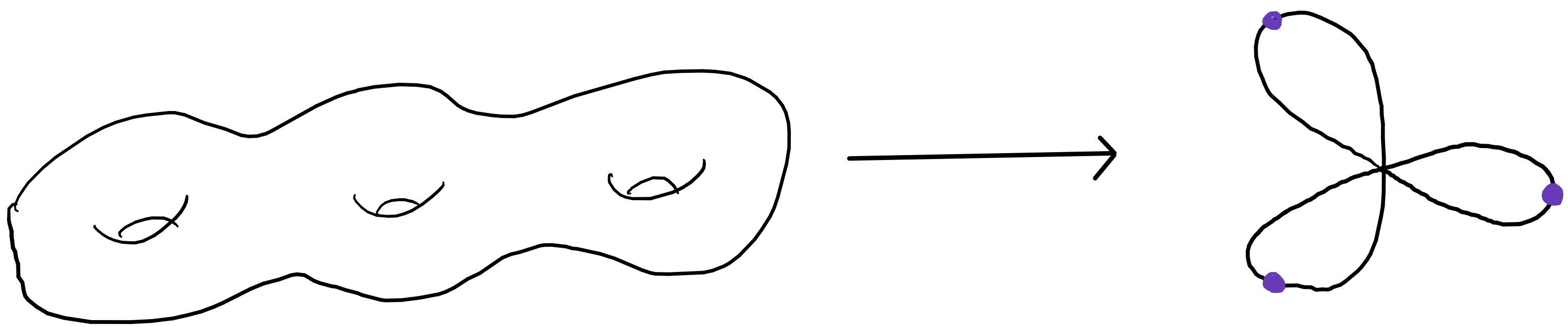


Folklore proof sketch:

Homomorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \mathbb{F}_g$ (surjective)

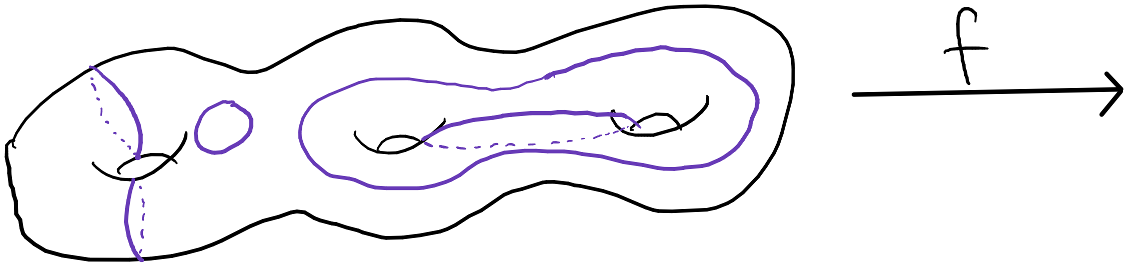
determines a unique map up to homotopy

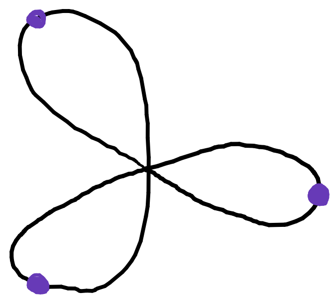
$$\begin{tikzcd} \Sigma_g \arrow[r, "f"] & \bigvee^g S^1 \\ K(\pi_1(\Sigma_g),1) \arrow[u, phantom, "\simeq"] & K(\mathbb{F}_g,1) \arrow[u, phantom, "\cong"] \end{tikzcd}$$



make map transverse to north poles

$$\Sigma_g \xrightarrow{f} \bigvee^g S^1$$





make map transverse to north poles

look at preimage f^{-1}(North poles)

Collection of simple closed curves in Σ_g which contains a cut system

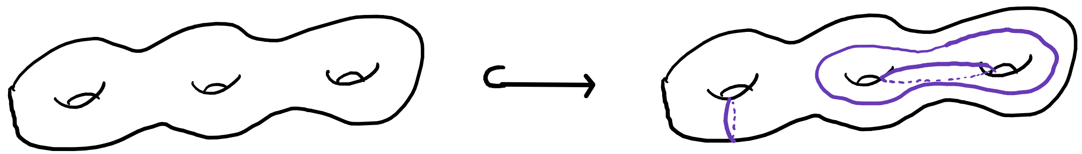
$\square$ (Folklore)

From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \twoheadrightarrow Fr_g$

surface group $\longrightarrow\!\!\!\!\twoheadrightarrow$ free group

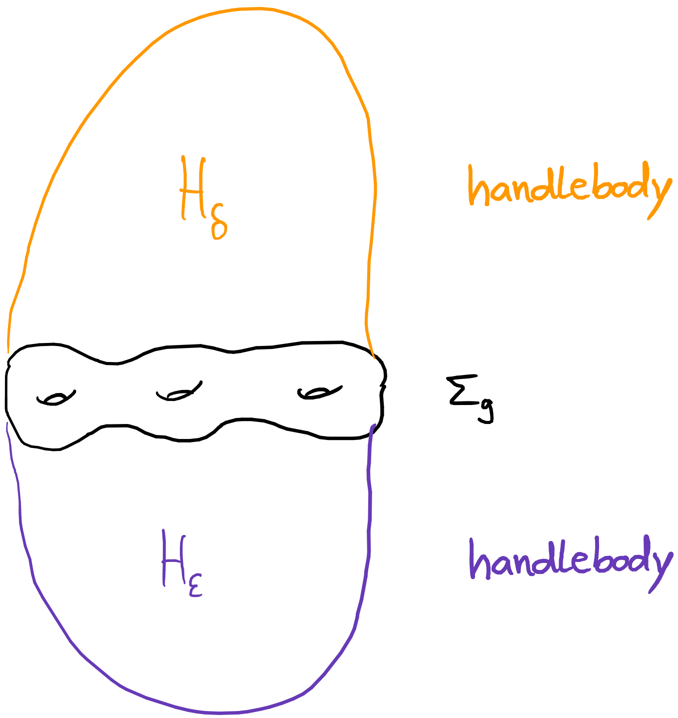
is realized geometrically by a handlebody (uniquely) ...



[Blackwell-Kirby-Klug-Longo-R, 2021]

... which can be computed algorithmically.

Heegaard splitting of a 3-manifold M^3



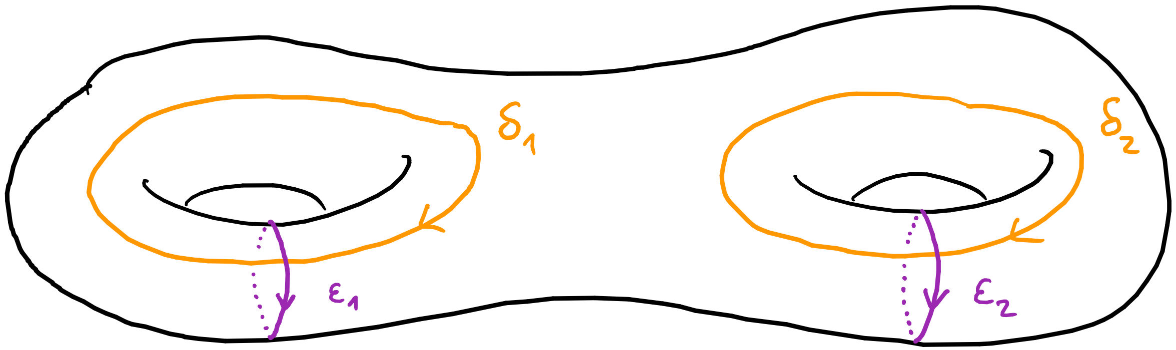
$$\begin{tikzcd}
 & \pi_1(H_\delta) \arrow[dr] & \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[dr, two heads] & \text{pushout} & \pi_1(M) \\
 & \pi_1(H_\varepsilon) \arrow[ur] &
\end{tikzcd}$$

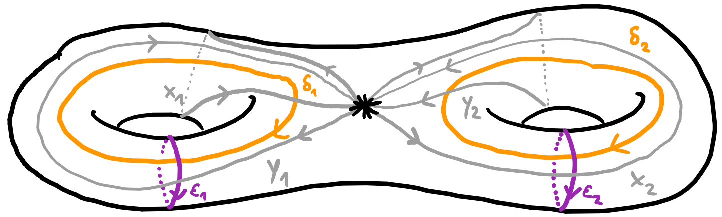
Splitting homomorphism

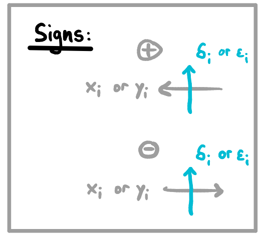
[Jaco : Heegaard splittings and splitting homomorphisms (1969)]

[Stallings : How not to prove the Poincaré conjecture (1966)]

Ex.: Splitting homomorphism for genus 2 splitting of S^3





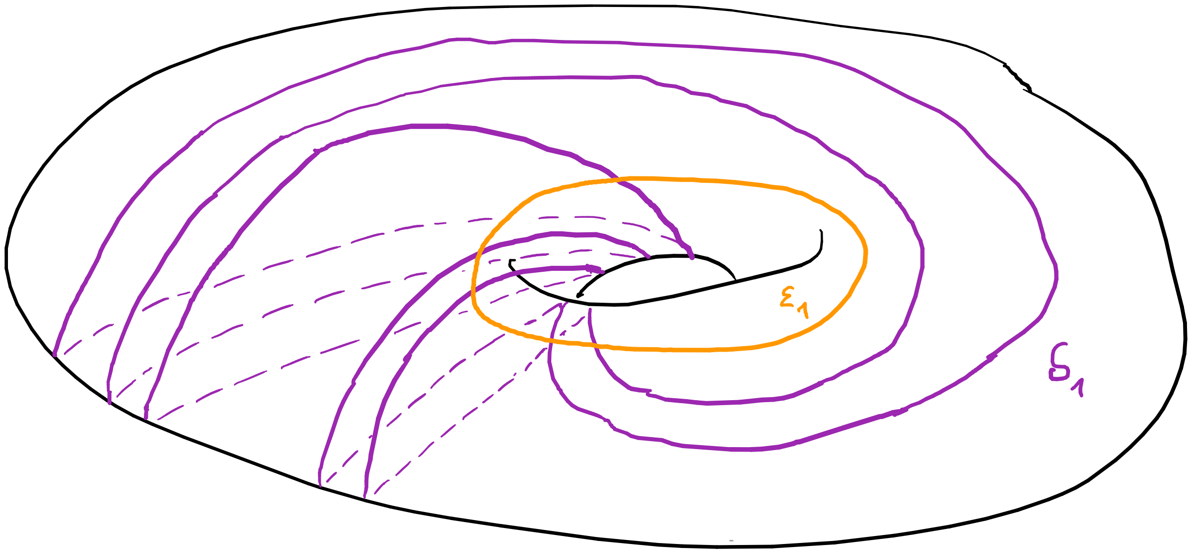


$$x_1 \mapsto d_1,\quad y_1 \mapsto 1 \ \text{(trivial)},\quad x_2 \mapsto 1,\quad y_2 \mapsto d_2$$

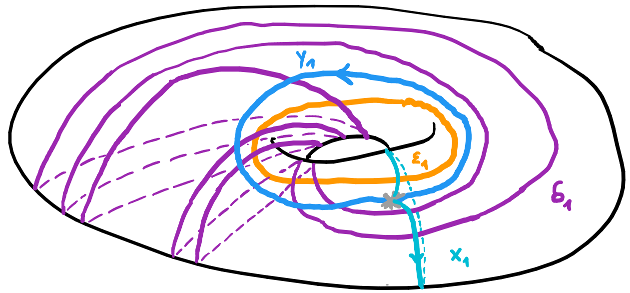
$$\begin{tikzcd}
 & \langle d_1, d_2 \rangle \arrow[dr] & \\
\langle x_1, y_1, x_2, y_2 \mid [x_1,y_1]\cdot[x_2,y_2] \rangle \arrow[ur] \arrow[dr] & & \langle d_1, d_2 \rangle *_{\pi_1(\Sigma)} \langle e_1, e_2 \rangle \cong \langle 1 \rangle \\
 & \langle e_1, e_2 \rangle \arrow[ur] &
\end{tikzcd}$$

$$x_1 \mapsto 1 \ \text{(trivial)},\quad y_1 \mapsto e_1^{-1},\quad x_2 \mapsto e_2,\quad y_2 \mapsto 1$$

Ex.: Splitting homomorphism for genus 1 splitting of $L(5,2)$



x_1, y_1 generators of $\pi_1(\Sigma)$



$x_1 \mapsto d_1^2$

$y_1 \mapsto d_1^5$

$x_1 \mapsto e_1$

$y_1 \mapsto 1$ (trivial)

$$\begin{tikzcd}
 & \langle d_1 \rangle \arrow[dr] & \\
\langle x_1, y_1 \mid [x_1, y_1] \rangle \arrow[ur] \arrow[dr] & & \langle d_1 \rangle *_{\pi_1(\Sigma)} \langle e_1 \rangle \cong \mathbb{Z}/5 \\
 & \langle e_1 \rangle \arrow[ur] &
\end{tikzcd}$$

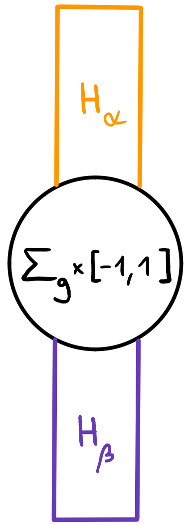
$$\left(\begin{matrix} x_1 = d_1^2 \\ y_1 = d_1^5 \\ x_1 = e_1 \\ y_1 = 1 \end{matrix} \right)$$

$d_1^5 = 1 \qquad e_1 = d_1^2$

(based, parameterized)

Heegaard splittings

of a 3-manifold Y^3



take π_1 of pieces $\downarrow$

$\boxed{\begin{matrix} 1:1 \\ \text{[Stallings, Jaco]} \end{matrix}}$

$\uparrow$ glue the handlebodies corresponding to the epimorphisms to $\Sigma_g \times \{-1\}$ and $\Sigma_g \times \{1\}$ respectively

group bisections of $\pi_1(Y,*)$

$$\begin{tikzcd} & \pi_1(H_\alpha) \arrow[r, two heads] & \pi_1(Y) \\ \pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] & \end{tikzcd}$$

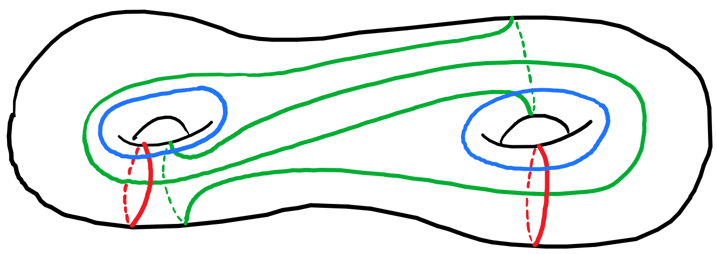
Plan:

-) Recall the 4-dim. closed case where

  triples of handlebodies determine 4-manifolds

  $\rightsquigarrow$ group trisection

  [Abrams, Gay, Kirby]

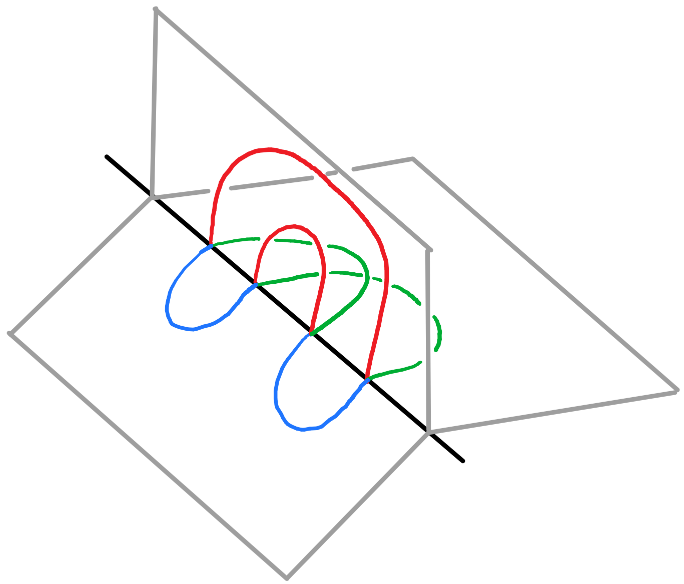


-) Relative case:

  bridge-trisected surface F^2

  $\cap$

  trisected 4-manifold X^4



Group trisections of a finitely presented group G:

Commutative cube

$$
\begin{tikzcd}
 & Fr_g \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_{k_1} \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_g \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & Fr_{k_3} \arrow[rr, two heads] & & G \\
Fr_g \arrow[ur, two heads] \arrow[rr, two heads] & & Fr_{k_2} \arrow[ur, two heads] &
\end{tikzcd}
$$

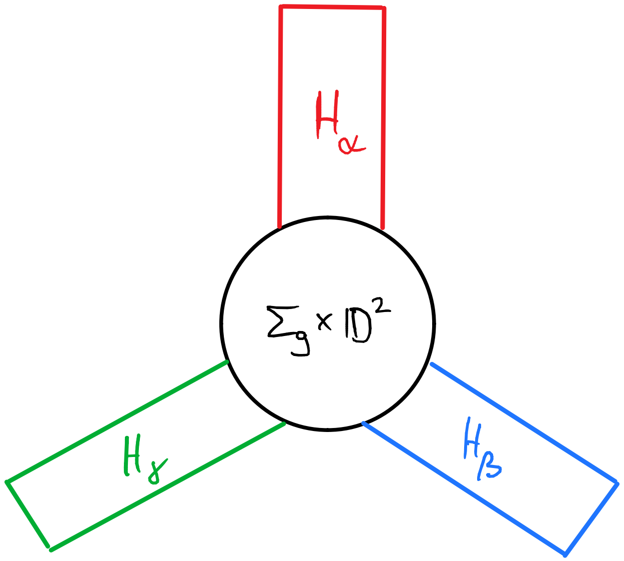
s.th. all maps are surjective

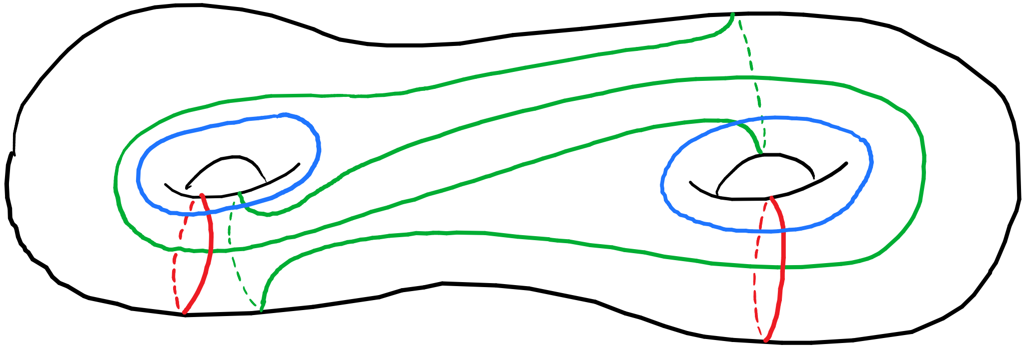
and all faces are push-outs

[Abrams, Gay, Kirby]

Group trisections of closed 4-manifolds:

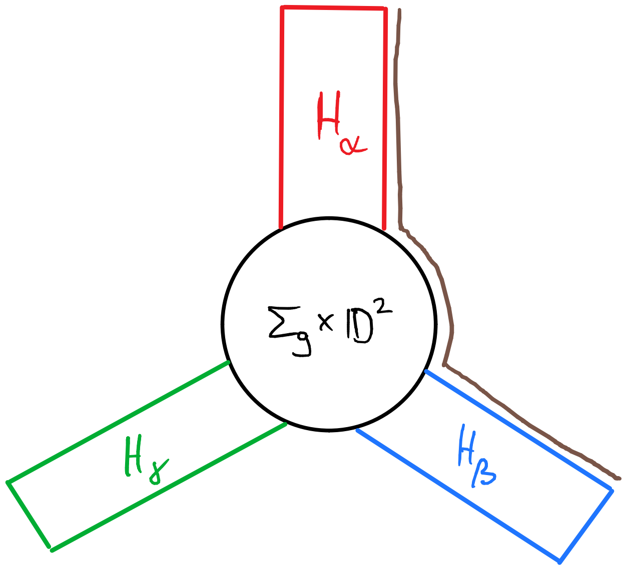
The handlebody-story three times

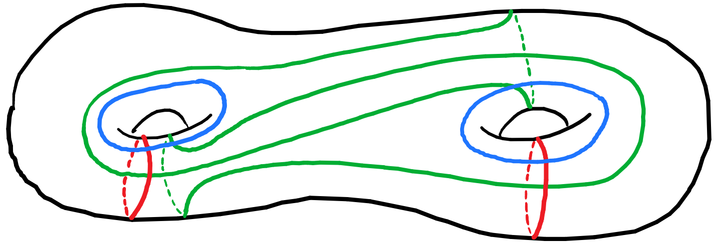




$$\begin{tikzcd}
 & \pi_1(H_\alpha) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(H_\beta) \\
\pi_1(H_\gamma) &
\end{tikzcd}$$

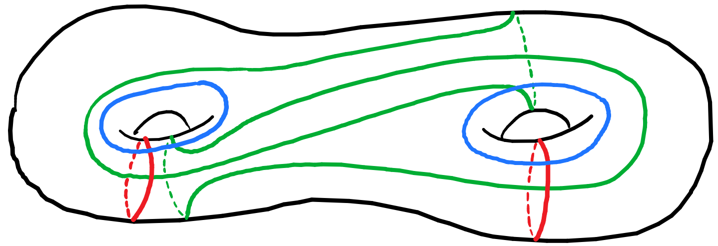
[Abrams, Gay, Kirby]

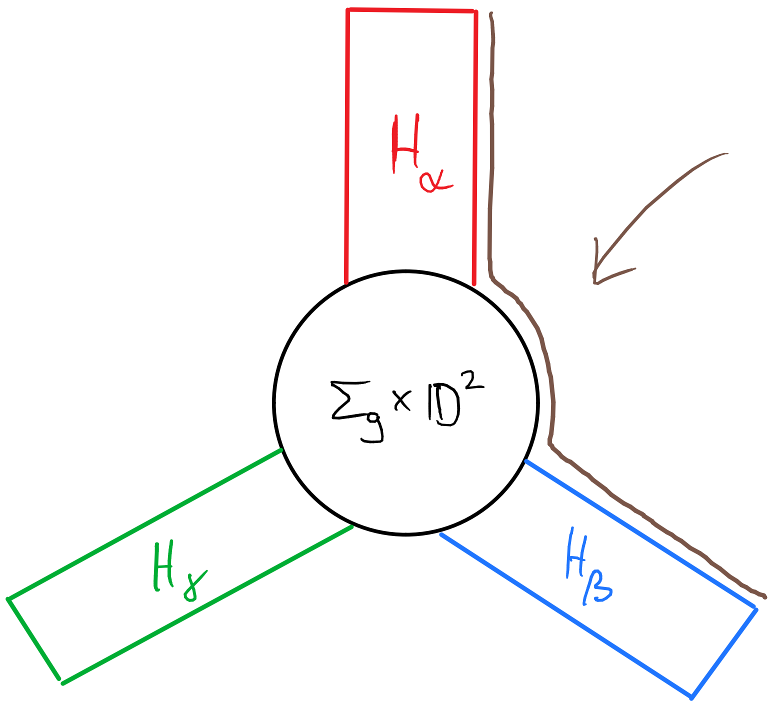




$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[r, two heads] & \pi_1(H_\alpha \cup_\Sigma H_\beta) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] & \\
\pi_1(H_\gamma) & &
\end{tikzcd}$$

[Abrams, Gay, Kirby]





from our algebra assumption:

this is a closed 3-manifold M

with $\pi_1(M) \cong Fr_k$ free

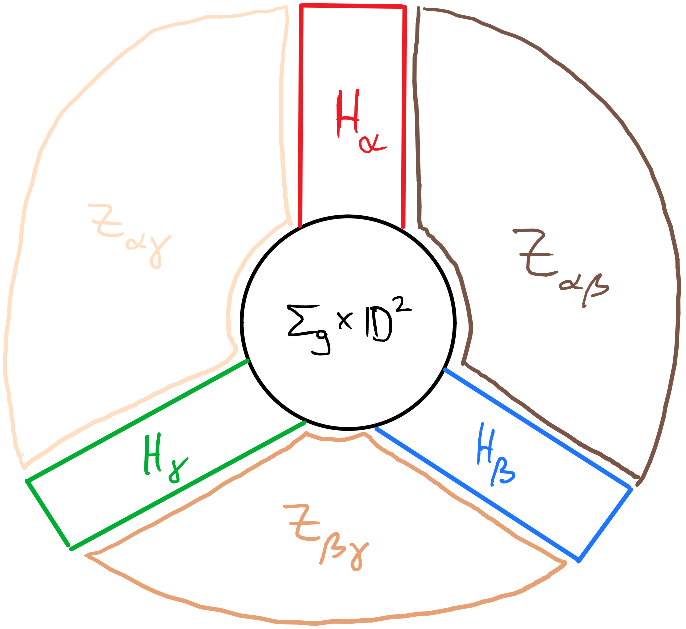
Kneser's thm. + 3D Poincaré conj.

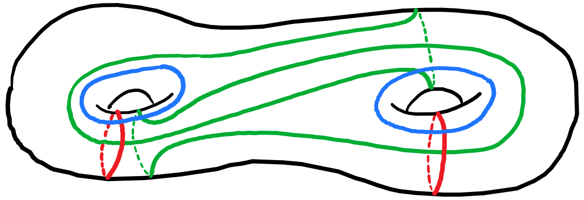
$$\Longrightarrow \qquad M \cong \#^k S^1 \times S^2$$

[Abrams, Gay, Kirby]

[Laudenbach-Poenaru] allows us to fill the sectors uniquely with $\natural^{k_i} S^1 \times \mathbb{D}^3$

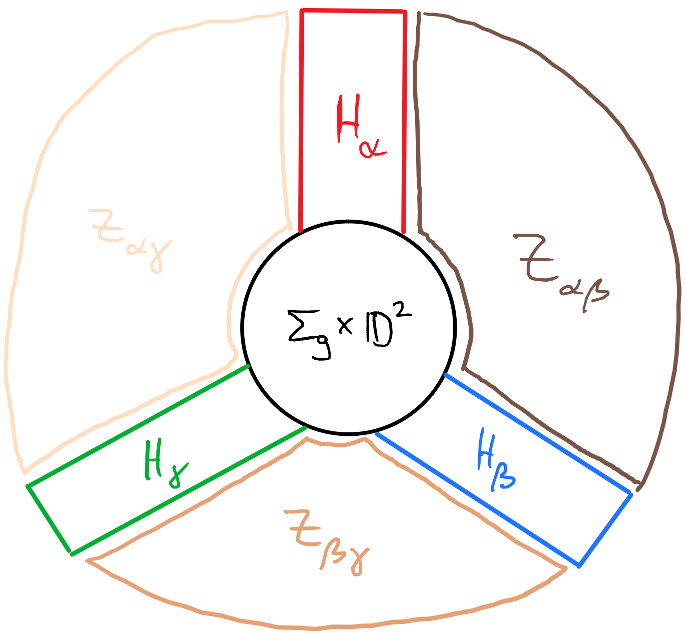
We can do this for all pairs of handlebodies





$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[r, two heads] \arrow[dd] & \pi_1(H_\alpha \cup_\Sigma H_\beta) \cong \pi_1(Z_{\alpha\beta}) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[ddd, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(H_\alpha \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\alpha\gamma}) & \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[r, two heads] & \pi_1(H_\beta \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\beta\gamma}) &
\end{tikzcd}
$$

[Abrams, Gay, Kirby]



4-manifold M^4 with $\pi_1(M^4) \cong G$

and group trisection corresponding to

the cube below

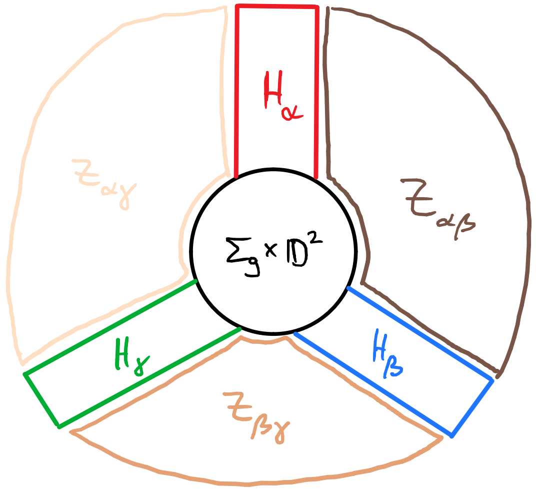
$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, twoheadrightarrow] \arrow[dd, twoheadrightarrow] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, twoheadrightarrow] \\
\pi_1(\Sigma_g) \arrow[ur, twoheadrightarrow] \arrow[rr, twoheadrightarrow] \arrow[dd, twoheadrightarrow] & & \pi_1(H_\beta) \arrow[ur, twoheadrightarrow] \arrow[dd, twoheadrightarrow] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, twoheadrightarrow] & & \pi_1(M) \cong G \\
\pi_1(H_\gamma) \arrow[ur, twoheadrightarrow] \arrow[rr, twoheadrightarrow] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, twoheadrightarrow] &
\end{tikzcd}$$

[Abrams, Gay, Kirby]

(based, parameterized)

trisections

of a 4-manifold X^4



take π_1 of pieces

1 : 1

[Abrams, Gay, Kirby]

the previously explained construction

group trisections of $\pi_1(X,*)$

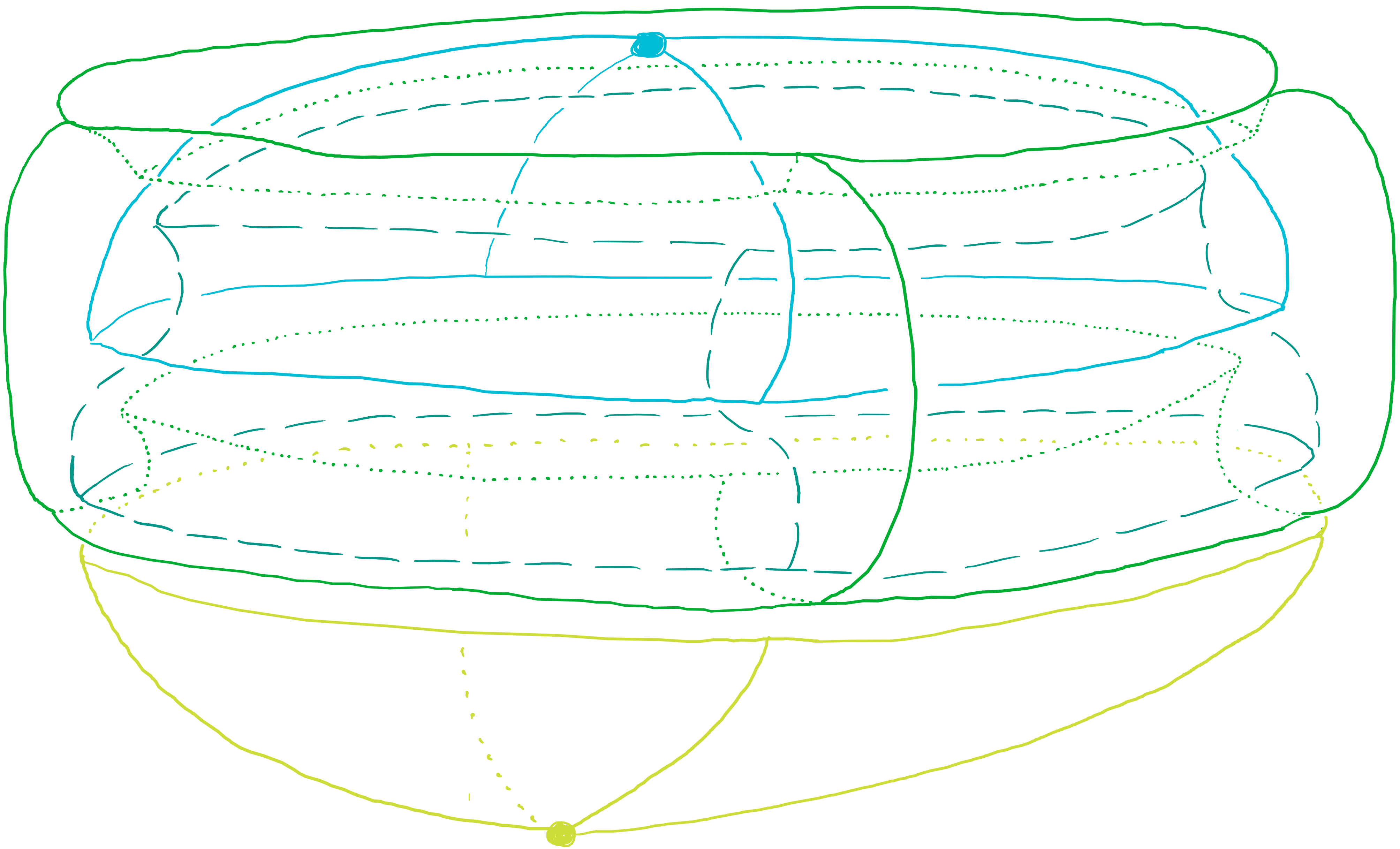
$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(X) \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}$$

<u>Now:</u> bridge-trisected surface F^2

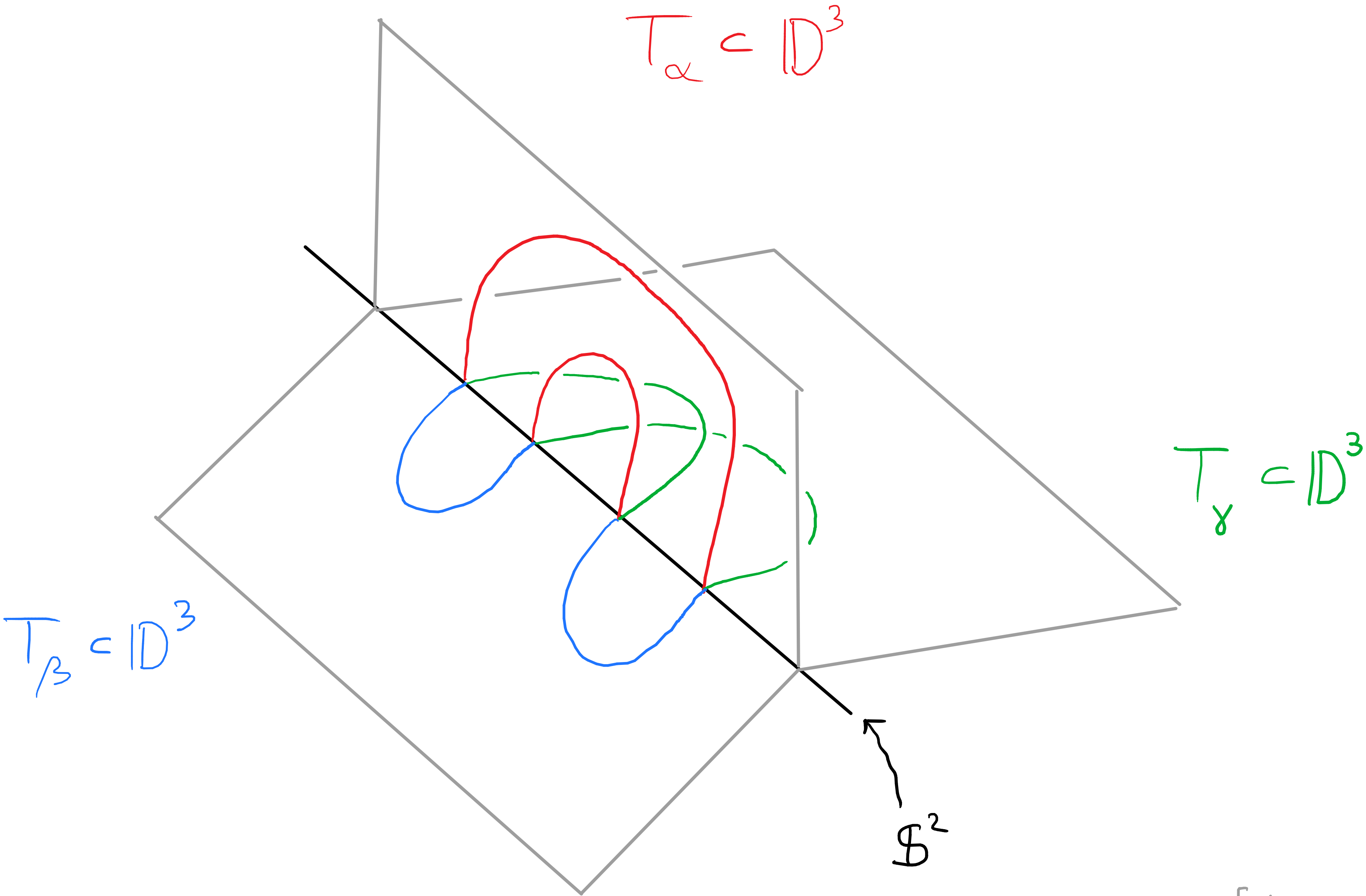
contained in

trisected 4-manifold X^4

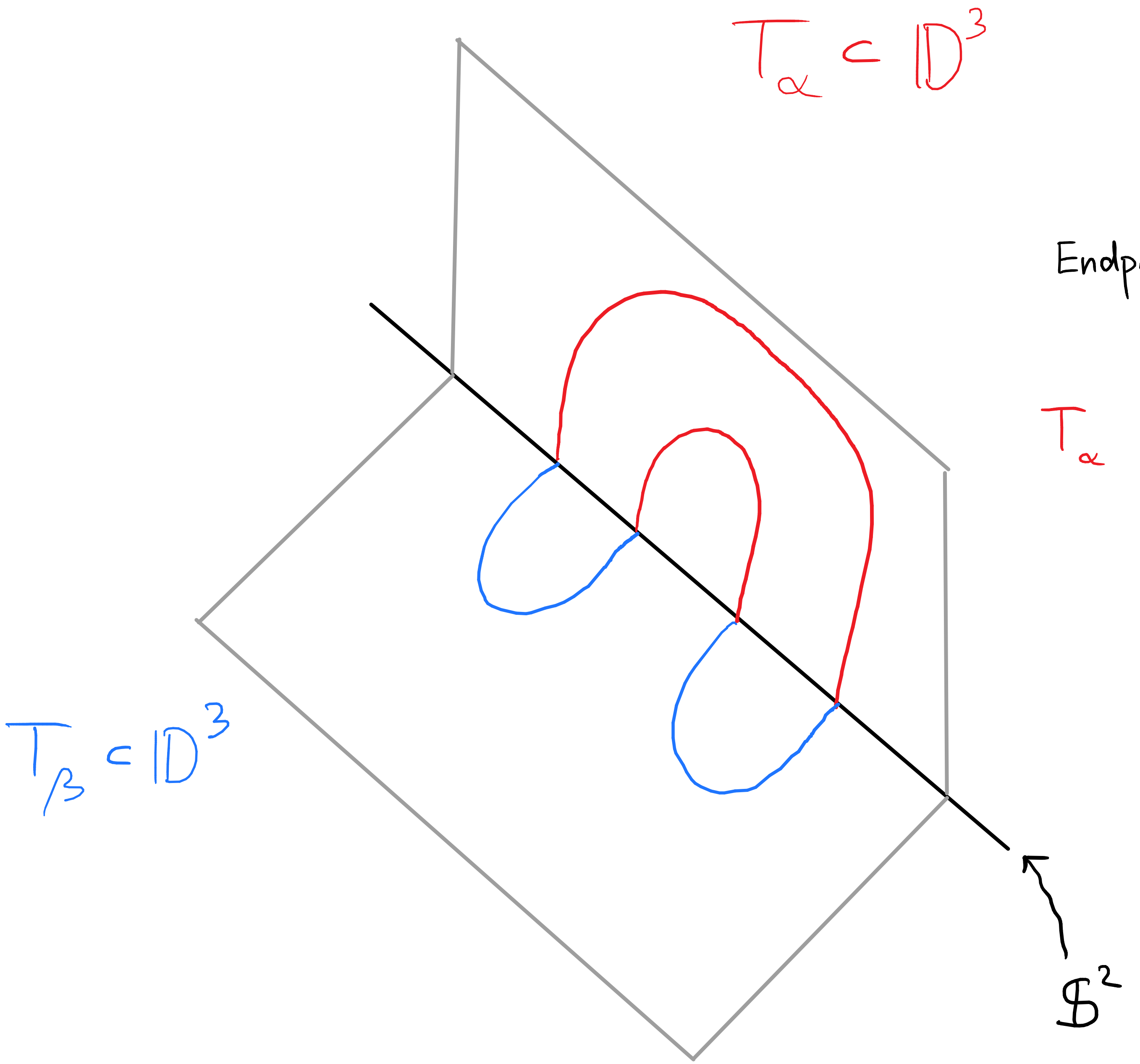
Spun trefoil - a knotted surface in S^4



Bridge-trisected surfaces in the 4-sphere



[Meier-Zupan]



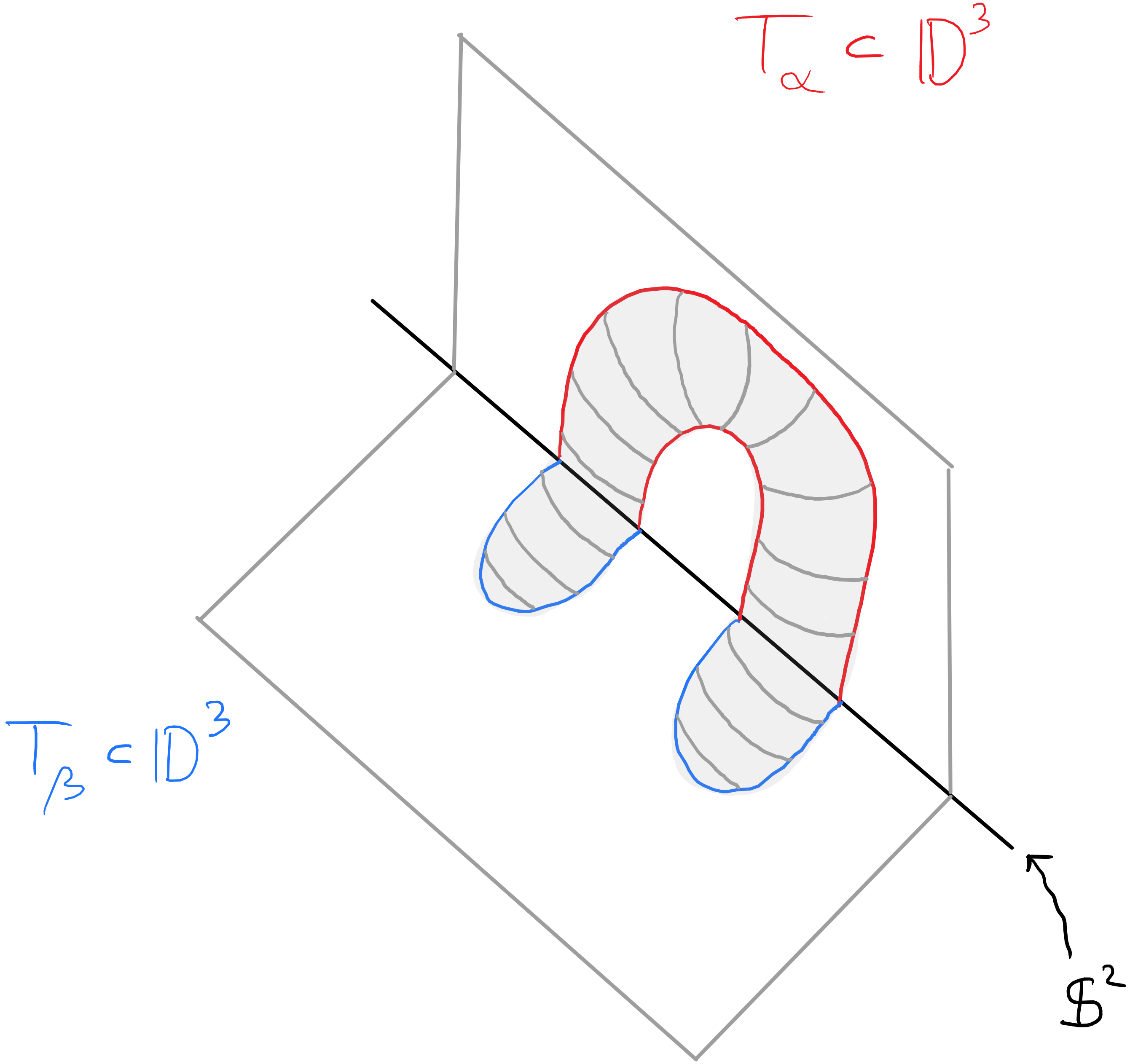
Endpoint-unions of trivial tangles form unlinks

$T_\alpha \cup_\partial T_\beta \subset \mathbb{D}^3 \cup_\partial \mathbb{D}^3 \cong \mathbb{S}^3$

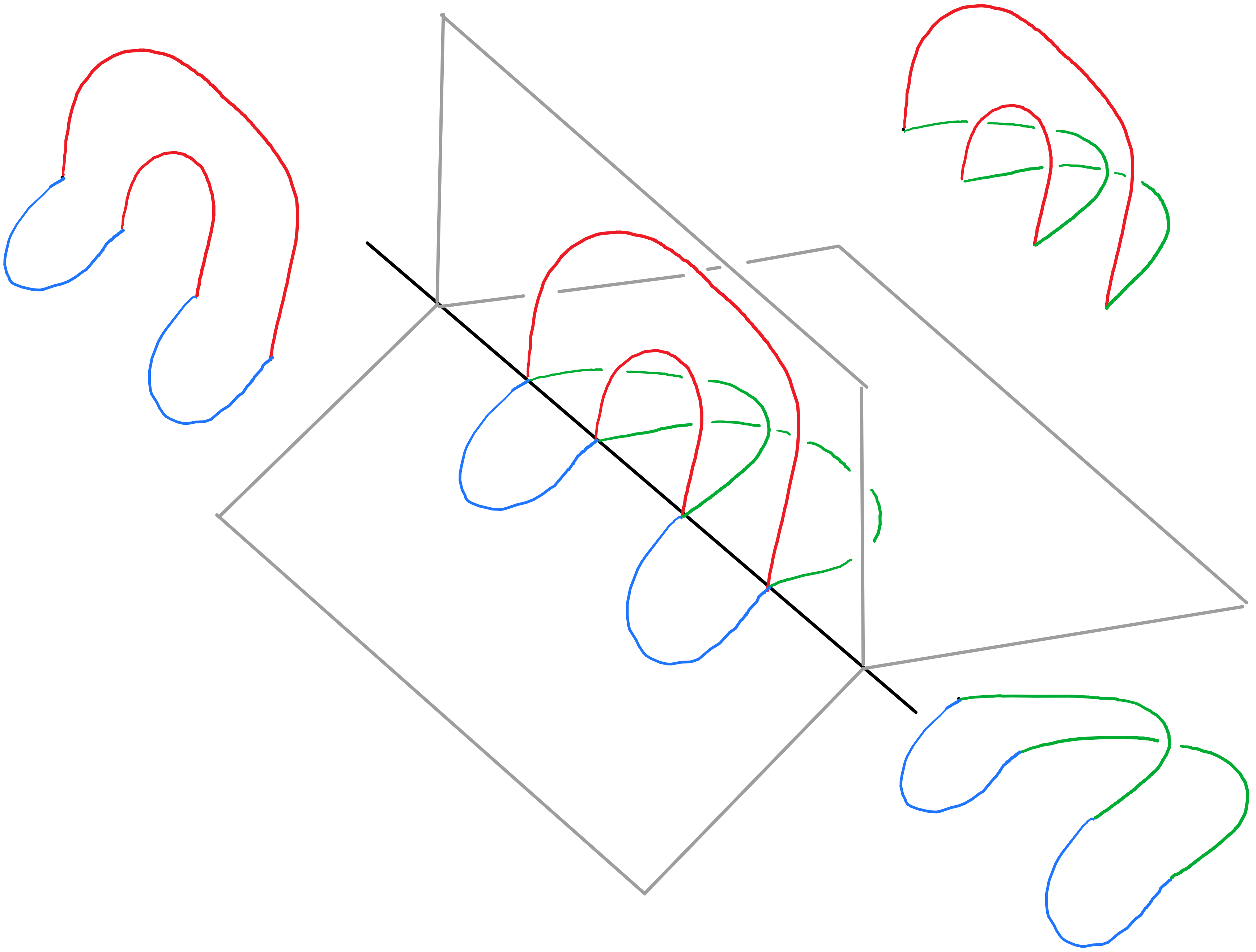
[Meier-Zupan]

Unlinks in S^3 bound (uniquely)

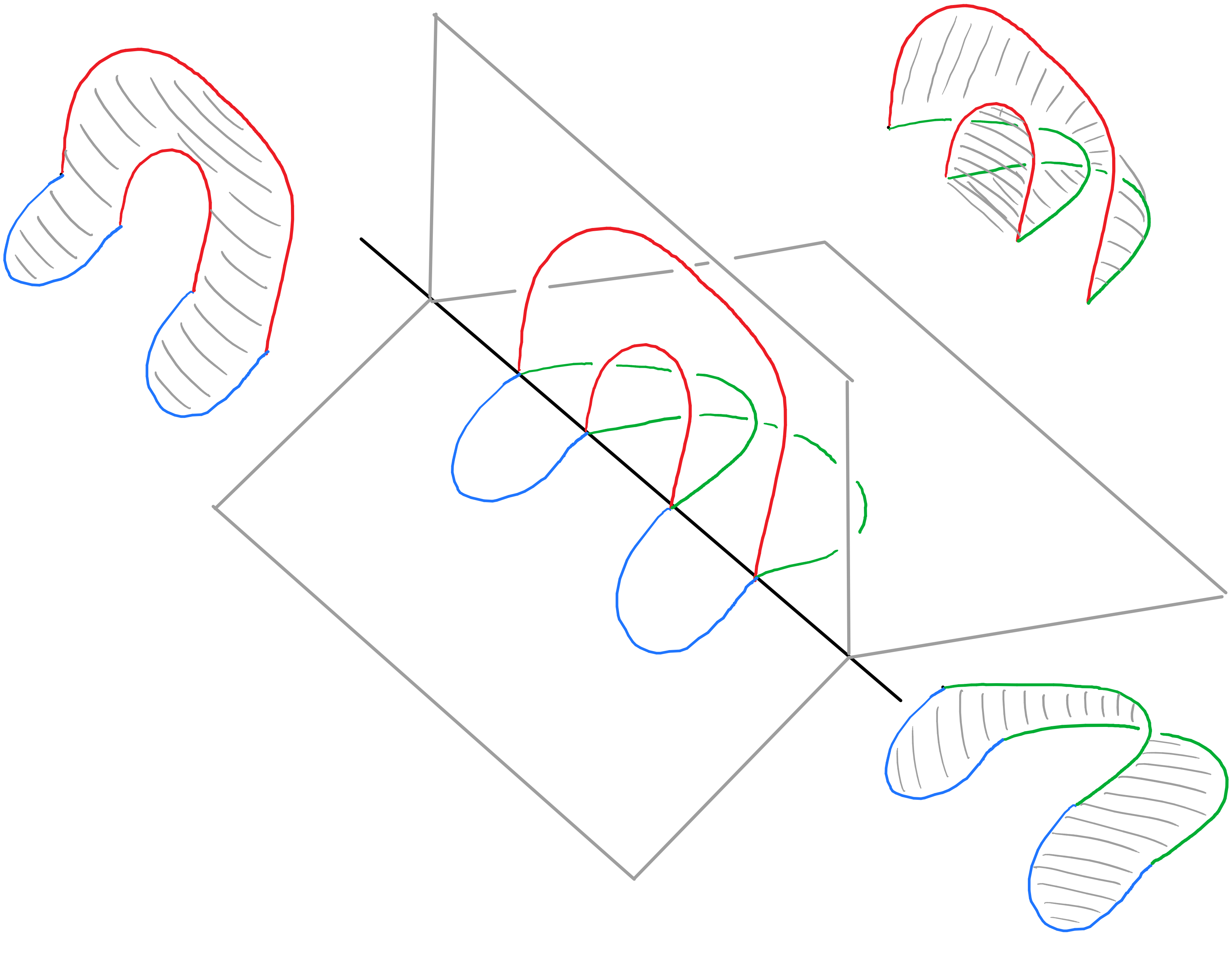
"undisks" in D^4



[Meier-Zupan]

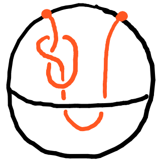


[Meier-Zupan]

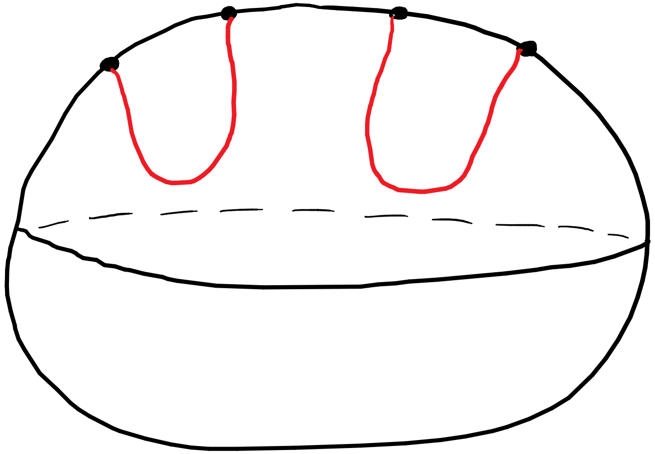


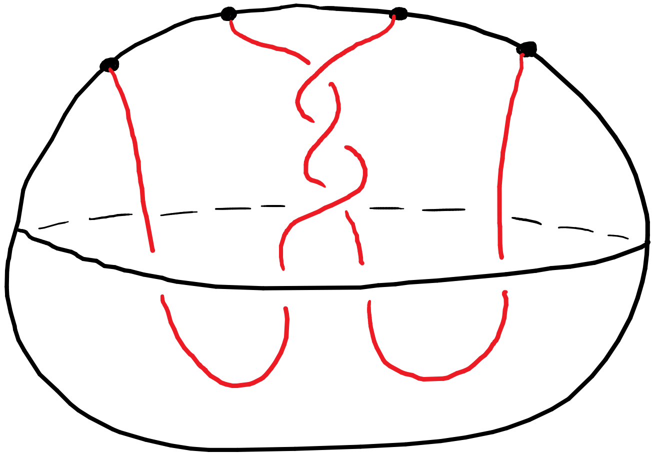
[Meier - Zupan]

Trivial tangles in 3-balls (and in handlebodies)

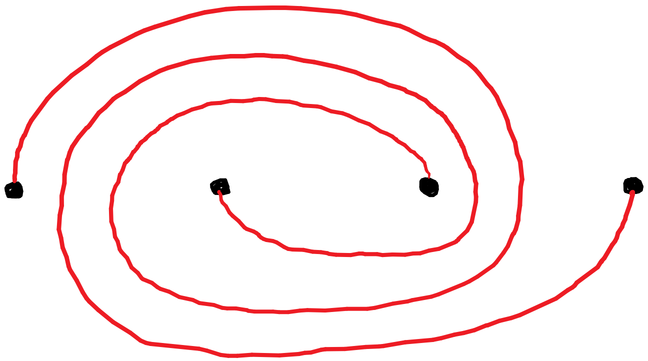


not a trivial tangle

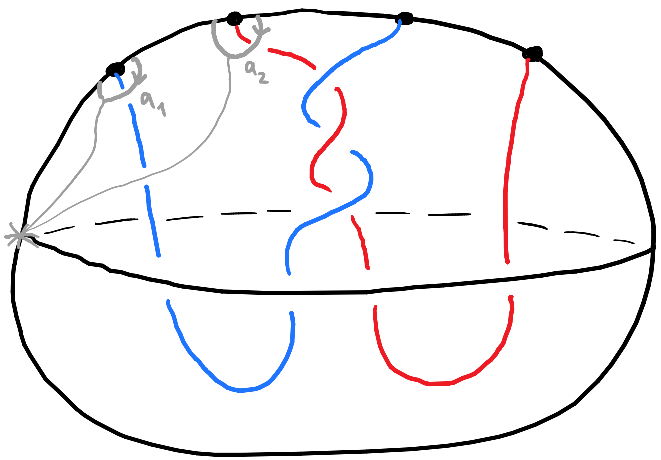


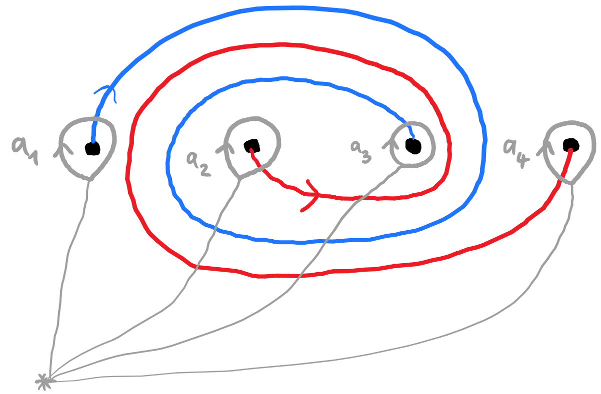


We like to draw the "shadows" of the tangles on a punctured plane:

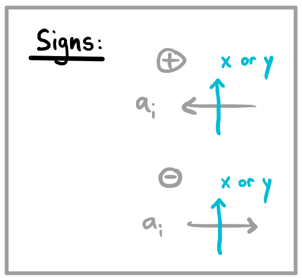


Topology





Algebra



$$\pi_1\left(\mathbb{S}^2 - \{\text{4 bridge points}\}\right) \longrightarrow \pi_1\left(\mathbb{D}^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

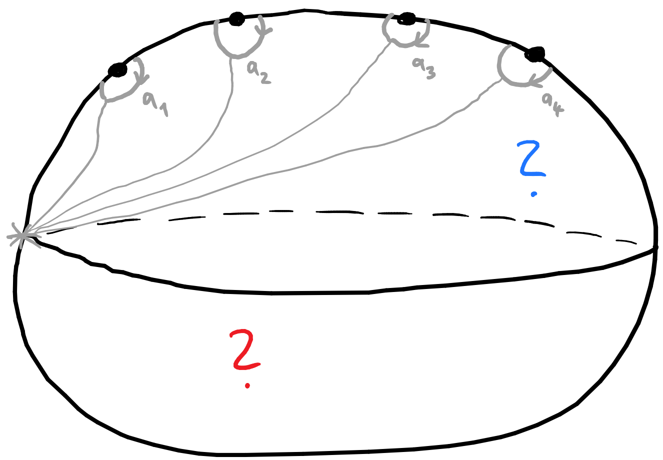
$$a_1 \longmapsto x^{-1}$$

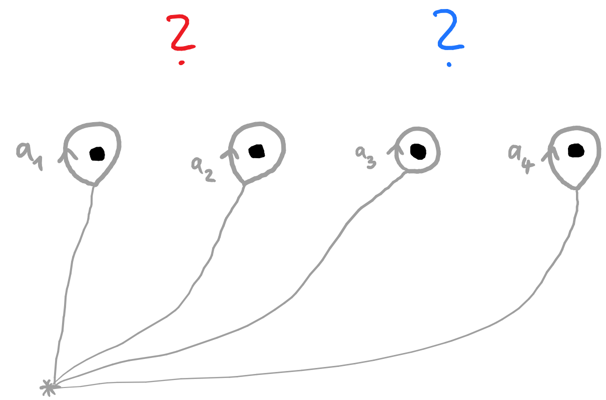
$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

$$a_4 \longmapsto y$$

Topology





Algebra

$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(\mathbb{D}^3 - \text{tangle})$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

$$a_1 \longmapsto x^{-1}$$

$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

$$a_4 \longmapsto y$$

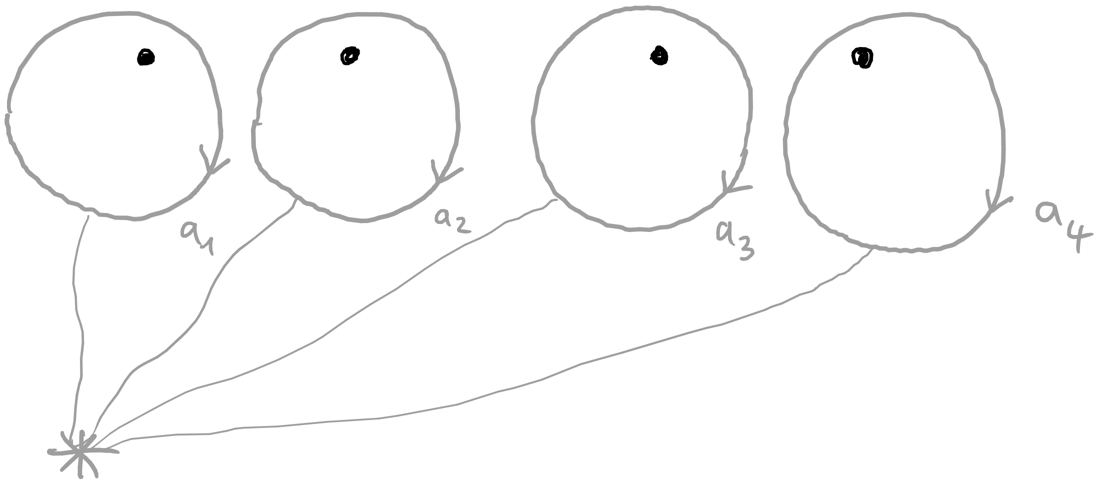
Punctured

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



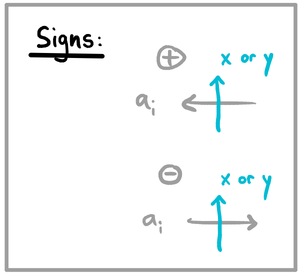
$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

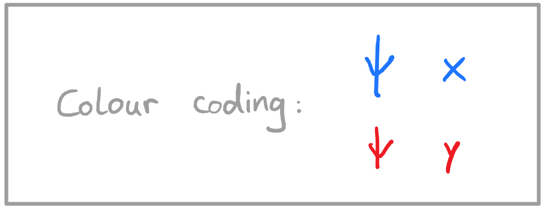
$$a_1 \longmapsto yxy^{-1}$$

$$a_2 \longmapsto yx^{-1}y^{-1}$$

$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$



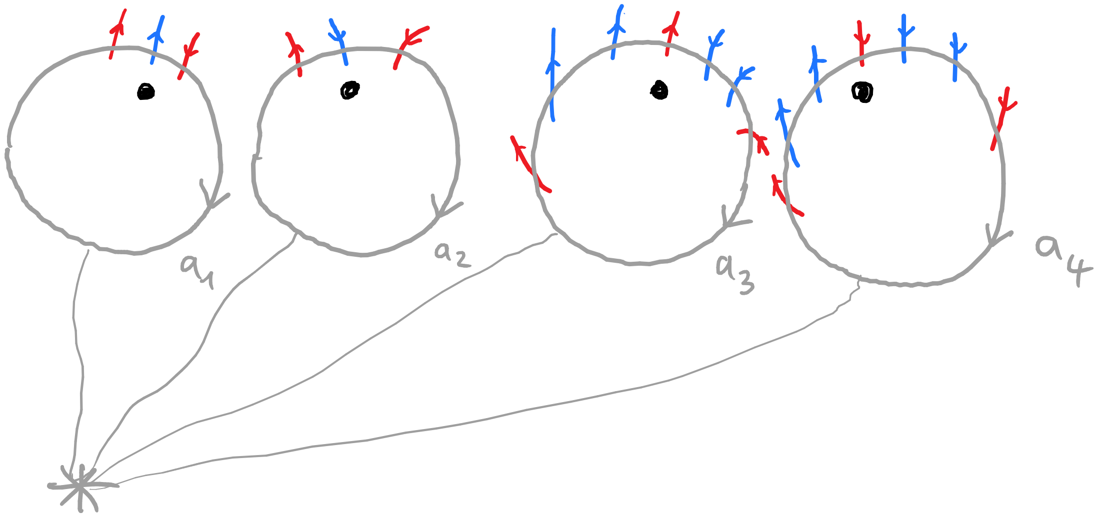


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

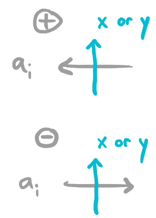
$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



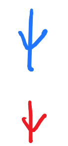
$$\pi_1\left(S^2 - \left\{ \begin{smallmatrix} 4 \text{ bridge} \\ \text{points} \end{smallmatrix} \right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto y x y^{-1}$$
$$a_2 \longmapsto y x^{-1} y^{-1}$$
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$
$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:



Colour coding:



x

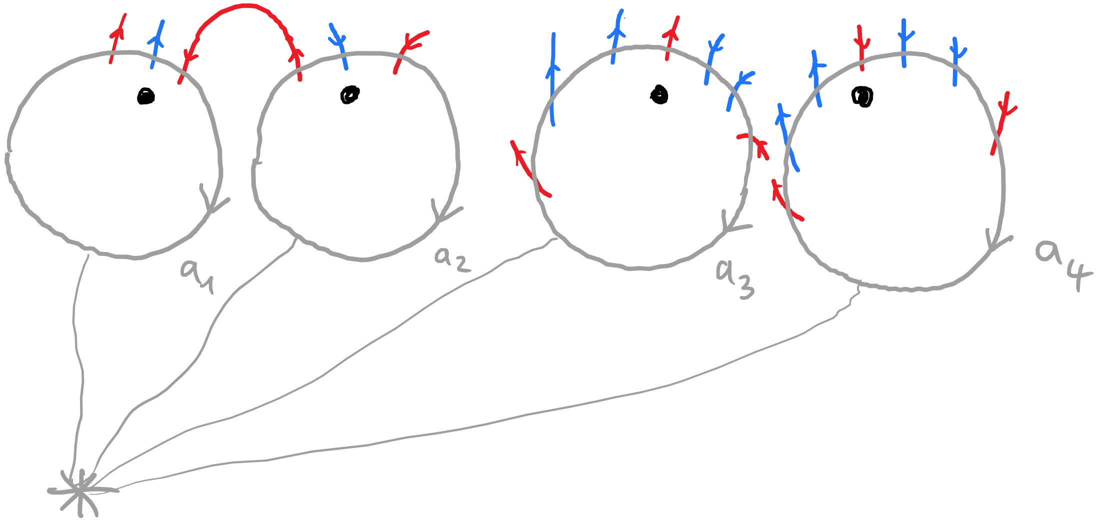
y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{matrix}4 \text{ bridge} \\ \text{points}\end{matrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

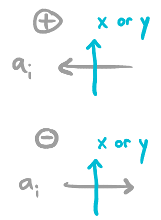
$$a_1 \longmapsto yxy^{-1}$$

$$a_2 \longmapsto yx^{-1}y^{-1}$$

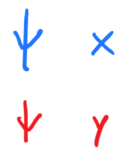
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$

$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



Colour coding:

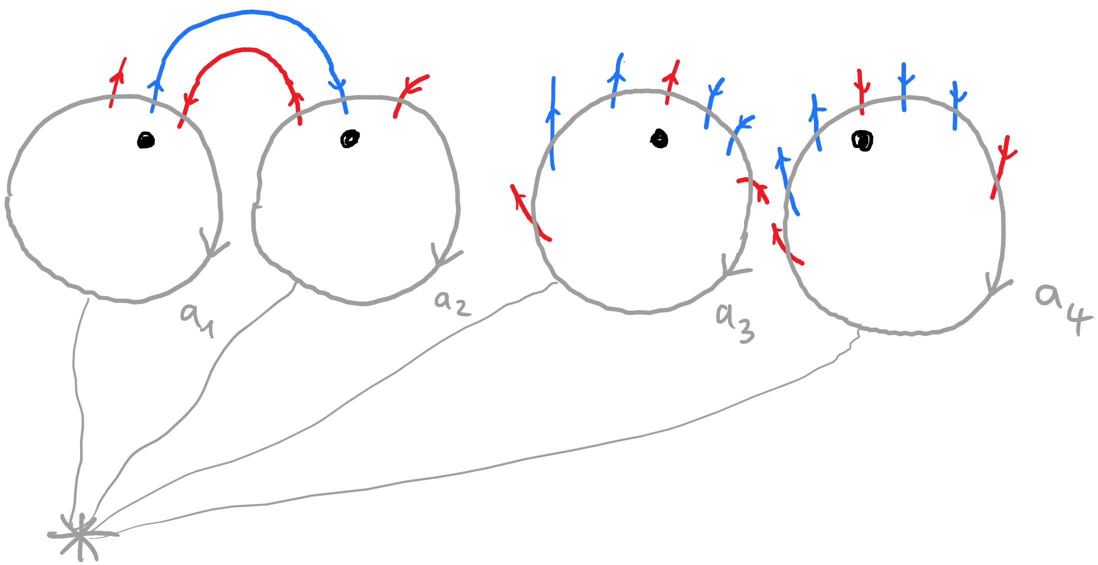


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

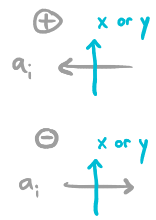
$$[\,y\,\cancel{x}\,\cancel{y^{-1}}\,]\,[\,\cancel{y}\,\cancel{x^{-1}}\,y^{-1}\,]\,[\,y\,x\,x\,y\,x^{-1}x^{-1}y^{-1}\,]\,[\,y\,x\,x\,y^{-1}\,x^{-1}x^{-1}y^{-1}\,]$$



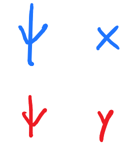
$$\pi_1\left(S^2 - \{\text{4 bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$a_1 \longmapsto y x y^{-1}$$
$$a_2 \longmapsto y x^{-1} y^{-1}$$
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$
$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:



Colour coding:

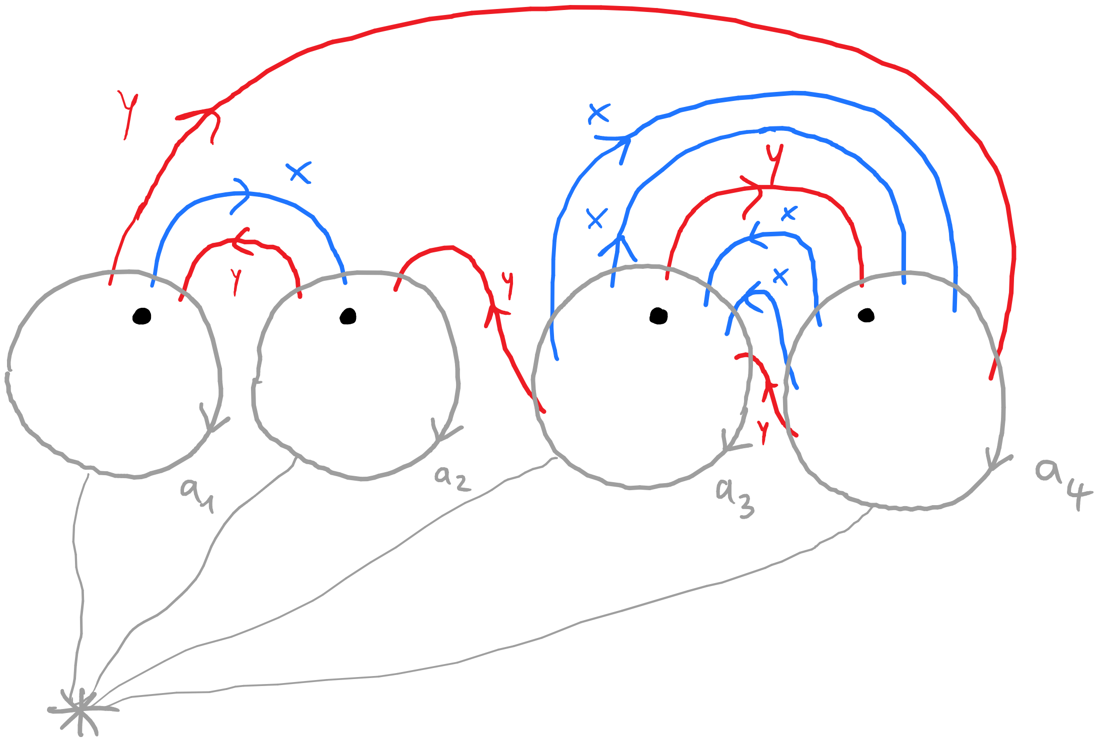


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{matrix}4 \text{ bridge}\\ \text{points}\end{matrix}\right\}\right) \longrightarrow \pi_1(\mathbb{D}^3 - \text{tangle})$$

$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:

$\oplus$ x or y, a_i

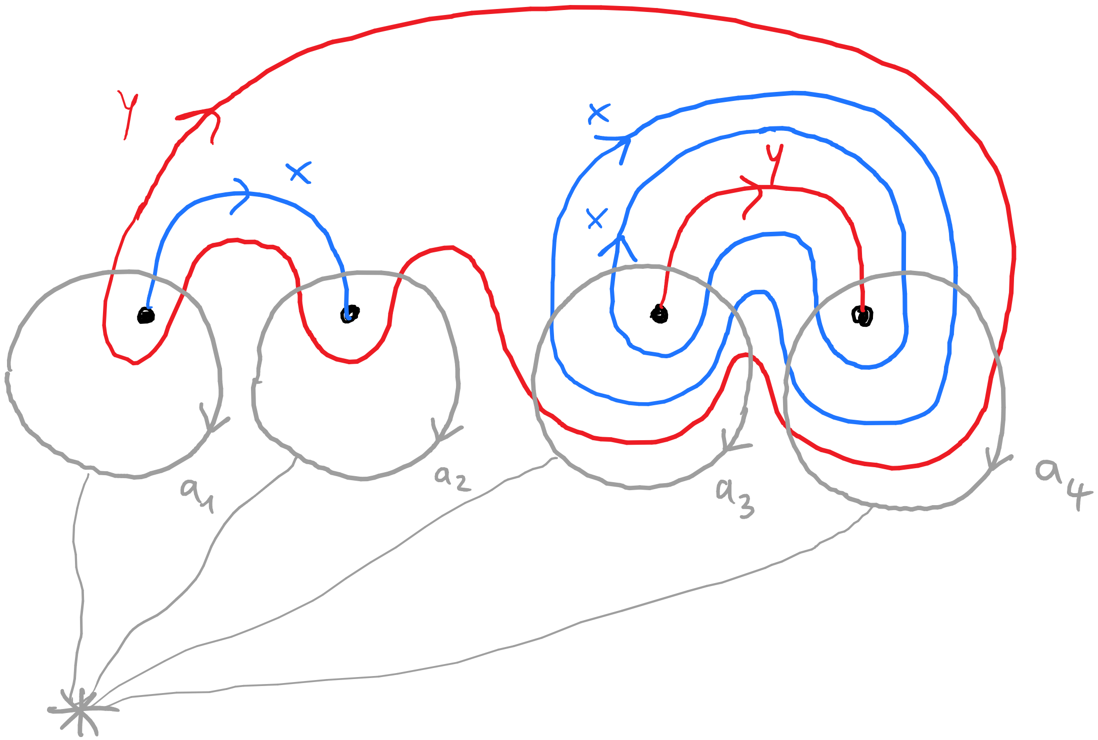
$\ominus$ x or y, a_i

Colour coding: x (blue), y (red)

Surface relation: $\quad a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



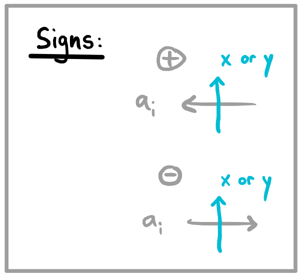
$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(\mathbb{D}^3 - \text{tangle})$$

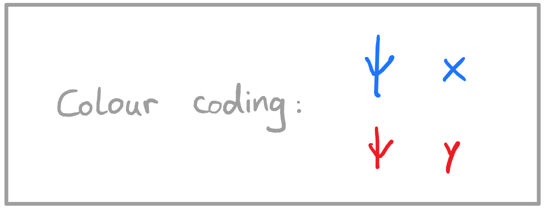
$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

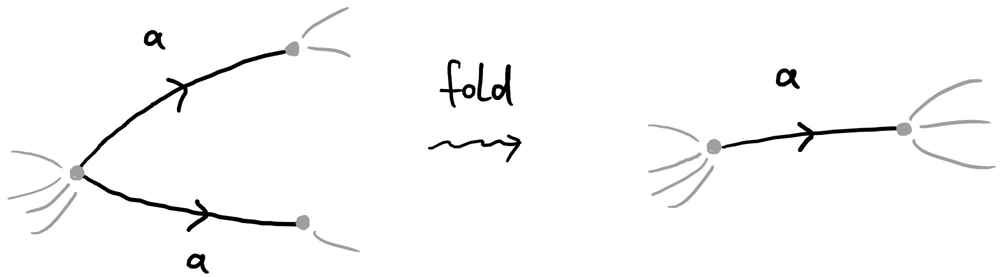
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$



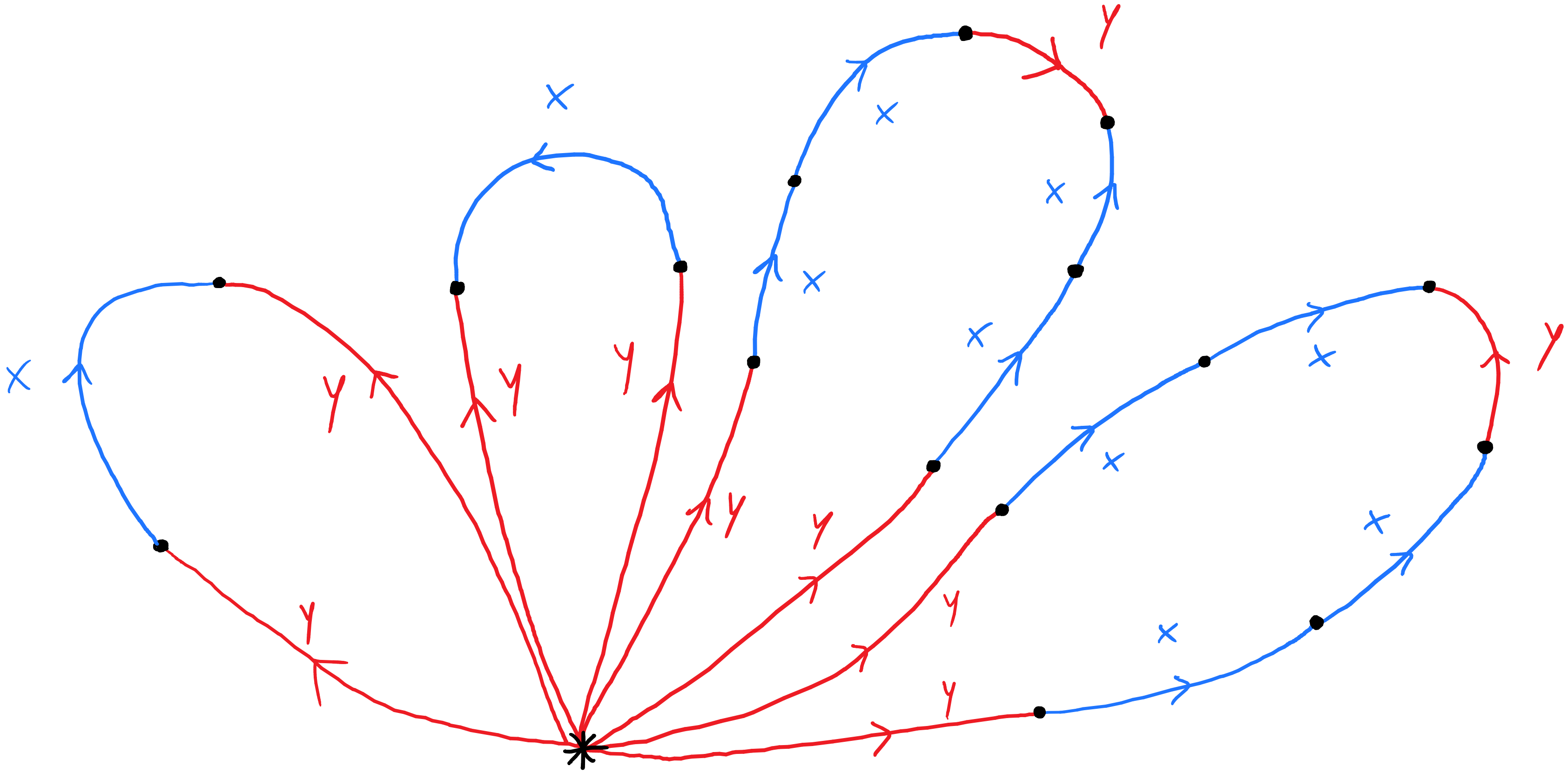


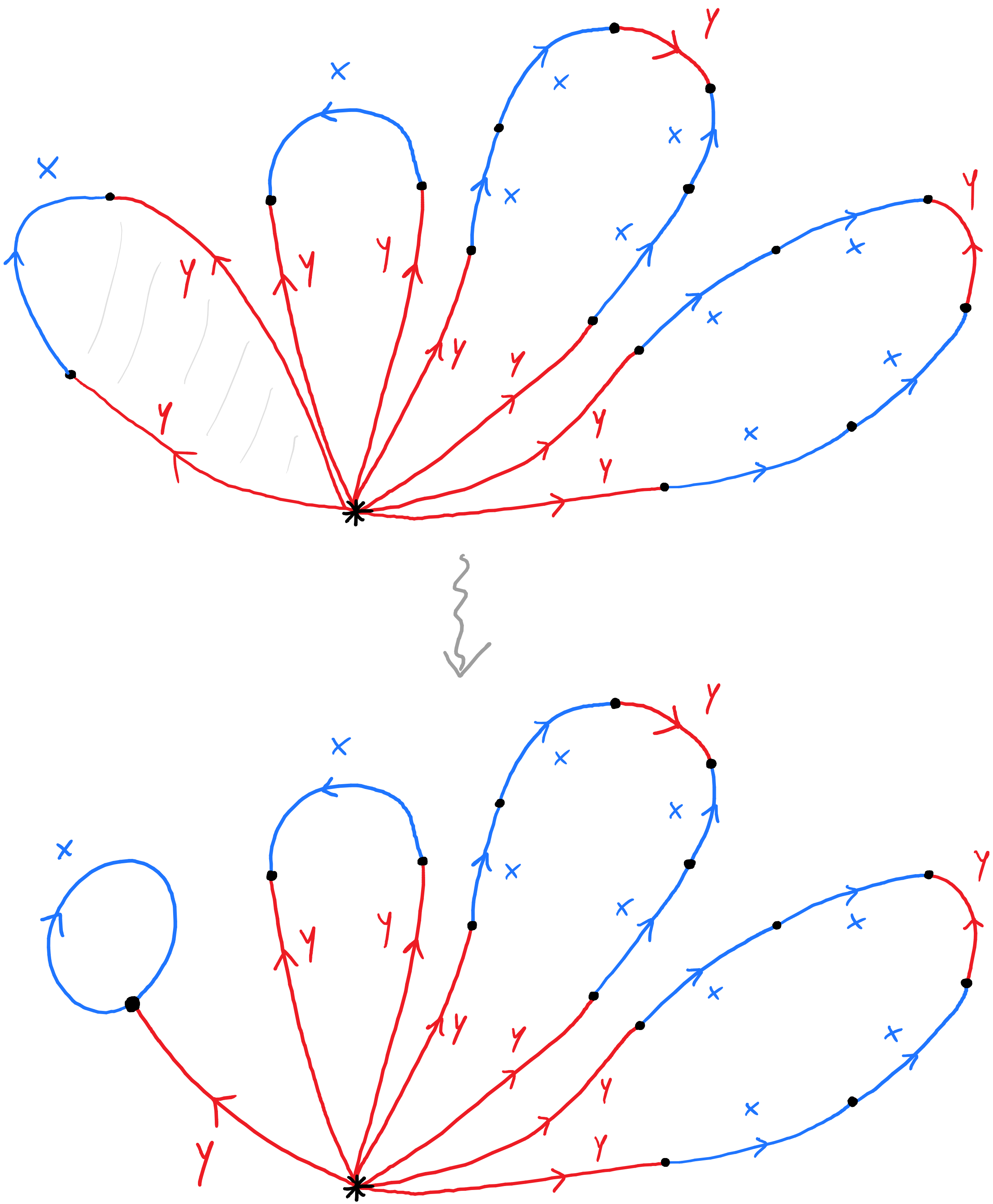
If there are closed circle components, we use band sums guided by <u>Stallings folding</u>

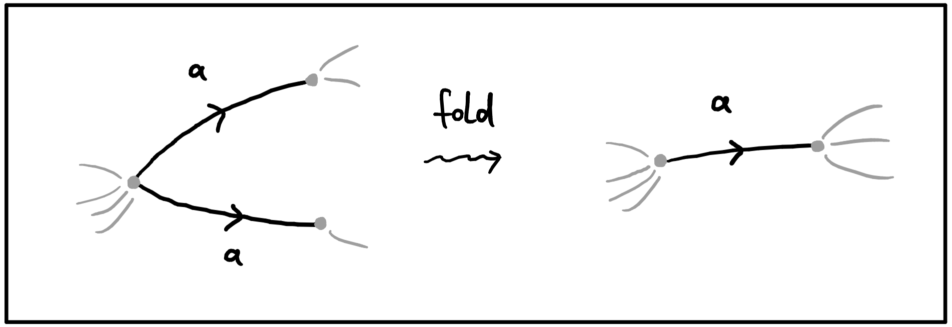


We would like to **check** whether $\langle yxy^{-1},\ yx^{-1}y^{-1},\ yxxyx^{-1}x^{-1}y^{-1},\ yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$

generates the free group $\langle x, y \rangle$





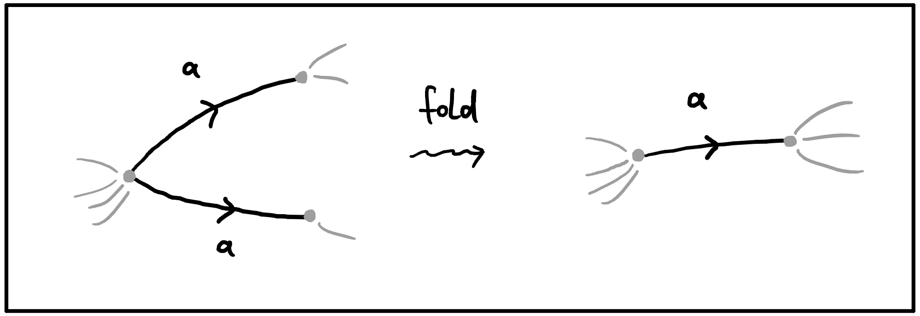


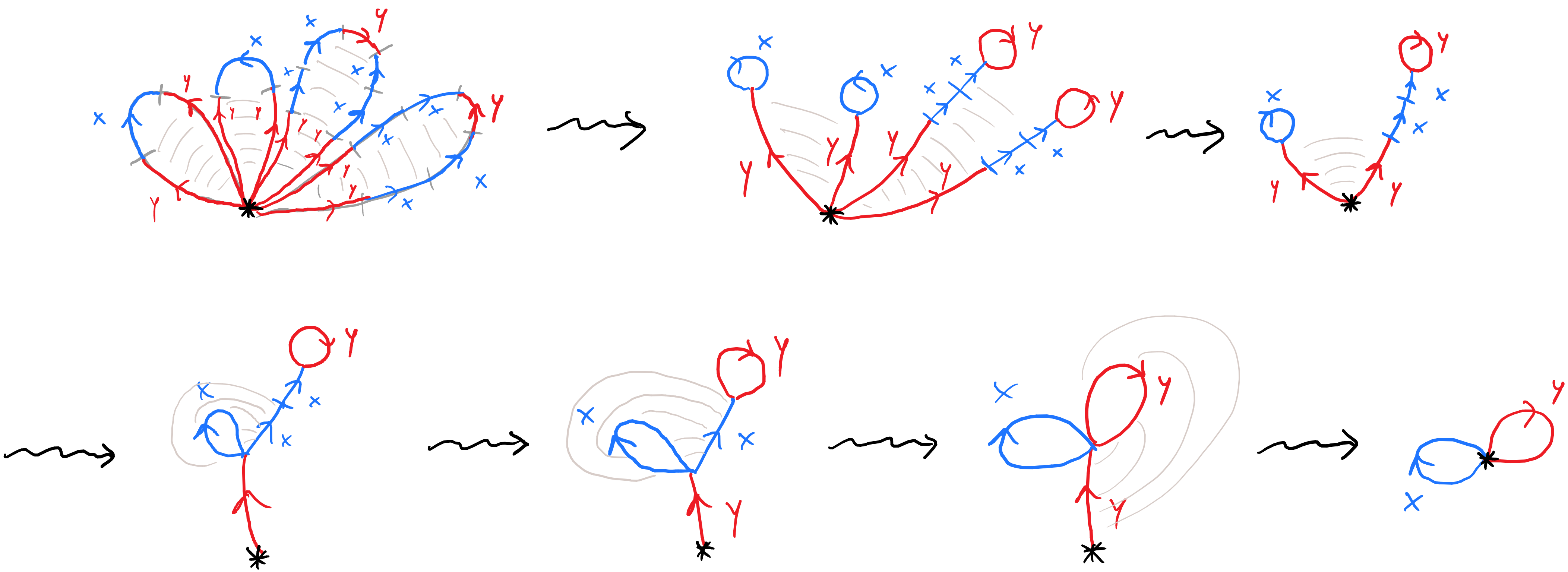
[Stallings]

Sequence of folds which show that

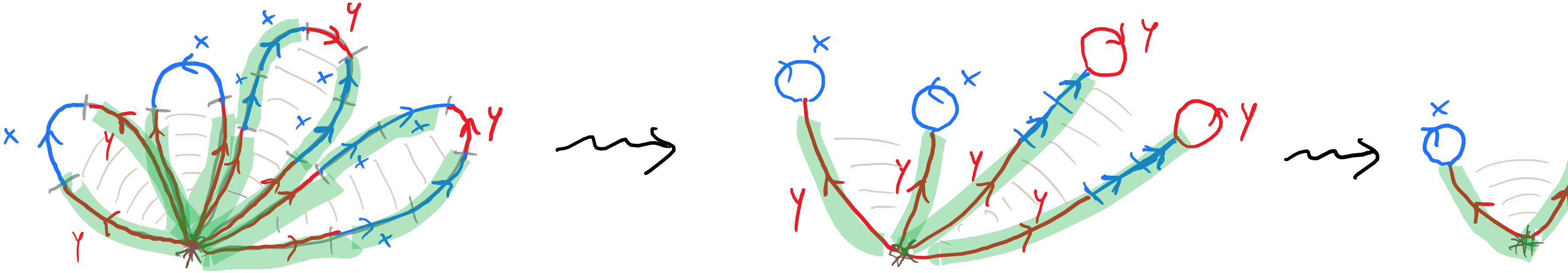
$$\langle yxy^{-1},\ yx^{-1}y^{-1},\ yxxyx^{-1}x^{-1}y^{-1},\ yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$$

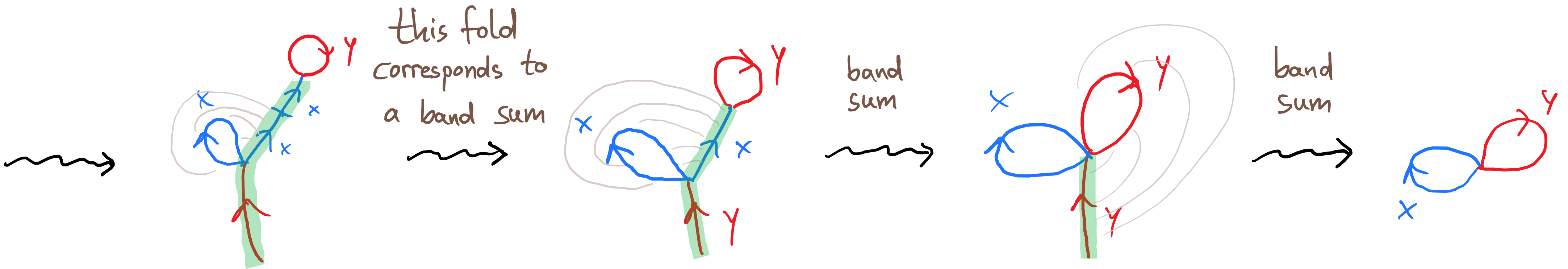
generates the free group $\langle x, y \rangle$

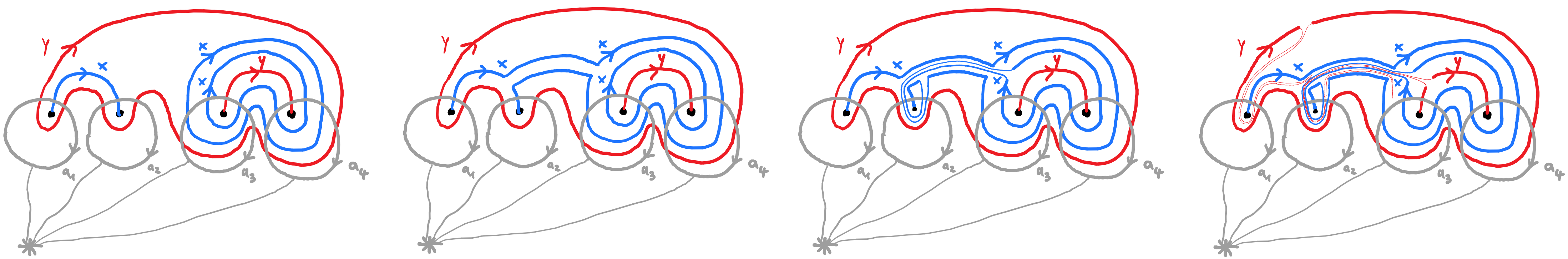




[Stallings]

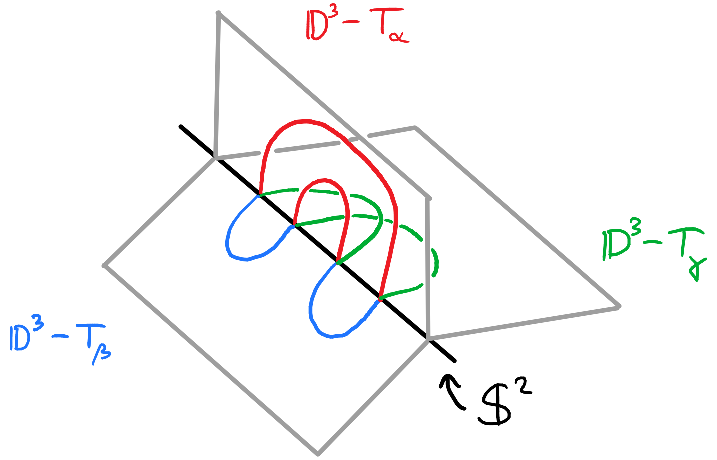

x
y


this fold corresponds to a band sum
band sum
band sum
x
y


y
x
a_1
a_2
a_3
a_4

(based, parameterized)

bridge trisections of a smoothly knotted surface $K^2 \subset S^4$



take π_1 of pieces

1 : 1

[Blackwell–Kirby–Klug–Longo–R, 2021]

trisected knotted surface group $\pi_1(S^4 - K)$

$$
\begin{tikzcd}
 & \pi_1(D^3 - T_\alpha) \arrow[rr, ->>] \arrow[dd, ->>] & & \overset{\text{free}}{\pi_1(S^3 - L_{\alpha\beta})} \arrow[dd, ->>] \\
\pi_1\left(S^2 - \{\text{26 bridge points}\}\right) \arrow[ur, ->>] \arrow[rr, ->>] \arrow[dd, ->>] & & \pi_1(D^3 - T_\beta) \arrow[ur, ->>] \arrow[dd, ->>] & \\
 & \underset{\text{free}}{\pi_1(S^3 - L_{\alpha\gamma})} \arrow[rr, ->>] & & \pi_1(S^4 - K) \\
\pi_1(D^3 - T_\gamma) \arrow[ur, ->>] \arrow[rr, ->>] & & \underset{\text{free}}{\pi_1(S^3 - L_{\beta\gamma})} \arrow[ur, ->>] &
\end{tikzcd}
$$

all faces are push-outs & conditions apply

We take inspiration from:

-) [Stallings: How not to prove the Poincaré conjecture (1965)]
-) [Jaco: Heegaard splittings and splitting homomorphisms (1968)]

  [Jaco: Stable equivalence of splitting homomorphisms (1970)]
-) [Abrams, Gay, Kirby: Group trisections and smooth 4-manifolds (2018)]

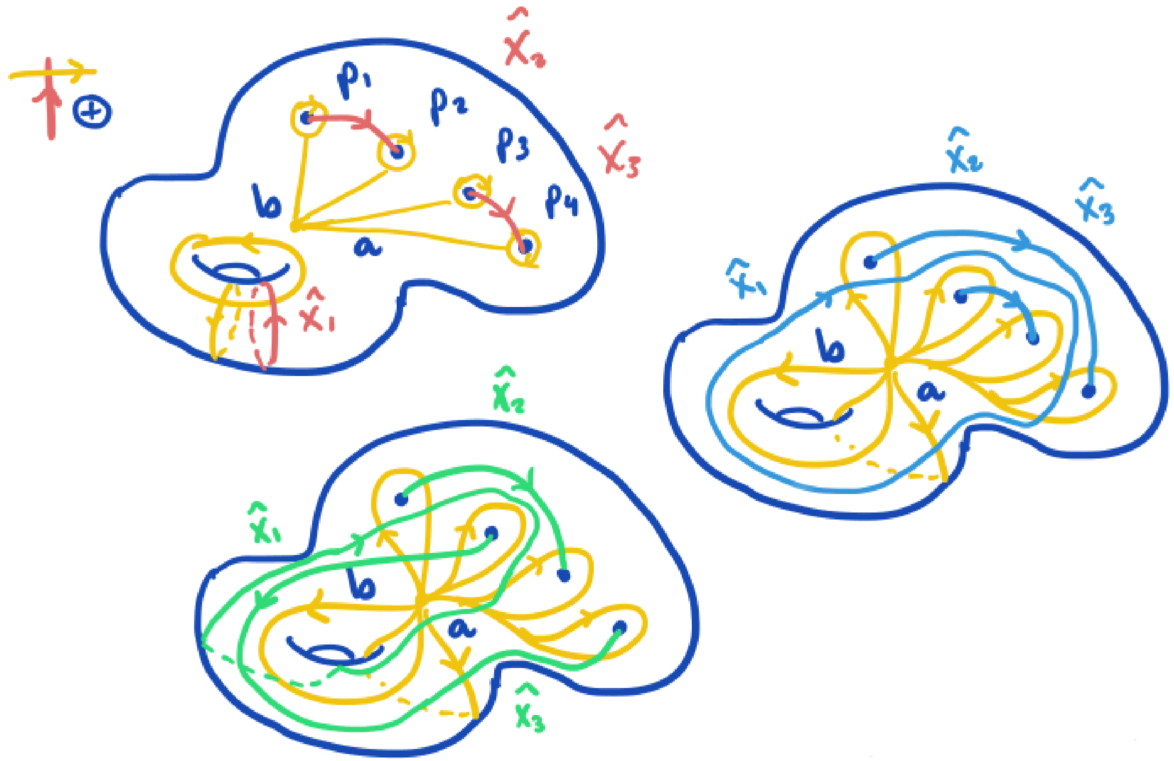
Thanks!

Example of a bridge trisected surface in a trisected 4-manifold:

bridge position of real $\mathbb{R}P^2$

$\cap$

genus 1 trisection of $\mathbb{C}P^2$



corresponding group trisection

$a \mapsto 1$	$a \mapsto x_1$	$a \mapsto \bar{x}_1 x_3$
$b \mapsto x_1$	$b \mapsto 1$	$b \mapsto x_1$
$p_1 \mapsto x_2$	$p_1 \mapsto \bar{x}_1 x_2 x_1$	$p_1 \mapsto x_3 \bar{x}_1 x_2 x_1 \bar{x}_3$
$p_2 \mapsto \bar{x}_2$	$p_2 \mapsto x_3$	$p_2 \mapsto x_3$
$p_3 \mapsto x_3$	$p_3 \mapsto \bar{x}_3$	$p_3 \mapsto \bar{x}_1 \bar{x}_2 x_1$
$p_4 \mapsto \bar{x}_3$	$p_4 \mapsto \bar{x}_1 \bar{x}_2 x_1$	$p_4 \mapsto \bar{x}_1 \bar{x}_3 x_1$