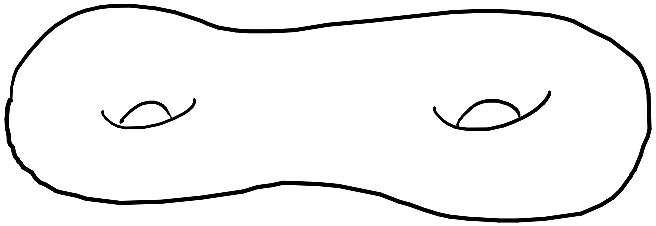
Top Flavours 2021, 17–18 June 2021, 25 min talk

Group trisections and smoothly knotted surfaces

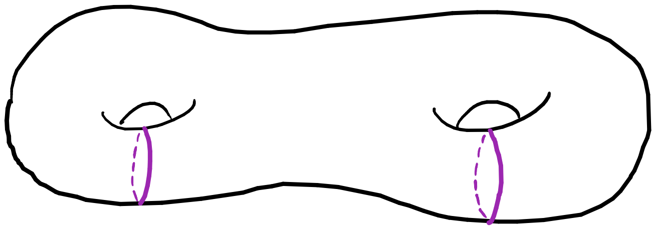
with Sarah Blackwell, Rob Kirby, Michael Klug and Vincent Longo

Benjamin Matthias Ruppik, 3rd year PhD student at the Max-Planck-Institute for Mathematics, Bonn

Handlebodies:



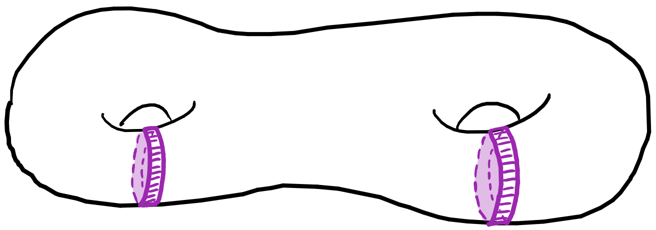
surface Σ_g



cut system of a handlebody:

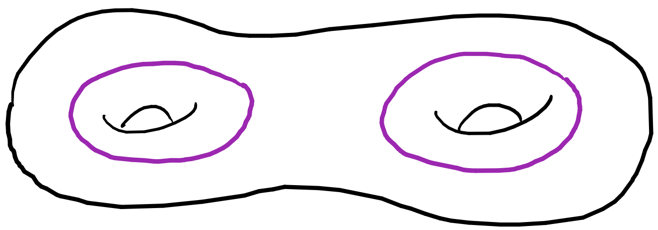
curves on Σ_g

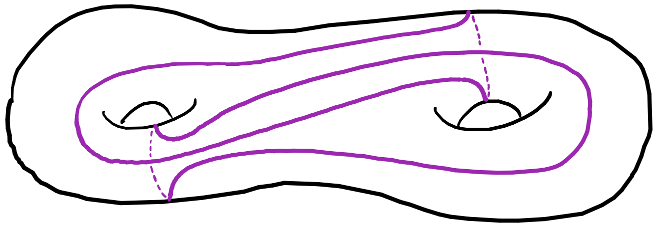
attach 2-handles along the curves

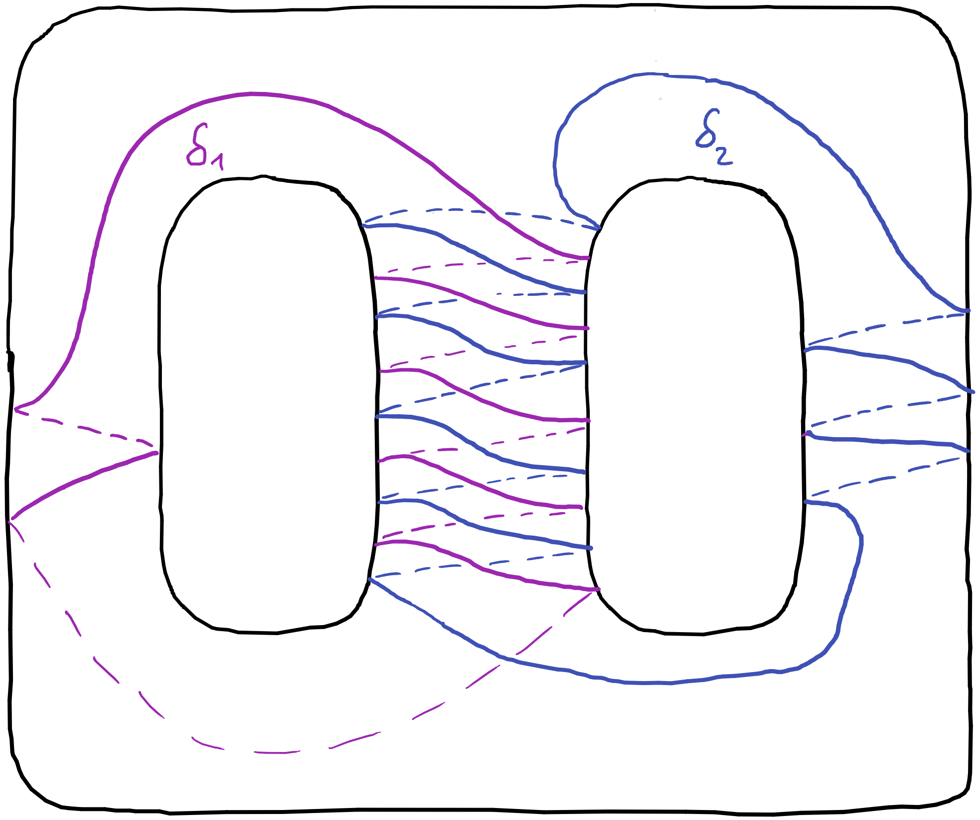


fill 2-sphere boundaries with 3-balls

Can you see the handlebodies?



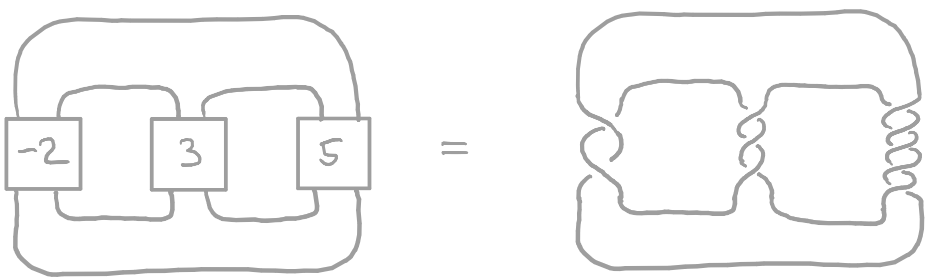


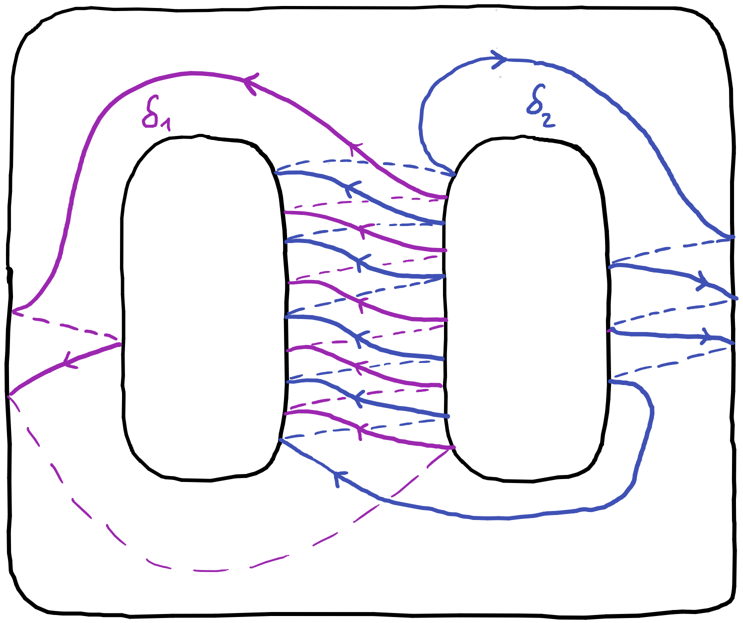


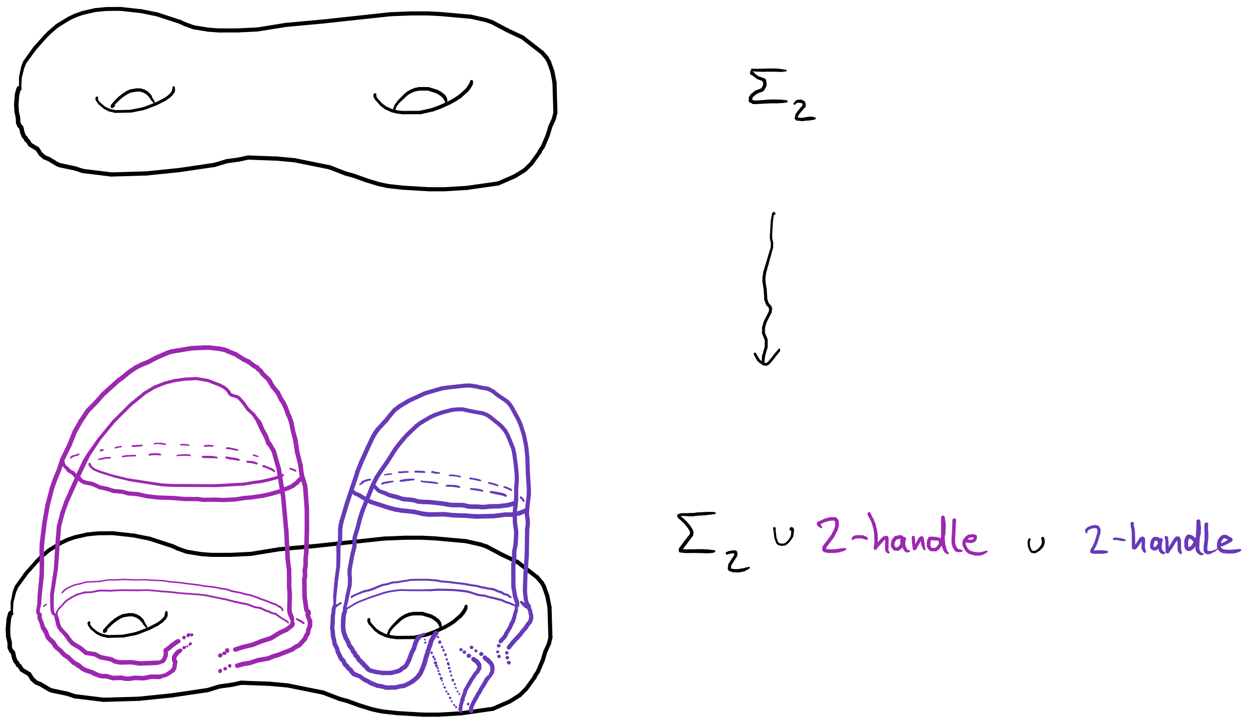
Side remark: This is one of the handlebodies in a genus 2 Heegaard diagram for the 3-mfld. P = Poincaré homology sphere

P = double branched cover $\Sigma_2(K)$ of S^3 branched over

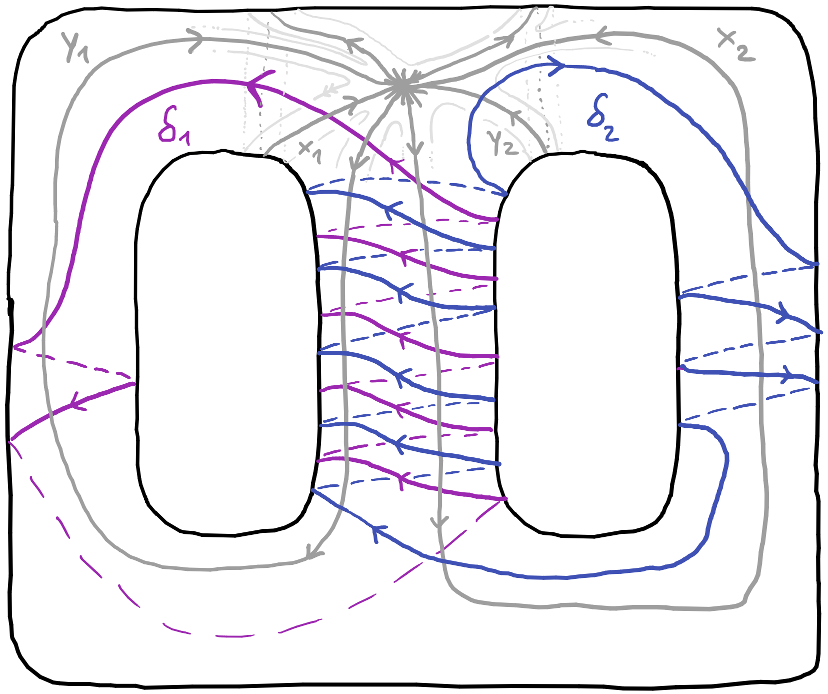
$K = (-2,3,5)$ Pretzel knot




δ_1
δ_2


Σ_2
$\Sigma_2 \cup$ 2-handle $\cup$ 2-handle

Topology



Algebra

$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

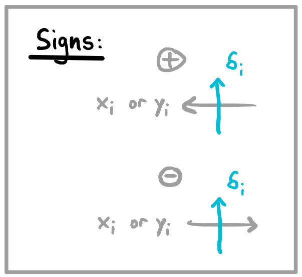
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

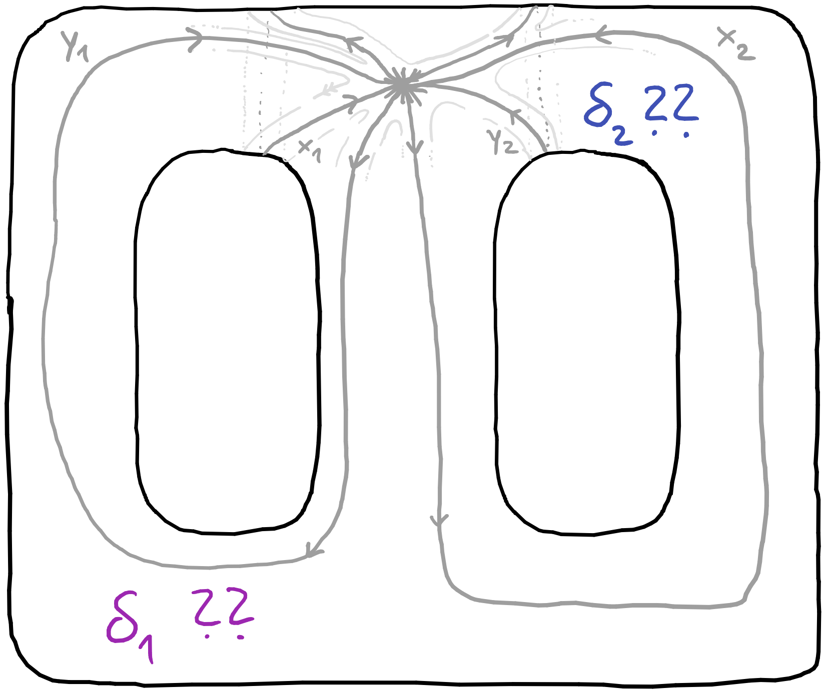
$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$$

$$y_2 \longmapsto d_2$$



Topology



Algebra

$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

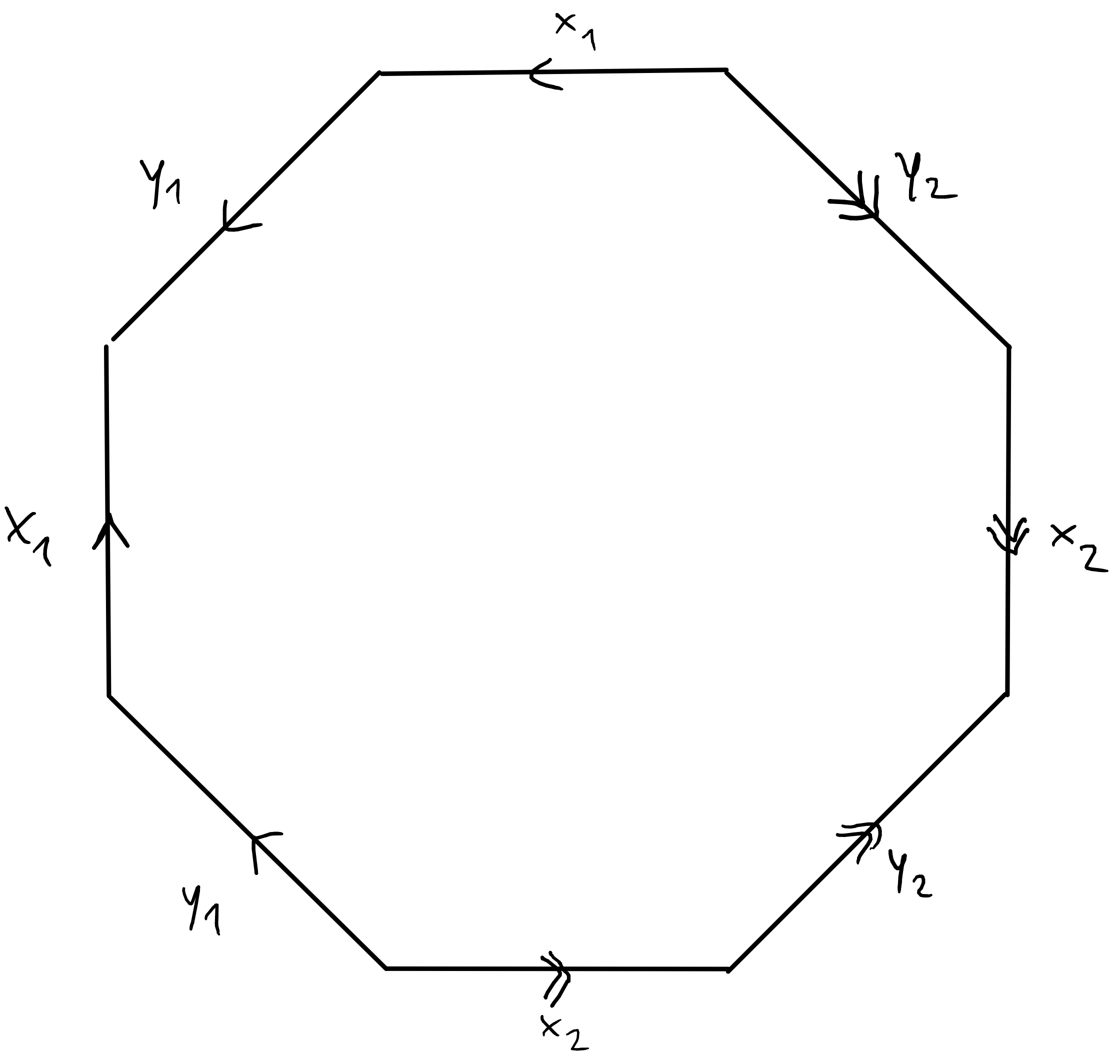
$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^{3}$$

$$y_2 \longmapsto d_2$$

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} \, x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



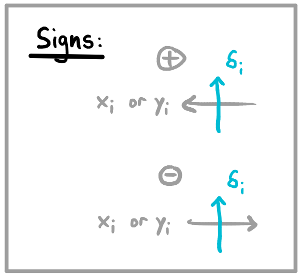
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^{3}$$

$$y_2 \longmapsto d_2$$

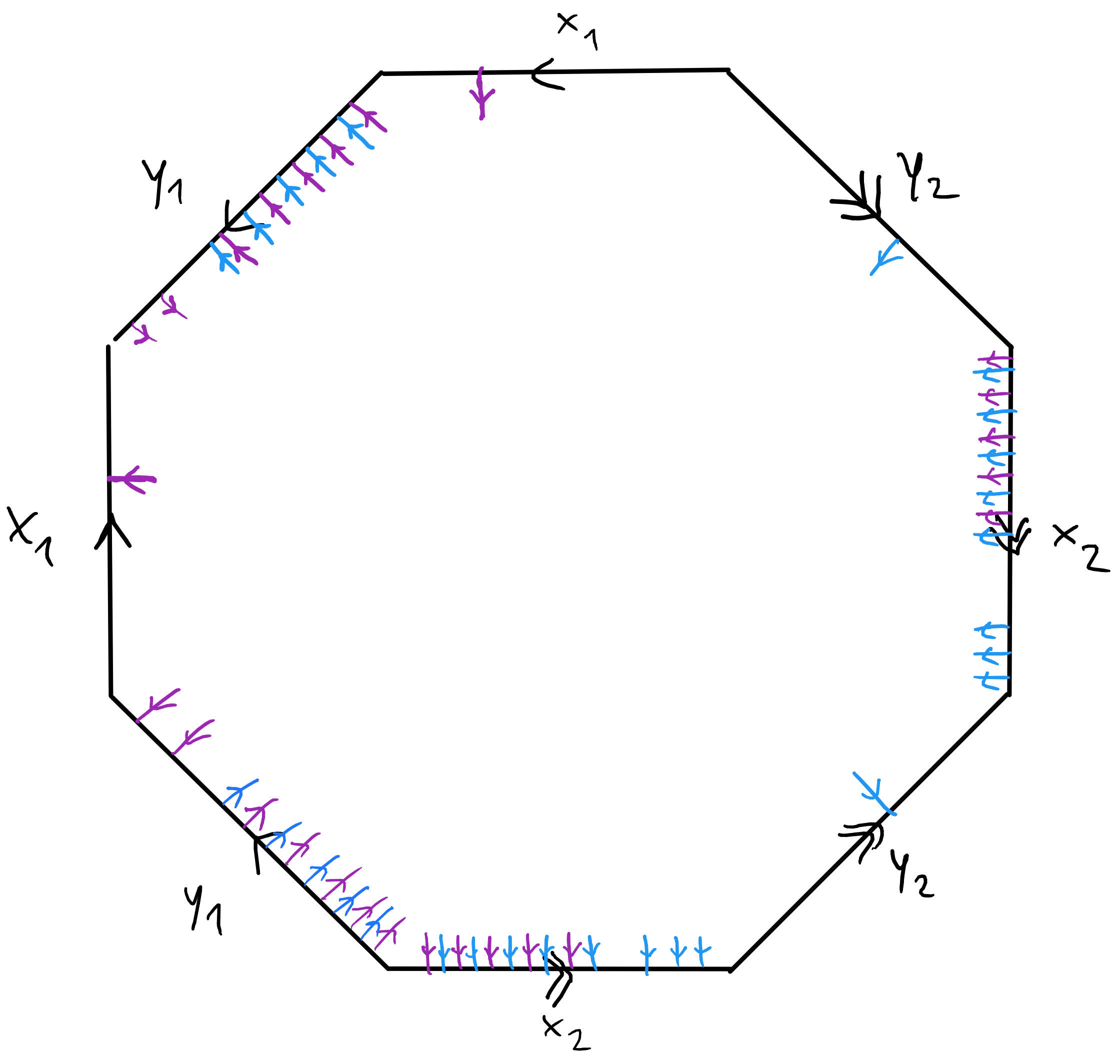


Colour coding: d_1 d_2

Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\Big\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



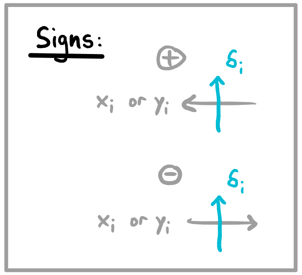
$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$$

$$y_2 \longmapsto d_2$$

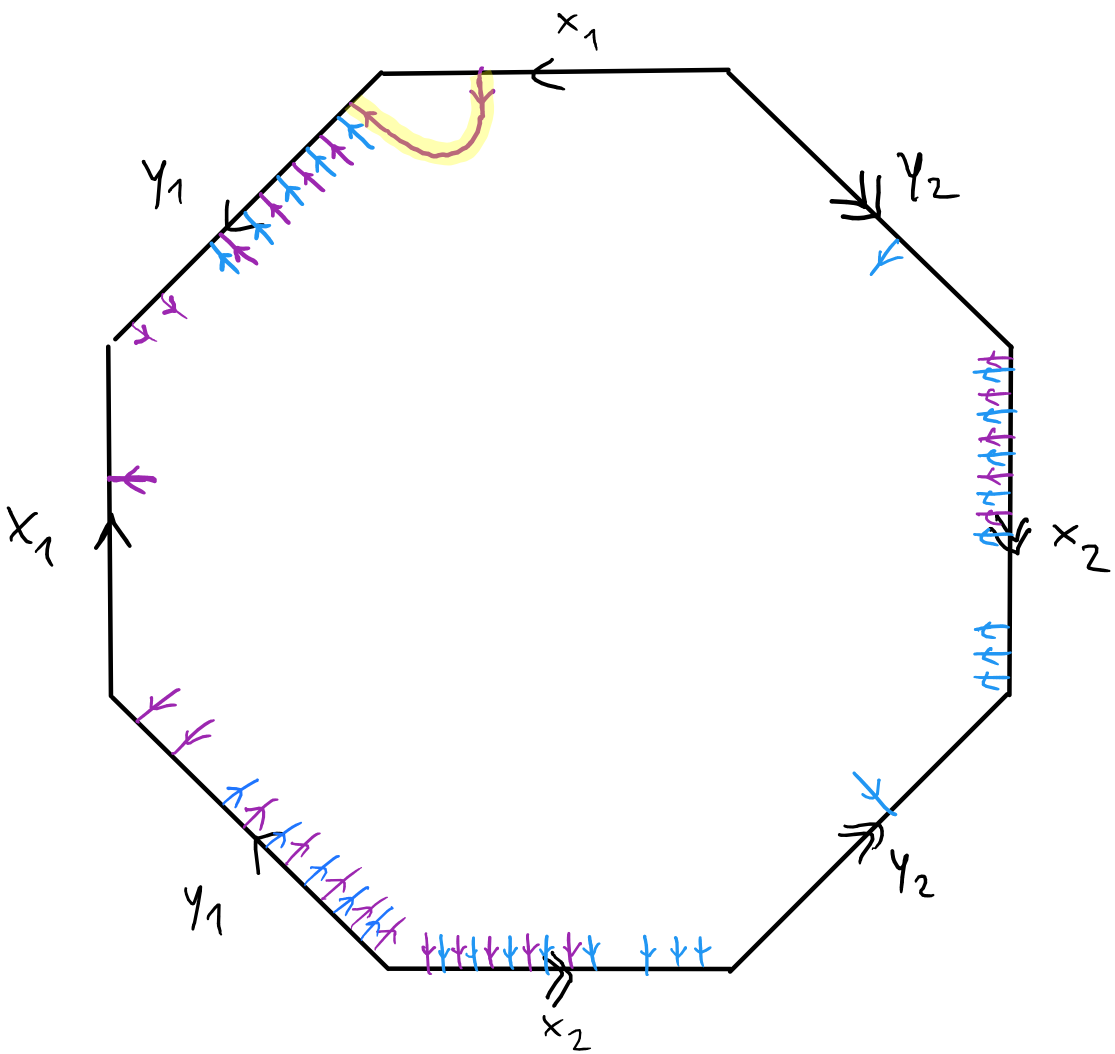


Colour coding: d_1 d_2

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[\cancel{d_1^{-1}}][(\cancel{d_1} d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

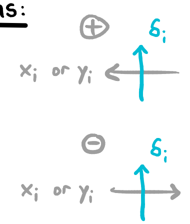
$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$$

$$y_2 \longmapsto d_2$$

Signs:

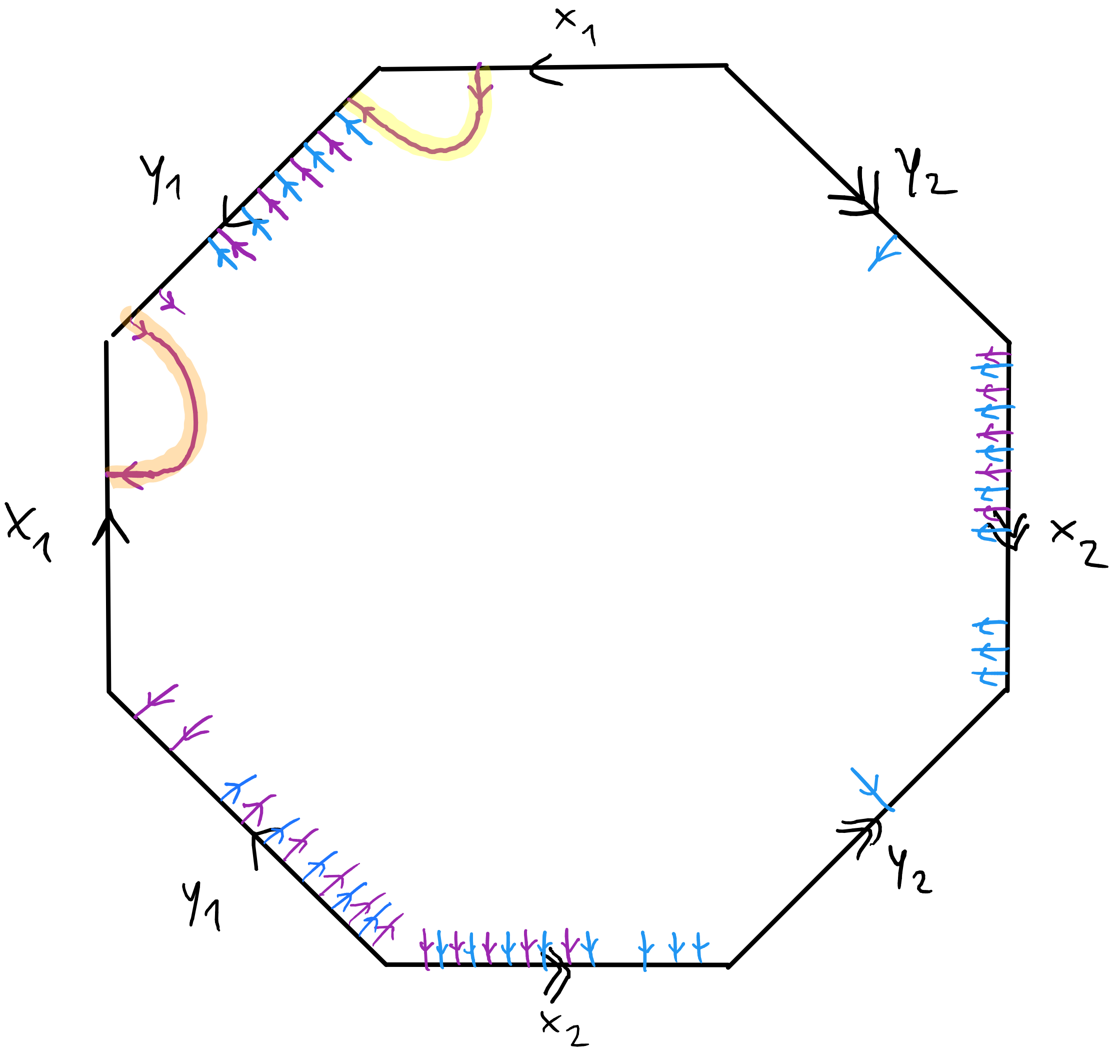


Colour coding: d_1 (purple), d_2 (blue)

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[\cancel{d_1^{-1}}][(\cancel{d_1} d_2)^5 d_1^{-2}][\cancel{d_1}][\cancel{d_1}\,{}^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$$

$$y_2 \longmapsto d_2$$

Signs:

$\oplus$ $\quad \delta_i$ $\quad x_i$ or y_i

$\ominus$ $\quad \delta_i$ $\quad x_i$ or y_i

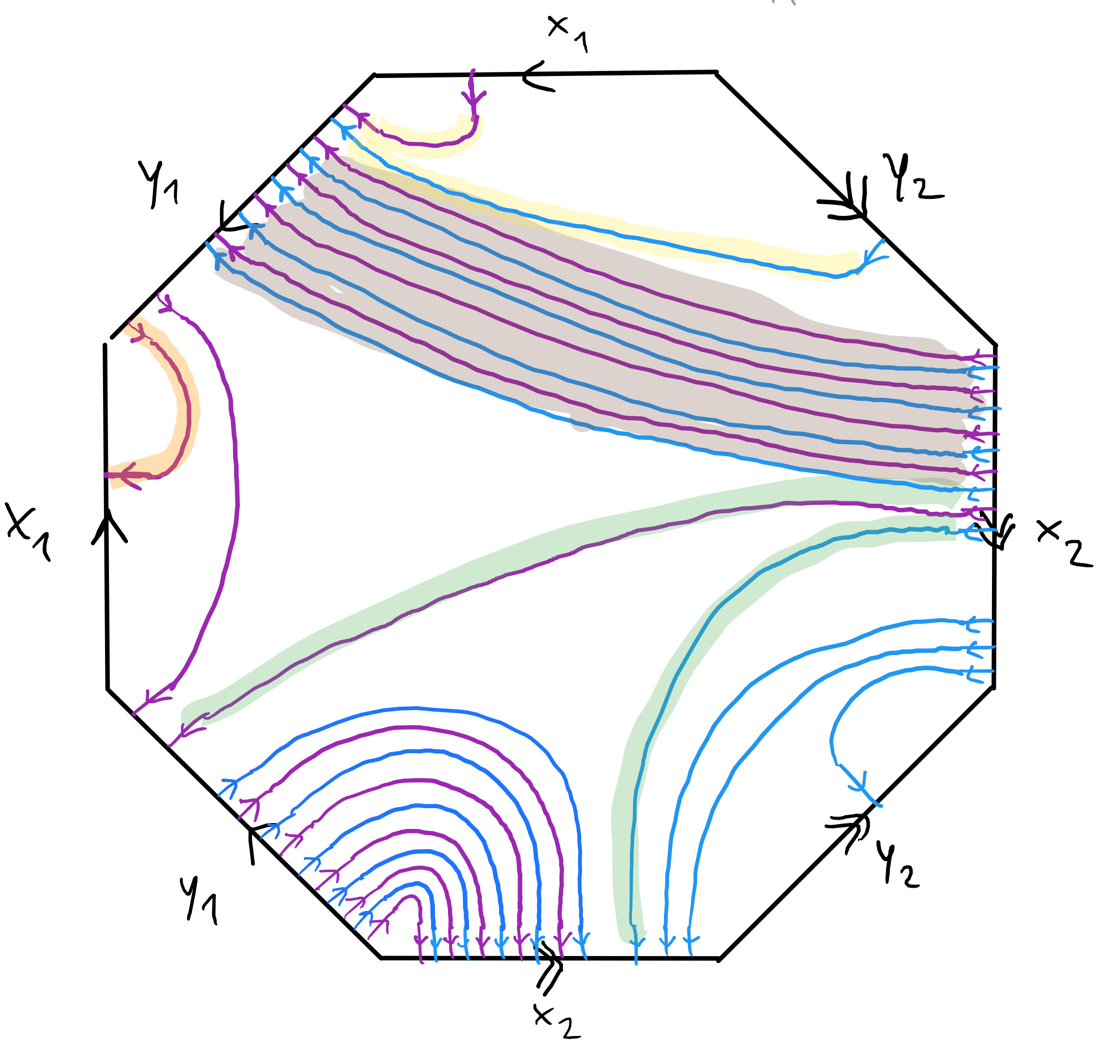
Colour coding: $\quad d_1 \quad d_2$

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\big\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^2 (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$

$$x_1 \qquad y_1 \qquad x_1^{-1} \qquad y_1^{-1} \qquad x_2 \qquad y_2 \qquad x_2^{-1} \qquad y_2^{-1}$$



$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$

$x_1 \longmapsto d_1^{-1}$

$y_1 \longmapsto (d_1 d_2)^5 d_1^{-2}$

$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$

$y_2 \longmapsto d_2$

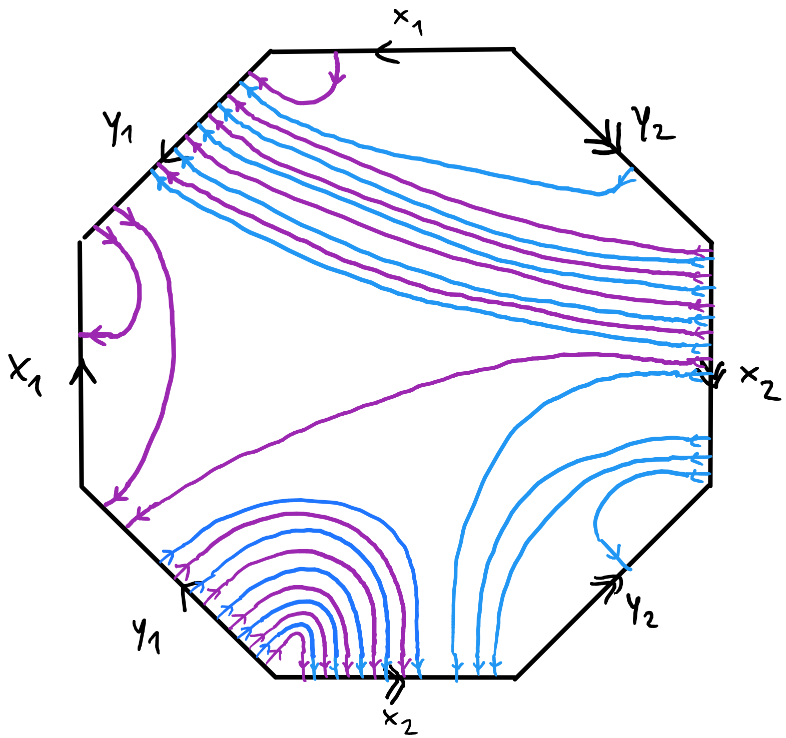
Signs:

$\oplus$ $\quad \delta_i$ $\quad x_i$ or y_i

$\ominus$ $\quad \delta_i$ $\quad x_i$ or y_i

Colour coding: $\quad d_1 \quad d_2$

Topology



Algebra

$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

$$x_2 \longmapsto (d_1 d_2)^5 \cdot d_2^3$$

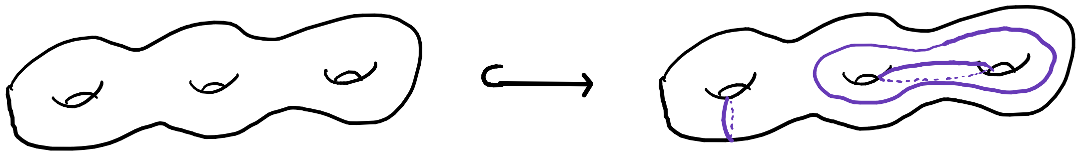
$$y_2 \longmapsto d_2$$

From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \twoheadrightarrow F_g$

surface group $\twoheadrightarrow$ free group

is uniquely realized geometrically by a handlebody.

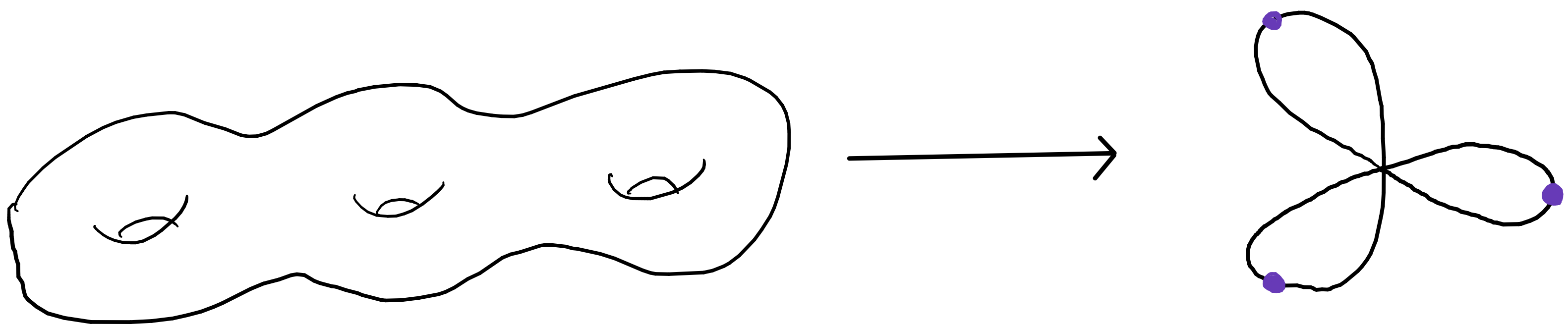


Folklore proof sketch:

Homomorphism $\pi_1(\Sigma_g) \overset{\varphi}{\twoheadrightarrow} F_g$

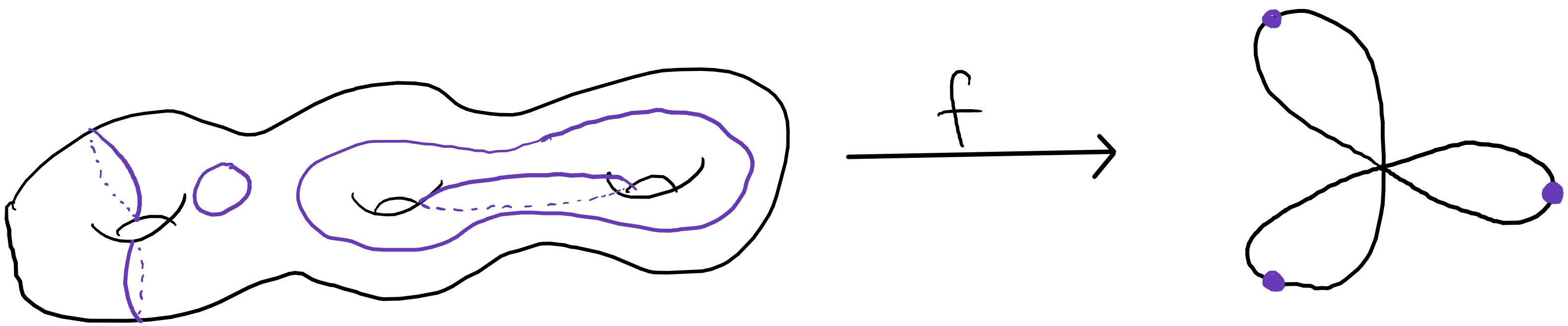
determines a unique map up to homotopy

$$\begin{tikzcd} \Sigma_g \arrow[r, "f"] \arrow[d, phantom, "\simeq" rotate=90] & \bigvee^g S^1 \arrow[d, phantom, "\simeq" rotate=90] \\ K(\pi_1(\Sigma_g), 1) & K(F_g, 1) \end{tikzcd}$$



make map transverse to north poles

$$\Sigma_g \xrightarrow{f} \bigvee^g S^1$$



look at preimage

f^{-1}(North poles)

make map transverse to north poles

Collection of simple closed curves in Σ_g, contains a cut system

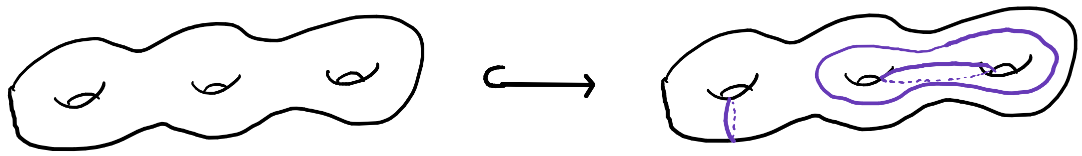
$\square$ (Folklore)

From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \mathrm{Fr}_g$

surface group $\twoheadrightarrow$ free group

is realized geometrically by a handlebody (uniquely) ...



[Blackwell–Kirby–Klug–Longo–R, 2021]

... which can be computed algorithmically.

Group trisections of a finitely presented group G:

Commutative cube

$$
\begin{tikzcd}
 & Fr_g \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_{k_1} \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_g \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & Fr_{k_3} \arrow[rr, two heads] & & G \\
Fr_g \arrow[ur, two heads] \arrow[rr, two heads] & & Fr_{k_2} \arrow[ur, two heads] &
\end{tikzcd}
$$

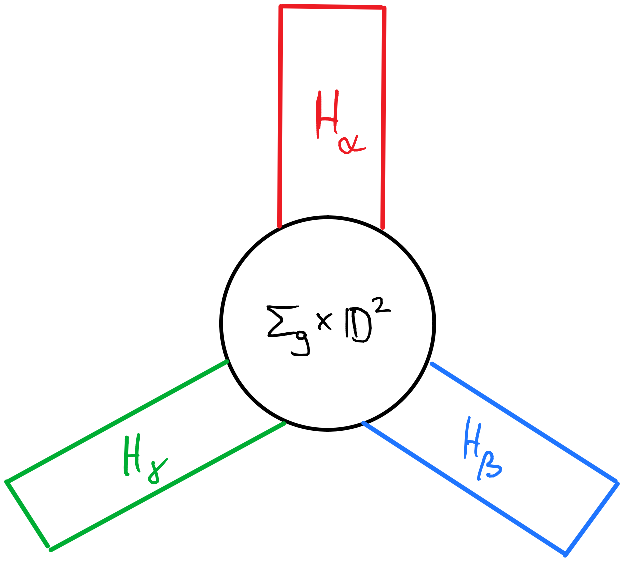
s.th. all maps are surjective

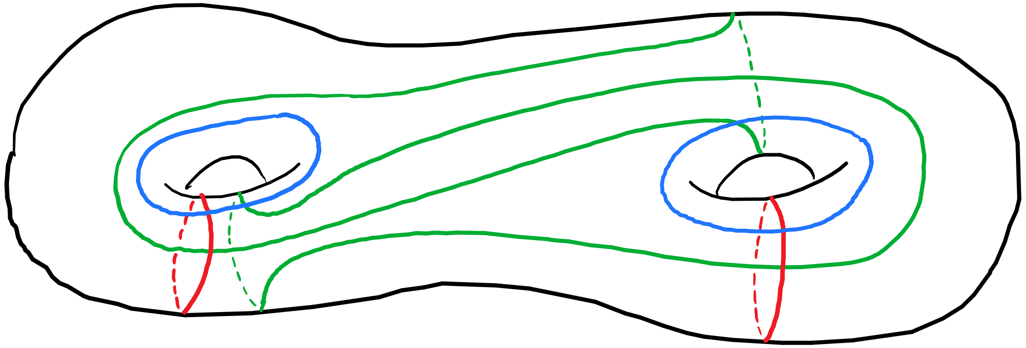
and all faces are push-outs

[Abrams, Gay, Kirby]

Group trisections of closed 4-manifolds:

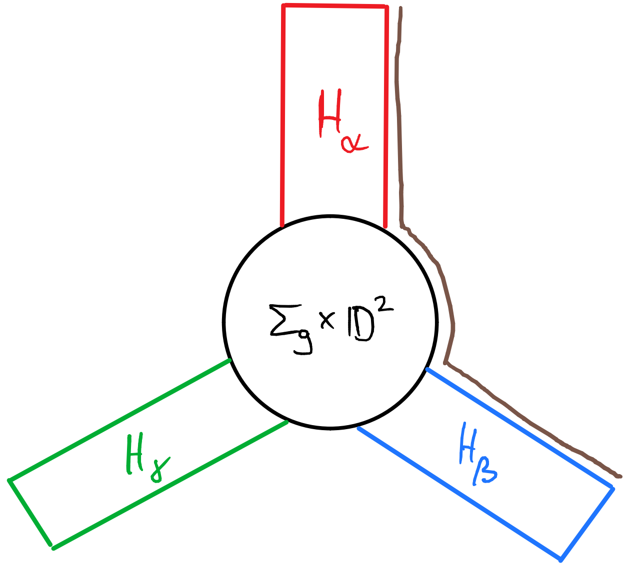
The handlebody-story three times

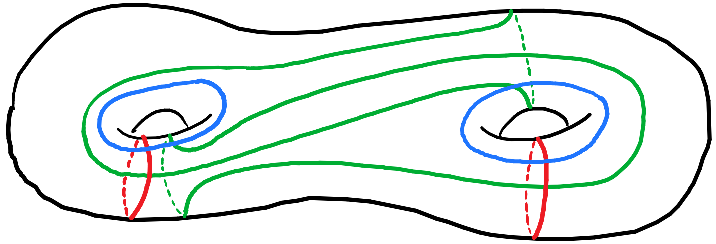




$$\begin{tikzcd}
 & \pi_1(H_\alpha) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(H_\beta) \\
\pi_1(H_\gamma) &
\end{tikzcd}$$

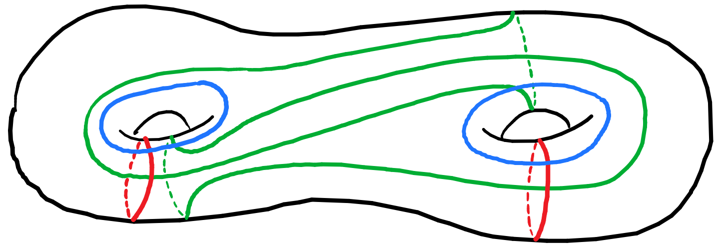
[Abrams, Gay, Kirby]

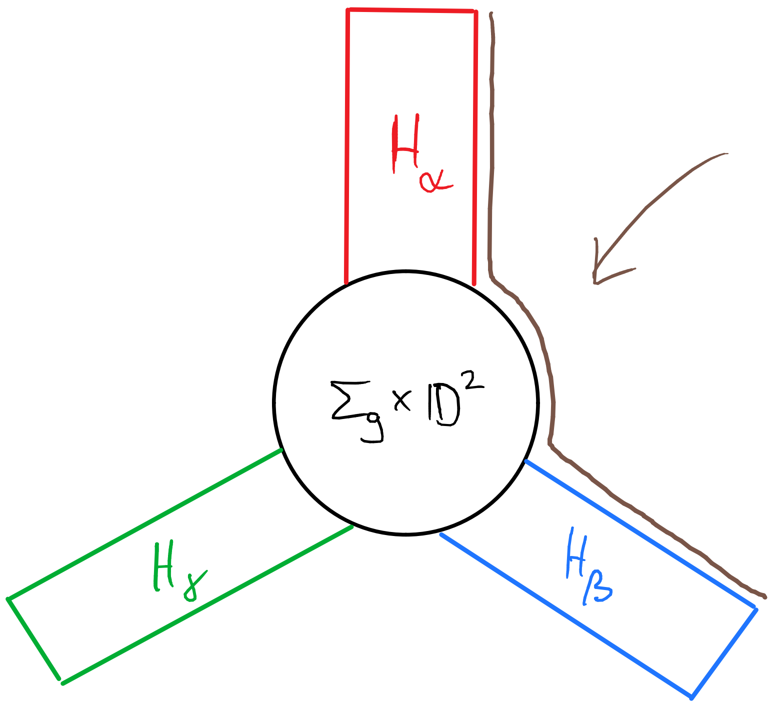




$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[r, two heads] & \pi_1(H_\alpha \cup_\Sigma H_\beta) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] & \\
\pi_1(H_\gamma) & &
\end{tikzcd}
$$

[Abrams, Gay, Kirby]





from our algebra assumption:

this is a closed 3-manifold M

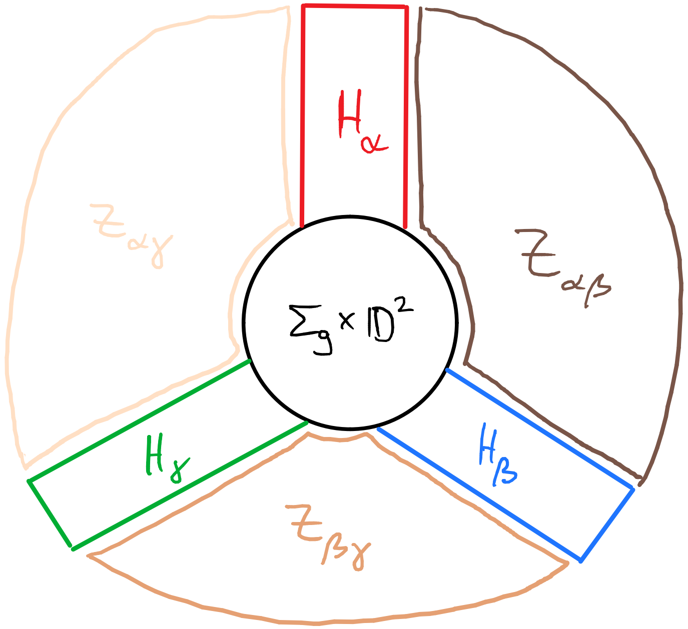
with $\pi_1(M) \cong Fr_k$ free

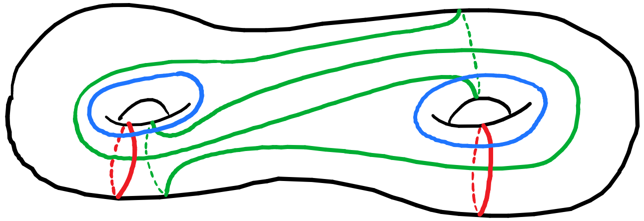
Kneser's thm. + 3D Poincaré conj.

$\Longrightarrow \qquad M \cong \#^k S^1 \times S^2$

[Abrams, Gay, Kirby]

[Laudenbach-Poenaru] allows us to fill the sectors uniquely with $\natural^{k_i} S^1 \times D^3$

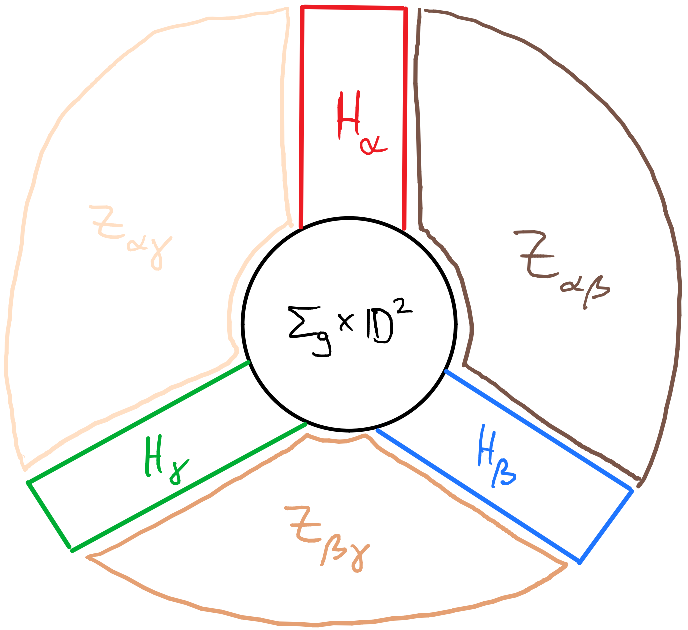




We can do this for all pairs of handlebodies

$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[r, two heads] \arrow[dd, two heads] & \pi_1(H_\alpha \cup_\Sigma H_\beta) \cong \pi_1(Z_{\alpha\beta}) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[dd, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(H_\alpha \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\alpha\gamma}) & \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[r, two heads] & \pi_1(H_\beta \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\beta\gamma}) &
\end{tikzcd}
$$

[Abrams, Gay, Kirby]

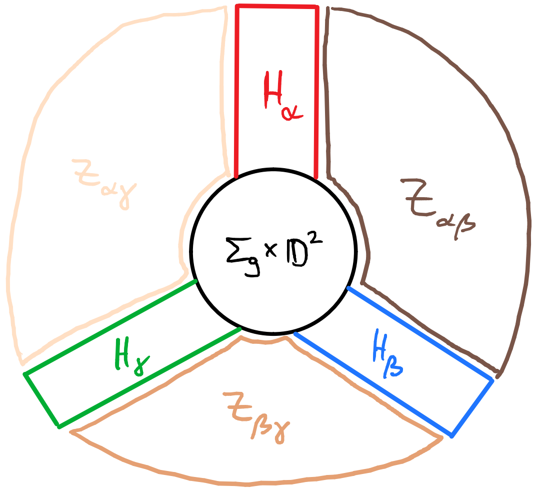


4-manifold M^4 with $\pi_1(M^4) \cong G$ and group trisection corresponding to the cube below

$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(M) \cong G \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}$$

[Abrams, Gay, Kirby]

$$\left\{ \begin{array}{c} \text{(based, parameterized)} \\ \text{trisections} \\ \text{of a 4-manifold } X^4 \end{array} \right\}$$



take π_1 of pieces

1 : 1

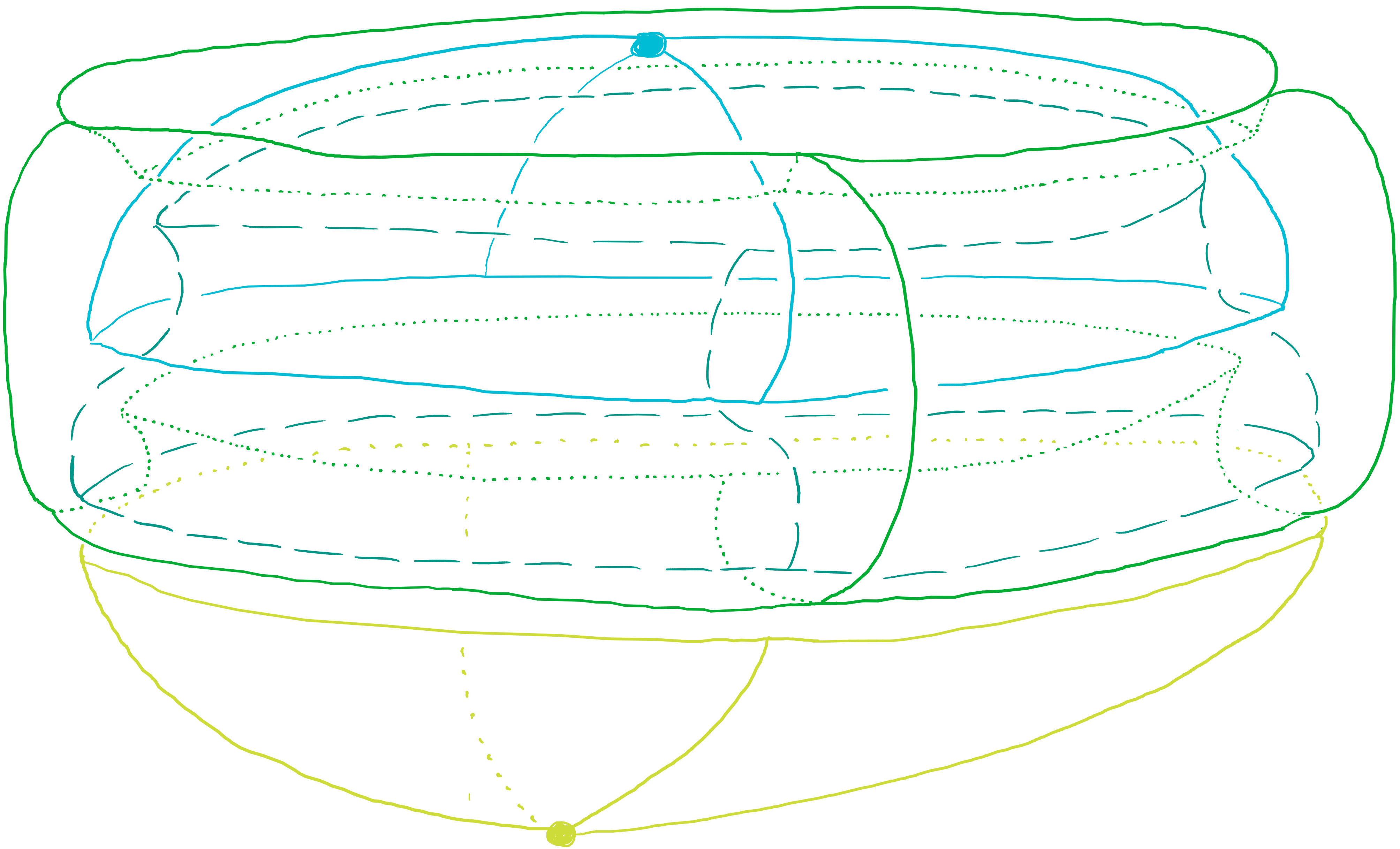
[Abrams, Gay, Kirby]

the previously explained construction

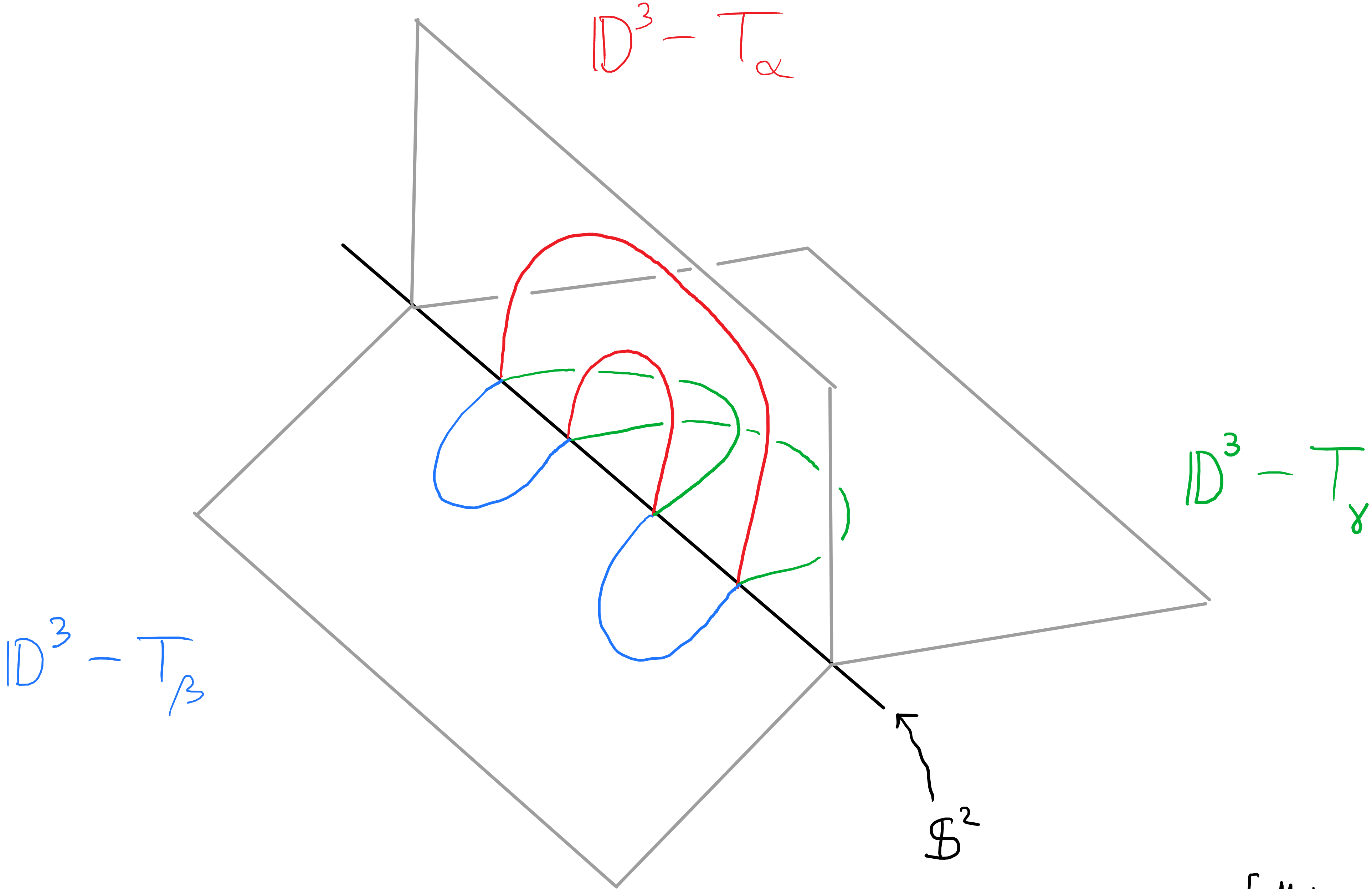
$$\left\{ \begin{array}{c} \text{group} \\ \text{trisections} \\ \text{of } \pi_1(X,*) \end{array} \right\}$$

$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(X) \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}$$

Spun trefoil – a knotted surface in S^4

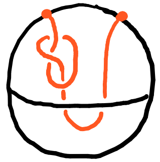


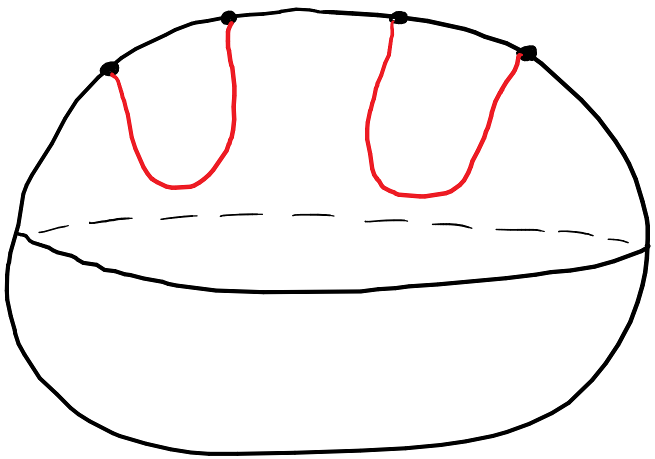
Bridge-trisected surfaces in the 4-sphere

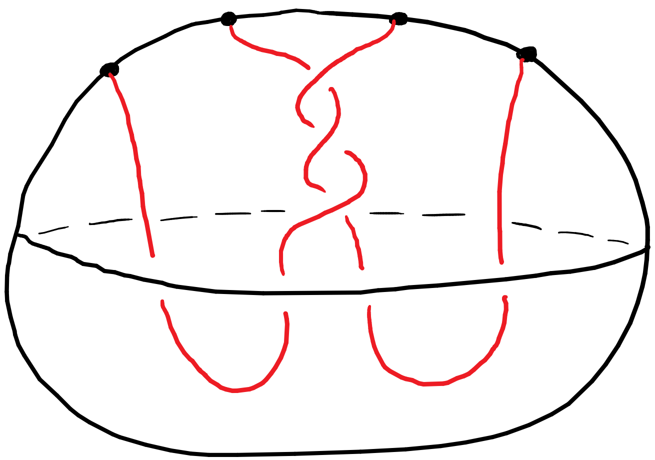


[Meier, Zupan]

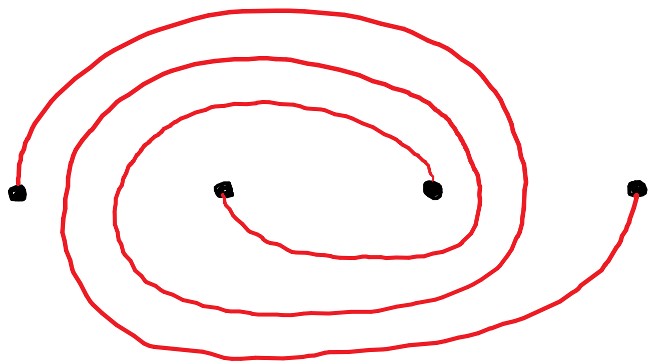
Trivial tangles in 3-balls (and in handlebodies)



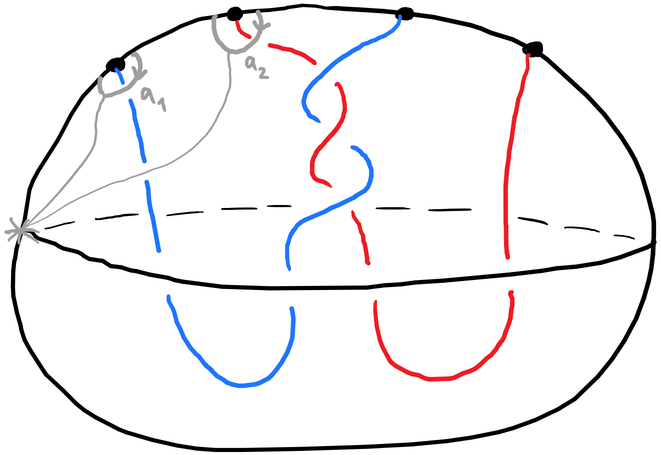


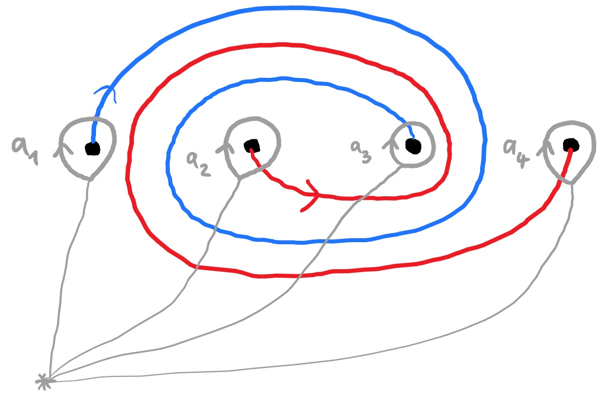


We like to draw the "shadows" of the tangles on a punctured plane:

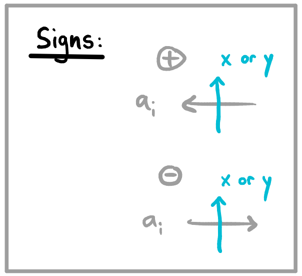


Topology





Algebra



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}\text{4 bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

$$a_1 \longmapsto x^{-1}$$

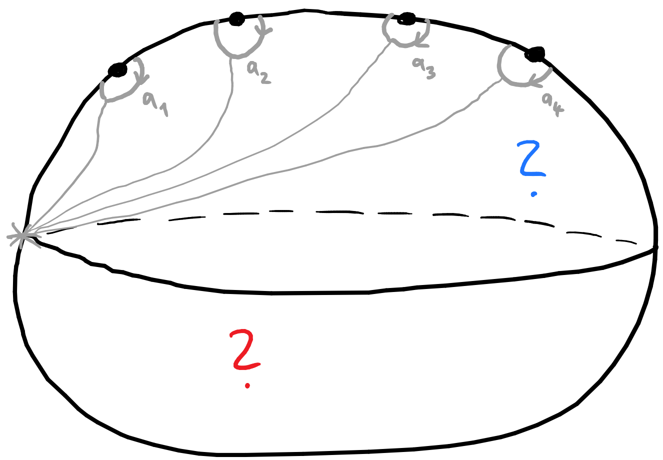
$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

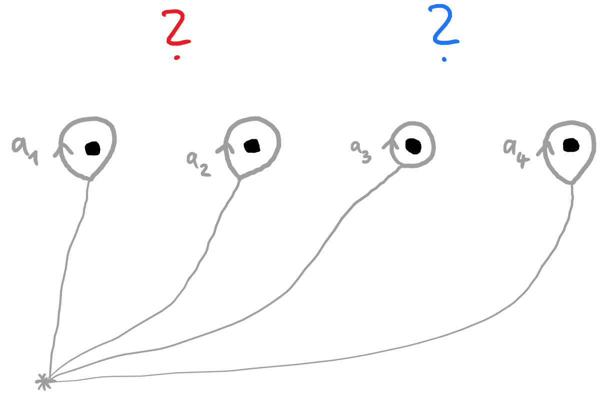
$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

$$a_4 \longmapsto y$$

Topology

Algebra





$$\pi_1\left(\mathbb{S}^2 - \left\{\begin{smallmatrix}\text{4 bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1\left(\mathbb{D}^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

$$a_1 \longmapsto x^{-1}$$

$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

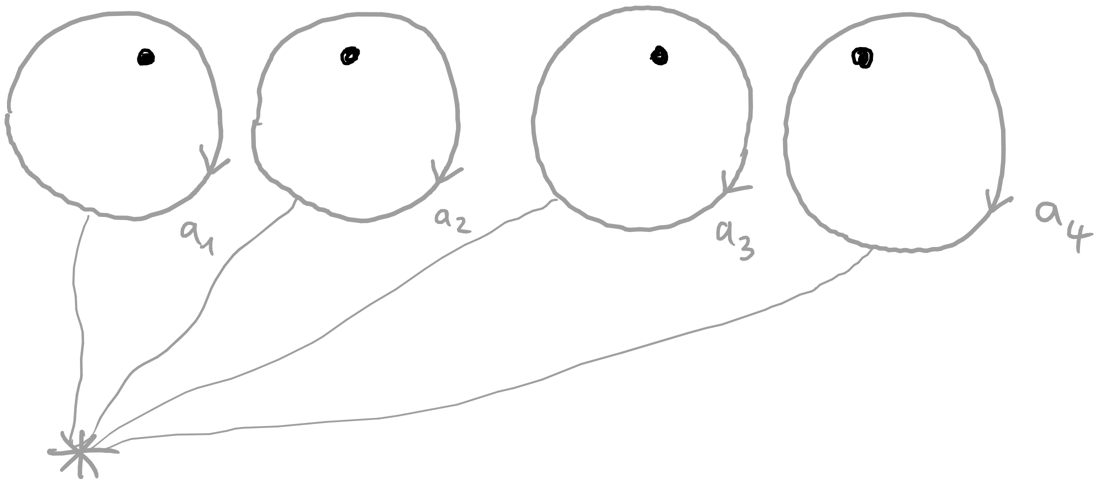
$$a_4 \longmapsto y$$

Punctured Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

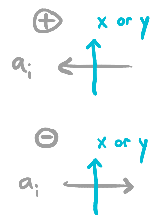
$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}\text{4 bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



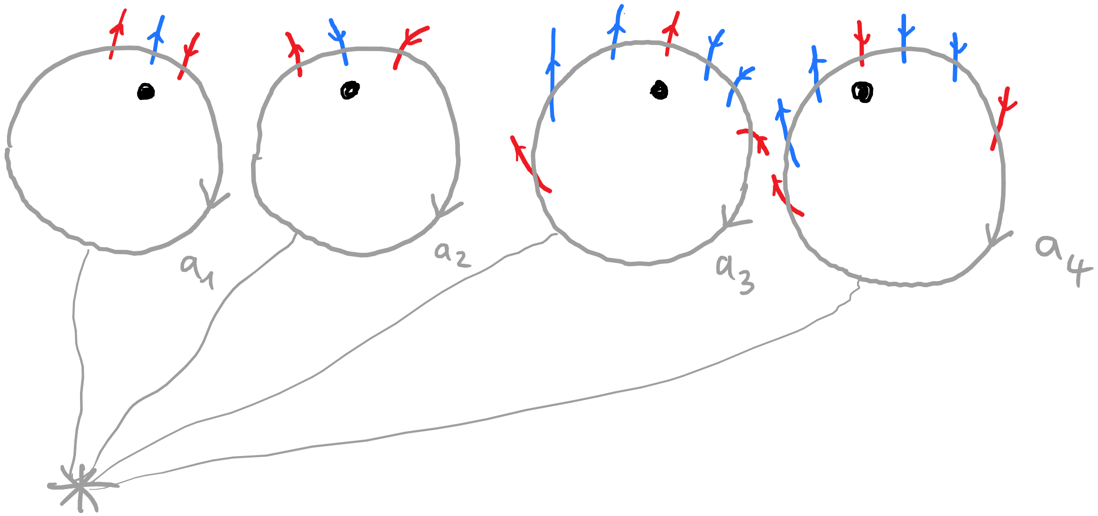
Colour coding: x y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

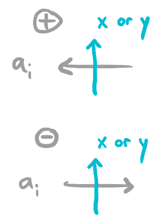
$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:



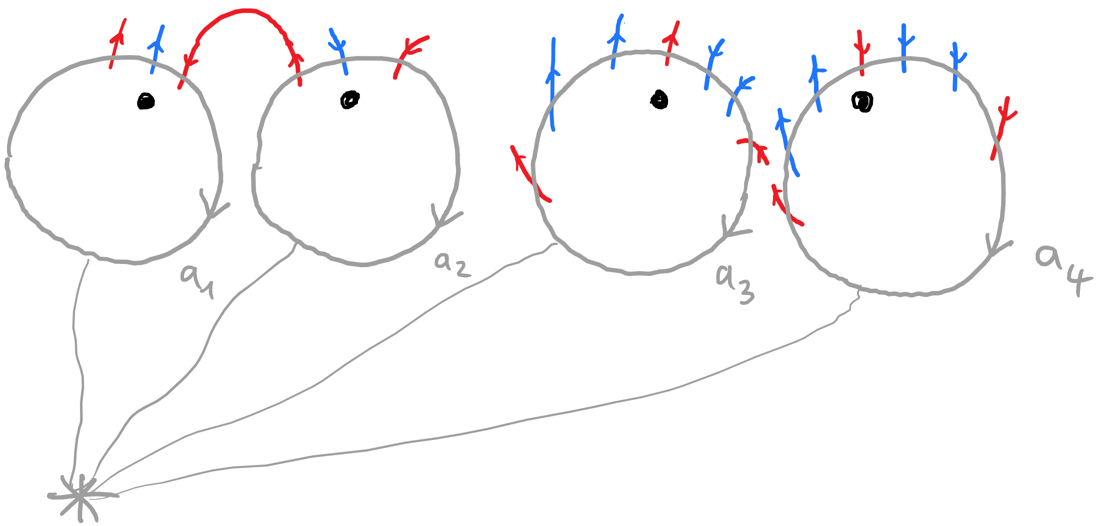
Colour coding: x y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\big\downarrow$$

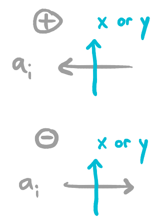
$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



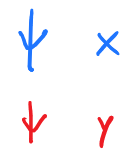
$$\pi_1\left(S^2 - \left\{\begin{smallmatrix} 4 \text{ bridge} \\ \text{points} \end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



Colour coding:

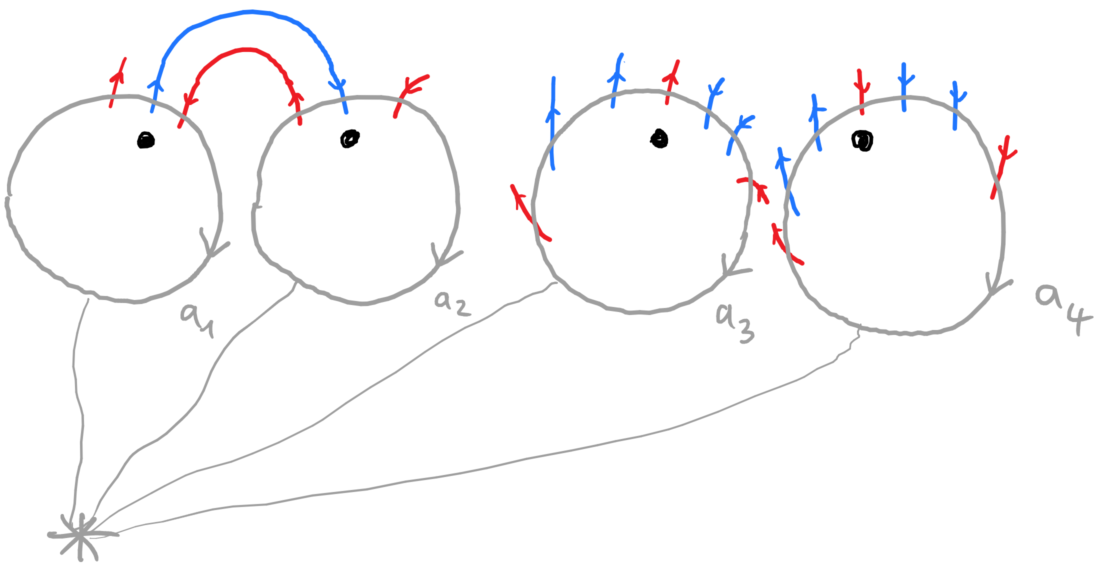


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[\,y\,\cancel{x}\,\cancel{y^{-1}}\,]\,[\,\cancel{y}\,\cancel{x^{-1}}\,y^{-1}\,]\,[\,y\,x\,x\,y\,x^{-1}x^{-1}y^{-1}\,]\,[\,y\,x\,x\,y^{-1}\,x^{-1}x^{-1}y^{-1}\,]$$



$$\pi_1\left(S^2 - \{\text{4 bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

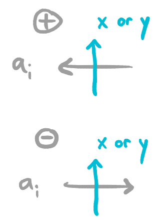
$$a_1 \longmapsto y x y^{-1}$$

$$a_2 \longmapsto y x^{-1} y^{-1}$$

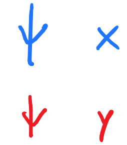
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$

$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:



Colour coding:

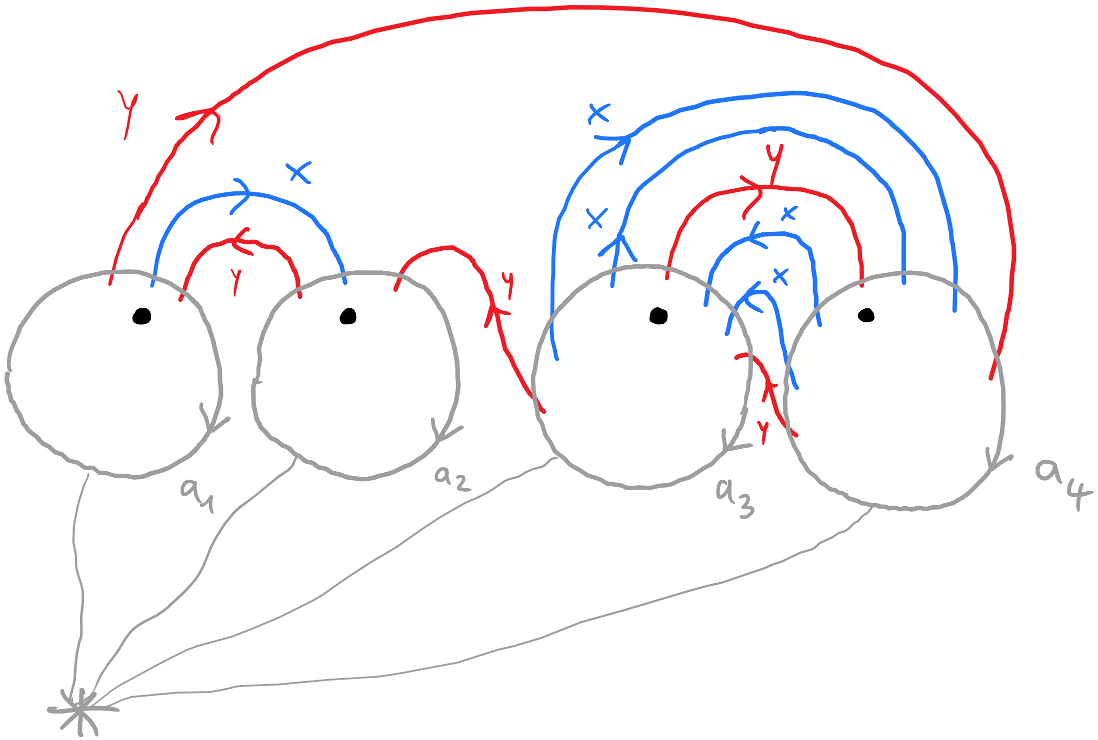


Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[\cancel{y}\cancel{x}\cancel{y^{-1}}][\cancel{y}\cancel{x^{-1}}\cancel{y^{-1}}][\cancel{y}\cancel{x}\cancel{x}\cancel{y}\cancel{x^{-1}}\cancel{x^{-1}}\cancel{y^{-1}}][\cancel{y}\cancel{x}\cancel{x}\cancel{y^{-1}}\cancel{x^{-1}}\cancel{x^{-1}}\cancel{y^{-1}}]$$



$$\pi_1\left(S^2 - \{4 \text{ bridge points}\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$a_1 \longmapsto y x y^{-1}$$
$$a_2 \longmapsto y x^{-1} y^{-1}$$
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$
$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:

$\oplus$ x or y, a_i

$\ominus$ x or y, a_i

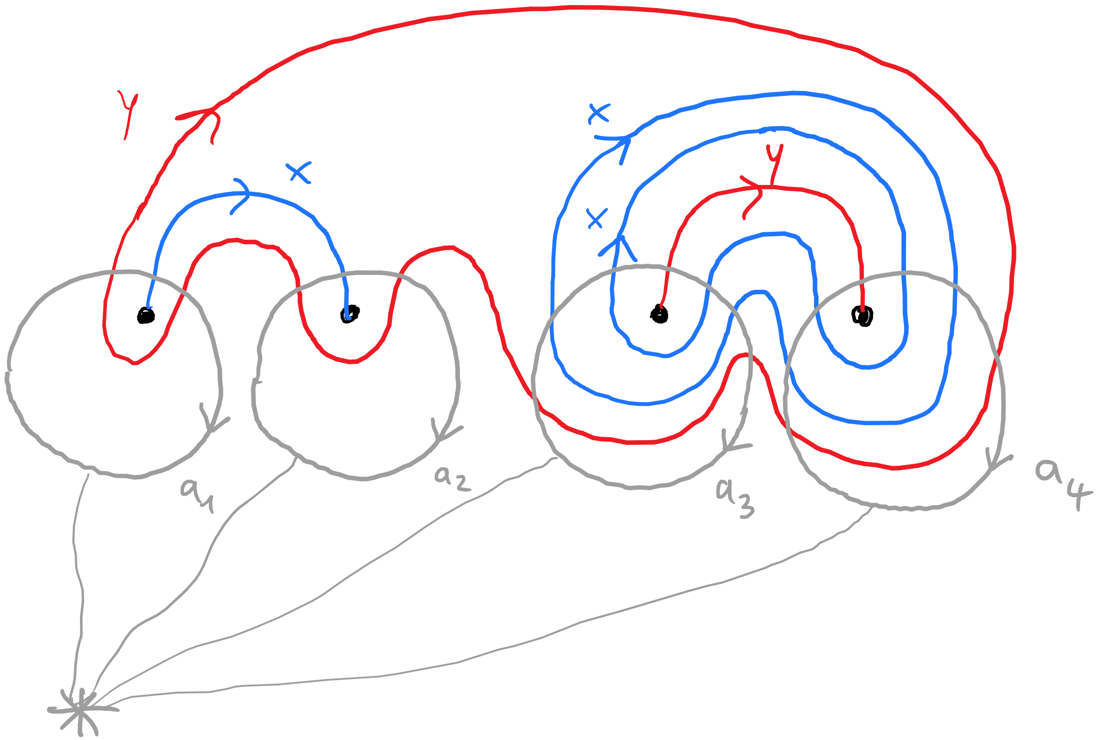
Colour coding: x (blue), y (red)

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\big\downarrow$$

$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

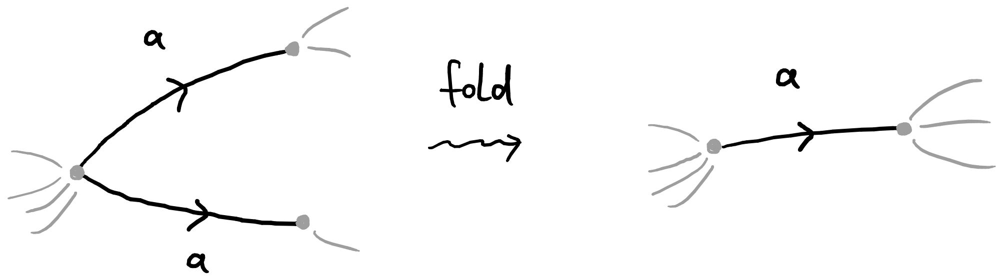
Signs:

⊕ x or y, a_i

⊖ x or y, a_i

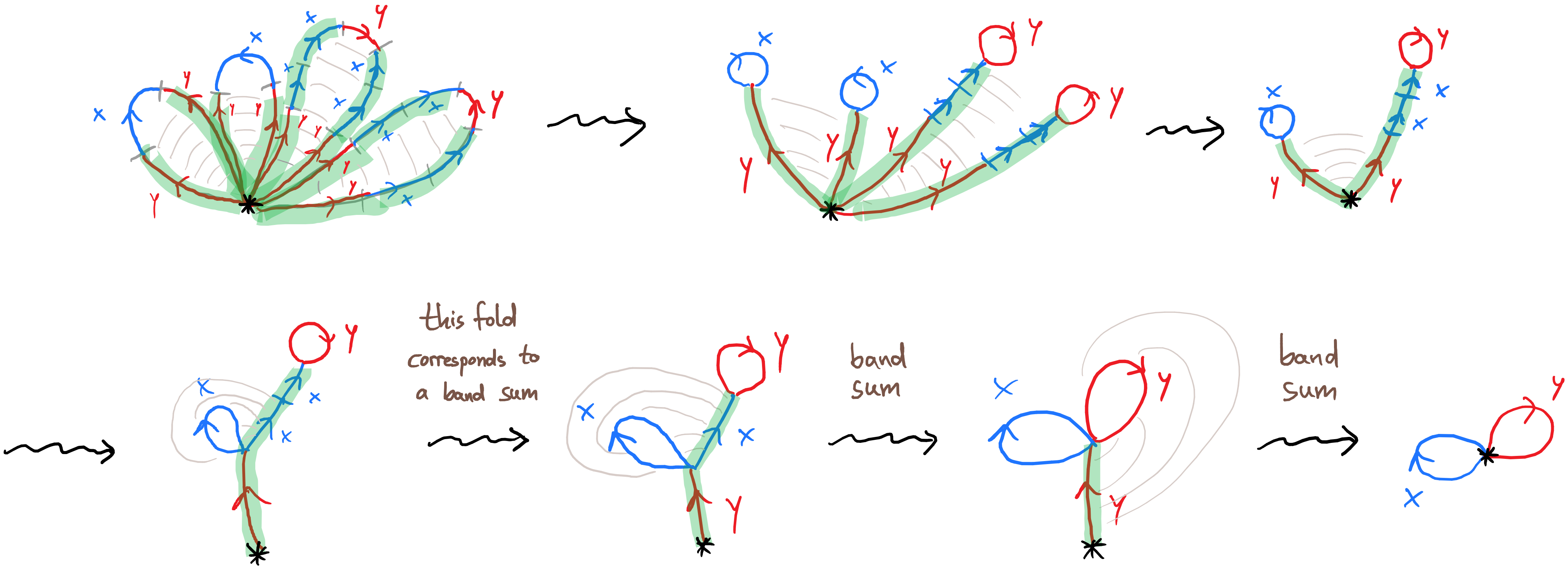
Colour coding: x (blue), y (red)

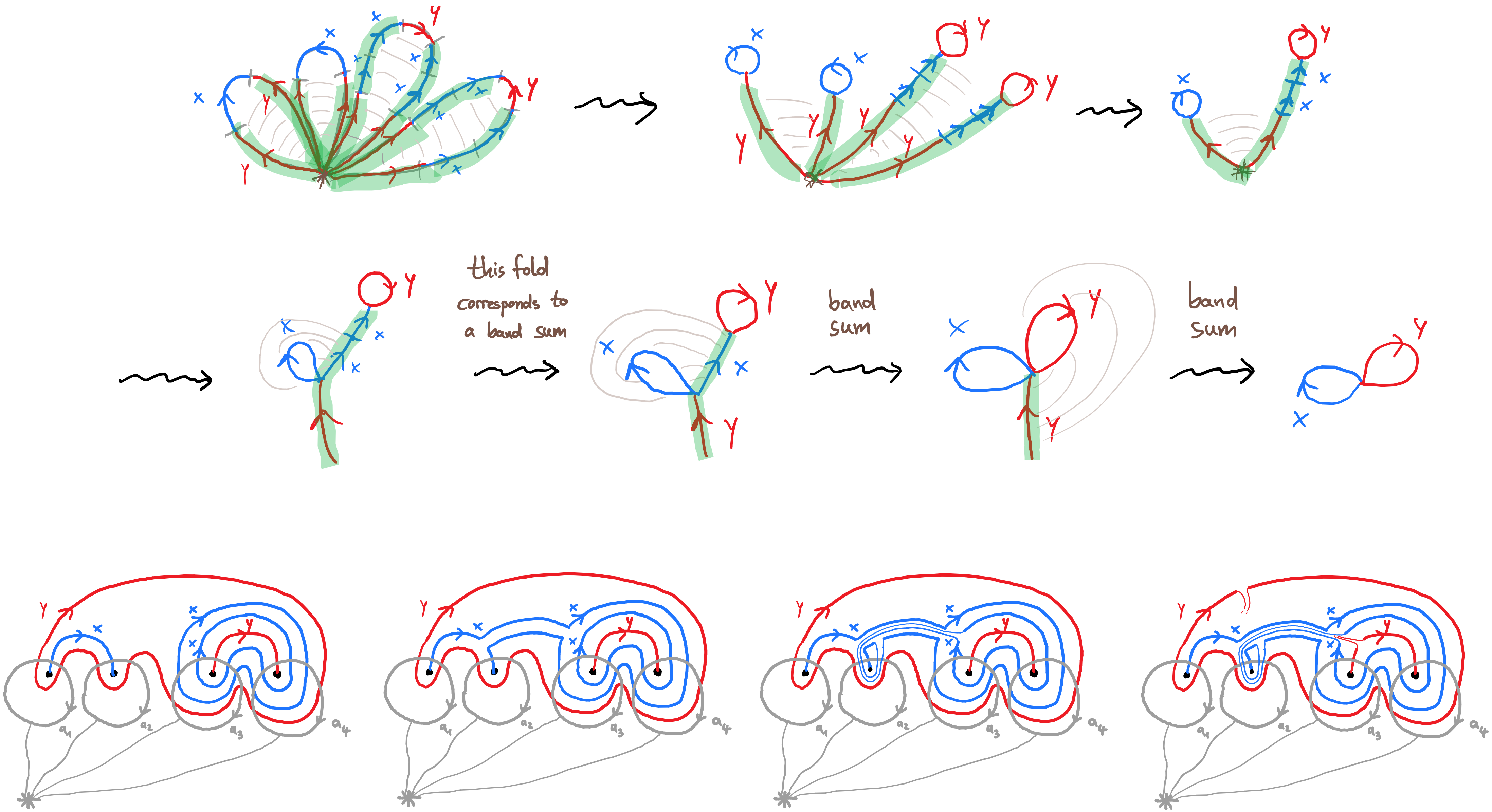
If there are closed circle components, we use band sums guided by **Stallings folding**



Sequence of folds which show that

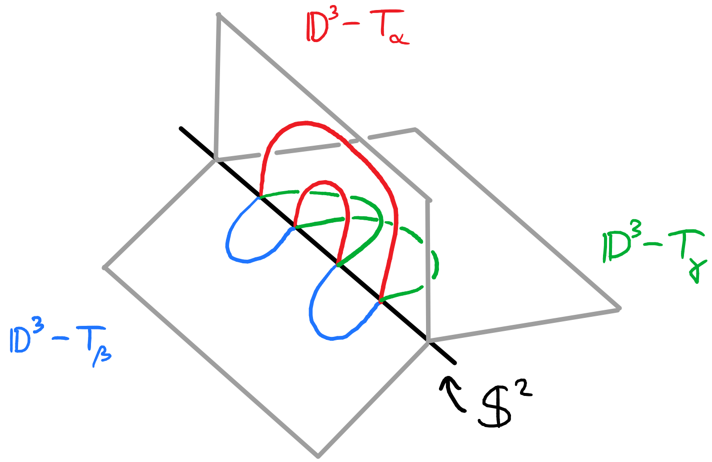
$\langle yxy^{-1}, yx^{-1}y^{-1}, yxxyx^{-1}x^{-1}y^{-1}, yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$ generates the free group $\langle x, y \rangle$




x
y
this fold corresponds to a band sum
band sum
band sum
a_1
a_2
a_3
a_4

(based, parameterized)

$$\left\{ \begin{array}{l} \text{bridge trisections} \\ \text{of a smoothly knotted} \\ \text{surface } K^2 \subset \mathbb{S}^4 \end{array} \right.$$



$\}$

take π_1 of pieces $\downarrow$

$\uparrow$ [Blackwell-Kirby-Klug-Longo-R, 2021]

$$\left\{ \begin{array}{l} \text{trisected} \\ \text{knotted surface} \\ \text{group } \pi_1(\mathbb{S}^4 - K) \end{array} \right.$$

$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3-T_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \overset{\text{free}}{\pi_1(\mathbb{S}^3-L_{\alpha\beta})} \arrow[dd, two heads] \\
\pi_1\left(\mathbb{S}^2-\left\{\begin{smallmatrix}\text{26 bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(\mathbb{D}^3-T_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \underset{\text{free}}{\pi_1(\mathbb{S}^3-L_{\alpha\gamma})} \arrow[rr, two heads] & & \pi_1(\mathbb{S}^4-K) \\
\pi_1(\mathbb{D}^3-T_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \underset{\text{free}}{\pi_1(\mathbb{S}^3-L_{\beta\gamma})} \arrow[ur, two heads] &
\end{tikzcd}$$

$\}$

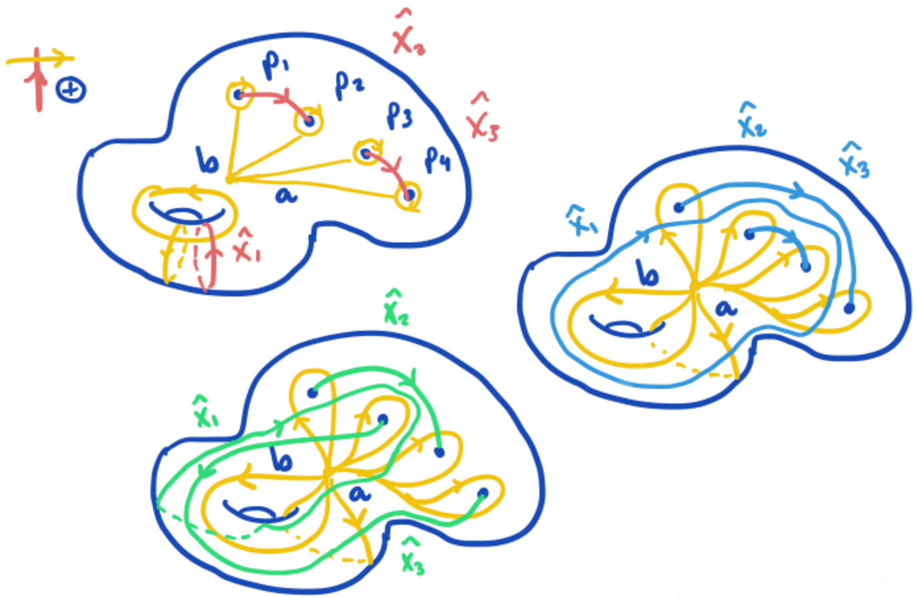
all faces are push-outs
& conditions apply

We take inspiration from:

- [Stallings: How not to prove the Poincaré conjecture (1965)]
- [Jaco: Heegaard splittings and splitting homomorphisms (1968)]

  [Jaco: Stable equivalence of splitting homomorphisms (1970)]
- [Abrams, Gay, Kirby: Group trisections and smooth 4-manifolds (2018)]

Thanks!



$a \mapsto 1$	$a \mapsto x_1$	$a \mapsto \bar{x}_1 x_3$
$b \mapsto x_1$	$b \mapsto 1$	$b \mapsto x_1$
$p_1 \mapsto x_2$	$p_1 \mapsto \bar{x}_1 x_2 x_1$	$p_1 \mapsto x_3 \bar{x}_1 x_2 x_1 \bar{x}_3$
$p_2 \mapsto \bar{x}_2$	$p_2 \mapsto x_3$	$p_2 \mapsto x_3$
$p_3 \mapsto x_3$	$p_3 \mapsto \bar{x}_3$	$p_3 \mapsto \bar{x}_1 \bar{x}_2 x_1$
$p_4 \mapsto \bar{x}_3$	$p_4 \mapsto \bar{x}_1 \bar{x}_2 x_1$	$p_4 \mapsto \bar{x}_1 \bar{x}_3 x_1$