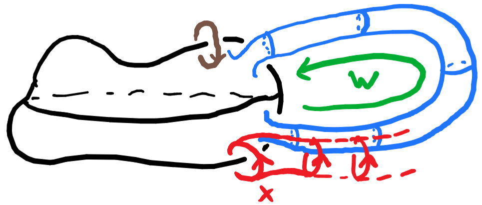
Effect of 1-handle addition on π_1(complement)

$$\pi_1(S^4 \setminus (S + h^1)) \cong \frac{\pi_1(S^4 \setminus S)}{\langle\langle [x, w] \rangle\rangle}$$

x = meridian

"guiding arc of 1-handle"



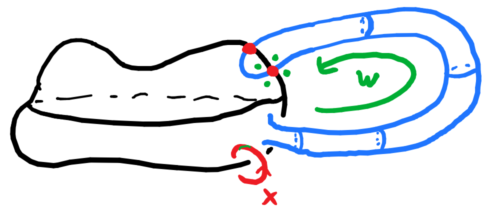
$x = w x w^{-1}$

Effect of finger move on π_1(complement)

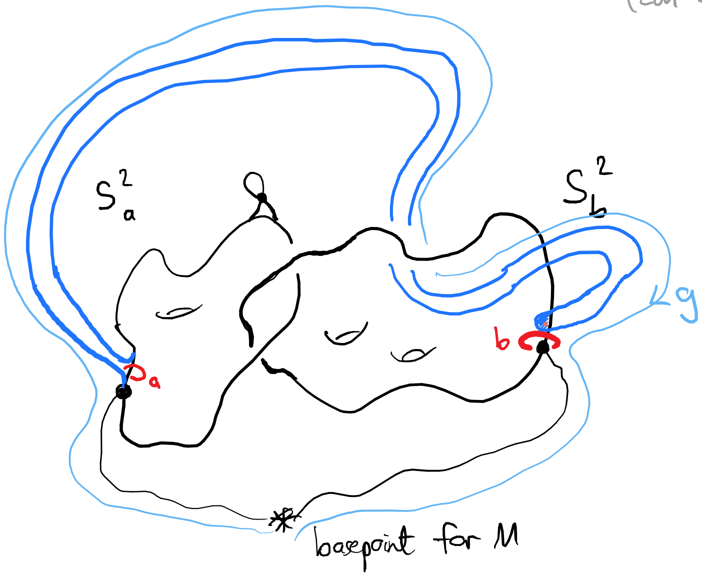
after finger move on S

$$\pi_1(S^4 \setminus S') \cong \frac{\pi_1(S^4 \setminus S)}{\langle\langle [x, \underbrace{w^{-1} x w}_{= x^w}] \rangle\rangle}$$

also a meridian



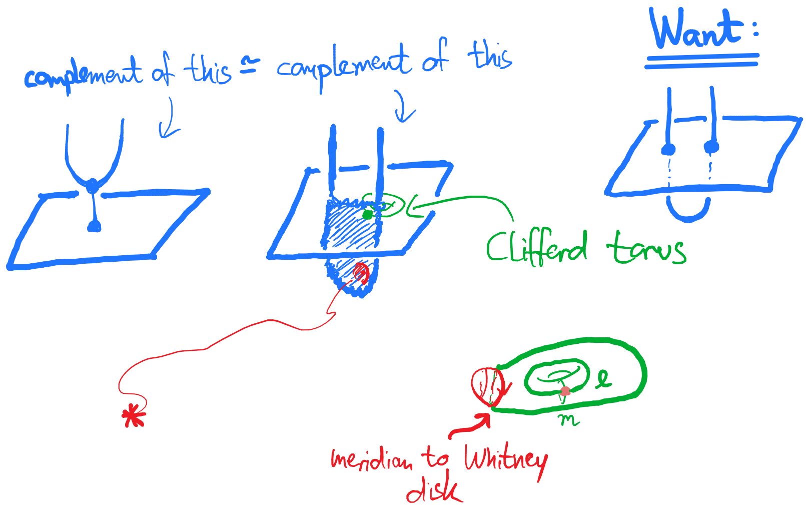
Lemma: Let $L: S^2_a \sqcup S^2_b \looparrowright M^4$ Link of surfaces in a 4-mfld.
(can be immersed)



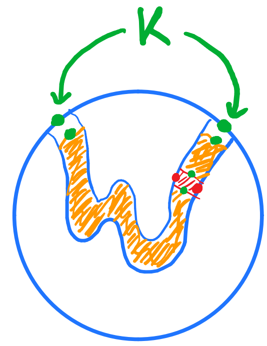
$$\pi_1(M - \underset{\sim}{L'}) = \frac{\pi_1(M-L)}{\langle\langle [g^{-1}ag, b] \rangle\rangle}$$

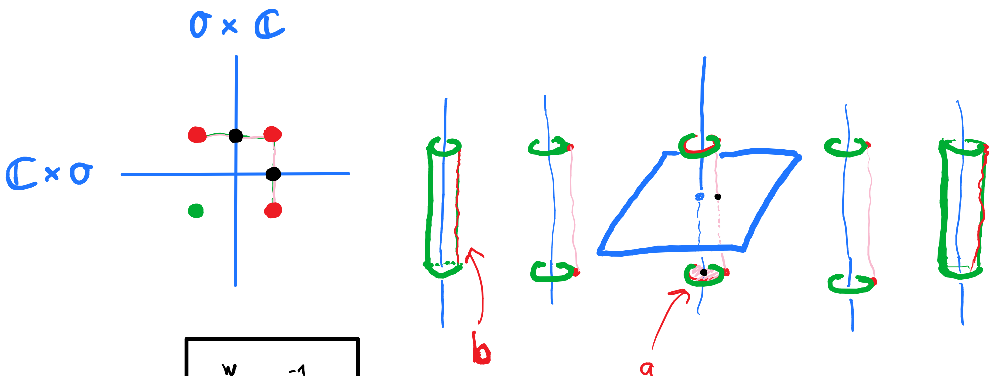
after finger move

$$\cong \frac{\pi_1(M-L)}{\langle\langle [m,\ell] \rangle\rangle}$$



Cartoon: Removing a slice disk from $\mathbb{D}^4$

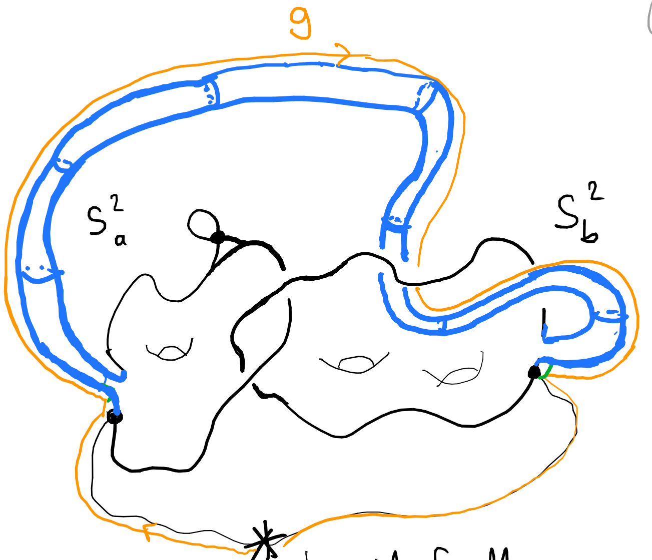




$$\boxed{m^w := w^{-1} m w}$$

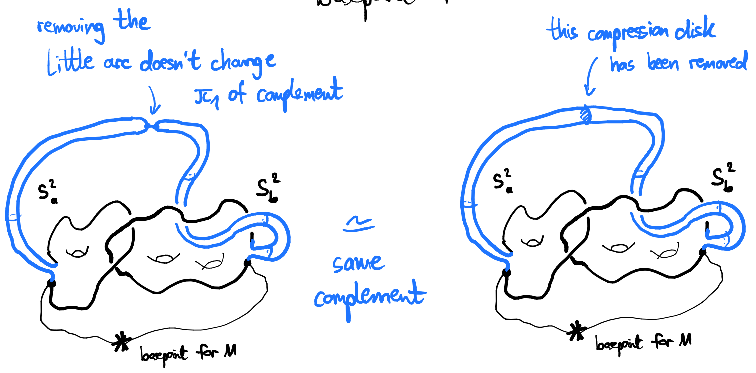
Lemma: Let $L: S^2_a \amalg S^2_b \looparrowright M^4$ Link of surfaces in a 4-mfld.

(can be immersed)

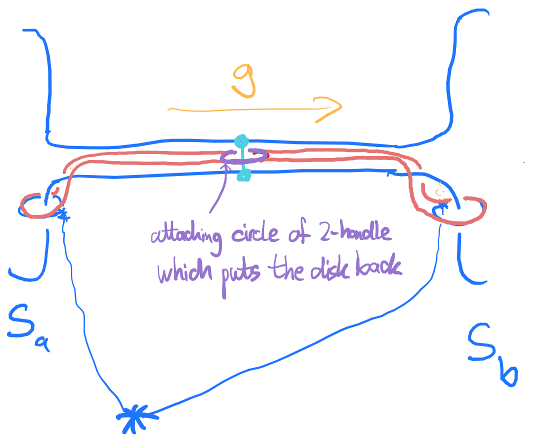


$$\boxed{\pi_1(M - \underbrace{L'}) \cong \frac{\pi_1(M-L)}{\langle\langle a = g b g^{-1} \rangle\rangle}}$$

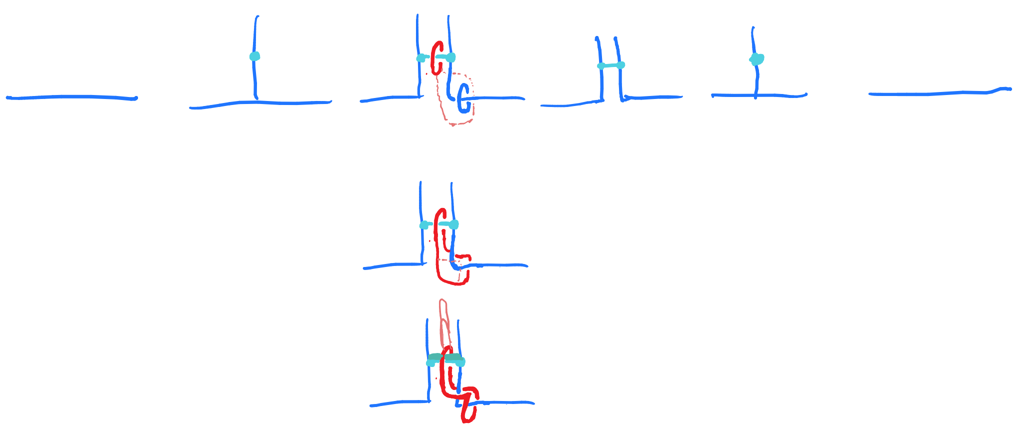
link after 1-handle attachment

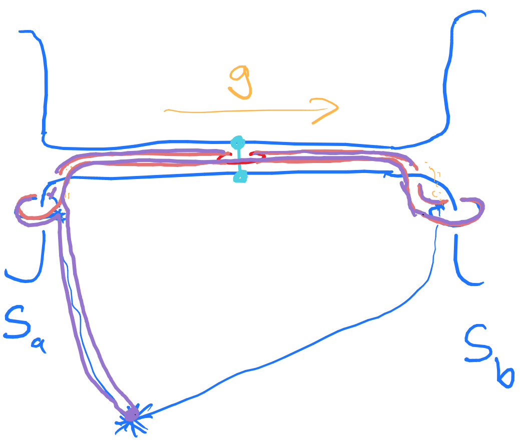


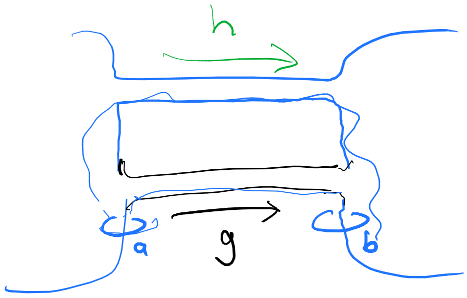
Have to understand how putting **disk** back changes π_1 of complement:



At the attaching regions of the 1-handle:







$b = hah^{-1}$

$b = gag^{-1}$

$$a^{-1}gbg^{-1} = e \iff \boxed{a = gbg^{-1}}$$

$$\updownarrow$$

meridian

$[a, gh]$

anything

$$\begin{aligned} a &= gbg^{-1} \\ &= ghah^{-1}g^{-1} \\ &= gha(gh)^{-1} \\ \iff e &= a^{-1}gha(gh)^{-1} \end{aligned}$$

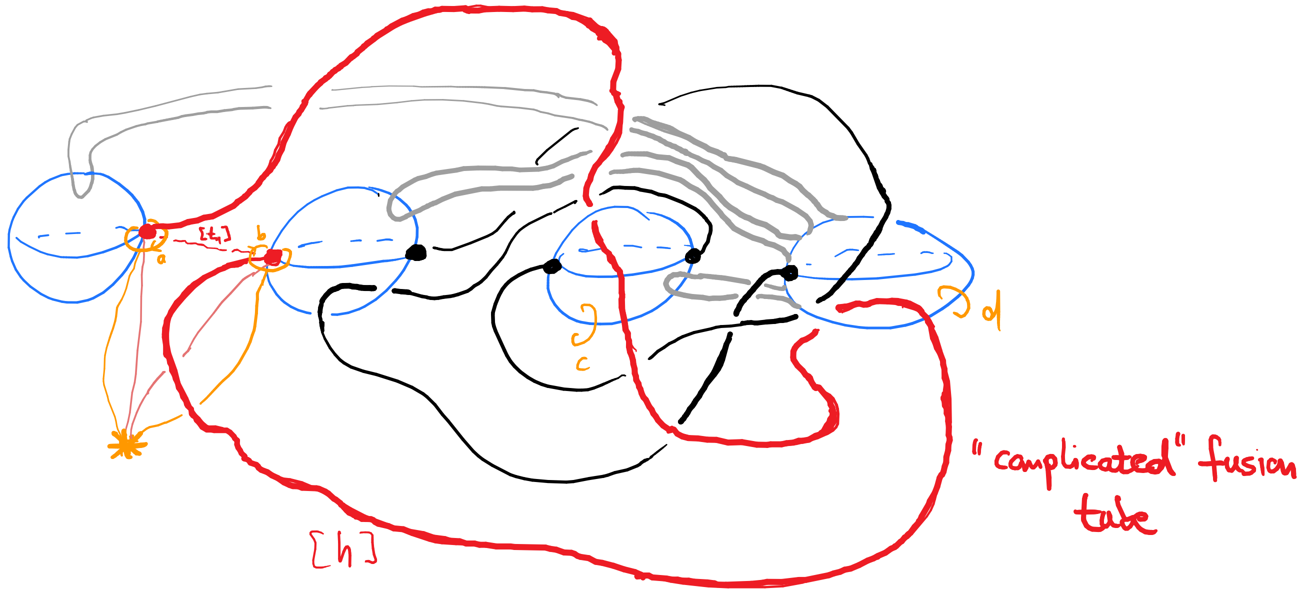
<u>On connected surfaces:</u>

Finger moves:	[meridian, meridian]
1-handle attachments:	[meridian, anything]

$u_{F.\text{-}Wh.}(K) \leq fus(K)$ for a ribbon 2-knot K:

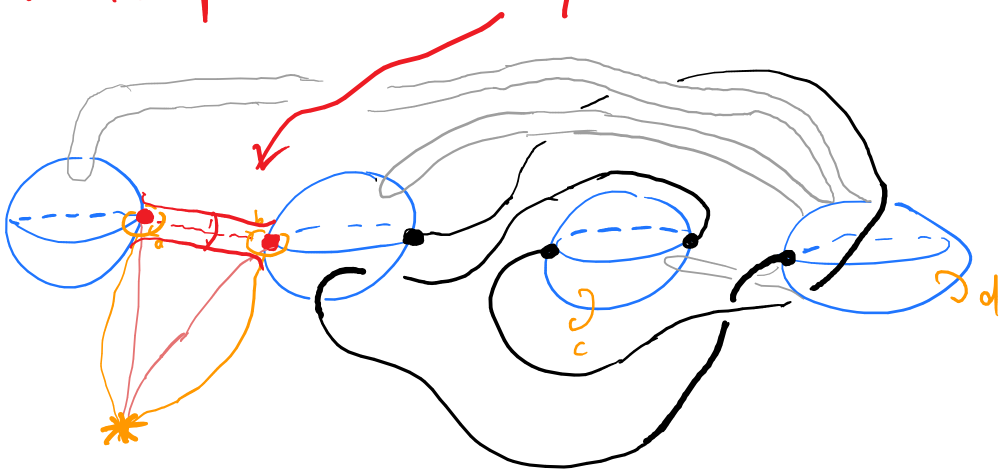
Group after "extending fingers" from the last minimum to all the others:

$$\langle a\rangle * \langle b,c,d \mid b = w_1 c w_1^{-1},\ c = w_2 d w_2^{-1}\rangle / [d,G] \cong \mathbb{Z}_a \oplus \mathbb{Z}_{b=c=d}$$



"twisting h about a & b" $\rightsquigarrow$ $[h] = [t] \in \pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}$

I.e. in the complement of the immersion, the "complicated" fusion tube is isotopic to this "easy" tube

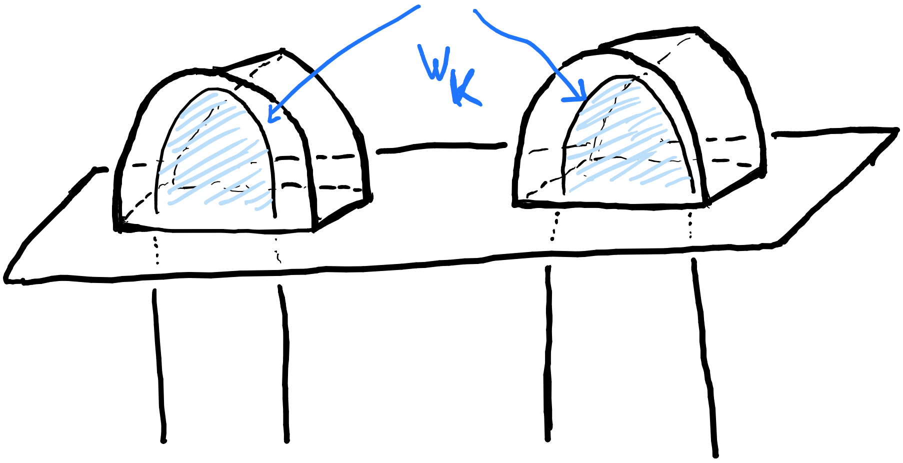


1-handle number $\overset{?}{\longleftrightarrow}$ tunnel #

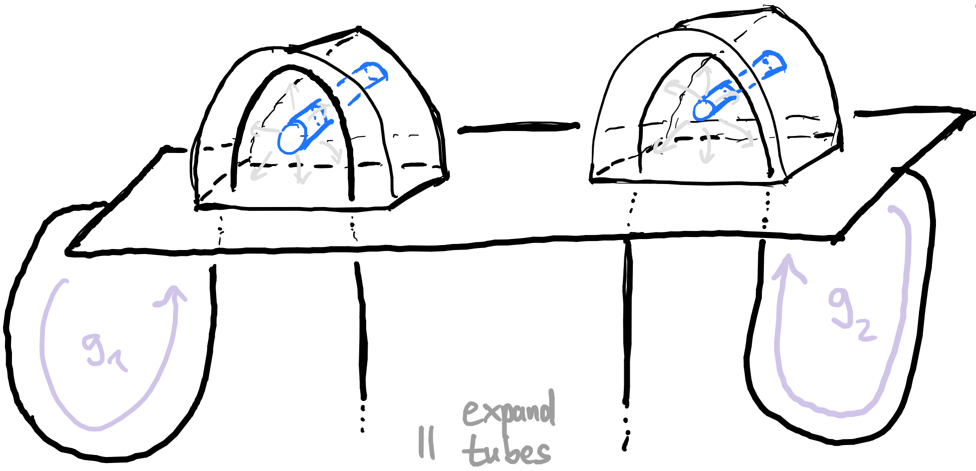
Finger-Wh. number $\overset{?}{\longleftrightarrow}$ unknotting #

Question: Is $u_{1\text{-}h.}(K) \overset{??}{\le} u_{F\text{-}Wh}(K)$

Interesting Whitney disks taking us to K

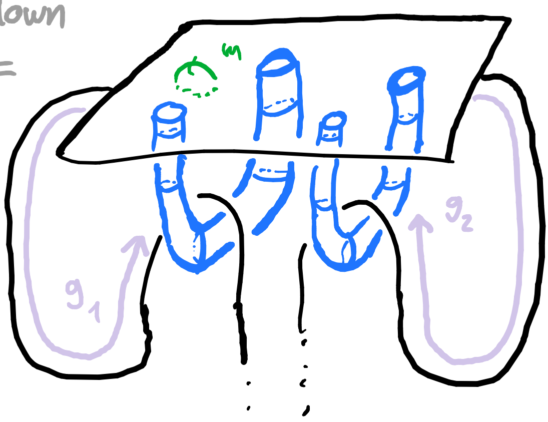


$$\frac{\pi_1(S^4-K)}{\langle\langle [m, g_1^{-1} m g_1], [m, g_2^{-1} m g_2] \rangle\rangle} \cong \mathbb{Z}$$

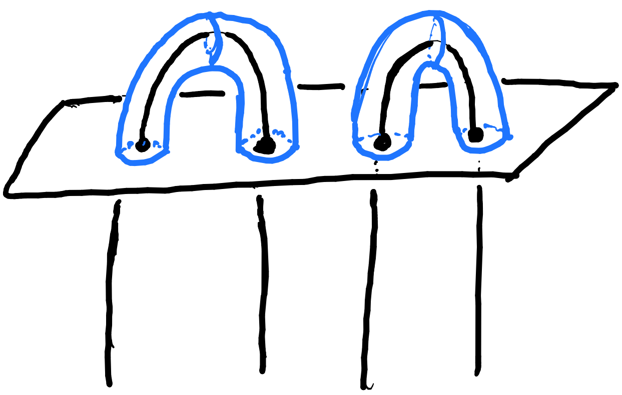


pull "fingernails" down

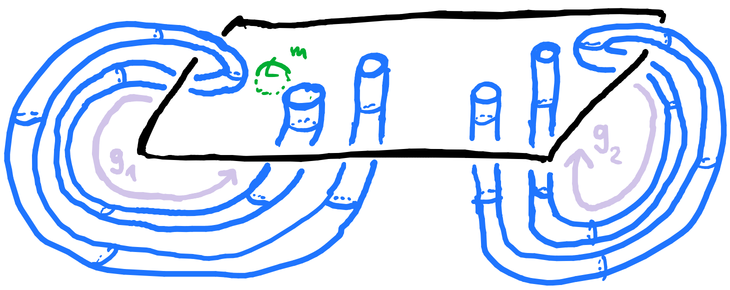
=



= expand tubes



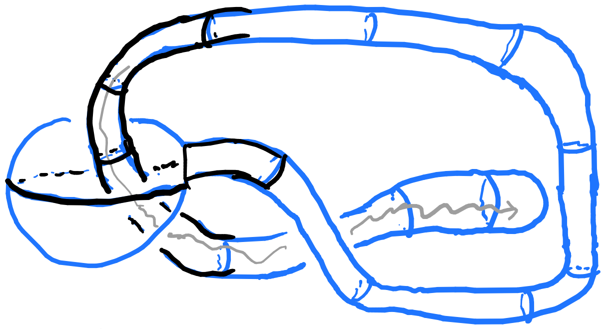
= "retract fingers"



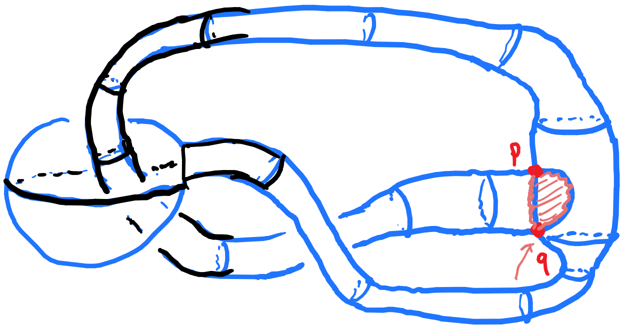
Example of a ribbon 2-knot via one finger & one Whitney move on unknot

un-2-knot

isotopy

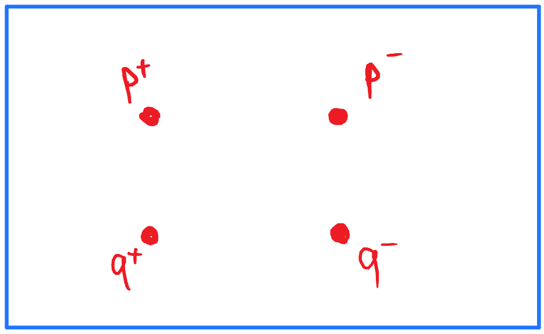


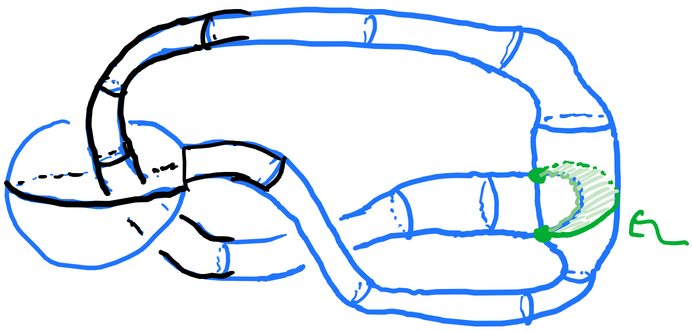
FM



"fingernail" disk back to unknot

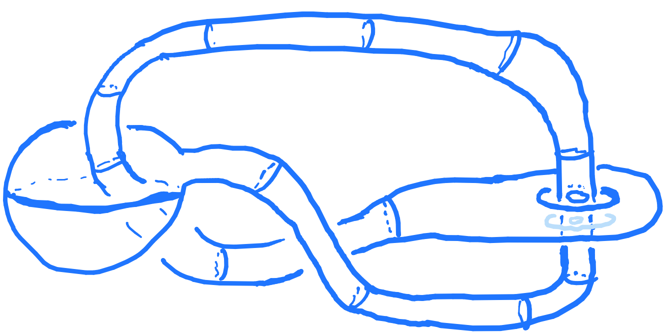
Preimage in S^2:



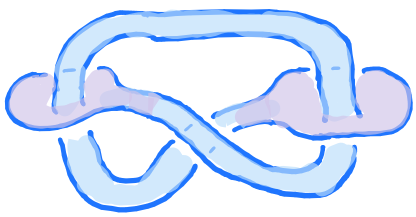


Whitney disk to nontrivial ribbon knot

WM



= Double of ribbon disk for stevedore



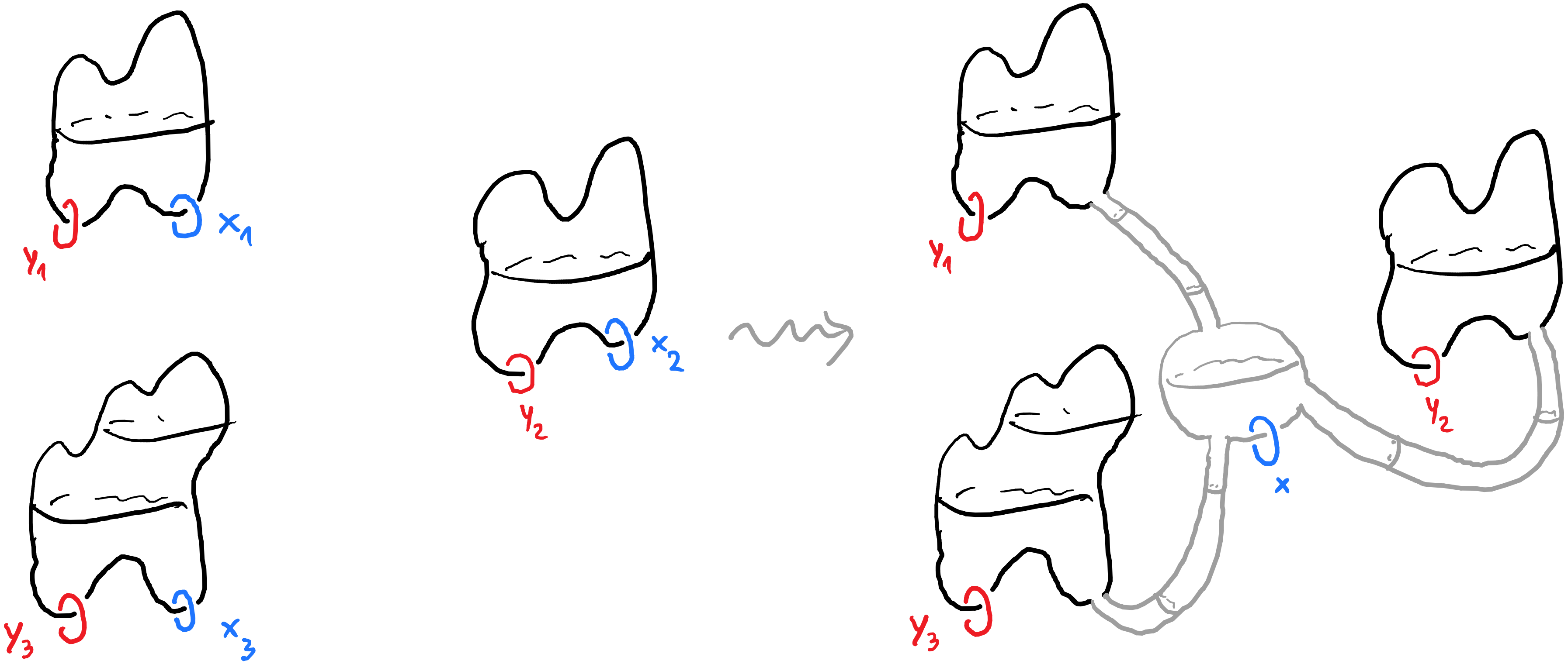
For k_i non-tr. 2-bridge & for natural number $r_i \geq 2$:

$$\boxed{u^{\mathbb{Z}}_{1\text{-h.}}\left(\mathrm{Spin}_{r_i}(k_i)\right) = 1}$$

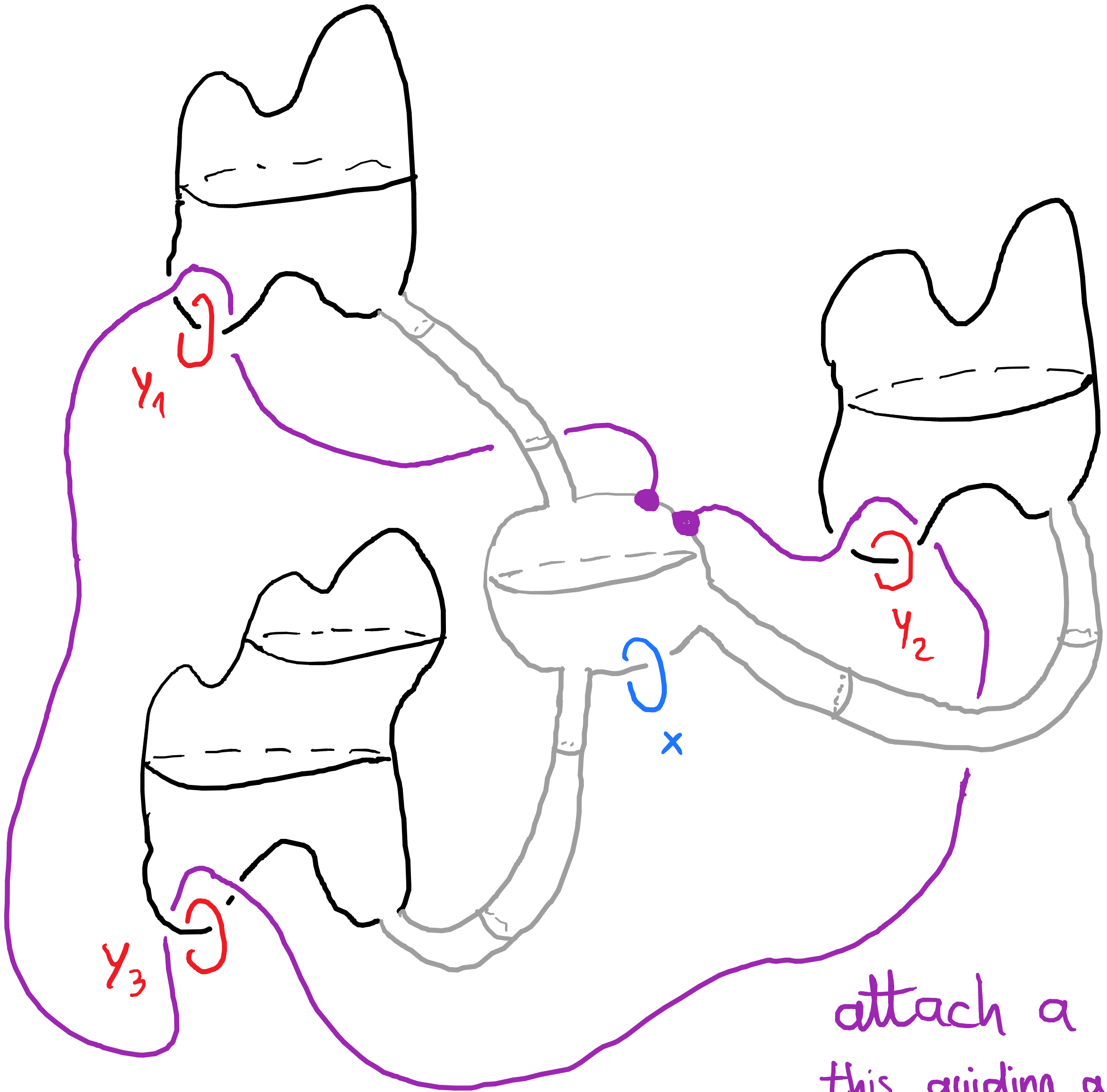
Lemma: For $r_1, \dots, r_n \geq 2$ coprime integers, $k_1, \dots, k_r$ 2-bridge

$$u^{\mathbb{Z}}_{1\text{-h.}}\left(\mathrm{Spin}_{r_1}(k_1) \# \dots \# \mathrm{Spin}_{r_n}(k_n)\right) = 1$$

Kanenobu's example:



Claim: Adding the relation $[x, y_1 \cdot y_2 \cdot \ldots \cdot y_n]$ abelianizes the group

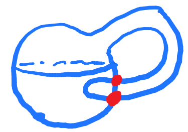


Similar π_1-calculation works for result of Finger move along this arc

(the immersion complement has $\pi_1 \cong \mathbb{Z}$)

Can we put it in the "standard position?"

(finger move on unknot)



attach a 1-handle along this guiding arc to get a torus with $\pi_1(\text{compl.}) \cong \mathbb{Z}$

– is it unknotted?