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*Introduction*

These notes are a collection of various things that I wanted to write down for myself to use as a reference.



# Homotopy theory

## CW-complexes

**Definition 1.** A map  $f: X \rightarrow Y$  is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

for all  $n \geq 0$  and all choices of basepoints  $x_0$  in  $X$ .

**Theorem 1** (Whitehead's Theorem). *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

**Proposition 1** (Geometric interpretation of  $n$ -connectedness). *If  $(X, A)$  is an  $n$ -connected CW-pair, then there exists a CW-pair  $(Z, A) \sim_{\text{rel } A} (X, A)$  such that all cells of  $Z \setminus A$  have dimension greater than  $n$ .*

## Homology

**Definition 2** (Acyclic). A space  $X$  is called *acyclic* if  $\tilde{H}_i(X) = 0$  for all  $i$ , i.e. if its reduced homology vanishes.

*Example 1.* Removing a point from a homology sphere yields an acyclic space. This example for the Poincaré homology sphere is described in (Hatcher, 2002, Example 2.38). TODO Insert proof.



# Knot Theory

## Constructions & Definitions

**Definition 3.** If  $K$  is an oriented knot, then

- the *reverse*  $\bar{K}$  is  $K$  with the opposite orientation
- the *obverse*  $rK$  is the reflection of  $K$  in a plane
- the *inverse*  $r\bar{K}$  is the concordance inverse of  $K$ .

**Proposition 2.** For  $K \subset \mathbb{S}^3$  we have that  $K \# r\bar{K}$  is slice, even ribbon.

**Definition 4** (Homotopically unlinked, (Rolfsen, 2003, 3.F.9.)). If  $L = L_1 \cup \dots \cup L_n$  is a link with  $n$  components, we say that  $L_i$  is *homotopically unlinked* from the remaining components if there is a homotopy  $h_t$  from the embedding of  $L_i$  to the constant map such that the images of  $h_t$  and  $L_j$  are disjoint at all times  $t \in \mathbb{I}$  and for all other components  $j \neq i$ .

*Example 2.* In the Whitehead link both components are homotopically unlinked from each other.

*Remark 1.* Homotopic linking (for two component links) is **not** a symmetric relation.

**Definition 5.** A link  $L = L_1 \cup L_2$  of two components in  $\mathbb{R}^n$  is *splittable* if there are disjoint, topological  $n$ -balls  $\mathbb{D}_1^n, \mathbb{D}_2^n \subset \mathbb{R}^n$  such that  $L_i$  lies in the interior of  $\mathbb{D}_i^n$ .

**Proposition 3.** If a link is splittable, then each component is homotopically unlinked from the other.

**Definition 6.** The lower central series of a group  $G$  is defined inductively by

$$\begin{aligned} G_0 &:= G \\ G_i &:= [G, G_{i-1}] = \langle [g, h] \mid g \in G, h \in G_{i-1} \rangle \end{aligned}$$

**Proposition 4.** This satisfies:

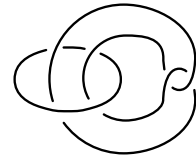


Figure 1: Positive Whitehead link, picture from (Meier, 2015).

The converse of 3 is not true, an example of van Kampen and Zeeman is discussed in (Rolfsen, 2003, 3.K.5.).

Observe that  $[G, G_{i-1}] = [G_{i-1}, G]$ .

- $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$
- each  $G_i$  is normal in  $G$
- the quotient  $G_i/G_{i+1}$  is in the center of  $G/G_{i+1}$ .

**Lemma 1.** If  $F$  is a free group, then  $\bigcap_{i=1}^{\infty} F_i$  is the trivial group.

*Characterizing the unknot*

**Proposition 5.** A knot is trivial if and only if its longitude represents the trivial element of the knot group.

*Proof.* This follows from Dehn's Lemma and the loop theorem.  $\square$

**Lemma 2** (Dehn's Lemma ([Rolfsen, 2003](#), 4.A.1)). Suppose  $M^3$  is a 3-manifold and  $f: \mathbb{D}^2 \rightarrow M^3$  is a piecewise-linear map of a disk with no singularities on the boundary, i.e.

$$x \in \partial\mathbb{D}^2, x \neq y \in \mathbb{D}^2 \Rightarrow f(x) \neq f(y).$$

Then there exists an embedding  $g: \mathbb{D}^2 \rightarrow M^3$  with  $g(\partial\mathbb{D}^2) = f(\partial\mathbb{D}^2)$ .

*Invariants*

*Fundamental group of knot and link complements*

**Proposition 6.** Knot complements  $\mathbb{S}^3 \setminus K$  are aspherical.

*Proof.* Uses the Sphere theorem to show that  $\pi_2$  is trivial.  $H_3$  of the universal cover vanishes because it is non-compact. Since the universal cover is a 3-dimensional manifold we conclude that all its homotopy groups are trivial, so it is contractible. TODO More details  $\square$

**Corollary 1.** Fundamental groups of knot complements are torsion-free.

*Proof.* The classifying space  $K(\pi_1(\mathbb{S}^3 \setminus K), 1)$  has a finite dimensional model. Now a standard argument using that the group homology of cyclic groups is nontrivial in infinitely many degrees. TODO  $\square$

**Proposition 7.** If  $L$  is a non-splittable link,  $\mathbb{S}^3 \setminus L$  is aspherical.

**Corollary 2.** The fundamental group of a link complement is torsion free.

*Proof.* The link group is a free product of the groups of the non-splittable parts of  $L$ . Now use that the free product of torsion free groups is torsion free. TODO  $\square$

Heuristic idea: The length of nonempty words in  $G_i$  increases with  $i$ , and so only the identity (which is the empty word in a free group) survives in all steps.

An space  $X$  is called *aspherical* if all its higher homotopy groups vanish, i.e.  $\pi_n(X) = 0$  for  $n \geq 2$ .

For a CW-complex  $X$  this is equivalent to the universal covering  $\tilde{X}$  being contractible.

By definition, an aspherical space in an Eilenberg-MacLane space of type  $K(\pi_1(X), 1)$ .

In general, all torsion in a free product is conjugated to torsion in one of the summands of the free product.

### Alexander polynomial

**Definition 7.**  $L$  oriented link with Seifert matrix  $A$ , then the first homology of the infinite cyclic covering of the link complement,  $H_1(X_\infty; \mathbb{Z})$ , has square presentation matrix  $tA - A^T$ .

The Alexander polynomial of  $L$  is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where  $\doteq$  means “up to a multiplication with a unit  $\{\pm t^{\pm n}\}$  of the Laurent ring  $\mathbb{Z}[t, t^{-1}]$ ”.

*Remark 2.*  $\mathbb{Z}[t^{\pm 1}]$  is **not** a PID.

**Definition 8.** The tunnel number  $t(K)$  of a knot  $K \subset \mathbb{S}^3$  is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in  $\mathbb{S}^3$  is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegaard splitting in that case?).

*Remark 3.* Every link has a tunnel number, this can be seen by adding a “vertical” tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number,  $t(K) \leq c(K)$ .

*Example 3.* • The unknot is the only knot with tunnel number 0.  
(Why?)

- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

### Arf invariant

**Theorem 2.** The Arf invariant of a knot  $K$  is related to the Alexander polynomial by

$$\text{Arf}(K) = \begin{cases} 0 & \text{if } \Delta_K(-1) \equiv \pm 1 \text{ modulo } 8 \\ 1 & \text{if } \Delta_K(-1) \equiv \pm 3 \text{ modulo } 8. \end{cases}$$

*Remark 4.* If  $K$  is a slice knot, we know that its determinant  $|\Delta_K(-1)|$  is an odd square integer. Thus we have  $\Delta_K(-1) \equiv \pm 1 \text{ modulo } 8$  and as such  $\text{Arf}(K) = 0$ ; Arf is a well defined concordance invariant.

$$(2k+1)^2 = 4k^2 + 4k + 1 = \underbrace{4k(k+1)}_{\text{even}} + 1 \equiv 1 \text{ modulo } 8$$

### Tristram-Levine $\omega$ -signatures

**Definition 9** ((Lickorish, 2012, Definition 8.8), (Kauffman, 1987, Definition 12.5)). Let  $L \subset \mathbb{S}^3$  be an oriented link and  $\omega \in \mathbb{S}^1 \subset \mathbb{C}$  a unit complex number with  $\omega \neq 1$ .

The  $\omega$ -signature is sometimes called the equivariant signature or Tristram-Levine-signature.

The  $\omega$ -signature  $\sigma_\omega(L)$  of  $L$  is defined to be the signature of the Hermitian matrix<sup>1</sup>

$$(1 - \omega)A + (1 - \bar{\omega})A^T$$

where  $A$  is any Seifert matrix for  $L$ .

**Definition 10.**  $\sigma_{-1}(L) = \sigma(A + A^T)$  is known as **the signature** of  $L$  or the *Murasugi signature*.

**Theorem 3.** The  $\omega$ -signature  $\sigma_\omega(L)$  is a well defined link invariant, i.e. it does not depend on the choice of Seifert surface.

*Proof.* Directly check that the signature does not change under S-equivalence. TODO Define S-equivalence  $\square$

TODO The  $\omega$ -signature can jump at the zeros of the Alexander polynomial because at those an eigenvalue can cross zero (changes sign) and is constant in between.

There are some notes on the  $\omega$ -signatures (also relating them to the 0-surgery  $S_0^3(K)$  on the knot  $K$ ) at (Conway, 2018).

<sup>1</sup> Any Hermitian matrix is diagonalizable with real eigenvalues, and the signature is defined as the number of positive minus the number of negative eigenvalues.

Sylvester's Law of Inertia states that the signature of a Hermitian matrix  $B$  is not changed by congruence  $C \cdot B \cdot C^T$ .

See this answer on MathOverflow: <https://mathoverflow.net/questions/85976/why-tristram-levine-signature-jumps-at-the-zeros-of-the-alexander-polynomial>

## Concordance

### Slice knots

**Definition 11** (Surgery on a knot in  $S^3$ ). The notation  $S_0^3(K)$  denotes the 0-surgery on a knot  $K \subset S^3$ , i.e. removing a tubular neighborhood  $S^1 \times \mathbb{D}^2$  of  $K$  and gluing in  $\mathbb{D}^2 \times S^1$  via a homeomorphism of the boundaries, which are both  $S^1 \times S^1$ . TODO

**Definition 12** (Trace of a knot). For  $n \in \mathbb{Z}$  the  $n$ -trace of a knot  $K \subset S^3$  is the 4-manifold  $X_n(K)$  obtained by attaching an  $n$ -framed 2-handle to the 4-ball along  $K$ , i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K \times \text{framing}: S^1 \times \mathbb{D}^2 \hookrightarrow S^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

**Theorem 4** ((Miller and Piccirillo, 2018, Thm. 1.8)). •  $K$  is smoothly slice if and only if  $X_0(K)$  smoothly embeds in  $S^4$ .

- Similarly,  $K$  is topologically slice if and only if  $X_0(K)$  topologically embeds in  $S^4$ .

*Remark 5* (Exotic  $\mathbb{R}^4$  from a topologically, but not smoothly slice knot).

References: (Davis, 2011) TODO

### Concordance of Links

**Definition 13.** A smooth link cobordism between the links  $L_0, L_1 \subset S^3$  is a smooth, compact, oriented surface  $\Sigma$  generically embedded in  $S^3 \times \mathbb{I}$  such that  $\partial\Sigma = \overline{L_0} \sqcup L_1$ , where  $\partial\Sigma \subset S^3 \times \{0, 1\}$ .

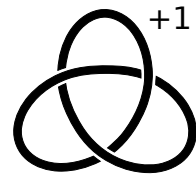


Figure 2: A Kirby diagram for  $X_n(K)$  is given just by the knot  $K$  with the framing  $n$  written next to it. For example, here is a Kirby diagram representing the 1-trace  $X_1$  (right handed trefoil). The boundary of this 4-manifold is the +1-surgery  $S_{+1}^3$  (right handed trefoil), a possible description of the Poincaré homology sphere.



**Proposition 8.** *Linking numbers are concordance invariants.*

*Remark 6* (The Hopf link is “the most non-slice link”, (Krushkal, 2015)). Any link in  $S^3$  bounds immersed smooth disks  $\coprod^n \mathbb{D}^2 \hookrightarrow \mathbb{D}^4$ .  
 TODO

**Definition 14** (Boundary link). A link  $L^n \subset S^{n+2}$  whose components bound disjoint Seifert surfaces is called a *boundary link*.

*Example 4.* The untwisted Bing double of any knot is a boundary link.  
 TODO Insert definition of Bing double  
 TODO Insert picture  
 Proof that Bing doubles are boundary links (draw the taco shells, need 4 copies of the Seifert surface of the knot)

**Proposition 9** ((Rolfsen, 2003, 5.E.1)). *If any two components of  $L^1 \subset S^3$  have nonzero linking number, then  $L$  is **not** a boundary link.*

*Proof.* Use the definition 15 of linking number where you count the intersection points of one component with a Seifert surface for the other.  $\square$

**Definition 15** (Linking number via intersections with a Seifert surface, (Rolfsen, 2003, 5.D.(2))). Let  $J$  and  $K$  be two disjoint oriented knots (e.g. link components). Pick a PL Seifert surface  $M^2$  for  $K$ , with a bicollar  $(N, N^+, N^-)$  of the interior  $\mathring{M}$ . Make  $J$  transverse to  $M$ , i.e. assume after a small homotopy of  $J$  in  $S^3 \setminus K$  that  $J$  meets  $M$  in a finite number of points and that at each point  $J$  passes locally from  $N^+$  to  $N^-$  or from  $N^-$  to  $N^+$ . Corresponding to this direction, weight the intersection types with  $+1$  or  $-1$ . The signed sum of these intersection points is the linking number  $\text{lk}(J, K) \in \mathbb{Z}$ .

**Proposition 10** ((Rolfsen, 2003, 5.E.8)). *If a link  $L$  is a boundary link, then each component represents an element in the second commutator subgroup<sup>2</sup> of the fundamental group of the complement of the remaining component(s).*

It is possible that each component of a link bounds a Seifert surface missing the other components, but still they do not bound disjoint surfaces.

On first sight this seems to depend on the choice of Seifert surface  $M$ .

<sup>2</sup> Also called *second derived subgroup* or  $G^{(2)}$ , it is generated by elements of the form  $[[x, y], [z, w]]$ .

*The Cappell-Shaneson way to slice a knot*

(Teichner et al., 2010, 4.1)

**Definition 16.** A knot  $K: S^1 \subset S^3$  which is slice in a homology 4-ball is called *homologically slice*.

*Heegaard Floer homology*

The concordance invariant  $\tau(K) \in \mathbb{Z}$  for a knot  $K \subset S^3$  is defined in (Ozsváth and Szabó, 2003), this yields a homomorphism  $\mathcal{C} \rightarrow \mathbb{Z}$ .

TODO Write up a definition

*Open question.* Find an obstruction which is able to tell the difference between homologically slice and actually slice. **(Still open?)**

## Braid groups

*Exercise 1.* Show that there is a presentation for the braid groups with just two generators.

## TQFTs - Topological Quantum Field Theories

TODO Write down axioms

A monoidal functor is supposed to preserve the identity objects for the tensor product. Since the empty set is the identity for the tensor product in the bordism category (given by disjoint union of the bordisms), the TQFT should send this to the identity object for  $\otimes_R$ , which is just the ground ring  $R$ .

## Open questions

*Open question 1.* Is the crossing number of a satellite knot bigger than that of its companion?

## Open Questions in Knot concordance

A list of knots whose sliceness status is (supposedly) not known:

1.  $(2,1)$ -cable on the figure eight
2. Negative Whitehead double of the right handed trefoil
3. Positive Whitehead double of the left handed trefoil
4. Any Whitehead double of the figure eight
5. ...

## Current developments/Recently solved

- The Conway knot  $11n34$  is not slice, as shown in (Piccirillo, 2018)
- Kronheimer and Mrowka found an error in their paper (Kronheimer and Mrowka, 2013) which (incorrectly?) claimed that Rasmussen's  $s$ -invariant could not detect an exotic 4-ball. In particular the methods of (Freedman et al., 2009) could possibly lead to an exotic 4-sphere.

# 4-manifolds

## Bordism

**Definition 17.** The  $n$ -dimensional oriented *cobordism group* over the space  $X$  is

$$\Omega_n[X] = \frac{\{f: M^n \rightarrow X \mid M^n \text{ oriented, closed, } n\text{-dim. manifold}\}}{\text{bordism}}$$

If  $\alpha: M^n \rightarrow X$  represents a bordism class,  $M^n$  is allowed to have more than one component.

**Proposition 11** ((Kauffman, 1987, 13.15, p. 319)). • *Pushing forward a fundamental class*

$$\begin{aligned} \Omega_n[X] &\rightarrow H_n(X; \mathbb{Z}) \\ [f: M^n \rightarrow X] &\mapsto f_*([M]) \end{aligned}$$

is an isomorphism for  $n \leq 3$ .

- The sequence

$$\Omega_4[*] \rightarrow \Omega_4[X] \rightarrow H_4(X; \mathbb{Z})$$

is exact.



# *Topics to study and Reading List*

## *Questions*

## *Reading List*

- List of open problems concerning quantum invariants is at ([Ohtsuki et al., 2002](#))
- Vassiliev knot invariants, for this ([Bar-Natan, 1995](#))
- Khovanov homology ([Bar-Natan, 2005](#))
- Kaufman's books, for example ([Kauffman, 2001](#))



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