Contents

Introduction

These notes are just a collection of various things that I wanted to write down to organize them better for myself.

Homotopy theory

CW-complexes

Definition 1. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1 (Whitehead's Theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n-connectedness)**.** *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\, A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

Homology

Definition 2 (Acyclic)**.** A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. This example for the Poincaré homology sphere is described in (Hatcher, 2002, Example 2.38). TODO Insert proof.

Knot Theory

Constructions & Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* $\overline{K}$ is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\overline{K}$ is the concordance inverse of K.

Proposition 2. *For $K \subset \mathbb{S}^3$ we have that $K\#r\overline{K}$ is slice, even ribbon.*

Definition 4 (Homotopically unlinked, (Rolfsen, 2003, 3.F.9.))**.** If $L = L_1 \cup \ldots \cup L_n$ is a link with n components, we say that L_i is *homotopically unlinked* from the remaining components if there is a homotopy h_t from the embedding of L_i to the constant map such that the images of h_t and L_j are disjoint at all times $t \in \mathbb{I}$ and for all other components $j \neq i$.

Example 2. In the Whitehead link both components are homotopically unlinked from each other.

Remark 1. Homotopic linking (for two component links) is **not** a symmetric relation.

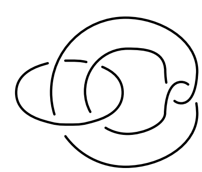

Figure 1: Positive Whitehead link, picture from (Meier, 2015).

Definition 5. A link $L = L_1 \cup L_2$ of two components in $\mathbb{R}^n$ is *splittable* if there are disjoint, topological n-balls $\mathbb{D}^n_1, \mathbb{D}^n_2 \subset \mathbb{R}^n$ such that L_i lies in the interior of $\mathbb{D}^n_i$.

Proposition 3. *If a link is splittable, then each component is homotopically unlinked from the other.*

The converse of 3 is not true, an example of van Kampen and Zeeman is discussed in (Rolfsen, 2003, 3.K.5.).

Definition 6. The lower central series of a group G is defined inductively by

Observe that $[G, G_{i-1}] = [G_{i-1}, G]$.

$$G_0 \coloneqq G$$
$$G_i \coloneqq [G, G_{i-1}] = \langle [g, h] \mid g \in G, h \in G_{i-1} \rangle$$

Proposition 4. *This satisfies:*

- $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$
- *each* G_i *is normal in* G
- *the quotient* G_i/G_{i+1} *is in the center of* G/G_{i+1}.

Lemma 1. *If F is a free group, then* $\bigcap_{i=1}^{\infty} F_i$ *is the trivial group.*

Heuristic idea: The length of nonempty words in G_i increases with i, and so only the identity (which is the empty word in a free group) survives in all steps.

Characterizing the unknot

Proposition 5. *A knot is trivial if and only if its longitude represents the trivial element of the knot group.*

Proof. This follows from Dehn's Lemma and the loop theorem. □

Lemma 2 (Dehn's Lemma (Rolfsen, 2003, 4.A.1))**.** *Suppose* M^3 *is a 3-manifold and* $f\colon \mathbb{D}^2 \to M^3$ *is a piecewise-linear map of a disk with no singularities on the boundary, i.e.*

$$x \in \partial\mathbb{D}^2, x \neq y \in \mathbb{D}^2 \Rightarrow f(x) \neq f(y).$$

Then there exists an embedding $g\colon \mathbb{D}^2 \to M^3$ *with* $g(\partial\mathbb{D}^2) = f(\partial\mathbb{D}^2)$.

Invariants

Fundamental group of knot and link complements

Proposition 6. *Knot complements* $\mathbb{S}^3 \setminus K$ *are aspherical.*

An space X is called *aspherical* if all its higher homotopy groups vanish, i.e. $\pi_n(X) = 0$ for $n \geq 2$.
For a CW-complex X this is equivalent to the universal covering $\widetilde{X}$ being contractible.
By definition, an aspherical space in an Eilenberg-MacLane space of type $K(\pi_1(X), 1)$.

Proof. Uses the Sphere theorem to show that π_2 is trivial. H_3 of the universal cover vanishes because it is non-compact. Since the universal cover is a 3-dimensional manifold we conclude that all its homotopy groups are trivial, so it is contractible. TODO More details □

Corollary 1. *Fundamental groups of knot complements are torsion-free.*

Proof. The classifying space $K(\pi_1(\mathbb{S}^3 \setminus K), 1)$ has a finite dimensional model. Now a standard argument using that the group homology of cyclic groups is nontrivial in infinitely many degrees. TODO □

Proposition 7. *If L is a non-splittable link,* $\mathbb{S}^3 \setminus L$ *is aspherical.*

Corollary 2. *The fundamental group of a link complement is torsion free.*

Proof. The link group is a free product of the groups of the non-splittable parts of L. Now use that the free product of torsion free groups is torsion free. TODO □

In general, all torsion in a free product is conjugated to torsion in one of the summands of the free product.

Alexander polynomial

Definition 7. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty;\mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 2. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

Definition 8. The *tunnel number* $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in $\mathbb{S}^3$ is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegaard splitting in that case?).

Remark 3. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 3.
- The unknot is the only knot with tunnel number 0. (Why?)
- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

Arf invariant

Theorem 2. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\mathrm{Arf}(K) = \begin{cases} 0 & \text{if } \Delta_K(-1) \equiv \pm 1 \text{ modulo } 8 \\ 1 & \text{if } \Delta_K(-1) \equiv \pm 3 \text{ modulo } 8. \end{cases}$$

Remark 4. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1$ modulo 8 and as such $\mathrm{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$(2k+1)^2 = 4k^2 + 4k + 1 = 4\underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1$ modulo 8

Tristram's ω-signatures

Definition 9 ((Lickorish, 2012, Definition 8.8), (Kauffman, 1987, Definition 12.5))**.** Let $L \subset \mathbb{S}^3$ be an oriented link and $\omega \in \mathbb{S}^1 \subset \mathbb{C}$ a unit complex number with $\omega \neq 1$.

The ω-signature $\sigma_\omega(L)$ of L is defined to be the signature or the Hermitian matrix

$$(1-\omega)A + (1-\overline{\omega})A^T$$

where A is any Seifert matrix for L.

Any Hermitian matrix is diagonalizable with real eigenvalues, and the signature is defined as the number of positive minus the number of negative eigenvalues.

Sylvester's Law of Inertia states that the signature of a Hermitian matrix B is not changed by congruence $C \cdot B \cdot C^T$.

Theorem 3. *The ω-signature $\sigma_\omega(L)$ is a well defined link invariant, i.e. it does not depend on the choice of Seifert surface.*

Proof. Directly check that the signature does not change under S-equivalence. TODO Define S-equivalence □

TODO Relation to zeros of the Alexander polynomial

Concordance

Slice knots

Definition 10 (Surgery on a knot in $\mathbb{S}^3$)**.** The notation $\mathbb{S}^3_0(K)$ denotes the 0-*surgery on a knot* $K \subset \mathbb{S}^3$, i.e. removing a tubular neighborhood $\mathbb{S}^1 \times \mathbb{D}^2$ of K and gluing in $\mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism of the boundaries, which are both $\mathbb{S}^1 \times \mathbb{S}^1$. TODO

Definition 11 (Trace of a knot)**.** For $n \in \mathbb{Z}$ the *n-trace of a knot* $K \subset \mathbb{S}^3$ is the 4-manifold $X_n(K)$ obtained by attaching an n-framed 2-handle to the 4-ball along K, i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K \times \text{framing}\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

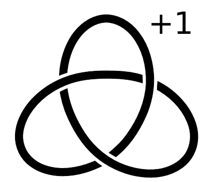

Figure 2: A Kirby diagram for $X_n(K)$ is given just by the knot K with the framing n written next to it. For example, here is a Kirby diagram representing the 1-trace X_1(right handed trefoil). The boundary of this 4-manifold is the $+1$-surgery $\mathbb{S}^3_{+1}$(right handed trefoil), a possible description of the Poincaré homology sphere.

Theorem 4 ((Miller and Piccirillo, 2018, Thm. 1.8))**.**
- *K is smoothly slice if and only if $X_0(K)$ smoothly embeds in $\mathbb{S}^4$.*
- *Similarly, K is topologically slice if and only if $X_0(K)$ topologically embeds in $\mathbb{S}^4$.*

Remark 5 (Exotic $\mathbb{R}^4$ from a topologically, but not smoothly slice knot). References: (Davis, 2011) TODO

Concordance of Links

Definition 12. A *smooth link cobordism* between the links $L_0, L_1 \subset \mathbb{S}^3$ is a smooth, compact, oriented surface Σ generically embedded in $\mathbb{S}^3 \times \mathbb{I}$ such that $\partial\Sigma = \overline{L_0} \coprod L_1$, where $\partial\Sigma \subset \mathbb{S}^3 \times \{0,1\}$.

Proposition 8. *Linking numbers are concordance invariants.*

Remark 6 (The Hopf link is "the most non-slice link", (Krushkal, 2015)). Any link in $\mathbb{S}^3$ bounds immersed smooth disks $\coprod^n \mathbb{D}^2 \looparrowright \mathbb{D}^4$. TODO

Definition 13 (Boundary link)**.** A link $L^n \subset \mathbb{S}^{n+2}$ whose components bound disjoint Seifert surfaces is called a *boundary link* .

It is possible that each component of a link bounds a Seifert surface missing the other components, but still they do not bound disjoint surfaces.

Example 4. The untwisted Bing double of any knot is a boundary link. TODO Insert definition of Bing double TODO Insert picture TODO Proof that Bing doubles are boundary links (draw the taco shells, need 4 copies of the Seifert surface of the knot)

Proposition 9 ((Rolfsen, 2003, 5.E.1))**.** *If any two components of $L^1 \subset \mathbb{S}^3$ have nonzero linking number, then L is **not** a boundary link.*

Proof. Use the definition 14 of linking number where you count the intersection points of one component with a Seifert surface for the other. □

Definition 14 (Linking number via intersections with a Seifert surface, (Rolfsen, 2003, 5.D.(2)))**.** Let J and K be two disjoint oriented knots (e.g. link components). Pick a PL Seifert surface M^2 for K, with a bicollar (N, N^+, N^-) of the interior $\mathring{M}$. Make J transverse to M, i.e. assume after a small homotopy of J in $\mathbb{S}^3 \setminus K$ that J meets M in a finite number of points and that at each point J passes locally from N^+ to N^- or from N^- to N^+. Corresponding to this direction, weight the intersection types with $+1$ or -1. The signed sum of these intersection points is the linking number $\mathrm{lk}(J, K) \in \mathbb{Z}$.

On first sight this seems to depend on the choice of Seifert surface M.

Proposition 10 ((Rolfsen, 2003, 5.E.8))**.** *If a link L is a boundary link, then each component represents an element in the second commutator subgroup*[1] *of the fundamental group of the complement of the remaining component(s).*

[1] Also called *second derived subgroup* or $G^{(2)}$, it is generated by elements of the form $[[x, y], [z, w]]$.

Heegaard Floer homology

The concordance invariant $\tau(K) \in \mathbb{Z}$ for a knot $K \subset \mathbb{S}^3$ is defined in (Ozsváth and Szabó, 2003), this yields a homomorphism $\mathcal{C} \to \mathbb{Z}$.

TODO Write up a definition

Braid groups

Exercise 1. Show that there is a presentation for the braid groups with just two generators.

TQFTs - Topological Quantum Field Theories

TODO Write down axioms

A monoidal functor is supposed to preserve the identity objects for the tensor product. Since the empty set is the identity for the tensor product in the bordism category (given by disjoint union of the bordisms), the TQFT should send this to the identity object for $\otimes_R$, which is just the ground ring R.

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

4-manifolds

TODO

Topics to study and Reading List

Questions

Reading List

- List of open problems concerning quantum invariants is at (Ohtsuki et al., 2002)
- Vassiliev knot invariants, for this could read (Bar-Natan, 1995)

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