Contents

Introduction

These notes are a collection of various topics that I wanted to write down and use as a reference.

Homotopy theory

CW-complexes

Definition 1. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1 (Whitehead's Theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n-connectedness)**.** *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\,A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

Homology

Definition 2 (Acyclic)**.** A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. If the dimension was at least 3 this does not change the fundamental group, so if we started with a nontrivial homology sphere (i.e. $\pi_1 \neq 1$) this will give an example of an acyclic, but non-contractible space.

This example for the Poincaré homology sphere is described in (Hatcher, 2002, Example 2.38). TODO Insert proof.

Knot Theory

Constructions & Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* $\overline{K}$ is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\overline{K}$ is the concordance inverse of K.

Proposition 2. *For $K \subset \mathbb{S}^3$ we have that $K \# r\overline{K}$ is slice, even ribbon.*

Definition 4 (Homotopically unlinked, (Rolfsen, 2003, 3.F.9.))**.** If $L = L_1 \cup \ldots \cup L_n$ is a link with n components, we say that L_i is *homotopically unlinked* from the remaining components if there is a homotopy h_t from the embedding of L_i to the constant map such that the images of h_t and L_j are disjoint at all times $t \in \mathbb{I}$ and for all other components $j \neq i$.

Example 2. In the Whitehead link both components are homotopically unlinked from each other.

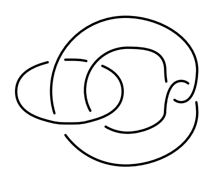

Figure 1: Positive Whitehead link, picture from (Meier, 2015).

Remark 1. Homotopic linking (for two component links) is **not** a symmetric relation.

Definition 5. A link $L = L_1 \cup L_2$ of two components in $\mathbb{R}^n$ is *splittable* if there are disjoint, topological n-balls $\mathbb{D}^n_1, \mathbb{D}^n_2 \subset \mathbb{R}^n$ such that L_i lies in the interior of $\mathbb{D}^n_i$.

Proposition 3. *If a link is splittable, then each component is homotopically unlinked from the other.*

The converse of 3 is not true, an example of van Kampen and Zeeman is discussed in (Rolfsen, 2003, 3.K.5.).

Definition 6. A link is called *algebraically split* if any pair of components has linking number zero.

The Whitehead link is algebraically split. An example of an algebraically split 3-component link are the Borromean rings.

Remark 2. Length 2 Milnor invariants are exactly the pairwise linking numbers $\mathrm{lk}(L_i, L_j)$ of a link. Thus for an algebraically split link the triple Milnor invariants are well defined integers.

Definition 7. The lower central series of a group G is defined inductively by

Observe that $[G, G_{i-1}] = [G_{i-1}, G]$.

$$G_0 := G$$
$$G_i := [G, G_{i-1}] = \langle [g,h] \mid g \in G, h \in G_{i-1} \rangle$$

Proposition 4. *This satisfies:*

- $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$
- *each* G_i *is normal in* G
- *the quotient* G_i/G_{i+1} *is in the center of* G/G_{i+1}.

Lemma 1. *If* F *is a free group, then* $\bigcap_{i=1}^{\infty} F_i$ *is the trivial group.*

Heuristic idea: The length of nonempty words in G_i increases with i, and so only the identity (which is the empty word in a free group) survives in all steps.

Definition 8. A *metabelian group* G is a group whose commutator subgroup $[G, G]$ is abelian.

Equivalently, G is metabelian if and only if there is an abelian normal subgroup $A \trianglelefteq G$ such that the quotient group G/A is abelian. In particular this means that metabelian groups are precisely the solvable groups of derived length[1] ≤ 2.

[1] The least n such that $G^{(n)} = \{1\}$ is called the *derived length* of a solvable group G.

Example 3.
- Any dihedral group is metabelian, because is contains a cyclic normal subgroup of index 2.
- The symmetric group S_4 is not metabelian, because its commutator subgroup is the non-abelian alternating group A_4.

Characterizing the unknot

Proposition 5. *A knot is trivial if and only if its longitude represents the trivial element of the knot group.*

Proof. This follows from Dehn's Lemma and the loop theorem. □

Lemma 2 (Dehn's Lemma (Rolfsen, 2003, 4.A.1))**.** *Suppose* M^3 *is a 3-manifold and* $f\colon \mathbb{D}^2 \to M^3$ *is a piecewise-linear map of a disk with no singularities on the boundary, i.e.*

$$x \in \partial\mathbb{D}^2, x \neq y \in \mathbb{D}^2 \Rightarrow f(x) \neq f(y).$$

Then there exists an embedding $g\colon \mathbb{D}^2 \to M^3$ *with* $g(\partial\mathbb{D}^2) = f(\partial\mathbb{D}^2)$.

Branched Coverings

Definition 9 (Cyclic covers branched over a knot)**.** TODO Insert definition, examples

Exercise 1 ((Rolfsen, 2003, 10.F.2)). Find a 600-fold branched noncyclic covering $\mathbb{S}^3 \to \mathbb{S}^3$ branched over the trefoil.

Solution: The Poincaré homology sphere P^3 appears the 5-fold cyclic covering of $\mathbb{S}^3$ branched over the trefoil. Moreover, $\mathbb{S}^3$ is the 120-sheeted universal covering of the Poincaré manifold, so by composing these we obtain a $5 \cdot 120 = 600$-sheeted branched covering of the 3-sphere:

$$\begin{tikzcd}
 & \mathbb{S}^3 \arrow[dl, "\text{5-fold branched cover}"'] \arrow[dd, "\text{600-fold branched cover}"] \\
P^3 \arrow[dr, "\text{120-fold cover}"'] & \\
 & \mathbb{S}^3
\end{tikzcd}$$

Invariants

Fundamental group of knot and link complements

Proposition 6. *Knot complements $\mathbb{S}^3 \setminus K$ are aspherical.*

An space X is called *aspherical* if all its higher homotopy groups vanish, i.e. $\pi_n(X) = 0$ for $n \geq 2$.
For a CW-complex X this is equivalent to the universal covering $\widetilde{X}$ being contractible.
By definition, an aspherical space in an Eilenberg-MacLane space of type $K(\pi_1(X), 1)$.

Proof. Uses the Sphere theorem to show that π_2 is trivial. H_3 of the universal cover vanishes because it is non-compact. Since the universal cover is a 3-dimensional manifold we conclude that all its homotopy groups are trivial, so it is contractible. TODO More details □

Corollary 1. *Fundamental groups of knot complements are torsion-free.*

Proof. The classifying space $K(\pi_1(\mathbb{S}^3 \setminus K), 1)$ has a finite dimensional model. Now a standard argument using that the group homology of cyclic groups is nontrivial in infinitely many degrees. TODO □

Proposition 7. *If L is a non-splittable link, $\mathbb{S}^3 \setminus L$ is aspherical.*

Corollary 2. *The fundamental group of a link complement is torsion free.*

Proof. The link group is a free product of the groups of the non-splittable parts of L. Now use that the free product of torsion free groups is torsion free. TODO □

In general, all torsion in a free product is conjugated to torsion in one of the summands of the free product.

Alexander polynomial

Definition 10. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 3. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID, for example the ideal $(2, 1+t)$ has height 2 and thus is not principal.

Definition 11. The *tunnel number* $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in $\mathbb{S}^3$ is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegaard splitting in that case?).

Remark 4. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 4. • The unknot is the only knot with tunnel number 0. (Why?)

- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

Unknotting number

Definition 12. The unknotting number $u(K)$ of a knot K is the minimum number crossing changes required to transform it into an unknot.

Proposition 8 ((Scharlemann and Thompson, 1989))**.** *Doubled knots are exactly those knots with three-genus and unknotting number* $= 1$*.*

Arf invariant

Theorem 2. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\mathrm{Arf}(K) = \begin{cases} 0 & \textit{if } \Delta_K(-1) \equiv \pm 1 \textit{ modulo } 8 \\ 1 & \textit{if } \Delta_K(-1) \equiv \pm 3 \textit{ modulo } 8. \end{cases}$$

Remark 5. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1$ modulo 8 and as such $\mathrm{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$(2k+1)^2 = 4k^2 + 4k + 1 = 4\underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1$ modulo 8

Tristram-Levine ω-signatures

Definition 13 ((Lickorish, 2012, Definition 8.8), (Kauffman, 1987, Definition 12.5))**.** Let $L \subset \mathbb{S}^3$ be an oriented link and $\omega \in \mathbb{S}^1 \subset \mathbb{C}$ a unit complex number with $\omega \neq 1$.

The ω-signature $\sigma_\omega(L)$ of L is defined to be the signature of the Hermitian matrix[2]

$$(1-\omega)A + (1-\overline{\omega})A^T$$

where A is any Seifert matrix for L.

The *ω-signature* is sometimes called the *equivariant signature* or *Tristram-Levine-signature*.

[2] Any Hermitian matrix is diagonalizable with real eigenvalues, and the signature is defined as the number of positive minus the number of negative eigenvalues.

Sylvester's Law of Inertia states that the signature of a Hermitian matrix B is not changed by congruence $C \cdot B \cdot C^T$.

Definition 14. $\sigma_{-1}(L) = \sigma(A + A^T)$ is known as **the** signature of L or the *Murasugi signature* .

Theorem 3. *The* ω*-signature* $\sigma_\omega(L)$ *is a well defined link invariant, i.e. it does not depend on the choice of Seifert surface.*

Proof. Directly check that the signature does not change under S-equivalence. TODO Define S-equivalence □

TODO The ω-signature can jump at the zeros of the Alexander polynomial because at those an eigenvalue can cross zero (changes sign) and is constant in between.

See this answer on MathOverflow: https://mathoverflow.net/questions/85976/why-tristram-levine-signature-jumps-at-the-zeros-o

There are some notes on the ω-signatures (also relating them to the 0-surgery $\mathbb{S}^3_0(K)$ on the knot K) at (Conway, 2018).

Milnor invariants

TODO

(Meilhan, 2018)

Concordance

Slice knots

Definition 15. A knot $K \subset \mathbb{S}^3$ is *topologically slice* if it bounds a locally flat 2-disk $\Delta^2 \subset \mathbb{D}^4$.

Definition 16. • TODO: Define smoothly slice and ribbon

- TODO: Draw pictures

Observation 1. One can connect sum a given slice disk with any knotted 2-sphere to obtain infinitely many non-isotopic slice disks for every slice knot. To see that this gives non-isotopic disks one can look at the fundamental groups of the slice disk exteriors, $\pi_1(\mathbb{D}^4 \setminus \Delta)$.

Definition 17 (Surgery on a knot in $\mathbb{S}^3$)**.** The notation $\mathbb{S}^3_0(K)$ denotes the 0-*surgery on a knot* $K \subset \mathbb{S}^3$, i.e. removing a tubular neighborhood $\mathbb{S}^1 \times \mathbb{D}^2$ of K and gluing in $\mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism of the boundaries, which are both $\mathbb{S}^1 \times \mathbb{S}^1$. TODO

Definition 18 (Trace of a knot)**.** For $n \in \mathbb{Z}$ the *n-trace of a knot* $K \subset \mathbb{S}^3$ is the 4-manifold $X_n(K)$ obtained by attaching an n-framed 2-handle to the 4-ball along K, i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K \times \text{framing}\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

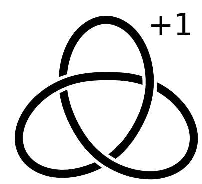

Figure 2: A Kirby diagram for $X_n(K)$ is given just by the knot K with the framing n written next to it. For example, here is a Kirby diagram representing the 1-trace X_1(right handed trefoil). The boundary of this 4-manifold is the $+1$-surgery $\mathbb{S}^3_{+1}$(right handed trefoil), a possible description of the Poincaré homology sphere.

Theorem 4 ((Miller and Piccirillo, 2018, Thm. 1.8))**.** • *K is smoothly slice if and only if $X_0(K)$ smoothly embeds in $\mathbb{S}^4$.*

- *Similarly, K is topologically slice if and only if $X_0(K)$ topologically embeds in $\mathbb{S}^4$.*

Remark 6 (Exotic $\mathbb{R}^4$ from a topologically, but not smoothly slice knot)*.* References: (Davis, 2011) TODO

Observe that this construction gives a *large* exotic $\mathbb{R}^4$ since it contains the compact subset $X_0(K)$ which does not embed smoothly in $\mathbb{R}^4_{\text{std}}$.

Definition 19. A knot K is called *doubly slice* if it occurs as an equatorial[3] cross section

$$K = S \cap \mathbb{S}^3$$

of an unknotted topologically locally flat 2-sphere $S^2 \subset \mathbb{S}^4$ embedded in the 4-sphere.

[3] $\mathbb{S}^3 \subset \mathbb{S}^4$ is the standard inclusion of the equator.

A double slice knot slices in two different ways.

Lemma 3 ((Friedl and Orson), (Friedl and Orson, 2015))**.** *$K \# -K$ is smoothly doubly slice.*

Remark 7. We can use the standard spinning of this knot to obtain a 2-sphere with cross-section $K\# -K$, but this $\mathbb{S}^2 \subset \mathbb{S}^4$ will be knotted in general. We can see this because its knot group is the same as the one of the knot we started with. So we cannot use this "obvious" sphere to prove that $K\# -K$ knot is doubly slice.

As an alternative, we can use the (more complicated) ± 1-twist spin of our knot to obtain an unknotted sphere in $\mathbb{S}^4$ with cross-section $K\# -K$ (Zeeman, 1965).

Definition 20. A knot $K \subset \mathbb{S}^3$ is *homotopy ribbon* if there is a slicing disk $\Delta^2 \subset \mathbb{D}^4$ for with the inclusion of the knot exterior into the slice disc exterior induces a surjection on fundamental groups,

$$\pi_1(\mathbb{S}^3 \setminus K) \twoheadrightarrow \pi_1(\mathbb{D}^4 \setminus \Delta)$$

Concordance of Links

Definition 21. A *smooth link cobordism* between the links $L_0, L_1 \subset \mathbb{S}^3$ is a smooth, compact, oriented surface Σ generically embedded in $\mathbb{S}^3 \times \mathbb{I}$ such that $\partial\Sigma = \overline{L_0} \coprod L_1$, where $\partial\Sigma \subset \mathbb{S}^3 \times \{0,1\}$.

Proposition 9. *Linking numbers are concordance invariants.*

Remark 8 (The Hopf link is "the most non-slice link", (Krushkal, 2015)). Any link in $\mathbb{S}^3$ bounds immersed smooth disks $\coprod^n \mathbb{D}^2 \looparrowright \mathbb{D}^4$. TODO

Definition 22 (Boundary link)**.** A link $L^n \subset \mathbb{S}^{n+2}$ whose components bound disjoint Seifert surfaces is called a *boundary link* .

It is possible that each component of a link bounds a Seifert surface missing the other components, but still they do not bound disjoint surfaces.

Example 5. The untwisted Bing double of any knot is a boundary link. TODO Insert definition of Bing double TODO Insert picture TODO Proof that Bing doubles are boundary links (draw the taco shells, need 4 copies of the Seifert surface of the knot)

Proposition 10 ((Rolfsen, 2003, 5.E.1))**.** *If any two components of $L^1 \subset \mathbb{S}^3$ have nonzero linking number, then L is **not** a boundary link.*

Proof. Use the definition 23 of linking number where you count the intersection points of one component with a Seifert surface for the other. □

Definition 23 (Linking number via intersections with a Seifert surface, (Rolfsen, 2003, 5.D.(2)))**.** Let J and K be two disjoint oriented knots (e.g. link components). Pick a PL Seifert surface M^2 for K, with a bicollar (N, N^+, N^-) of the interior $\mathring{M}$. Make J transverse to M, i.e. assume after a small homotopy of J in $\mathbb{S}^3 \setminus K$ that J meets M in a finite number of points and that at each point J passes locally from N^+ to N^- or from N^- to N^+. Corresponding to this direction, weight the intersection types with $+1$ or -1. The signed sum of these intersection points is the linking number $\operatorname{lk}(J, K) \in \mathbb{Z}$.

On first sight this seems to depend on the choice of Seifert surface M.

Proposition 11 ((Rolfsen, 2003, 5.E.8))**.** *If a link L is a boundary link, then each component represents an element in the second commutator subgroup*[4] *of the fundamental group of the complement of the remaining component(s).*

[4] Also called *second derived subgroup* or $G^{(2)}$, it is generated by elements of the form $[[x, y], [z, w]]$.

The Cappell-Shaneson way to slice a knot

(Teichner et al., 2010, 4.1)

Definition 24. A knot $K\colon \mathbb{S}^1 \subset \mathbb{S}^3$ which is slice in a homology 4-ball is called *homologically slice*.

Open question ((Still open?)). Find an obstruction which is able to tell the difference between homologically slice and actually slice.

Heegaard Floer homology

The concordance invariant $\tau(K) \in \mathbb{Z}$ for a knot $K \subset \mathbb{S}^3$ is defined in (Ozsváth and Szabó, 2003), this yields a homomorphism $\mathcal{C} \to \mathbb{Z}$.

TODO Explain this definition

Definition 25 (Ozsváth and Szabó's τ-invariant)**.**

$$\tau(K) = \min\{i \in \mathbb{Z} \mid H_*(\mathcal{F}(K, i)) \to \widehat{HF}(\mathbb{S}^3) \text{ is nontrivial}\}$$

Properties of τ:

- It provides a lower bound for the four-ball genus,

$$|\tau(K)| \leq g_4(K)$$

- TODO

Definition 26 ((Hom, 2011))**.** An *L-space* is a rational homology 3-sphere[5] with the simplest possible Heegaard-Floer homology,

$$\dim \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$

[5] Y^3 is a rational homology sphere iff $H_1(Y; \mathbb{Q}) = 0$, which means that the first homology with integer coefficients is a finite group.

For any rational homology 3-sphere $\operatorname{rk} \widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|$, and L-spaces are those for which this bound is sharp.

A knot admitting a non-trivial L-space surgery is called an *L-space knot* .

Example 6.
- Lens spaces are L-spaces (and this is also where the name comes from).
- Torus knots are L-space knots, as $pq \pm 1$ surgery on the (p, q)-torus knot yields a lens space.

TODO Why?

The Algebraic Concordance Group

Definition 27. Let A, B be square integral matrices. A is called *cobordant* to B if $A \oplus (-B)$ is hyperbolic.

A congruent to B implies A cobordant to B.

Remark 9. • Cobordism of matrices is reflexive and symmetric

- For transitivity one has to impose the extra restriction that $\det(A - A^T) \neq 0$ (but this is always satisfied for Seifert matrices A because $A - A^T$ represents the intersection form on a basis for the first homology) and the following lemma

Lemma 4 (Stable hyperbolic implies hyperbolic / Cancellation)**.** *Suppose A, N are square and $\det(N - N^T) \neq 0$. If $A \oplus N$ and N are hyperbolic, then A is hyperbolic.*

Property P and R

Definition 28 (Property P, (Kirby and Melvin, 1978))**.** If (nontrivial) Dehn surgery on a knot cannot yield a homotopy 3-sphere then the knot is said to satisfy *Property P* .

Theorem 5 (Kronheimer, Mrowka)**.** *All knots, except the unknot, have Property P.*

Definition 29 (Property R)**.** If 0-framed surgery on a knot cannot yield $\mathbb{S}^1 \times \mathbb{S}^2$ then it satisfies *Property R* .

We could also drop the condition that the framing is assumed to be trivial: It is clear from looking at the intersection pairing on the trace of the surgery that if the result is $\mathbb{S}^1 \times \mathbb{S}^2$, the framing had to be 0.

Theorem 6 (Gabai)**.** *If* 0*-framed surgery on* $K \subset \mathbb{S}^3$ *is* $\mathbb{S}^1 \times \mathbb{S}^2$ *then* K *is the unknot*[6]*.*

[6] In other words: All knots, except the unknot, have Property R.

Conjecture 1 (Generalized Property R, (Scharlemann and Thompson, 2009), (Kirby, 1995, Problem 1.82))*.* Suppose $L \subset \mathbb{S}^3$ is an n-component link with integer framings, and surgery on L gives the connected sum $\#^n(\mathbb{S}^1 \times \mathbb{S}^2)$.

Then there is a sequence of handle slides on L that converts L into a 0-framed unlink.

Remark 10*.*
- For n=1, i.e. a single component, there are no slides possible and the statement is Gabai's theorem.
- For $n > 1$ it is necessary to include the possibility of handle slides into the theorem. For an example, you can use Kirby moves on a zero framed unlink to complicate the link.
- The only framed links that are relevant for the conjecture are those where all framings and linking numbers are trivial.

Satellite operators

(Cochran et al., 2014)

Let P be an oriented knot in the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, this is called a *pattern knot* . For K an oriented knot in $\mathbb{S}^3$ we denote by $P(K)$ the (untwisted) *satellite* of K with pattern P. The knot K is sometimes called a companion[7] of $P(K)$ (Lickorish, 2012, p. 10).

An important special case of satellites are

[7] You can imagine that the satellite knot is orbiting its companion.

Definition 30 ((p,q)-cables)**.** The (p,q)-*cable* of a knot K , denoted by $K_{p,q}$ is the satellite knot with pattern the (p,q)-torus knot. p is the number of times $K_{p,q}$ transverses the longitudinal direction of K, and q the meridional number.

We can view a pattern as a function

$$P \colon \mathcal{K} \to \mathcal{K}$$

on the set of isotopy classes of knots. These descent to yield functions (*satellite operators*)

$$\mathcal{K}/\sim \to \mathcal{K}/\sim$$

$\mathcal{K} = \frac{\{\text{knots } \mathbb{S}^1 \hookrightarrow \mathbb{S}^3\}}{\text{isotopy}}$

for various equivalence relations $\sim$ on $\mathcal{K}$, for example concordance.

If K and J are concordant, their satellites will be concordant via a concordance which "follows along" the concordance between K and J.

Braid groups

Exercise 2. Show that there is a presentation for the braid groups with just two generators.

Definition 31 ((Hedden, 2010))**.** A knot which has a diagram with only positive crossings is called a *positive knot* .

Equivalently, a knot is positive if it has a projection for which the writhe is equal to the crossing number.

Positive braids are those knots and links which can be obtained as the closure of a word in the braid group containing only positive generators[8].

A *quasi-positive braid* is a braid that is a product of conjugates of positive generators[9].

A knot is called *quasi-positive* if it is the closure of a quasi-positive braid.

[8] I.e. closures of products $\prod_{k=1}^{m} \sigma_{i_k}$, where there are no σ_i^{-1}, and σ_i is a positive Artin generator.

[9] I.e. as products of positive bands $\alpha \cdot \sigma_j \cdot \alpha^{-1}$, where $\alpha \in B_n$ is an arbitrary braid word

Quasipositive knots are closures of braids consisting of arbitrary conjugates of positive generators.

Strongly quasipositive knots require these conjugates to be of a special form useful for constructing Seifert surfaces.

TQFTs - Topological Quantum Field Theories

TODO Write down axioms

A monoidal functor is supposed to preserve the identity objects for the tensor product. Since the empty set is the identity for the tensor product in the bordism category (given by disjoint union of the bordisms), the TQFT should send this to the identity object for $\otimes_R$, which is just the ground ring R.

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

Open question 2 (Smooth unknotting conjecture). A 2-knot is trivial if and only if the fundamental group is infinite cyclic.

Remark 11.
- It is proved that a 2-knot exterior is homotopy equivalent to S^1 if and only if the fundamental group is infinite cyclic
- Freedman proved the unknotting conjecture in the topological category, but it is still open in the piecewise-linear or smooth category

Open Questions in Knot concordance

A list of knots whose sliceness status is (supposedly) not known:

1. $(2, 1)$-cable on the figure eight
2. Negative Whitehead double of the right handed trefoil
3. Positive Whitehead double of the left handed trefoil
4. Any Whitehead double of the figure eight
5. ...

Current developments/Recently solved

- The Conway knot $11n34$ is not slice, as shown in (Piccirillo, 2018)
- Kronheimer and Mrowka found an error in their paper (Kronheimer and Mrowka, 2013) which (incorrectly?) claimed that Rasmussen's s-invariant could not detect an exotic 4-ball.
  In particular the methods of (Freedman et al., 2009) could possibly lead to an exotic 4-sphere.

4-Manifolds

Bordism

Definition 32. The n-dimensional oriented *cobordism group* over the space X is

$$\Omega_n[X] = \frac{\{f\colon M^n \to X \mid M^n \text{ oriented, closed, } n\text{-dim. manifold}\}}{\text{bordism}}$$

If $\alpha\colon M^n \to X$ represents a bordism class, M^n is allowed to have more than one component.

Proposition 12 ((Kauffman, 1987, 13.15, p. 319)).
- *Pushing forward a fundamental class*

  $$\begin{aligned} \Omega_n[X] &\to H_n(X;\mathbb{Z}) \\ [f\colon M^n \to X] &\mapsto f_*([M]) \end{aligned}$$

  is an isomorphism for $n \leq 3$.

- *The sequence*

  $$\Omega_4[*] \to \Omega_4[X] \to H_4(X;\mathbb{Z})$$

  is exact.

Mazur manifolds

References:

- Homology spheres: (Saveliev, 2013)

- (Akbulut et al., 1979)

  TODO

Andrews-Curtis Conjecture

(SCP)

Definition 33. A *balanced presentation* is a presentation

$$\langle g_1, \dots, g_n \mid r_1, \dots, r_n \rangle$$

with the same number of generators and relations.

The Andrew-Curtis moves on a balanced presentation are

(i) **1-handle slides:** Replace a pair of generators $\{x, y\}$ by $\{x, xy\}$

(ii) **2-handle slides:** Replace a pair of relations $\{r, s\}$ by $\{grg^{-1}s, s\}$, where g is any word in the generators

(iii) **1-2-handle cancellations:** Add a generator together with a new relation killing it

In particular, you are <u>not</u> allowed to make a copy of a relation to keep for later use (which would correspond to **2-3-handle creation/cancellation**).

Conjecture (Andrews-Curtis conjecture (Probably false?)). Any balanced presentation of the trivial group can be transformed by Andrews-Curtis moves to the trivial presentation.

Example 7. TODO

$$\langle x, y \mid xyx = yxy, x^5 = y^4 \rangle$$

is a balanced presentation of the trivial group, but until now nobody was able to find a sequence of Andrews-Curtis moves to transform it into the trivial presentation

$$\langle x, y \mid x, y \rangle$$

TODO Explain how to construct homotopy 4-spheres from balanced presentations of the trivial group (Akbulut and Kirby)

In (Kirby, 2006, I.3) Kirby suspected that the *Dolgachev surface* $E(1)_{2,3}$ (an exotic copy of the rational elliptic surface $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$) might require 1- and/or 3-handles. But in (Akbulut, 2008) Akbulut found a handlebody presentation without those.

Open question ((Still open?)). Does a simply connected, closed, smooth 4-manifold need 1-handles and/or 3-handles?

Observation 2. If exotic $\mathbb{S}^4$ exist, their handle decomposition must contain 1- or 3-handles.

For this suppose you have a handle decomposition of a manifold with the homology of $\mathbb{S}^4$ and no 1- and 3-handles, then there could not be any 2-handles either because these would give nontrivial second homology.

Definition 34. Use the following notation to denote the abelian group

$$\Gamma_n = \frac{\text{orientation preserving diffeomorphisms of } \mathbb{S}^{n-1}}{\text{those that extend to a diffeomorphism of } \mathbb{D}^n}$$

- $\Gamma_1 = \Gamma_2 = 0$
- Munkres, Smale: $\Gamma_3 = 0$
- Cerf: $\Gamma_4 = 0$
- Kervaire, Milnor: $\Gamma_5 = \Gamma_6 = 0$, $\Gamma_7 = \mathbb{Z}/28$

Proposition 13. *For $n \geq 5$ we can identify Γ_n with the set of oriented smooth structures of the topological n-sphere. I.e. in dimension ≥ 5 all exotic spheres can be obtained by using a diffeomorphism of $\mathbb{S}^{n-1}$ to glue two n-disks along their boundary.*

Theorem 7 (Cerf, (Geiges and Zehmisch, 2010))**.** *Any diffeomorphism of the 3-sphere $\mathbb{S}^3$ extends over the 4-ball $\mathbb{D}^4$, in other words*

$$\Gamma_4 = 0.$$

Observation 3. Cerf's theorem implies that there are no exotic structures on $\mathbb{S}^4$ that can be obtained by gluing two 4-disks along their boundary.

Trisections

"TRISECTIONS ARE TO 4-MANIFOLDS AS HEEGAARD SPLITTINGS ARE TO 3-MANIFOLDS"

References

- Original paper: (Gay and Kirby, 2016)
- Lecture notes: (Gay, 2019)

Definitions

Definition 35. • The standard genus g surface is

$$\Sigma_g = \#^g(\mathbb{S}^1 \times \mathbb{S}^1)$$

- The standard genus g solid handlebody is

$$H_g = \natural^g(\mathbb{S}^1 \times \mathbb{D}^2)$$

  with $\partial H_g = \Sigma_g$
- The standard 4-dimensional 1-handlebody (of "genus k") is

$$Z_k = \natural^k(\mathbb{S}^1 \times \mathbb{D}^3)$$

  i.e. a 4-ball to which we attach k-many 4-dimensional 1-handles.

- $\#^n A$ is the connected sum of n copies of A, with $\#^0 A = \mathbb{S}^m$
- TODO

$\partial(A \natural B) = (\partial A)\#(\partial B)$

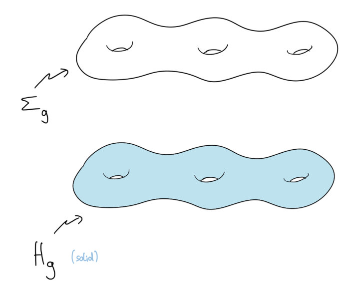

Figure 3: Standard manifolds of dimension 2, 3

Topics to study and Reading List

Questions

- TODO Collect questions that come up, and preferably with an answer

Reading List

- List of open problems concerning quantum invariants is at (Ohtsuki et al., 2002)
- Vassiliev knot invariants, for this (Bar-Natan, 1995)
- Khovanov homology (Bar-Natan, 2005)
- Kaufman's books, for example (Kauffman, 2001)
- Baez's book on gauge theory (Baez and Muniain, 1994)

Possible Projects

Using Machine Learning to calculate Knot Invariants

Literature:

- (Jejjala et al., 2019): The authors used a neural network to investigate a relationship between the Jones polynomial of a knot and the hyperbolic volume of the knot complement in case that the knot is hyperbolic
- (Hughes, 2016)
- (Bull et al., 2018)

Ideas:

- Use similar machine learning techniques to approximate concordance invariants of knots and links
- Use Khovanov homology groups as inputs for the network in the hope that this captures more combinatorial content about the knot than merely the coefficients of knot polynomials. One could also try to apply this to other knot homology theories
- Use a convolutional neural network to encode and process 2-dimensional input: For example the Khovanov and Heegaard-Floer complexes of a knot can be equipped with various gradings which could give extra information
- In the case that there seems to be a previously unknown relationship between invariants one could then try to prove this exactly

Visualizing Slice Disks in Trisection diagrams

There is a relative setting where trisections can be used to describe embedded 2-dimensional submanifolds of 4-manifolds. Maybe from this one could try to get new statements about concordance?
TODO:

- Learn more about trisections and in particular about representing surfaces in 4-manifolds
- Find out what people have already tried with questions about concordance using trisections
- There should be a calculus for modifying trisections? Find out more about this. Would it be feasible to implement this in a computer program, in the spirit of something like Frank Swenton's KirbyCalculator? (Swenton)

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