

\mathbb{CP}^2 versus $S^2 \vee S^4$

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\mathbb{CP}^2	$S^2 \vee S^4$
$\mathbb{Z}[x]/x^3$	$\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta)$
$ x = 2$	$ \alpha = 2, \beta = 4$

Table 1: Cup product structures on $H^*(_, \mathbb{Z})$.

The Hopf map

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$$\eta: \partial\mathbb{D}^4 = \mathbb{S}^3 \rightarrow \mathbb{S}^2 = \text{2-skeleton of } \mathbb{CP}^2$$

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- ▶ In other words, $\mathbb{CP}^2 = \text{Cone}(\eta)$ is the mapping cone of the Hopf map.
- ▶ The argument on the last slide shows that η is not nullhomotopic, $0 \neq [\eta] \in \pi_3(\mathbb{S}^2)$, because if it were zero its mapping cone would be homotopy equivalent to $\mathbb{S}^2 \vee \mathbb{S}^4$.

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- ▶ One explicit way to think of this map is by writing it as

$$\begin{aligned} \mathbb{C}^2 \supset \mathbb{S}^3 &\rightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 \\ (z_0, z_1) &\mapsto \frac{z_0}{z_1} \end{aligned}$$

Hopf fibration

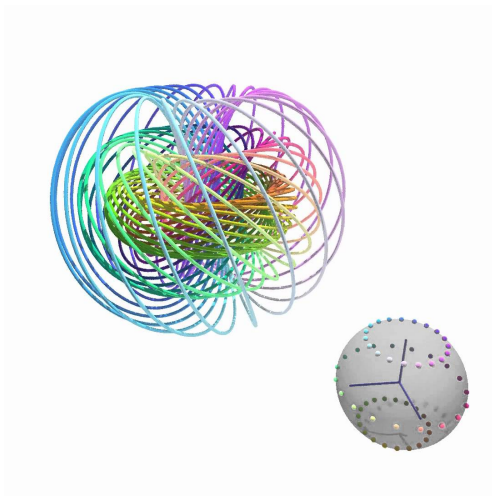


Figure 1: Some fibres of $\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2$ drawn in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$

Source: <https://nilesjohnson.net/hopf-production.html>

Outlook on the next semester: Stable phenomena

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We will also look at the (iterated) suspensions:

$$\begin{aligned}\eta: \mathbb{S}^3 &\rightarrow \mathbb{S}^2 \in \pi_3(\mathbb{S}^2) \\ \rightsquigarrow \Sigma\eta: \mathbb{S}^4 &\rightarrow \mathbb{S}^3 \in \pi_4(\mathbb{S}^3) \\ \rightsquigarrow \Sigma^2\eta: \mathbb{S}^5 &\rightarrow \mathbb{S}^4 \in \pi_5(\mathbb{S}^4) \\ \rightsquigarrow &\dots\end{aligned}$$

Homotopy groups of spheres

$\pi_i(S^n)$													
		$i \rightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0
\downarrow	2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2
	5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}
	6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2
	7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0
	8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0

Figure 2: Some homotopy groups of spheres.

Source: Hatcher, Algebraic Topology, Section 4.1, p.339