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- Question: What is the difference between $\mathbb{S}^2 \vee \mathbb{S}^4$ and $\mathbb{CP}^2$?
- How can we prove that they are **not** homotopy equivalent?

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$\mathbb{CP}^2$	$\mathbb{S}^2 \vee \mathbb{S}^4$
$\mathbb{Z}[x]/x^3$	$\Lambda[\alpha, \beta]$
$\lvert x \rvert = 2$	$\lvert\alpha\rvert = 2, \lvert\beta\rvert = 4$

Table 1: Cup product structures on $H^*(\_\_; \mathbb{Z})$.

The Hopf map

- The attaching map of the 4-cell,

$$\eta\colon \partial\mathbb{D}^4 = \mathbb{S}^3 \to \mathbb{S}^2 = \text{2-skeleton of } \mathbb{CP}^2$$

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- In other words, $\mathbb{CP}^2 = \mathsf{Cone}(\eta)$ is the mapping cone of the Hopf map.
- The argument on the last slide shows that η is not nullhomotopic, $0 \neq [\eta] \in \pi_3(\mathbb{S}^2)$, because if it were zero its mapping cone would be homotopy equivalent to $\mathbb{S}^2 \vee \mathbb{S}^4$.

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- One explicit way to think of this map is by writing it as

$$\begin{aligned} \mathbb{C}^2 \supset \mathbb{S}^3 &\to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 \\ (z_0, z_1) &\mapsto \frac{z_0}{z_1} \end{aligned}$$

Hopf fibration

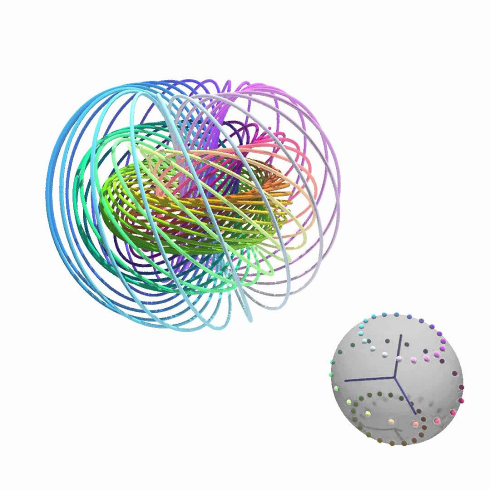

Figure 1: Some fibres of $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$ drawn in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$
Source: `https://nilesjohnson.net/hopf-production.html`

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We will also look at the (iterated) suspensions:

$$\begin{aligned} \eta\colon \mathbb{S}^3 \to \mathbb{S}^2 &\in \pi_3(\mathbb{S}^2) \\ \rightsquigarrow \Sigma\eta\colon \mathbb{S}^4 \to \mathbb{S}^3 &\in \pi_4(\mathbb{S}^3) \\ \rightsquigarrow \Sigma^2\eta\colon \mathbb{S}^5 \to \mathbb{S}^4 &\in \pi_5(\mathbb{S}^4) \\ \rightsquigarrow \dots \end{aligned}$$

Homotopy groups of spheres

$\pi_i(S^n)$

$n \downarrow$ \ $i \longrightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Figure 2: Source: Hatcher, Algebraic Topology, Section 4.1, p.339