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Introduction

Homotopy theory

CW-complexes

Definition 1. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1 (Whitehead's Theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n-connectedness)**.** *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\, A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

Homology

Definition 2 (Acyclic)**.** A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. This example for the Poincare homology sphere is described in Example 2.38 [1]. TODO Insert proof.

[1] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002

Knot Theory

Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* $\overline{K}$ is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\overline{K}$ is the concordance inverse of K.

Proposition 2. *For $K \subset \mathbb{S}^3$ we have that $K \# r\overline{K}$ is slice, even ribbon.*

Alexander polynomial

Definition 4. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 1. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

Definition 5 (Surgery on a knot in $\mathbb{S}^3$)**.** The notation $\mathbb{S}^3_0(K)$ denotes the 0-surgery on a knot $K \subset \mathbb{S}^3$, i.e. removing a tubular neighborhood $\mathbb{S}^1 \times \mathbb{D}^2$ of K and gluing in $\mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism of the boundaries, which are both $\mathbb{S}^1 \times \mathbb{S}^1$. TODO

Definition 6 (Trace of a knot)**.** For $n \in \mathbb{Z}$ the n-trace of a knot $K \subset \mathbb{S}^3$ is the 4-manifold $X_n(K)$ obtained by attaching an n-framed 2-handle to the 4-ball along K, i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K \times \text{framing}\colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

A Kirby diagram for $X_n(K)$ is given just by the knot K with the framing n written next to it.

Invariants

Definition 7. The *tunnel number* $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in S^3 is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegard splitting in that case?).

Remark 2. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2.
- The unknot is the only knot with tunnel number 0. (Why?)
- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

Arf invariant

Theorem 2. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\operatorname{Arf}(K) = \begin{cases} 0 & \textit{if } \Delta_K(-1) \equiv \pm 1 \textit{ modulo } 8 \\ 1 & \textit{if } \Delta_K(-1) \equiv \pm 3 \textit{ modulo } 8. \end{cases}$$

Remark 3. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1$ modulo 8 and as such $\operatorname{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$(2k+1)^2 = 4k^2 + 4k + 1 = 4\underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1$ modulo 8

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

4-manifolds

TODO

Bibliography

[Hat02] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

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