

Chapter 1

Homotopy theory

1.1 CW-complexes

Definition 1.1.1. A map $f: X \rightarrow Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X .

Theorem 1.1.2 (Whitehead's Theorem). *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1.1.3 (Geometric interpretation of n -connectedness, [Hat02, Proposition 4.15]). *If (X, A) is an n -connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\text{rel } A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n .*

1.2 Homology

Definition 1.2.1 (Acyclic). A space X is called *acyclic* if $\tilde{H}_i(X) = 0$ for all i , i.e. if its reduced homology vanishes.

Example 1.2.2. *Removing a point from a homology sphere yields an acyclic space. This example for the Poincare homology sphere is described in [Hat02, Example 2.38] TODO Insert proof.*

Chapter 2

Knot Theory

2.1 Definitions

Definition 2.1.1. If K is an oriented knot, then

- the *reverse* \overline{K} is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\overline{K}$ is the concordance inverse of K .

Proposition 2.1.2. For $K \subset \mathbb{S}^3$ we have that $K \# r\overline{K}$ is slice, even ribbon.

2.1.1 Alexander polynomial

Definition 2.1.3. L oriented link with Seifert matrix A , then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where \doteq means “up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$ ”.

Remark 2.1.4. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

2.1.2 Invariants

Definition 2.1.5. The tunnel number $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in \mathbb{S}^3 is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegaard splitting in that case?).

Remark 2.1.6. Every link has a tunnel number, this can be seen by adding a “vertical” tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2.1.7. • The unknot is the only knot with tunnel number 0. (Why?)

- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

2.2 Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

Chapter 3

4-manifolds

Bibliography

[Hat02] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

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