Contents

Homotopy theory

CW-complexes

Definition 1. A map $f: X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1 (Whitehead's Theorem). *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n-connectedness). *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\, A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

Homology

Definition 2 (Acyclic). A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. This example for the Poincaré homology sphere is described in (Hatcher, 2002, Example 2.38). TODO Insert proof.

Knot Theory

Constructions & Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* $\overline{K}$ is K with the opposite orientation

- the *obverse* rK is the reflection of K in a plane

- the *inverse* $r\overline{K}$ is the concordance inverse of K.

Proposition 2. *For $K \subset S^3$ we have that $K \# r\overline{K}$ is slice, even ribbon.*

Invariants

Alexander polynomial

Definition 4. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 1. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

Definition 5. The *tunnel number* $t(K)$ of a knot $K \subset S^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in S^3 is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegard splitting in that case?).

Remark 2. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2. • The unknot is the only knot with tunnel number 0. (Why?)

• The trefoil knot has tunnel number 1.

• The figure eight knot has tunnel number 1.

Arf invariant

Theorem 2. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\mathrm{Arf}(K) = \begin{cases} 0 & \text{if } \Delta_K(-1) \equiv \pm 1 \text{ modulo } 8 \\ 1 & \text{if } \Delta_K(-1) \equiv \pm 3 \text{ modulo } 8. \end{cases}$$

Remark 3. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1$ modulo 8 and as such $\mathrm{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4\underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1 \text{ modulo } 8$$

Concordance

Slice knots

Definition 6 (Surgery on a knot in $\mathbb{S}^3$)**.** The notation $\mathbb{S}_0^3(K)$ denotes the 0-*surgery on a knot* $K \subset \mathbb{S}^3$, i.e. removing a tubular neighborhood $\mathbb{S}^1 \times \mathbb{D}^2$ of K and gluing in $\mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism of the boundaries, which are both $\mathbb{S}^1 \times \mathbb{S}^1$. TODO

Definition 7 (Trace of a knot)**.** For $n \in \mathbb{Z}$ the *n-trace of a knot* $K \subset \mathbb{S}^3$ is the 4-manifold $X_n(K)$ obtained by attaching an n-framed 2-handle to the 4-ball along K, i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K \times \text{framing} : \, \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

Theorem 3 ((Miller and Piccirillo, 2018, Thm. 1.8))**.** • *K is smoothly slice if and only if* $X_0(K)$ *smoothly embeds in* $\mathbb{S}^4$.

• *Similarly, K is topologically slice if and only if* $X_0(K)$ *topologically embeds in* $\mathbb{S}^4$.

Remark 4 (Exotic $\mathbb{R}^4$ from a topologically, but not smoothly slice knot)*.* References: ((https://mathoverflow.net/users/1400/kelly-davis)) TODO

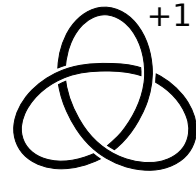A Kirby diagram for $X_n(K)$ is given just by the knot K with the framing n written next to it.

Figure 1: A Kirby diagram representing the 1-trace X_1(right handed trefoil). The boundary of this 4-manifold is the +1-surgery $\mathbb{S}_{+1}^3$(right handed trefoil), a possible description of the Poincaré homology sphere.

Concordance of Links

Definition 8. A *smooth link cobordism* between the links $L_0, L_1 \subset \mathbb{S}^3$ is a smooth, compact, oriented surface Σ generically embedded in $\mathbb{S}^3 \times \mathbb{I}$ such that $\partial \Sigma = \overline{L_0} \coprod L_1$, where $\partial \Sigma \subset \mathbb{S}^3 \times \{0, 1\}$.

Proposition 3. *Linking numbers are concordance invariants.*

Remark 5 (The Hopf link is "the most non-slice link", (Krushkal, 2015)). Any link in $\mathbb{S}^3$ bounds immersed smooth disks $\coprod^n \mathbb{D}^2 \hookrightarrow \mathbb{D}^4$. TODO

Definition 9 (Boundary link). A link $L^n \subset \mathbb{S}^{n+2}$ whose components bound disjoint Seifert surfaces is called a *boundary link* .

> It is possible that each component of a link bounds a Seifert surface missing the other components, but still they do not bound disjoint surfaces.

Example 3. The untwisted Bing double of any knot is a boundary link. TODO Insert definition of Bing double TODO Insert picture TODO Proof that Bing doubles are boundary links (draw the taco shells, need 4 copies of the Seifert surface of the knot)

Proposition 4 ((Rolfsen, 2003, 5.E.1)). *If any two components of $L^1 \subset \mathbb{S}^3$ have nonzero linking number, then L is **not** a boundary link.*

Proof. Use the definition 10 of linking number where you count the intersection points of one component with a Seifert surface for the other. $\square$

Definition 10 (Linking number via intersections with a Seifert surface, (Rolfsen, 2003, 5.D.(2))). Let J and K be two disjoint oriented knots (e.g. link components). Pick a PL Seifert surface M^2 for K, with a bicollar (N, N^+, N^-) of the interior $\mathring{M}$. Make J transverse to M, i.e. assume after a small homotopy of J in $\mathbb{S}^3 \setminus K$ that J meets M in a finite number of points and that at each point J passes locally from N^+ to N^- or from N^- to N^+. Corresponding to this direction, weight the intersection types with $+1$ or -1. The signed sum of these intersection points is the linking number $\operatorname{lk}(J, K) \in \mathbb{Z}$.

> On first sight this seems to depend on the choice of Seifert surface M.

Proposition 5 ((Rolfsen, 2003, 5.E.8)). *If a link L is a boundary link, then each component represents an element in the second commutator subgroup[1] of the fundamental group of the complement of the remaining component(s).*

> [1] Also called *second derived subgroup* or $G^{(2)}$, it is generated by elements of the form $[[x, y], [z, w]]$.

Braid groups

Exercise 1. Show that there is a presentation for the braid groups with just two generators.

TQFTs - Topological Quantum Field Theories

TODO Write down axioms

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

> A monoidal functor is supposed to preserve the identity objects for the tensor product. Since the empty set is the identity for the tensor product in the bordism category (given by disjoint union of the bordisms), the TQFT should send this to the identity object for $\otimes_R$, which is just the ground ring R.

4-manifolds

TODO

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