Chapter 1

Homotopy theory

1.1 CW-complexes

Definition 1.1.1. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1.1.2 (Whitehead's Theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1.1.3 (Geometric interpretation of n-connectedness, [Hat02, Proposition 4.15])**.** *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\, A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

1.2 Homology

Definition 1.2.1 (Acyclic)**.** A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1.2.2. *Removing a point from a homology sphere yields an acyclic space. This example for the Poincare homology sphere is described in [Hat02, Example 2.38] TODO Insert proof.*

Chapter 2

Knot Theory

2.1 Definitions

2.1.1 Alexander polynomial

Definition 2.1.1. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 2.1.2. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

2.1.2 Invariants

Definition 2.1.3. The tunnel number $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in $\mathbb{S}^3$ is a handlebody. The same definition is valid for links.
The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegard splitting in that case?).

Remark 2.1.4. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2.1.5.

- *The unknot is the only knot with tunnel number 0. (Why?)*
- *The trefoil knot has tunnel number 1.*
- *The figure eight knot has tunnel number 1.*

2.2 Open questions

Open question 1. *Is the crossing number of a satellite knot bigger than that of its companion?*

Chapter 3

4-manifolds

Bibliography

[Hat02] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

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