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Introduction

Homotopy theory

CW-complexes

Definition 1. A map $f: X \rightarrow Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X .

Theorem 1 (Whitehead's Theorem). *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n -connectedness). *If (X, A) is an n -connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\text{rel } A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n .*

Homology

Definition 2 (Acyclic). A space X is called *acyclic* if $\tilde{H}_i(X) = 0$ for all i , i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. This example for the Poincare homology sphere is described in Example 2.38¹. TODO Insert proof.

¹ Allen Hatcher. Algebraic topology. 2002. Cambridge UP, Cambridge, 606(9), 2002

Knot Theory

Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* \bar{K} is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\bar{K}$ is the concordance inverse of K .

Proposition 2. For $K \subset S^3$ we have that $K \# r\bar{K}$ is slice, even ribbon.

Alexander polynomial

Definition 4. L oriented link with Seifert matrix A , then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty; \mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where \doteq means “up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$ ”.

Remark 1. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

Invariants

Definition 5. The tunnel number $t(K)$ of a knot $K \subset S^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in S^3 is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegaard splitting in that case?).

Remark 2. Every link has a tunnel number, this can be seen by adding a “vertical” tunnel at every crossing in a link diagram. This shows

that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2. • The unknot is the only knot with tunnel number 0.
(Why?)

- The trefoil knot has tunnel number 1.
- The figure eight knot has tunnel number 1.

Arf invariant

Theorem 2. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\text{Arf}(K) = \begin{cases} 0 & \text{if } \Delta_K(-1) \equiv \pm 1 \text{ modulo } 8 \\ 1 & \text{if } \Delta_K(-1) \equiv \pm 3 \text{ modulo } 8. \end{cases}$$

Remark 3. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1 \text{ modulo } 8$ and as such $\text{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4 \underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1 \text{ modulo } 8$$

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

4-manifolds

TODO

Bibliography

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