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Introduction

Homotopy theory

CW-complexes

Definition 1. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms

$$\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and all choices of basepoints x_0 in X.

Theorem 1 (Whitehead's Theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proposition 1 (Geometric interpretation of n-connectedness)**.** *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \sim_{\mathrm{rel}\, A} (X, A)$ such that all cells of $Z \setminus A$ have dimension greater than n.*

Homology

Definition 2 (Acyclic)**.** A space X is called *acyclic* if $\widetilde{H}_i(X) = 0$ for all i, i.e. if its reduced homology vanishes.

Example 1. Removing a point from a homology sphere yields an acyclic space. This example for the Poincare homology sphere is described in Example 2.38 [1]. TODO Insert proof.

[1] Allen Hatcher. Algebraic topology. *Cambridge UP, Cambridge*, 606(9), 2002

Knot Theory

Constructions & Definitions

Definition 3. If K is an oriented knot, then

- the *reverse* $\overline{K}$ is K with the opposite orientation
- the *obverse* rK is the reflection of K in a plane
- the *inverse* $r\overline{K}$ is the concordance inverse of K.

Proposition 2. *For $K \subset \mathbb{S}^3$ we have that $K\#r\overline{K}$ is slice, even ribbon.*

Invariants

Alexander polynomial

Definition 4. L oriented link with Seifert matrix A, then the first homology of the infinite cyclic covering of the link complement, $H_1(X_\infty;\mathbb{Z})$, has square presentation matrix $tA - A^T$.

The *Alexander polynomial* of L is given by

$$\Delta_L(t) \doteq \det(tA - A^T)$$

where $\doteq$ means "up to a multiplication with a unit $\{\pm t^{\pm n}\}$ of the Laurent ring $\mathbb{Z}[t, t^{-1}]$".

Remark 1. $\mathbb{Z}[t^{\pm 1}]$ is **not** a PID.

Definition 5 (Surgery on a knot in $\mathbb{S}^3$)**.** The notation $\mathbb{S}^3_0(K)$ denotes the 0-*surgery on a knot* $K \subset \mathbb{S}^3$, i.e. removing a tubular neighborhood $\mathbb{S}^1 \times \mathbb{D}^2$ of K and gluing in $\mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism of the boundaries, which are both $\mathbb{S}^1 \times \mathbb{S}^1$. TODO

Definition 6 (Trace of a knot)**.** For $n \in \mathbb{Z}$ the *n-trace of a knot* $K \subset \mathbb{S}^3$ is the 4-manifold $X_n(K)$ obtained by attaching an n-framed 2-handle to the 4-ball along K, i.e.

$$X_n(K) = \mathbb{D}^4 \cup_{K\times\text{framing}:\, \mathbb{S}^1\times\mathbb{D}^2 \hookrightarrow \mathbb{S}^3} (\mathbb{D}^2 \times \mathbb{D}^2).$$

A Kirby diagram for $X_n(K)$ is given just by the knot K with the framing n written next to it.

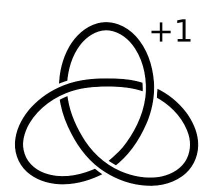

Figure 1: A Kirby diagram representing the 1-trace X_1(right handed trefoil). The boundary of this 4-manifold is the $+1$-surgery $\mathbb{S}^3_{+1}$(right handed trefoil), a possible description of the Poincare homology sphere.

Theorem 2. • *K is smoothly slice if and only if $X_0(K)$ smoothly embeds in $\mathbb{S}^4$.*

• *Similarly, K is topologically slice if and only if $X_0(K)$ topologically embeds in $\mathbb{S}^4$.*

Definition 7. The *tunnel number* $t(K)$ of a knot $K \subset \mathbb{S}^3$ is the minimal number of arcs that must be added to the knot (forming a graph with three edges at a vertex) so that its complement in $\mathbb{S}^3$ is a handlebody. The same definition is valid for links.

The boundary will be a minimal Heegaard splitting of the knot complement (The knot complement is a manifold with boundary, so what is the definition of a Heegard splitting in that case?).

Remark 2. Every link has a tunnel number, this can be seen by adding a "vertical" tunnel at every crossing in a link diagram. This shows that the tunnel number of a knot is always less than or equal to the crossing number, $t(K) \leq c(K)$.

Example 2. • The unknot is the only knot with tunnel number 0. (Why?)

• The trefoil knot has tunnel number 1.

• The figure eight knot has tunnel number 1.

Arf invariant

Theorem 3. *The Arf invariant of a knot K is related to the Alexander polynomial by*

$$\mathrm{Arf}(K) = \begin{cases} 0 & \textit{if } \Delta_K(-1) \equiv \pm 1 \textit{ modulo } 8 \\ 1 & \textit{if } \Delta_K(-1) \equiv \pm 3 \textit{ modulo } 8. \end{cases}$$

Remark 3. If K is a slice knot, we know that its determinant $|\Delta_K(-1)|$ is an odd square integer. Thus we have $\Delta_K(-1) \equiv \pm 1$ modulo 8 and as such $\mathrm{Arf}(K) = 0$; Arf is a well defined concordance invariant.

$(2k+1)^2 = 4k^2 + 4k + 1 = 4\underbrace{k(k+1)}_{\text{even}} + 1 \equiv 1$ modulo 8

Concordance

Definition 8. A *smooth link cobordism* between the links $L_0, L_1 \subset \mathbb{S}^3$ is a smooth, compact, oriented surface Σ generically embedded in $\mathbb{S}^3 \times \mathbb{I}$ such that $\partial\Sigma = \overline{L_0} \coprod L_1$, where $\partial\Sigma \subset \mathbb{S}^3 \times \{0, 1\}$.

Proposition 3. *Linking numbers are concordance invariants.*

Remark 4 (The Hopf link is "the most non-slice link"). [2] Any link in $\mathbb{S}^3$ bounds immersed smooth disks $\coprod^n \mathbb{D}^2 \looparrowright \mathbb{D}^4$. TODO

[2] Vyacheslav Krushkal. "Slicing" the Hopf link. *Geometry & Topology*, 19(3):1657–1683, 2015

TQFTs - Topological Quantum Field Theories

TODO Write down axioms

A monoidal functor is supposed to preserve the identity objects for the tensor product. Since the empty set is the identity for the tensor product in the bordism category (given by disjoint union of the bordisms), the TQFT should send this to the identity object for $\otimes_R$, which is just the ground ring R.

Open questions

Open question 1. Is the crossing number of a satellite knot bigger than that of its companion?

4-manifolds

TODO

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[Hat02] Allen Hatcher. Algebraic topology. *Cambridge UP, Cambridge*, 606(9), 2002.

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