ON THE TRAFFIC-LIGHT QUEUE

By J. N. Darroch[1]

University of Adelaide

0. Summary. A formal solution for the stationary distribution of queue-length at a fixed-cycle traffic light is found for a fairly general distribution of arrivals and for a single stream of vehicles which either all turn left or else all go straight on or turn right. (We assume that the vehicles are driving on the right of the road.) Some inequalities are derived for the expected queue-length and for the expected delay per vehicle.

1. Introduction. Consider a single stream of vehicles approaching an intersection controlled by traffic-lights and forming a queue there. For our purposes, one *cycle* of the lights comprises one green and one red phase since the amber phases can be thought of as effectively green or effectively red. For vehicles which have just been queueing and are going straight on or turning right we suppose that, as they cross the stop-line during the green phase, they are separated by *constant* time intervals of *unit* length. Further, we assume that the green and red phases are of *fixed* durations equal to g and r units respectively, where g and r are integers. Let $X_{k,n}$ denote the queue-length at time $t = k$, $k = 0, 1, 2, \cdots, g + r - 1$, during the nth cycle, where $t = 0$ at the beginning of the green phase, and let $X_{k,n}$ have p.g.f.

$$\phi_{k,n}(z) = E[z^{X_{k,n}}].$$

Let $Y_{k,n}$ denote the number of vehicles arriving at the end of the queue (or, if there is no queue, at the stop line) during the interval $k < t \leqq k + 1$ in the nth cycle. We shall assume that the variables $Y_{k,n}$ are independently and identically distributed with p.g.f.

$$\psi_1(z) = E[z^{Y_{k,n}}]$$

and mean

$$\alpha = E[Y_{k,n}] = \psi_1'(1).$$

Winsten [1] and Newell [3] made the same assumptions as the above and took

$$\psi_1(z) = 1 - \alpha + \alpha z. \tag{1}$$

Winsten pointed out that, on this assumption of 0 or 1 arrivals per unit time,

$$X_{g,n+1} = \max \{X_{g,n} + \sum_{k=0}^{g+r-1} Y_{k,n} - g, 0\} \tag{2}$$

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[1] Work done at the University of Manchester.

and showed that in equilibrium the average delay per vehicle is expressible in terms of g, r, α and $E[X_g]$ as

$$[r/(1-\alpha)(g+r)][E[X_g]+\alpha(r+1)/2]. \tag{3}$$

(We omit the subscript n when referring to stationary random variables, probabilities and probability generating functions.) Using (2) Newell proved that in equilibrium

$$\phi_g(z) = \prod_{i=1}^{r} [(1-z_i)/(z-z_i)] \tag{4}$$

where $z_1, z_2, \cdots, z_r$ are the r zeros of $z^g - (1-\alpha+\alpha z)^{g+r}$ outside the unit circle of the complex plane. He also derived some asymptotic formulae for $E[X_g]$ and, using (3), for the expected delay.

In this paper we adopt an approach which does not assume (2) and allows a general $\psi_1(z)$ and a flexible service mechanism (see Section 2). This approach leads to inequalities for the expected queue-length and the expected delay. Moreover, the removal of the restriction to 0 or 1 arrivals per unit time which, though reasonably satisfactory for a traffic-light queue is not so for other queues, permits interpretation in the wider context of any queue with geometrically distributed service-time (see Section 2) and with service interrupted at regular intervals. With this wider context in mind, we note that by taking

$$\psi_1(z) = e^{\gamma(\eta(z)-1)},$$

the assumption of independent $Y_{k,n}$ is consistent with customers arriving in batches of random size (with p.g.f. $\eta(z)$), the time-intervals between batches having independent exponential distributions with means equal to γ^{-1}.

2. The service mechanism. We suppose that, for $k = 0, 1, \cdots, g-1$,

$$\begin{aligned} X_{k+1,n} &= X_{k,n} + Y_{k,n} + U_{k,n} - 1 \qquad &\text{if } X_{k,n} > 0 \\ &= V_{k,n} &\text{if } X_{k,n} = 0. \end{aligned} \tag{5}$$

(During the red phase, $X_{k+1,n} = X_{k,n} + Y_{k,n}$, $k = g, g+1, \cdots, g+r-1$.) The $U_{k,n}$ are defined to have p.g.f. $1-\lambda+\lambda z$ and to be independent of each other and of the $X_{k,n}$ and of the $Y_{k,n}$. The $V_{k,n}$ are defined to be independent of each other and of the $X_{k,n}$ but $V_{k,n}$ is related to $Y_{k,n}$ by the condition that

$$V_{k,n} \leqq Y_{k,n}. \tag{6}$$

We assume that the $V_{k,n}$ have a common distribution with p.g.f. $\theta(z)$. One consequence of (6) is that

$$\theta(0) \geqq \psi_1(0). \tag{7}$$

Let us distinguish by the labels L and $\bar{L}$ those vehicles which are going to turn left and those which are going straight on or turning right. The inclusion of the variables $U_{k,n}$ in the model provides for the fact that, in an L queue, the front vehicle has to get across the oncoming stream of traffic and may have to wait to do so. However the assumption that these variables are independent and have constant probabilities $1 - \lambda$, λ of equalling 0, 1, is only a first approximation of what actually happens. We note that, on this assumption, the "service" time (where "service" means permission to cross the halt line) has a geometric distribution interrupted, of course, by the red phases. Still considering an L queue, it would be reasonable to assume that, if $Y_{k,n} > 0$, $V_{k,n} = Y_{k,n} - 1$ with probability $1 - \lambda$ and $V_{k,n} = Y_{k,n}$ with probability λ, and this mechanism would also apply to queues other than at traffic lights.

For an $\bar{L}$ queue, which has the right of way during the green phase, it would be reasonable to take $\lambda = 0$, that is to put $U_{k,n}$ identically equal to zero. It may also be reasonable to put $V_{k,n}$ identically equal to zero, but the weaker assumption allows for the fact that oncoming L vehicles or pedestrians may hinder the free passage of the arriving $Y_{k,n}$ vehicles and cause some of them to stop.

As we have suggested, the service mechanism given by (5) is capable of providing a reasonable model either for L queues or for $\bar{L}$ queues and these often occur in practice, side by side on a multi-lane road or at intersections with restricted entries. Unfortunately (5) cannot be made to describe a single mixed stream of L and $\bar{L}$ vehicles but, in this connection, we refer to Newell [4] who considered the behaviour of two opposing single-stream mixed queues during those cycles in which they both remain non-empty.

3. The stationary distribution.

THEOREM 3.1. *A sufficient condition for the distribution of queue-length to converge to a stationary distribution is*

$$(g + r)\alpha + g\lambda < g. \tag{8}$$

PROOF. Let N denote the number of cycles (in the present context we think of a cycle as starting with a red phase) from one occurrence of the event $E_g = [X_g = 0]$ to the next. Suppose first that $\theta(z) = 1$. The variables $U_{k,n}$ are defined during the whole of all green phases in which the queue remains non-empty and $[N > \nu] = \bigcap_{m=1}^{\nu} [S_m > 0]$, where

$$S_m = \sum_{n=1}^{m} \left(\sum_{k=g}^{g+r-1} Y_{k,n} + \sum_{k=0}^{g-1} Y_{k,n+1} + \sum_{k=0}^{g-1} U_{k,n+1} - g \right).$$

Since $E[S_m] = m[(g + r)\alpha + g\lambda - g] < 0$, a fourth-moment one-sided Chebycheff inequality gives $P[S_m > 0] < c/m^2$. Therefore,

$$P[N > \nu] < P[S_\nu > 0] < c/\nu^2 \to 0 \quad \text{as} \quad \nu \to \infty.$$

Moreover,

$$E[N] = 1 + \sum_{\nu=1}^{\infty} P[N > \nu] < 1 + c\sum_{\nu=1}^{\infty} 1/\nu^2 < \infty.$$

Thus the event E_g has finite mean recurrence time. It follows (see Feller [2] for instance) that, as the Markov chain formed by the states $X_g = i$, $i = 0, 1, 2, \cdots$, and consecutive cycles is clearly irreducible and aperiodic, it is also ergodic. It follows that the queue-length distribution is ergodic at all points of the cycle.

When $\theta(z) \neq 1$, a slight modification of the above argument shows that the event $E = [X_1 X_2 \cdots X_g = 0]$ has finite mean recurrence time. Consider the subsequence of cycles at which E occurs and let us imbed in this subsequence the Markov chain whose states are

$$E_k = [X_k = 0, X_{k+1} > 0, \cdots, X_g > 0], \quad k = 1, 2, \cdots, g - 1,$$

and E_g. Because this chain is finite it follows that the number of cycles in the subsequence between two occurrences of E_g has finite mean. Therefore the recurrence time of $X_g = 0$ in the full sequence has a finite mean and, once again, the queue-length distribution is ergodic.

THEOREM 3.2. *The* p.g.f. *of the stationary distribution of X_g is related to the stationary probabilities $Q_k = P[X_k = 0]$, $k = 0, 1, \cdots, g$ by*

$$\phi_g(z) = \frac{[\theta(z)\zeta - 1]\left(\sum_{k=0}^{g-1} Q_k \zeta^k\right)}{z^g - (\psi_2(z))^g(\psi_1(z))^r}[\psi_2(z)]^g \tag{9}$$

where $\psi_2(z) = (1 - \lambda + \lambda z)\psi_1(z)$ and $\zeta = \zeta(z) = z/\psi_2(z)$.

PROOF. From (5) we obtain

$$\phi_{k+1,n}(z) = [(1 - \lambda + \lambda z)\psi_1(z)/z](\phi_{k,n}(z) - \phi_{k,n}(0)) + \theta(z)\phi_{k,n}(0), \tag{10}$$

and thence

$$\phi_{g,n}(z) = (\psi_2(z)/z)^g \phi_{0,n}(z) + [\theta(z) - \zeta^{-1}]\left[\sum_{k=0}^{g-1} \phi_{k,n}(0)\zeta^{k-g+1}\right].$$

Consideration of the red phase gives $\phi_{0,n+1}(z) = \phi_{g,n}(z)(\psi_1(z))^r$ and in equilibrium we have $\phi_{0,n+1}(z) = \phi_{0,n}(z)$. Formula (9) follows from the above relationships by writing $\phi_{g,n}(z) = \phi_g(z)$ and $\phi_{k,n}(0) = Q_k$.

THEOREM 3.3. *Condition* (8) *is also necessary for a stationary distribution to exist and*

$$\sum_{k=0}^{g-1} Q_k = [g - (g + r)\alpha - g\lambda]/[1 - \alpha - \lambda + \theta'(1)] = K, \quad \text{say}. \tag{11}$$

PROOF. Since $\phi_g(z)$ is a p.g.f., $\phi_g(1-) = 1$, and this gives (11). Clearly $0 < Q_k < 1$, $k = 0, 1, \cdots, g - 1$ in equilibrium from which it follows that $(g + r)\alpha + g\lambda < g$.

The complete solution for $\phi_g(z)$ is given in Theorem 3.4, the proof of which requires the following lemma which is very similar to some theorems in Takács [5].

LEMMA 3.1. *Let $\Psi(z)$ denote a* p.g.f. *for which*

(i) *$\Psi(z)$ is analytic in $|z| < 1 + \Delta$ for some $\Delta > 0$,*

(ii) $|\Psi(z)| = 1$ *on the unit circle only at* $z = 1$.
Then provided that

$$\Psi'(1) < g \tag{12}$$

the equation

$$z^g - \Psi(z) = 0 \tag{13}$$

has g *distinct roots,* $z_0, z_1, \cdots, z_{g-1}$ *say, within the unit circle, where* $z_0 = 1$ *and* $|z_k| < 1$ *for* $k \geqq 1$.

PROOF. Conditions (i) and (12) imply that for some $\delta > 0$,

$$(1 + \delta)^g > \Psi(1 + \delta).$$

Since $\Psi(z)$ is a p.g.f., $|\Psi(z)| \leqq \Psi(|z|)$ and therefore

$$|z| > |\Psi(z)|^{1/g} \quad \text{on} \quad |z| = 1 + \delta.$$

By Rouche's theorem it follows that the equation

$$z - e^{2\pi ik/g}[\Psi(z)]^{1/g} = 0 \tag{14_k}$$

has exactly one root z_k in $|z| < 1 + \delta$, $k = 0, 1, \cdots, g - 1$. Clearly $z_0 = 1$ and, by Condition (ii), $|z_k| < 1$, $k = 1, 2, \cdots, g - 1$. The proof is completed by noting that $z_{k_1} \neq z_{k_2}$ for $k_1 \neq k_2$ since $e^{2\pi ik_1/g} \neq e^{2\pi ik_2/g}$, and that the roots of (13) are those of the Equations (14_k).

Before applying Lemma 3.1 to

$$\Psi(z) = [\psi_2(z)]^g[\psi_1(z)]^r = (1 - \lambda + \lambda z)^g[\psi_1(z)]^{g+r},$$

we note that it is difficult to imagine Condition (i) placing any practical limitation on $\psi_1(z)$ and Condition (ii) certainly does not. For (ii) is contravened only if $\lambda = 0$ and the values of j for which $P[Y_{k,n} = j] > 0$ are confined to a sequence of the form $a, a + b, a + 2b, \cdots, b > 1$, and in any practical example this is not so because $P[Y_k = 0] > 0$ and $P[Y_k = 1] > 0$.

THEOREM 3.4. *Subject to the conditions*
(i) $\psi_1(z)$ *is analytic in* $|z| < 1 + \Delta$ *for some* $\Delta > 0$,
(ii) $|\psi_1(z)| = 1$ *on the unit circle only at* $z = 1$,
(iii) *inequality* (8) *is satisfied,*

$$\phi_g(z) = K \frac{[\theta(z)\zeta - 1][\psi_2(z)]^g}{z^g - [\psi_2(z)]^g[\psi_1(z)]^r} \prod_{k=1}^{g-1} \left(\frac{\zeta - \zeta_k}{1 - \zeta_k}\right), \tag{15}$$

where K *is given by* (11), $z_1, z_2, \cdots, z_{g-1}$ *are the* $g - 1$ *zeros of* $z^g - [\psi_2(z)]^g[\psi_1(z)]^r$ *which are strictly inside the unit circle, and* $\zeta_k = \zeta(z_k) = z_k/\psi_2(z_k)$.

PROOF. Applying Lemma 3.1 to $\Psi(z) = [\psi_2(z)]^g[\psi_1(z)]^r$ we deduce that, since $\Psi'(1) = (g + r)\alpha + g\lambda$, the denominator of (9) has zeros $z_0 = 1$ and z_k, $|z_k| < 1$, $k = 1, 2, \cdots, g - 1$. We note that $|\zeta_k| < 1$ since $\zeta_k^g = [\psi_1(z_k)]^r$ and $|\psi_1(z_k)| < 1$. Because $\phi_g(z)$ is a p.g.f., it is analytic within the unit circle and the numerator of (9) must therefore have the same zeros as the denominator. Clearly $z_0 = 1$

is a zero of $\theta(z)\zeta - 1$ and no z_k, $k \geqq 1$, is a zero of $\theta(z)\zeta - 1$ because $|\theta(z_k)| < 1$ and $|\zeta_k| < 1$. Let us write

$$\sum_{k=0}^{g-1} Q_k\zeta^k = Q_{g-1}\prod_{k=1}^{g-1}(\zeta - a_k).$$

Then z_1 must be a zero of one of the factors $\zeta - a_k$. Suppose, without loss of generality, that it is the first; then $a_1 = \zeta_1$. Now $\zeta - \zeta_1 = [z - \zeta_1\psi_2(z)]/\psi_2(z)$ and, by a simple application of Rouches theorem with the unit circle as contour, we deduce that $z - \zeta_1\psi_2(z)$ has only one zero, namely z_1, within the unit circle. Therefore z_2 must be a zero of one of the factors $\zeta - a_2, \cdots, \zeta - a_{g-1}$. Continuing in this way we have $a_2 = \zeta_2, \cdots, a_{g-1} = \zeta_{g-1}$. On imposing the condition that $\phi_g(1-) = 1$, (15) now follows.

Corollary 3.4.1. *The ratios of the probabilities $Q_0, Q_1, \cdots, Q_g$, are independent of $\theta(z)$.*

Proof. From the following identity in ζ,

$$\sum_{k=0}^{g-1} Q_k\zeta^k = K\prod_{k=1}^{g-1}[(\zeta - \zeta_k)/(1 - \zeta_k)],$$

we deduce that the numbers Q_k/K, $k = 0, 1, \cdots, g - 1$, are functions of $\zeta_1, \zeta_2, \cdots, \zeta_k$. But the latter depend only on λ and $\psi_1(z)$ and are independent of $\theta(z)$. The proof is completed by observing that $Q_g = Q_0[\psi_1(0)]^r$.

Corollary 3.4.2. *The probabilities $Q_0, Q_1, \cdots, Q_g$ satisfy the inequalities $Q_0 < Q_1 < \cdots < Q_g$.*

Proof. This intuitively obvious property follows by putting $\theta(z) = 1$ and obtaining from (10) that

$$Q_{k+1} = \psi_1(0)(1 - \lambda)P[X_k = 1] + Q_k, \quad k = 0, 1, \cdots, g - 1.$$

Now $\psi_1(0)(1 - \lambda) > 0$ and clearly $P[X_k = 1] > 0$ in equilibrium. Therefore $Q_{k+1} > Q_k$ and Corollary 3.4.1 shows that this inequality remains true for general $\theta(z)$.

When $\psi_1(z) = 1 - \alpha + \alpha z$, $\lambda = 0$ and $\theta(z) = 1$, it is straightforward to deduce Newell's solution, namely (4), from (15).

4. The expected delay per vehicle. Winsten's formula (3) relating $E[D]$, the expected delay per vehicle, to $E[X_g]$ will now be generalised. We follow Winsten in first finding the expected amount of waiting done per cycle, but in other respects the present derivation differs from his as he made explicit use of the assumption that $Y_k = 0, 1$.

If at time $t = k$ there are X_k vehicles waiting, we may, to a good approximation, say that the amount of waiting done during the interval $k - \frac{1}{2} < t \leqq k + \frac{1}{2}$ is X_k. Therefore the expected amount of waiting done per cycle, $E[W]$ say, is given by $E[W] = \sum_{k=0}^{g+r-1} E[X_k]$. From (10)

$$E[X_{k+1}] = E[X_k] + Q_k[1 - \alpha - \lambda + \theta'(1)] - (1 - \alpha - \lambda), \qquad k = 0, 1, \cdots, g - 1, \tag{16}$$

and during the red phase we have

$$E[X_{k+1}] = E[X_k] + \alpha, \qquad k = g, g+1, \cdots, g+r-1. \tag{17}$$

Using (16) and (17) we find that

$$E[W] = (g+r)E[X_g] - [1 - \alpha - \lambda + \theta'(1)]\sum_{k=1}^{g-1} kQ_k + A \tag{18}$$

where

$$A = \tfrac{1}{2}g(g-1)(1-\alpha-\lambda) + \tfrac{1}{2}r(r+1)\alpha. \tag{19}$$

Now differentiate (9) and put $z = 1$ to obtain the following linear relation between $E[X_g]$ and $\sum kQ_k$.

$$\begin{aligned}[g - (g+r)\alpha - g\lambda]&E[X_g] \\ &= [1-\alpha-\lambda][1-\alpha-\lambda+\theta'(1)]\sum_{k=1}^{g-1} kQ_k + B\end{aligned} \tag{20}$$

where

$$\begin{aligned}B = \tfrac{1}{2}[\theta''(1) &+ 2\theta'(1)(1-\alpha-\lambda) - 2(\alpha+\lambda)(1-\alpha-\lambda) \\ &- 2\lambda\alpha - \psi_1''(1)]K - \tfrac{1}{2}[g^2(1-\lambda)(1-\lambda-2\alpha) \\ &- g(1-\lambda^2) + (g^2 - r^2)\alpha^2 + (g+r)(\alpha^2 - \psi_1''(1))].\end{aligned} \tag{21}$$

Substitute from (20) into (18) to obtain

$$E[W] = [r(1-\lambda)/(1-\alpha-\lambda)]E[X_g] + A + B/(1-\alpha-\lambda). \tag{22}$$

Clearly in equilibrium we have $E[W] = E[D]E[\sum_{k=0}^{g+r-1} Y_k]$ and consequently

$$E[D] = E[W]/(g+r)\alpha. \tag{23}$$

5. Inequalities for the expected queue length and the expected delay. Formulae (20), (22) and (30) show that inequalities for $E[X_g]$ and $E[D]$ can be found by first finding inequalities for $\sum kQ_k$, and we now do this.

THEOREM 5.1. *Let K' denote the integral part of K (defined by (11)), and let $q = P[Y_{k,n} = 0]$, $p_1 = P[Y_{k,n} = 1]$. Then*

$$\begin{aligned}\tfrac{1}{2}(g+1)K - \tfrac{1}{2}q^r[(g-1)(\theta(0) + rp_1(1-\lambda)) + g + 1] &< \sum_{k=1}^{g-1} kQ_k \\ &< \tfrac{1}{2}K'(2g - K' - 1) + (g - K' - 1)(K - K').\end{aligned} \tag{24}$$

PROOF. We have $0 < Q_k < 1$ and $\sum_{k=0}^{g-1} Q_k = K$. The supremum of $\sum kQ_k$ subject to these conditions is clearly obtained by putting

$$Q_0 = Q_1 = \cdots = Q_{g-K'-2} = 0, \qquad Q_{g-K'-1} = K - K',$$
$$Q_{g-K'} = Q_{g-K'+1} = \cdots = Q_{g-1} = 1,$$

and these values produce the upper bound in (24).

To derive the lower bound we make use of the three following additional inequalities for the Q_k.

$$Q_0 < Q_1 < \cdots < Q_{g-1}. \tag{25}$$

$$Q_0 < q^r. \tag{26}$$

$$Q_1 < q^r[\theta(0) + rp_1(1 - \lambda)]. \tag{27}$$

We proved (25) in Corollary 3.4.2. The relation $Q_0 = q^r Q_g$ gives (26).

To obtain (27), first equate coefficients of z^1 in (9) to give

$$(1 - \lambda)q^{r+1}P[X_g = 1] = Q_1 - Q_0[\theta(0) + rp_1(1 - \lambda)]. \tag{28}$$

When we combine $P[X_g = 1] < 1 - Q_g = 1 - Q_0 q^{-r}$ with (28) we have

$$Q_1 < (1 - \lambda)q^{r+1} + Q_0[\theta(0) + rp_1(1 - \lambda) - (1 - \lambda)q]. \tag{29}$$

Now recall (7) which tells us that the coefficient of Q_0 in (29) is positive, and apply (26) to yield (27). To obtain the infimum of $\sum kQ_k$ subject to (25), (26) and (27), put

$$Q_0 = q^r, \qquad Q_1 = q^r[\theta(0) + rp_1(1 - \lambda)]$$

$$Q_2 = Q_3 = \cdots = Q_{g-1} = \{K - q^r[1 + \theta(0) + rp_1(1 - \lambda)]\}/(g - 2).$$

These values produce the lower bound in (24). (We assume that these values are such that $Q_0 \leqq Q_1 \leqq Q_2$, since this is invariably the case. If it is not the case, the appropriate adjustments are obvious and result in a slight improvement in the lower bound for $\sum kQ_k$.)

The bounds for $\sum kQ_k$ given in Theorem 5.1 can be improved by using various additional inequalities relation $Q_0, Q_1, \cdots, Q_{g-1}$ (obtained, for instance, from $P[X_k = 2] > 0$). However these improvements are not sufficiently substantial to be worth including here.

Whereas the actual value of $\sum kQ_k$ depends on g, r, λ, $\theta(z)$ and $\psi_1(z)$, the bounds given in Theorem 5.1 virtually depend only on g and K (the second term in the lower bound is usually small). Therefore, to illustrate their use in finding bounds for $E[X_g]$ and $E[D]$ we consider a few typical values of g and K in conjunction with the model studied by Winsten and Newell, defined by

TABLE 1

Bounds for $E[X_g]$, $E[D]$ when $g = r, \lambda = 0, \theta(z) = 1, \psi_1(z) = 1 - \alpha + \alpha z$

g	α		
	.20	.40	.49
10	(0, .153)	(0, .980)	(9.76, 11.60)
	(3.44, 3.92)	(4.58, 6.63)	(24.92, 28.59)
20	(0, .167)	(0, .990)	(7.86, 12.20)
	(6.56, 7.08)	(8.75, 10.81)	(26.02, 34.69)

$\lambda = 0, \theta(z) = 1, \psi_1(z) = 1 - \alpha + \alpha z$. We further suppose that $g = r$. In Table 1, two values of g are combined with three values of α (which, if the unit of time is taken as 2 seconds, correspond respectively to 360, 720 and 882 vehicles per hour) and the six values of K range from .391 to 15. In four of the six entries, substitution of the lower bound for $\sum kQ_k$ in (20) gives a negative lower bound for $E[X_g]$ and this was replaced by zero in deriving the lower bound for $E[D]$.

Newell derived three asymptotic formulae for $E[X_g]$, the first of which ([3], formula (3.6)) is appropriate for low traffic intensities and is consistent with the bounds in the table for the two entries with $\alpha = .20$. His formula (5.6) is appropriate for the other four entries and is consistent with them. His third formula, which appears in two places ((5.3) and (5.5)), gives values of $E[X_g]$ which lie above the upper bounds for all six entries in the table.

Finally we compare Table 1 with Table 2 which gives the bounds for $E[X_g]$ and $E[D]$ corresponding to the model defined by $g = r; \lambda = 0, \theta(z) = 1, \psi_1(z) = e^{\alpha(z-1)}$. This model was the basis of Webster's Monte-Carlo Study [6]. We note that when the traffic intensity (which may be defined as 2α, or, more generally, as $(g + r)\alpha/g(1 - \lambda)$) is near to 1, $E[X_g]$ and $E[D]$ depend crucially on $\psi_1(z)$.

TABLE 2

Bounds for $E[X_g]$, $E[D]$ when $g = r$, $\lambda = 0$, $\theta(z) = 1$, $\psi_1(z) = e^{\alpha(z-1)}$

g	α		
	.20	.40	.49
10	(0, .195)	(.237, 1.647)	(21.53, 23.37)
	(3.65, 4.13)	(6.25, 8.30)	(48.94, 52.61)
20	(0, .208)	(0, 1.657)	(20.63, 24.97)
	(6.77, 7.29)	(10.42, 12.48)	(50.04, 58.71)

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