Quantum Algorithms and Learning Theory
Notes and Exercises

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1 Nielsen & Chuang: Chapter 2

1.1 Postulates of Quantum Mechanics

First, we cover the fundamental postulates of quantum mechanics.

1.1.1 State Space and State Vector

Associated with an isolated physical system is a Hilbert space, H. A Hilbert space is a complete inner-product vector space. Note that completeness holds trivially in a finite-dimensional vector space because we have closure with respect to all sequences (and hence any Cauchy sequence in the vector space must converge to a vector in the same space). Nevertheless, the state space of a physical system may be infinite-dimensional.

A system is completely described by a unit vector $u \in H$ called the state vector.

For example, consider a system given by a single qubit, which has a two-dimensional state space. Let $|0\rangle$ and $|1\rangle$ be an orthonormal basis for this space. Hence, a state vector in this space is given by

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

1.1.2 Evolution

The evolution of a closed quantum system is described by a unitary transformation. Recall that an operator U is unitary iff $U^\dagger U = I = UU^\dagger$ (and hence preserves inner products[1]).

So, let the state of a system at time t_1 be given by $|\psi\rangle$ and $|\psi'\rangle$ at t_2. Hence,

$$|\psi'\rangle = U\,|\psi\rangle$$

[1] and furthermore has a spectral decomposition because it is normal

1.1.3 Evolution in Continuous Time

Schrodinger's equation provides the time evolution of the state of a quantum system

$$i\hbar \frac{d\left|\psi\right\rangle}{dt} = H\left|\psi\right\rangle \tag{1}$$

where H is the (Hermitian) Hamiltonian of the closed system. Because the Hamiltonian is Hermitian it has spectral decomposition

$$H = \sum_E E\left|E\right\rangle\left\langle E\right|$$

where E is the energy eigenvalue corresponding to energy eigenstate $\left|E\right\rangle$.

For example, consider the Hamiltonian $H = \hbar\omega X$ (recall that $X = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Hence, we solve for its eigenvalues and eigenvectors

$$\begin{aligned}
\det\left\{\hbar\omega\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\right\} &= 0 \\
\lambda^2 - 1^2 &= 0 \\
\lambda &= \pm 1 \\
\Rightarrow E_{\pm} &= \pm\hbar\omega \\
\hbar\omega\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\left|E_+\right\rangle &= 0 \\
\left|E_+\right\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} := \left|+\right\rangle \\
\left|E_-\right\rangle &= \left|-\right\rangle
\end{aligned}$$

Now, notice that we can solve Schrodinger's equation (1) and have

$$\left|\psi(t_2)\right\rangle = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]\left|\psi(t_1)\right\rangle$$

and equivalently from 1.1.2 we can represent this transformation with unitary operator $U = \exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]$. This holds in general and so we can consider the two descriptions from 1.1.2 and 1.1.3 interchangeably (the authors prefer the latter).

1.1.4 Quantum Measurement

Quantum measurements are described by a collection of measurements operators $\{M_m\}$ (where m refers to the potential measurement outcomes of the experiment) which act on the state space of the system being observed.

Hence, if the pre-measurement state is $|\psi\rangle$, then

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

and the post-measurement state is

$$\frac{M_m |\psi\rangle}{\sqrt{p(m)}}$$

Furthermore, $\{M_m\}$ satisfy the completeness equation

$$\sum_m M_m^\dagger M_m = I$$

Now, we see an interesting implication. If we seek to distinguish our physical system from a set of orthogonal states, then we can reliably do so by simply defining each measurement operator to be the outer product of our states of interest. We add a final operator defined to be the remaining complement of the identity in order to satisfy the completeness equation.

On the flipside, two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ necessarily share a parallel component in their orthogonal decomposition. Hence, any observable that corresponds to the pre-measurement state being $|\psi_1\rangle$ with probability $p > 0$ has a probability $p' > 0$ of having been in state $|\psi_2\rangle$.

1.1.5 Projective Measurements

There exists a special class of quantum measurements known as projective measurements. These measurements can be described by an observable M, a hermitian operator on the state space being observed. M has spectral decomposition

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace of M with eigenvalues m.

Furthermore, if the pre-measurement state is $|\psi\rangle$, then

$$p(m) = \langle\psi| P_m |\psi\rangle$$

and the post-measurement state is

$$\frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

This simplifies the formula for the expected value of a measurement

$$\begin{aligned}\langle M\rangle &= \sum_m m p(m) \\ &= \langle\psi| (\sum_m m P_m) |\psi\rangle \\ &= \langle\psi| M |\psi\rangle\end{aligned}$$

For example, consider the system given by single qubits with observable Pauli matrix Z. Hence, Z has eigenvalues $+1$ and -1 and eigenstates $|0\rangle$ and $|1\rangle$, respectively. So, consider state $|\psi\rangle = |+\rangle \Rightarrow p(+1) = \langle +|0\rangle \langle 0|+\rangle = \frac{1}{2}$. Similarly, $p(-1) = \frac{1}{2}$.

1.1.6 POVM measurements

POVMs are best viewed as a special case of the general measurement formalism, providing the simplest means to study post-measurement statistics without knowledge of the post measurement state.

From above, $p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$ so if we define $E_m := M_m^\dagger M_m$ then these E_m'sd are sufficient for the purpose of computing probabilities. We denote $\{E_m\}$ as a POVM.

Note that projective operators are the special case of being equivalent to their respective POVM element because $E_m = P_m^\dagger P_m = P_m$.

Nevertheless, the POVM formalism is a useful guide in for our intuition in quantum information. Consider if Alice prepares some state for Bob that is either $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Recall from 1.1.4, Bob can't determine which state was prepared with full certainty (because of the shared orthogonal component $|0\rangle$). Still, we can define a POVM[2]

$$\begin{aligned}E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1| \\ E_2 &= \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \\ E_3 &= I - E_1 - E_2\end{aligned}$$

Now, notice what happens.

[2]verify that completeness and these being positive operators holds

$$
\begin{aligned}
\langle\psi_1| E_1 |\psi_1\rangle &= \langle 0| \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1|0\rangle \\
&= 0 \\
\langle\psi_2| E_1 |\psi_2\rangle &= \frac{\langle 0| + \langle 1|}{\sqrt{2}} \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1| \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2\sqrt{2}+2} > 0
\end{aligned}
$$

Hence, if we observe E_1 after the measurement described by $\{E_1, E_2, E_3\}$, then Alice must've prepared $|\psi_2\rangle$. Similarly,

$$
\begin{aligned}
\langle\psi_1| E_2 |\psi_1\rangle &= \langle 0| \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} |0\rangle \\
&= \frac{\sqrt{2}}{2\sqrt{2}+2} > 0 \\
\langle\psi_2| E_2 |\psi_2\rangle &= \frac{\langle 0| + \langle 1|}{\sqrt{2}} \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= 0
\end{aligned}
$$

so if we observe E_2, then Bob concludes that Alice prepared $|\psi_1\rangle$. Our routine is imperfect because we may observe E_3 and hence would infer nothing of the original state. Still, we would never *incorrectly* guess given that we allow ourselves to abstain when we see E_3.

Exercise 1.1. *(2.64) Suppose Bob is given a quantum state chosen from a set* $S = |\psi_1\rangle, \cdots, |\psi_m\rangle$ *of linearly independent states. Construct a POVM* $\{E_1, \cdots, E_{m+1}\}$ *such that if outcome* E_i *occurs,* $1 \leq i \leq m$*, then Bob knows with certainty that he was given state* $|\psi_i\rangle$*.*

To distinguish the states we require $\langle\psi_i| E_j |\psi_i\rangle = p_i\delta_{ij}$ where $p_i > 0$ and $1 \leq i, j \leq m$.

So, we can use the Gram-Schmidt process using S as our linearly independent set. This will give us an orthonormal set $U = |\varphi_1\rangle, \cdots, |\varphi_m\rangle$ that spans the same subspace as S. Next, we can represent each $|\psi_i\rangle$ in this orthonormal basis, U. Finally, for each i we can find a vector $|\psi_i'\rangle$ in the span of U that is orthogonal to all $|\psi_j\rangle, j \neq i$. Hence, we can define $E_i = |\psi_i'\rangle \langle\psi_i'|, 1 \leq i \leq m$. Finally, take $E_{m+1} = I - \sum_m E_i$.

Creating an optimal POVM is much trickier (in the sense of minimizing the probability p_{m+1}).

From this exercise, we see that POVMs present a reliable way to distinguish non-orthogonal states given that we allow for the slack of an "inconclusive" measurement (E_{m+1}).

1.1.7 Composite Systems

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.

Exercise 1.2. *(2.66) Show that the average value of the observable $X_1 Z_2$ (X acting on the first qubit and Z on the second) for a two qubit system measured in the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is zero.*

Proof. Let observable $M = X_1 Z_2$. Hence,

$$\begin{aligned}
\langle M \rangle &= \frac{\langle 00| + \langle 11|}{\sqrt{2}} M \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
&= \frac{\langle 00| + \langle 11|}{\sqrt{2}} \frac{X_1 |0\rangle Z_2 |0\rangle + X_1 |1\rangle Z_2 |1\rangle}{\sqrt{2}} \\
&= \frac{\langle 00| + \langle 11|}{\sqrt{2}} \frac{|1\rangle |0\rangle - |0\rangle |1\rangle}{\sqrt{2}} \\
&= 0
\end{aligned}$$

$\square$

Interestingly, we can show that a general quantum measurement (as described in 1.1.4) can be implemented as a projective measurement coupled with unitary dynamics.

Consider a quantum system with state space Q and measurements M_m on this system. We can introduce an *ancilla* system M with orthonormal basis $|m\rangle$ which is in one-to-one correspondence with the possible outcomes of the measurement we wish to implement.

So, let $|0\rangle$ be a fixed state of M and define an operator U on $|\psi\rangle |0\rangle$ (with $|\psi\rangle$ as a state of Q) by

$$U |\psi\rangle |0\rangle := \sum_m M_m |\psi\rangle |m\rangle$$

Hence,

$$\langle \varphi| \langle 0| U^\dagger U |\psi\rangle |0\rangle = \sum_m \sum_{m'} \langle \varphi| M_m^\dagger M_{m'} |\psi\rangle \langle m|m'\rangle$$

So, because the states $|m\rangle$ are orthonormal

$$= \sum_m \langle \varphi| M_m^\dagger M_m |\psi\rangle$$

and finally by the completeness of M_m

$$= \langle \varphi | \psi \rangle$$

We can show that U can be extended to a unitary operator on $Q \otimes M$ (exercise). Furthermore, consider the two systems given by projectors $P_m := I_Q \otimes |m\rangle \langle m|$.

1.2 Superdense Coding

1.3 The Density Operator

1.4 EPR and the Bell Inequality

2 Nielsen & Chuang: Chapter 4

3 Nielsen & Chuang: Chapter 5

1. Quantum Fourier Transform
2. Quantum Phase Estimation Algorithm
3. Shor's algorithm for Factoring

4 Nielsen & Chuang: Chapter 6

Quantum Search algorithms

5 Algorithms for solving linear systems of equations

`https://arxiv.org/abs/0811.3171`

6 Density Matrix Exponentiation Algorithms

`https://www.nature.com/articles/nphys3029`

7 Review of Quantum Machine Learning

`https://www.nature.com/articles/nature23474`

8 Learnability of Quantum States

1. https://arxiv.org/abs/quant-ph/0608142
2. https://arxiv.org/abs/1711.01053
3. https://arxiv.org/abs/1801.05721

9 Appendix

9.1 Pauli Matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$