

# Quantum-inspired $\ell^2$ sampling and applications to machine learning

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# Machine Learning

## Quantum Machine Learning

## Classical $\ell^2$ sampling

## Remarks

# Machine Learning

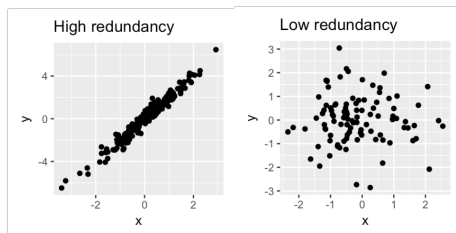
## Introduction

- ▶ Machine learning is a broad term for algorithms which are capable of finding patterns in data.
- ▶ Fundamental goal: capture these patterns in a "model" that *generalizes* to unseen data.
- ▶ These algorithms have two components:
  1. A learning element. Updates the model depending on its performance on the considered dataset.
  2. A performance element. Provides the measure of performance.
- ▶ Bottom line: "machine learning" is a somewhat hollow term. Many ML algorithms are in fact familiar linear algebraic techniques.

# PCA

## Motivation: Singular value transformation

- ▶ "Training" dataset  $\mathcal{T}$  consists of the accessible samples of data.  $\mathcal{T}$  is drawn from a subset of  $\Omega \subset \mathbb{R}^d$  where each component represents a "feature".
- ▶ Samples from  $\Omega$  are assumed to be drawn according to some distribution  $\mathcal{D}$ .
- ▶ Example: data is collected on the heights and lengths of cherry blossom petals.

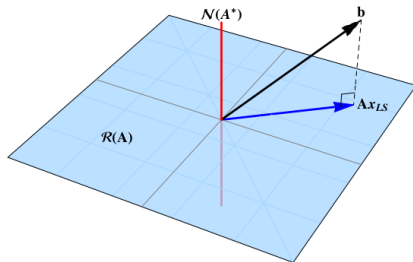


- ▶ How and why may it make sense to reduce the dimensionality of the feature space?

# Moore-Penrose Pseudoinverse

## Motivation: Singular value transformation

- ▶ Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$  unit vector. In machine learning,  $A$  is the matrix with rows given by the samples of  $\mathcal{T}$ .
- ▶ We wish to find the  $x_{LS}$  which satisfies
$$x_{LS} = \arg \min_x \|Ax - b\|_2$$
- ▶ Notation:  $x_{LS} = A^+b$
- ▶ Common strategy uses SVD: write  $A = UDV^\dagger$  and then  $A^+ = VD^+U^\dagger$  where  $D^+$  simply inverts the non-zero diagonal entries.



Machine Learning

Quantum Machine Learning

Classical  $\ell^2$  sampling

Remarks

# Moore-Penrose Pseudoinverse

Harrow, Hassidim, Lloyd (orig.) Wiebe, Braun

- ▶ HHL algorithm: application of phase estimation and Hamiltonian simulation to solve linear system.
- ▶ We can use HHL as a subroutine to compute  $A^+ |b\rangle = |x\rangle$  in ( $|x\rangle$  is the least-square solution).
- ▶ Note that  $|x\rangle$  is a quantum state. Hence, we may efficiently measure an expectation value  $x^T M x$  where  $M$  is some p.s.d operator.
- ▶ Runtime bound  $\tilde{O}(\log(N)(s^3 \kappa^6)/\epsilon)$  time (query complexity)
- ▶ Assumption:  $A$  is sparse with low condition number  $\kappa$ . Hamiltonian ( $\hat{H}$ ) simulation is efficient when  $\hat{H}$  is sparse. No low-rank assumptions are necessary.
- ▶ "Key" assumption: the quantum state  $|b\rangle$  can be prepared efficiently.

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Quantum Machine Learning

Classical  $\ell^2$  sampling

Remarks



## In search of a "fair" comparison

- ▶ How can we compare the speed of quantum algorithms with quantum input and quantum output to classical algorithms with classical input and classical output?
- ▶ Quantum machine learning algorithms can be exponentially faster than the best standard classical algorithms for similar tasks, but this comparison is unfair because the quantum algorithms get outside help through input state preparation.
- ▶ We want a classical model that helps its algorithms stand a chance against quantum algorithms, while still ensuring that they can be run in nearly all circumstances one would run the quantum algorithm.
- ▶ Solution (Tang): compare quantum algorithms with quantum state preparation to classical algorithms with sample and query access to input.

# Classical $\ell^2$ Sampling Model

## Definition

We have "query access" to  $x \in \mathbb{C}^n$  if, given  $i \in [n]$ , we can efficiently compute  $x_i$ . We say that  $x \in \mathcal{Q}$ .

## Definition

We have sample **and** query access to  $x \in \mathbb{C}^n$  if

1. We have query access to  $x$  i.e.  $x \in \mathcal{Q}$  ( $\Rightarrow \mathcal{SQ} \subset \mathcal{Q}$ )
2. can produce independent random samples  $i \in [n]$  where we sample  $i$  with probability  $|x_i|^2 / \|x\|^2$  and can query for  $\|x\|$ .

We say that  $x \in \mathcal{SQ}$ .

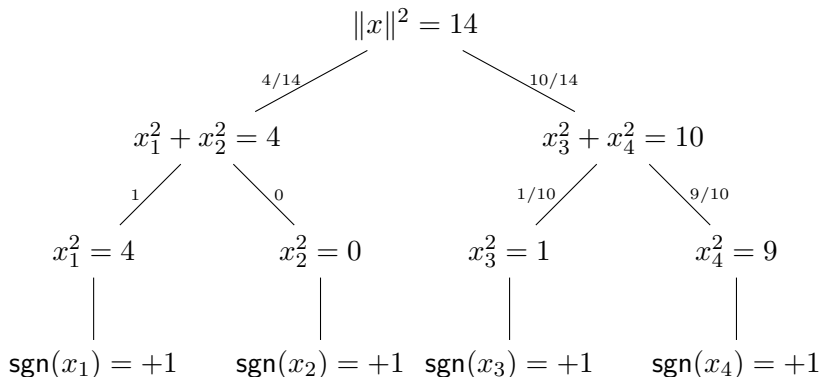
## Definition

For  $A \in \mathbb{C}^{m \times n}$ ,  $A \in \mathcal{SQ}$  (abuse) if

1.  $A_i \in \mathcal{SQ}$  where  $A_i$  is the  $i$ th row of  $A$
2.  $\tilde{A} \in \mathcal{SQ}$  for  $\tilde{A}$  the vector of row norms (so  $\tilde{A}_i = \|A_i\|$ ).

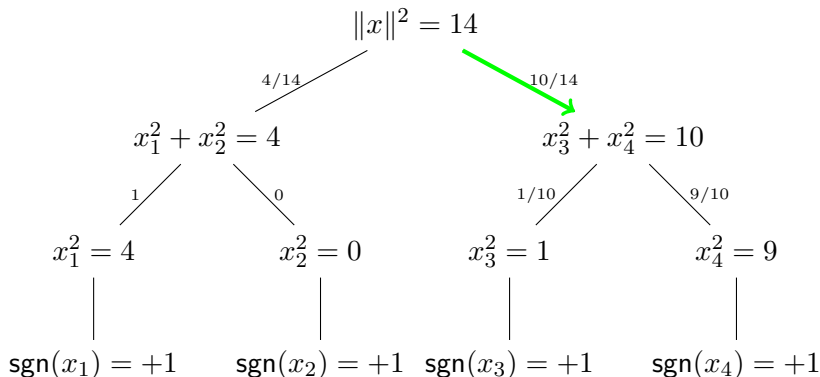
## Example Data Structure

Say we have the vector  $\vec{x} = (2, 0, 1, 3)$  and  $\vec{x} \in \mathcal{SQ}$ . Consider the following binary tree data structure.



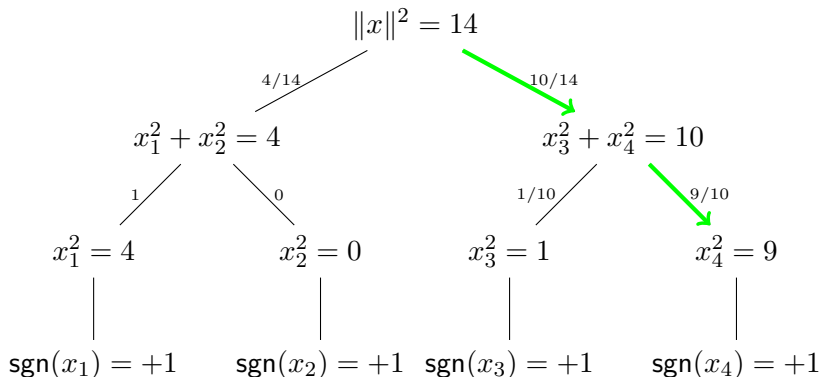
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## Example Data Structure

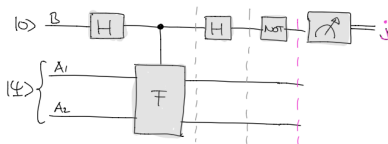
Say we have the vector  $\vec{x} = (2, 0, 1, 3)$  and  $\vec{x} \in \mathcal{SQ}$ . Consider the following binary tree data structure.



# Dequantization Toolbox

## Method 1: Inner product estimation (Tang, 2018)

- ▶ For  $x, y \in \mathbb{C}^n$ , if we are given that  $x \in \mathcal{SQ}$  and  $y \in \mathcal{Q}$ , then we can estimate  $\langle x, y \rangle$  with probability  $\geq 1 - \delta$  and error  $\epsilon \|x\| \|y\|$
- ▶ Quantum analog: SWAP test



# Dequantization Toolbox

## Method 1: Inner product estimation (Tang, 2018)

### Fact

For  $\{X_{i,j}\}$  i.i.d random variables with mean  $\mu$  and variance  $\sigma^2$ , let

$$Y := \operatorname{median}_{j \in [6 \log 1/\delta]} \operatorname{mean}_{i \in [6/\epsilon^2]} X_{i,j}$$

Then  $|Y - \mu| \leq \epsilon\sigma$  with probability  $\geq 1 - \delta$ , using only  $O(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$  samples.

- ▶ In words: We may create a mean estimator from  $6/\epsilon^2$  samples of  $X$ . Here we compute the median of  $6 \log 1/\delta$  such estimators
- ▶ Catoni (2012) shows that Chebyshev's inequality is the best guarantee one can provide when considering pure empirical mean estimators for an unknown distribution (and finite  $\mu, \sigma$ )
- ▶ "Median of means" provides an exponential improvement in probability of success  $(1 - \delta)$  guarantee

# Dequantization Toolbox

Method 1: Inner product estimation (Tang, 2018)

## Corollary

*For  $x, y \in \mathbb{C}^n$ , given  $x \in \mathcal{SQ}$  and  $y \in \mathcal{Q}$ , we can estimate  $\langle x, y \rangle$  to  $\epsilon \|x\| \|y\|$  error with probability  $\geq 1 - \delta$  with query complexity  $O(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$*

## Proof.

Sample an **index**  $s$  from  $x$ . Then, define  $Z := x_s v_s \frac{\|v\|^2}{|v_s|^2}$ . Apply the Fact with  $X_{i,j}$  being independent samples  $Z$ . □



# Dequantization Toolbox

## Method 2: Thin Matrix-Vector (Tang, 2018)

- ▶ For  $V \in \mathbb{C}^{n \times k}$ ,  $w \in \mathbb{C}^k$ , given  $V^\dagger \in \mathcal{SQ}$  (*column-wise sampling of  $V$* ) and  $w \in \mathcal{Q}$ , we can simulate  $Vw \in \mathcal{SQ}$  with  $\text{poly}(k)$  queries
- ▶ In words: if we can least-square sample the columns of matrix  $V$  and query the entries of vector  $w$ , then
  1. We can query entries of their multiplication ( $Vw$ )
  2. We can least-square sample from a distribution that emulates their multiplication
- ▶ Hence, as long as  $k \ll n$ , we can perform each using a number of steps polynomial in the number of columns of  $V$ .

# Dequantization Toolbox

## Method 2: Thin Matrix-Vector (Tang, 2018)

### Definition

Rejection sampling

### Algorithm

*Input: Samples from distribution  $P$*

*Output: Samples from distribution  $Q$*

- ▶ *Sample  $s$  from  $P$*
- ▶ *Compute  $r_s = \frac{1}{N} \frac{Q(s)}{P(s)}$ , for fixed constant  $N$*
- ▶ *Output  $s$  with probability  $r_s$  and restart otherwise*

### Fact

*Fact. If  $r_i \leq 1, \forall i$ , then the above procedure is well-defined and outputs a sample from  $Q$  in  $N$  iterations in expectation.*

# Dequantization Toolbox

## Method 2: Thin Matrix-Vector (Tang, 2018)

### Proposition

*For  $V \in \mathbb{R}^{n \times k}$  and  $w \in \mathbb{R}^k$ , given  $V^\dagger \in \mathcal{SQ}$  and  $w \in \mathcal{Q}$ , we can simulate  $Vw \in \mathcal{SQ}$  with expected query complexity  $O(k^2 C(V, w))$ , where*

$$C(V, w) := \frac{\sum_{i=1}^k \|V_{(\cdot, i)} w_i\|^2}{\|Vw\|^2}$$

*We can compute entries  $(Vw)_i$  with  $O(k)$  queries.*

*We can sample using rejection sampling:*

- ▶  *$P$  is the distribution formed by sampling from  $V_{(\cdot, j)}$  with probability proportional to  $\|V_{(\cdot, j)} w_j\|^2$*
- ▶  *$Q$  is the target  $Vw$ .*

$$r_i = \frac{(Vw)_i^2}{k \sum_{j=1}^k (V_{ij} w_j)^2} = \frac{Q(i)}{k C(V, w) P(i)}$$

# Dequantization Toolbox

## Method 2: Thin Matrix-Vector (Tang, 2018)

- ▶ Notice that we can compute these  $r_i$ 's (in fact, despite that we cannot compute probabilities from the target distribution), and that the rejection sampling guarantee is satisfied (via Cauchy-Schwarz).
- ▶ The probability of success is  $\frac{\|Vw\|^2}{k \sum_{i=1}^k \|w_i V^{(i)}\|^2}$ . Thus, to estimate the norm of  $Vw$ , it suffices to estimate the probability of success of this rejection sampling process.
- ▶ Through a Chernoff bound, we see that the average of  $O(kC(V, w)(\frac{1}{\epsilon^2} \log \frac{1}{\delta}))$  "coin flips" is in  $[(1 - \epsilon)\|Vw\|, (1 + \epsilon)\|Vw\|]$  with probability  $\geq 1 - \delta$ , where each coin flip costs  $k$  queries and samples.

# Dequantization Toolbox

## Method 3: Low-Rank Approximation (Frieze, Kannan, Vempala, 1998)

- ▶ For  $A \in \mathbb{C}^{m \times n}$ , given  $A \in \mathcal{SQ}$  and some threshold  $k$ , we can output a description of a low-rank approximation of  $A$  with  $\text{poly}(k)$  queries.
- ▶ Specifically, we output two matrices  $S, \hat{U} \in \mathcal{SQ}$  where  $S \in \mathbb{C}^{\ell \times n}$ ,  $\hat{U} \in \mathbb{C}^{\ell \times k}$  ( $\ell = \text{poly}(k, \frac{1}{\epsilon})$ ), and this implicitly describes the low-rank approximation to  $A$ ,  
 $D := A(S^\dagger \hat{U})(S^\dagger \hat{U})^\dagger$  ( $\Rightarrow \text{rank } D \leq k$ ).
- ▶ This matrix satisfies the following low-rank guarantee with probability  $\geq 1 - \delta$ : for  $\sigma := \sqrt{2/k} \|A\|_F$ , and  $A_\sigma := \sum_{\sigma_i \geq \sigma} \sigma_i u_i v_i^\dagger$  (using SVD),

$$\|A - D\|_F^2 \leq \|A - A_\sigma\|_F^2 + \epsilon^2 \|A\|_F^2$$

- ▶ Pay special attention to the  $\|A - A_\sigma\|_F^2$  term. This says that our guarantee is weak if  $A$  has no large singular values.
- ▶ Quantum analog: phase estimation

# Moore-Penrose Pseudoinverse (low-rank)

Application (Lloyd, Tang, 2018)

## Problem

For a low-rank matrix  $A \in \mathbb{R}^{m \times n}$  and a vector  $x \in \mathbb{R}^n$ , given  $x, A \in \mathcal{SQ}$ , (approximately) simulate  $A^+x \in \mathcal{SQ}$ , where  $A^+$  is the pseudoinverse of  $A$ .

## Algorithm

- ▶ Low-rank approximation (3) gives us  $S, \hat{U} \in \mathcal{SQ}$ .
- ▶ Applying thin-matrix vector (2), we get  $\hat{V} \in \mathcal{SQ}$ , where  $\hat{V} := S^T \hat{U}$ ; we can show that the columns of  $\hat{V}$  behave like the right singular vectors of  $A$ .
- ▶ Low-rank approximation (3) also outputs their approximate singular values  $\hat{\sigma}_i$
- ▶ Hence, we can approximate the vector we wish to sample:

$$A^+x = (A^T A)^+ A^T x \approx \sum_{i=1}^k \frac{1}{\hat{\sigma}_i^2} \hat{v}_i \hat{v}_i^T A^T x$$

# Moore-Penrose Pseudoinverse (low-rank) cont.

Application (Lloyd, Tang, 2018)

- ▶ We approximate  $\hat{v}_i^T A^T x$  to additive error for all  $i$  by noticing that  $\hat{v}_i^T A^T x = \text{tr}(A^T x \hat{v}_i^T)$  is an inner product of the order two tensors  $A^T$  and  $x \hat{v}_i^T$ .
- ▶ Thus, we can apply (1), since being given  $A \in \mathcal{SQ}$  implies  $A^T \in \mathcal{SQ}$  for  $A^T$  viewed as a long vector.
- ▶ Finally, using (2), sample from the linear combination using these estimates and  $\hat{\sigma}_i$ .

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# Thoughts

- ▶ Claim (Tang): For machine learning problems, SQ assumptions are more reasonable than state preparation assumptions.
- ▶ We discussed pseudo-inverse which inverts singular values, but in principle we could have applied any function to the singular values
- ▶ Gilyen et. al (2018) show that many quantum machine learning algorithms indeed apply polynomial functions to singular values
- ▶ Our discussion suggests that exponential quantum speedups are tightly related to problems where high-rank matrices play a crucial role (e.g. Hamiltonian simulation or QFT)

Thank you for listening!

Questions? [fms15@duke.edu](mailto:fms15@duke.edu)

# Read the Fine Print

- ▶ In general QML algorithms convert quantum input states to the desired quantum output state.
- ▶ In practice, data is initially stored classically and the algorithm's output must be accessed classically as well.
- ▶ This poses two problems if seek to use these algorithms: the "state preparation" and "readout" problems.
- ▶ Even if we ignore the readout problem, can we at least find a state preparation routine that maintains a speedup for the discussed quantum algorithms? Open question!
- ▶ See "Quantum Machine Learning Algorithms: Read the Fine Print" by Aaronson

# "Dequantization" (Tang)

## Definition

Let  $\mathcal{A}$  be a quantum algorithm with input  $|\varphi_1\rangle, \dots, |\varphi_C\rangle$  and output either a state  $|\psi\rangle$  or a value  $\lambda$ . We say we dequantize  $\mathcal{A}$  if we describe a classical algorithm that, given  $\varphi_1, \dots, \varphi_C \in \mathcal{SQ}$ , can evaluate queries to  $\psi \in \mathcal{SQ}$  or output  $\lambda$ , with similar guarantees to  $\mathcal{A}$  and query complexity  $\text{poly}(C)$ .