Chapter 2
Estimates on Grassmann Manifolds

Abstract The main result of this chapter, called the *Avalanche Principle* (AP), relates the expansion of a long product of matrices with the product of expansions of the individual matrices. This principle was introduced by M. Goldstein and W. Schlag in the context of $\mathrm{SL}(2, \mathbb{C})$ matrices. Besides extending the AP to matrices of arbitrary dimension and possibly non-invertible, the geometric approach we use here provides a relation between the most expanding (singular) directions of such a long product of matrices and the corresponding singular directions of the first and last matrices in the product. The AP along with other estimates on the action of matrices on Grassmann manifolds will play a fundamental role in the next chapters, when we establish the continuity the LE and of the Oseledets decomposition.

2.1 Grassmann Geometry

Grassmann geometry is the geometric study of manifolds of linear subspaces of an Euclidean space and of the action of linear groups (and algebras) on them. Its foundations were laid in the masterpiece 'Die lineale Ausdehnungslehre' of Hermann Grassmann, whose genius is still not fully understood, as explained in the survey [2].

2.1.1 Projective Spaces

The projective space is the simplest compact model to study the action of a linear map. Given an n-dimensional Euclidean space V, consider the equivalence relation defined on $V \setminus \{0\}$ by $u \equiv v$ if and only if $u = \lambda\, v$ for some $\lambda \neq 0$. For $v \in V \setminus \{0\}$, the set $\hat{v} := \{\lambda\, v : \lambda \in \mathbb{R} \setminus \{0\}\}$ is the equivalence class of the vector v relative to this relation. The *projective space of* V is the quotient $\mathbb{P}(V) := \{\hat{v} : v \in V \setminus \{0\}\}$ of $V \setminus \{0\}$ by this equivalence relation. It is a compact topological space when endowed with the quotient topology.

P. Duarte and S. Klein, *Lyapunov Exponents of Linear Cocycles*,
Atlantis Studies in Dynamical Systems 3, DOI 10.2991/978-94-6239-124-6_2

The unit sphere $\mathbb{S}(V) := \{v \in V : \|v\| = 1\}$ is a compact Riemannian manifold of constant curvature 1 and diameter π. The natural projection $\hat{\pi} : \mathbb{S}(V) \to \mathbb{P}(V)$, $\hat{\pi}(v) = \hat{v}$, is a (double) covering map. Hence the projective space $\mathbb{P}(V)$ has a natural smooth Riemannian structure for which the covering map $\hat{\pi}$ is a local isometry. Thus $\mathbb{P}(V)$ is a compact Riemannian manifold with constant curvature 1 and diameter $\frac{\pi}{2}$.

Given a linear map $g \in \mathscr{L}(V)$ define $\mathbb{P}(g) := \{\hat{v} \in \mathbb{P}(V) : g\,v \neq 0\}$. We refer to the linear map $\varphi_g : \mathbb{P}(g) \subset \mathbb{P}(V) \to \mathbb{P}(V)$, $\varphi_g(\hat{v}) := \hat{\pi}(\frac{g\,v}{\|g\,v\|})$, as the *projective action of g on $\mathbb{P}(V)$*. If g is invertible then $\varphi_g : \mathbb{P}(V) \to \mathbb{P}(V)$ is a diffeomorphism with inverse $\varphi_{g^{-1}} : \mathbb{P}(V) \to \mathbb{P}(V)$. Through these maps, the group $\mathrm{GL}(V)$, of all linear automorphisms on V, acts transitively on the projective space $\mathbb{P}(V)$.

We will consider three different metrics on the projective space $\mathbb{P}(V)$. The Riemannian distance, ρ, measures the length of an arc connecting two points on the sphere. More precisely, given $u, v \in \mathbb{S}(V)$,

$$\rho(\hat{u}, \hat{v}) := \min\{\angle(u, v), \angle(u, -v)\}. \tag{2.1}$$

The second metric, d, corresponds to the Euclidean distance. More precisely, given $u, v \in \mathbb{S}(V)$,

$$d(\hat{u}, \hat{v}) := \min\{\|u - v\|, \|u + v\|\} \tag{2.2}$$

measures the smallest chord of the arcs between u and v and between u and $-v$. The third metric, δ, measures the sine of the arc between two points on the sphere. More precisely, given $u, v \in \mathbb{S}(V)$,

$$\delta(\hat{u}, \hat{v}) := \frac{\|u \wedge v\|}{\|u\|\,\|v\|} = \sin(\angle(u, v)). \tag{2.3}$$

The fact that δ is a metric on $\mathbb{P}(V)$ follows from the sine addition law, which implies that $\sin(\theta + \theta') \leq \sin\theta + \sin\theta'$, for all $\theta, \theta' \in [0, \frac{\pi}{2}]$.

These three distances are equivalent. For all $\hat{u}, \hat{v} \in \mathbb{P}(V)$,

$$\delta(\hat{u}, \hat{v}) = \sin\rho(\hat{u}, \hat{v}) \quad \text{and} \quad d(\hat{u}, \hat{v}) = \operatorname{chord}\rho(\hat{u}, \hat{v}). \tag{2.4}$$

The inequalities

$$\frac{2\theta}{\pi} \leq \sin\theta \leq \operatorname{chord}\theta = 2\,\sin(\theta/2) \leq \theta \qquad \forall\, 0 \leq \theta \leq \frac{\pi}{2}$$

imply that

$$\frac{2}{\pi}\,\rho(\hat{u}, \hat{v}) \leq \delta(\hat{u}, \hat{v}) \leq d(\hat{u}, \hat{v}) \leq \rho(\hat{u}, \hat{v}). \tag{2.5}$$

Because of (2.4), these three metrics determine the same group of isometries on the projective space.

2.1.2 Exterior Algebra

Exterior Algebra was introduced by H. Grassmann in the 'Ausdehnungslehre'. We present here an informal description of some of its properties. See the book of Stenberg [8] for a rigorous treatment of the subject.

Let V be a finite n-dimensional Euclidean space. Given k vectors $v_1, \dots, v_k \in V$, their kth exterior product is a formal skew-symmetric product $v_1 \wedge \cdots \wedge v_k$, in the sense that for any permutation $\sigma = (\sigma_1, \dots, \sigma_k) \in \mathrm{S}_k$,

$$v_{\sigma_1} \wedge \cdots \wedge v_{\sigma_k} = (-1)^{\mathrm{sgn}(\sigma)} v_1 \wedge \cdots \wedge v_k.$$

These formal products are elements of an anti-commutative and associative graded algebra $(\wedge_* V, +, \wedge)$, called the *exterior algebra* of V. Formal products $v_1 \wedge \cdots \wedge v_k$ are called *simple k-vectors* of V. The kth *exterior power of* V, denoted by $\wedge_k V$, is the linear span of all simple k vectors of V. Elements of $\wedge_k V$ are called *k-vectors*.

An easy consequence of this formal definition is that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \dots, v_k$ are linearly dependent. Another simple consequence is that given two bases $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ of the same k-dimensional linear subspace of V, if for some real matrix $A = (a_{ij})$ we have $w_i = \sum_{j=1}^{k} a_{ij}\, v_j$ for all $i = 1, \dots, k$, then

$$w_1 \wedge \cdots \wedge w_k = (\det A)\, v_1 \wedge \cdots \wedge v_k.$$

More generally, two families $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ of linearly independent vectors span the same k-dimensional subspace if and only if for some real number $\lambda \neq 0$, $w_1 \wedge \cdots \wedge w_k = \lambda\, v_1 \wedge \cdots \wedge v_k$. Hence we identify the line spanned by a simple k-vector $v = v_1 \wedge \cdots \wedge v_k$, i.e., the projective point $\hat{v} \in \mathbb{P}(\wedge_k V)$ determined by v, with the k-dimensional subspace spanned by the vectors $\{v_1, \dots, v_k\}$, denoted hereafter by $\langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle$.

The subspaces $\wedge_k V$ induce the grading structure of the exterior algebra $\wedge_* V$, i.e., we have the direct sum decomposition $\wedge_* V = \oplus_{k=0}^{\dim V} \wedge_k V$ with $(\wedge_k V) \wedge (\wedge_{k'} V) \subset \wedge_{k+k'} V$ for all $0 \leq k, k' \leq \dim V$. Geometrically, the exterior product operation $\wedge : \wedge_k V \times \wedge_{k'} V \to \wedge_{k+k'} V$ corresponds to the algebraic sum of linear subspaces, in the sense that given families $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ of linearly independent vectors such that $\langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle \cap \langle\!\langle w_1 \wedge \cdots \wedge w_{k'} \rangle\!\rangle = 0$, then

$$\langle\!\langle v_1 \wedge \cdots \wedge v_k \ \wedge\ w_1 \wedge \cdots \wedge w_{k'} \rangle\!\rangle = \langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle + \langle\!\langle w_1 \wedge \cdots \wedge w_{k'} \rangle\!\rangle.$$

Let Λ_k^n be the set of all k-subsets $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, with $i_1 < \cdots < i_k$, and order it lexicographically. Given a basis $\{e_1, \dots, e_n\}$ of V, define for each $I \in \Lambda_k^n$, the kth exterior product $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then the ordered family $\{e_I : I \in \Lambda_k^n\}$ is a basis of $\wedge_k V$. In particular $\dim \wedge_k V = \binom{n}{k}$.

The exterior algebra $\wedge_* V$ inherits an Euclidean structure from V. More precisely, there is a unique inner product on $\wedge_* V$ such that for any orthonormal basis

$\{e_1, \ldots, e_n\}$ of V, the family $\{e_I : I \in \Lambda^n_k,\ 0 \le k \le n\}$ is an orthonormal basis of the exterior algebra $\wedge_* V$.

Given vectors $v_1, \ldots, v_k \in V$ let us call *parallelepiped* generated by these vectors the set

$$P(v_1, \ldots, v_k) := \left\{ \sum_{j=1}^{k} t_j v_j : t_j \in [0, 1],\ j = 1, \ldots, k \right\}.$$

Interestingly, the norm of the simple k-vector $v_1 \wedge \cdots \wedge v_k$ is equal to the k-dimensional volume of the parallelepiped generated by its factors v_j. More precisely,

$$\|v_1 \wedge \cdots \wedge v_k\| = \mathrm{Vol}_k(P(v_1, \ldots, v_k)), \tag{2.6}$$

where Vol_k stands for the k-dimensional Hausdorff measure. To explain this fact first notice that if the vectors $v_1, \ldots, v_k$ are pairwise orthogonal then

$$\frac{\|v_1 \wedge \cdots \wedge v_k\|}{\|v_1\| \cdots \|v_k\|} = \| \frac{v_1}{\|v_1\|} \wedge \cdots \wedge \frac{v_k}{\|v_k\|} \| = 1$$

because the vectors $\{v_j/\|v_j\| : j = 1, \ldots, k\}$ are orthonormal. This shows that $\|v_1 \wedge \cdots \wedge v_k\| = \|v_1\| \cdots \|v_k\|$ and establishes (2.6) in this case. In general we use the Gram-Schmidt orthogonalization method, defining recursively

$$v'_1 = v_1 \ \text{ and } \ v'_j = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, v'_i \rangle}{\|v'_i\|^2} v'_i \ \text{ for } j = 2, \ldots, k.$$

At each step, when we replace v_j by v'_j, both wedge products and k-volumes are preserved. Hence $v'_1 \wedge \cdots \wedge v'_k = v_1 \wedge \cdots \wedge v_k$ and

$$\begin{aligned} \|v_1 \wedge \cdots \wedge v_k\| &= \|v'_1 \wedge \cdots \wedge v'_k\| = \|v'_1\| \ldots \|v'_k\| \\ &= \mathrm{Vol}_k(P(v'_1, \ldots, v'_k)) = \mathrm{Vol}_k(P(v_1, \ldots, v_k)). \end{aligned}$$

Formula (2.6) also implies that for any simple vectors $e = e_1 \wedge \cdots \wedge e_r$ and $f = f_1 \wedge \cdots \wedge f_s$ in V,

$$\|e \wedge f\| \le \|e\| \, \|f\|. \tag{2.7}$$

Moreover, equality holds if and only if $\langle e_i, f_j \rangle = 0$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

A simple k-vector $v_1 \wedge \cdots \wedge v_k$ of norm one is called a *unit k-vector*. From the previous considerations the correspondence $v_1 \wedge \cdots \wedge v_k \mapsto \langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle$ is one-to-one, between the set of unit k-vectors in $\wedge_k V$ and the set of oriented k-dimensional linear subspaces of V. In particular, if V is an oriented Euclidean space then the 1-dimensional space $\wedge_n V$ has a canonical unit n-vector, denoted by ω, and called the *volume element* of $\wedge_n V$. In this case there is a unique operator, called the *Hodge star* operator, $* : \wedge_* V \to \wedge_* V$ defined by

$$v \wedge (*w) = \langle v, w \rangle\, \omega, \quad \text{for all } v, w \in \wedge_* V.$$

The Hodge star operator maps $\wedge_k V$ isomorphically, and isometrically, onto $\wedge_{n-k} V$, for all $0 \le k \le n$. Geometrically it corresponds to the orthogonal complement operation on linear subspaces, i.e., for any simple k-vector,

$$\langle\!\langle *(v_1 \wedge \cdots \wedge v_k) \rangle\!\rangle = \langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle^{\perp}.$$

A dual product operation $\vee : \wedge_* V \times \wedge_* V \to \wedge_* V$ can be defined by

$$v \vee w := *((*v) \wedge (*w)), \quad \text{for all } v, w \in \wedge_* V.$$

This operation maps $\wedge_k V \times \wedge_{k'} V$ to $\wedge_{k+k'-n} V$, and describes the intersection operation on linear subspaces, in the sense that given families $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ of linearly independent vectors with $\langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle + \langle\!\langle w_1 \wedge \cdots \wedge w_{k'} \rangle\!\rangle = V$, then

$$\langle\!\langle (v_1 \wedge \cdots \wedge v_k) \vee (w_1 \wedge \cdots \wedge w_{k'}) \rangle\!\rangle = \langle\!\langle v_1 \wedge \cdots \wedge v_k \rangle\!\rangle \cap \langle\!\langle w_1 \wedge \cdots \wedge w_{k'} \rangle\!\rangle.$$

The geometric meaning of the $\vee$-operation reduces by duality to that of the sum $\wedge$-operation and the complement $*$-operation.

Any linear map $g : V \to V$ induces a linear map $\wedge_k g : \wedge_k V \to \wedge_k V$, called the kth *exterior power* of g, such that for all $v_1, \ldots, v_k \in V$,

$$\wedge_k g(v_1 \wedge \cdots \wedge v_k) = g(v_1) \wedge \cdots \wedge g(v_k).$$

This construction is functorial in the sense that for all linear maps $g, g' : V \to V$,

$$\wedge_k \mathrm{id}_V = \mathrm{id}_{\wedge_k V}, \quad \wedge_k (g' \circ g) = \wedge_k g' \circ \wedge_k g \quad \text{and} \quad \wedge_k\, g^* = (\wedge_k g)^*,$$

where $g^* : V \to V$ denotes the *adjoint operator*.

A clear consequence of these properties is that if $g : V \to V$ is an *orthogonal automorphism*, i.e., $g^* \circ g = \mathrm{id}_V$, then so is $\wedge_k g : \wedge_k V \to \wedge_k V$.

Consider a matrix $A \in \mathrm{Mat}(n, \mathbb{R})$. Given $I, J \in \Lambda^n_k$, we denote by $A_{I \times J}$ the square sub-matrix of A indexed in $I \times J$. If a linear map $g : V \to V$ is represented by A relative to a basis $\{e_1, \ldots, e_n\}$, then the kth exterior power $\wedge_k g : \wedge_k V \to \wedge_k V$ is represented by the matrix $\wedge_k A := (\det A_{I \times J})_{I,J}$ relative to the basis $\{e_I : I \in \Lambda^n_k\}$. The matrix $\wedge_k A$ is called the kth *exterior power* of A. Obviously, matrix exterior powers satisfy the same functorial properties as linear maps, i.e., for all $A, A' \in \mathrm{Mat}(n, \mathbb{R})$,

$$\wedge_k \mathrm{I}_n = \mathrm{I}_{\binom{n}{k}}, \quad \wedge_k (A' A) = (\wedge_k A')(\wedge_k A) \quad \text{and} \quad \wedge_k\, A^* = (\wedge_k A)^*,$$

where A^* denotes the *transpose* matrix of A.

Let $n = \dim V$ and $\{e_i : i = 1, \ldots, n\}$ be an eigen-basis of a linear endomorphism $g : V \to V$ with eigenvalues $\{\lambda_i : i = 1, \ldots, n\}$, i.e., $ge_i = \lambda_i e_i$ for all $i = 1, \ldots, n$. Then the family $\{e_I : I \in \Lambda^n_k\}$ is an eigen-basis of $\wedge_k g : \wedge_k V \to \wedge_k V$ with eigenvalues

$$\lambda_I = \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}, \quad I = \{i_1, \ldots, i_k\} \in \Lambda^n_k.$$

In other words, $(\wedge_k g)e_I = \lambda_I e_I$ for all $I \in \Lambda^n_k$.

2.1.3 Grassmann Manifolds

Grassmannians, like projective spaces, are compact Riemannian manifolds which stage the action of linear maps. For each $0 \leq k \leq n$, the *Grassmannian* $\mathrm{Gr}_k(V)$ is the space of all k-dimensional linear subspaces of V. Notice that the projective space $\mathbb{P}(V)$ and the Grassmannian $\mathrm{Gr}_1(V)$ are the same object if we identify each point $\hat{v} \in \mathbb{P}(V)$ with the line $\langle v\rangle = \{\lambda\, v : \lambda \in \mathbb{R}\}$. The full Grassmannian $\mathrm{Gr}(V)$ is the union of all Grassmannians $\mathrm{Gr}_k(V)$ with $0 \leq k \leq n$. Denote by $\mathscr{L}(V)$ the algebra of linear endomorphisms on V, and consider the map $\pi : \mathrm{Gr}(V) \to \mathscr{L}(V)$, $E \mapsto \pi_E$, that assigns the orthogonal projection π_E onto E, to each subspace $E \in \mathrm{Gr}(V)$. This map is one-to-one, and we endow $\mathrm{Gr}(V)$ with the unique topology that makes the map $\pi : \mathrm{Gr}(V) \to \pi(\mathrm{Gr}(V))$ a homeomorphsim. With it, $\mathrm{Gr}(V)$ becomes a compact space, and each Grassmannian $\mathrm{Gr}_k(V)$ is a closed connected subspace of $\mathrm{Gr}(V)$.

The group $\mathrm{GL}(V)$ acts transitively on each Grassmannian. The action of $\mathrm{GL}(V)$ on $\mathrm{Gr}_k(V)$ is given by $\cdot : \mathrm{GL}(V) \times \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V)$, $(g, E) \mapsto g\,E$. The special orthogonal group $\mathrm{SO}(V)$, of orientation preserving orthogonal automorphisms, acts transitively on Grassmannians too. All Grassmannians are compact homogeneous spaces.

For each $0 \leq k \leq n$, the *Plücker* embedding is the map $\psi : \mathrm{Gr}_k(V) \to \mathbb{P}(\wedge_k V)$ that to each subspace E in $\mathrm{Gr}_k(V)$ assigns the projective point $\hat{v} \in \mathbb{P}(\wedge_k V)$, where $v = v_1 \wedge \cdots \wedge v_k$ is any simple k-vector formed as exterior product of a basis $\{v_1, \ldots, v_k\}$ of E. This map is one-to-one and equivariant, i.e., for all $g \in \mathrm{GL}(V)$ and $E \in \mathrm{Gr}(V)$,

$$\psi(g\,E) = \varphi_{\wedge_k g}\,\psi(E). \tag{2.8}$$

We will consider the metrics $\rho, d, \delta : \mathrm{Gr}_k(V) \times \mathrm{Gr}_k(V) \to [0, +\infty)$ defined for any given $E, F \in \mathrm{Gr}_k(V)$ by

$$\rho(E, F) := \rho(\psi(E), \psi(F)), \tag{2.9}$$

$$d(E, F) := d(\psi(E), \psi(F)), \tag{2.10}$$

$$\delta(E, F) := \delta(\psi(E), \psi(F)). \tag{2.11}$$

which assign diameter $\frac{\pi}{2}$, $\sqrt{2}$ and 1, respectively, to the manifold $\mathrm{Gr}_k(V)$. These distances are preserved by the action of orthogonal linear maps in $\mathrm{SO}(V)$.

Given $k, k' \geq 0$ such that $k + k' \geq n = \dim V$, the intersection of subspaces is an operation $\cap : \mathrm{Gr}_{k,k'}(\cap) \subset \mathrm{Gr}_k(V) \times \mathrm{Gr}_{k'}(V) \to \mathrm{Gr}_{k+k'-n}(V)$ where:

Definition 2.1 The domain is defined by

$$\mathrm{Gr}_{k,k'}(\cap) := \{(E, E') \in \mathrm{Gr}_k(V) \times \mathrm{Gr}_{k'}(V) : \ E + E' = V\}.$$

Similarly, given $k, k' \geq 0$ such that $k + k' \leq n = \dim V$, the algebraic sum of subspaces is operation $+ : \mathrm{Gr}_{k,k'}(+) \subset \mathrm{Gr}_k(V) \times \mathrm{Gr}_{k'}(V) \to \mathrm{Gr}_{k+k'-n}(V)$ where:

Definition 2.2 The domain is defined by

$$\mathrm{Gr}_{k,k'}(+) := \{(E, E') \in \mathrm{Gr}_k(V) \times \mathrm{Gr}_{k'}(V) : \ E \cap E' = \{0\}\}.$$

The considerations in Sect. 2.1.2 show that the Plücker embedding satisfies the following relations:

Proposition 2.1 *Given $E \in \mathrm{Gr}_k(V)$, $E' \in \mathrm{Gr}_{k'}(V)$, consider unit vectors $v \in \Psi(E)$ and $v' \in \Psi(E')$.*

(a) If $(E, E') \in \mathrm{Gr}_{k,k'}(\cap)$ then $\psi(E \cap E') = \widehat{v \vee v'}$.
(b) If $(E, E') \in \mathrm{Gr}_{k,k'}(+)$ then $\psi(E + E') = \widehat{v \wedge v'}$.

A duality between sums and intersections stems from these facts.

Proposition 2.2 *The orthogonal complement operation $E \mapsto E^\perp$ is a d-isometric involution on $\mathrm{Gr}(V)$ which maps $\mathrm{Gr}_{k,k'}(+)$ to $\mathrm{Gr}_{n-k,n-k'}(\cap)$ and satisfies for all $(E, E') \in \mathrm{Gr}_{k,k'}(+)$,*

$$(E + E')^\perp = (E^\perp) \cap (E')^\perp.$$

The composition semigroup $\mathscr{L}(V)$ has two partial actions on Grassmannians, called the *push-forward action* and the *pull-back action*. Before introducing them, a couple of facts are needed.

Definition 2.3 Given $g \in \mathscr{L}(V)$, we denote by $\mathrm{K}g := \{v \in V : \ g\, v = 0\}$ the *kernel of* g, and by $\mathrm{R}g := \{g\, v : \ v \in V\}$ the *range of* g.

Lemma 2.1 *Given $g \in \mathscr{L}(V)$ and $E \in \mathrm{Gr}(V)$,*

1. *if $E \cap (\mathrm{K}g) = \{0\}$ then the linear map $g|_E : E \to g(E)$ is an isomorphism, and in particular $\dim g(E) = \dim E$.*
2. *if $E + (\mathrm{R}g) = V$ then the linear map $g^*|_{E^\perp} : E^\perp \to g^{-1}(E)^\perp$ is an isomorphism, and in particular $\dim g^{-1}(E) = \dim E$.*

Proof The first statement is obvious because if $E \cap (\mathrm{K}g) = \{0\}$ then $\mathrm{K}(g|_E) = \{0\}$. If $E + (\mathrm{R}g) = V$ then, since $\mathrm{K}g^* = (\mathrm{R}g)^\perp$, we have $E^\perp \cap (\mathrm{K}g^*) = E^\perp \cap (\mathrm{R}g)^\perp = (E + \mathrm{R}g)^\perp = \{0\}$. Hence by 1, the linear map $g^*|_{E^\perp} : E^\perp \to g^*(E^\perp)$ is an isomorphism. It is now enough to remark that $g^*(E^\perp) = g^{-1}(E)^\perp$. In fact, the inclusion $g^*(E^\perp) \subset$

$g^{-1}(E)^{\perp}$ is clear. Since $g^*|E^{\perp}$ is injective, $\dim g^*(E^{\perp}) = \dim(E^{\perp})$. On the other hand, by the transversality condition, $g^{-1}(E)$ has dimension

$$\begin{aligned}\dim g^{-1}(E) &= \dim\left((g|_{(\mathrm{K}g)^{\perp}})^{-1}(E \cap \mathrm{R}g)\right) + \dim(\mathrm{K}g)\\ &= \dim(E \cap \mathrm{R}g) + \dim(\mathrm{K}g)\\ &= \dim(E) + \dim(\mathrm{R}g) - n + \dim(\mathrm{K}g) = \dim(E).\end{aligned}$$

Hence both $g^*(E^{\perp})$ and $g^{-1}(E)^{\perp}$ have dimension equal to $\dim(E^{\perp})$, and the equality follows. □

Given $g \in \mathscr{L}(V)$ and $k \geq 0$ such that $k + \dim(\mathrm{K}g) \leq n = \dim V$, the *push-forward by g* is the map $\varphi_g : \mathrm{Gr}_k(g) \subset \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V)$, $E \mapsto gE$, where:

Definition 2.4 The domain is defined by

$$\mathrm{Gr}_k(g) := \{E \in \mathrm{Gr}_k(V) : \; E \cap (\mathrm{K}g) = \{0\}\}.$$

We warn the reader that the notation φ_g is used for both the projective and the Grassmannian actions of $g \in \mathscr{L}(V)$.

Similarly, given $k \geq 0$ such that $k + \dim(\mathrm{R}g) \geq n = \dim V$, the *pull-back by g* is the map $\varphi_{g^{-1}} : \mathrm{Gr}_k(g^{-1}) \subset \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V)$, $E \mapsto g^{-1}E$, where:

Definition 2.5 The domain is defined by

$$\mathrm{Gr}_k(g^{-1}) := \{E \in \mathrm{Gr}_k(V) : \; E + (\mathrm{R}g) = V\}.$$

From the proof of Proposition 2.1 we obtain a duality between push-forwards and pull-backs which can be expressed as follows.

Proposition 2.3 *Given $g \in \mathscr{L}(V)$ and $k \geq 0$ such that $k + \dim(\mathrm{R}g) \geq n = \dim V$, we have $\mathrm{Gr}_k(g^{-1}) = \mathrm{Gr}_{n-k}(g^*)^{\perp}$ and for all $E \in \mathrm{Gr}_k(g^{-1})$,*

$$(g^{-1}E)^{\perp} = g^*(E^{\perp}).$$

In Sect. 2.3 we derive a modulus of Lipschitz continuity, w.r.t. the metric δ, for the sum and intersection operations.

2.1.4 Flag Manifolds

Let V be a finite n-dimensional Euclidean space. Any strictly increasing sequence of linear subspaces $F_1 \subset F_2 \subset \cdots \subset F_k \subset V$ is called a *flag* in the Euclidean space V. Formally, flags are denoted as lists $F = (F_1, \ldots, F_k)$. The sequence $\tau = (\tau_1, \ldots, \tau_k)$ of dimensions $\tau_j = \dim F_j$ is called the *signature* of the flag F. The integer k is called the *length* of the flag F, and the *length* of the signature τ. Let $\mathcal{F}(V)$ be the set of all flags in V, and define $\mathcal{F}_\tau(V)$ to be the space of flags with a given signature τ. Two special cases of flag spaces are the projective space $\mathbb{P}(V) = \mathcal{F}_\tau(V)$, when $\tau = (1)$, and the Grassmannian $\mathrm{Gr}_k(V) = \mathcal{F}_\tau(V)$, when $\tau = (k)$.

The general linear group $\mathrm{GL}(V)$ acts naturally on $\mathcal{F}(V)$. Given $g \in \mathrm{GL}(V)$ the action of g on $\mathcal{F}_\tau(V)$ is given by the map $\varphi_g : \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(V)$, $\varphi_g F = (gF_1, \dots, gF_k)$. The special orthogonal subgroup $\mathrm{SO}(V) \subset \mathrm{GL}(V)$ acts transitively on $\mathcal{F}_\tau(V)$. Hence, all flag manifolds $\mathcal{F}_\tau(V)$ are compact homogeneous spaces. Each of them is a compact connected Riemannian manifold where the group $\mathrm{SO}(V)$ acts by isometries. Since $\mathcal{F}_\tau(V) \subset \mathrm{Gr}_{\tau_1}(V) \times \mathrm{Gr}_{\tau_2}(V) \times \cdots \times \mathrm{Gr}_{\tau_k}(V)$, the product distances

$$\rho_\tau(F, F') = \max_{1 \le j \le k} \rho(F_j, F'_j) \tag{2.12}$$

$$d_\tau(F, F') = \max_{1 \le j \le k} d(F_j, F'_j) \tag{2.13}$$

$$\delta_\tau(F, F') = \max_{1 \le j \le k} \delta(F_j, F'_j) \tag{2.14}$$

are equivalent to the Riemannian distance on $\mathcal{F}_\tau(V)$. With these metrics, the flag manifold $\mathcal{F}_\tau(V)$ has diameter $\frac{\pi}{2}$, $\sqrt{2}$ and 1, respectively. The group $\mathrm{SO}(V)$ acts isometrically on $\mathcal{F}_\tau(V)$ with respect to these distances.

Given a signature $\tau = (\tau_1, \dots, \tau_k)$, if $n = \dim V$, we define

$$\tau^\perp := (n - \tau_k, \dots, n - \tau_1).$$

When $\tau = (\tau_1, \dots, \tau_k)$ we will write $\tau^\perp = (\tau_1^\perp, \dots, \tau_k^\perp)$, where $\tau_j^\perp = n - \tau_{k+1-i}$.

Definition 2.6 Given a flag $F = (F_1, \dots, F_k) \in \mathcal{F}_\tau(V)$, its *orthogonal complement* is the $\tau^\perp$-flag $F^\perp := (F_k^\perp, \dots, F_1^\perp)$.

The map $\cdot^\perp : \mathcal{F}(V) \to \mathcal{F}(V)$ is an isometric involution on $\mathcal{F}(V)$, mapping $\mathcal{F}_\tau(V)$ onto $\mathcal{F}_{\tau^\perp}(V)$. The involution character, $(F^\perp)^\perp = F$ for all $F \in \mathcal{F}(V)$, is clear. As explained in Sect. 2.1.2, the Hodge star operator $* : \wedge_k V \to \wedge_{n-k} V$ is an isometry between these Euclidean spaces. By choice of metrics on the Grassmannians, see (2.10), the Plücker embeddings are isometries. Finally, the Plücker embedding conjugates the orthogonal complement map $\cdot^\perp : \mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V)$ with the Hodge star operator. Hence for each $0 \le k \le n$, the map $\cdot^\perp : \mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V)$ is an isometry. The analogous conclusion for flags follows from the definition of distance d_τ.

Given $g \in \mathscr{L}(V)$ and a signature τ such that $\tau_i + \dim(\mathrm{K}g) \le n$ for all i, the *push-forward by* g on flags is the map $\varphi_g : \mathcal{F}_\tau(g) \subset \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(V)$, $\varphi_g F := (g\,F_1, \dots, g\,F_k)$, where:

Definition 2.7 The domain of φ_g is defined by

$$\mathcal{F}_\tau(g) := \{F \in \mathcal{F}_\tau(V) : \; F_k \cap (\mathrm{K}g) = \{0\}\}.$$

Similarly, given a signature τ such that $\tau_i + \dim(Rg) \geq n$ for all i, the *pull-back by g* on flags is the map $\varphi_{g^{-1}} : \mathfrak{F}_\tau(g^{-1}) \subset \mathfrak{F}_\tau(V) \to \mathfrak{F}_\tau(V)$, $\varphi_{g^{-1}} F := (g^{-1}F_1, \ldots, g^{-1}F_k)$, where:

Definition 2.8 The domain of $\varphi_{g^{-1}}$ is defined by

$$\mathfrak{F}_\tau(g^{-1}) := \{F \in \mathfrak{F}_\tau(V) : \; F_1 + (Rg) = V\}.$$

The duality between duality between push-forwards and pull-backs is expressed as follows.

Proposition 2.4 *Given $g \in \mathscr{L}(V)$, $\mathfrak{F}_\tau(g^{-1}) = \mathfrak{F}_{\tau^\perp}(g^*)^\perp$ and for all $F \in \mathfrak{F}_\tau(g^{-1})$,*

$$(\varphi_{g^{-1}} F)^\perp = \varphi_{g^*}(F^\perp).$$

2.2 Singular Value Geometry

Singular value geometry refers here to the geometry of the *singular value decomposition* (SVD) of a linear endomorphism $g : V \to V$ on some Euclidean space V. It also refers to some geometric properties of the action of g on Grassmannians and flag manifolds related to the singular value decomposition of g.

2.2.1 Singular Value Decomposition

Let V be a Euclidean space of dimension n.

Definition 2.9 Given $g \in \mathscr{L}(V)$, the *singular values* of g are the square roots of the eigenvalues of the quadratic form $Q_g : V \to \mathbb{R}$, $Q_g(v) = \|g\, v\|^2 = \langle gv, gv\rangle$, i.e., the eigenvalues of the positive semi-definite self-adjoint operator $\sqrt{g^* g}$.

Given $g \in \mathscr{L}(V)$, let

$$s_1(g) \geq s_2(g) \geq \cdots \geq s_n(g) \geq 0,$$

denote the sorted singular values of g. The adjoint g^* has the same singular values as g because the operators $\sqrt{g^* g}$ and $\sqrt{g\, g^*}$ are conjugate.

The largest singular value, $s_1(g)$, is the square root of the maximum value of Q_g over the unit sphere, i.e., $s_1(g) = \max_{\|v\|=1} \|g\, v\| = \|g\|$ is the operator norm of g. Likewise, the least singular value, $s_n(g)$, is the square root of the minimum value of Q_g over the unit sphere, i.e., $s_n(g) = \min_{\|v\|=1} \|g\, v\|$. This number, also denoted by $\mathfrak{m}(g)$, is called the *least expansion* of g. If g is invertible then $\mathfrak{m}(g) = \|g^{-1}\|^{-1}$, while otherwise $\mathfrak{m}(g) = 0$.

Definition 2.10 The eigenvectors of the quadratic form Q_g, i.e., of the positive semi-definite self-adjoint operator $\sqrt{g^* g}$, are called the *singular vectors* of g.

By the spectral theory of self-adjoint operators, for any $g \in \mathscr{L}(V)$ there exists an orthonormal basis consisting of singular vectors of g.

Proposition 2.5 *Given $g \in \mathscr{L}(V)$, let $v \in V$ be such that $g^* g\, v = \lambda^2 v$ with $\lambda \geq 0$ and $\|v\| = 1$, i.e., v is a unit singular vector of g with singular value λ. Then there exists a unit vector $w \in V$ such that*

(a) $g\, v = \lambda\, w$,
(b) $g\, g^*\, w = \lambda^2\, w$, *i.e., w is a singular vector of g^*.*

Proof Let $v \in V$ be a unit singular vector of g. Then $g^* g\, v = \lambda^2 v$ with $\lambda \geq 0$ and $\lambda^2 = \langle \lambda^2 v, v\rangle = \langle g^* g\, v, v\rangle = \|g\, v\|^2$, which implies that $\lambda = \|g\, v\|$. Since $(g\, g^*)\,(g\, v) = g\,(g^* g)\, v = \lambda^2\, g\, v$, if $\lambda \neq 0$ then setting $w = g\, v/\|g\, v\| = \lambda^{-1}\, g\, v$, we have $(g\, g^*)\, w = \lambda^2\, w$, which proves that w is a singular vector of g^*. By definition $g\, v = \lambda\, w$. When $\lambda = 0$, take w to be any unit vector in $\mathrm{K}g^*$. Notice that $\dim(\mathrm{K}g) = \dim(\mathrm{K}g^*)$. In this case v and w are singular vectors of g and g^*, respectively, such that $g\, v = 0 = \lambda\, w$. $\square$

By the previous proposition, given $g \in \mathscr{L}(V)$ there exist two orthonormal singular vector basis of V, $\{v_1(g), \ldots, v_n(g)\}$ and $\{v_1(g^*), \ldots, v_n(g^*)\}$ for g and g^*, respectively, such that

$$g\, v_j(g) = s_j(g)\, v_j(g^*) \quad \text{for all } \; 1 \leq j \leq n.$$

Denote by D_g the diagonal matrix with diagonal entries $s_j(g)$, $1 \leq j \leq n$, seen as an operator $D_g \in \mathscr{L}(\mathbb{R}^n)$. Define the linear maps $U_g, U_{g^*} : \mathbb{R}^n \to V$ by $U_g(e_j) = v_j(g)$ and $U_{g^*}(e_j) = v_j(g^*)$, for all $1 \leq j \leq n$, where the e_j are the vectors of the canonical basis in $\mathbb{R}^n$. By construction U_g and U_{g^*} are isometries and the following decomposition holds

$$g = U_{g^*}\, D_g\, (U_g)^*,$$

known as the *singular value decomposition* (SVD) of g.

We say that g has a *simple singular spectrum* if its n singular values are all distinct. When g has simple singular spectrum, the singular vectors $v_j(g)$ and $v_j(g^*)$ above are uniquely determined up to a sign, and in particular they determine well-defined projective points $\bar{\mathfrak{v}}_j(g), \bar{\mathfrak{v}}_j(g^*) \in \mathbb{P}(V)$.

Definition 2.11 Given $g \in \mathscr{L}(V)$, we call *singular basis* of g any orthonormal basis $\{v_1, \ldots, v_n\}$ of V formed by singular vectors of g ordered in such a way that $\|g\, v_i\| = s_i(g)$ for all $i = 1, \ldots, n$.

Given $g \in \mathscr{L}(V)$, consider singular bases $\{v_1, \ldots, v_n\}$ and $\{v_1^*, \ldots, v_n^*\}$ for g and g^*, respectively, such that

$$g\, v_j = s_j\, v_j^* \;\text{ with }\; s_j = s_j(g) \quad \text{for all } \; 1 \leq j \leq n.$$

For any $I = \{i_1, \dots, i_k\} \in \Lambda_k^n$ we have

$$(\wedge_k g)(v_{i_1} \wedge \cdots \wedge v_{i_k}) = (s_{i_1} \dots s_{i_k})\,(v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*).$$

Therefore, by the considerations at the end of Sect. 2.1.2, the families of k-vectors $\{v_I = v_{i_1} \wedge \cdots \wedge v_{i_k} : I \in \Lambda_k^n\}$ and $\{v_I^* = v_{i_1}^* \wedge \cdots \wedge v_{i_k}^* : I \in \Lambda_k^n\}$ form two singular bases for $\wedge_k g$ and $\wedge_k g^*$, respectively, while the products $s_I = s_{i_1} \dots s_{i_k}$ are the singular values of both $\wedge_k g$ and $\wedge_k g^*$.

Proposition 2.6 *For any* $1 \le k \le \dim V$, $\|\wedge_k g\| = s_1(g) \ \dots \ s_k(g)$.

Proof The maximum product s_I is attained when $I = \{1, \dots, k\} \in \Lambda_k^n$. Hence $\|\wedge_k g\| = s_1 \ \dots \ s_k$. □

The *volume expansion factor* of a linear map $g : V \to V'$ between two Euclidean spaces V and V' is defined by

$$\det_+(g) := \sqrt{\det(g^* g)}.$$

This name is justified by the following fact.

Proposition 2.7 *Given a linear map* $g \colon V \to V'$ *between Euclidean spaces, with* $n = \dim V$*, for any Borel set* $B \subset V$*,*

$$\mathrm{Vol}_n(g(B)) = \det_+(g)\,\mathrm{Vol}_n(B),$$

where Vol_n *denotes the n-dimensional Hausdorff measure.*

Proof Let $\{v_1, \dots, v_n\}$ be any basis of V and consider the parallelipiped $B = P(v_1, \dots, v_n)$. By Proposition 2.9 below and formula (2.6),

$$\begin{aligned}\mathrm{Vol}_n(g(B)) &= \|(g v_1) \wedge \cdots \wedge (g v_n)\| = \|(\wedge_n g)(v_1 \wedge \cdots \wedge v_n)\| \\ &= \|\wedge_n g\|\,\|v_1 \wedge \cdots \wedge v_n\| = \det_+(g)\mathrm{Vol}_n(B).\end{aligned}$$

On the third step we have used the fact that $\wedge_n V$ has dimension 1. □

Because of this property the volume expansion factor behaves multiplicatively.

Proposition 2.8 *Given Euclidean spaces* V*,* V' *and* V''*, if* $g : V \to V'$ *is an isomorphism and* $g' : V' \to V''$ *any linear map then*

$$\det_+(g' \circ g) = \det_+(g')\,\det_+(g).$$

Proposition 2.9 *Let* V *and* V' *be Euclidean spaces with* $n = \dim V \le \dim V'$. *Then for any linear map* $g \colon V \to V'$

$$\det_+(g) = s_1(g) \dots s_n(g) = \|\wedge_n g\|.$$

Proof The squares $s_i^2 = s_i(g)^2$ $(1 \le i \le n)$ are the eigenvalues of $g^* g$. □

Next proposition provides a method to compute the volume expansion factor.

Proposition 2.10 *Let $g\colon V \to V'$ be a linear map between Euclidean spaces. Given orthonormal bases $\{v_i\colon i = 1, \ldots, n\}$ of V and $\{v_i'\colon i = 1, \ldots, n\}$ of the range gV,*

$$\det{}_+(g) = \left|\det\left(\langle g v_i, v_j'\rangle\right)_{i,j}\right|.$$

Proof The matrix $A \in \mathrm{Mat}(n, \mathbb{R})$ with entries $a_{ij} = \langle g v_i, v_j'\rangle$ represents the linear map g in the given orthonormal bases. Consider the isometries $U\colon \mathbb{R}^n \to V$ and $U'\colon \mathbb{R}^n \to V'$ respectively defined by $U e_i = v_i$ and $U' e_i = v_i'$ for all $i = 1, \ldots, n$. Then $g = U' A U^*$ and

$$\begin{aligned} \det{}_+(g)^2 &= \det(g^* g) = \det(U A^* A U^*) \\ &= \det(A^* A) = \det(A)^2. \end{aligned}$$

This proves that $\det_+(g) = \left|\det A\right|$. □

2.2.2 *Gaps and Most Expanding Directions*

Consider a linear map $g \in \mathscr{L}(V)$ and a number $1 \le k < \dim V$.

Definition 2.12 The kth *gap ratio* of g is defined to be

$$\mathrm{gr}_k(g) := \frac{s_k(g)}{s_{k+1}(g)} \ge 1.$$

We will also write $\mathrm{gr}(g)$ instead of $\mathrm{gr}_1(g)$.

Definition 2.13 We say that g *has a first singular gap* when $\mathrm{gr}(g) > 1$. More generally, we say that g *has a k singular gap* when $\mathrm{gr}_k(g) > 1$.

In some occasions it is convenient to work with the inverse quantity, denoted by

$$\sigma_k(g) := \mathrm{gr}_k(g)^{-1} \le 1. \tag{2.15}$$

Proposition 2.11 *For any* $1 \le k < \dim V$,

$$\mathrm{gr}_k(g) = \frac{\|\wedge_k g\|^2}{\|\wedge_{k-1} g\|\, \|\wedge_{k+1} g\|} = \mathrm{gr}_1(\wedge_k g).$$

Proof The first equality follows from Proposition 2.6. The two first singular values of $\wedge_k g$ are $s_1(\wedge_k g) = s_1(g) \ldots s_{k-1}(g) s_k(g)$ and $s_2(\wedge_k g) = s_1(g) \ldots s_{k-1}(g) s_{k+1}(g)$. Hence

$$\mathrm{gr}_1(\wedge_k g) = \frac{s_1(\wedge_k g)}{s_2(\wedge_k g)} = \frac{s_k(g)}{s_{k+1}(g)} = \mathrm{gr}_k(g). \qquad \square$$

Given $g \in \mathscr{L}(V)$, if $\mathrm{gr}(g) > 1$ then the singular value $s_1(g) = \|g\|$ is simple.

Definition 2.14 In this case we denote by $\overline{\mathfrak{v}}(g) \in \mathbb{P}(V)$ the associated singular direction, and refer to it as the *g-most expanding direction*.

By definition we have

$$\varphi_g \overline{\mathfrak{v}}(g) = \overline{\mathfrak{v}}(g^*). \tag{2.16}$$

More generally, given $1 \leq k < \dim V$, we have:

Definition 2.15 If $\mathrm{gr}_k(g) > 1$ we define the *g-most expanding k-subspace* to be

$$\overline{\mathfrak{v}}_k(g) := \Psi^{-1}\left(\overline{\mathfrak{v}}(\wedge_k g)\right),$$

where Ψ stands for the Plücker embedding defined in Sect. 2.1.3.

The subspace $\overline{\mathfrak{v}}_k(g)$ is the direct sum of all singular directions associated with the singular values $s_1(g), \ldots, s_k(g)$. We have

$$\varphi_g \overline{\mathfrak{v}}_k(g) = \overline{\mathfrak{v}}_k(g^*). \tag{2.17}$$

Analogously, let $n = \dim V$ and assume $\mathrm{gr}_{n-k}(g) > 1$.

Definition 2.16 We define the *g-least expanding k-subspace* as

$$\underline{\mathfrak{v}}_k(g) := \overline{\mathfrak{v}}_{n-k}(g)^{\perp}.$$

The subspace $\underline{\mathfrak{v}}_k(g)$ is the direct sum of all singular directions associated with the singular values $s_{n-k+1}(g), \ldots, s_n(g)$. Again we have

$$\varphi_g \underline{\mathfrak{v}}_k(g) = \underline{\mathfrak{v}}_k(g^*). \tag{2.18}$$

Let $\tau = (\tau_1, \ldots, \tau_k)$ be a signature with $1 \leq \tau_1 < \cdots < \tau_k < \dim V$.

Definition 2.17 We define the *τ-gap ratio* of g to be

$$\mathrm{gr}_\tau(g) := \min_{1 \leq j \leq k} \mathrm{gr}_{\tau_j}(g).$$

When $\mathrm{gr}_\tau(g) > 1$ we say that g *has a τ-gap pattern*.

Note that $\mathrm{gr}_\tau(g) > 1$ means that g has a τ_j singular gap for $1 \leq j \leq k$. Recall that $\mathcal{F}_\tau(V)$ denotes the space of all τ-flags, i.e., flags $F = (F_1, \ldots, F_k)$ such that $\dim(F_j) = \tau_j$ for $j = 1, \ldots, k$.

Definition 2.18 If $\mathrm{gr}_\tau(g) > 1$ then the *most expanding* τ*-flag* is

$$\overline{\mathfrak{v}}_\tau(g) := (\overline{\mathfrak{v}}_{\tau_1}(g), \ldots, \overline{\mathfrak{v}}_{\tau_k}(g)) \in \mathcal{F}_\tau(V).$$

Given $g \in \mathscr{L}(V)$ the domain of its push-forward action on $\mathcal{F}_\tau(V)$ is

Definition 2.19 $\mathcal{F}_\tau(g) := \{F \in \mathcal{F}_\tau(V) : \ F_k \cap \mathrm{K}_g = \{0\}\}$.

The *push-forward* of a flag $F \in \mathcal{F}_\tau(g)$ by g is

$$\varphi_g F = g\, F := (g\, F_1, \ldots, g\, F_k).$$

Proposition 2.12 *Given* $g \in \mathscr{L}(V)$ *such that* $\mathrm{gr}_\tau(g) > 1$*, the push-forward induces a map* $\varphi_g : \mathcal{F}_\tau(g) \to \mathcal{F}_\tau(g^*)$ *such that* $\varphi_g \overline{\mathfrak{v}}_\tau(g) = \overline{\mathfrak{v}}_\tau(g^*)$.

Proof Given $F \in \mathcal{F}_\tau(g)$, we have $F_j \cap \mathrm{K}_g = \{0\}$ for all $j = 1, \ldots, k$. Hence $\dim gF_j = \dim F_j = \tau_j$ for all j, which proves that $\varphi_g F \in \mathcal{F}_\tau(V)$. To check that $\varphi_g F \in \mathcal{F}_\tau(g^*)$ we need to show that $gF_k \cap \mathrm{K}_{g^*} = \{0\}$. Assume $g\,v \in \mathrm{K}_{g^*}$, with $v \in F_k$, and let us see that $g\,v = 0$. By assumption $g^* g\,v = 0$, which implies $(g\,g^*)\,g\,v = 0$. Since the self-adjoint map $g\,g^*$ induces an automorphism on R_g, we conclude that $g\,v = 0$.

The second statement follows from (2.17). □

Given $g \in \mathscr{L}(V)$, the domain of its pull-back action on $\mathcal{F}_\tau(V)$ is

Definition 2.20 $\mathcal{F}_\tau^{-1}(g) := \{F \in \mathcal{F}_\tau(V) : \ F_1 + \mathrm{R}_g = V\}$.

The *pull-back* of a flag $F \in \mathcal{F}_\tau(g)$ by g is

$$\varphi_{g^{-1}} F = g^{-1} F := (g^{-1} F_1, \ldots, g^{-1} F_k).$$

Definition 2.21 If $\mathrm{gr}_{\tau^\perp}(g) > 1$ the *least expanding* τ*-flag* is

$$\underline{\mathfrak{v}}_\tau(g) := (\underline{\mathfrak{v}}_{\tau_1}(g), \ldots, \underline{\mathfrak{v}}_{\tau_k}(g)) \in \mathcal{F}_\tau(V).$$

Proposition 2.13 *If* $\mathrm{gr}_\tau(g) > 1$ *then* $\underline{\mathfrak{v}}_{\tau^\perp}(g) = \overline{\mathfrak{v}}_\tau(g)^\perp$.

Proof Let $\{v_1, \ldots, v_n\}$ be a singular basis of g. Since this basis is orthonormal,

$$\underline{\mathfrak{v}}_{n-k}(g) = \langle v_{k+1}, \ldots, v_n \rangle = \langle v_1, \ldots, v_k \rangle^\perp = \overline{\mathfrak{v}}_k(g)^\perp.$$

Hence

$$\underline{\mathfrak{v}}_{\tau^\perp}(g) = (\underline{\mathfrak{v}}_{n-\tau_k}(g), \ldots, \underline{\mathfrak{v}}_{n-\tau_1}(g)) = (\overline{\mathfrak{v}}_{\tau_1}(g), \ldots, \overline{\mathfrak{v}}_{\tau_k}(g))^\perp = \overline{\mathfrak{v}}_\tau(g)^\perp. \qquad \square$$

Proposition 2.14 *Given* $g \in \mathscr{L}(V)$ *such that* $\mathrm{gr}_{\tau^\perp}(g) > 1$*, the pull-back induces a map* $\varphi_{g^{-1}} : \mathcal{F}_\tau^{-1}(g) \to \mathcal{F}_\tau^{-1}(g^*)$ *such that* $\varphi_{g^{-1}} \underline{\mathfrak{v}}_\tau(g) = \underline{\mathfrak{v}}_\tau(g^*)$.

Proof Given $F \in \mathfrak{F}_\tau^{-1}(g)$, we have $F_j + \mathrm{R}_g = V$ for all $j = 1, \ldots, k$. Hence $\dim g^{-1}F_j = \dim F_j = \tau_j$ for all j, which proves that $\varphi_{g^{-1}}F \in \mathfrak{F}_\tau(V)$. To check that $\varphi_{g^{-1}}F \in \mathfrak{F}_\tau^{-1}(g^*)$ just notice that $g^{-1}F_1 + \mathrm{R}_{g^*} \supseteq \mathrm{K}_g + \mathrm{K}_g^\perp = V$.

The second statement follows from (2.18) and Proposition 2.13. □

We end this section proving that the orthogonal complement involution conjugates the push-forward action by $g \in \mathscr{L}(V)$ with the pull-back action by the adjoint map g^*.

Proposition 2.15 *Given $g \in \mathscr{L}(V)$ such that $\mathrm{gr}_{\tau^\perp}(g) > 1$, the action of $\varphi_{g^{-1}}$ on $\mathfrak{F}_\tau(V)$ is conjugated to the action of φ_{g^*} on $\mathfrak{F}_{\tau^\perp}(V)$ by the orthogonal complement involution. More precisely, we have $\mathfrak{F}_\tau^{-1}(g) = \mathfrak{F}_{\tau^\perp}(g^*)^\perp$ and $\mathfrak{F}_\tau^{-1}(g^*) = \mathfrak{F}_{\tau^\perp}(g)^\perp$, and the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{F}_{\tau^\perp}(g^*) & \xrightarrow{\varphi_{g^*}} & \mathfrak{F}_{\tau^\perp}(g) \\ {\scriptstyle .^\perp}\big\downarrow & & \big\downarrow{\scriptstyle .^\perp} \\ \mathfrak{F}_\tau^{-1}(g) & \xrightarrow[\varphi_{g^{-1}}]{} & \mathfrak{F}_\tau^{-1}(g^*) \end{array} .$$

Proof To see that $\mathfrak{F}_\tau^{-1}(g) = \mathfrak{F}_{\tau^\perp}(g^*)^\perp$, notice that the following equivalences hold:

$$\begin{aligned} F \in \mathfrak{F}_\tau^{-1}(g) \quad &\Leftrightarrow \quad F_1 + \mathrm{R}_g = V \\ &\Leftrightarrow \quad F_1^\perp \cap \mathrm{K}_{g^*} = \{0\} \quad \Leftrightarrow \quad F^\perp \in \mathfrak{F}_{\tau^\perp}(g^*). \end{aligned}$$

Exchanging the roles of g and g^* we obtain the relation $\mathfrak{F}_\tau^{-1}(g^*) = \mathfrak{F}_{\tau^\perp}(g)^\perp$.

Finally, notice that it is enough to prove the diagram's commutativity at the Grassmannian level. For that use Proposition 2.3. □

2.2.3 Angles and Expansion

Throughout this section let $\hat{p}, \hat{q} \in \mathbb{P}(V)$, and $p \in \hat{p}$, $q \in \hat{q}$ denote representative vectors. The projective distance $\delta(\hat{p}, \hat{q})$ was defined by

$$\delta(\hat{p}, \hat{q}) := \sqrt{1 - \frac{\langle p, q\rangle^2}{\|p\|^2 \|q\|^2}} = \frac{\|p \wedge q\|}{\|p\| \, \|q\|} = \sin \rho(\hat{p}, \hat{q}).$$

We also define the *minimum distance* between any two subspaces $E, F \in \mathrm{Gr}(V)$,

$$\delta_{\min}(E, F) := \min_{u \in E \setminus \{0\}, v \in F \setminus \{0\}} \delta(\hat{u}, \hat{v}), \tag{2.19}$$

and the *Hausdorff distance* between subspaces $E, F \in \mathrm{Gr}_k(V)$,

$$\delta_{\mathrm{H}}(E, F) := \max\left\{ \max_{u\in E\setminus\{0\}} \delta_{\min}(\hat{u}, F),\ \max_{v\in F\setminus\{0\}} \delta_{\min}(\hat{v}, E) \right\}.$$

Given a unit vector $v \in V$, $\|v\| = 1$, denote by $\pi_v, \pi_v^{\perp} : V \to V$ the orthogonal projections $\pi_v(x) := \langle v, x\rangle\, v$, respectively $\pi_v^{\perp}(x) := x - \langle v, x\rangle\, v$.

Lemma 2.2 *Given $u, v \in V$ non-collinear with $\|u\| = \|v\| = 1$, denote by P the plane spanned by u and v. Then*

(a) $\pi_v - \pi_u$ *is a self-adjoint endomorphism,*
(b) $\mathrm{K}(\pi_v - \pi_u) = P^{\perp}$,
(c) the restriction $\pi_v - \pi_u : P \to P$ *is anti-conformal with similarity factor* $\left|\sin\angle(u, v)\right|$,
(d) $\|\pi_v^{\perp} - \pi_u^{\perp}\| = \|\pi_v - \pi_u\| = \delta(\hat{u}, \hat{v})$.

Proof Item (a) follows because orthogonal projections are self-adjoint operators.

Given $w \in P^{\perp}$, we have $\pi_u(w) = \pi_v(w) = 0$, which implies $w \in \mathrm{K}(\pi_u - \pi_v)$. Hence $P^{\perp} \subset \mathrm{K}(\pi_u - \pi_v)$. Since u and v are non-collinear, $\pi_u - \pi_v$ has rank 2. Thus $\mathrm{K}(\pi_u - \pi_v) = P^{\perp}$, which proves (b).

For (c) we may assume that $V = \mathbb{R}^2$ and consider $u = (u_1, u_2)$, $v = (v_1, v_2)$, with $u_1^2 + u_2^2 = v_1^2 + v_2^2 = 1$. The projections π_u and π_v are represented by the matrices

$$U = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_1^2 & v_1v_2 \\ v_1v_2 & v_2^2 \end{pmatrix}$$

w.r.t. the canonical basis. Hence $\pi_v - \pi_u$ is given by

$$V - U = \begin{pmatrix} v_1^2 - u_1^2 & v_1v_2 - u_1u_2 \\ v_1v_2 - u_1u_2 & v_2^2 - u_2^2 \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix}$$

where $\alpha = v_1v_2 - u_1u_2$ and $\beta = v_1^2 - u_1^2 = -(v_2^2 - u_2^2)$. This proves that the restriction of $\pi_v - \pi_u$ to the plane P is anti-conformal. The similarity factor of this map is

$$\|\pi_v - \pi_u\| = \|\pi_v(u) - u\| = \|\pi_v^{\perp}(u)\| = \left|\sin\angle(u, v)\right|$$

Finally, since $u - \langle v, u\rangle\, v \perp v$,

$$\begin{aligned}\|\pi_v^{\perp} - \pi_u^{\perp}\|^2 &= \|\pi_v - \pi_u\|^2 = \|\pi_v^{\perp}(u)\|^2 = \|u - \langle v, u\rangle\, v\|^2 \\ &= \|u \wedge v\|^2 = \delta(\hat{u}, \hat{v})^2.\end{aligned}$$ □

Lemma 2.3 *Let V be a Euclidean space of even dimension $2k$ and let $E, F \in \mathrm{Gr}_k(V)$ be subspaces such that $V = E \oplus F$. Then the linear map $\pi_E - \pi_F$ admits an invariant decomposition $V = P_1 \oplus \cdots \oplus P_k$ into pairwise orthogonal planes P_j such that*

(1) *each P_j is invariant under π_E and π_F,*
(2) $P_j = E_j \oplus F_j$, *where* $E_j = E \cap P_j$, $F_j = F \cap P_j$ *and* $\dim E_j = \dim F_j = 1$,
(3) $(\pi_E - \pi_F)|_{P_j} : P_j \to P_j$ *is anti-conformal.*

Proof Choose unit vectors $u_0 \in E$ and $v_0 \in F$ such that $\angle(u_0, v_0) = \max\{\angle(u, v) : u \in E \setminus \{0\}, v \in F \setminus \{0\}\}$. Then the function $f(x) = \|u - v_0\|^2$ defined over the unit sphere in E attains its maximum value at u_0. By the method of Lagrange multipliers, $\pi_E(u_0 - v_0)$ is collinear with u_0, which implies that $\pi_E(v_0)$ is also collinear with u_0. Therefore $\pi_E(v_0) = \langle u_0, v_0\rangle u_0$. By a similar argument, $\pi_F(u_0) = \langle u_0, v_0\rangle v_0$. The plane P spanned by the vectors u_0 and v_0 is invariant under both projections π_E and π_F. Hence, by Lemma 2.2 the restriction $\pi_E - \pi_F : P \to P$ is anti-conformal. Now the orthogonal complement $P^\perp$ is also invariant under π_E, π_F and $\pi_E - \pi_F$. Defining $E_0 = E \cap P^\perp$ and $F_0 = F \cap P^\perp$, we have $P^\perp = E_0 \oplus F_0$ and $\pi_E - \pi_F = \pi_{E_0} - \pi_{F_0}$ over $P^\perp$, where π_{E_0} and π_{F_0} denote orthogonal projections on $P^\perp$. The claim of this lemma follows proceeding inductively with $\pi_{E_0} - \pi_{F_0}$. □

Definition 2.22 Given $E, F \in \mathrm{Gr}(V)$, we denote by $\pi_F : V \to V$ the orthogonal projection onto F, and by $\pi_{E,F} : E \to F$ the restriction of π_F to E.

Proposition 2.16 *Given* $E, F \in \mathrm{Gr}_k(V)$,

(a) $\delta(E, F) = \sqrt{1 - \det_+(\pi_{E,F})^2} = \sqrt{1 - \det_+(\pi_{F,E})^2}$,
(b) $\delta_{\mathrm{H}}(E, F) = \|\pi_{E,F^\perp}\| = \|\pi_{F,E^\perp}\| = \|\pi_E - \pi_F\|$,
(c) $\delta_{\mathrm{H}}(E, F) \leq \delta(E, F)$.

Proof Consider the unit k-vectors $e = \Psi(E)$ and $f = \Psi(F)$.

For (a) notice first that $\delta(E, F) = \delta(e, f) = \sqrt{1 - \langle e, f\rangle^2}$. Since the exterior power $\wedge_k \pi_{F,E} : \wedge_k F \to \wedge_k E$ is also an orthogonal projection we have $\langle e, f\rangle = \langle e, \wedge_k \pi_{F,E}(f)\rangle = \|\wedge_k \pi_{F,E}\| = \det_+(\pi_{F,E})$.

Take an orthogonal reflexion $g \in \mathrm{O}(V)$ such that $g(F) = E$ and $g(E) = F$. We have $g^{-1}(E^\perp) = F^\perp$ and $\pi_{E,F^\perp} = g^{-1} \circ \pi_{F,E^\perp} \circ g$. Therefore $\|\pi_{E,F^\perp}\| = \|\pi_{F,E^\perp}\|$.

We have $\delta_{\mathrm{H}}(E, F) = \|\pi_{E,F^\perp}\|$ because for any unit vector $u \in \hat{u}$, with $\hat{u} \in \mathbb{P}(E)$,

$$\|\pi_{E,F^\perp}(u)\| = \min_{v \in F \setminus \{0\}} \delta(\hat{u}, \hat{v}).$$

To finish (b) we still have to prove that $\|\pi_E - \pi_F\| = \|\pi_{E,F^\perp}\|$. Restricting our attention to the subspace $V_0 = (E \cap (E \cap F)^\perp) \oplus (F \cap (E \cap F)^\perp)$, because $\pi_E - \pi_F$ vanishes on $V_0^\perp$ we can assume that $V = E \oplus F$. In particular $\dim V = 2k$. Consider the orthogonal invariant decomposition of Lemma 2.3. It is enough to check that the relation $\|\pi_E - \pi_F\| = \|\pi_{E,F^\perp}\|$ holds on each plane P_j. Therefore we may as well assume that $k = 1$. Notice that over the subspace E we have $\pi_E - \pi_F = \pi_{E,F^\perp}$. Since the linear map $\pi_E - \pi_F$ is anti-conformal, the norm $\|\pi_E - \pi_F\|$ is attained along E, which implies that $\|\pi_E - \pi_F\| = \|\pi_{E,F^\perp}\|$. This proves item (b).

Since $\pi_{E,F}$ is an orthogonal projection all its singular values are in the range $[0, 1]$. Hence, for any unit vector $u \in E$, $\|\pi_{E,F}(u)\| \geq \mathfrak{m}(\pi_{E,F}) \geq \det_+(\pi_{E,F})$. Thus

$$\|\pi_{E,F^\perp}(u)\|^2 = 1 - \|\pi_{E,F}(u)\|^2 \le 1 - \det_+(\pi_{E,F})^2.$$

Item (c) follows taking the maximum over all unit vectors $u \in E$. $\square$

The following complementary quantity to the distance $\delta(\hat{p}, \hat{q})$ plays a special role in the sequel.

Definition 2.23 The α-angle between $\hat{p}$ and $\hat{q}$ is defined to be

$$\alpha(\hat{p}, \hat{q}) := \frac{|\langle p, q\rangle|}{\|p\| \, \|q\|} = \cos\rho(\hat{p}, \hat{q}).$$

In order to give a geometric meaning to this angle we define the *projective orthogonal hyperplane* of $\hat{p} \in \mathbb{P}(V)$ as

$$\Sigma(\hat{p}) := \{\hat{x} \in \mathbb{P}(V) : \ \langle x, p\rangle = 0 \quad \text{for} \ \ x \in \hat{x}\}.$$

The number $\alpha(\hat{p}, \hat{q})$ is the sine of the minimum angle between $\hat{p}$ and $\Sigma(\hat{q})$. As in Definition (2.19), given a subspace $F \subset V$ we write

$$\rho_{\min}(\hat{p}, F) := \min_{q \in F\setminus\{0\}} \rho(\hat{p}, \hat{q}).$$

Proposition 2.17 *For any* $\hat{p}, \hat{q} \in \mathbb{P}(V)$,

$$\alpha(\hat{p}, \hat{q}) = \sin\rho_{\min}(\hat{p}, \Sigma(\hat{q})) = \delta_{\min}(\hat{p}, \Sigma(\hat{q})) \tag{2.20}$$

$$\alpha(\hat{p}, \hat{q}) = 0 \ \Leftrightarrow \ \delta(\hat{p}, \hat{q}) = 1 \ \Leftrightarrow \ p \perp q. \tag{2.21}$$

These concepts extend naturally to Grassmannians and flag manifolds.

Definition 2.24 Given $E, F \in \mathrm{Gr}_k(V)$, we define the α-angle between them

$$\alpha(E, F) = \alpha_k(E, F) := \alpha(\Psi(E), \Psi(F)),$$

where $\Psi : \mathrm{Gr}_k(V) \to \mathbb{P}(\wedge_k V)$ denotes the Plücker embedding (see Sect. 2.1.3).

Definition 2.25 We say that two k-subspaces $E, F \in \mathrm{Gr}_k(V)$ *are orthogonal*, and we write $E \perp F$, iff $\alpha(E, F) = 0$.

The *Grassmannian orthogonal hyperplane* of F is defined as

$$\Sigma(F) := \{E \in \mathrm{Gr}_k(V) : \ \alpha(E, F) = 0\}.$$

As before, the number $\alpha(E, F)$ equals the sine of the minimum angle between E and $\Sigma(F)$.

Proposition 2.18 *For any* $E, F \in \mathrm{Gr}_k(V)$,

$$\alpha(E, F) = \sin\rho_{\min}(E, \Sigma(F)) = \delta_{\min}(E, \Sigma(F)).$$

Next we characterize the angle $\alpha(E, F)$. Consider the notation of Definition 2.22.

Proposition 2.19 *Given* $E, F \in \mathrm{Gr}_k(V)$*,*

(a) $\alpha(E, F) = \alpha(E^\perp, F^\perp)$*,*
(b) $\alpha(E, F) = \det_+(\pi_{E,F}) = \det_+(\pi_{F,E})$*,*
(c) $E \perp F$ *iff there exists a pair* (e, f) *of unit vectors such that* $e \in E \cap F^\perp$ *and* $f \in F \cap E^\perp$*,*
(d) $\alpha(E, F) \le \|\pi_{E,F}\| = \sqrt{1 - \delta_{\min}(E, F^\perp)^2}$.

Proof Given $E, F \in \mathrm{Gr}_k(V)$, take orthonormal bases $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ of E and F, respectively, and consider the associated unit k-vectors $u = u_1 \wedge \cdots \wedge u_k$ and $v = v_1 \wedge \cdots \wedge v_k$, so that $u \in \Psi(E)$ and $v \in \Psi(F)$.

Using the Hodge star operator we obtain unit vectors $*u \in \Psi(E^\perp)$ and $*v \in \Psi(F^\perp)$. Hence

$$\alpha(E^\perp, F^\perp) = \left|\langle *u, *v\rangle\right| = \left|\langle u, v\rangle\right| = \alpha(E, F),$$

which proves (a). Also

$$\begin{aligned}
\alpha(E, F) &:= \left|\langle\, u_1 \wedge \cdots \wedge u_k,\ v_1 \wedge \cdots \wedge v_k \,\rangle\right| \\
&= \left|\det \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \ldots & \langle u_1, v_k\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \ldots & \langle u_2, v_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_k, v_1\rangle & \langle u_k, v_2\rangle & \ldots & \langle u_k, v_k\rangle \end{pmatrix}\right| \\
&= \det_+(\pi_{E,F}).
\end{aligned}$$

For the second equality above write $u_i = w_i + \sum_{j=1}^k \langle u_i, v_j\rangle\, v_j$ with $w_i \in F^\perp$ and use the anti-symmetry of the exterior product. For the third equality remark that the matrix with entries $\langle u_i, v_j\rangle$ represents $\pi_{E,F}$ w.r.t. the given orthonormal bases. By symmetry, $\alpha(E, F) = \det_+(\pi_{F,E})$. This proves (b).

From these relations, $\alpha(E, F) = 0 \Leftrightarrow \mathrm{K}(\pi_{E,F}) \ne \{0\} \Leftrightarrow \mathrm{K}(\pi_{F,E}) \ne \{0\}$, which explains (c).

Finally, because all singular values of $\pi_{E,F}$ are in $[0, 1]$,

$$\begin{aligned}
\alpha(E, F) = \det_+(\pi_{E,F}) &\le \|\pi_{E,F}\| \\
&= \max_{u \in E,\, \|u\|=1} \|\pi_{E,F}(u)\| \\
&= \max_{u \in E,\, \|u\|=1} \sqrt{1 - \|\pi_{E,F^\perp}(u)\|^2} \\
&= \sqrt{1 - \delta_{\min}(E, F^\perp)^2},
\end{aligned}$$

which proves (d). □

Next we extend α-angles to flags. Consider a signature τ of length k.

Definition 2.26 Given flags $F, G \in \mathcal{F}_\tau(V)$, define

$$\alpha(F, G) = \alpha_\tau(F, G) := \min_{1\leq j\leq k} \alpha(F_j, G_j).$$

Definition 2.27 We say that two τ-flags $F, G \in \mathcal{F}_\tau(V)$ *are orthogonal*, and we write $F \perp G$, if $F_j \perp G_j$ for some $j = 1, \ldots, k$.

Comparing the two definitions, for all $F, G \in \mathcal{F}_\tau(V)$

$$\alpha_\tau(F, G) = 0 \quad \Leftrightarrow \quad G \perp F.$$

Hence, the *orthogonal flag hyperplane* of F is defined as

$$\Sigma(F) := \{G \in \mathcal{F}_\tau(V) \; : \; \alpha(G, F) = 0\}.$$

As in the previous cases, the number $\alpha_\tau(F, G)$ equals the sine of the minimum angle between F and $\Sigma(G)$.

Proposition 2.20 *For any $F, G \in \mathcal{F}_\tau(V)$,*

$$\alpha(E, F) = \sin \rho_{\min}(F, \Sigma(G)) = \delta_{\min}(F, \Sigma(G)).$$

Consider a sequence of linear maps $g_0, g_1, \ldots, g_{n-1} \in \mathscr{L}(V)$. The following quantities, called *expansion rifts*, measure the break of expansion in the composition $g_{n-1} \ldots g_1\, g_0$ of the maps g_j.

Definition 2.28 The first expansion rift of the sequence above is the number

$$\rho(g_0, g_1, \ldots, g_{n-1}) := \frac{\|g_{n-1} \cdots g_1 g_0\|}{\|g_{n-1}\| \cdots \|g_1\| \|g_0\|} \in [1, +\infty)\,.$$

Given $1 \leq k \leq \dim V$, the kth expansion rift is

$$\rho_k(g_0, g_1, \ldots, g_{n-1}) := \rho(\wedge_k g_0, \wedge_k g_1, \ldots, \wedge_k g_{n-1})\,.$$

Given a signature $\tau = (\tau_1, \ldots, \tau_k)$, the τ-expansion rift is defined as

$$\rho_\tau(g_0, g_1, \ldots, g_{n-1}) := \min_{1\leq j\leq k} \rho_{\tau_j}(g_0, g_1, \ldots, g_{n-1})\,.$$

The key concept of this section is that of angle between linear maps. The quantity $\alpha(g, g')$, for instance, is the sine of the angle between $\varphi_g(\overline{\mathfrak{v}}(g)) = \overline{\mathfrak{v}}(g^*)$ and $\Sigma(\overline{\mathfrak{v}}(g'))$. As we will see, this angle is a lower bound on the expansion rift of two linear maps g and g'.

Definition 2.29 Given $g, g' \in \mathscr{L}(V)$, we define

$$\begin{aligned} \alpha(g, g') &:= \alpha(\overline{\mathfrak{v}}(g^*), \overline{\mathfrak{v}}(g')) && \text{if } g \text{ and } g' \text{ have a first gap ratio} \\ \alpha_k(g, g') &:= \alpha(\overline{\mathfrak{v}}_k(g^*), \overline{\mathfrak{v}}_k(g')) && \text{if } g \text{ and } g' \text{ have a } k \text{ gap ratio} \\ \alpha_\tau(g, g') &:= \alpha(\overline{\mathfrak{v}}_\tau(g^*), \overline{\mathfrak{v}}_\tau(g')) && \text{if } g \text{ and } g' \text{ have a } \tau \text{ gap pattern.} \end{aligned}$$

The following exotic operation is introduced to obtain an upper bound on the expansion rift $\rho(g, g')$. Consider the algebraic operation $a \oplus b := a + b - a\,b$ on the set $[0, 1]$. Clearly $([0, 1], \oplus)$ is a commutative semigroup isomorphic to $([0, 1], \cdot)$. In fact, the transformation $\Phi : ([0, 1], \oplus) \to ([0, 1], \cdot)$, $\Phi(x) := 1 - x$, is a semigroup isomorphism. We summarize some properties of this operation.

Proposition 2.21 *For any $a, b, c \in [0, 1]$,*

(1) $0 \oplus a = a$,
(2) $1 \oplus a = 1$,
(3) $a \oplus b = (1 - b)\,a + b = (1 - a)\,b + a$,
(4) $a \oplus b < 1 \;\Leftrightarrow\; a < 1$ *and* $b < 1$,
(5) $a \le b \;\Rightarrow\; a \oplus c \le b \oplus c$,
(6) $b > 0 \;\Rightarrow\; (ab^{-1} \oplus c)\, b \le a \oplus c$,
(7) $a\,c + b\,\sqrt{1 - a^2}\,\sqrt{1 - c^2} \le \sqrt{a^2 \oplus b^2}$.

Proof Items (1)–(6) are left as exercises. For the last item consider the function $f : [0, 1] \to [0, 1]$ defined by $f(c) := a\,c + b\,\sqrt{1 - a^2}\,\sqrt{1 - c^2}$. A simple computation shows that

$$f'(c) = a - \frac{b\,c\,\sqrt{1 - a^2}}{\sqrt{1 - c^2}}$$

The derivative f' has a zero at $c = a/\sqrt{a \oplus b}$, and one can check that this zero is a global maximum of f. Since $f(a/\sqrt{a \oplus b}) = \sqrt{a^2 \oplus b^2}$, item (7) follows. □

Definition 2.30 Given $g, g' \in \mathscr{L}(V)$ with τ-gap patterns, the upper τ-angle between g and g' is defined to be

$$\beta_\tau(g, g') := \sqrt{\mathrm{gr}_\tau(g)^{-2} \oplus \alpha_\tau(g, g')^2 \oplus \mathrm{gr}_\tau(g')^{-2}}.$$

We will write $\beta_k(g, g')$ when $\tau = (k)$, and $\beta(g, g')$ when $\tau = (1)$.

The next proposition relates norm expansion by the linear map g, and distance contraction by the projective map φ_g, with angles and gap ratios.

Proposition 2.22 *Given $g \in \mathscr{L}(V)$ with $\sigma(g) < 1$, a point $\hat{w} \in \mathbb{P}(V)$ and a unit vector $w \in \hat{w}$,*

(a) $\alpha(\hat{w}, \overline{\mathfrak{v}}(g))\,\|g\| \le \|g\,w\| \le \|g\|\,\sqrt{\alpha(\hat{w}, \overline{\mathfrak{v}}(g))^2 \oplus \sigma(g)^2}$,
(b) $\delta(\varphi_g(\hat{w}), \overline{\mathfrak{v}}(g^*)) = \delta(\varphi_g(\hat{w}), \varphi_g(\overline{\mathfrak{v}}(g))) \le \dfrac{\sigma(g)}{\alpha(\hat{w}, \overline{\mathfrak{v}}(g))}\,\delta(\hat{w}, \overline{\mathfrak{v}}(g))$.

Proof Let us write $\alpha = \alpha(\hat{w}, \overline{\mathfrak{v}}(g))$ and $\sigma = \sigma(g)$. Take a unit vector $v \in \overline{\mathfrak{v}}(g)$ such that $\angle(v, w)$ is non obtuse. Then $w = \alpha\, v + u$ with $u \perp v$ and $\|u\| = \sqrt{1-\alpha^2}$. Choosing a unit vector $v^* \in \overline{\mathfrak{v}}(g^*)$, we have $gw = \alpha\, \|g\|\, v^* + gu$ with $gu \perp v^*$ and $\|gu\| \leq \sqrt{1-\alpha^2}\, s_2(g) = \sqrt{1-\alpha^2}\, \sigma\, \|g\|$. We define the number $0 \leq \kappa \leq \sigma$ so that $\|gu\| = \sqrt{1-\alpha^2}\, \kappa\, \|g\|$. Hence

$$\alpha^2\, \|g\|^2 \leq \alpha^2\, \|g\|^2 + \|gu\|^2 = \|gw\|^2,$$

and also

$$\begin{aligned} \|gw\|^2 &= \alpha^2\, \|g\|^2 + \|gu\|^2 = \|g\|^2\, \left(\alpha^2 + (1-\alpha^2)\kappa^2\right) \\ &= \|g\|^2\, \left(\alpha^2 \oplus \kappa^2\right) \leq \|g\|^2\, \left(\alpha^2 \oplus \sigma^2\right), \end{aligned}$$

which proves (a).

Item (b) follows from

$$\begin{aligned} \delta\left(\varphi_g(\hat{w}), \overline{\mathfrak{v}}(g^*)\right) &= \frac{\|g\, v \wedge gw\|}{\|gv\|\, \|gw\|} = \frac{\|g\, v \wedge gu\|}{\|g\|\, \|gw\|} = \frac{\|v^* \wedge gu\|}{\|gw\|} \\ &= \frac{\|gu\|}{\|gw\|} \leq \frac{\sigma\, \sqrt{1-\alpha^2}\, \|g\|}{\alpha\, \|g\|} = \frac{\sigma\, \delta(\hat{w}, \overline{\mathfrak{v}}(g))}{\alpha}. \end{aligned}$$ □

Next proposition relates the expansion rift $\rho(g, g')$ with the angle $\alpha(g, g')$ and the upper angle $\beta(g, g')$.

Proposition 2.23 *Given $g, g' \in \mathscr{L}(V)$ with a (1)-gap pattern,*

$$\alpha(g, g') \leq \frac{\|g'\, g\|}{\|g'\|\, \|g\|} \leq \beta(g, g')$$

Proof Let $\alpha := \alpha(g, g') = \alpha(\overline{\mathfrak{v}}(g^*), \overline{\mathfrak{v}}(g'))$ and take unit vectors $v \in \overline{\mathfrak{v}}(g)$, $v^* \in \overline{\mathfrak{v}}(g^*)$ and $v' \in \overline{\mathfrak{v}}(g')$ such that $\langle v^*, v' \rangle = \alpha > 0$ and $g\, v = \|g\|\, v^*$.

Since $\varphi_g(\overline{\mathfrak{v}}(g)) = \overline{\mathfrak{v}}(g^*)$, $w = \frac{g\, v}{\|g\, v\|}$ is a unit vector in $\hat{w} = \overline{\mathfrak{v}}(g^*)$. Hence, applying Proposition 2.22(a) to g' and $\hat{w}$, we get

$$\alpha(g, g')\, \|g'\| = \alpha(\hat{w}, \overline{\mathfrak{v}}(g'))\, \|g'\| \leq \|\frac{g'\, g\, v}{\|g\, v\|}\| \leq \frac{\|g'\, g\|}{\|g\|},$$

which proves the first inequality.

For the second inequality, consider any $\hat{w} \in \mathbb{P}(g)$ and a unit vector $w \in \hat{w}$ such that $a := \langle w, v \rangle = \alpha(\hat{w}, \overline{\mathfrak{v}}(g)) \geq 0$. Then $w = a\, v + \sqrt{1-a^2}\, u$, where u is a unit vector orthogonal to v. It follows that $g\, w = a\, \|g\|\, v^* + \sqrt{1-a^2}\, g\, u$ with $g\, u \perp v^*$, and $\|g\, u\| = \kappa\, \|g\|$ for some $0 \leq \kappa \leq \sigma(g)$. Therefore

$$\frac{\|g\, w\|^2}{\|g\|^2} = a^2 + (1-a^2)\, \kappa^2 = a^2 \oplus \kappa^2.$$

and

$$\frac{g\,w}{\|g\,w\|} = \frac{a}{\sqrt{a^2 \oplus \kappa^2}}\, v^* + \frac{\sqrt{1-a^2}}{\sqrt{a^2 \oplus \kappa^2}}\, \frac{g\,u}{\|g\|}.$$

The vector v' can be written as $v' = \alpha\, v^* + w'$ with $w' \perp v^*$ and $\|w'\| = \sqrt{1-\alpha^2}$. Set now $b := \alpha(\varphi_g(\hat{w}), \overline{\mathfrak{v}}(g'))$. Then

$$\begin{aligned} b = \left|\langle \frac{g\,w}{\|g\,w\|}, v'\rangle\right| &\le \frac{\alpha\, a}{\sqrt{a^2 \oplus \kappa^2}} + \frac{\sqrt{1-a^2}}{\sqrt{a^2 \oplus \kappa^2}}\, \frac{\left|\langle g\,u, v'\rangle\right|}{\|g\|} \\ &\le \frac{\alpha\, a}{\sqrt{a^2 \oplus \kappa^2}} + \frac{\kappa\,\sqrt{1-a^2}}{\sqrt{a^2 \oplus \kappa^2}}\, \left|\langle \frac{g\,u}{\|g\,u\|}, w'\rangle\right| \\ &\le \frac{\alpha\, a}{\sqrt{a^2 \oplus \kappa^2}} + \frac{\kappa\,\sqrt{1-a^2}}{\sqrt{a^2 \oplus \kappa^2}}\, \|w'\| \\ &= \frac{\alpha\, a}{\sqrt{a^2 \oplus \kappa^2}} + \frac{\kappa\,\sqrt{1-a^2}\,\sqrt{1-\alpha^2}}{\sqrt{a^2 \oplus \kappa^2}} \le \frac{\sqrt{\alpha^2 \oplus \kappa^2}}{\sqrt{a^2 \oplus \kappa^2}}. \end{aligned}$$

We use Lemma 2.21 (7) on the last inequality. Finally, by Proposition 2.22(a) applied to $g' \in \mathscr{L}(V)$ and the unit vector $gw/\|gw\| \in \varphi_g(\hat{w})$,

$$\begin{aligned} \|g'\,g\,w\| &\le \|g'\|\,\sqrt{b^2 \oplus \sigma(g')^2}\,\|g\,w\| \\ &\le \|g'\|\,\|g\|\,\sqrt{b^2 \oplus \sigma(g')^2}\,\sqrt{a^2 \oplus \kappa^2} \\ &\le \|g'\|\,\|g\|\,\sqrt{\kappa^2 \oplus \alpha^2 \oplus \sigma(g')^2} \le \beta(g, g')\,\|g'\|\,\|g\|, \end{aligned}$$

where on the two last inequalities use items (6) and (5) of Lemma 2.21. $\square$

Corollary 2.1 *Given $g, g' \in \mathscr{L}(V)$ with a (k)-gap pattern,*

$$\alpha_k(g, g') \le \frac{\|\wedge_k(g'\,g)\|}{\|\wedge_k g'\|\,\|\wedge_k g\|} \le \beta_k(g, g')$$

Proof Apply Proposition 2.23 to the composition $(\wedge_k g')\,(\wedge_k g)$. Notice that by Definition 2.15, the Plücker embedding satisfies $\Psi(\overline{\mathfrak{v}}_k(g)) = \overline{\mathfrak{v}}(\wedge_k g)$. Hence

$$\alpha_k(g, g') = \alpha(\overline{\mathfrak{v}}_k(g^*), \overline{\mathfrak{v}}_k(g')) = \left|\langle \overline{\mathfrak{v}}(\wedge_k g), \overline{\mathfrak{v}}(\wedge_k g')\rangle\right| = \alpha(\wedge_k g, \wedge_k g'). \qquad \square$$

The next results show how close the bounds $\alpha(g, g')$ and $\beta(g, g')$ are to each other and to the rift $\rho(g, g')$.

Lemma 2.4 *Given $g, g' \in \mathscr{L}(V)$ with* (1)*-gap patterns,*

$$1 \leq \frac{\beta(g, g')}{\alpha(g, g')} \leq \sqrt{1 + \frac{\mathrm{gr}(g)^{-2} \oplus \mathrm{gr}(g')^{-2}}{\alpha(g, g')^2}}.$$

Proof Just notice that

$$\frac{\sqrt{\kappa^2 \oplus \alpha^2 \oplus (\kappa')^2}}{\alpha} \leq \sqrt{\frac{\alpha^2 + (\kappa^2 \oplus (\kappa')^2)}{\alpha^2}} = \sqrt{1 + \frac{\kappa^2 \oplus (\kappa')^2}{\alpha^2}}. \qquad \square$$

Proposition 2.24 *Given $g, g' \in \mathscr{L}(V)$ with a* (1)*-gap pattern*

$$\alpha(g, g') \geq \rho(g, g') \sqrt{1 - \frac{\mathrm{gr}(g)^{-2} + \mathrm{gr}(g')^{-2}}{\rho(g, g')^2}}.$$

Proof By Proposition 2.23

$$\rho(g, g')^2 \leq \beta(g, g')^2 \leq \alpha(g, g')^2 + \sigma(g)^2 + \sigma(g')^2, \qquad \square$$

which implies the claimed inequality.

These inequalities then imply the following more general fact.

Proposition 2.25 *Given $g_0, g_1, \ldots, g_{n-1} \in \mathscr{L}(V)$, if for all $1 \leq i \leq n-1$ the linear maps g_i and $g^{(i)} = g_{i-1} \ldots g_0$ have* (1)*-gap patterns, then*

$$\prod_{i=1}^{n-1} \alpha(g^{(i)}, g_i) \leq \frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_{n-1}\| \ldots \|g_1\| \|g_0\|} \leq \prod_{i=1}^{n-1} \beta(g^{(i)}, g_i)$$

Proof By definition $g^{(n-1)} = g_{n-1} \ldots g_1 g_0$, and by convention $g^{(0)} = \mathrm{id}_V$. Hence $\|g_{n-1} \ldots g_1 g_0\| = \prod_{i=0}^{n-1} \frac{\|g^{(i+1)}\|}{\|g^{(i)}\|}$. This implies that

$$\begin{aligned} \frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_{n-1}\| \ldots \|g_1\|} &= \left(\prod_{i=0}^{n-1} \frac{1}{\|g_i\|} \right) \left(\prod_{i=0}^{n-1} \frac{\|g^{(i+1)}\|}{\|g^{(i)}\|} \right) \\ &= \prod_{i=0}^{n-1} \frac{\|g_i \, g^{(i)}\|}{\|g_i\| \, \|g^{(i)}\|}. \end{aligned}$$

It is now enough to apply Proposition 2.23 to each factor. $\square$

2.3 Lipschitz Estimates

In this section we will derive some inequalities describing quantities such as the contracting behavior of a linear endomorphism on the projective space, the Lipschitz dependence of a projective action on the acting linear endomorphism, the continuity of most expanding directions as functions of a linear map, and the Lipschitz modulus of continuity for sum and intersection operations on flag manifolds. Except for Propositions 2.28 and 2.29, the content of this section will be only used in Chaps. 4 and 5.

2.3.1 Projective Action

Proposition 2.26 *Given $p, q \in V \setminus \{0\}$,*

$$\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \| \leq \max\{\frac{1}{\|p\|}, \frac{1}{\|q\|}\} \, \|p - q\|.$$

Proof Given to vectors $u, v \in V$ with $\|u\| \geq \|v\| = 1$ we have

$$\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \| \leq \|u - v\|.$$

Assume for instance that $\|p\| \geq \|q\|$, so that

$$\max\{\|p\|^{-1}, \|q\|^{-1}\} = \|q\|^{-1}.$$

Applying the previous inequality with $u = \frac{p}{\|q\|}$ and $v = \frac{q}{\|q\|}$, we get

$$\begin{aligned} \| \frac{p}{\|p\|} - \frac{q}{\|q\|} \| &= \| \frac{u}{\|u\|} - \frac{v}{\|v\|} \| \leq \|u - v\| = \| \frac{p}{\|q\|} - \frac{q}{\|q\|} \| \\ &= \|q\|^{-1} \, \|p - q\| = \max\{\|p\|^{-1}, \|q\|^{-1}\} \, \|p - q\|. \end{aligned}$$ □

Given a linear map $g \in \mathscr{L}(V)$, the projective action of g is given by the map $\varphi_g : \mathbb{P}(g) \to \mathbb{P}(g^*)$, $\varphi_g(\hat{p}) := \widehat{g\,p}$.

For any non collinear vectors $p, q \in V$ with $\|p\| = \|q\| = 1$, define

$$v_p(q) := \frac{q - \langle p, q\rangle\, p}{\|q - \langle p, q\rangle\, p\|}.$$

This is the normalized unit vector of the orthogonal projection of q onto $p^{\perp}$.

Proposition 2.27 *Given $g \in \mathscr{L}(V)$, and points $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,*

$$\frac{\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q}))}{\delta(\hat{p}, \hat{q})} = \frac{\|gp \wedge gv_p(q)\|}{\|g\,p\|\,\|g\,q\|}.$$

Proof Let $p \in \hat{p}$ and $q \in \hat{q}$ be unit vectors such that $\theta = \angle(p, q) \in [0, \frac{\pi}{2}]$. We can write $q = (\cos\theta)\,p + (\sin\theta)\,v_p(q)$. Hence

$$\delta(\hat{p}, \hat{q}) = \|p \wedge q\| = (\sin\theta)\,\|p \wedge v_p(q)\| = \sin\theta,$$

and

$$\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q})) = \frac{\|g\,p \wedge g\,q\|}{\|g\,p\|\,\|g\,q\|} = (\sin\theta)\,\frac{\|gp \wedge gv_p(q)\|}{\|g\,p\|\,\|g\,q\|}. \qquad \square$$

Given a point $\hat{p} \in \mathbb{P}(V)$, we identify the tangent to the projective space at $\hat{p}$ as $T_{\hat{p}}\mathbb{P}(V) = p^{\perp}$, for any representative $p \in \hat{p}$.

Proposition 2.28 *Given $g \in \mathscr{L}(V)$, $\hat{x} \in \mathbb{P}(g)$, and a representative $x \in \hat{x}$, the derivative of the map $\varphi_g : \mathbb{P}(g) \to \mathbb{P}(g^*)$ at $\hat{x}$ is given by*

$$(D\varphi_g)_{\hat{x}}\,v = \frac{g\,v - \langle \frac{g\,x}{\|g\,x\|},\, g\,v\rangle\,\frac{g\,x}{\|g\,x\|}}{\|g\,x\|} = \frac{1}{\|g\,x\|}\,\pi^{\perp}_{gx/\|gx\|}(g\,v)$$

Proof The sphere $\mathbb{S}(V) := \{v \in V : \|v\| = 1\}$ is a double covering space of $\mathbb{P}(V)$, whose covering map is the canonical projection $\hat{\pi} : \mathbb{S}(V) \to \mathbb{P}(V)$. With the identification $T_{\hat{p}}\mathbb{P}(V) = p^{\perp}$, the derivative of $\hat{\pi}$, $D\hat{\pi}_x : T_x\mathbb{S}(V) \to T_{\hat{x}}\mathbb{P}(V)$, is the identity linear map. The map φ_g lifts to the map defined on the sphere by $\widetilde{\varphi}_g(x) := \frac{g\,x}{\|g\,x\|}$. Hence we can identify the derivatives $(D\varphi_g)_{\hat{x}}$ and $(D\widetilde{\varphi}_g)_x$. A simple calculation leads to the explicit expression above for $(D\widetilde{\varphi}_g)_x v$. $\square$

We will use the following closed ball notation

$$B^{(d)}(\hat{p}, r) := \{\hat{x} \in \mathbb{P}(V) :\ d(\hat{x}, \hat{p}) \leq r\},$$

where the superscript emphasizes the distance in matter. Given a projective map $f : X \subset \mathbb{P}(V) \to \mathbb{P}(V)$, we denote by $\mathrm{Lip}_d(f)$ the least Lipschitz constant of f with respect to the distance d. Next proposition refers to the projective metrics δ and ρ defined in Sect. 2.1.1.

Proposition 2.29 *Given $0 < \kappa < 1$ and $g \in \mathscr{L}(V)$ such that $\mathrm{gr}(g) \geq \kappa^{-1}$,*

(1) $\varphi_g\left(B^{(\delta)}(\overline{\mathfrak{v}}(g), r)\right) \subset B^{(\delta)}(\overline{\mathfrak{v}}(g^*), \kappa\, r/\sqrt{1-r^2})$, *for any* $0 < r < 1$,
(2) $\varphi_g\left(B^{(\rho)}(\overline{\mathfrak{v}}(g), a)\right) \subset B^{(\rho)}(\overline{\mathfrak{v}}(g^*), \kappa\, \tan a)$, *for any* $0 < a < \frac{\pi}{2}$,
(3) $\mathrm{Lip}_\rho(\varphi_g|_{B^{(\delta)}(\overline{\mathfrak{v}}(g), r)}) \leq \kappa\, \frac{r+\sqrt{1-r^2}}{1-r^2}$, *for any* $0 < r < 1$.

Proof Item (1) of this proposition follows from Proposition 2.22(b), because

$$\delta(\hat{w}, \overline{\mathfrak{v}}(g)) < r \quad \text{implies} \quad \alpha(\hat{w}, \overline{\mathfrak{v}}(g)) = \sqrt{1 - \delta(\hat{w}, \overline{\mathfrak{v}}(g))^2} \geq \sqrt{1 - r^2}.$$

Item (2) reduces to (1), because we have $\delta(\hat{u}, \hat{v}) = \sin \rho(\hat{u}, \hat{v})$, which implies that $B^{(\rho)}(\hat{v}, a) = B^{(\delta)}(\hat{v}, \sin a)$.

To prove (3), take unit vectors $v \in \overline{\mathfrak{v}}(g)$ and $v^* \in \overline{\mathfrak{v}}(g^*)$ such that $g\, v = \|g\|\, v^*$. Because v is a g-most expanding vector, $\|\pi_{v^*}^{\perp} \circ g\| = \|g \circ \pi_v^{\perp}\| \leq s_2(g) \leq \kappa\, \|g\|$. Given $\hat{x}$ such that $\delta(\hat{x}, \overline{\mathfrak{v}}(g)) < r$, and a unit vector $x \in \hat{x}$, by Proposition 2.22(a)

$$\frac{\|g\|}{\|gx\|} \leq \frac{1}{\alpha(\hat{x}, \overline{\mathfrak{v}}(g))} \leq \frac{1}{\sqrt{1 - r^2}}.$$

Using item (b) of the same proposition we get

$$\delta(\varphi_g(\hat{x}), \overline{\mathfrak{v}}(g^*)) \leq \frac{\sigma(g)}{\alpha(\hat{x}, \overline{\mathfrak{v}}(g))}\, \delta(\hat{x}, \overline{\mathfrak{v}}(g)) \leq \frac{\kappa\, r}{\sqrt{1 - r^2}}$$

By Proposition 2.28 we have

$$(D\varphi_g)_x\, v = \frac{1}{\|gx\|}\, \pi_{v^*}^{\perp}(g\, v) + \frac{1}{\|gx\|} \left(\pi_{\tilde{\varphi}_g(x)}^{\perp} - \pi_{v^*}^{\perp} \right) (g\, v).$$

Thus, by Lemma 2.2(d),

$$\begin{aligned} \|(D\varphi_g)_x\| &\leq \frac{\kappa\, \|g\|}{\|gx\|} + \frac{\delta(\varphi_g(\hat{x}), \overline{\mathfrak{v}}(g^*))\, \|g\|}{\|gx\|} \\ &\leq \frac{\kappa}{\sqrt{1 - r^2}} + \frac{\kappa\, r}{1 - r^2} = \frac{\kappa\, (r + \sqrt{1 - r^2})}{1 - r^2}. \end{aligned}$$

Since $B^{(\delta)}(\overline{\mathfrak{v}}(g), r)$ is a convex Riemannian disk, by the mean value theorem $\varphi_g|_{B^{(\delta)}(\overline{\mathfrak{v}}(g), r)}$ has Lipschitz constant $\leq \frac{\kappa\, (r + \sqrt{1 - r^2})}{1 - r^2}$ with respect to distance ρ. $\square$

2.3.2 *Operations on Flag Manifolds*

As before let V be a finite n-dimensional Euclidean space. Recall that the Grassmann manifold $\mathrm{Gr}_k(V)$ identifies through the Plücker embedding with a submanifold of $\mathbb{P}(\wedge_k V)$. Up to a sign, $E \in \mathrm{Gr}_k(V)$ is identified with the unit k-vector $e = e_1 \wedge \cdots \wedge e_k$ associated to any orthonormal basis $\{e_1, \ldots, e_k\}$ of E. Recall that the Grassmann distance (2.10) on $\mathrm{Gr}_k(V)$ can be characterized by

$$d(E_1, E_2) := \min\{\|e_1 - e_2\|, \|e_1 + e_2\|\},$$

where e_j is a unit k-vector of E_j, for $j = 1, 2$.

Definition 2.31 Given $E, F \in \mathrm{Gr}(V)$, we say that E and F are $(\cap)$ transversal if $E + F = V$. Analogously, we say that E and F are $(+)$ transversal if $E \cap F = \{0\}$.

The following numbers quantify the transversality of two linear subspaces.

Definition 2.32 Given $E \in \mathrm{Gr}_r(V)$ and $F \in \mathrm{Gr}_s(V)$, consider a unit r-vector e of E, a unit s-vector f of F, a unit $(n-r)$-vector $e^\perp$ of $E^\perp$ and a unit $(n-s)$-vector $f^\perp$ of $F^\perp$. We define

$$\begin{aligned}\theta_+(E, F) &:= \|e \wedge f\|,\\ \theta_\cap(E, F) &:= \|e^\perp \wedge f^\perp\|.\end{aligned}$$

Since the chosen unit vectors are unique up to a sign, these quantities are well-defined.

Remark 2.1 If $r + s > n$ then $\theta_+(E, F) = 0$. Similarly, if $r + s < n$ then $\theta_\cap(E, F) = 0$.

Remark 2.2 Given $E, F \in \mathrm{Gr}(V)$, $\theta_\cap(E, F) = \theta_+(E^\perp, F^\perp)$.

Next proposition establishes a Lispchitz modulus of continuity for the sum and intersection operations on Grassmannians in terms of the previous quantities.

Proposition 2.30 *Given* $r, s \in \mathbb{N}$ *and* $E, E' \in \mathrm{Gr}_r(V)$, $F, F' \in \mathrm{Gr}_s(V)$,

(1) $d(E + F, E' + F') \leq \max\left\{\dfrac{1}{\theta_+(E, F)}, \dfrac{1}{\theta_+(E', F')}\right\}(d(E, E') + d(F, F'))$,

(2) $d(E \cap F, E' \cap F') \leq \max\left\{\dfrac{1}{\theta_\cap(E, F)}, \dfrac{1}{\theta_\cap(E', F')}\right\}(d(E, E') + d(F, F'))$.

Proof (1) Consider unit r-vectors e and e' representing the subspaces E and E' respectively. Consider also unit s-vectors f and f' representing the subspaces F and F' respectively. By Proposition 2.26

$$\begin{aligned}d(E + F, E' + F') &= \left\|\frac{e \wedge f}{\|e \wedge f\|} - \frac{e' \wedge f'}{\|e' \wedge f'\|}\right\|\\ &\leq K\,\|e \wedge f - e' \wedge f'\|\\ &\leq K\,(\|e \wedge (f - f')\| + \|(e - e') \wedge f'\|)\\ &\leq K\,(\|e - e'\| + \|f - f'\|)\end{aligned}$$

where $K = \max\{\|e \wedge f\|^{-1}, \|e' \wedge f'\|^{-1}\} = \max\{\theta_+(E, F)^{-1}, \max\{\theta_+(E', F')^{-1}\}$.
(2) reduces to (1) by duality (see Proposition 2.2). □

Next proposition gives an alternative characterization of the transversality measurements $\theta_+(E, F)$ and $\theta_\cap(E, F)$. Let, as before, $\pi_E : V \to E$ denote the orthogonal projection onto a subspace $E \subset V$, and define the restriction $\pi_{E,F} := \pi_F|_E : E \to F$.

Proposition 2.31 *Given $E \in \mathrm{Gr}_r(V)$ and $F \in \mathrm{Gr}_s(V)$,*

(1) $\theta_+(E, F) = \det_+(\pi_{E,F^\perp}) = \det_+(\pi_{F,E^\perp})$.
(2) $\theta_\cap(E, F) = \det_+(\pi_{E^\perp,F}) = \det_+(\pi_{F^\perp,E})$.

Proof Notice that $E \cap F = \mathrm{K}(\pi_{E,F^\perp}) = \mathrm{K}(\pi_{F,E^\perp})$. If $E \cap F \neq \{0\}$ then the three terms in (1) vanish. Otherwise $\pi_{E,F^\perp}$ and $\pi_{F,E^\perp}$ are isomorphisms onto their ranges $\mathrm{R}(\pi_{E,F^\perp}) = F^\perp \cap (E + F)$ and $\mathrm{R}(\pi_{F,E^\perp}) = E^\perp \cap (E + F)$. Take an orthonormal basis $\{f_1, \dots, f_s, f_{s+1}, \dots, f_{s+r}, \dots, f_n\}$ such that $\{f_1, \dots, f_s\}$ spans F and the family of vectors $\{f_1, \dots, f_r, f_{s+1}, \dots, f_{s+r}\}$ spans $E + F$. Consider the unit s-vector $f = f_1 \wedge \cdots \wedge f_s$ of F, and a unit r-vector $e = e_1 \wedge \cdots \wedge e_r$ of E. Hence $\{f_{s+1}, \dots, f_{s+r}\}$ is a basis of $\mathrm{R}(\pi_{E,F^\perp})$ and

$$\begin{aligned}\theta_+(E, F) &= \|(e_1 \wedge \cdots \wedge e_r) \wedge (f_1 \wedge \cdots \wedge f_s)\| \\ &= \|\pi_{E,F^\perp}(e_1) \wedge \cdots \wedge \pi_{E,F^\perp}(e_r) \wedge f_1 \wedge \cdots \wedge f_s\| \\ &= \det_+(\pi_{E,F^\perp})\, \|f_{s+1} \wedge \cdots \wedge f_{s+r} \wedge f_1 \wedge \cdots \wedge f_s\| = \det_+(\pi_{E,F^\perp}).\end{aligned}$$

Reversing the roles of E and F, and because $\|e \wedge f\|$ is symmetric in e and f, we obtain $\theta_+(E, F) = \det_+(\pi_{F,E^\perp})$, which proves (1).

By duality and Remark 2.2, item (2) reduces to (1). □

The measurement on the $(\cap)$ transversality admits the following lower bound in terms of the angle in Definition 2.24.

Proposition 2.32 *Given $E \in \mathrm{Gr}_r(V)$ and $F \in \mathrm{Gr}_s(V)$, if $E + F = V$ then*

$$\theta_\cap(E, F) \geq \alpha_r(E, E \cap F + F^\perp).$$

Proof Combining Lemmas 2.5 and 2.6 below we have

$$\begin{aligned}\theta_\cap(E, F) &\geq \theta_\cap(E, F \cap (E \cap F)^\perp) = \alpha_r(E, (F \cap (E \cap F)^\perp)^\perp) \\ &= \alpha_r(E, (E \cap F) + F^\perp).\end{aligned}$$ □

Lemma 2.5 *Given $E \in \mathrm{Gr}_r(V)$, $E' \in \mathrm{Gr}_{r'}(V)$ and $F \in \mathrm{Gr}_s(V)$ such that $r + s \geq n$ and $E \subseteq E'$ then $\theta_\cap(E', F) \geq \theta_\cap(E, F)$.*

Proof Because $E \subset E'$, we have $\pi_{F^\perp,E} = \pi_{E',E} \circ \pi_{F^\perp,E'}$. Hence by Proposition 2.8

$$\begin{aligned}\theta_\cap(E, F) &= \det_+(\pi_{F^\perp,E}) = \det_+(\pi_{\pi_{E'}(F^\perp),E}) \det_+(\pi_{F^\perp,E'}) \\ &\leq \det_+(\pi_{F^\perp,E'}) = \theta_\cap(E', F),\end{aligned}$$

where $\det_+(\pi_{\pi_{E'}(F^\perp),E}) \leq 1$ because $\|\pi_E\| \leq 1$. □

Lemma 2.6 *Given $E, E' \in \mathrm{Gr}_r(V)$, $\theta_\cap(E', E^\perp) = \alpha_r(E', E)$.*

Proof Given orthonormal bases $\{v_1, \dots, v_r\}$ of E, and $\{v'_1, \dots, v'_r\}$ of E',

$$\begin{aligned}\theta_\cap(E', E^\perp) &= \det{}_+(\pi_{E',E})\\ &= \left|\langle \wedge_r \pi_{E,E'}(v_1 \wedge \cdots \wedge v_r), v'_1 \wedge \cdots \wedge v'_r\rangle\right|\\ &= \left|\langle \pi_{E'}(v_1) \wedge \cdots \wedge \pi_{E'}(v_r), v'_1 \wedge \cdots \wedge v'_r\rangle\right|\\ &= \left|\langle v_1 \wedge \cdots \wedge v_r, v'_1 \wedge \cdots \wedge v'_r\rangle\right| = \alpha_r(E, E'). \end{aligned}$$

□

Next proposition gives a modulus of lower semi-continuity for the transversality measurement $\theta_\cap$.

Proposition 2.33 *Given $E, E_0 \in \mathrm{Gr}_r(V)$ and $F, F_0 \in \mathrm{Gr}_s(V)$,*

$$\theta_\cap(E, F) \geq \theta_\cap(E_0, F_0) - d(E, E_0) - d(F, F_0).$$

Proof Consider unit vectors $e \in \Psi(E^\perp), f \in \Psi(F^\perp)$, $e_0 \in \Psi(E_0^\perp)$ and $f_0 \in \Psi(F_0^\perp)$, chosen so that

$$\begin{aligned} d(E, E_0) &= d(E^\perp, E_0^\perp) = \|e - e_0\|,\\ d(F, F_0) &= d(F^\perp, F_0^\perp) = \|f - f_0\|. \end{aligned}$$

Hence

$$\begin{aligned}\theta_\cap(E, F) = \|e \wedge f\| &\geq \|e_0 \wedge f_0\| - \|e \wedge f - e_0 \wedge f_0\|\\ &\geq \theta_\cap(E_0, F_0) - \|e \wedge (f - f_0)\| - \|(e - e_0) \wedge f_0\|\\ &\geq \theta_\cap(E_0, F_0) - \|f - f_0\| - \|e - e_0\|\\ &\geq \theta_\cap(E_0, F_0) - d(F, F_0) - d(E, E_0). \end{aligned}$$

□

Next proposition refines inequality (2.7).

Proposition 2.34 *Given $E, F \in \mathrm{Gr}_k(V)$, and families of vectors $\{u_1, \dots, u_k\} \subset E$ and $\{u_{k+1}, \dots, u_{k+i}\} \subset F^\perp$ with $1 \leq i \leq m - k$,*

(a) $\|u_1 \wedge \cdots \wedge u_k \wedge u_{k+1} \wedge \cdots \wedge u_{k+i}\| \leq \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\|$,
(b) $\|u_1 \wedge \cdots \wedge u_k \wedge u_{k+1} \wedge \cdots \wedge u_{k+i}\| \geq \alpha(E, F) \, \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\|$.

Proof Since $\pi_{F^\perp,E^\perp}$ is an orthogonal projection, all its singular values are in [0, 1]. Thus, because $\det_+(\pi_{F^\perp,E^\perp})$ is the product of all singular values, while $\mathfrak{m}(\wedge_i \pi_{F^\perp,E^\perp})$ is the product of the i smallest singular values, we have

$$\det{}_+(\pi_{F^\perp,E^\perp}) \leq \mathfrak{m}(\wedge_i \pi_{F^\perp,E^\perp}) \leq \|\wedge_i \pi_{F^\perp,E^\perp}\| = 1.$$

Hence

$$\begin{aligned}\|u_1 \wedge \cdots \wedge u_k \wedge u_{k+1} \wedge \cdots \wedge u_{k+i}\| &= \|u_1 \wedge \cdots \wedge u_k \wedge \pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \cdots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \\ &= \|u_1 \wedge \ldots u_k\| \, \|\pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \cdots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \\ &\leq \|\wedge_i \pi_{F^\perp,E^\perp}\| \, \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\| \\ &= \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\|,\end{aligned}$$

which proves (a). By Proposition 2.19 we have

$$\alpha(E, F) = \alpha(F^\perp, E^\perp) = \det_+(\pi_{F^\perp,E^\perp}) \leq \mathfrak{m}(\wedge_i(\pi_{F^\perp,E^\perp})).$$

Thus

$$\begin{aligned}\|u_1 \wedge \cdots \wedge u_k \wedge u_{k+1} \wedge \cdots \wedge u_{k+i}\| &= \|u_1 \wedge \cdots \wedge u_k \wedge \pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \cdots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \\ &= \|u_1 \wedge \ldots u_k\| \, \|\pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \cdots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \\ &\geq \mathfrak{m}(\wedge_i \pi_{F^\perp,E^\perp}) \, \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\| \\ &\geq \alpha(E, F) \, \|u_1 \wedge \cdots \wedge u_k\| \, \|u_{k+1} \wedge \cdots \wedge u_{k+i}\|,\end{aligned}$$

which proves (b). □

The angle α is a Lipschitz continuous function.

Proposition 2.35 *Given $u, u', v, v' \in \mathbb{P}(V)$,*

$$\left|\alpha(u, v) - \alpha(u', v')\right| \leq d(u, u') + d(v, v').$$

Proof Exercise. □

The intersection of complementary flags satisfying the appropriate transversality conditions determines a decomposition of the Euclidean space V. We end this section defining by this operation and proving a modulus of continuity for it.

Consider a signature $\tau = (\tau_1, \ldots, \tau_k)$ of length k with $\tau_k < \dim V$. We make the convention that $\tau_0 = 0$ and $\tau_{k+1} = \dim V$.

Definition 2.33 A τ-decomposition is a family of linear subspaces $E_\cdot = \{E_i\}_{1 \leq i \leq k+1}$ in $\mathrm{Gr}(V)$ such that $V = \oplus_{i=1}^{k+1} E_i$ and $\dim E_i = \tau_i - \tau_{i-1}$ for all $1 \leq i \leq k+1$.

Let $\mathcal{D}_\tau(V)$ denote the space of all τ-decompositions, which we endow with the following metric

$$d_\tau(E_\cdot, E'_\cdot) = \max_{1 \leq i \leq k+1} d_{\tau_i - \tau_{i-1}}(E_i, E'_i),$$

where $d_{\tau_i - \tau_{i-1}}$ stands for the distance (2.10) in $\mathrm{Gr}_{\tau_i - \tau_{i-1}}(V)$.

Given two flags $F \in \mathcal{F}_\tau(V)$ and $F' \in \mathcal{F}_{\tau^\perp}(V)$, we will define a decomposition, denoted by $F \sqcap F'$, formed out by intersecting the components of these flags. For that we introduce the following measurement.

Definition 2.34 Given two flags $F \in \mathcal{F}_\tau(V)$ and $F' \in \mathcal{F}_{\tau^\perp}(V)$, let

$$\theta_\sqcap(F, F') := \min_{1\leq i\leq k} \theta_\cap(F_i, F'_{k-i+1}).$$

Notice that $\dim F_i = \tau_i$ and $\dim F'_{k-i+1} = \tau^\perp_{k-i+1} = \dim V - \tau_i$, i.e., the subspaces F_i and F'_{k-i+1} have complementary dimensions. We will refer to this quantity as the *transversality measurement* between the flags F and F'.

In the next proposition we complete F and F' to full flags of length $k+1$ setting $F_{k+1} = F'_{k+1} = V$. Set also $\tau_0 = 0$ and $\tau_{k+1} = \dim V$.

Proposition 2.36 *If* $\theta_\sqcap(F, F') > 0$ *then the following is a direct sum decomposition in the space* $\mathcal{D}_\tau(V)$*,*

$$V = \bigoplus_{i=1}^{k+1} F_i \cap F'_{k-i+2},$$

with $\dim(F_i \cap F'_{k-i+2}) = \tau_i - \tau_{i-1}$ *for all* $1 \leq i \leq k+1$.

Proof Since the subspaces F_i and F'_{k-i+1} have complementary dimensions, the relation $\theta_\cap(F_i, F'_{k-i+1}) > 0$ implies that

$$V = F_i \oplus F'_{k-i+1}. \tag{2.22}$$

By Lemma 2.5, $\theta_\cap(F_i, F'_{k-i+2}) \geq \theta_\cap(F_i, F'_{k-i+1}) > 0$. Therefore $F_i + F'_{k-i+2} = V$ and

$$\begin{aligned}\dim(F_i \cap F'_{k-i+2}) &= \tau_i + \tau^\perp_{k-i+2} - \dim V \\ &= \tau_i + (\dim V - \tau_{i-1}) - \dim V = \tau_i - \tau_{i-1}.\end{aligned}$$

We prove by finite induction in $i = 1, \ldots, k+1$ that

$$F_i = \bigoplus_{j\leq i} F_j \cap F'_{k-j+2}. \tag{2.23}$$

Since $F_{k+1} = V$ the proposition follows from this relation at $i = k+1$.

For $i = 1$, (2.23) reduces to $F_1 = F_1 \cap V$. The induction step follows from

$$F_{i+1} = F_i \oplus \left(F_{i+1} \cap F'_{k-i+1}\right).$$

Since the following dimensions add up

$$\begin{aligned}\dim F_{i+1} = \tau_{i+1} &= \tau_i + (\tau_{i+1} - \tau_i) \\ &= \dim F_i + \dim(F_{i+1} \cap F'_{k-i+1}),\end{aligned}$$

it is enough to see that

$$F_i \cap \left(F_{i+1} \cap F'_{k-i+1}\right) = F_i \cap F'_{k-i+1} = \{0\},$$

which holds because of (2.22). □

Hence, by the previous proposition we can define:

Definition 2.35 Given flags $F \in \mathcal{F}_\tau(V)$ and $F' \in \mathcal{F}_{\tau^\perp}(V)$ such that $\theta_\sqcap(F, F') > 0$ we define $F \sqcap F' := \{F_i \cap F'_{k-i+2}\}_{1\leq i \leq k+1}$ and call it the intersection decomposition of the flags F and F'.

Next proposition provides a modulus of lower semi-continuity for the transversality measurement $\theta_\sqcap$.

Proposition 2.37 *Given $F, F_0 \in \mathcal{F}_\tau(V)$ and $F', F'_0 \in \mathcal{F}_{\tau^\perp}(V)$,*

$$\theta_\sqcap(F, F') \geq \theta_\sqcap(F_0, F'_0) - d_\tau(F, F_0) - d_{\tau^\perp}(F', F'_0).$$

Proof Apply Proposition 2.33 at each subspace of the τ-decompositions. □

The modulus of continuity for the intersection map $\sqcap : \mathcal{F}_\tau(V) \times \mathcal{F}_{\tau^\perp}(V) \to \mathcal{D}_\tau(V)$ is established below.

Proposition 2.38 *Given flags $F_1, F_2 \in \mathcal{F}_\tau(V)$ and $F'_1, F'_2 \in \mathcal{F}_{\tau^\perp}(V)$,*

$$d_\tau(F_1 \sqcap F'_1, F_2 \sqcap F'_2) \leq \max\left\{\frac{1}{\theta_\sqcap(F_1, F'_1)}, \frac{1}{\theta_\sqcap(F_2, F'_2)}\right\} (d_\tau(F_1, F_2) + d_{\tau^\perp}(F'_1, F'_2)).$$

Proof Apply Proposition 2.30 at each subspace of the τ-decompositions. □

Given two linear maps $g_0, g_1 \in \mathscr{L}(V)$ with τ-gap ratios such that $\alpha_\tau(g_0, g_1) > 0$, they determine a τ-decomposition of V as intersection of the image by φ_{g_0} of the most expanding τ-flag for g_0 with the least expanding $\tau^\perp$-flag for g_1 (see Definitions 2.18 and 2.21). The corresponding intersection transversality measurement is bounded from below by the angle $\alpha_\tau(g_0, g_1)$.

Proposition 2.39 *Given $g_0, g_1 \in \mathscr{L}(V)$, if $\mathrm{gr}_\tau(g_0) > 1$ and $\mathrm{gr}_\tau(g_1) > 1$ then*

$$\theta_\sqcap(\underline{\mathfrak{v}}_{\tau^\perp}(g_1), \overline{\mathfrak{v}}_\tau(g_0^*)) \geq \alpha_\tau(g_0, g_1).$$

In particular, if $\alpha_\tau(g_0, g_1) > 0$ then the flags $\overline{\mathfrak{v}}_\tau(g_0^*)$ and $\underline{\mathfrak{v}}_{\tau^\perp}(g_1)$ determine the decomposition $\overline{\mathfrak{v}}_\tau(g_0^*) \sqcap \underline{\mathfrak{v}}_{\tau^\perp}(g_1) \in \mathcal{D}_\tau(V)$.

Proof Let $n = \dim V$. Consider the flags $F = \overline{\mathfrak{v}}_\tau(g_0^*)$ and $F' = \underline{\mathfrak{v}}_{\tau^\perp}(g_1)$. We have $F_i = \overline{\mathfrak{v}}_{\tau_i}(g_0^*)$ and $F_{k-i+1} = \underline{\mathfrak{v}}_{\tau^\perp_{k-i+1}}(g_1) = \underline{\mathfrak{v}}_{n-\tau_i}(g_1) = \overline{\mathfrak{v}}_{\tau_i}(g_1)^\perp$. Hence by Lemma 2.6,

$$\theta_\cap(F_i, F'_{k-i+1}) = \theta_\cap(\overline{\mathfrak{v}}_{\tau_i}(g_0^*), \overline{\mathfrak{v}}_{\tau_i}(g_1)^\perp) = \alpha_{\tau_i}(\overline{\mathfrak{v}}_{\tau_i}(g_0^*), \overline{\mathfrak{v}}_{\tau_i}(g_1)) = \alpha_{\tau_i}(g_0, g_1),$$

and taking the minimum, $\theta_\sqcap(F, F') \geq \alpha_\tau(g_0, g_1)$. □

2.3.3 Dependence on the Linear Map

We establish a modulus of Lipschitz continuity for the most expanding direction of a linear endomorphism with a first singular gap. For any $0 < \kappa < 1$, consider the set $\mathscr{L}_\kappa := \{g \in \mathscr{L}(V) \ : \ \mathrm{gr}(g) \geq \frac{1}{\kappa}\}$. We denote by $\overline{\mathfrak{v}} : \mathscr{L}_\kappa \to \mathbb{P}(V)$ the map that assigns the g-most expanding direction to each $g \in \mathscr{L}_\kappa$.

The *relative distance* between linear maps $g, g' \in \mathscr{L}(V) \setminus \{0\}$ is defined as

$$d_{\mathrm{rel}}(g, g') := \frac{\|g - g'\|}{\max\{\|g\|, \|g'\|\}}.$$

Notice that this relative distance is not a metric. It does not satisfy the triangle inequality. We introduce it just to lighten the notation.

Proposition 2.40 *The map $\overline{\mathfrak{v}} : \mathscr{L}_\kappa \to \mathbb{P}(V)$ is locally Lipschitz.*

More precisely, given $0 < \kappa < 1$ there exists $\varepsilon_0 = \varepsilon_0(\kappa) > 0$, which increases as κ decreases, such that for any $g_1, g_2 \in \mathscr{L}_\kappa$ satisfying $d_{\mathrm{rel}}(g_1, g_2) \leq \varepsilon_0$,

$$d(\overline{\mathfrak{v}}(g_1), \overline{\mathfrak{v}}(g_2)) \leq \frac{16}{1 - \kappa^2}\, d_{\mathrm{rel}}(g_1, g_2).$$

Proof Let $g \in \mathscr{L}_\kappa$ and $\lambda > 0$. The singular values (resp. singular vectors) of g are the eigenvalues (resp. eigenvectors) of $\sqrt{g^*\, g}$. Hence $s_j(\lambda\, g) = \lambda\, s_j(g)$, for all j. We also have $\overline{\mathfrak{v}}(\lambda g) = \overline{\mathfrak{v}}(g)$ and $\mathrm{gr}(\lambda\, g) = \mathrm{gr}(g)$.

Consider the subspace $\mathscr{L}_\kappa(1) := \{g \in \mathscr{L}_\kappa \ : \ \|g\| = 1\}$. The projection $g \mapsto g/\|g\|$ takes $\mathscr{L}_\kappa$ to $\mathscr{L}_\kappa(1)$. It also satisfies $\overline{\mathfrak{v}}(g/\|g\|) = \overline{\mathfrak{v}}(g)$ and

$$\left\| \frac{g_1}{\|g_1\|} - \frac{g_2}{\|g_2\|} \right\| \leq 2\, d_{\mathrm{rel}}(g_1, g_2).$$

Hence we can focus our attention on the restricted map $\overline{\mathfrak{v}} : \mathscr{L}_\kappa(1) \to \mathbb{P}(V)$.

Let $\mathscr{L}_\kappa^+(1)$ denote the subspace of $g \in \mathscr{L}_\kappa(1)$ such that $g = g^* \geq 0$, i.e., g is positive semi-definite.

Given $g \in \mathscr{L}_\kappa(1)$, we have $\|g^*\, g\| = 1 = \|g\|$, $\mathrm{gr}(g^* g) = \mathrm{gr}(g)^2$ and $\overline{\mathfrak{v}}(g^* g) = \overline{\mathfrak{v}}(g)$. Also, for all $g_1, g_2 \in \mathscr{L}_\kappa(1)$,

$$\begin{aligned}\|g_1^*\, g_1 - g_2^*\, g_2\| &\leq \|g_1^*\|\, \|g_1 - g_2\| + \|g_1^* - g_2^*\|\, \|g_2\| \\ &= (\|g_1^*\| + \|g_2\|)\, \|g_1 - g_2\| \leq 2\, \|g_1 - g_2\|.\end{aligned}$$

Hence, the mapping $g \mapsto g^* g$ takes $\mathscr{L}_\kappa(1)$ to $\mathscr{L}^+_{\kappa^2}(1)$ and has Lispschitz constant 2. Therefore, it is enough to prove that the restricted map $\overline{\mathfrak{v}} : \mathscr{L}^+_{\kappa^2}(1) \to \mathbb{P}(V)$ has (locally) Lipschitz constant $4\,(1-\kappa^2)^{-1}$.

Let δ_0 be a small positive number and take $0 < \varepsilon_0 \ll \frac{\delta_0}{4}$. The size of δ_0 will be fixed throughout the rest of the proof according to necessity. Take $h_1, h_2 \in \mathscr{L}^+_{\kappa^2}(1)$ such that $\|h_1 - h_2\| < \varepsilon_0$ and set $\hat{p}_0 := \overline{\mathfrak{v}}(h_1)$. By Proposition 2.29 we have

$$\varphi_{h_1}\left(B(\hat{p}_0, \delta_0)\right) \subset B\left(\hat{p}_0, \frac{\kappa^2 \delta_0}{\sqrt{1-\delta_0^2}}\right) \subset B(\hat{p}_0, \delta_0),$$

where all balls refer to the projective sine-metric δ defined in (2.3). The second inclusion holds if δ_0 is chosen small enough. Take any $\hat{p} \in B(\hat{p}_0, \delta_0)$ and choose unit vectors $p \in \hat{p}$ and $p_0 \in \hat{p}_0$ such that $\langle p, p_0\rangle > 0$. Then $p = \langle p, p_0\rangle\, p_0 + w$, with $w \in p_0^\perp$, $h_1(p_0) = p_0$ and $h_1(w) \in p_0^\perp$. Hence

$$\begin{aligned}\|h_1(p)\| &= \|\langle p, p_0\rangle\, p_0 + h_1(w)\| \geq \langle p, p_0\rangle \\ &= \sqrt{1 - \|p \wedge p_0\|^2} \geq \sqrt{1-\delta_0^2} \geq 1/2,\end{aligned}$$

and again, assuming δ_0 is small,

$$\|h_2(p)\| \geq \|h_1(p)\| - \|h_1 - h_2\| \geq \sqrt{1-\delta_0^2} - \varepsilon_0 \geq 1/2.$$

Thus, by Lemma 2.9 below, for all $\hat{p} \in B(\hat{p}_0, \delta_0)$,

$$d(\varphi_{h_1}(\hat{p}), \varphi_{h_2}(\hat{p})) \leq 2\,\|h_1 - h_2\|.$$

Choosing ε_0 small enough, $\frac{\kappa^2\,\delta_0}{\sqrt{1-\delta_0^2}} + 2\,\varepsilon_0 < \delta_0$. This implies that

$$\varphi_{h_2}\left(B(\hat{p}_0, \delta_0)\right) \subset B(\hat{p}_0, \delta_0).$$

By Proposition 2.29 we know that $T_1 = \varphi_{h_1}|_{B(\hat{p}_0,\delta_0)}$ has Lispchitz constant $\kappa' = \kappa^2\,\frac{\delta_0+\sqrt{1-\delta_0^2}}{1-\delta_0^2} \approx \kappa^2$, and assuming δ_0 is small enough we have $\frac{1}{1-\kappa'} \leq \frac{2}{1-\kappa^2}$. Notice that although the Lispchitz constant in this proposition refers to the Riemannian metric ρ, since the ratio $\mathrm{Lip}_\delta(T_1)/\mathrm{Lip}_\rho(T_1)$ approaches 1 as δ_0 tends to 0, we can assume that $\mathrm{Lip}_\delta(T_1) \leq \kappa'$. Thus, by Lemma 2.7 below applied to T_1 and $T_2 = \varphi_{h_2}|_{B(\hat{p}_0,\delta_0)}$, we have $d(T_1, T_2) \leq 2\,\|h_1 - h_2\|$ and

$$d(\overline{\mathfrak{v}}(h_1), \overline{\mathfrak{v}}(h_2)) \leq \frac{1}{1-\kappa'}\, d(T_1, T_2) \leq \frac{4}{1-\kappa^2}\,\|h_1 - h_2\|. \qquad \square$$

Lemma 2.7 *Let* (X, d) *be a complete metric space,* $T_1 : X \to X$ *a Lipschitz contraction with* $\mathrm{Lip}(T_1) < \kappa < 1$, $x_1^* = T_1(x_1^*)$ *a fixed point, and* $T_2 : X \to X$ *any other map with a fixed point* $x_2^* = T_2(x_2^*)$*. Then*

$$d(x_1^*, x_2^*) \leq \frac{1}{1-\kappa}\, d(T_1, T_2),$$

where $d(T_1, T_2) := \sup_{x \in X} d(T_1(x), T_2(x))$.

Proof

$$\begin{aligned} d(x_1^*, x_2^*) &= d(T_1(x_1^*), T_2(x_2^*)) \\ &\leq d(T_1(x_1^*), T_1(x_2^*)) + d(T_1(x_2^*), T_2(x_2^*)) \\ &\leq \kappa\, d(x_1^*, x_2^*) + d(T_1, T_2), \end{aligned}$$

which implies that

$$d(x_1^*, x_2^*) \leq \frac{1}{1-\kappa}\, d(T_1, T_2). \qquad \square$$

Lemma 2.8 *Given* $g_1, g_2 \in \mathscr{L}(V)$*, for any* $1 \leq i \leq \dim V$*,*

$$\|\wedge_i g_1 - \wedge_i g_2\| \leq i\, \max\{1, \|g_1\|, \|g_2\|\}^{i-1}\, \|g_1 - g_2\|.$$

Proof Given any unit i-vector $v_1 \wedge \cdots \wedge v_i \in \wedge_i V$, determined by an orthonormal family of vectors $\{v_1, \ldots, v_i\}$,

$$\begin{aligned} &\|(\wedge_i g_1)(v_1 \wedge \cdots \wedge v_i) - (\wedge_i g_2)(v_1 \wedge \cdots \wedge v_i)\| \\ &\quad = \|(g_1 v_1) \wedge \cdots \wedge (g_1 v_i) - (g_2 v_1) \wedge \cdots \wedge (g_2 v_i)\| \\ &\quad \leq \sum_{j=1}^{i} \|(g_1 v_1) \wedge \cdots \wedge (g_1 v_{j-1}) \wedge (g_1 v_j - g_2 v_j) \wedge (g_2 v_{j+1}) \wedge \cdots \wedge (g_2 v_i)\| \\ &\quad \leq \sum_{j=1}^{i} \|g_1\|^{j-1}\, \|g_2\|^{i-j}\, \|g_1 v_j - g_2 v_j\| \\ &\quad \leq i\, \max\{1, \|g_1\|, \|g_2\|\}^{i-1}\, \|g_1 - g_2\|. \qquad \square \end{aligned}$$

Given a dimension $1 \leq l \leq \dim V$ and $0 < \kappa < 1$, consider the set

$$\mathscr{L}_{l,\kappa} := \{g \in \mathscr{L}(V) \,:\, \mathrm{gr}_l(g) \geq \kappa^{-1}\},$$

and define

$$C_l(g_1, g_2) := \frac{l \max\{1, \|g_1\|, \|g_2\|\}^{l-1}}{\max\{\|\wedge_l g_1\|, \|\wedge_l g_2\|\}}.$$

Corollary 2.2 *The map* $\overline{\mathfrak{v}} : \mathscr{L}_{l,\kappa} \to \mathrm{Gr}_l(V)$ *is locally Lipschitz.*

More precisely, given $0 < \kappa < 1$ *there exists* $\varepsilon_0 > 0$ *such that for any* $g_1, g_2 \in \mathscr{L}_{l,\kappa}$ *such that* $\|g_1 - g_2\| \leq \varepsilon_0 \, C_l(g_1, g_2)^{-1}$*, we have*

$$d(\overline{\mathfrak{v}}_l(g_1), \overline{\mathfrak{v}}_l(g_2)) \leq \frac{16}{1-\kappa^2} \, C_l(g_1, g_2) \, \|g_1 - g_2\|.$$

Proof By Lemma 2.8, $d_{\mathrm{rel}}(\wedge_l g_1, \wedge_l g_2) \leq C_l(g_1, g_2) \, \|g_1 - g_2\|$. Apply Proposition 2.40 to the linear maps $\wedge_l g_j : \wedge_l V \to \wedge_l V$, $j = 1, 2$. □

Given $g \in \mathscr{L}(V)$ having k and $k+r$ gap ratios, if a subspace $E \in \mathrm{Gr}_k(V)$ is close to the g most expanding subspace $\overline{\mathfrak{v}}_k(g)$ then the restriction $g|_{E^\perp}$ has an r-gap ratio and the most expanding r-dimensional subspace of $g|_{E^\perp}$ is close to the intersection of $\overline{\mathfrak{v}}_{k+r}(g)$ with $E^\perp$. Next proposition expresses this fact in a quantitative way.

Proposition 2.41 *Given* $0 < \varkappa < \frac{1}{2}$ *and integers* $1 \leq k < k + r \leq \dim V$*, there exists* $\delta_0 > 0$ *such that for all* $g \in \mathscr{L}(V)$ *and* $E \in \mathrm{Gr}_k(V)$*, if*

(a) $\sigma_k(g) < \varkappa$ *and* $\sigma_{k+r}(g) < \varkappa$*,*
(b) $\delta(E, \overline{\mathfrak{v}}_k(g)) < \delta_0$

then

(1) $\sigma_r(g|_{E^\perp}) \leq 2\,\varkappa$*,*
(2) $\delta\left(\overline{\mathfrak{v}}_r(g|_{E^\perp}),\ \overline{\mathfrak{v}}_{k+r}(g) \cap E^\perp\right) \leq \dfrac{20r}{1 - 4\,\varkappa^2}\, \delta(E, \overline{\mathfrak{v}}_k(g)).$

Proof Consider the compact space

$$\mathscr{K}_r = \{h \in \mathscr{L}(V) : \ \|h\| \leq 1 \ \text{ and } \ \sigma_r(h) \leq \varkappa\}.$$

By uniform continuity of σ_r on $\mathscr{K}_r$ there exists $\delta_0 > 0$ such that for all $h \in \mathscr{L}(V)$ if there exists $h_0 \in \mathscr{K}_r$ with $\|h - h_0\| < \delta_0$ then $\sigma_r(h) \leq 2\,\varkappa$.

Recall that π_F denotes the orthogonal projection onto a linear subspace $F \subset V$.

Given $g \in \mathscr{L}(V)$ such that (a) holds, consider the map $h = \frac{g}{\|g\|} \circ \pi_{\overline{\mathfrak{v}}_k(g)^\perp}$. We have $h \in \mathscr{K}_r$ because $\sigma_r(h) = \sigma_r(g \circ \pi_{\overline{\mathfrak{v}}_k(g)^\perp}) = \sigma_{k+r}(g) < \varkappa$.

Given $E \in \mathrm{Gr}_k(V)$ such that (b) holds, we define $h_E = \frac{g}{\|g\|} \circ \pi_{E^\perp}$. Then by items (b) and (c) of Proposition 2.16

$$\|h - h_E\| \leq \|\pi_{\overline{\mathfrak{v}}_k(g)^\perp} - \pi_{E^\perp}\| \leq \delta(\overline{\mathfrak{v}}_k(g)^\perp, E^\perp) = \delta(E, \overline{\mathfrak{v}}_k(g)) < \delta_0,$$

which implies that $\sigma_r(g|_{E^\perp}) = \sigma_r(h_E) \leq 2\,\varkappa$, and hence proves (1).

To prove item (2) we use the triangle inequality

$$\begin{aligned}
\delta(\overline{\mathfrak{v}}_r(g|_{E^\perp}), \overline{\mathfrak{v}}_{k+r}(g) \cap E^\perp) \leq\ & \delta(\overline{\mathfrak{v}}_r(h_E), \overline{\mathfrak{v}}_r(h)) \\
& + \delta(\overline{\mathfrak{v}}_r(h), \overline{\mathfrak{v}}_{k+r}(g) \cap \overline{\mathfrak{v}}_k(g)^\perp) \\
& + \delta(\overline{\mathfrak{v}}_{k+r}(g) \cap \overline{\mathfrak{v}}_k(g)^\perp,\ \overline{\mathfrak{v}}_{k+r}(g) \cap E^\perp) \\
\leq & \left(\frac{16\, r}{1 - 4\, \varkappa^2} + 0 + 2\right) \delta(E, \overline{\mathfrak{v}}_k(g)) \\
\leq & \frac{20r}{1 - 4\, \varkappa^2}\, \delta(E, \overline{\mathfrak{v}}_k(g)).
\end{aligned}$$

By Corollary 2.2, with $C_r(h_E, h) = r$, we get a bound on $\delta(\overline{\mathfrak{v}}_r(h_E), \overline{\mathfrak{v}}_r(h))$. The second distance is zero because $\overline{\mathfrak{v}}_r(h) = \overline{\mathfrak{v}}_{k+r}(g) \cap \overline{\mathfrak{v}}_k(g)^\perp$. Finally we use item (2) of Proposition 2.30 to derive a bound on the third distance. Notice that although the conclusion of Proposition 2.30 is stated in terms of the distance d, the ratio between the metrics d and δ is very close to 1 when δ_0 is small. Finally notice that $\overline{\mathfrak{v}}_k(g) \subset \overline{\mathfrak{v}}_{k+r}(g)$ implies $\theta_\cap(\overline{\mathfrak{v}}_{k+r}(g), \overline{\mathfrak{v}}_k(g)^\perp) = 1$. ☐

Lemma 2.9 *Given $g_1, g_2 \in \mathscr{L}(V)$, $\hat{p} \in \mathbb{P}(g_1) \cap \mathbb{P}(g_2)$ and any unit vector $p \in \hat{p}$,*

$$d(\varphi_{g_1}(\hat{p}), \varphi_{g_2}(\hat{p})) \leq \max\{\frac{1}{\|g_1\, p\|}, \frac{1}{\|g_2\, p\|}\}\, \|g_1 - g_2\|.$$

Proof Applying Proposition 2.26 to the non-zero vectors $g_1\, p$ and $g_2\, p$, we get

$$\begin{aligned}
d(\varphi_{g_1}(\hat{p}), \varphi_{g_2}(\hat{p})) &\leq \| \frac{g_1\, p}{\|g_1\, p\|} - \frac{g_2\, p}{\|g_2\, p\|} \| \\
&\leq \max\{\|g_1\, p\|^{-1}, \|g_2\, p\|^{-1}\}\, \|g_1\, p - g_2\, p\| \\
&\leq \max\{\|g_1\, p\|^{-1}, \|g_2\, p\|^{-1}\}\, \|g_1 - g_2\|.
\end{aligned}$$

☐

The final four lemmas of this section apply to invertible linear maps in $\mathrm{GL}(V)$. They express the continuity of the map $g \mapsto \varphi_g$ with values in the space of Lipschitz or Hölder continuous maps on the projective space. These facts will be needed in Chap. 5.

Lemma 2.10 *Given $g_1, g_2 \in \mathrm{GL}(V)$, and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,*

$$\left|\frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} - \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})}\right| \leq C(g_1, g_2)\, \|g_1 - g_2\|,$$

where $C(g_1, g_2) := (\|g_1^{-1}\|^2 + \|g_2\|^2\, \|g_1^{-1}\|^2\, \|g_2^{-1}\|^2)\, (\|g_1\| + \|g_2\|)$.

Proof Given $p \in \hat{p}$ and $q \in \hat{q}$, by Proposition 2.27

$$\begin{aligned}
&\left|\frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} - \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})}\right| = \left|\frac{\|g_1 p \wedge g_1 v_p(q)\|}{\|g_1 p\|\,\|g_1 q\|} - \frac{\|g_2 p \wedge g_2 v_p(q)\|}{\|g_2 p\|\,\|g_2 q\|}\right| \\
&\quad \leq \frac{\|g_1 p \wedge g_1 v_p(q) - g_2 p \wedge g_2 v_p(q)\|}{\|g_1 p\|\,\|g_1 q\|} \\
&\qquad + \left|\frac{1}{\|g_1 p\|\,\|g_1 q\|} - \frac{1}{\|g_2 p\|\,\|g_2 q\|}\right| \|g_2 p \wedge g_2 v_p(q)\| \\
&\quad \leq \|g_1^{-1}\|^2\, \|g_1 p \wedge (g_1 v_p(q) - g_2 v_p(q))\| + \|g_1^{-1}\|^2\, \|(g_1 p - g_2 p) \wedge g_2 v_p(q)\| \\
&\qquad + \|g_1^{-1}\|^2\, \|g_2^{-1}\|^2\, (\|g_1 p\|\, \big|\|g_1 q\| - \|g_2 q\|\big| + \|g_2 q\|\, \big|\|g_1 p\| - \|g_2 p\|\big|)\, \|g_2\|^2 \\
&\quad \leq \|g_1^{-1}\|^2\, (\|g_1\| + \|g_2\|)\, \|g_1 - g_2\| \\
&\qquad + \|g_2\|^2\, \|g_1^{-1}\|^2\, \|g_2^{-1}\|^2\, (\|g_1\| + \|g_2\|)\, \|g_1 - g_2\| \\
&\quad = (\|g_1^{-1}\|^2 + \|g_2\|^2\, \|g_1^{-1}\|^2\, \|g_2^{-1}\|^2)\, (\|g_1\| + \|g_2\|)\, \|g_1 - g_2\|. \qquad \square
\end{aligned}$$

Lemma 2.11 *Given $g \in \mathrm{GL}(V)$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,*

$$\frac{1}{\|g\|^2\, \|g^{-1}\|^2} \leq \frac{\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q}))}{\delta(\hat{p}, \hat{q})} \leq \|g\|^2\, \|g^{-1}\|^2.$$

Proof Given $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$ consider unit vectors $p \in \hat{p}$, $q \in \hat{q}$ and set $v = v_p(q)$. We have $\|p\| = \|q\| = \|v\| = 1$ and $\langle p, v\rangle = 0$. This last relation implies $\|p \wedge v\| = 1$. Hence

$$\|gp \wedge gv\| = \|(\wedge_2 g)(p \wedge v)\| \geq \|(\wedge_2 g)^{-1}\|^{-1} \geq \|g^{-1}\|^{-2}.$$

Analogously

$$\|gp \wedge gv\| = \|(\wedge_2 g)(p \wedge v)\| \leq \|\wedge_2 g\| \leq \|g\|^2.$$

We also have

$$\|g^{-1}\|^{-2} \leq \|g\, p\|\, \|g\, q\| \leq \|g\|^2.$$

To finish the proof combine these inequalities with Proposition 2.27. $\square$

Given $g \in \mathrm{GL}(V)$, we define

$$\ell(g) := \max\{\log\|g\|, \log\|g^{-1}\|\}. \tag{2.24}$$

Lemma 2.12 *For every $g \in \mathrm{GL}(V)$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,*

$$-4\,\ell(g) \leq \log\left[\frac{\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q}))}{\delta(\hat{p}, \hat{q})}\right] \leq 4\,\ell(g).$$

Proof It follows from Lemma 2.11.

Lemma 2.13 *Given $g_1, g_2 \in \mathrm{GL}(V)$, $0 < \alpha \leq 1$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,*

$$\left| \left(\frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right)^{\alpha} - \left(\frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right)^{\alpha} \right| \leq C_1(g_1, g_2)\, \|g_1 - g_2\|,$$

where $C_1(g_1, g_2) = \alpha\ \max\{\|g_1\|\,\|g_1^{-1}\|, \|g_2\|\,\|g_2^{-1}\|\}^{2(1-\alpha)}\, C(g_1, g_2)$, and $C(g_1, g_2)$ stands for the constant in Lemma 2.10.

Proof Setting $\Delta_1 := \frac{\delta(\varphi_{g_1}\hat{p}, \varphi_{g_1}\hat{q})}{\delta(\hat{p},\hat{q})}$ and $\Delta_2 := \frac{\delta(\varphi_{g_2}\hat{p}, \varphi_{g_2}\hat{q})}{\delta(\hat{p},\hat{q})}$, from Lemmas 2.10 and 2.11 we get

$$\begin{aligned} \left|\Delta_1^{\alpha} - \Delta_2^{\alpha}\right| &\leq \alpha\ \max\{\Delta_1^{\alpha-1}, \Delta_2^{\alpha-1}\}\left|\Delta_1 - \Delta_2\right| \\ &\leq \alpha\ \max\{\|g_1\|\,\|g_1^{-1}\|, \|g_2\|\,\|g_2^{-1}\|\}^{2(1-\alpha)}\left|\Delta_1 - \Delta_2\right| \\ &\leq \alpha\ \max\{\|g_1\|\,\|g_1^{-1}\|, \|g_2\|\,\|g_2^{-1}\|\}^{2(1-\alpha)}\, C(g_1, g_2)\, \|g_1 - g_2\|. \quad \square \end{aligned}$$

2.4 Avalanche Principle

Consider a long chain of n linear maps $g_0 : V_0 \to V_1$, $g_1 : V_1 \to V_2$, etc., between Euclidean spaces V_i of the same dimension m. The AP relates the expansion $\|g_{n-1} \ldots g_1\, g_0\|$ of the composition $g_{n-1} \ldots g_1\, g_0$ with the product of the individual expansions $\|g_{n-1}\| \ldots \|g_1\|\,\|g_0\|$. Given two quantities M_n and N_n depending on a large number $n \in \mathbb{N}$, we say in rough terms that they are ε-asymptotic, and write $M_n \overset{\varepsilon}{\asymp} N_n$, when $e^{-n\,\varepsilon} \leq M_n/N_n \leq e^{n\,\varepsilon}$. In general it is not true that $\|g_{n-1} \ldots g_1\, g_0\| \overset{\varepsilon}{\asymp} \|g_{n-1}\| \ldots \|g_1\|\,\|g_0\|$ for some small $\varepsilon > 0$, unless some atypically sharp alignment of the singular directions of the linear maps g_j occurs. Given the chain of linear maps $g_0, g_1, \ldots, g_{n-1}$, its *rift* $\rho(g_0, \ldots, g_{n-1}) := \frac{\|g_{n-1} \cdots g_0\|}{\|g_{n-1}\| \cdots \|g_0\|} \in [0, 1]$ measures the break of expansion in the composition $g_{n-1} \ldots g_1\, g_0$. The AP says that given any such chain $g_0, g_1, \ldots, g_{n-1}$, where the gap ratio of each map g_j is large, and the rift of any pair of consecutive maps is never too small, the rift of the composition behaves multiplicatively, in the sense that for some small number $\varepsilon > 0$,

$$\rho(g_0, g_1, \ldots, g_{n-1}) \overset{\varepsilon}{\asymp} \rho(g_0, g_1)\, \rho(g_1, g_2) \ldots \rho(g_{n-2}, g_{n-1}),$$

or, equivalently,

$$\frac{\|g_{n-1} \ldots g_1\, g_0\|\, \|g_1\| \ldots \|g_{n-2}\|}{\|g_1\, g_0\| \ldots \|g_{n-1}\, g_{n-2}\|} \overset{\varepsilon}{\asymp} 1.$$

The AP was introduced by Goldstein and Schlag [6, Proposition 2.2] as a technique to obtain Höder continuity of the integrated density of states for quasi-periodic

Schrödinger cocycles. In its original version, the AP applies to chains of unimodular matrices in $\mathrm{SL}(2, \mathbb{C})$, and the length of the chain is assumed to be less than some lower bound on the norms of the matrices. Note that for unimodular matrices, the gap ratio and the norm are two equivalent measurements. Still in this unimodular setting, for matrices in $\mathrm{SL}(2, \mathbb{R})$, Bourgain and Jitomirskaya [4, Lemma 5] relaxed the constraint on the length of the chain of matrices, and later Bourgain [3, Lemma 2.6] removed it, at the cost of slightly weakening the conclusion of the AP.

Later, Schlag [7, Lemma 1] generalized the AP to invertible matrices in $\mathrm{GL}(m, \mathbb{C})$. Recently, C. Sadel has shared with the authors an earlier draft of [1], containing his version of the AP for $\mathrm{GL}(m, \mathbb{C})$ matrices. Both of these higher dimensional APs assume some bound on the length of the chains of matrices. A higher dimensional AP without this assumption was proven by the authors [5, Theorem 3.1] for invertible real matrices.

We present here a more general AP, which holds for (possibly non-invertible) matrices in $\mathrm{Mat}(m, \mathbb{R})$. As a by-product of the geometric approach used in the proof, we also obtain a quantitative control on the most expanding directions of the matrix product, something essential in the proof of the continuity of the Oseledets decomposition.

2.4.1 Contractive Shadowing

Here we prove a *shadowing lemma* saying that under some conditions, a loose pseudo-orbit of a chain of contracting maps is shadowed by a true orbit of the mapping sequence. In particular, a closed pseudo-orbit is shadowed by a periodic orbit of the mapping chain.

Given a metric space (X, d), denote the closed ε-ball around $x \in X$ by

$$B(x, \varepsilon) := \{z \in X : \ d(z, x) \leq \varepsilon\}.$$

Given an open set $X^0 \subset X$, define

$$X^0(\varepsilon) := \{x \in X^0 : \ d(x, \partial X^0) \geq \varepsilon\},$$

where ∂X^0 denotes the topological boundary of X^0 in (X, d).

Lemma 2.14 (shadowing lemma) *Consider* $\varepsilon > 0$ *and* $0 < \delta < \kappa < 1$ *such that* $\delta/(1-\kappa) < \varepsilon < 1/2$.

Given a family $\{(X_j, d_j)\}_{0 \leq j \leq n}$ *of compact metric spaces with diameter* 1, *a chain of continuous mappings* $\{g_j : X_j^0 \to X_{j+1}\}_{0 \leq j \leq n-1}$ *defined on open sets* $X_j^0 \subset X_j$, *and a sequence of points* $x_j \in X_j$, *assume that for every* $0 \leq j \leq n-1$*:*

(a) $x_j \in X_j^0$ *and* $d(x_j, \partial X_j^0) = 1$,
(b) g_j *has Lipschitz constant* $\leq \kappa$ *on* $X_j^0(\varepsilon)$,
(c) $g_j(x_j) \in X_{j+1}^0(2\,\varepsilon)$,
(d) $g_j(X_j^0(\varepsilon)) \subset B(g_j(x_j), \delta)$.

Then, setting $g^{(n)} := g_{n-1} \circ \cdots \circ g_1 \circ g_0$, *the following hold:*

(1) *the composition* $g^{(n)}$ *is defined on* $B(x_0, \varepsilon)$ *and* $\mathrm{Lip}(g^{(n)}|_{B(x_0,\varepsilon)}) \leq \kappa^n$,
(2) $d(\,g_{n-1}(x_{n-1}),\; g^{(n)}(x_0)\,) \leq \frac{\delta}{1-\kappa}$,
(3) *if* $x_0 = g_{n-1}(x_{n-1})$ *then* $g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)$ *and there is a point* $x^* \in B(x_0, \varepsilon)$ *such that* $g^{(n)}(x^*) = x^*$ *and* $d\,(x_0, x^*) \leq \frac{\delta}{(1-\kappa)(1-\kappa^n)}$.

Proof The proof's inductive scheme is better understood with the help of Fig. 2.1 (see also Fig. 2.2), where we set $z_j^i := (g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i)(x_i)$ for $i \leq j \leq n$, with the convention that this composition is the identity when $i = j$. Of course we have to prove that all points z_j^i are well-defined.

The boxed expressions represent upper bounds on the distance between the points respectively above and below the box. The ith row represents the orbit of $x_i \in X_i$ by the chain of mappings $\{g_j\}_{j\geq i}$. All points in the jth column belong to the space X_j.

To explain the last upper bound at the bottom of each column, first notice that $z_i^i = x_i$. By (a), $z_i^{i-1} = g_{i-1}(x_{i-1})$ is well-defined, and by (c), $z_i^{i-1} \in X_i^0(2\,\varepsilon) \subset X_i^0(\varepsilon)$. Likewise $z_{i-1}^{i-2} \in X_{i-1}^0(\varepsilon)$, and $z_i^{i-2} = g_{i-1}(g_{i-2}(x_{i-2}))$ is well-defined. Then by (d) we have

$$d(z_i^{i-1}, z_i^{i-2}) = d(g_{i-1}(x_{i-1}), g_{i-1}(g_{i-2}(x_{i-2}))) \leq \delta. \tag{2.25}$$

$$
\begin{array}{ccccccccccccc}
X_0 & & X_1 & & X_2 & & X_3 & & \ldots & & X_{n-1} & & X_n \\
\hline
z_0^0 & \xrightarrow{g_0} & z_1^0 & \xrightarrow{g_1} & z_2^0 & \xrightarrow{g_2} & z_3^0 & \xrightarrow{g_3} & \ldots & \xrightarrow{g_{n-2}} & z_{n-1}^0 & \xrightarrow{g_{n-1}} & z_n^0 \\
 & & & & \boxed{\delta} & & \boxed{\kappa\,\delta} & & & & \boxed{\kappa^{n-3}\delta} & & \boxed{\kappa^{n-2}\delta} \\
 & & z_1^1 & \xrightarrow{g_1} & z_2^1 & \xrightarrow{g_2} & z_3^1 & \xrightarrow{g_3} & \ldots & \xrightarrow{g_{n-2}} & z_{n-1}^1 & \xrightarrow{g_{n-1}} & z_n^1 \\
 & & & & & & \boxed{\delta} & & & & \boxed{\kappa^{n-4}\delta} & & \boxed{\kappa^{n-3}\delta} \\
 & & & & z_2^2 & \xrightarrow{g_2} & z_3^2 & \xrightarrow{g_3} & \ldots & \xrightarrow{g_{n-2}} & z_{n-1}^2 & \xrightarrow{g_{n-1}} & z_n^2 \\
 & & & & & & & & & & \boxed{\kappa^{n-5}\delta} & & \boxed{\kappa^{n-4}\delta} \\
 & & & & & & z_3^3 & \xrightarrow{g_3} & \ldots & \xrightarrow{g_{n-2}} & z_{n-1}^3 & \xrightarrow{g_{n-1}} & z_n^3 \\
 & & & & & & & \ddots & & & \vdots & & \vdots \\
 & & & & & & & & z_{n-2}^{n-2} & \xrightarrow{g_{n-2}} & z_{n-1}^{n-2} & \xrightarrow{g_{n-1}} & z_n^{n-2} \\
 & & & & & & & & & & & & \boxed{\delta} \\
 & & & & & & & & & & z_{n-1}^{n-1} & \xrightarrow{g_{n-1}} & z_n^{n-1} \\
 & & & & & & & & & & & & z_n^n \\
\hline
\end{array}
$$

Fig. 2.1 Family of orbits for the chain of mappings $\{g_j : X_j^0 \to X_{j+1}\}_j$

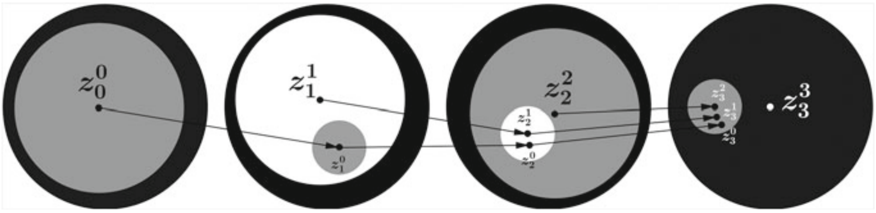

Fig. 2.2 Shadowing property for a chain of contractive mappings

All other bounds are obtained applying (b) inductively. More precisely, we prove by induction in the column index j that

(i) all points z_j^i in the jth column are well-defined and belong to $X_j^0(\varepsilon)$,
(ii) distances between consecutive points in the column j are bounded by the expressions in Fig. 2.1, i.e., for all $1 \le i \le j-1$,

$$d(z_j^{i-1}, z_j^i) \le \kappa^{j-i-1}\,\delta. \tag{2.26}$$

The initial inductive steps, $j = 0, 1, 2$, follow from (a), (c) and (2.25). Assume now that the points z_j^i in jth column satisfy (i) and (ii). Then their images $z_{j+1}^i = g_j(z_j^i)$ are well-defined. By (b) we have for all $1 \le i \le j-1$,

$$d(z_{j+1}^{i-1}, z_{j+1}^i) = d(g_j(z_j^{i-1}), g_j(z_j^i)) \le \kappa\, d(z_j^{i-1}, z_j^i) \le \kappa^{j-i}\,\delta.$$

Together with (2.25) this proves (ii) for the column $j+1$. To prove (i) consider any $1 \le i \le j$. By (c) and the triangle inequality,

$$\begin{aligned} d(z_{j+1}^i, \partial X_{j+1}^0) &\ge d(z_{j+1}^j, \partial X_{j+1}^0) - d(z_{j+1}^i, z_{j+1}^j) \\ &\ge d(g_j(x_j), \partial X_{j+1}^0) - \sum_{l=i+1}^{j} d(z_{j+1}^{l-1}, z_{j+1}^l) \\ &\ge 2\,\varepsilon - \sum_{l=i+1}^{j} \kappa^{j-l}\,\delta \ge 2\,\varepsilon - \frac{\delta}{1-\kappa} \ge \varepsilon. \end{aligned}$$

This proves (i) for the column $j+1$, and concludes the induction.

Conclusion (1) follows from (b) and the following claim, to be proved by induction in i.

For every $i = 0, 1, \ldots, n-1$, $g^{(i)}(B(x_0, \varepsilon)) \subset X_i^0(\varepsilon)$, where $g^{(i)} = g_{i-1} \circ \cdots \circ g_0$.

Consider first the case $i = 0$. Given $x \in B(x_0, \varepsilon)$,

$$d(x, \partial X_0^0) \ge d(x_0, \partial X_0^0) - d(x, x_0) \ge 1 - \varepsilon > \varepsilon.$$

This implies that $d(g_0(x), g_0(x_0)) \le \kappa\, d(x, x_0)$. Thus

$$
\begin{aligned}
d(g_0(x), \partial X_1^0) &\geq d(g_0(x_0), \partial X_1^0) - d(g_0(x_0), g_0(x)) \geq 2\,\varepsilon - d(g_0(x_0), g_0(x)) \\
&\geq 2\,\varepsilon - \kappa\, d(x_0, x) \geq 2\,\varepsilon - \kappa\,\varepsilon > \varepsilon
\end{aligned}
$$

which proves that $g_0(B(x_0, \varepsilon)) \subset X_1^0(\varepsilon)$.

Assume now that for every $l \leq i - 1$,

$$(g_l \circ \cdots \circ g_0)(B(x_0, \varepsilon)) \subset X_{l+1}^0(\varepsilon).$$

By (b), $g^{(i)}$ acts as a κ^i contraction on $B(x_0, \varepsilon)$ and $g^{(i)}(B(x_0, \varepsilon)) \subset X_i^0(\varepsilon)$. Thus for every $x \in B(x_0, \varepsilon)$,

$$
\begin{aligned}
d(g^{(i+1)}(x), \partial X_{i+1}^0) &\geq d(g_i(x_i), \partial X_{i+1}^0) - d(g_i(x_i), g^{(i+1)}(x)) \\
&\geq 2\,\varepsilon - d(z_{i+1}^0, z_{i+1}^i) - d(z_{i+1}^0, g^{(i+1)}(x)) \\
&\geq 2\,\varepsilon - \sum_{l=0}^{i-1} d(z_{i+1}^l, z_{i+1}^{l+1}) - d(g^{(i+1)}(x_0), g^{(i+1)}(x)) \\
&\geq 2\,\varepsilon - (\delta + \kappa\,\delta + \cdots + \kappa^{i-1}\,\delta) - \kappa^i\, d(x_0, x) \\
&\geq 2\,\varepsilon - (\delta + \kappa\,\delta + \cdots + \kappa^{i-1}\,\delta) - \kappa^i\,\varepsilon \\
&\geq 2\,\varepsilon - (1 - \kappa)\,\varepsilon\,(1 + \kappa + \cdots + \kappa^{i-1}) - \kappa^i\,\varepsilon = \varepsilon
\end{aligned}
$$

which proves that $g^{(i+1)}(B(x_0, \varepsilon)) \subset X_{i+1}^0(\varepsilon)$, and establishes the claim above.

Thus $g^{(n)}$ is well-defined on $B(x_0, \varepsilon)$, and, because of assumption (b), $g^{(n)}$ is a κ^n Lipschitz contraction on this ball. This proves (1).

Item (2) follows by (2.26). In fact

$$d(g_{n-1}(x_{n-1}), g^{(n)}(x_0)) = d(z_n^{n-1}, z_n^0) \leq \sum_{l=1}^{n-1} d(z_n^l, z_n^{l-1}) \leq \sum_{l=1}^{n-1} \kappa^{n-l-1}\,\delta \leq \frac{\delta}{1 - \kappa}.$$

Finally we prove (3). Assume $x_0 = g_{n-1}(x_{n-1})$.

It is enough to see that $g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)$, because by (1) $g^{(n)}$ acts as a κ^n-contraction in the closed ball $B(x_0, \varepsilon)$. The conclusion on the existence of a fixed point, as well as the proximity bound, follow from the classical fixed point theorem for Lipschitz contractions.

Given $x \in B(x_0, \varepsilon)$, we know from the previous calculation that

$$d(x_0, g^{(n)}(x_0)) < \delta + \kappa\,\delta + \cdots + \kappa^{n-2}\,\delta.$$

Hence

$$\begin{aligned}
d(g^{(n)}(x), x_0) &\le d(g^{(n)}(x), g^{(n)}(x_0)) + d(g^{(n)}(x_0), x_0)\\
&\le \kappa^{n-1}\, d(x, x_0) + \delta + \kappa\,\delta + \cdots + \kappa^{n-2}\,\delta\\
&\le \delta + \kappa\,\delta + \cdots + \kappa^{n-2}\,\delta + \kappa^{n-1}\,\varepsilon\\
&\le (1-\kappa)\,\varepsilon\,(1 + \kappa + \cdots + \kappa^{n-2}) + \kappa^{n-1}\,\varepsilon\\
&= (1-\kappa)\,\varepsilon\,\frac{1-\kappa^{n-1}}{1-\kappa} + \kappa^{n-1}\,\varepsilon = \varepsilon.
\end{aligned}$$

Thus $g^{(n)}(x) \in B(x_0, \varepsilon)$, which proves that $g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)$. □

2.4.2 Statement and Proof of the AP

In the statement and proof of the AP we will use the notation introduced in Sect. 2.2.3. Given a chain of linear mappings $\{g_j : V_j \to V_{j+1}\}_{0\le j\le n-1}$ we denote the composition of the first i maps by $g^{(i)} := g_{i-1}\ \ldots\ g_1\, g_0$. Throughout this chapter, $a \lesssim b$ will stand for $a \le C\, b$ for some absolute constant C.

Theorem 2.1 (Avalanche Principle) *There exists a constant $c > 0$ such that given $0 < \varepsilon < 1,\ 0 < \kappa \le c\,\varepsilon^2$ and a chain of linear mappings $\{g_j : V_j \to V_{j+1}\}_{0\le j\le n-1}$ between Euclidean spaces V_j, if*

(a) $\sigma(g_i) \le \kappa$, *for* $0 \le i \le n-1$, *and*
(b) $\alpha(g_{i-1}, g_i) \ge \varepsilon$, *for* $1 \le i \le n-1$,

then

(1) $d(\overline{\mathfrak{v}}(g^{(n)}), \overline{\mathfrak{v}}(g_0)) \lesssim \kappa\,\varepsilon^{-1}$,
(2) $d(\overline{\mathfrak{v}}(g^{(n)*}), \overline{\mathfrak{v}}(g^*_{n-1})) \lesssim \kappa\,\varepsilon^{-1}$,
(3) $\sigma(g^{(n)}) \lesssim \kappa\, \left(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2}\right)^{n-1}$,
(4) $\left|\log\|g^{(n)}\| + \sum_{i=1}^{n-2} \log\|g_i\| - \sum_{i=1}^{n-1} \log\|g_i\, g_{i-1}\|\right| \lesssim n\,\frac{\kappa}{\varepsilon^2}$.

Remark 2.3 (On the assumptions) Assumption (a) says that the (first) gap ratio of each g_j is large, $\mathrm{gr}(g_j) \ge \kappa^{-1}$. Given (a), by Propositions 2.23 and 2.24, assumption (b) is equivalent to a condition on the rift, $\rho(g_{j-1}, g_j) \gtrsim \varepsilon$ for all $j = 1, \ldots, n-1$.

Remark 2.4 (On the conclusions) Conclusions (1) and (2) say that the most expanding direction $\overline{\mathfrak{v}}(g^{(n)})$ of the product $g^{(n)}$, and its image $\varphi_{g^{(n)}}\overline{\mathfrak{v}}(g^{(n)})$, are respectively κ/ε-close to the most expanding direction $\overline{\mathfrak{v}}(g_0)$ of g_0, and to the image $\varphi_{g_{n-1}}\overline{\mathfrak{v}}(g_{n-1})$ of the most expanding direction of g_{n-1}. Conclusion (3) says that the composition map $g^{(n)}$ has a large gap ratio. Finally, conclusion (4) is equivalent to

$$e^{-n\,C\,\kappa\,\varepsilon^{-2}} \leq \frac{\|g_{n-1}\,\dots\,g_1\,g_0\|\,\|g_1\|\,\dots\,\|g_{n-2}\|}{\|g_1\,g_0\|\,\dots\,\|g_{n-1}\,g_{n-2}\|} \leq e^{n\,C\,\kappa\,\varepsilon^{-2}},$$

for some universal constant $C > 0$. These inequalities describe the asymptotic almost multiplicative behavior of the rifts

$$\rho(g_0, g_1, \dots, g_{n-1}) \overset{C\,\kappa/\varepsilon^2}{\asymp} \rho(g_0, g_1)\,\rho(g_1, g_2)\,\dots\,\rho(g_{n-2}, g_{n-1}).$$

Proof The strategy of the proof is to look at the contracting action of linear mappings g_j on the projective space.

For each $j = 0, 1, \dots, n$ consider the compact metric space $X_j = \mathbb{P}(V_j)$ with the normalized Riemannian distance, $\mathrm{d}(\hat{u}, \hat{v}) = \frac{2}{\pi}\,\rho(\hat{u}, \hat{v})$. The reader should be warned of the notational similarity between this projective metric and the one defined in (2.2). We do not refer to the metric (2.2) in this proof. However, the distance in the statement of the AP can be understood as any of the four equivalent projective distances δ, d, ρ or d. For $0 \leq j < n$ define

$$X_j^0 := \{\hat{v} \in X_j :\ \alpha(\hat{v}, \overline{\mathfrak{v}}(g_j)) > 0\},$$
$$Y_j^0 := \{\hat{v} \in X_j :\ \alpha(\hat{v}, \overline{\mathfrak{v}}(g_{j-1}^*)) > 0\}.$$

The domain of the projective map $\varphi_{g_j} : \mathbb{P}(g_j) \subset X_j \to X_{j+1}$ clearly contains the open set X_j^0. Analogously, the domain of $\varphi_{g_{j-1}^*} : \mathbb{P}(g_j^*) \subset X_j \to X_{j-1}$ contains Y_j^0. We will apply Lemma 2.14 to chains of projective maps formed by the mappings $\varphi_{g_j} : X_j^0 \to X_{j+1}$ and their adjoints $\varphi_{g_{j-1}^*} : Y_j^0 \to X_{j-1}$.

Take positive numbers ε and κ such that $0 < \kappa \ll \varepsilon^2$, let $r := \sqrt{1 - \varepsilon^2/4}$, and define the following input parameters for the application of Lemma 2.14,

$$\varepsilon_{\mathrm{sh}} := \frac{1}{\pi}\,\arcsin\varepsilon,$$
$$\kappa_{\mathrm{sh}} := \kappa\,\frac{r + \sqrt{1 - r^2}}{1 - r^2} \asymp \frac{4\,\kappa}{\varepsilon^2},$$
$$\delta_{\mathrm{sh}} := \frac{\kappa\,r}{\sqrt{1 - r^2}} \asymp \frac{2\,\kappa}{\varepsilon}.$$

A simple calculation shows that there exists $0 < c < 1$ such that for any $0 < \varepsilon < 1$ and $0 < \kappa \leq c\,\varepsilon^2$, the pre-conditions $0 < \delta_{\mathrm{sh}} < \kappa_{\mathrm{sh}} < 1$ and $\frac{\delta_{\mathrm{sh}}}{1-\kappa_{\mathrm{sh}}} < \varepsilon_{\mathrm{sh}} < 1/2$ of the shadowing lemma are satisfied.

Define $x_j = \overline{\mathfrak{v}}(g_j)$ and $x_j^* = \overline{\mathfrak{v}}(g_{j-1}^*)$. This lemma is going to be applied to the following chains of maps and sequences of points

(A) $\varphi_{g_0}, \dots, \varphi_{g_{n-1}}, \varphi_{g_{n-1}^*}, \dots, \varphi_{g_0^*}, \quad x_0, \dots, x_{n-1}, x_n^*, \dots, x_1^*,$

(B) $\varphi_{g_{n-1}^*}, \dots, \varphi_{g_0^*}, \varphi_{g_0}, \dots, \varphi_{g_{n-1}}, \quad x_n^*, \dots, x_1^*, x_0, \dots, x_{n-1},$

from which we will infer the conclusions (1) and (2). Let us check now that assumptions (a)–(d) of Lemma 2.14 hold in both cases (A) and (B).

By definition $\partial X_j^0 := \{\hat{v} \in X_j : \alpha(\hat{v}, x_j) = 0\} = \{\hat{v} \in X_j : \hat{v} \perp x_j\}$. Hence, if $\hat{v} \in \partial X_j^0$ then $\mathrm{d}(x_j, \hat{v}) = 1$, which proves that $\mathrm{d}(x_j, \partial X_j^0) = 1$. Analogously, $\partial Y_j^0 = \{\hat{v} \in X_j : \hat{v} \perp x_j^*\}$ and $\mathrm{d}(x_j^*, \partial Y_j^0) = 1$. Therefore assumption (a) holds.

By definition of $X_j^0(\varepsilon)$,

$$\begin{aligned}\hat{v} \in X_j^0(\varepsilon) \;\Leftrightarrow\; \mathrm{d}(\hat{v}, \partial X_j^0) \geq \varepsilon \;\Leftrightarrow\; \rho(\hat{v}, \partial X_j^0) \geq \frac{\pi}{2}\,\varepsilon \\ \Leftrightarrow\; \delta(\hat{v}, \partial X_j^0) = \alpha(\hat{v}, x_j) \geq \sin\left(\frac{\pi}{2}\,\varepsilon\right) \\ \Leftrightarrow\; \delta(\hat{v}, x_j) \leq \cos\left(\frac{\pi}{2}\,\varepsilon\right).\end{aligned}$$

Similarly, by definition of $Y_j^0(\varepsilon)$,

$$\hat{v} \in Y_j^0(\varepsilon) \;\Leftrightarrow\; \delta(\hat{v}, x_j^*) \leq \cos\left(\frac{\pi}{2}\,\varepsilon\right).$$

Thus, because

$$\cos\left(\frac{\pi}{2}\,\varepsilon_{\mathrm{sh}}\right) = \cos\left(\frac{1}{2}\,\arcsin\varepsilon\right) \leq \sqrt{1 - \frac{\varepsilon^2}{4}} = r,$$

we have $X_j^0(\varepsilon_{\mathrm{sh}}) \subset B^{(\delta)}(x_j, r)$ and $Y_j^0(\varepsilon_{\mathrm{sh}}) \subset B^{(\delta)}(x_j^*, r)$, and assumption (b) holds by Proposition 2.29 (3).

By the gap assumption,

$$\alpha(\varphi_{g_j}(x_j), x_{j+1}) = \alpha(\overline{\mathfrak{v}}(g_j^*), \overline{\mathfrak{v}}(g_{j+1})) = \alpha(g_j, g_{j+1}) \geq \varepsilon.$$

Therefore

$$\begin{aligned}\mathrm{d}(\varphi_{g_j}(x_j), \partial X_{j+1}^0) &= \frac{2}{\pi}\,\arcsin\delta(\varphi_{g_j}(x_j), \partial X_{j+1}^0) = \frac{2}{\pi}\,\arcsin\alpha(\varphi_{g_j}(x_j), x_{j+1}) \\ &\geq \frac{2}{\pi}\,\arcsin\varepsilon = 2\,\varepsilon_{\mathrm{sh}}.\end{aligned}$$

Similarly, by the gap assumption,

$$\alpha(\varphi_{g_{j-1}^*}(x_j^*), x_{j-1}^*) = \alpha(\overline{\mathfrak{v}}(g_{j-1}), \overline{\mathfrak{v}}(g_{j-1}^*)) = \alpha(g_{j+1}^*, g_j^*) = \alpha(g_j, g_{j+1}) \geq \varepsilon,$$

and in the same way we infer that

$$\mathrm{d}(\varphi_{g_{j-1}^*}(x_j^*), \partial Y_{j-1}^0) \geq \frac{2}{\pi}\,\arcsin\varepsilon = 2\,\varepsilon_{\mathrm{sh}}.$$

This proves that (c) of the shadowing lemma holds. Notice that in both cases (A) and (B), the assumption (c) holds trivially for the middle points, because $\varphi_{g_{n-1}}(x_{n-1}) = x_n^* \in Y_n^0(2\,\varepsilon_{\rm sh})$ and $\varphi_{g_0^*}(x_1^*) = x_0 \in X_0^0(2\,\varepsilon_{\rm sh})$.

It was proved above that $X_j^0(\varepsilon_{\rm sh}) \subset B^{(\delta)}(x_j, r)$ and $Y_j^0(\varepsilon_{\rm sh}) \subset B^{(\delta)}(x_j^*, r)$. By (2.5) we have $\mathrm{d}(\hat{u}, \hat{v}) \le \delta(\hat{u}, \hat{v})$. Thus by Proposition 2.29 (1),

$$\varphi_{g_j}(X_j^0(\varepsilon_{\rm sh})) \subset B^{(\delta)}(x_j^*, \delta_{\rm sh}) \subset B^{(\mathrm{d})}(x_j^*, \delta_{\rm sh}) \;\text{ with }\; x_j^* = \varphi_{g_j}(x_j),$$

and analogously,

$$\varphi_{g_{j-1}^*}(Y_j^0(\varepsilon_{\rm sh})) \subset B^{(\delta)}(x_{j-1}, \delta_{\rm sh}) \subset B^{(\mathrm{d})}(x_{j-1}, \delta_{\rm sh}) \;\text{ with }\; x_{j-1} = \varphi_{g_{j-1}^*}(x_j^*).$$

Hence, (d) of Lemma 2.14 holds.

Therefore, because $\varphi_{g_0^*}(x_1^*) = x_0$ and $\varphi_{g_{n-1}}(x_{n-1}) = x_n^*$, conclusion (3) of Lemma 2.14 holds for both chains (A) and (B). The projective points $\overline{\mathfrak{v}}(g^{(n)})$ and $\overline{\mathfrak{v}}(g^{(n)*})$ are the unique fixed points of the chains of mappings (A) and (B), respectively. Hence, by the shadowing lemma both distances $\mathrm{d}(x_0, \overline{\mathfrak{v}}(g^{(n)}))$ and $\mathrm{d}(x_n^*, \overline{\mathfrak{v}}(g^{(n)*}))$ are bounded above by

$$\frac{\delta_{\rm sh}}{(1-\kappa_{\rm sh})\,(1-\kappa_{\rm sh}^{2n})} \asymp \delta_{\rm sh} \asymp \frac{\kappa}{\varepsilon}.$$

This proves conclusions (1) and (2) of the AP.

From Proposition 2.28 we infer that for any $g \in \mathscr{L}(V)$,

$$\|(D\varphi_g)_{\overline{\mathfrak{v}}(g)}\| = \frac{s_2(g)}{\|g\|} = \sigma(g).$$

Hence, by conclusion (1) of the shadowing lemma

$$\begin{aligned}\sigma(g^{(n)}) = \|(D\varphi_{g^{(n)}})_{\overline{\mathfrak{v}}(g^{(n)})}\| &\le \mathrm{Lip}(\varphi_{g^{(n)}}|_{B(\overline{\mathfrak{v}}(g_0),\varepsilon_{\rm sh})}) \\ &\le (\kappa_{\rm sh})^n \le \left(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2}\right)^n.\end{aligned}$$

On the other hand, by (1) the distance from $\overline{\mathfrak{v}}(g^{(n)})$ to $\overline{\mathfrak{v}}(g_0)$ is of order $\kappa\,\varepsilon^{-1} \ll \varepsilon$ and

$$\mathrm{Lip}(\varphi_{g_0}|_{B(\overline{\mathfrak{v}}(g_0),\kappa\,\varepsilon^{-1})}) \lesssim \|(D\varphi_{g_0})_{\overline{\mathfrak{v}}(g_0)}\| = \sigma(g_0) \le \kappa.$$

Therefore

$$\sigma(g^{(n)}) \lesssim \kappa\,(\kappa_{\rm sh})^{n-1} \le \kappa\,\left(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2}\right)^{n-1},$$

which proves conclusion (3) of the AP.

Before proving (4), notice that applying (3) to the chain of linear maps $g_0, \ldots, g_{i-1}$ we get that $g^{(i)} := g_{i-1} \ldots g_0$ has a first gap ratio for all $i = 1, \ldots, n$.

We claim that

$$\left|\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)\right| \lesssim \kappa\, \varepsilon^{-1}. \tag{2.27}$$

By (2) of the AP, applied to the chain of linear maps $g_0, \ldots, g_{i-1}$,

$$\mathrm{d}(\overline{\mathfrak{v}}(g^{(i)*}), \overline{\mathfrak{v}}(g^*_{i-1})) \leq \frac{\delta_{\mathrm{sh}}}{(1-\kappa_{\mathrm{sh}})(1-\kappa_{\mathrm{sh}}^{2i})} \lesssim \kappa\, \varepsilon^{-1}.$$

Hence, by Proposition 2.35

$$\begin{aligned}\left|\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)\right| &= \left|\alpha(\overline{\mathfrak{v}}(g^{(i)*}), \overline{\mathfrak{v}}(g_i)) - \alpha(\overline{\mathfrak{v}}(g^*_{i-1}), \overline{\mathfrak{v}}(g_i))\right| \\ &\leq \mathrm{d}(\overline{\mathfrak{v}}(g^{(i)*}), \overline{\mathfrak{v}}(g^*_{i-1})) \lesssim \kappa\, \varepsilon^{-1}.\end{aligned}$$

For any i, the logarithm of any ratio between the four factors $\alpha(g^{(i)}, g_i)$, $\beta(g^{(i)}, g_i)$, $\alpha(g_{i-1}, g_i)$ and $\beta(g_{i-1}, g_i)$ is of order $\kappa\, \varepsilon^{-2}$. In fact, by (2.27)

$$\left|\log \frac{\alpha(g^{(i)}, g_i)}{\alpha(g_{i-1}, g_i)}\right| \lesssim \frac{1}{\varepsilon} \left|\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)\right| \leq \kappa\, \varepsilon^{-2}.$$

By hypothesis (a), $\sigma(g_i) \leq \kappa$. From conclusion (3) we also have $\sigma(g^{(i)}) < \kappa$, provided we make the constant c small enough. Hence by Lemma 2.4,

$$\left|\log \frac{\beta(g_{i-1}, g_i)}{\alpha(g_{i-1}, g_i)}\right| \lesssim \frac{\kappa^2}{\varepsilon^2} \quad \text{and} \quad \left|\log \frac{\beta(g^{(i)}, g_i)}{\alpha(g^{(i)}, g_i)}\right| \lesssim \frac{\kappa^2}{\varepsilon^2}.$$

Since $\kappa^2 \varepsilon^{-2} \ll \kappa\, \varepsilon^{-2}$, the logarithms of the other ratios between the factors above are all $\lesssim \kappa\, \varepsilon^{-2}$. Thus, for some universal constant $C > 0$, each of these ratios is inside the interval $[e^{-C\kappa\,\varepsilon^{-2}}, e^{C\kappa\,\varepsilon^{-2}}]$.

Finally, applying Proposition 2.25 to the rifts $\rho(g_0, \ldots, g_{n-1})$, $\rho(g_0, g_1)$, $\rho(g_1, g_2)$, etc., we have

$$e^{-nC\kappa\,\varepsilon^{-2}} \leq \prod_{i=1}^{n-1} \frac{\alpha(g^{(i)}, g_i)}{\beta(g_{i-1}, g_i)} \leq \frac{\rho(g_0, \ldots, g_{n-1})}{\prod_{i=1}^{n-1} \rho(g_{i-1}, g_i)} \leq \prod_{i=1}^{n-1} \frac{\beta(g^{(i)}, g_i)}{\alpha(g_{i-1}, g_i)} \leq e^{nC\kappa\,\varepsilon^{-2}},$$

which by Remark 2.4 is equivalent to (4). $\square$

Next proposition is a practical reformulation of the Avalanche Principle.

Proposition 2.42 *There exists $c > 0$ such that given $0 < \varepsilon < 1$, $0 < \kappa \leq c\,\varepsilon^2$ and $g_0, g_1, \ldots, g_{n-1} \in \mathrm{Mat}(m, \mathbb{R})$, if*

$$\begin{aligned} &\text{(gaps)}\ \ \mathrm{gr}(g_i) > \frac{1}{\kappa} && \textit{for all}\ \ 0 \leq i \leq n-1 \\ &\text{(angles)}\ \ \frac{\|g_i\, g_{i-1}\|}{\|g_i\|\,\|g_{i-1}\|} > \varepsilon && \textit{for all}\ \ 1 \leq i \leq n-1 \end{aligned}$$

then

$$\max\left\{ d(\overline{\mathfrak{v}}(g^{(n)*}), \overline{\mathfrak{v}}(g^*_{n-1})),\ d(\overline{\mathfrak{v}}(g^{(n)}), \overline{\mathfrak{v}}(g_0)) \right\} \lesssim \kappa\,\varepsilon^{-1}$$

$$\left| \log\|g^{(n)}\| + \sum_{i=1}^{n-2} \log\|g_i\| - \sum_{i=1}^{n-1} \log\|g_i g_{i-1}\| \right| \lesssim n\,\frac{\kappa}{\varepsilon^2}.$$

Proof Consider the constant $c > 0$ in Theorem 2.1, let $c' := c\,(1 - 2\,c^2)$ and assume $0 < \kappa \leq c'\,\varepsilon^2$.

Assumption (gaps) here is equivalent to assumption (a) of Theorem 2.1. By Proposition 2.24, the assumption (angles) here implies

$$\begin{aligned} \alpha(g_{i-1}, g_i) &\geq \rho(g_{i-1}, g_i)\sqrt{1 - \frac{2\,\kappa^2}{\rho(g_{i-1}, g_i)^2}} \\ &\geq \varepsilon\sqrt{1 - \frac{2\,\kappa^2}{\varepsilon^2}} \geq \varepsilon\,\sqrt{1 - 2\,c^2\,\varepsilon^2} =: \varepsilon', \end{aligned}$$

Since $0 < \kappa \leq c'\,\varepsilon^2$, and $c'\,\varepsilon^2 \leq c\,(1 - 2\,c^2\,\varepsilon^2)\,\varepsilon^2 = c\,(\varepsilon')^2$ we have $0 < \kappa \leq c\,(\varepsilon')^2$. Thus, because $\varepsilon \asymp \varepsilon'$, this proposition follows from conclusions (1), (2) and (4) of Theorem 2.1. □

2.4.3 Consequences of the AP

Given a chain of linear maps $\{g_j : V_j \to V_{j+1}\}_{0 \leq j \leq n-1}$ between Euclidean spaces V_j, and integers $0 \leq i < j \leq n$ we define

$$g^{(j,i)} := g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i.$$

With this notation the following relation holds for $0 \leq i < k < j \leq n$,

$$g^{(j,i)} = g^{(j,k)} \circ g^{(k,i)}.$$

Next proposition states, in a quantified way, that the most expanding directions $\overline{\mathfrak{v}}(g^{n,i)}) \in \mathbb{P}(V_i)$ are almost invariant under the adjoints of the chain mappings.

Proposition 2.43 *Under the assumptions of Theorem 2.1, where* $0 < \kappa \ll \varepsilon^2$,

$$d(\varphi_{g_i^*}\,\overline{\mathfrak{v}}(g^{(n,i+1)}),\overline{\mathfrak{v}}(g^{(n,i)})) \lesssim \frac{\kappa}{\varepsilon}\,(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2})^{n-i}.$$

Proof Consider κ, ε, $\kappa_{\rm sh}$ and $\varepsilon_{\rm sh}$ as in Theorem 2.1. From the proof of item (3) of the AP, applied to the chain of mappings $g_{n-1}^*,\ldots,g_i^*$, we conclude that the composition $g^{(n,i)} = g_i^*\circ\cdots\circ g_{n-1}^*$ is a $(\kappa_{\rm sh})^{n-i}$-Lipschitz contraction on the ball $B(\overline{\mathfrak{v}}(g_{n-1}^*),\varepsilon_{\rm sh})$. On the other hand, by (2) of the AP we have $\mathrm{d}(\,\overline{\mathfrak{v}}(g^{(n,i+1)*},\overline{\mathfrak{v}}(g_{n-1}^*)\,) \lesssim \kappa\,\varepsilon^{-1}$ and $\mathrm{d}(\,\overline{\mathfrak{v}}(g_{n-1}^*),\overline{\mathfrak{v}}(g^{(n,i)*}\,) \lesssim \kappa\,\varepsilon^{-1}$. Since $\kappa\,\varepsilon^{-1} \ll \varepsilon \asymp \varepsilon_{\rm sh}$, both projective points $\overline{\mathfrak{v}}(g^{(n,i)*})$ and $\overline{\mathfrak{v}}(g^{(n,i+1)*})$ belong to the ball $B(\overline{\mathfrak{v}}(g_{n-1}^*),\varepsilon_{\rm sh})$. Thus,

$$\begin{aligned}
&\mathrm{d}(\varphi_{g_i^*}\,\overline{\mathfrak{v}}(g^{(n,i+1)}),\overline{\mathfrak{v}}(g^{(n,i)}))\\
&\quad = \mathrm{d}(\,\varphi_{g_i^*}\circ\varphi_{g^{(n,i+1)*}}\,\overline{\mathfrak{v}}(g^{(n,i+1)*}),\ \varphi_{g^{(n,i)*}}\,\overline{\mathfrak{v}}(g^{(n,i)*})\,)\\
&\quad = \mathrm{d}(\,\varphi_{g^{(n,i)*}}\,\overline{\mathfrak{v}}(g^{(n,i+1)*}),\ \varphi_{g^{(n,i)*}}\,\overline{\mathfrak{v}}(g^{(n,i)*})\,)\\
&\quad \le (\kappa_{\rm sh})^{n-i}\,\mathrm{d}(\,\overline{\mathfrak{v}}(g^{(n,i+1)*},\overline{\mathfrak{v}}(g^{(n,i)*}\,)\\
&\quad \le (\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2})^{n-i}\,\big(\mathrm{d}(\,\overline{\mathfrak{v}}(g^{(n,i+1)*},\overline{\mathfrak{v}}(g_{n-1}^*)\,)+\mathrm{d}(\,\overline{\mathfrak{v}}(g_{n-1}^*),\overline{\mathfrak{v}}(g^{(n,i)*}\,)\big)\\
&\quad \lesssim \frac{2\,\kappa}{\varepsilon}\,(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2})^{n-i}.
\end{aligned}$$

which proves the proposition. □

Most expanding directions and norms of products of chains matrices under an application of the AP admit the following modulus of continuity.

Proposition 2.44 *Let* $c>0$ *be the universal constant in Theorem 2.1. Given numbers* $0<\varepsilon<1$ *and* $0<\kappa<c\,\varepsilon^2$, *and given two chains of matrices* $g_0,\ldots,g_{n-1}$ *and* $g_0',\ldots,g_{n-1}'$ *in* $\mathrm{Mat}(m,\mathbb{R})$, *both satisfying the assumptions of the AP for the given parameters* κ *and* ε, *if* $d_{\rm rel}(g_i,g_i')<\delta$ *for all* $i=0,1,\ldots,n-1$, *then*

(a) $d(\,\overline{\mathfrak{v}}(g_{n-1}\,\ldots\,g_0),\ \overline{\mathfrak{v}}(g_{n-1}'\,\ldots\,g_0')\,) \lesssim \frac{\kappa}{\varepsilon}+8\,\delta$,

(b) $\left|\log\dfrac{\|g_{n-1}\,\ldots\,g_0\|}{\|g_{n-1}'\,\ldots\,g_0'\|}\right| \lesssim n\left(\dfrac{\kappa}{\varepsilon^2}+\dfrac{\delta}{\varepsilon}\right)$.

Proof Item (a) follows from conclusion (1) of Theorem 2.1, and Proposition 2.40,

$$\begin{aligned}
d(\,\overline{\mathfrak{v}}(g_{n-1}\,\ldots\,g_0),\ \overline{\mathfrak{v}}(g_{n-1}'\,\ldots\,g_0')\,) &\le d(\,\overline{\mathfrak{v}}(g_{n-1}\,\ldots\,g_0),\ \overline{\mathfrak{v}}(g_0)\,)\\
&\quad + d(\overline{\mathfrak{v}}(g_0),\overline{\mathfrak{v}}(g_0'))+d(\,\overline{\mathfrak{v}}(g_0'),\ \overline{\mathfrak{v}}(g_{n-1}'\,\ldots\,g_0')\,)\\
&\lesssim 2\,\frac{\kappa}{\varepsilon}+\frac{16\,\delta}{1-\kappa^2} \lesssim \frac{\kappa}{\varepsilon}+8\,\delta.
\end{aligned}$$

Assuming $\|g_i\| \geq \|g_i'\|$, we have

$$\frac{\|g_i\|}{\|g_i'\|} \leq 1 + \frac{\|g_i - g_i'\|}{\|g_i'\|} \leq 1 + \frac{\|g_i\|}{\|g_i'\|}\, d_{\mathrm{rel}}(g_i, g_i') \leq 1 + \delta\, \frac{\|g_i\|}{\|g_i'\|}$$

which implies

$$\frac{\|g_i\|}{\|g_i'\|} \leq \frac{1}{1-\delta}.$$

Because the case $\|g_i\| \leq \|g_i'\|$ is analogous, we conclude that

$$\left|\log \frac{\|g_i\|}{\|g_i'\|}\right| \leq \log\left(\frac{1}{1-\delta}\right) \leq \frac{\delta}{1-\delta} \asymp \delta.$$

Since the two chains of matrices satisfy the assumptions of the AP we have

$$\frac{\|g_i\, g_{i-1}\|}{\|g_i\|\,\|g_{i-1}\|} \geq \alpha(g_{i-1}, g_i) \geq \varepsilon \quad \text{and} \quad \frac{\|g_i'\, g_{i-1}'\|}{\|g_i'\|\,\|g_{i-1}'\|} \geq \alpha(g_{i-1}', g_i') \geq \varepsilon.$$

A simple calculation gives

$$\begin{aligned} d_{\mathrm{rel}}(\, g_i\, g_{i-1},\; g_i'\, g_{i-1}'\,) &\leq \frac{\|g_i\|\|g_{i-1}\|}{\|g_i g_{i-1}\|} \max\left\{1, \frac{\|g_i'\|}{\|g_i\|}\right\} d_{\mathrm{rel}}(g_i, g_i') \\ &\quad + \frac{\|g_i'\|\|g_{i-1}'\|}{\|g_i' g_{i-1}'\|} \max\left\{1, \frac{\|g_{i-1}\|}{\|g_{i-1}'\|}\right\} d_{\mathrm{rel}}(g_{i-1}, g_{i-1}') \\ &\leq \frac{2}{(1-\delta)^2}\, \frac{\delta}{\varepsilon} \asymp \frac{\delta}{\varepsilon}. \end{aligned}$$

Therefore, arguing as above,

$$\left|\log \frac{\|g_i\, g_{i-1}\|}{\|g_i'\, g_{i-1}'\|}\right| \lesssim \frac{\delta}{\varepsilon}.$$

Hence, by conclusion (4) of the AP we have

$$\begin{aligned} \left|\log \frac{\|g_{n-1}\, \cdots\, g_0\|}{\|g_{n-1}'\, \cdots\, g_0'\|}\right| &\leq \left|\log \frac{\|g_{n-1}\, \cdots\, g_0\|\, \|g_1\|\, \cdots\, \|g_{n-2}\|}{\|g_1\, g_0\|\, \cdots\, \|g_{n-1}\, g_{n-2}\|}\right| \\ &\quad + \left|\log \frac{\|g_1'\, g_0'\|\, \cdots\, \|g_{n-1}'\, g_{n-2}'\|}{\|g_{n-1}'\, \cdots\, g_0'\|\, \|g_1'\|\, \cdots\, \|g_{n-2}'\|}\right| \\ &\quad + \sum_{i=1}^{n-2} \left|\log \frac{\|g_i'\|}{\|g_i\|}\right| + \sum_{i=1}^{n-1} \left|\log \frac{\|g_i\, g_{i-1}\|}{\|g_i'\, g_{i-1}'\|}\right| \end{aligned}$$

$$\lesssim 2\,n\,\frac{\kappa}{\varepsilon^2} + (n-2)\,\delta + (n-1)\,\frac{\delta}{\varepsilon}$$
$$\lesssim n\,\left(\frac{\kappa}{\varepsilon^2} + \frac{\delta}{\varepsilon}\right),$$

which proves (b). □

The next proposition is a flag version of the AP.

Let $\tau = (\tau_1, \ldots, \tau_k)$ be a signature with $0 < \tau_1 < \tau_2 < \cdots < \tau_k < m$.

We call τ*-block product* any of the functions $\pi_{\tau,j} : \mathrm{Mat}(m, \mathbb{R}) \to \mathbb{R}$,

$$\pi_{\tau,j}(g) := s_{\tau_{j-1}+1}(g)\,\ldots\,s_{\tau_j}(g), \qquad 1 \le j \le k,$$

where by convention $\tau_0 = 0$. A τ*-singular value product*, abbreviated τ-s.v.p., is any product of distinct τ-block products. By definition, τ-block products are τ-singular value products. Other examples of τ-singular value products are the functions

$$p_{\tau_j}(g) = s_1(g)\,\ldots\,s_{\tau_j}(g) = \|\wedge_{\tau_j} g\|.$$

Note that for every $1 \le j \le k$ we have:

$$\pi_{\tau,j}(g) = \frac{p_{\tau_j}(g)}{p_{\tau_{j-1}}(g)},$$

and

$$p_{\tau_j}(g) = \pi_{\tau,1}(g)\,\ldots\,\pi_{\tau,j}(g).$$

Proposition 2.45 (Flag AP) *Let $c > 0$ be the universal constant in Theorem 2.1. Given numbers $0 < \varepsilon < 1$, $0 < \kappa \le c\,\varepsilon^2$ and a chain of matrices $g_j \in \mathrm{Mat}(m, \mathbb{R})$, with $j = 0, 1, \ldots, n-1$, if*

(a) $\sigma_\tau(g_i) \le \kappa$, *for* $0 \le i \le n-1$, *and*
(b) $\alpha_\tau(g_{i-1}, g_i) \ge \varepsilon$, *for* $1 \le i \le n-1$,

then

(1) $d(\overline{\mathfrak{v}}_\tau(g^{(n)*}), \overline{\mathfrak{v}}_\tau(g^*_{n-1})) \lesssim \kappa\,\varepsilon^{-1}$
(2) $d(\overline{\mathfrak{v}}_\tau(g^{(n)}), \overline{\mathfrak{v}}_\tau(g_0)) \lesssim \kappa\,\varepsilon^{-1}$
(3) $\sigma_\tau(g^{(n)}) \le \left(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2}\right)^n$
(4) *for any τ-s.v.p. function π,*

$$\left|\log \pi(g^{(n)}) + \sum_{i=1}^{n-2} \log \pi(g_i) - \sum_{i=1}^{n-1} \log \pi(g_i\,g_{i-1})\right| \lesssim n\,\frac{\kappa}{\varepsilon^2}.$$

Proof For each $j = 1, \ldots, k$, consider the chain of matrices $\wedge_{\tau_j} g_0, \wedge_{\tau_j} g_1, \ldots, \wedge_{\tau_j} g_{n-1}$. Assumptions (a) and (b) here imply the corresponding assumptions of

Theorem 2.1 for all these chains of exterior power matrices. Hence, by (1) of the AP

$$\begin{aligned} d(\overline{\mathfrak{v}}_{\tau_j}(g^{(n)*}), \overline{\mathfrak{v}}_{\tau_j}(g^*_{n-1})) &= d(\Psi(\overline{\mathfrak{v}}_{\tau_j}(g^{(n)*})), \Psi(\overline{\mathfrak{v}}_{\tau_j}(g^*_{n-1}))) \\ &= d(\overline{\mathfrak{v}}(\wedge_{\tau_j} g^{(n)*}), \overline{\mathfrak{v}}(\wedge_{\tau_j} g^*_{n-1})) \lesssim \kappa\, \varepsilon^{-1}. \end{aligned}$$

Thus, taking the maximum in j we get $d(\overline{\mathfrak{v}}_\tau(g^{(n)*}), \overline{\mathfrak{v}}_\tau(g^*_{n-1})) \lesssim \kappa\, \varepsilon^{-1}$, which proves (1). Conclusion (2) follows in the same way.

Similarly, from (3) of Theorem 2.1, we infer the corresponding conclusion here

$$\sigma_\tau(g^{(n)}) = \max_{1\le j\le k} \sigma_{\tau_j}(g^{(n)}) = \max_{1\le j\le k} \sigma(\wedge_{\tau_j} g^{(n)}) \le \left(\frac{\kappa\,(4+2\,\varepsilon)}{\varepsilon^2}\right)^n.$$

Let us now prove (4).

For the τ-s.v.p. $\pi(g) = p_{\tau,j}(g) = \|\wedge_{\tau_j} g\|$ conclusion (4) is a consequence of the corresponding conclusion of Theorem 2.1.

For the τ-block product $\pi = \pi_{\tau,j}$, since

$$\log \pi(g) = \log\|\wedge_{\tau_j} g\| - \log\|\wedge_{\tau_{j-1}} g\|,$$

conclusion (4) follows again from Theorem 2.1 (4).

Finally, since any τ-s.v.p. is a finite product of τ-block products we can reduce (4) to the previous case. $\square$

We finish this section with a version of the AP for complex matrices.

The *singular values* of a complex matrix $g \in \mathrm{Mat}(m, \mathbb{C})$ are defined to be the eigenvalues of the positive semi-definite hermitian matrix $g^*\, g$, where g^* stands for the transjugate of g, i.e., the conjugate transpose of g. Similarly, the *singular vectors* of g are defined as the eigenvectors of $g^*\, g$. The sorted singular values of $g \in \mathrm{Mat}(m, \mathbb{C})$ are denoted by $s_1(g) \ge s_2(g) \ge \cdots \ge s_m(g)$. The top singular value of g coincides with its norm, $s_1(g) = \|g\|$.

The (first) gap ratio of g is the quotient $\sigma(g) := s_2(g)/s_1(g) \le 1$. We say that $g \in \mathrm{Mat}(m, \mathbb{C})$ has a (first) gap ratio when $\sigma(g) < 1$. When this happens the complex eigenspace

$$\{v \in \mathbb{C}^m :\ g^*\, g\, v = \|g\|\, v\} = \{v \in \mathbb{C}^m :\ \|g\, v\| = \|g\|\, \|v\|\}$$

has complex dimension one and determines a point in $\mathbb{P}(\mathbb{C}^m)$, denoted by $\overline{\mathfrak{v}}(g)$ and referred to as the *g-most expanding direction*.

Given points $\hat{v}, \hat{u} \in \mathbb{P}(\mathbb{C}^m)$, we set

$$\alpha(\hat{v}, \hat{u}) := \frac{|\langle v, u\rangle|}{\|v\|\,\|u\|} \qquad \text{where} \quad v \in \hat{v},\ u \in \hat{u}. \tag{2.28}$$

Given $g, g' \in \mathrm{Mat}(m, \mathbb{C})$, both with (first) gap ratios, we define the *angle between* g and g' to be

$$\alpha(g, g') := \alpha(\overline{\mathfrak{v}}(g^*), \overline{\mathfrak{v}}(g')).$$

With these definitions, the real version of the AP leads in a straightforward manner to a slightly weaker complex version, stated and proved below. However, adapting the original proof to the complex case, replacing each real concept by its complex analog, would lead to the same stronger estimates as in Theorem 2.1.

Proposition 2.46 (Complex AP) *Let $c > 0$ be the universal constant in Theorem 2.1. Given numbers $0 < \varepsilon < 1$, $0 < \kappa \le c\,\varepsilon^4$ and a chain of matrices $g_j \in \mathrm{Mat}(m, \mathbb{C})$, with $j = 0, 1, \ldots, n-1$, if*

(a) $\sigma(g_i) \le \kappa$, for $0 \le i \le n-1$, and
(b) $\alpha(g_{i-1}, g_i) \ge \varepsilon$, for $1 \le i \le n-1$,

then

(1) $d(\overline{\mathfrak{v}}(g^{(n)*}), \overline{\mathfrak{v}}(g^*_{n-1})) \lesssim \kappa\,\varepsilon^{-2}$
(2) $d(\overline{\mathfrak{v}}(g^{(n)}), \overline{\mathfrak{v}}(g_0)) \lesssim \kappa\,\varepsilon^{-2}$
(3) $\sigma(g^{(n)}) \le \left(\frac{\kappa\,(4+2\,\varepsilon^2)}{\varepsilon^4}\right)^n$
(4) $\left|\log\|g^{(n)}\| + \sum_{i=1}^{n-2} \log\|g_i\| - \sum_{i=1}^{n-1} \log\|g_i\, g_{i-1}\|\right| \lesssim n\,\frac{\kappa}{\varepsilon^4}$.

Proof Make the identification $\mathbb{C}^m \equiv \mathbb{R}^{2m}$, and given $g \in \mathrm{Mat}(m, \mathbb{C}^m)$ denote by $g^{\mathbb{R}} \in \mathrm{Mat}(2m, \mathbb{R})$ the matrix representing the linear operator $g : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ in the canonical basis.

We make explicit the relationship between gap ratios and angles of the complex matrices and $g, g' \in \mathrm{Mat}(m, \mathbb{C})$, and the gap ratios and angles of their real analogues $g^{\mathbb{R}}$ and $(g')^{\mathbb{R}}$.

Given $g \in \mathrm{Mat}(m, \mathbb{C})$, for each eigenvalue λ of g, the matrix $g^{\mathbb{R}}$ has a corresponding pair of eigenvalues $\lambda, \overline{\lambda}$. Since $g \mapsto g^{\mathbb{R}}$ is a C^*-algebra homomorphism, we have $(g^*\,g)^{\mathbb{R}} = (g^{\mathbb{R}})^*\,(g^{\mathbb{R}})$. Therefore, for all $i = 1, \ldots, m$, $s_i(g) = s_{2i-1}(g^{\mathbb{R}}) = s_{2i}(g^{\mathbb{R}})$. In particular, considering the signature $\tau = (2)$,

$$\sigma_{(2)}(g^{\mathbb{R}}) = \frac{s_3(g^{\mathbb{R}})}{s_1(g^{\mathbb{R}})} = \frac{s_2(g)}{s_1(g)} = \sigma(g). \tag{2.29}$$

The g-most expanding direction $\overline{\mathfrak{v}}(g) \in \mathbb{P}(\mathbb{C}^m)$ is a complex line which we can identify with the real 2-plane $\overline{\mathfrak{v}}_{(2)}(g^{\mathbb{R}})$. This identification, $\overline{\mathfrak{v}}(g) \equiv \overline{\mathfrak{v}}_{(2)}(g^{\mathbb{R}})$, comes from a natural isometric embedding $\mathbb{P}(\mathbb{C}^m) \hookrightarrow \mathrm{Gr}_2(\mathbb{R}^{2m})$.

Consider two points $\hat{v}, \hat{u} \in \mathbb{P}(\mathbb{C}^m)$ and take unit vectors $v \in \hat{v}$ and $u \in \hat{u}$. Denote by $U, V \subset \mathbb{C}^m$ the complex lines spanned by these vectors, which are planes in $\mathrm{Gr}_2(\mathbb{R}^{2m})$. Consider the complex orthogonal projection onto the complex line V, $\pi_{u,v} : U \to V$, defined by $\pi_{u,v}(x) := \langle x, v\rangle\, v$. By (2.28) we have $\alpha(\hat{v}, \hat{u}) = \|\pi_{u,v}\|$. On the other hand, since the adjoints $\pi^*_{u,v} : V \to U$ of $\pi_{u,v}$ both as a complex and as a real linear maps coincide, it follows that $\pi_{u,v} = \pi_{U,V}$ is the restriction to U of the (real) orthogonal projection onto the 2-plane V. Thus, by Proposition 2.19(b),

$$\alpha_2(U, V) = \sqrt{\det{}_{\mathbb{R}}(\pi_{u,v}^* \pi_{u,v})} = \det{}_{\mathbb{C}}(\pi_{u,v}^* \pi_{u,v}) = \|\pi_{u,v}\|^2 = \alpha(\hat{v}, \hat{u})^2.$$

In particular,

$$\alpha_{(2)}(g^{\mathbb{R}}, (g')^{\mathbb{R}}) = \alpha_{(2)}(\overline{\mathfrak{v}}((g^{\mathbb{R}})^*), \overline{\mathfrak{v}}((g')^{\mathbb{R}})) = \alpha(\overline{\mathfrak{v}}(g^*), \overline{\mathfrak{v}}(g'))^2 = \alpha(g, g')^2. \quad (2.30)$$

Take $\kappa, \varepsilon > 0$ such that $\kappa < c\,\varepsilon^4$, $0 < \varepsilon < 1$, and consider a chain of matrices $g_j \in \mathrm{Mat}(m, \mathbb{C})$, $j = 0, 1, \ldots, n-1$ satisfying the assumptions (a) and (b) of the complex AP. By (2.29) and (2.30), the assumptions (a) and (b) of Proposition 2.45 hold for the chain of real matrices $g_j^{\mathbb{R}} \in \mathrm{Mat}(2m, \mathbb{R})$, $j = 0, 1, \ldots, n-1$, with parameters κ and ε^2, and with $\tau = (2)$. Therefore conclusions (1)–(4) of the complex AP follow from the corresponding conclusions of Proposition 2.45. In conclusion (4) we use the (2)-singular value product $\pi(g) := \|g\|^2 = \|\wedge_2 g^{\mathbb{R}}\|$. $\square$

References

1. A. Ávila, S. Jitomirskaya, C. Sadel, Complex one-frequency cocycles. J. Eur. Math. Soc. (JEMS) **16**(9), 1915–1935 (2014). MR 3273312
2. M. Barnabei, A. Brini, G.-C. Rota, On the exterior calculus of invariant theory. J. Algebra **96**(1), 120–160 (1985). MR 808845 (87j:05002)
3. J. Bourgain, Positivity and continuity of the Lyapounov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. J. Anal. Math. **96**, 313–355 (2005). MR 2177191 (2006i:47064)
4. J. Bourgain, S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. J. Statist. Phys. **108**(5–6), 1203–1218 (2002), Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR 1933451 (2004c:47073)
5. P. Duarte, S. Klein, Continuity of the lyapunov exponents for quasiperiodic cocycles. Comm. Math. Phys. **332**(3), 1113–1166 (2014). MR 3262622
6. M. Goldstein, W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. Ann. Math. **154**(2)(1), 155–203 (2001). MR 1847592 (2002h:82055)
7. W. Schlag, Regularity and convergence rates for the Lyapunov exponents of linear cocycles. J. Mod. Dyn. **7**(4), 619–637 (2013). MR 3177775
8. S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall Inc, Englewood Cliffs, 1964). MR 0193578 (33 #1797)

Lyapunov Exponents of Linear Cocycles
Continuity via Large Deviations
Duarte, P.; Klein, S.
2016, XIII, 263 p. 4 illus., Hardcover
ISBN: 978-94-6239-123-9
A product of Atlantis Press