4CM20 - Hybrid Systems and Control

Solving Linear Matrix Inequalities using the Multi Parametric Toolbox (MPT3)

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Motivating Examples on Canvas

- Quadratic Lyapunov function

$$\text{Find } P \text{ s.t.} \begin{cases} A^\top P + PA < 0 \\ P > 0 \end{cases}$$

- Common quadratic Lyapunov function

$$\text{Find } P \text{ s.t.} \begin{cases} A_1^\top P + PA_1 < 0 \\ A_2^\top P + PA_2 < 0 \\ P > 0 \end{cases}$$

- Piecewise quadratic Lyapunov function (continuity and S-procedure)

Find $T = T^\top$, and U_i, W_i symm. with nonneg. entries s.t.

$$\begin{cases} A_i^\top P_j + P_j A_i + E_j^\top U_j E_j < 0 \\ P_i - E_i^\top W_i E_i > 0 \\ P_i = F_i^\top T F_i & (i,j) \in \mathcal{J} \end{cases}$$

Example: Quadratic Lyapunov Function

Given

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \; Q = 0.1I$$

Find P such that

$$\begin{cases} A^\top P + PA + Q \leq 0 & (1) \\ P > 0 & (2) \end{cases}$$

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L = Lf + Lp;                        %combine constraints
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disp(diagnostics.info);              %check feasibility
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diagnostics=optimize(L);             %solve the LMIs
disp(diagnostics.info);              %check feasibility
P = value(Pvar)                      %convert P to double
```

Always check if P really satisfies all constraints!

General Procedure

- Define unknowns: `sdpvar(n,m,'type')`
- Define constraints `Li` for `i = 1,...,N`
- Combine constraints by addition (`L = L1+L2`) or concatenation (`L=[L1, L2]`)
- Solve for unknowns: `diagnostics = optimize(L)`
- Check solver info for feasibility: `diagnostics.info` and `diagnostics.problem`. When infeasible, check nonnegativity of residuals of inequality constraints: `check(L)`
- Convert solutions: `value(..)`
- Check solutions: substitute in constraints/LMIs and check manually! (eigenvalues, nonnegativity, etc.)

Commands and Technicalities

- Equality operator: $==$
- Inequality operators (non-strict only!): $<=$, $>=$
- Condition on matrix (eigenvalues): `L=[M<=0]`
  Only when M is symmetric!
- Condition on matrix elements: `L=[M(:)<=0]`
- Solve while minimizing an objective function (e.g., parameter, matrix elements or eigenvalues): `optimize(L,Obj)`
- Specify additional options, such as SDPT3 solver (recommended!):

```
opts = sdpsettings('solver','sdpt3')
optimize(L,[],opts)
```

Final Notes

For installation files and instructions, see
MPT3: https://www.mpt3.org/Main/Installation
SDPT3: https://blog.nus.edu.sg/mattohkc/softwares/sdpt3/

For more info, check the YALMIP Wiki
https://yalmip.github.io/tutorial/semidefiniteprogramming/

Or in your command window in Matlab type

- `help sdpvar`
- `help optimize`

On Canvas

- Examples
- This presentation