Nonlocal UFL: Finite elements for Helmholtz equations with a nonlocal boundary condition

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Order of Presentation

Motivating Problem: Helmholtz scattering

A nonlocal boundary condition

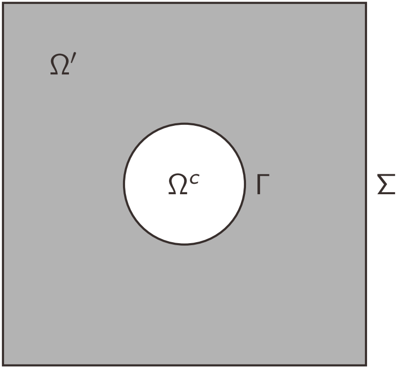
Nonlocal UFL

Numerical Results

Thanks to...

- NSF 1525697, 1909176
- The U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Department of Energy Computational Science Graduate Fellowship under Award Number DE-SC0021110
- Luke Olson (UIUC)

Exterior scattering



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$$\begin{cases} -\Delta u + \kappa^2 u = 0, & \mathbb{R}^d \setminus \Omega \\ -\frac{\partial u}{\partial n} = f, & \Gamma \end{cases}$$

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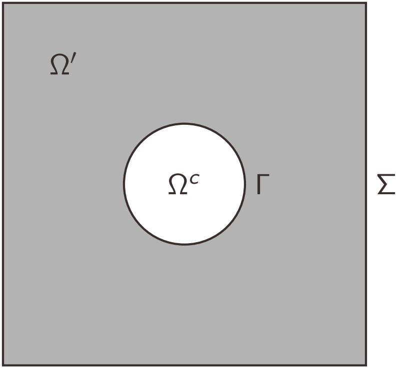

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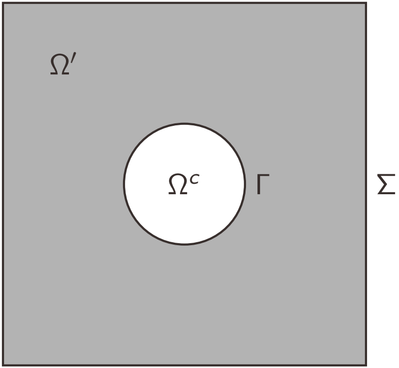

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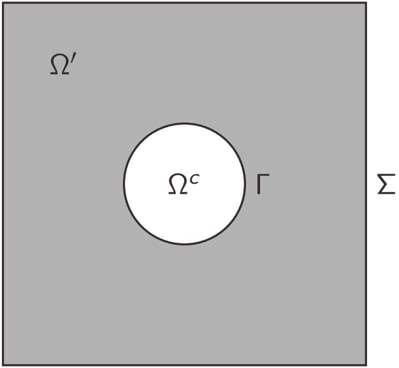
- In some finite domain of interest $\Omega' \subseteq \mathbb{R}^d \setminus \Omega'$ bounded by Σ.

Exterior scattering: computational problem



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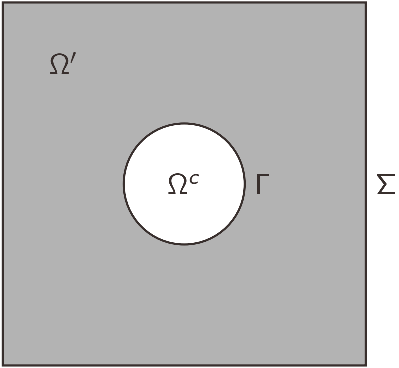

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- Problem we can actually solve

$$\begin{cases} -\Delta u + \kappa^2 u = 0, & \Omega' \\ -\frac{\partial u}{\partial n} = f, & \Gamma \\ ?????, & \Sigma \end{cases}$$

Integral form of the solution

With $\mathcal{K}$ the Green's function, the *true* solution satisfies:

$$u(x) = D(u)(x) - S(\frac{\partial}{\partial \boldsymbol{n}})(x), \quad x \in \Omega'$$

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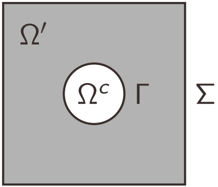

Exact boundary conditions

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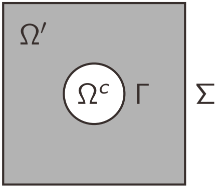

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Variational Form:

$$a(u, v) = F(v) \text{ for all } v \in H^2(\Omega')$$

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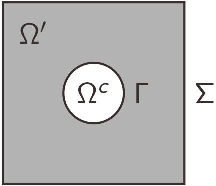

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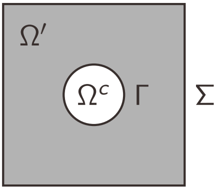

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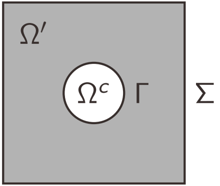

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  - *Solution:* Use pytential to evaluate layer potentials with fast multiple methods (FMM)

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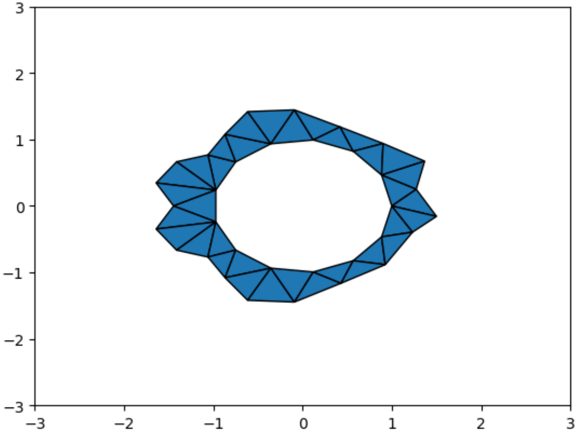

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  - ✓ Firedrake evaluates inner product

Solving the system with Firedrake

Extend UFL:

$$a(u,v) = (\nabla u, \nabla v) - \kappa^2 (u,v) - i\kappa\langle u, v\rangle_\Sigma + \langle \left(i\kappa - \frac{\partial}{\partial \mathbf{n}}\right) D(u) v\rangle_\Sigma$$

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Will be written as:

```
Du = DoubleLayerPotential(u, HelmholtzKernel(dim=2),
                          source=gamma, target=sigma)
a = inner(grad(u), grad(v)) * dx - \
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Can currently be written as:

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Du = DoubleLayerPotential(u, HelmholtzKernel(dim=2),
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ddnDu = DoubleLayerPotential(u, pytential.ddn(HelmholtzKernel(dim=2)),
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Numerical results: 2D

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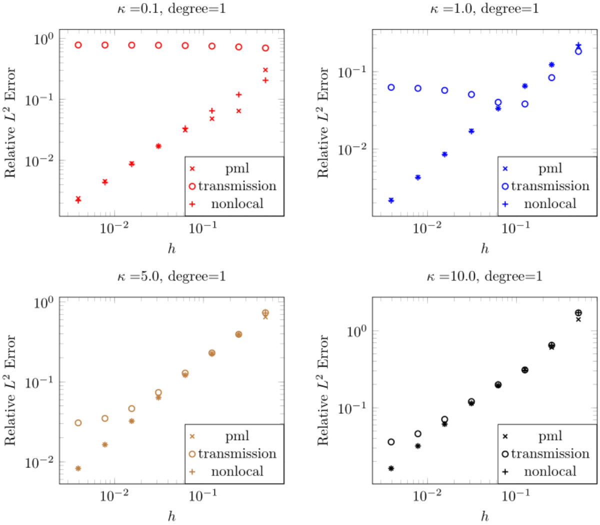

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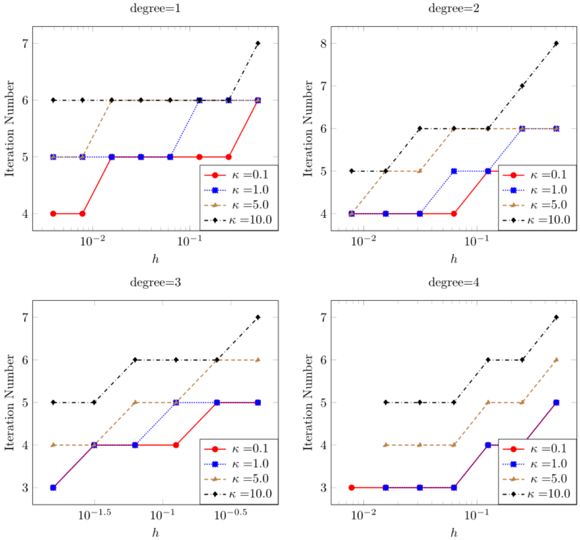

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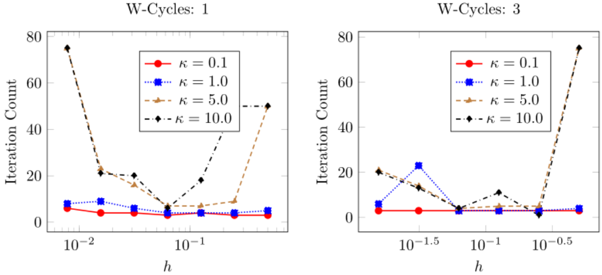
- PyAMG: Precondition with plane waves



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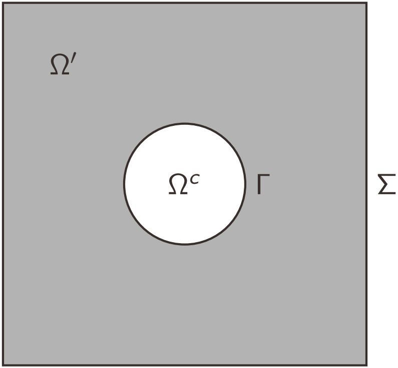
- *Coming soon*: `VolumePotentials` in UFL
- Leveraging nonlocal UFL to solve more FEM-BEM problems in Firedrake!
  - Further investigation into preconditioners

Thank you!

Questions?

Backup Slides

Exterior scattering: PML



- Perfectly Matched Layers:

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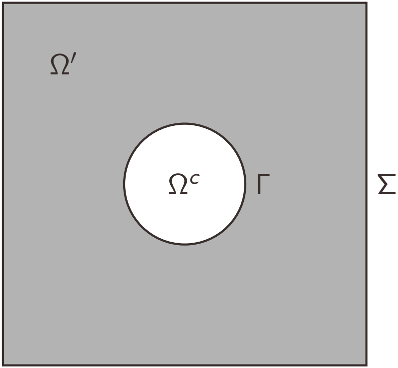

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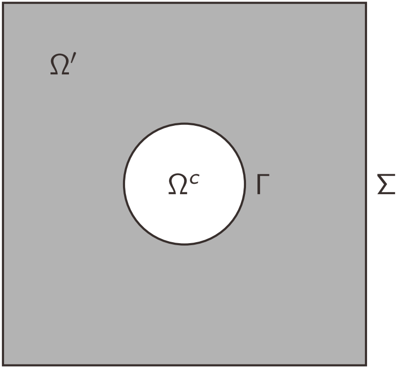

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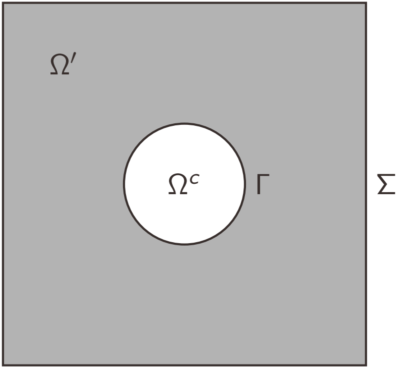

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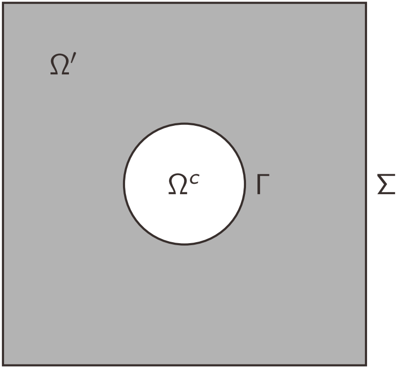

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- **Solvers are a pain!**

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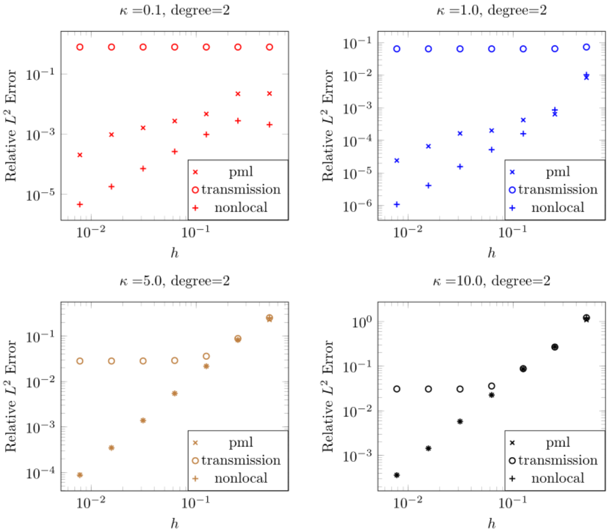
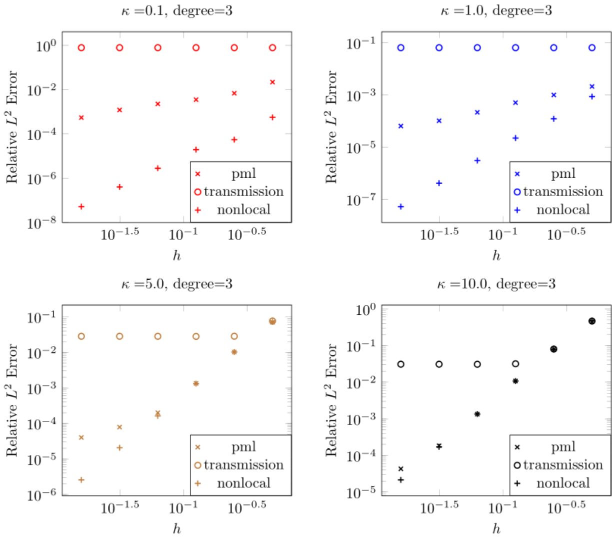
- For $h \leq h_0$, we have optimal-order H^1 and L^2 error estimaes.

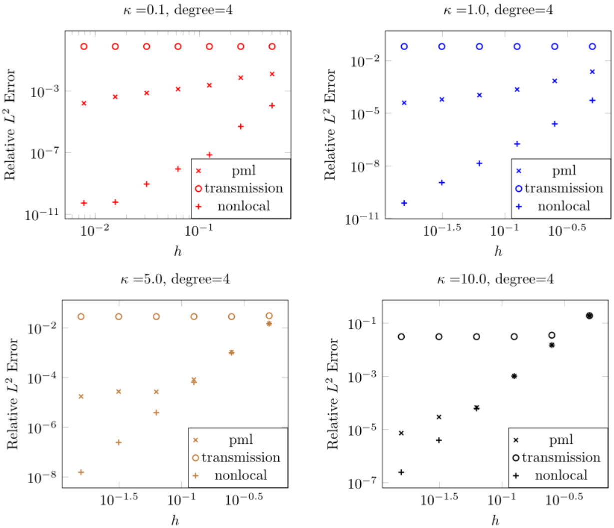
Numerical results: 2D, degree 2



Numerical results: 2D, degree 3



Numerical results: 2D, degree 4



Numerical results: 3D, degree 1