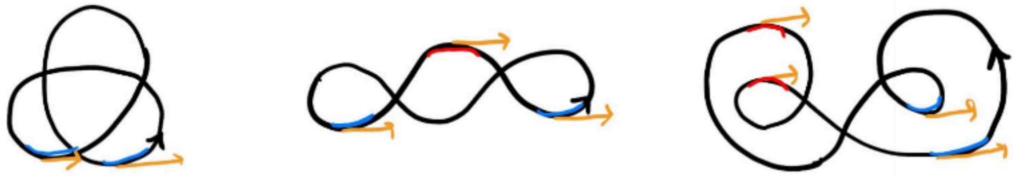
Problem 1. *Determine the rotation number of the following curves. Please show some work.*



$2 - 0 = 2 = TN$ $\qquad$ $2 - 1 = 1 = TN$ $\qquad$ $2 - 2 = 0 = TN$

Solution. $\square$

Problem 2. *Write down a linear system of differential equations in functions $f_1, \dots, f_6$ that is satisfied by $f_1 = T \cdot N$, $f_2 = T \cdot B$, $f_3 = N \cdot B$, $f_4 = T \cdot T$, $f_5 = N \cdot N$, $f_6 = B \cdot B$ when T, N, B are a Frenet frame.*[1] *Verify that the functions $f_1 = f_2 = f_3 = 0$ and $f_4 = f_5 = f_6 = 1$ is a solution to your system of differential equations.*[2]

Solution. Recall the Frenet equations give that $T' = \kappa N$, $N' = -\kappa T - \tau B$, $B' = \tau N$ Then if we differentiate,

$$f_1' = T' \cdot N + T \cdot N' = \kappa N \cdot N + T \cdot (-\kappa T - \tau B) = \kappa(N \cdot N) - \kappa(T \cdot T) - \tau(T \cdot B) = \kappa f_5 - \kappa f_4 - \tau f_2$$

$$f_2' = T' \cdot B + T \cdot B' = \kappa N \cdot B + T \cdot \tau N = \kappa(N \cdot B) + \tau(T \cdot N) = \kappa f_3 + \tau f_1$$

$$f_3' = N' \cdot B + N \cdot B' = (-\kappa T - \tau B) \cdot B + N \cdot \tau N = -\kappa(T \cdot B) - \tau(B \cdot B) + \tau(N \cdot N) = -\kappa f_2 - \tau f_6 + \tau f_5$$

$$f_4' = T' \cdot T + T \cdot T' = 2T \cdot T' = 2T \cdot \kappa N = 2\kappa(T \cdot N) = 2\kappa f_1$$

$$f_5' = N' \cdot N + N \cdot N' = 2N \cdot N' = 2N \cdot (-\kappa T - \tau B) = -2\kappa(T \cdot N) - 2\tau(N \cdot B) = -2\kappa f_1 - 2\tau f_3$$

$$f_6' = B' \cdot B + B \cdot B' = 2B \cdot B' = 2B \cdot \tau N = 2\tau(N \cdot B) = 2\tau f_3$$

Then linear system of differential equations are

$$\begin{cases} f_1' = \kappa f_5 - \kappa f_4 - \tau f_2 \\ f_2' = \kappa f_3 + \tau f_1 \\ f_3' = -\kappa f_2 - \tau f_6 + \tau f_5 \\ f_4' = 2\kappa f_1 \\ f_5' = -2\kappa f_1 - 2\tau f_3 \\ f_6' = 2\tau f_3 \end{cases} . \tag{1}$$

Now if we plug in $f_1 = f_2 = f_3 = 0$ and $f_4 = f_5 = f_6 = 1$, sice all the derivatives are 0,

$$\begin{cases} 0 = \kappa(1) - \kappa(1) - \tau(0) = 0 \\ 0 = \kappa(0) + \tau(0) \\ 0 = -\kappa(0) - \tau(1) + \tau(1) \\ 0 = 2\kappa(0) \\ 0 = -2\kappa(0) - 2\tau(0) \\ 0 = 2\tau(0) \end{cases} . \tag{2}$$

Thus they satisfy my linear system of differential equations. □

[1] Hint: differentiate these dot product functions, and express the answer back in terms of these functions. The final answer should be a system of differential equations involving $f_1, \dots, f_6$, not T, N, B.

[2] Remark: recall this fact is used to prove the fundamental theorem of space curves.

Problem 3. *Let $F : V \to V$ be a linear operator on a finite-dimensional inner-product space. Prove the following are equivalent.*

(a) F preserves the inner product.

(b) F preserves lengths of vectors.

(c) F preserves orthonormality (i.e. it sends an ONB to another ONB).

(d) F preserves orthonormality of some orthonormal basis (i.e. there exists an ONB that is sent to an ONB under F).

A map satisfying these conditions is called a linear isometry or an orthogonal map.[3]

Solution.

(a $\Rightarrow$ c) To say $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is an orthonormal basis for V means $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$. If F preserves the inner product, then it preserves this property.

(c $\Rightarrow$ d) If F preserves the orthonormality of all orthonormal bases, then indeed there exists such a basis that is preserved, because all finite-dimensional inner product spaces have an orthonormal basis.

(d $\Rightarrow$ b) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the orthonormal basis that F sends to another orthonormal basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$. Any vector $\mathbf{v}$ in V can be written as a linear combination $\mathbf{v} = \sum a_i \mathbf{v}_i$, and its image is $F(\mathbf{v}) = \sum a_i \mathbf{w}_i$. So

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\sum a_i^2} = \sqrt{F(\mathbf{v}) \cdot F(\mathbf{v})} = |F(\mathbf{v})|.$$

(b $\Rightarrow$ a) If F preserves the lengths of vectors, then for any two vectors $v, w \in V$,

$$\begin{aligned}
|v + w|^2 &= |F(v + w)|^2 \\
&= |Fv + Fw|^2, \\
(v + w) \cdot (v + w) &= (Fv + Fw) \cdot (Fv + Fw), \\
(v \cdot v) + 2(v \cdot w) + (w \cdot w) &= (Fv \cdot Fv) + 2(Fv \cdot Fw) + (Fw \cdot Fw), \\
|v|^2 + 2(v \cdot w) + |w|^2 &= |Fv|^2 + 2(Fv \cdot Fw) + |Fw|^2 \\
&= |v|^2 + 2(Fv \cdot Fw) + |w|^2, \\
2(v \cdot w) &= 2(Fv \cdot Fw).
\end{aligned}$$

So F preserves the inner product in general.

□

[3]Remark: for $\mathbb{R}^2$ with the standard inner product, the orthogonal maps are rotations and reflections.

Problem 4. *Show that the curvature and torsion of a curve are invariant under rigid motions.* [4] [5] [6]

Solution. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, and assume WLOG that it is unit speed (we previously proved that any regular curve has a unit-speed reparametrization). Let $\beta : I \to \mathbb{R}^3$ be α under a rigid transformation, that is, $\beta := R \circ \alpha = Q\alpha + p$ where Q is an orthogonal matrix and $p \in \mathbb{R}^3$, both constant. We want to show that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \tau_\beta$.

Firstly, since Q is constant, notice that $T_\beta := \beta' = Q\alpha' = QT_\alpha$ by the chain rule. (This also shows that β is also unit speed since lengths are preserved by the orthogonal map $x \mapsto Qx$). This implies

$$T'_\beta = QT'_\alpha = Q(\kappa_\alpha N_\alpha) = \kappa_\alpha (QN_\alpha)$$

where N_α is a unit vector. But Q preserves lengths, so $|QN_\alpha| = 1$. So $T'_\beta := \kappa_\beta N_\beta = \kappa_\alpha(QN_\alpha)$. It follows that $|T'_\beta| = |\kappa_\beta|$ and

$$|T'_\beta| = |\kappa_\alpha(QN_\alpha)| = |\kappa_\alpha||(QN_\alpha)| = |\kappa_\alpha| \implies \kappa_\alpha = \kappa_\beta.$$

since κ is non-negative by definition. Furthermore, this implies $N_\beta = QN_\alpha$.
Secondly, $B_\beta := T_\beta \times N_\beta = QT_\alpha \times QN_\alpha$.

- Claim: $Qv \times Qw = Q(v \times w)$ for all $v, w \in \mathbb{R}^3$ and orthogonal matrices $Q \in \mathbb{R}^{3\times 3}$. Proof: Suppose $v, w \in \mathbb{R}^3$. Then $(v \times w) \cdot v = (v \times w) \cdot w = 0$ by the definition of the cross product. Since Q is orthogonal, it preserves the inner product, so $Q(v\times w)\cdot Qv = Q(v\times w)\cdot Qw = 0$. This means $Q(v \times w)$ is orthogonal to both Qv and Qw, that is, $Q(v \times w) \in \text{span}\{Qv, Qw\}^\perp = \text{span}\{Qv \times Qw\}$. Now, Q also preserves lengths, so $|Q(v \times w)| = |v \times w| = |v||w|\sin\theta = |Qv||Qw|\sin\theta = |Qv \times Qw|$. It follows that $Q(v \times w) = \pm Qv \times Qw$. But we know that Q must have positive determinant (i.e. preserve the orientation of any basis), so $\{v, w, v \times w\}$ has the same orientation as $\{Qv, Qw, Q(v \times w)\}$. Therefore $Q(v \times w) = +Qv \times Qw$.

Using the claim, we have $B_\beta = Q(T_\alpha \times N_\alpha)$. By the Frenet equations,

$$B'_\beta = Q(T'_\alpha \times N_\alpha + T_\alpha \times N'_\alpha) = Q(\kappa_\alpha(N_\alpha \times N_\alpha) + T_\alpha \times (-\kappa_\alpha T_\alpha - \tau_\alpha B_\alpha)) = Q(0 - \kappa_\alpha(T_\alpha \times T_\alpha) - \tau_\alpha(T_\alpha \times B_\alpha))$$

$$= Q(0 - 0 + \tau_\alpha N_\alpha) = \tau_\alpha(QN_\alpha) = \tau_\alpha N_\beta$$

because $B := T \times N \implies N = B \times T$. But we also know that $B'_\beta := \tau_\beta N_\beta$. Combining these gives

$$B'_\beta = \tau_\beta N_\beta = \tau_\alpha N_\beta \implies \tau_\alpha = \tau_\beta.$$

□

[4] Recall that (by our definition) a rigid motion is a composition of translations and orthogonal maps.
[5] Remark: this is the converse of the fundamental theorem of space curves.
[6] Hint: You'll probably need to show something about rigid motions and the cross product...

Problem 5. *For points a, b, c in the plane, write $C(a,b,c)$ for the center of this circle that passes through a, b, c.*[7] *The osculating circle*[8] *at $\alpha(t)$ is defined as the circle through $\alpha(t)$ with center*

$$C = \lim_{s \to 0} C(\alpha(t-s), \alpha(t), \alpha(t+s)).$$

(i) *Fix $\lambda > 0$ and define $\beta(s) = (s, \lambda s^2)$. For $s \neq 0$, compute the center of the circle that passes through $\beta(s), \beta(0)$, and $\beta(-s)$.* [9]

(ii) *Assume α satisfies $\alpha(0) = (0,0)$ and $\alpha'(0) = (1,0)$. Use the preceding part and the Taylor expansion of $\alpha(t)$ to show the radius of the osculating circle at $\alpha(0)$ is $1/\kappa$, where $\kappa = \kappa(0)$ is the curvature.* [10] [11]

Solution. **(i)** Given a circle that goes through points $(-s, \lambda s^2), (0,0)$, and $(s, \lambda s^2)$, we know by symmetry that the center of the circle is on some point $(0, y)$. By definition of a circle, the distance from the center to all of these points are the same, allowing us to derive the center in terms of λ and s.

$$\sqrt{s^2 + \lambda^2 s^4 - 2\lambda s^2 y + y^2} = y \implies s^2 + \lambda^2 s^4 - 2\lambda s^2 y = 0 \implies y = \frac{s^2 + \lambda^2 s^4}{2\lambda s^2} = \frac{1 + \lambda^2 s^2}{2\lambda}$$

This implies that our circle has center $\left(0, \frac{1+\lambda^2 s^2}{2\lambda}\right)$.
(ii) Given a small enough value of s, $\alpha(t)$ is well approximated by its quadratic taylor polynomial on the interval $(t-s, t+s)$.

$$\alpha(t) \approx \alpha(0) + \alpha'(0)t + \frac{\alpha''(0)}{2}t^2 = (0,0) + (t,0) + \left(0, \frac{\kappa}{2}t^2\right) = \left(t, \frac{\kappa}{2}t^2\right)$$

This is the same form as our β from part (i) with $\lambda = \frac{\kappa}{2}$, allowing us to use the formula for the circle's center derived earlier. For any three points $\alpha(-s), \alpha(0)$, and $\alpha(s)$, the osculating circle will have center $\left(0, \frac{1}{\kappa} + \frac{\kappa}{4}s^2\right)$. This implies the radius of this circle is

$$\lim_{s \to 0} \frac{1}{\kappa} + \frac{\kappa}{4}s^2 = \frac{1}{\kappa}.$$

To address the remark, since circles have a uniform curvature equal to $\frac{1}{r}$ where r is the radius of the circle, the osculating circle at the point $\alpha(t)$ has curvature $\kappa(t)$. This shows how the osculating circle matches the curvature at a point in the same way a tangent line matches the slope at a point. □

[7] Fact: if a, b, c do not lie on a line, then there is a unique circle passing through these points.

[8] Remark: the tangent line is the line that best approximates a curve at a point. Similarly, the osculating circle is the circle that best approximates a plane curve at a point.

[9] Hint: First write the equation of a circle through a point (a,b) with radius r. Then plug in $\beta(\pm s), \beta(0)$ and solve a system of equations to find a, b, r.

[10] Hint: use specifically the degree-2 Taylor approximation.

[11] Remark: this problem gives a geometric interpretation for the curvature.