

SURFACE MAPPING CLASS GROUP ACTIONS ON 3-MANIFOLDS

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ABSTRACT. For each circle bundle $S^1 \rightarrow X \rightarrow \Sigma_g$ over a surface with genus $g \geq 2$, there is a natural surjection $\pi : \text{Homeo}^+(X) \rightarrow \text{Mod}(\Sigma_g)$. When X is the unit tangent bundle $U\Sigma_g$, it is well-known that π splits. On the other hand π does not split when the Euler number $e(X)$ is not divisible by the Euler characteristic $\chi(\Sigma_g)$ by [CT23]. In this paper we show that this homomorphism does not split in many cases where $\chi(\Sigma_g)$ divides $e(X)$.

1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus $g \geq 2$, and let $X_{g,e}$ denote the oriented S^1 -bundle over Σ_g with Euler number e . Let $\text{Homeo}^+(X_{g,e})$ be the group of orientation-preserving homeomorphisms of $X_{g,e}$ that act trivially on the center of $\pi_1(X_{g,e})$, and let $\text{Mod}(X_{g,e}) := \pi_0(\text{Homeo}^+(X_{g,e}))$ denote the mapping class group.

The (generalized) Nielsen realization problem for $X_{g,e}$ asks whether the surjective homomorphism

$$\text{Homeo}^+(X_{g,e}) \rightarrow \text{Mod}(X_{g,e})$$

splits over subgroups of $\text{Mod}(X_{g,e})$. In this paper we study a closely related problem. For each g, e there is a surjection $\text{Mod}(X_{g,e}) \rightarrow \text{Mod}(\Sigma_g)$. Consider the composition

$$\pi_{g,e} : \text{Homeo}^+(X_{g,e}) \twoheadrightarrow \text{Mod}(X_{g,e}) \twoheadrightarrow \text{Mod}(\Sigma_g).$$

Problem 1.1. Does $\pi_{g,e} : \text{Homeo}^+(X_{g,e}) \twoheadrightarrow \text{Mod}(\Sigma_g)$ split?

If $e = \pm(2g - 2)$, then $X_{g,e}$ is the unit (co)tangent bundle, and $\pi_{g,e}$ does split; see [Sou10, §1]. On the other hand, if e is not divisible by $2g - 2$, then the surjection $\text{Mod}(X_{g,e}) \rightarrow \text{Mod}(\Sigma_g)$ does not split by work of the second two authors [CT23], so $\pi_{g,e}$ also does not split in these cases. Given this, it remains to study the case when e is divisible by $2g - 2$ and $e \neq \pm(2g - 2)$. In these cases $\text{Mod}(X_{g,e}) \rightarrow \text{Mod}(\Sigma_g)$ does split [CT23], but we prove $\pi_{g,e}$ does not split in many cases.

Theorem A. *Fix a surface Σ_g of genus g and $e \in \mathbb{Z}$. Assume that $g = 4k - 1$ where $k \geq 3$ and k is not a power of 2, and assume that e is divisible by $(2g - 2)2p$ where p is an odd prime dividing k . Then the natural surjective homomorphism $\pi_{g,e} : \text{Homeo}^+(X_{g,e}) \rightarrow \text{Mod}(\Sigma_g)$ does not split.*

Date: November 13, 2023.

For example, if $e = 0$, we find that $\pi_{g,e} : \text{Homeo}^+(\Sigma_g \times S^1) \rightarrow \text{Mod}(\Sigma_g)$ does not split when $g = 11, 19, 23, 27, 35, 39, 43, 47, \dots$

Theorem A solves the Nielsen realization problem for $\text{Mod}(\Sigma_g)$ subgroups of $\text{Mod}(X_{g,e})$ in the cases of the theorem. Specifically, if e is divisible by $2g - 2$, then $\text{Mod}(X_{g,e}) \cong H^1(\Sigma_g; \mathbb{Z}) \rtimes \text{Mod}(\Sigma_g)$ [CT23], and every $\text{Mod}(\Sigma_g)$ subgroup of $\text{Mod}(X_{g,e})$ is the image of a splitting of $\text{Mod}(X_{g,e}) \rightarrow \text{Mod}(\Sigma_g)$. By Theorem A, $\text{Homeo}^+(X_{g,e}) \rightarrow \text{Mod}(X_{g,e})$ does not split over any of these $\text{Mod}(\Sigma_g)$ subgroups.

Short proof sketch of Theorem A. The proof strategy is similar to an argument of Chen–Salter [CS22] that shows that $\text{Homeo}^+(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$ does not split when $g \geq 2$. Theorem A is proved by contradiction: assuming the existence of a splitting $\text{Mod}(\Sigma_g) \rightarrow \text{Homeo}^+(X_{g,e})$, first we obtain, by lifting, an action of the based mapping class group $\text{Mod}(\Sigma_g, *)$ on the cover $\widehat{X}_{g,e} \cong \mathbb{R}^2 \times S^1$ corresponding to the center of $\pi_1(X_{g,e})$. The conditions on g and e in Theorem A guarantee the existence of a $\mathbb{Z}/2p\mathbb{Z}$ subgroup of $\text{Mod}(\Sigma_g, *)$ for which we can show the action on $\widehat{X}_{g,e}$ has a fixed circle. Denoting a generator of $\mathbb{Z}/2p\mathbb{Z}$ by α , we show that $\text{Mod}(\Sigma_g, *)$ is generated by the centralizers of α^2 and α^p . This shows that the entire group $\text{Mod}(\Sigma_g, *)$ acts on $\widehat{X}_{g,e}$ with a fixed circle, which contradicts the fact that the point-pushing subgroup $\pi_1(\Sigma_g) < \text{Mod}(\Sigma_g, *)$ acts freely (by deck transformations) on $\widehat{X}_{g,e}$.

Other questions. Related to the $\text{Mod}(\Sigma_g)$ action on the unit tangent bundle $U\Sigma_g$, we pose the following question.

Question 1.2. Do either of the following surjections split?

$$\text{Diff}^+(U\Sigma_g) \rightarrow \text{Mod}(\Sigma_g) \quad \text{or} \quad \text{Homeo}(U\Sigma_g) \rightarrow \text{Mod}(U\Sigma_g)$$

If one includes orientation-reversing diffeomorphisms and mapping classes, then if $g \geq 12$, then $\text{Diff}(U\Sigma_g) \rightarrow \text{Mod}^\pm(\Sigma_g)$ does not split by Souto [Sou10, Thm. 1].

Acknowledgement. The authors LC and BT are supported by NSF grants DMS2203178, DMS-2104346, and DMS-2005409.

2. PROOF OF THEOREM A

Fix $g = 4k - 1$ and e as in the theorem statement, and set $\Sigma = \Sigma_g$ and $X = X_{g,e}$. Suppose for a contradiction that there is a homomorphism

$$\sigma : \text{Mod}(\Sigma) \rightarrow \text{Homeo}(X)$$

whose composition with $\pi = \pi_{g,e} : \text{Homeo}(X) \rightarrow \text{Mod}(\Sigma)$ is the identity.

2.1. Step 1: lifting argument. Consider the covering space $\widehat{X} = \widetilde{\Sigma} \times S^1$ of X , where $\widetilde{\Sigma} \cong \mathbb{R}^2$ is the universal cover. This is the covering corresponding to the center ζ of $\pi_1(X)$. Given the action of $\text{Mod}(\Sigma)$ on X , we consider the set of all lifts of homeomorphisms in this action to \widehat{X} . This is an action of the pointed mapping class group $\text{Mod}(\Sigma, *)$ on \widehat{X} . To explain this, we start with the following general proposition.

Proposition 2.1. *Let Y be a closed manifold. Let $\zeta < \pi_1(Y)$ be the center of the fundamental group, and denote $\Delta = \pi_1(Y)/\zeta$. Let $\widehat{Y} \rightarrow Y$ be the covering space with $\pi_1(\widehat{Y}) = \zeta$. Fix a basepoint $*$ in Y . Assume that the evaluation map*

$$\text{Homeo}(Y) \rightarrow Y, \quad f \mapsto f(*)$$

induces a surjection $\pi_1(\text{Homeo}(Y)) \twoheadrightarrow \zeta < \pi_1(Y)$. Then there is a commutative diagram

(1)

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \text{Homeo}(\widehat{Y})^\Delta & \xrightarrow{q} & \text{Homeo}(Y) \longrightarrow 1 \\ & & \parallel & & \downarrow \widehat{p} & & \downarrow p \\ 1 & \longrightarrow & \Delta & \longrightarrow & \text{Mod}(Y, *) & \longrightarrow & \text{Mod}(Y) \longrightarrow 1 \end{array}$$

whose rows are exact, where the bottom row is the (generalized) Birman exact sequence. Furthermore, this diagram is a pullback diagram.

A version of Proposition 2.1 when $Y = \Sigma_g$ (whose center is trivial) is used in [CS22].

We prove Proposition 2.1 after explaining how it gives the desired lifting. In our situation, the center of $\pi_1(X)$ is the kernel of $\pi_1(X) \rightarrow \pi_1(\Sigma)$ since $\pi_1(\Sigma)$ has trivial center. Thus Proposition 2.1 gives us the following diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \text{Homeo}(\widehat{X})^\Delta & \xrightarrow{q} & \text{Homeo}(X) \longrightarrow 1 \\ & & \parallel & & \downarrow \widehat{p} & & \downarrow p \\ 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \text{Mod}(X, *) & \longrightarrow & \text{Mod}(X) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \text{Mod}(\Sigma, *) & \longrightarrow & \text{Mod}(\Sigma) \longrightarrow 1 \end{array}$$

The splitting σ defines a subgroup $\text{Mod}(\Sigma) < \text{Mod}(X)$ and a splitting of p over this subgroup. Since the top row is a pullback of the middle row, it follows that \widehat{p} splits over $\text{Mod}(\Sigma, *)$ (this uses only general facts about pullbacks). Denote this splitting by

$$\widehat{\sigma} : \text{Mod}(\Sigma, *) \rightarrow \text{Homeo}(\widehat{X})^\Delta.$$

Under this splitting the point-pushing subgroup $\pi_1(\Sigma)$ acts by deck transformations.

Remark 2.2. If $G < \text{Mod}(\Sigma)$ and $\sigma(G)$ has a fixed point $*$, then after choosing a lift $\hat{*}$ of $*$, one can lift canonically elements of $\sigma(G)$ to \hat{X} by choosing the unique lift that fixes $\hat{*}$. This implies that $G < \text{Mod}(\Sigma)$ can be lifted to $G < \text{Mod}(\Sigma, *)$ so that $\hat{\sigma}(G)$ has a fixed point.

Proof of Proposition 2.1. First we recall the construction of the bottom row of diagram (1). Evaluation at $* \in Y$ defines a fibration

$$\text{Homeo}(Y, *) \rightarrow \text{Homeo}(Y) \xrightarrow{\epsilon} Y.$$

The long exact sequence of homotopy groups gives an exact sequence

$$\pi_1(\text{Homeo}(Y)) \xrightarrow{\epsilon_*} \pi_1(Y) \rightarrow \text{Mod}(Y, *) \rightarrow \text{Mod}(Y) \rightarrow 1.$$

In general the image of ϵ_* is contained in the center of $\pi_1(Y)$; see e.g. [Hat02, §1.1, Exer. 20]. By assumption, ϵ_* surjects onto the center, so we obtain the short exact sequence in the bottom row of (1). The homomorphism $\pi_1(Y) \rightarrow \text{Mod}(Y, *)$ is the so-called “point-pushing” homomorphism. It sends $\eta \in \pi_1(Y)$ (basepoint = $*$) to the time-1 map of an isotopy that pushes $*$ around η in reverse (this follows directly from the definition of the connecting homomorphism in the long exact sequence; note that it makes sense for the reverse of η to appear in defining this homomorphism since concatenation of paths is left-to-right, while composition of functions is right-to-left).

Next we define $\hat{p} : \text{Homeo}(\hat{Y})^\Delta \rightarrow \text{Mod}(Y, *)$. Fix a point $\hat{*} \in \hat{Y}$ that covers the basepoint $* \in Y$. Given $f \in \text{Homeo}(\hat{Y})^\Delta$. Choose a path $[0, 1] \rightarrow \hat{Y}$ from $\hat{*}$ to $f(\hat{*})$ and let γ_f denote the composition $[0, 1] \rightarrow \hat{Y} \rightarrow Y$. By isotopy extension, there exists an isotopy $h_t : Y \rightarrow Y$ where $h_0 = \text{id}_Y$ and $h_t(*) = \gamma_f(t)$ for each $t \in [0, 1]$. Define

$$\hat{p}(f) = [h_1 \circ q(f)].$$

The map \hat{p} is well-defined. The choice of γ_f is unique only up to an element of $\pi_1(\hat{Y}) = \zeta$. This implies that the isotopy class $[h_1 \circ f]$ is only well-defined up to composition by a point-pushing mapping class by an element of ζ , but such a point-push is trivial by assumption.

It is a straightforward exercise to check that \hat{p} is a homomorphism. The right square in diagram (1) commutes because $q(f)$ and $h_1 \circ q(f)$ are isotopic by construction. It is easy to see that the left square in the diagram commutes by applying the definition of \hat{p} to deck transformations.

Finally, regarding the claim that the diagram is a pullback, we show that the map to the fibered product

$$\hat{p} \times q : \text{Homeo}(\hat{Y})^\Delta \rightarrow \text{Mod}(Y, *) \times_{\text{Mod}(Y)} \text{Homeo}(Y)$$

is an isomorphism. The codomain consists of pairs $(\phi, g) \in \text{Mod}(Y, *) \times \text{Homeo}(Y)$ such that g is isotopic to a representative of the isotopy class ϕ .

We define an inverse ι to $\hat{p} \times q$. Given (ϕ, g) in the fibered product, choose an isotopy g_t from g to a homeomorphism representing ϕ . Lift g_t to an isotopy \tilde{g}_t such that \tilde{g}_1 fixes $\hat{*}$, and define $\iota(\phi, g) = \tilde{g}_0$. The reader can check that the maps ι and $\hat{p} \times q$ are inverses. \square

2.2. Step 2: finite group action rigidity. Recall that $g = 4k - 1$ and $k \geq 3$ is not a power of 2; let p be an odd prime dividing k . From Step 1, we have homomorphism $\widehat{\sigma} : \text{Mod}(\Sigma, *) \rightarrow \text{Homeo}(\widehat{X})$ that descends to a splitting $\sigma : \text{Mod}(\Sigma) \rightarrow \text{Homeo}(X)$. In this section we describe the action of a particular finite subgroup of $\text{Mod}(\Sigma, *)$ on \widehat{X} .

Proposition 2.3. *There exists an element $\alpha \in \text{Mod}(\Sigma, *)$ of order $2p$ such that the fixed sets of $\widehat{\sigma}(\alpha)$, $\widehat{\sigma}(\alpha)^2$, and $\widehat{\sigma}(\alpha)^p$ coincide and are equal to an embedded circle $c \subset \widehat{X}$.*

It is worth noting that the fixed set of a finite-order, orientation-preserving homeomorphism of a 3-manifold can be wildly embedded [MZ54].

In order to prove Proposition 2.3 we first construct the specific element α . Then we prove (Proposition 2.5) a weaker version of Proposition 2.3 with the additional assumption that the action is smooth. Finally, we combine this with a result of Pardon [Par21] and Smith theory to prove Proposition 2.3.

Construction of α . We obtain α as an element in a dihedral subgroup D_{4k} of $\text{Mod}(\Sigma)$, where D_{4k} denotes the dihedral group of order $8k$. The dihedral action $D_{4k} \curvearrowright \Sigma$ we use has quotient Σ/D_{4k} homeomorphic to T^2 and the quotient $\Sigma \rightarrow \Sigma/D_{4k}$ has a single branch point; it is determined by the homomorphism

$$\langle x, y \rangle = F_2 \cong \pi_1(T^2 \setminus \text{pt}) \rightarrow D_{4k} = \langle a, b \mid a^{4k} = b^2 = 1, bab = a^{-1} \rangle$$

$$x \mapsto a, \quad y \mapsto b.$$

By Riemann–Hurwitz, the genus of Σ is $4k - 1$. The orbifold $O = \Sigma/D_{4k}$ has fundamental group

$$\pi_1^{\text{orb}}(O) = \langle x, y, h \mid h^{2k} = 1, h = [x, y] \rangle,$$

and there is a short exact sequence

$$(2) \quad 1 \rightarrow \pi_1(\Sigma) \rightarrow \pi_1^{\text{orb}}(O) \rightarrow D_{4k} \rightarrow 1.$$

This sequence induces a homomorphism $\pi_1^{\text{orb}}(O) \rightarrow \text{Mod}(\Sigma, *)$. We take $\alpha = h^{k/p}$, where p , as defined above, is an odd prime dividing k , which exists by assumption. Then α is an element of order $2p$ in the subgroup $\langle h \rangle \cong \mathbb{Z}/2k\mathbb{Z}$ of $\pi_1^{\text{orb}}(O) < \text{Mod}(\Sigma, *)$.

Remark 2.4. The argument that follows works equally well when Σ/D_{4k} is a genus- g surface and $\Sigma \rightarrow \Sigma/D_{4k}$ has a single branched point. This provides more values of g, e for which the conclusion of Theorem A holds.

Smooth case. Here we prove the following proposition.

Proposition 2.5. *Fix $D_{4k} < \text{Mod}(\Sigma)$ as above. Suppose that $\sigma : D_{4k} \rightarrow \text{Diff}^+(X)$ and is a splitting of $\pi : \text{Homeo}^+(X) \rightarrow \text{Mod}(\Sigma)$ over D_{4k} . Then $\widehat{\sigma}(\alpha)$ fixes a unique circle on $\widehat{X} = \mathbb{H}^2 \times S^1$. Consequently, the fixed set of $\sigma(a^{2k/p})$ is nonempty.*

The last part of the statement of Proposition 2.5 follows from the preceding statement because the image of α under $\pi_1^{orb}(O) \rightarrow D_{4k}$ is $a^{2k/p}$.

Proof of Proposition 2.5. First we reduce to a more geometric setting. By Meeks–Scott [MS86, Thm. 2.1], the smooth(!) action $\sigma(D_{4k}) \curvearrowright X$ preserves some geometric metric on X . There are two possibilities for the geometry: if $e(X) = 0$, then X has $\mathbb{H}^2 \times \mathbb{R}$ -geometry, and if $e(X) \neq 0$, then X has $\mathrm{PSL}_2(\mathbb{R})$ -geometry. We treat these cases in parallel.

The universal cover \widetilde{X} (with the induced geometric structure) is either $\mathbb{H}^2 \times \mathbb{R}$ or $\mathrm{PSL}_2(\mathbb{R})$. In either case, \widetilde{X} has an isometric foliation by lines whose leaf space is isometric to \mathbb{H}^2 , and this foliation is preserved by $\mathrm{Isom}(\widetilde{X})$, so there is a homomorphism $\mathrm{Isom}(\widetilde{X}) \rightarrow \mathrm{Isom}(\mathbb{H}^2)$. Let $\mathrm{Isom}^+(\widetilde{X}) < \mathrm{Isom}(\widetilde{X})$ be the group whose action on the leaves and on the leaf space are both orientation preserving. There is an exact sequence

$$(3) \quad 1 \rightarrow \mathbb{R} \rightarrow \mathrm{Isom}^+(\widetilde{X}) \xrightarrow{F} \mathrm{Isom}^+(\mathbb{H}^2) \rightarrow 1.$$

See also [Sco83, §4].

Next consider the group Λ of all lifts of elements of $\sigma(D_{4k}) < \mathrm{Isom}(X)$ to $\mathrm{Isom}^+(\widetilde{X})$. This yields an exact sequence

$$1 \rightarrow \pi_1(X) \rightarrow \Lambda \rightarrow D_{4k} \rightarrow 1.$$

The action of Λ on \widetilde{X} induces an action of Λ/ζ on $\widetilde{X}/\zeta = \widehat{X} \cong \mathbb{H}^2 \times S^1$, where ζ is the center of $\pi_1(X)$. This action extends to an action of $\mathrm{Isom}^+(\widetilde{X})/\zeta$, and there is a homomorphism

$$\rho : \Lambda/\zeta \rightarrow \mathrm{Isom}^+(\widetilde{X})/\zeta \xrightarrow{\cong} \mathrm{Isom}(\widehat{X}).$$

The last map is an isomorphism by the general formula $\mathrm{Isom}(\widehat{X}/\Lambda) = N_{\mathrm{Isom}(\widehat{X})}(\Lambda)/\Lambda$ for discrete subgroups $\Lambda < \mathrm{Isom}(\widehat{X})$.

To prove the proposition, we first identify Λ/ζ with $\pi_1^{orb}(O)$ (Claim 2.6). Then it is a formal consequence of our setup that $\rho(h^{k/p}) = \widehat{\sigma}(\alpha)$, and after showing $\mathrm{Isom}^+(\widetilde{X})/\zeta \cong \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2)$ (Claim 2.7), we show that $\rho(h^{k/p})$ fixes a unique circle in \widehat{X} (Claim 2.8).

Claim 2.6. The restriction of the sequence (3) to Λ is a short exact sequence

$$1 \rightarrow \zeta \rightarrow \Lambda \rightarrow \pi_1^{orb}(O) \rightarrow 1$$

where ζ is the center of $\pi_1(X)$.

Proof of Claim 2.6. Recall the map $F : \mathrm{Isom}^+(\widetilde{X}) \rightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ from (3). First we identify $F(\Lambda) < \mathrm{Isom}^+(\mathbb{H}^2)$ with $\pi_1^{orb}(O)$. For this, it suffices to show that $F(\Lambda)$ fits into a short exact sequence

$$(4) \quad 1 \rightarrow \pi_1(\Sigma) \rightarrow F(\Lambda) \rightarrow D_{4k} \rightarrow 1,$$

where the “monodromy” $D_{4k} \rightarrow \mathrm{Out}^+(\pi_1(\Sigma)) \cong \mathrm{Mod}(\Sigma)$ has image the given subgroup $D_{4k} < \mathrm{Mod}(\Sigma)$. This implies that $F(\Lambda) \cong \pi_1^{orb}(O)$ because

$\pi_1^{orb}(O)$ is an extension of the same form (see (2)), and extensions of $\pi_1(\Sigma)$ are determined by their monodromy [Bro82, §IV.3].

To construct the extension (4), first note that the restriction of (3) to $\pi_1(X)$ is the short exact sequence

$$1 \rightarrow \zeta \rightarrow \pi_1(X) \rightarrow \pi_1(\Sigma) \rightarrow 1.$$

The group $\pi_1(\Sigma) = F(\pi_1(X))$ is normal in $F(\Lambda)$ because $\pi_1(X)$ is normal in Λ . Furthermore, the surjection $\Lambda \rightarrow F(\Lambda)$ induces a surjection $D_{4k} = \Lambda/\pi_1(X) \rightarrow F(\Lambda)/\pi_1(\Sigma)$.

The quotient map $\tilde{X} \rightarrow \mathbb{H}^2$, which is equivariant with respect to $\Lambda \rightarrow F(\Lambda)$ descends to a map $X = \tilde{X}/\pi_1(X) \rightarrow \mathbb{H}^2/\pi_1(\Sigma) = \Sigma$ that's equivariant with respect to $D_{4k} = \Lambda/\pi_1(X) \rightarrow F(\Lambda)/\pi_1(\Sigma)$.

Since σ is a realization, the induced action of $\sigma(D_{4k})$ on Σ is a realization of the $D_{4k} < \text{Mod}(\Sigma)$, and in particular the D_{4k} action on Σ is faithful. Therefore, $F(\Lambda)/\pi_1(\Sigma) \cong D_{4k}$, and the monodromy of the associated extension

$$1 \rightarrow \pi_1(\Sigma) \rightarrow F(\Lambda) \rightarrow D_{4k} \rightarrow 1,$$

is the given inclusion $D_{4k} < \text{Mod}(\Sigma)$. This concludes the proof that $F(\Lambda)$ is isomorphic to $\pi_1^{orb}(O)$.

To finish the proof of Claim 2.6, it remains to show that the intersection of Λ with $\mathbb{R} = \ker(F)$ is ζ . We do this by showing (i) $\Lambda \cap \mathbb{R}$ is the center of Λ , and (ii) the center of Λ is contained in $\pi_1(X)$. Together with the obvious containment $\zeta < \Lambda \cap \mathbb{R}$, (i) and (ii) imply $\Lambda \cap \mathbb{R} = \zeta$.

(i): First note that $\Lambda \cap \mathbb{R}$ is central because \mathbb{R} is central in $\text{Isom}(\tilde{X})$. On the other hand, the center of Λ is contained in $\Lambda \cap \mathbb{R}$ because the center of $\Lambda/(\Lambda \cap \mathbb{R}) \cong \pi_1^{orb}(O)$ has trivial center.

(ii): To show the center of Λ is contained in $\pi_1(X)$, we show that the center of Λ projects trivially to $D_{4k} = \Lambda/\pi_1(X)$. This is true because $\Lambda \rightarrow D_{4k}$ factors through $\pi_1^{orb}(O)$, which has trivial center. \square

We summarize the relation between the relevant groups in Diagram (5).

$$(5) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & \zeta & \xlongequal{\quad} & \zeta & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X) & \longrightarrow & \Lambda & \longrightarrow & D_{4k} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \pi_1^{orb}(O) & \longrightarrow & D_{4k} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

By Claim 2.6, $\Lambda/\zeta \cong \pi_1^{orb}(O)$, so ρ takes the form

$$\rho : \pi_1^{orb}(O) \rightarrow \text{Isom}^+(\tilde{X})/\zeta \cong \text{Isom}(\hat{X})$$

By construction, this homomorphism is the restriction of $\hat{\sigma} : \text{Mod}(\Sigma, *) \rightarrow \text{Homeo}(\hat{X})$ to $\pi_1^{orb}(O)$. Since $\alpha = h^{k/p}$, to show the fixed set of $\hat{\sigma}(\alpha)$ is a circle, it suffices to show the same statement for $\rho(h^{k/p})$. To prove this, we first compute $\text{Isom}(\hat{X}) \cong \text{Isom}^+(\tilde{X})/\zeta$.

Claim 2.7. The group $\text{Isom}^+(\tilde{X})/\zeta$ is isomorphic to $\text{Isom}^+(\mathbb{H}^2) \times \text{SO}(2)$.

Proof of Claim 2.7. First note that there is an extension

$$1 \rightarrow \text{SO}(2) \rightarrow \text{Isom}^+(\tilde{X})/\zeta \rightarrow \text{Isom}^+(\mathbb{H}^2) \rightarrow 1$$

induced from (3). This sequence is obviously split when $\tilde{X} = \mathbb{H}^2 \times \mathbb{R}$ since $\text{Isom}^+(\tilde{X}) \cong \text{Isom}^+(\mathbb{H}^2) \times \mathbb{R}$ is a product.

Assume now that $\tilde{X} = \widetilde{\text{PSL}_2(\mathbb{R})}$, and write $e = (2g - 2)n$ where n is a nonzero integer. Let K denote the kernel of the universal cover homomorphism $\widetilde{\text{PSL}_2(\mathbb{R})} \rightarrow \text{PSL}_2(\mathbb{R})$.

We claim that $\zeta = \frac{1}{n}K$. To see this, note that the extension

$$1 \rightarrow K \rightarrow \widetilde{\text{PSL}_2(\mathbb{R})} \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1$$

pulled back under a Fuchsian representation $\pi_1(\Sigma) \rightarrow \text{PSL}_2(\mathbb{R})$ induces the extension of the unit tangent bundle group $\pi_1(U\Sigma)$, which has Euler number $2 - 2g$, and there is an n -fold fiberwise cover $U\Sigma \rightarrow X$, so the center of $\pi_1(U\Sigma) < \pi_1(X)$ is generated by the n -th power of the generator of the center of $\pi_1(X)$, i.e. $\zeta = \frac{1}{n}K$.

The inclusion of $\text{PSL}_2(\mathbb{R})$ in $\text{Isom}(\widetilde{\text{PSL}_2(\mathbb{R})})$ (given by left-multiplication) descends to a homomorphism

$$\text{PSL}_2(\mathbb{R}) = \widetilde{\text{PSL}_2(\mathbb{R})}/K \rightarrow \text{Isom}(\widetilde{\text{PSL}_2(\mathbb{R})})/K \twoheadrightarrow \text{Isom}(\widetilde{\text{PSL}_2(\mathbb{R})})/\zeta$$

that defines a splitting of the sequence

$$1 \rightarrow \text{SO}(2) \rightarrow \text{Isom}(\widetilde{\text{PSL}_2(\mathbb{R})})/\zeta \rightarrow \text{PSL}_2(\mathbb{R}) \rightarrow 1. \quad \square$$

The following Claim 2.8 is the last step in the proof of Proposition 2.5.

Claim 2.8. Let p be an odd prime dividing k . If e is divisible by $(2g - 2)2p$, then the fixed set of $\rho(h^{k/p})$ is a circle.

Before proving the claim, we explain how the factors of $\text{Isom}^+(\mathbb{H}^2) \times \text{SO}(2) \cong \text{Isom}^+(\hat{X})$ act on $\hat{X} = \tilde{X}/\zeta$.

Remark 2.9. Consider the isomorphism $\text{Isom}(\hat{X}) \cong \text{Isom}^+(\mathbb{H}^2) \times \text{SO}(2)$ from Claim 2.7. In each case ($e = 0$ or $e \neq 0$) the action of $\text{SO}(2)$ on \hat{X} covers the identity of \mathbb{H}^2 and acts freely by rotation on the circle fibers of $X \rightarrow \mathbb{H}^2$. For the $\text{Isom}^+(\mathbb{H}^2)$ action, when $e = 0$, then $\hat{X} \cong \mathbb{H}^2 \times S^1$ is a

metric product, and the action of $\text{Isom}^+(\mathbb{H}^2)$ is trivial on the S^1 factor and is the natural action on \mathbb{H}^2 . If $e = (2g - 2)n$ is nonzero, then

$$\widehat{X} \cong \widetilde{\text{PSL}_2(\mathbb{R})} / \zeta \cong \text{PSL}_2(\mathbb{R}) / (\mathbb{Z}/n\mathbb{Z}),$$

and with respect to this isomorphism, the action of $\text{Isom}^+(\mathbb{H}) \cong \text{PSL}_2(\mathbb{R})$ on \widehat{X} is induced from left multiplication of $\text{PSL}_2(\mathbb{R})$ on $\text{PSL}_2(\mathbb{R})$. Identifying $\text{PSL}_2(\mathbb{R})$ with the unit tangent $U\mathbb{H}^2$, we can also view $\text{PSL}_2(\mathbb{R})/(\mathbb{Z}/n\mathbb{Z})$ as the quotient of $U\mathbb{H}^2$ by the $\mathbb{Z}/n\mathbb{Z}$ action that covers the identity of \mathbb{H}^2 and rotates each fiber.

Proof of Claim 2.8. Write $e = (2g - 2)2pm$ for some integer m .

First note that since $\rho(h)$ has finite order, the induced isometry of \mathbb{H}^2 has a unique fixed point, so $\rho(h)$ preserves a unique circle C of the fibering $\widehat{X} \rightarrow \mathbb{H}^2$. The same is true for $\rho(h^{k/p})$, and we will show that $\rho(h^{k/p})$ acts trivially on C .

Since $h = [x, y]$ is a commutator in $\pi_1^{orb}(O)$ and $\text{SO}(2)$ is abelian, we find that the projection

$$\rho(h) \in \text{Isom}^+(\widehat{X}) \cong \text{Isom}^+(\mathbb{H}^2) \times \text{SO}(2) \rightarrow \text{SO}(2)$$

is trivial. Therefore, the action of $\rho(h)$ on \widehat{X} factors through $\text{Isom}^+(\mathbb{H}^2)$ acting on \widehat{X} . This action is described in Remark 2.9. If $e = 0$, since $\text{Isom}^+(\mathbb{H}^2)$ acts trivially on the S^1 factor of $\widehat{X} \cong \mathbb{H}^2 \times S^1$, we conclude that $\rho(h)$ acts trivially on C . If $e \neq 0$, then $\rho(h)$ acts as a rotation by $2\pi(pm/k)$ on C , so $\rho(h^{k/p})$ acts as a rotation by $2\pi m$, which is trivial. \square

This completes the proof of Proposition 2.5. \square

Homeomorphism case. Here prove Proposition 2.3.

Proof of Proposition 2.3. By Pardon [Par21, Thm. 1.1], there is a sequence of smooth D_{4k} actions converging in $\text{Hom}(D_{4k}, \text{Homeo}(X))$ to the given action of $\sigma(D_{4k})$ on X . Sufficiently close approximates also give a splitting of π over $D_{4k} < \text{Mod}(\Sigma)$ because $\text{Homeo}(X)$ is locally path connected [EK71].

For each of the smooth approximations of $\sigma(D_{4k})$, the fixed set of $a^{2k/p}$ is nonempty by Proposition 2.5. This implies that $\sigma(a^{2k/p})$ has a fixed point (a sequence of fixed points, one for each smooth action, sub-converges to a fixed point of the $\sigma(a^{2k/p})$ action). By Remark 2.2, there exists a lift of $a^{2k/p} \in D_{4k} < \text{Mod}(\Sigma)$ to a finite order element $\alpha' \in \pi_1^{orb}(O) < \text{Mod}(\Sigma, *)$ so that $\widehat{\sigma}(\alpha')$ has a fixed point. Since $\pi_1^{orb}(O)$ has a unique conjugacy class of finite subgroup of order $2p$, the subgroups $\langle \alpha' \rangle$ and $\langle \alpha \rangle$ are conjugate, so the fixed set of $\widehat{\sigma}(\alpha)$ is nonempty.

It remains to show the fixed set of $\widehat{\sigma}(\alpha)$ is a circle, and that this circle is the same as the fixed sets of $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$.

First we show (using Smith theory) that both $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$ have fixed set a single circle (we are not yet claiming/arguing that the fixed sets of $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$ are the same). To see this, we focus on $\widehat{\sigma}(\alpha)^2$ for concreteness.

Consider the group Λ_0 of all lifts of powers of $\widehat{\sigma}(\alpha)^2$ to the universal cover \widetilde{X} . This group is an extension

$$1 \rightarrow \mathbb{Z} \rightarrow \Lambda_0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 1,$$

which is central and split; hence $\Lambda_0 \cong \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. It is central because α acts orientation-preservingly on fibers of $X \rightarrow \Sigma$ (otherwise, the action of α on Σ would reverse orientation, contrary to the construction); it splits because $\widehat{\sigma}(\alpha)$ has a fixed point. The (lifted) action of $\widehat{\sigma}(\alpha)^2$ on \widetilde{X} has fixed set a line (i.e. embedded copy of \mathbb{R}) by Smith theory and local Smith theory [Bre12, Theorem 20.1], and this line is preserved and acted properly by $\mathbb{Z} < \Lambda_0$; thus $\widehat{\sigma}(\alpha)^2$ acts on \widehat{X} with a circle in its fixed set. Furthermore, each component of the fixed set of $\widehat{\sigma}(\alpha)^2$ acting on \widehat{X} corresponds to a distinct conjugacy class of order- p subgroup of Λ_0 . Since there is only one $\mathbb{Z}/p\mathbb{Z}$ subgroup of Λ_0 , the fixed set of $\widehat{\sigma}(\alpha)^2$ is connected, i.e. a single circle. The same argument¹ works for $\widehat{\sigma}(\alpha)^p$.

Now we determine the fixed set of $\widehat{\sigma}(\alpha)$. First observe that $\widehat{\sigma}(\alpha)$ preserves the fixed set of $\widehat{\sigma}(\alpha)^2$ and has a fixed point there (the fixed set of $\widehat{\sigma}(\alpha)$ is nonempty and contained in the fixed set of $\widehat{\sigma}(\alpha)^2$). The only $\mathbb{Z}/2p\mathbb{Z}$ action on the circle with a fixed point is the trivial action, so in fact the fixed sets of $\widehat{\sigma}(\alpha)$ and $\widehat{\sigma}(\alpha)^2$ are the same. The same argument applies to $\widehat{\sigma}(\alpha)$ and $\widehat{\sigma}(\alpha)^p$. This proves Proposition 2.3. \square

2.3. Step 3: centralizer argument. Recall that we have defined α as an element of order $2p$ in $\pi_1^{orb}(O) < \text{Mod}(\Sigma, *)$. In this step we prove that $\text{Mod}(\Sigma, *)$ is generated by the centralizers of α^2 and α^p .

Proposition 2.10 (centralizer property). *Let $\alpha \in \text{Mod}(\Sigma, *)$ be the element of order $2p$ constructed above. Then*

$$\text{Mod}(\Sigma, *) = \langle C(\alpha^2), C(\alpha^p) \rangle,$$

where $C(-)$ denotes the centralizer in $\text{Mod}(\Sigma, *)$.

Strategy for proving Proposition 2.10. Set $\Gamma = \langle C(\alpha^2), C(\alpha^p) \rangle$. Our method for showing $\Gamma = \text{Mod}(\Sigma, *)$, which is similar to the proof of [CS22, Thm. 1.1], is to inductively build subsurfaces

$$(6) \quad S_0 \subset S_1 \subset \cdots \subset S_N \subset \Sigma \setminus \{*\}$$

such that $\text{Mod}(S_n) \subset \Gamma$ for each n and S_N fills $\Sigma \setminus \{*\}$ (i.e. each boundary component of S_N is inessential in $\Sigma \setminus \{*\}$). The fact that S_N fills implies that $\text{Mod}(S_N) = \text{Mod}(\Sigma, *)$, so then $\text{Mod}(\Sigma, *) \subset \Gamma$ by the last step in the inductive argument.

In order to ensure that $\text{Mod}(S_n) \subset \Gamma$, the subsurface S_n is obtained from S_{n-1} by an operation known as *subsurface stabilization*. If $S \subset \Sigma$ is a subsurface and $c \subset \Sigma$ is a simple closed curve that intersects S in a single arc,

¹Smith theory applies to prime-order finite cyclic group actions, so we cannot apply this argument directly to $\widehat{\sigma}(\alpha)$.

then the *stabilization of S along c* is the subsurface $S \cup N(c)$, where $N(c)$ is a regular neighborhood of c . It is easy to show that $\text{Mod}(S \cup N(c))$ is generated by $\text{Mod}(S)$ and the Dehn twist τ_c [CS22, Lem. 4.2], so if $\text{Mod}(S) \subset \Gamma$ and $\tau_c \in \Gamma$, then $\text{Mod}(S \cup N(c)) \subset \Gamma$. Therefore, for the proof, it suffices to find a sequence of subsurface stabilizations along curves whose Dehn twist belongs to $\Gamma = \langle C(\alpha^2), C(\alpha^p) \rangle$.

Model for the α action. Our proof of Proposition 2.10 makes use of an explicit model for Σ with its α action, which is pictured below in the case $k = 6$ and $p = 3$ (recall that $g = 4k - 1$ and p is an odd prime dividing k).

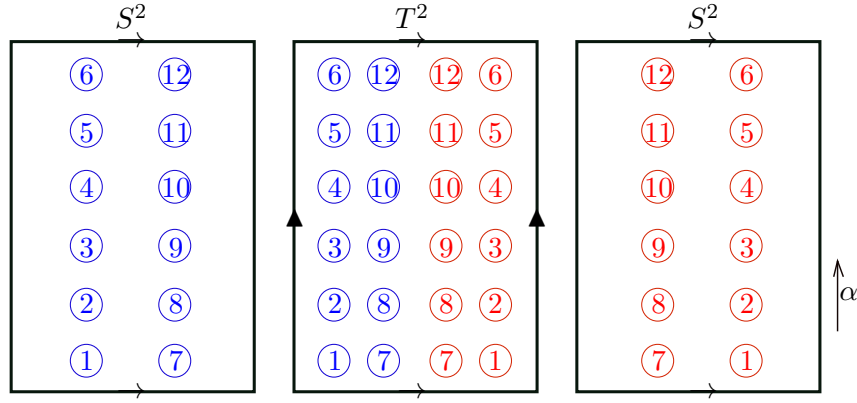


FIGURE 1. The model of the surface Σ_{23} where $p = 3$.

The surface Σ is built out of two copies of the standard action of $\mathbb{Z}/2p\mathbb{Z}$ on S^2 and one copy of a free action of $\mathbb{Z}/2p\mathbb{Z}$ on T^2 . We glue each copy of S^2 to T^2 along k/p free orbits by an equivariant connected sum. In the figure, α acts by vertical translation. Note that the fixed points of α on S^2 are not pictured in the figure – they are at $\pm\infty$ along the x -axis.

To derive this model, recall that $D_{4k} = \langle a, b \mid a^{4k} = 1 = b^2, bab = a^{-1} \rangle$ has abelianization $D_{4k} \rightarrow (\mathbb{Z}/2\mathbb{Z})^2$ with kernel $\langle a^2 \rangle \cong \mathbb{Z}/2k\mathbb{Z}$. Then there is a sequence of regular covers

$$\Sigma \xrightarrow{\langle a^2 \rangle} \Sigma / \langle a^2 \rangle \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^2} \Sigma / D_{4k}.$$

The cover $\Sigma / \langle a^2 \rangle \rightarrow \Sigma / D_{4k}$ is unbranched and is the $\mathbb{Z}/2\mathbb{Z}$ -homology cover of T^2 (in particular, $\Sigma / \langle a^2 \rangle$ is also a torus). The cover $\Sigma \rightarrow \Sigma / \langle a^2 \rangle$ is branched over four points; the local monodromy around the branched points is a^2 at two of the branched points and a^{-2} at the other two. Choosing branched cuts joining branched points in $a^{\pm 2}$ pairs gives a model for Σ , and one can check that this model is equivalent to the one described above. (The spheres in Figure 1 arise from pre-images under $\Sigma \rightarrow \Sigma / \langle a^2 \rangle$ of neighborhoods of the branch cuts.)

By Remark 2.2, since $\sigma(a^{2k/p})$ has a fixed point, the subgroup $\langle \alpha \rangle \subset \text{Mod}(\Sigma, *)$, which lifts $\langle a^{2k/p} \rangle$, has a fixed point. The different lifts of $\langle a^{2k/p} \rangle$

to a finite subgroup of $\text{Mod}(\Sigma, *)$ are in one-to-one correspondence to fixed points of $a^{2k/p}$. Since these fixed points are permuted transitively by the action of D_{4k} , the different lifts of $\langle a^{2k/p} \rangle$ are conjugate. Consequently, for the purpose of our argument, we can choose $*$ to be any one of the four fixed points of α and prove Proposition 2.10 for this choice, without loss of generality. (It will also be evident from the argument that a similar argument applies if $*$ is changed to another fixed point.)

Remark 2.11. We do not know how generally the relation $\text{Mod}(\Sigma, *) = \langle C(\alpha^2), C(\alpha^p) \rangle$ holds. For example, it may hold for every $\mathbb{Z}/2p\mathbb{Z}$ subgroup of $\text{Mod}(\Sigma, *)$. We do not know a general (abstract) approach to this problem.

Proof of Proposition 2.10.

Symmetry breaking. In preparation for constructing a sequence of sub-surface stabilizations, in this paragraph we find a suitable collection of Dehn twists that belong to Γ . The obvious way for τ_c to belong to Γ is if c is preserved by either α^2 or α^p . More generally, we use a process that we call *symmetry breaking* to show $\tau_c \in \Gamma$ for certain c . We formulate this in the following lemma, which is similar to [CS22, Lem. 3.2].

Lemma 2.12 (Symmetry breaking). *Assume that $c, d \subset \Sigma$ are simple closed curves that intersect once and $\tau_d \in \Gamma$. Suppose that either (i) $\alpha^p(c)$ is disjoint from c and d or (ii) the curves $\alpha^2(c), \alpha^4(c), \dots, \alpha^{2p-2}(c)$ are disjoint from d and the curves $c, \alpha^2(c), \alpha^4(c), \dots, \alpha^{2p-2}(c)$ are pairwise disjoint. Then $\tau_c \in \Gamma$.*

Proof of Lemma 2.12. We prove case (i) of the statement; case (ii) is similar. Since Dehn twists about disjoint curves commute,

$$(\tau_c \tau_{\alpha^p(c)}) \tau_d (\tau_c \tau_{\alpha^p(c)})^{-1} = \tau_c \tau_d \tau_c^{-1}.$$

The left hand side of the equation is in Γ because $\tau_d \in \Gamma$ by assumption and $\tau_c \tau_{\alpha^p(c)} \in \Gamma$ because the curves $c, \alpha^p(c)$ are permuted by α^p and are disjoint (so their twists commute), and thus $\tau_c \tau_{\alpha^p(c)} \in C(\alpha^p) \subset \Gamma$. Since c and d intersect once, the braid relation implies that $\tau_d \tau_c \tau_d^{-1} = \tau_c \tau_d \tau_c^{-1}$ also belongs to Γ . Since $\tau_d \in \Gamma$ this implies that $\tau_c \in \Gamma$, as desired. \square

Remark 2.13. When applying Lemma 2.12(i) or (ii) we refer to it as the α^p - or α^2 -symmetry breaking, respectively.

Lemma 2.14. *Dehn twists about the curves in Figure 2 are in Γ .*

In Figure 2, we illustrate the case $k = 6, p = 3$. The corresponding curves in the general case belong to Γ by the exact same argument.

Proof of Lemma 2.14. First observe that τ_{c_1} and τ_{c_6} are in Γ because each is invariant under α^p . We deduce that $\tau_{c_2} \in \Gamma$ using α^2 -symmetry breaking with $d = c_1$. Each of τ_{c_3} and τ_{c_4} are in Γ by α^p -symmetry breaking with $d = c_2$. Finally, both τ_{c_5} and τ_{c_7} are in Γ by α^p -symmetry breaking with $d = c_3$. \square

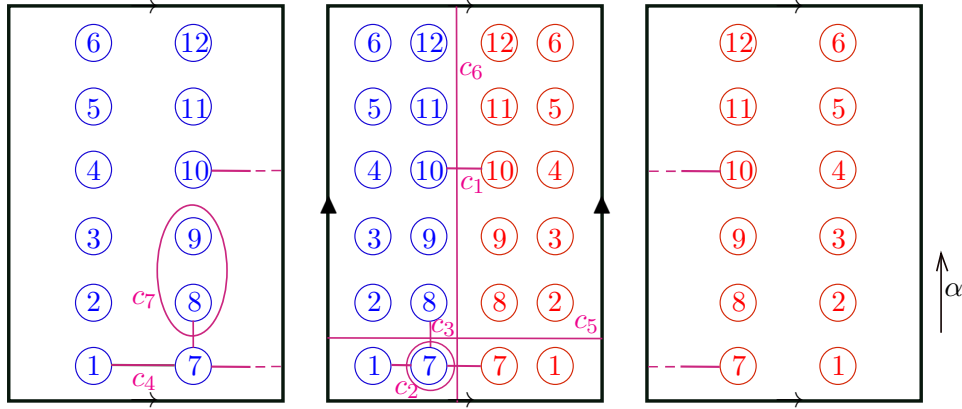


FIGURE 2. The curves used in the proof of Lemma 2.14. Here we are using the model for Σ_{23} , but the proof follows in the same way for similar types of curves on any Σ .

Surface stabilization sequence. We stabilize with the sequence of curves represented in Figure 3. To get the initial subsurface S_0 we can take the subsurface spanned by the chain of curves c_0, c_1, \dots, c_4 . This subsurface has genus 2 and one boundary component, and these curves are Humphries generators for S_0 [FM12, Fig. 4.10]. Next we extend this chain with the curves c_5, \dots, c_{4k-1} ; at this point the left genus-0 subsurface has been filled. Next we stabilize with c_{4k} and the curves $(c_{4k+1}, c_{4k+2}, \dots, c_{8k-2})$ that fill the right genus-0 subsurface; there is some choice in the order of curves we stabilize, but this is not important. Finally we stabilize with the curves c_{8k-1} and c_{8k} that generate $\pi_1(T^2)$.

All of the twists about the curves used are in Γ . In each case, this can be seen either directly from the statement of Lemma 2.14 or by an argument that is a small variation of its proof. Since this collection of curves fills Σ , we have shown that $\Gamma = \text{Mod}(\Sigma, *)$. This proves Proposition 2.10. \square

2.4. Step 4: conclusion. By Proposition 2.3, $\hat{\sigma}(\alpha)$, $\hat{\sigma}(\alpha^2)$ and $\hat{\sigma}(\alpha^p)$ all have the same fixed set, which is a circle $c \subset \hat{X}$. The centralizers $C(\alpha^2)$ and $C(\alpha^p)$ preserve c . By Proposition 2.10, $\text{Mod}(\Sigma, *) = \langle C(\alpha^2), C(\alpha^p) \rangle$, so $\hat{\sigma}(\text{Mod}(\Sigma, *))$ preserves c . This contradicts the fact that $\hat{\sigma}(\pi_1(\Sigma_g))$ acts as the deck group, which as a properly discontinuous action does not preserve any compact set.

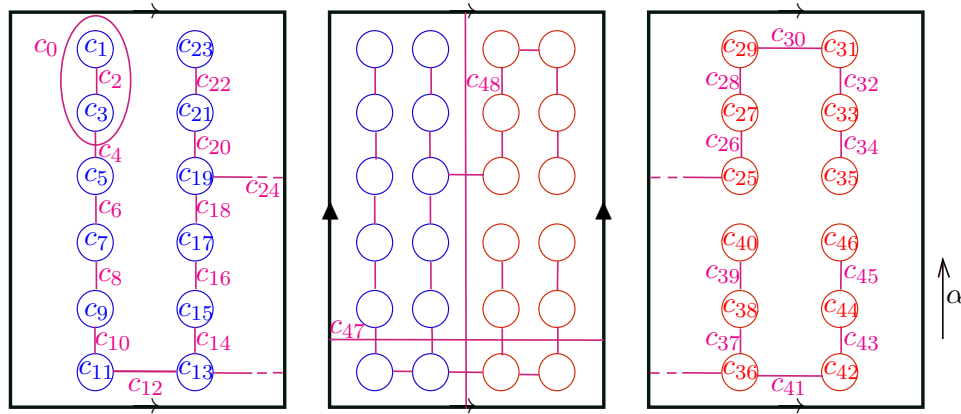


FIGURE 3. The stabilization sequence we use for the case $k = 6$ and $p = 3$.

REFERENCES

- [Bre12] Glen E Bredon. *Sheaf theory*, volume 170. Springer Science & Business Media, 2012.
- [Bro82] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
- [CS22] L. Chen and N. Salter. Global fixed points of mapping class group actions and a theorem of Markovic. *J. Topol.*, 15(3):1311–1324, 2022.
- [CT23] Lei Chen and Bena Tshishiku. Mapping class groups of circle bundles over a surface, 2023.
- [EK71] R. Edwards and R. Kirby. Deformations of spaces of imbeddings. *Ann. of Math. (2)*, 93:63–88, 1971.
- [FM12] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [MS86] W. H. Meeks, III and P. Scott. Finite group actions on 3-manifolds. *Invent. Math.*, 86(2):287–346, 1986.
- [MZ54] D. Montgomery and L. Zippin. Examples of transformation groups. *Proc. Amer. Math. Soc.*, 5:460–465, 1954.
- [Par21] J. Pardon. Smoothing finite group actions on three-manifolds. *Duke Math. J.*, 170(6):1043–1084, 2021.
- [Sco83] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.
- [Sou10] J. Souto. A remark on the action of the mapping class group on the unit tangent bundle. *Ann. Fac. Sci. Toulouse Math. (6)*, 19(3-4):589–601, 2010.

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