Homework 1

Math 106

Due Friday, Sept 15 by 5pm

Name:

Topics covered: curves, arclength

Instructions:

- This assignment must be submitted on Gradescope by the due date. Gradescope Entry Code: XXV57E.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Let* $\alpha : I \to \mathbb{R}^3$ *and* $\beta : I \to \mathbb{R}^3$ *be two curves. Let* $\alpha \cdot \beta : I \to \mathbb{R}$ *be the function defined by* $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$ *(dot product). Prove that*

$$(\alpha \cdot \beta)'(t) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t).$$

Solution. Either use the limit definition, or write in coordinates and use the product rule. $\square$

Problem 2. *Let* $\alpha : I \to \mathbb{R}^3$ *be a curve. Prove that* $|\alpha(t)|$ *is constant if and only if* $\alpha(t)$ *is orthogonal to* $\alpha'(t)$ *for all* $t \in I$.

Solution. Recall that $|\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. The derivative of this function is $2\alpha(t) \cdot \alpha'(t)$. Then $|\alpha(t)|$ is constant if and only if $|\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ is constant if and only if its derivative $2\alpha(t) \cdot \alpha'(t)$ is zero if and only if $\alpha(t)$ and $\alpha'(t)$ are orthogonal for all t. $\square$

Problem 3. *Find a parameterized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ whose trace is the cycloid. Compute the length of one period of the cycloid.*[1]

Solution. The cycloid is parameterized by $\alpha(t) = (t - \sin(t), 1 - \cos(t))$. One way to obtain this is to first consider a translated version where the circle is centered around the origin. If we want to be at $(0, -1)$ at time 0 and travel clockwise, we can parameterize the unit circle by $t \mapsto (-\sin(t), -\cos(t))$. Now we get the cycloid by translating by $(t, 1)$ at time t.

Now we want to compute $\int_0^{2\pi} |\alpha'(t)|\, dt$. First compute

$$\alpha'(t) = (1 - \cos(t), \sin(t))$$

and

$$|\alpha'(t)|^2 = (1 - \cos(t))^2 + \sin^2(t) = 1 - 2\cos(t) + \cos^2(t) + \sin^2(t) = 2 - 2\cos(t).$$

By the half-angle formula $\sin(t/2) = \sqrt{(1 - \cos(t))/2}$; equivalently $\sqrt{2 - 2\cos(t)} = 2\sin(t/2)$.

Now integrate with u-substitution.

$$\int_0^{2\pi} \sin(t/2)\, dt = 2\int_0^{\pi} \sin(u)\, du = 2(-\cos(\pi) + \cos(0)) = 4.$$

Putting it all together, the length is equal to

$$\int_0^{2\pi} |\alpha'(t)|\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt = \int_0^{2\pi} 2\sin(t/2)\, dt = 8.$$

□

[1] Hint: at some point you may want to use the half-angle formula.

Problem 4. *The curve* $\alpha(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$ *for* $t \in [0, \infty)$ *is called the logarithmic spiral. Plot this curve (by hand or with computer). Compute its length (is it finite or infinite?).*

Solution. We compute

$$\alpha'(t) = \big(- e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t)\big)$$

and find that $|\alpha'(t)|^2 = 2e^{-2t}$, so $|\alpha'(t)| = \sqrt{2}e^{-t}$. Then

$$\int_0^\infty |\alpha'(t)|\, dt = \sqrt{2}\int_0^\infty e^{-t}\, dt = \sqrt{2}(-e^{-t})_{t=0}^{t=\infty} = \sqrt{2}.$$

□

Problem 5. *Let $\alpha : [a, b] \to \mathbb{R}^3$ be a curve. Let v be a unit vector. Prove that*

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

Choose v appropriately to deduce that the shortest path between any two points is a straight line.

Solution. The equality holds by FTC and because the derivative of $\alpha(t) \cdot v$ is $\alpha'(t) \cdot v$. The inequality holds by the Schwarz inequality

$$\alpha'(t) \cdot v \leq |\alpha'(t) \cdot v| \leq |\alpha'(t)| \, |v| \, |\cos(\theta)| \leq |\alpha'(t)|.$$

(Note that $|v| = 1$.)

Plugging in $v = (\alpha(b) - \alpha(a))/|\alpha(b) - \alpha(a)|$ we find

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt.$$

Then if $p = \alpha(a)$ and $q = \alpha(b)$, we conclude that the preceding inequality holds for any path between p and q. Thus the length of any path is at least as long as the length of the straight line. □

Problem 6. *Fix a curve $\alpha : I \to \mathbb{R}^3$ and fix $[a, b] \subset I$. For a partition*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

we defined $L(\alpha, P) = \sum |\alpha(t_{i+1}) - \alpha(t_i)|$ and $|P| = \max(t_{i+1} - t_i)$. Prove that for each $\epsilon > 0$, there exists $\delta > 0$ so that if $|P| < \delta$, then

$$\left| \int_a^b |\alpha'(t)|\, dt - L(\alpha, P) \right| < \epsilon.$$

[2] [3]

Solution. By definition of the integral, there exists $\delta' > 0$ so that if $|P| < \delta'$, then

$$\left| \int_a^b |\alpha'(t)|\, dt - \sum |\alpha'(t_i)|(t_{i+1} - t_i) \right| < \epsilon/2.$$

Therefore, it suffices to show we can choose $|P|$ small enough so that

$$\left| \sum |\alpha'(t_i)|(t_{i+1} - t_i) - |\alpha(t_{i+1}) - \alpha(t_i)| \right| \le \sum \left| \alpha'(t_i)(t_{i+1} - t_i) - (\alpha(t_{i+1}) - \alpha(t_i)) \right| < \epsilon/2.$$

(Here we use the triangle inequality and the reverse triangle.) By the mean-value theorem for vector valued functions applied to the function $f(t) = \alpha(t) - \alpha'(t_i)t$, there exists $t_i \le s_i \le t_{i+1}$ so that

$$|\alpha(t_{i+1}) - \alpha(t_i) - \alpha'(t_i)(t_{i+1} - t_i)| = |f(t_{i+1}) - f(t_i)| \le |f'(s_i)|(t_{i+1} - t_i) = |\alpha'(s_i) - \alpha'(t_i)|(t_{i+1} - t_i).$$

Since α' is uniformly continuous on $[a, b]$, there exists $\delta'' > 0$ so that if $|s - t| < \delta''$ then $|\alpha'(s) - \alpha'(t)| \le \frac{\epsilon}{2(b-a)}$. Now we have

$$\begin{aligned} \sum \left| \alpha'(t_i)(t_{i+1} - t_i) - (\alpha(t_{i+1}) - \alpha(t_i)) \right| &\le \sum |\alpha'(s_i) - \alpha'(t_i)|(t_{i+1} - t_i) \\ &\le \sum \tfrac{\epsilon}{2(b-a)}(t_{i+1} - t_i) \\ &= \epsilon/2 \end{aligned}$$

Then if we choose $\delta = \min\{\delta', \delta''\}$, we conclude that

$$\begin{aligned} \left| \int_a^b |\alpha'(t)|\, dt - L(\alpha, P) \right| &< \left| \int_a^b |\alpha'(t)|\, dt - \sum |\alpha'(t_i)|(t_{i+1} - t_i) \right| \\ &\quad + \left| \sum |\alpha'(t_i)|(t_{i+1} - t_i) - \sum |\alpha(t_{i+1}) - \alpha(t_i)| \right| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

□

We also give a second proof, which is a bit more clever.

[2]Suggestion: First replace $\int_a^b |\alpha'(t)|\, dt$ by a Riemann sum, and compare the Riemann sum to $L(\alpha, P)$. For the latter, it may help to use the mean value theorem for vector-valued functions (you will need to figure out which function to apply it to).

[3]This problem is probably harder than most HW problems for the course. Please ask for help if you are stuck.

Solution. First we define a function $F : [a,b]^3 \to \mathbb{R}$ be the function $(s_1, s_2, s_3) \mapsto |(x'(s_1), y'(s_2), z'(s_3))|$. By applying the mean value theorem coordinate-wise, we find

$$|\alpha(t_{i+1}) - \alpha(t_i)| = F(s_{i1}, s_{i2}, s_{i3})(t_{i+1} - t_i)$$

On the other hand, by the mean value theorem for integrals gives

$$\int_{t_i}^{t_{i+1}} |\alpha'(t)|\, dt = |\alpha'(s_i)|(t_{i+1} - t_i) = F(s_i, s_i, s_i)(t_{i+1} - t_i).$$

Now we deduce the result directly from the fact that F is uniformly continuous. $\square$