Problem 1. *Show that the set $\{(x,y,z) : x^2+y^2=1\}$ is a surface by covering it with coordinate charts. (What surface is this?)*

Solution. To show the set $\{(x,y,z) : x^2+y^2=1\}$ is a surface, we must prove it is possible to cover it with coordinate charts.

First, we construct a coordinate chart for the set $\{(x,y,z) : x^2+y^2=1\}$ in terms of θ and v.

$$\phi(\theta, v) = (cos\theta, sin\theta, v), \theta \in (0, 2\pi), v \in (-\infty, \infty)$$

The surface described by the set is a cylinder with a unit circle base stretching infinitely in the positive and negative z-direction.

To prove that the set is a surface, we must prove that the parameterization of the set is a valid coordinate chart.

First, we prove that $\phi(\theta, v)$ is injective. Suppose (θ_0, v_0) and (θ_1, v_1) are two points where $\theta_0, \theta_1 \in (0, 2\pi), v_0, v_1 \in (-\infty, \infty)$ and that $\phi(\theta_0, v_0) = \phi(\theta_1, v_1)$.

If $\phi(\theta_0, v_0) = \phi(\theta_1, v_1)$, then $(cos\theta_0, sin\theta_0, v_0) = (cos\theta_1, sin\theta_1, v_1)$. Of the possible values of $\theta \in (0, 2\pi)$, there is only one distinct θ that maps to any combination of $cos\theta$ and $sin\theta$. Therefore, $\theta_0 = \theta_1$. In addition, only one distinct $v \in (-\infty, \infty)$ maps to that specific v. Therefore, $v_0 = v_1$. This means $(cos\theta_0, sin\theta_0, v_0) = (cos\theta_1, sin\theta_1, v_1)$ if $(\theta_0, v_0) = (\theta_1, v_1)$. Since $\phi(\theta_0, v_0) = \phi(\theta_1, v_1) \to (\theta_0, v_0) = (\theta_1, v_1)$, we can conclude that ϕ is injective.

Then, we prove that ϕ is smooth. Since $x^2+y^2=1$ is smooth, the parameterization of $x^2+y^2=1$ given by ϕ is also smooth. Therefore, ϕ is smooth.

We must also prove that ϕ is regular by proving $D\phi$ is injective.

$$\begin{aligned} D\phi &= \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} -sin\theta & 0 \\ cos\theta & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

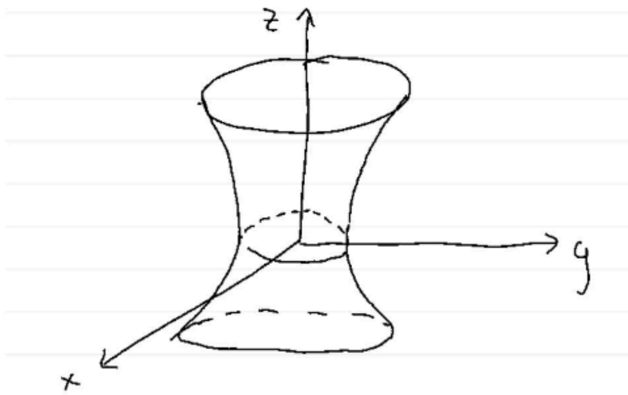
Since $D\phi$ is linearly independent, $D\phi$ is injective. Therefore ϕ is regular.

By proving that ϕ is injective, smooth, and regular, we can conclude it is a valid coordinate chart. However, since ϕ is defined on the open sets $\theta \in (0, 2\pi), v \in (-\infty, \infty)$, it does not include the part of the set when $\theta = 0$. To account for this, we must cover the surface with two coordinate charts, $\phi(\theta, v), \theta \in (0, 2\pi), v \in (-\infty, \infty)$ and $\phi(\theta, v), \theta \in (-\pi, \pi), v \in (-\infty, \infty)$ to fully cover the surface.

Since the set can be covered with coordinate charts, we can conclude it is a surface.

□

Problem 2. *The equation $x^2+y^2-z^2=1$ defines a surface. Give a plot of this surface. Determine the tangent planes of the surface at the points $(x, y, 0)$ and show that they contain the z-axis.*



Solution.

Let S denote the set of points on the surface. Consider the chart $\phi : U \subset \mathbb{R}^2 \to S$ defined by $\phi(x, z) = (x, \sqrt{1+z^2-x^2}, z)$ where $x \in (-1, 1)$, $z \in (-\infty, \infty)$. Then $\phi_x(x, z) = (1, -\frac{x}{\sqrt{1+z^2-x^2}}, 0)$ and $\phi_z(x, z) = (0, \frac{1}{\sqrt{1+z^2-x^2}}, 1)$. So for any point p on the surface of the form $(x, y, 0)$ where $y > 0$, we have $\phi_x(x, 0) = (1, -\frac{x}{1-x^2}, 0)$ and $\phi_z(x, 0) = (0, 0, 1)$. The normal vector field of ϕ for these points is

$$N(x) = \phi_x(x, 0) \times \phi_z(x, 0) = \left(-\frac{x}{\sqrt{1-x^2}}, 1, 0\right).$$

As the z-coordinate of N is 0, we see that the normal vector of p is in the xy-plane. We also have $T_pS = \text{span}(N(x))^\perp = \{v \in \mathbb{R}^3 \mid v \cdot N(x) = 0\}$. Now, notice that $k = (0, 0, 1) \in T_pS$. But the z-axis is the line spanned by k, thus T_pS contains the z-axis. We arrive at the same conclusion for points p where $y < 0$ by using $\phi_2(x, z) := (x, -\sqrt{1+z^2-x^2}, z)$.

Next, consider the chart $\phi_3(y, z) = (\sqrt{1+z^2-y^2},\, y,\, z)$. Then

$$N(y) = \phi_{3y}(y, 0) \times \phi_{3z}(y, 0) = \left(-\frac{y}{\sqrt{1+y^2}}, 1, 0\right) \times (0, 0, 1) = \left(1, -\frac{y}{\sqrt{1+y^2}}, 0\right)$$

which gives $T_pS = \{v \in \mathbb{R}^3 \mid v \cdot N(y) = 0\}$ for $p = (x, y, 0)$ where $x > 0$. Similarly, $k = (0, 0, 1) \in T_pS$, so T_pS contains the z-axis. We also arrive at the same conclusion with $\phi_4(y, z) = (-\sqrt{1+z^2-y^2},\, y,\, z)$ for p where $x < 0$. Thus the tangent planes of the surface at all points $(x, y, 0)$ contain the z-axis. $\square$

Problem 3. *Let $N = (0,0,1) \in S^2$ be the north pole. Define stereographic projection $\pi : S^j \backslash \{N\} \to \mathbb{R}^2$ as follows. Given $p = (x,y,z)$, define $\pi(p)$ as the intersection of the line between N and p with the xy-plane. Determine a formula for the inverse of π.* [1] [2]

Solution. Consider an arbitrary point $q = (u,v,0) \in \mathbb{R}^3$ on the xy-plane. The line that connects this point to the north pole is given by

$$r(t) = N + t(q - N) = (ut, vt, 1-t), \tag{1}$$

where $t \in \mathbb{R}$. For every $p = (x,y,z) \in S^2$, we have $x^2 + y^2 + z^2 = 1$. Thus the points on r that intersect S^2 satisfy

$$\begin{aligned} 1 &= (ut)^2 + (vt)^2 + (1-t)^2 \\ &= u^2t^2 + v^2t^2 + t^2 - 2t + 1. \end{aligned}$$

As $r(0) = N$, consider when $t \neq 0$. Then the equation above yields $u^2t + v^2t + t - 2 = 0$. Solving for t yields $t = 2/(u^2 + v^2 + 1)$. Substituting into (1) gives

$$\pi^{-1}(u,v,0) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

□

[1] Hint: Given $(u,v) \in \mathbb{R}^2$, find a formula for the line between $(u,v,0)$ and $(0,0,1)$ and compute the intersection of this line with S^2.

[2] This problem gives yet another way to cover the sphere by coordinate charts.

Problem 4. *Let $\alpha : [0,1] \to \mathbb{R}^2$ be a plane curve that lies in the right-half plane and meets the y axis at its endpoints $\alpha(0), \alpha(1)$. What conditions should α satisfy to ensure that the set obtained by rotating this curve about the y-axis is a surface? Your explanation should be rigorous.*

Solution. First we require

- α is be simple (no self-intersection).

This is necessary since surfaces don't have self-intersections by assumption (as discussed in class).

We write $\alpha(t) = (\alpha_1(t), \alpha_2(t), 0)$. Next we require

- $\alpha_1'(0) \neq 0$.

With this assumption, after re-parameterizing we can write $\alpha(r) = (r, f(r))$ at least r near 0 (this is the inverse function theorem applied to α_1; here $f = \alpha_2 \circ \alpha_1^{-1}$).

Next we require

- The function $(x,z) \mapsto f\big(\sqrt{x^2+z^2}\big)$ is smooth.

If we assume this, then $(x,z) \mapsto \big(x, f(\sqrt{x^2+z^2}), z\big)$ defines a chart around $\alpha(0)$. We make a similar assumption to get a chart around $\alpha(1)$.

We could stop here, but it's a bit unsatisfying. If we're given f, how do we decide if $(x,z) \mapsto f\big(\sqrt{x^2+z^2}\big)$ is smooth? Since this function is radially symmetric, we reduce to studying the smoothness of

$$g(x) = f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0. \end{cases}$$

Claim. g is smooth if and only if the odd-order derivatives of f vanish.

Proof of Claim. We can compute the derivatives of g piecewise (away from 0). In order for the left-and right-hand derivatives at 0 to agree, the odd derivative of g must vanish. Conversely, if the odd order derivatives of f vanish, then we see that g is smooth.

Thus our final condition is

- The odd order derivatives of $f = \alpha_2 \circ \alpha_1^{-1}$ vanish.

$\square$

Problem 5. *Give a coordinate system for the Möbius strip. Do the same for an annulus with two twists. Include an image of your surfaces, plotted using* `ParametricPlot3D` *in Mathematica, or similar.*

Solution. We will use two coordinate charts Φ_1, Φ_2, defined on

$$\Phi_1 : (-1,1) \times (0, 2\pi) \to S \qquad \Phi_2 : (-1,1) \times (-\pi, \pi) \to S$$

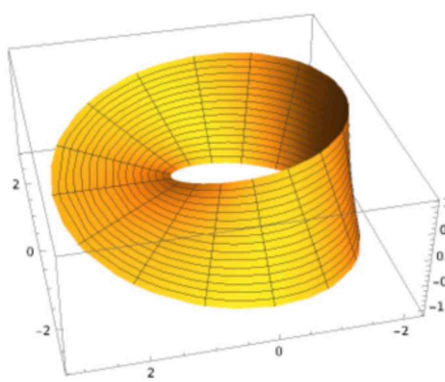
Note: We're only really using two functions to hit the "extra line" at the boundary that cannot be touched by an injective coordinate map as we wish. The maps are both defined as follows:

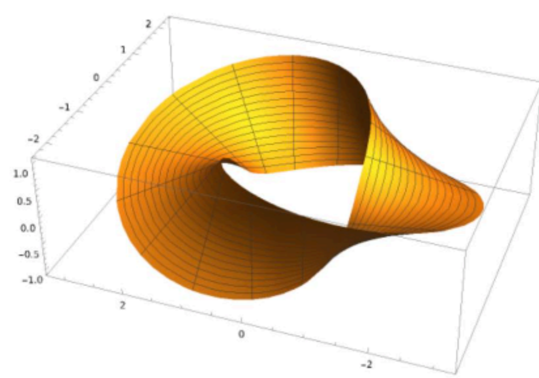
$$\begin{aligned} \Phi_i(u,v) &= 2(\sin v, \cos v, 0) + u\left(\cos\left(\frac{v}{2}\right)\sin v, \cos\left(\frac{v}{2}\right)\cos v, \sin\left(\frac{v}{2}\right)\right) \\ &= \left(2\sin v + u\cos\left(\frac{v}{2}\right)\sin v, 2\cos v + u\cos\left(\frac{v}{2}\right)\cos v, u\sin\left(\frac{v}{2}\right)\right). \end{aligned}$$

Here, the first term will be rotating around the unit circle in the xy plane, which will form the center of our mobius strip. The second term is the "offset" from this center, multiplied by u, which corresponds to the position width-wise on the strip. The $\sin(v/2), \cos(v/2)$ terms correspond to the single flip, as it will be a 180 degree rotation after going around the unit circle. For two twists, we replace these terms with $\sin v$ and $\cos v$. Using the same domains as before, we get maps $\Phi_i'(u,v)$ defined as follows:

$$\Phi_i'(u,v) = (2\sin v + u\cos(v)\sin v, 2\cos v + u\cos(v)\cos v, u\sin(v)).$$

Plugging these into Mathematica, we get the following two graphs:

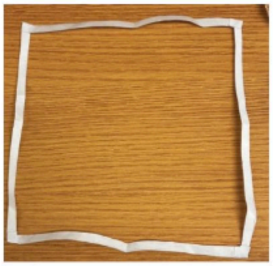




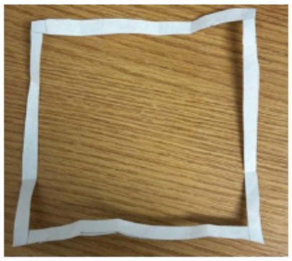
$\square$

Problem 6 (Bonus). *Form a surface by gluing two cylinders along a square in such a way that the central curves of the two cylinders meet at a right angle. Cut each cylinder down the middle. What is the result? Repeat with gluing a cylinder and a Möbius band, and with gluing two Möbius bands. Include pictures of the results.*

Solution. The result of gluing 2 cylinders and cutting down the middle will be a rectangle (or a square if 2 cylinders have the same diameters.

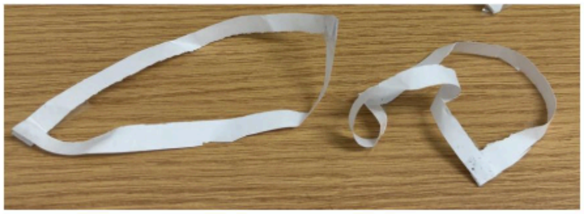


The result of gluing a cylinder and a Möbius band is also a rectangle.



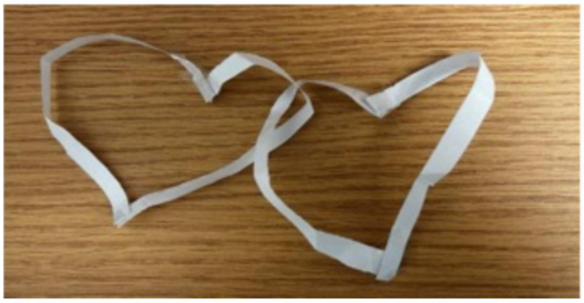
The result of gluing 2 Möbius bands depends on the direction in which each band is twisted.

- If the 2 bands are twisted in the same direction, the result will be 2 separate pieces, one twisted and one with boat-like shape.



Bravo!

- If the 2 bands are twisted in opposite direction, the result will be 2 hearts linking.



□