Problem 1. *Determine which trees have Prüfer codes that*

1. *contain only one value;*
2. *contain exactly two values;*
3. *have distinct values.*

You should explain your answer, but you don't need to give careful proof.

Solution. Let T be a tree with n vertices satisfying one of the given Prüfer code properties given

1. Such a tree must have a single vertex v connected to every other node. This is because our Prufer code indicates that all non-v nodes are leaves (else they would be recorded in the code)
2. As in the above logic, we would this time require two nodes, v, w, such that $\{v, w\} \in E$ and all other edges are either connections to v or w. This is because all other nodes are required to be leaves, by the above logic, but our tree is connected, and so the only possible connection is $\{v, w\}$.
3. Such trees would be exactly the trees given by P_i. If no vertex appears twice in the code, we can have no vertex with degree ¿2. If such a thing were to happen with vertex v, we would need for only one of the neighbors to get "deleted" in our prufer code creation (as when a vertex gets deleted, its neighbor, in this case v, gets recorded). Therefore the two remaining nodes at the end of the deletion process would be connected (this occurs at the end of every Prufer code creation process), and we would have a three cycle, as both of these two elements would be connected to v as well. Therefore, as the degree of each element would be no greater than 2, and our graph would be connected and without cycles (for it would be a tree), we would have a path.

□

[redacted]

Problem 2. *Prove that if $T_1, \ldots, T_k$ are pairwise-intersecting subtrees of a tree T, then T has a vertex that belongs to all of $T_1, \ldots, T_k$.* [1] [2]

Before completing the proof, it is helpful to introduce a lemma.

Lemma: For $T_1 = (V_1, T_1), T_2 = (V_2, T_2)$ subtrees of a tree $T = (V, E)$ such that $T_1 \cap T_2 \neq \emptyset$, the intersection of T_1 and T_2, $T_1 \cap T_2 = (V_1 \cap V_2, E_1 \cap E_2)$, is a subtree of T.

Notice that $T_1 \cap T_2$ is contained in T and that it cannot be the case that a cycle is contained in $T_1 \cap T_2$ because that would imply that a cycle is contained in T.

Therefore, all that remains to be shown is that $T_1 \cap T_2$ is connected.

Aiming for proof by contradiction, assume that $T_1 \cap T_2$ is disconnected. Then there exists $v_1, v_2 \in V_1 \cap V_2$ such that there does not exist a path between v_1 and v_2 in $T_1 \cap T_2$. Yet, T_1 is connected and contains v_1 and v_2. Therefore, there is a path between v_1 and v_2 in T_1. The same holds for T_2. However, these paths are not identical because, if they were, they would be contained in $T_1 \cap T_2$ and v_1 and v_2 would not be disconnected. Since there are two distinct pathes between v_1 and v_2 contained in T, T is not a tree, and we arrive at contradiction. Therefore, it cannot be the case that $T_1 \cap T_2$ is disconnect. Thus, $T_1 \cap T_2$ must be a connected graphs with no cycles, indicating that it is a tree as desired. This completes the proof of the lemma.

Solution.

Let $T = (V, E)$, and let $T_1 = (V_1, E_1), \ldots, T_k = (V_k, E_k)$ be a collection of subtrees of T such that $V_i \cap V_j \neq \emptyset$ for all $1 \leq i < j \leq k$.

Using proof by induction on k, I aim to show that T has a vertex that belongs to all $T_1, \ldots, T_k$.

Base cases:

If $k = 1$ a vertex of T belongs to T_1 by the definition of a subtree

If $k = 2$ we have that $T_1 \cap T_2 \neq \emptyset$ since T_1, T_2 are pairwise intersecting. This implies that some vertex v of T is shared by T_1, T_2 by definition.

Now, assume that $k = 3$. Aiming for proof by contradiction, assume that $V_1 \cap V_2 \cap V_3 = \emptyset$. Then there exists $u_3 \in V_1 \cap V_2$ such that $u_3 \notin V_3$, $u_2 \in V_1 \cap V_3$ such that $u_2 \notin V_2$, and $u_1 \in V_2 \cap V_3$ such that $u_1 \notin V_1$.

Since T_1 is tree such that $v_2, v_3 \in V_1$, there exists a single path in T_1 starting form v_2 and running to v_3.

Now, let $U = T_1 \cup T_2$ and notice that U is disjoint from T_1 by assumption. Since T_2 is a tree such that $v_1, v_3 \in T_2$ there exists a single path in U starting from v_1 and running to v_3. Since T_3

[1] Remark: This is a graph-theoretic analog of Helly's theorem.

[2] Hint: use induction on k.

is a tree such that $v_1, v_2 \in T_3$, there exists a single path in U starting from v_2 and running to v_1.

Thus, we see that $v_2 \to v_3$ is a path between v_2 and v_3 in T and we see that $v_2 \to v_1$ joined with $v_1 \to v_3$ is a walk from v_2 to v_3 in T. In particular, these two routes are distinct since the first is contained in T_1 and the second is contained in U which is disjoint from T_1

Therefore, there are two distinct paths between v_2 and v_3 in T. This implies that T contains a cycle which contradicts the fact that T is a tree.

Therefore, it cannot be the case that $V_1 \cap V_2 \cap V_3$ is disjoint, completing the proof for $k = 3$.

Inductive Step: Assume that T has at least one vertex that belongs to all $T_1, \ldots, T_k$ for some collection of pairwise-intersecting subtrees of a tree T up to some $k \in \mathbf{N}$. We will show that for the statement holds for $k + 1$ as well.

Let $T_1, \ldots, T_k, T_{k+1}$ be a collection of pairwise intersecting subtrees of a tree T. Now, consider $T_k' = T_k \cap T_{k+1}$. By the lemma, T_k' is a subtree of T. Now, consider $V_k' \cap V_i = V_{k+1} \cap V_k \cap V_i$ for any $1 \leq i \leq k - 1$. By the base case for $k = 3$, we have that $V_k' \cap V_i \neq \emptyset$. Hence $T_1, \ldots, T_{k-1}, T_k'$ is a collection of pairwise intersecting subtrees of a tree T and the inductive hypothesis applies, giving that

$$\bigcap_{i=1}^{k} V_i = \left(\bigcap_{i=1}^{k-1} V_i \right) \cap V_k \cap V_{k+1} = \left(\bigcap_{i=1}^{k-1} V_i \right) \cap V_k' \neq \emptyset$$

which demonstrates that T has a vertex which belongs to all of $T_1, \ldots, T_{k+1}$ as desired. $\square$

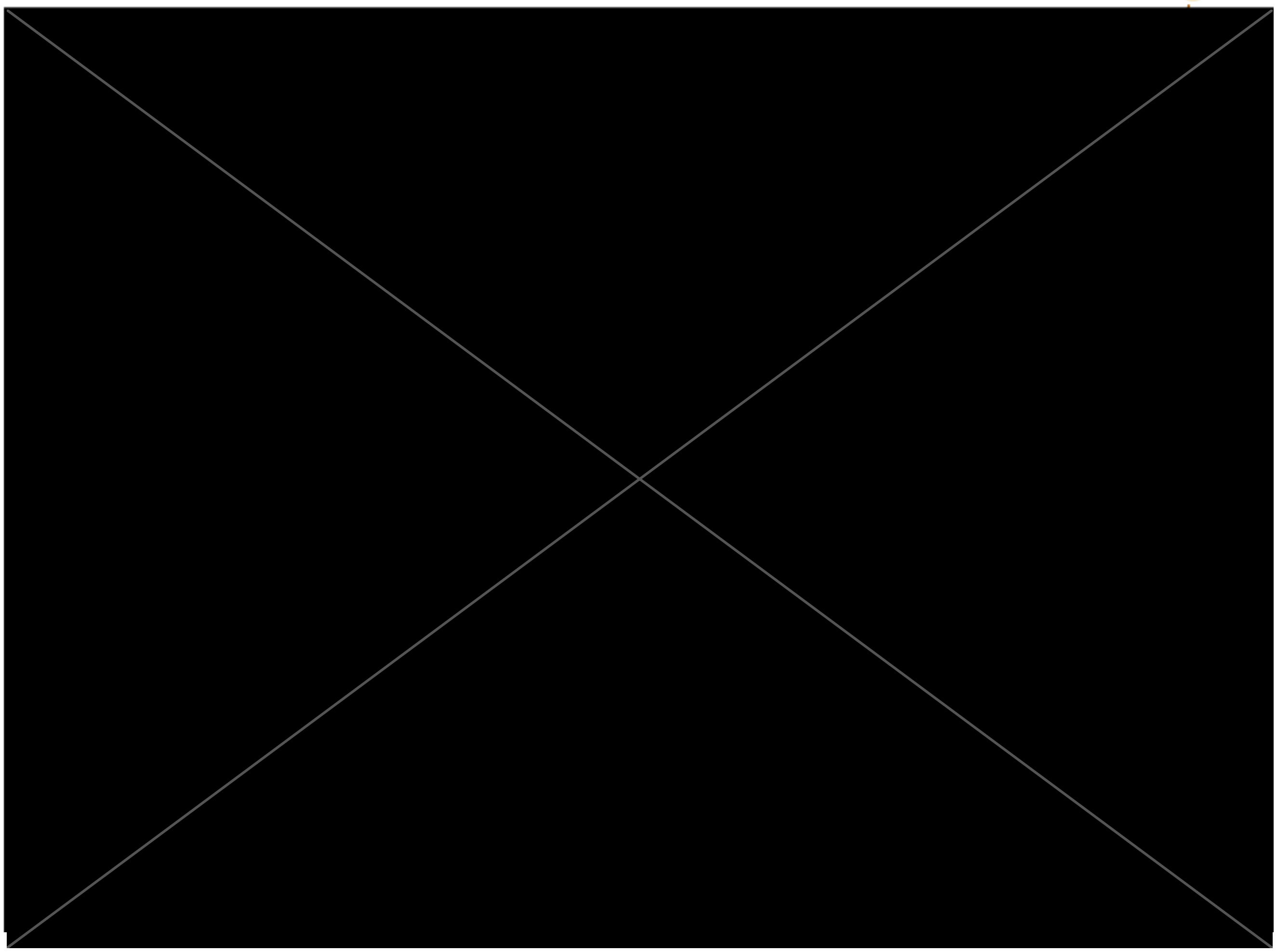
[redacted]

Problem 2. *Prove that if $T_1, \dots, T_k$ are pairwise-intersecting subtrees of a tree T, then T has a vertex that belongs to all of $T_1, \dots, T_k$.* [1] [2]

Solution. I will prove with induction. Consider the base case with a single vertex, all subtrees are a single vertex, so trivially that vertex belongs to all of any non-empty subtrees. Let us now consider a tree T with n vertices and subtrees $T_1, \dots, T_k$. Every tree has at least one vertex of degree 1

[1] Remark: This is a graph-theoretic analog of Helly's theorem.
[2] Hint: use induction on k.

(proved in class and on last problem set). Let us call this vertex v. Consider the tree $T \setminus v$ and the subtrees $T_1 \setminus v, \ldots, T_k \setminus v$ because v has degree 1 these are all still connected and still trees. $T_1 \setminus v, \ldots, T_k \setminus v$ still are pairwise-intersecting because any two trees which intersect on v must also intersect on u, for u being the only vertex which v is connected to. The only exception to this rule would be if there exists $T_i = v$ if this is the case then every subtree intersects pairwise with T_i so every subtree includes v so there is a vertex belonging to all $T_1 \setminus v, \ldots, T_k \setminus v$. Assuming no such tree exists, $T \setminus v$ has size $n - 1$ and $T_1 \setminus v, \ldots, T_k \setminus v$ all intersect pairwise so from our induction hypothesis, we know there is a vertex belonging to all $T_1 \setminus v, \ldots, T_k \setminus v$. Because $T_i \setminus v \subset T_i$, this vertex also belongs to all $T_1, \ldots, T_k$. So for any set of pairwise-intersecting subtrees of a tree T, there exists a vertex belonging to all of them. $\square$



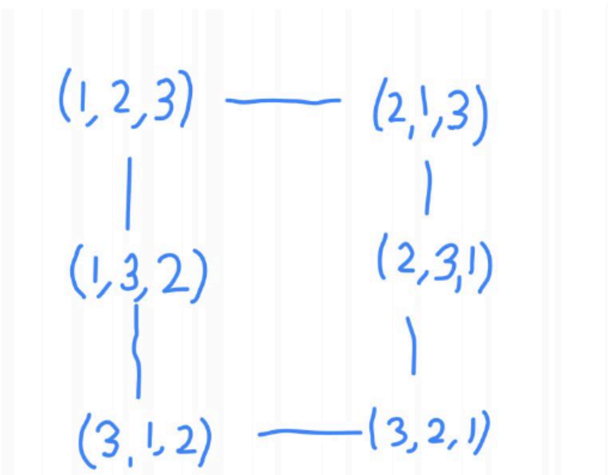
Problem 3. *Let G_n be the graph whose vertices are orderings of the elements of $\{1, \ldots, n\}$ with $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ adjacent if they differ by switching a pair of adjacent entries.*[3]

(a) The graph G_3 is isomorphic to a familiar graph. Which one is it?

(b) Show that G_n is connected. [4]

Solution. Collaborated with Alex Duchnowski

(a)



The above is clearly isomorphic to the familiar graph C_6.

(b)

For a particular ordering $(a_1, \ldots, a_n)$, we define $f(a_1, \ldots, a_n)$ to be the number of pairs (a_i, a_j) such that $i < j$ but $a_i > a_j$. If a particular ordering is not already in ascending order, then let k be the smallest index such that $a_k > a_{k+1}$ (the lack of existence of such an k is exactly to say the ordering is in order). Then, swapping a_i and a_j reduces the value of f by exactly 1 (as the pair (a_k, a_{k+1}) is removed from f's count and no other pairs (a_i, a_j) are affected by this swap). Repeatedly iterating this process must eventually yield an ordering for which f is zero (as each step reduces the value of f by 1, and this value is initially bounded by n^2); this value of f occurs at the ordering $(1, \ldots, n)$.

This iterative process is exactly constructing a path from any ordering $(a_1, \ldots, a_n)$ to the ordering $(1, \ldots, n)$. So all vertices are in the connected component of $(1, \ldots, n)$ and the graph is connected.

□

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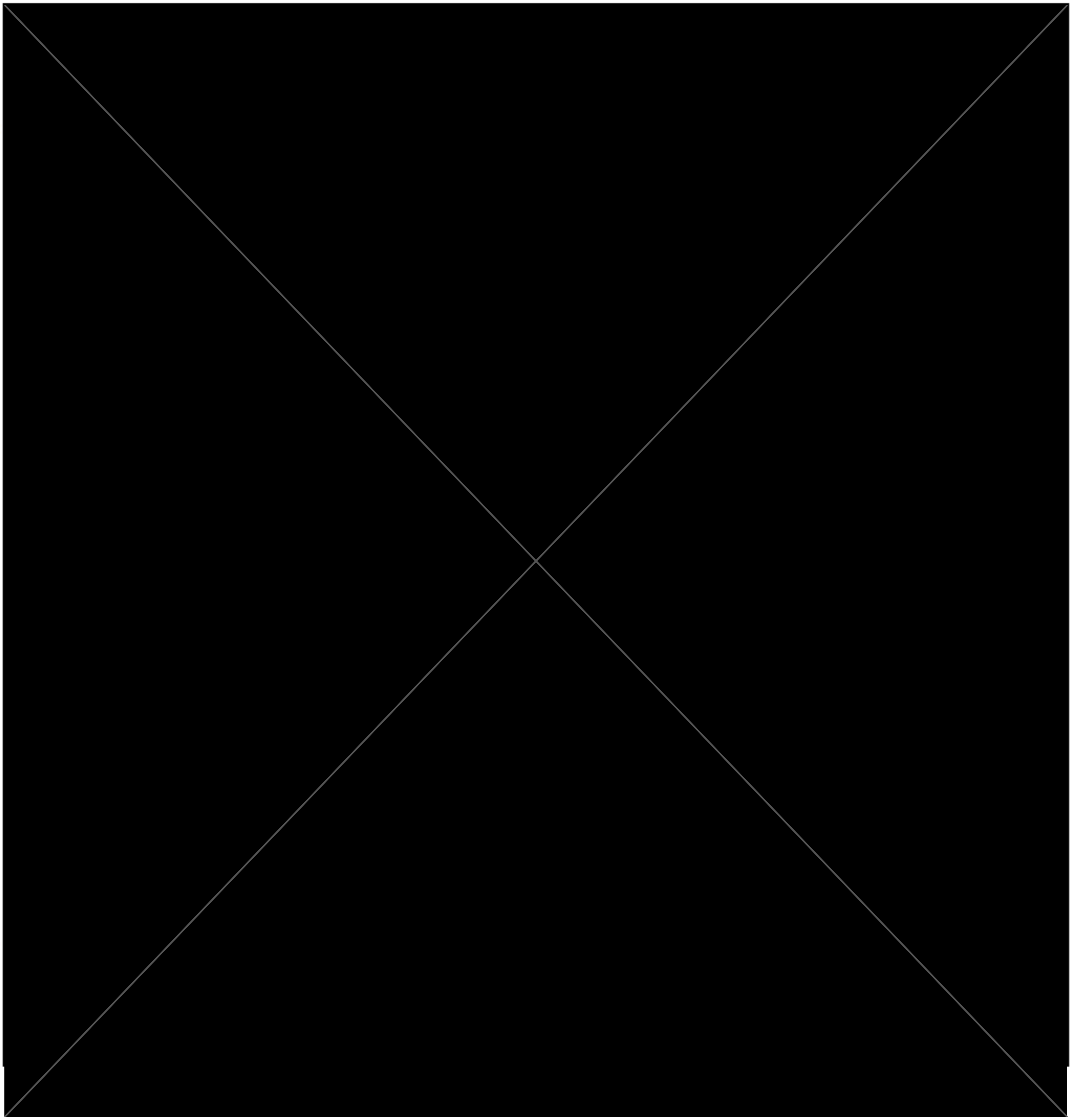
Problem 4. *Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.*

Solution. First observe that due to the special symmetric property of complete graphs, $\forall e \in E(K_n)$: the number of spanning trees that contain e is constant. Let this constant be k. Observe that $|E(K_n)| = \frac{n(n-1)}{2}$. Then, we know that the sum of all edges in all spanning trees is $\sum_{i=1}^{\frac{n(n-1)}{2}} k = \frac{n(n-1)}{2}k$. On the other hand, by the Cayley's theorem, K_n has n^{n-2} different spanning trees. By the definition of tree, each tree has exactly $n-1$ edges, so the sum of all edges in all spanning trees is $\sum_{i=1}^{n^{n-2}}(n-1) = n^{n-2}(n-1)$.

$$\frac{n(n-1)}{2}k = n^{n-2}(n-1)$$

$$k = 2n^{n-3}$$

This means that there are $2n^{n-3}$ spanning trees that contain our removed edge, so subtracting k from the total number of spanning trees for K_n, we get $n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$. $\square$



Problem 5. *Call a graph "even" if every vertex has even degree. Prove that the number of even graphs with vertex set* $\{1, \ldots, n\}$ *is* $2^{\binom{n-1}{2}}$. 5

Solution. Say the set of even graphs with n vertices is E_n, and any graph with n vertices is A_n. We construct the following functions:

1. $f : E_n \mapsto A_{n-1}$ removes the vertex n
2. $g : A_{n-1} \mapsto E_n$ adds a vertex n and connects all odd-degree vertices to n.

We can prove that g maps to an even graph because all the odd-degree vertices in $\{1, \cdots, n-1\}$ will become even-degree vertices when an edge is added. n can only be odd if there were an odd number of odd-degree edges. This is not possible by the degree sum formula, which dictates that the sum of degrees is even.

It is sufficient to prove that f and g are inverses of each other:

1. $\forall G \in E_n, f(g(G)) = G$:

   Since both the vertex added and removed was v, and all edges added and removed connect to v. The subgraph that does not contain v stays unchanged during the transformation.

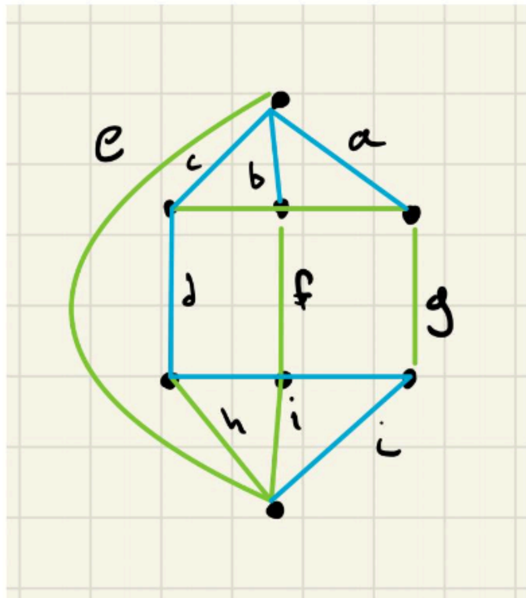
2. $\forall G \in A_{n-1}, g(f(G)) = G$:

   When vertex n is removed, the only vertices that become odd are the ones directly connecting to it, therefore they are the only ones connected back to n when we add n.

Therefore, there exists a bijection between E_n and A_{n-1}. The number of graphs with $n-1$ edges is $2^{\binom{n-1}{2}}$, so the same number of even graphs exist. $\square$

Problem 6. *Consider an alternative version of Bridg-it, where the player that forms a path connecting their ends loses. Give a strategy that shows that Player 2 can always win.* [6]

Solution. We have the same set up as the setup that we had in class: two spanning trees. See diagram:



Except, this time player two does not remove the "invisible edge e that connected the bottom and top point. So player one goes first adding an edge to one of the trees T (green) or T' (blue). By the statement in this footnote, player two can remove some single edge (not doubled), such that the tree that just "gained an edge" is still spanning. We only run into a problem with this if player two is forced to remove this imaginary edge e. We would need to remove this edge if the tree T has another path going to the top, making a cycle that includes e.
Case 1: these edges are all already doubled. In this case we are done because player 1 has already lost
Case 2: There is at least one edge that is not e in T that isn't double. In this case, we can remove this edge as player 2. We can do this because it forms part of a cycle, so the two points on that edge are still connected vie the other direction around the cycle. Thus tree must still be spanning. Thus, the Player one will necessarily end with a spanning tree, and will lose. □