IMPA minicourse exercises

Constructions of exotic spheres

(1) For each of the constructions of exotic 7-sphere given in lecture give a variation of the construction that produces the standard sphere S^7.

(2) Show that the inclusion of a disk $D^n \hookrightarrow S^n$ induces an isomorphism $\pi_0 \operatorname{Diff}(D^n, \partial D^n) \cong \pi_0 \operatorname{Diff}(S^n)$.[13] Use this to show that $\Sigma_{\phi\circ\psi} \cong \Sigma_\phi \# \Sigma_\psi$, where Σ_ϕ denotes $D^n \cup_\phi D^n$ and $\phi, \psi \in \operatorname{Diff}^+(S^{n-1})$. (This shows the natural "clutching" map $\pi_0 \operatorname{Diff}^+(S^n) \to \Theta_n$ is a homomorphism.)

(3) (Challenge) Use one of the constructions of exotic 7-spheres to give an explicit construction of a diffeomorphism of S^6 that is not isotopic to the identity. (Which construction seems most useful?)

(4) (Reading project) Check out the paper "Eight faces of the Poincaré homology sphere" by Kirby–Scharlemann, which gives eight constructions of the Poincaré homology 3-sphere and explains why they are equivalent. Some of these are similar to the constructions of exotic 7-spheres from lecture.

Warped product metrics

(1) Check that the warped product $\mathbb{R} \times_{\cosh t} \mathbb{R}$ is a model for hyperbolic space. Describe geodesics in this model.

(2) Prove that a surface of revolution is isometric to a warped product and derive the Bishop–O'Neill formula for sectional curvature in this case.[14]

(3) Let $\mathbb{D}$ be the Poincaré disk model of the hyperbolic plane, so the metric is $ds^2 = 4\frac{dx^2+dy^2}{(1-r^2)^2}$, where $r^2 = x^2 + y^2$. Consider the exponential map $\mathbb{R}^2 = T_o\mathbb{D} \to \mathbb{D}$, where o is the origin. The pullback of the hyperbolic metric on $\mathbb{D}$ to $\mathbb{R}^2$ is a warped product metric. Compute the warping function.[15]

Gluing and homeomorphisms

(1) Fix manifolds A, B and assume ∂A and ∂B are diffeomorphic. For a diffeomorphism $\phi : \partial A \to \partial B$, write $M_\phi = A \cup_\phi B$.
   (a) Prove that if ϕ, ψ are isotopic then M_ϕ and M_ψ are diffeomorphic.

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[13] Hint: there is a fibration $\operatorname{Diff}^+(S^n) \to \operatorname{Emb}^+(D^n, S^n)$. This embedding space is homotopy equivalent to the frame bundle, but here we only need π_0 and π_1 of the embedding space, which aren't so hard to compute.

[14] Hint: to get the right formulas, use a unit-speed parameterization of the curve that is being revolved. This should work better than the parameterization of the curve viewed as a graph of a function..

[15] Hint: it may help to determine parameterizations for the unit speed geodesics through the origin in $\mathbb{D}$. Being unit speed gives a differential equation that is solved by a hyperbolic trig function.

   (b) Prove that if $\psi = f_B \circ \phi \circ f_A$, where $f_A : \partial A \to \partial A$ is a diffeomorphism that extends to A and similarly for f_B, then M_ϕ is diffeomorphic to M_ψ.
(2) Prove that $M \# \Sigma$ is homeomorphic to M for any manifold M and homotopy sphere Σ.

The group Θ_n

(1) Let X be a simply connected n-manifold. Show that X is h-cobordant to S^n if and only if X bounds a contractible manifold. [16] [17]
(2) Let Σ be a homotopy sphere. Show that $\Sigma \# \overline{\Sigma}$ bounds a contractible manifold. Deduce that $\overline{\Sigma}$ is the inverse of Σ in Θ_n.[18] [19]
(3) Use the h-cobordism theorem to show that the following are equivalent.
   (a) Σ is smoothly concordant to S^n.
   (b) Σ is (smoothly) h-cobordant to S^n.
   (c) Σ is orientation-preservingly diffeomorphic to S^n.

   Deduce that there is a bijection between Θ_n and $\mathbb{S}(S^n)$.

Smoothing theory

(1) The following statements were made during lecture.
   - $\Theta_7 = \mathbb{Z}/28\mathbb{Z}$
   - S^7 has 15 smooth structures

   How can these both be true?
(2) For based spaces A, B, there is a homotopy equivalence

$$\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$$

   where $\Sigma(-)$ denote the suspension and $(-) \vee (-)$ denotes the wedge and $(-) \wedge (-)$ is the smashed product. Use this to deduce the homotopy type of $\Sigma(S^n \times S^n)$ and to compute $\mathbb{S}(S^n \times S^n) \cong [S^n \times S^n, \mathrm{Top/O}]$.[20]
(3) Using the same strategy as the previous exercise, compute $[T^n, \mathrm{Top/O}]$. (I stated this computation in lecture, but didn't prove it.)
(4) Look up a statement of the s-cobordism theorem, and use it to deduce that concordance implies diffeomorphism.[21]

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[16] Hint: use the h-cobordism theorem.

[17] Hint:This exercise has a short solution where you judiciously add or subtract disks.

[18] Consider $(\Sigma \setminus D^n) \times I$.

[19] For this and the previous exercise, the details are in Kervaire–Milnor "Groups of homotopy spheres I," Section 2.

[20] You will want to use that Top/O is an infinite loop space and the adjunction $[X, \Omega Y] = [\Sigma X, Y]$. This was also used in lecture.

[21] Remark: I believe I said h-cobordism theorem in class, but our manifolds are not necessarily simply connected...

(5) Assume M is a noncompact n-manifold. Prove that $M\#\Sigma$ is concordant (hence diffeomorphic) to M for every homotopy n-sphere Σ.[22] [23]

(6) Give an example of two smooth structures that are diffeomorphic but not concordant.[24]

(7) (Reading project) Check out the article "Minicourse on smoothing theory" by Jim Davis (you can google it). It contains a short exposition of smoothing theory and might help you better understand the discussion of smoothing theory from last lecture (and a bit for the next lecture).

Hyperbolic manifolds

(1) Show that orientable surfaces are stably parallelizable.

(2) Let M a closed hyperbolic manifold with residually finite fundamental group.[25] Prove that for any R, there is a finite cover of M that has injectivity radius $> R$. (In the process of finding a proof, figure out the correct assumptions – you shouldn't need much hyperbolic geometry.)

(3) (Reading project) Check out the paper "Virtually spinning hyperbolic manifolds" by Long–Reid, where they prove that a closed hyperbolic manifold has a finite cover that is spin (i.e. the second Stiefel–Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is zero).

(4) (Challenge) Find a proof that closed hyperbolic manifolds are virtually stably parallelizable. Less ambitious would be to show that characteristic classes (Stiefel–Whitney, Pontryagin) vanish in a finite cover. Or show that M is virtually almost complex (this is also implied by virtually stably parallelizable).

[22] Use the fact (mentioned in lecture) that the map $\mathbb{S}(S^n) \to \mathbb{S}(M)$ defined by $\Sigma \mapsto M\#\Sigma$ is induced by a map $M \to S^n$ that collapses the complement of a ball in M to a point.

[23] Hint: recall (or look up) the Hopf degree theorem.

[24] This can be done with either S^n or $S^n \times S^n$ or T^n. We will also discuss this in the next lecture.

[25] This means that the intersection of finite index subgroups is trivial. Equivalently, for any element of the group, there is a surjection to a finite group where that element survives.