Problem 1. *True or false: For a surface $S \subset \mathbb{R}^3$ and $p \in S$, if the Gauss curvature $K(p)$ is negative, then there exists a nonzero vector $w \in T_pS$ whose normal curvature is zero.*

Solution. True. Identify the circle S^1 with the set of unit-length vectors in T_pS. Consider the map $f : w \mapsto$ (Normal curvature of f). Because $K(p) = \kappa_1(p) \cdot \kappa_2(p) < 0$, and $\kappa_1 < \kappa_2$, we have that $\kappa_1(p) < 0$ and $\kappa_2(p)$. We know that there exist principal directions w_1 and w_2 on our copy of S^1 such $f(w_1) = \kappa_1(p) < 0$ and $f(w_2) = \kappa_2(p) > 0$. Note now that f is continuous, as the normal curvature can be defined entirely in terms of other continuous functions. If we now trace the arc along S^1 between w_1, w_2, the value of f will vary from $\kappa_1(p)$ to $\kappa_2(p)$. By the intermediate value theorem, this value must pass through $w \in S^1$ such that $f(w) = 0$, which is what we wished to show.

$\square$

Problem 2 (dC, 3.5.12). *Prove that every minimal surfaces $S \subset \mathbb{R}^3$ is unbounded.* [1]

Solution. From class we proved that if a surface is closed and bounded, then the gauss curvature has elliptical points, which means that $K(p) > 0$. Therefore the principal curvatures are either both positive or both negative.

For S to be a minimal surface its mean curvature is 0. We know that $H = \dfrac{\kappa_1(p) + \kappa_2(p)}{2}$.

If S is closed and bounded, then $\kappa_1(p)$ and $\kappa_2(p)$ will have the same sign, so they won't ever cancel each other out, so H won't ever be 0. Therefore a closed bounded surface will never be a minimal surface. So in order to be a minimal surface S can't be closed and bounded.

□

[1] Hint: We proved a helpful fact about curvature of closed, bounded surfaces in class.

Problem 3 (dC, 4.2.4). *Show that the stereographic projection chart $\mathbb{R}^2 \to S^2$ is conformal.*

Solution. From HW4, we know the sterographic projection chart $\mathbb{R}^2 \to S^2$:

$$\phi(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, -\frac{2}{u^2+v^2+1}+1)$$

□

We proceed by calculating the first fundamental form of this chart:

$$\begin{aligned}
\phi_u &= (\frac{2(-u^2+v^2+1)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}) \\
\phi_v &= (\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(u^2-v^2+1)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}) \\
E &= \frac{4(-u^2+v^2+1)^2}{(u^2+v^2+1)^4} + \frac{16u^2v^2}{(u^2+v^2+1)^4} + \frac{16u^2}{(u^2+v^2+1)^4} \\
&= \frac{4v^4+4u^4+8v^2+8u^2+8u^2v^2+4}{(u^2+v^2+1)^4} \\
&= \frac{4(u^2+v^2+1)^2}{(u^2+v^2+1)^4} \\
&= \frac{4}{(u^2+v^2+1)^2} \\
F &= \frac{-8uv(-u^2+v^2+1)}{(u^2+v^2+1)^2} + \frac{-8uv(u^2-v^2+1)}{(u^2+v^2+1)^2} + \frac{16uv}{(u^2+v^2+1)^2} = 0 \\
G &= \frac{16u^2v^2}{(u^2+v^2+1)^4} + \frac{4(u^2-v^2+1)^2}{(u^2+v^2+1)^4} + \frac{16v^2}{(u^2+v^2+1)^4} \\
&= \frac{4v^4+4u^4+8v^2+8u^2+8u^2v^2+4}{(u^2+v^2+1)^4} \\
&= \frac{4(u^2+v^2+1)^2}{(u^2+v^2+1)^4} \\
&= \frac{4}{(u^2+v^2+1)^2}
\end{aligned}$$

To apply the definition of a conformal chart, let $\phi' : U \to \mathbb{R}^2$ be a chart to $\mathbb{R}^2$. The first fundamental form of this chart is the identity matrix. Thus, ϕ is conformal if it follows the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We see that it does, so ϕ is conformal.

Problem 4 (dC, 4.2.19). *Consider the cylinder $C = \{x^2 + y^2 = 1\}$, and let M be S^2 without the north and south poles. Given $p = (x, y, z) \in M$, let R_p be the ray based at $(0, 0, z)$ and going through p. Define $f : M \to C$ by $f(p) = R_p \cap C$. Prove that f is an area preserving map. Use this to give a quick computation of the area of S^2.*

Solution. To show f is area-preserving, we must find a chart ϕ for S^2 and show that the first fundamental forms of ϕ and $f \circ \phi$ have equal determinants everywhere. The canonical chart for S^2 is

$$\phi(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi), \quad \theta \in [0, 2\pi], \quad \varphi \in (0, \pi),$$

whose first fundamental form is

$$\mathrm{I}_\phi = \begin{bmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{bmatrix}.$$

The composition $f \circ \phi$ has the same z-coordinate as ϕ, and its x- and y-coordinates are just proportionally scaled so the sum of their squares always equals 1. Thus, it must be defined

$$(f \circ \phi)(\theta, \varphi) = (\cos\theta, \sin\theta, \cos\varphi), \quad \theta \in [0, 2\pi], \quad \varphi \in (0, \pi).$$

The first fundamental form of $f \circ \phi$ is calculated as follows:

$$\begin{aligned} (f \circ \phi)_\theta &= (-\sin\theta, \cos\theta, 0), \\ (f \circ \phi)_\varphi &= (0, 0, -\sin\varphi), \\ E_{f\circ\phi} = (f \circ \phi)_\theta \cdot (f \circ \phi)_\theta &= \sin^2\theta + \cos^2\theta = 1, \\ F_{f\circ\phi} = (f \circ \phi)_\theta \cdot (f \circ \phi)_\varphi &= 0, \\ G_{f\circ\phi} = (f \circ \phi)_\varphi \cdot (f \circ \phi)_\varphi &= \sin^2\varphi. \end{aligned}$$

So, both fundamental forms have determinant $\sin^2\varphi$. The area of a region on a surface is the integral of the square root of the determinant of the first fundamental form over the preimage of that region. Since f preserves this determinant, the area of any region $\phi(Q)$ of S^2 is the same as the area of the region $f(\phi(Q))$ of the cylinder, and f is area-preserving. The area of the unit sphere thus equals the area of the unit-radius cylinder with height 2 (since $\cos\varphi$ ranges from 1 to -1). That is,

$$A(S^2) = A(\text{cylinder}) = 2\pi r h = 2\pi(1)(2) = \boxed{4\pi}.$$

□

Problem 5. *Let S_1, S_2 be surfaces. Let $\phi : U \to S_1$ be a chart, and let $f : S_1 \to S_2$ be a smooth bijection. Let E_1, F_1, G_1 and E_2, F_2, G_2 be the first fundamental forms of S_1 and S_2 with respect to the charts ϕ and $f \circ \phi$, respectively. Prove that if f is area preserving, then $E_1G_1 - F_1^2 = E_2G_2 - F_2^2$.*

Solution. Fix some point (u_0, v_0). Then, we have that

$$\int_{u_0}^{u}\int_{v_0}^{v}(E_1G_1 - F_1^2)dvdu = Area(\phi([u_0, u] \times [v_0, v])) = Area(f(\phi([u_0, u] \times [v_0, v]))) = \int_{u_0}^{u}\int_{v_0}^{v}(E_2G_2 - F_2^2)dvdu$$

Taking the mixed partial with respect to u and v of both sides at $(u, v) = (u_0, v_0)$ gives

$$\frac{\partial\partial}{\partial v\partial u}|_{(u_0,v_0)}\int_{u_0}^{u}\int_{v_0}^{v}(E_1G_1 - F_1^2)dvdu = \frac{\partial\partial}{\partial v\partial u}|_{(u_0,v_0)}\int_{u_0}^{u}\int_{v_0}^{v}(E_2G_2 - F_2^2)dvdu$$
$$E_1(u_0, v_0)G_1(u_0, v_0) - F_1(u_0, v_0)^2 = E_2(u_0, v_0)G_2(u_0, v_0) - F_2(u_0, v_0)^2$$

, by the FTOC. $\square$