

# SYMMETRIES OF EXOTIC SMOOTHINGS OF ASPHERICAL SPACE FORMS

MAURICIO BUSTAMANTE AND BENA TSHISHIKU

**ABSTRACT.** We study finite group actions on smooth manifolds of the form  $W \# \Sigma$ , where  $\Sigma$  is an exotic  $n$ -sphere and  $W$  is either a hyperbolic or a flat manifold. We classify the finite cyclic groups that act freely on an exotic torus  $T^n \# \Sigma$ . For hyperbolic manifolds  $M$ , we produce examples  $M \# \Sigma$  that admit no nontrivial smooth action of a finite group, while  $\text{Isom}(M)$  is arbitrarily large.

## 1. INTRODUCTION

This paper is motivated by the following question.

**Question 1.1.** Let  $W$  be a closed smooth  $n$ -manifold and  $W'$  a manifold which is homeomorphic but not diffeomorphic to  $W$ . To what extent does  $W'$  support the same symmetries as  $W$ ? If a Lie group  $G$  acts on  $W$ , does  $G$  also act on  $W'$ ? How does the answer depend on the choice  $W'$ ?

Question 1.1 is difficult in general, so it's reasonable to focus on particular classes of manifolds. When  $W = S^n$ , several structural results are known by the work of Hsiang–Hsiang [Hsi67, HH69]. For example, the standard smooth structure on  $S^n$  is characterized as the unique smooth structure with a faithful action of  $\text{SO}(n+1)$ . The current paper seeks to address Question 1.1 when  $W$  is either the  $n$ -dimensional torus  $T^n$  or a hyperbolic  $n$ -manifold and  $W' = W \# \Sigma$  for some exotic sphere  $\Sigma$ .

**Conventions.** All manifolds are oriented. All actions preserve the orientation, unless stated otherwise. For the remainder of the paper,  $G$  will denote a finite group. Throughout, we use  $\Sigma$  to denote an exotic  $n$ -sphere, that is, a manifold which is homeomorphic but not diffeomorphic to the  $n$ -sphere  $S^n$ . We denote by  $\Theta_n$  the group of homotopy  $n$ -spheres [KM63]. Note that every nontrivial element in  $\Theta_n$  can be represented by an exotic  $n$ -sphere, as long as  $n \geq 5$ . The topological groups  $\text{Homeo}(X)$  and  $\text{Diff}(X)$  of homeomorphisms and diffeomorphisms of a compact smooth manifold  $X$  are understood to be equipped with the compact-open and  $C^\infty$ -topology respectively.

**1.1. Classifying actions on  $W \# \Sigma$ .** Our first results classify (fixed-point) free actions  $G \curvearrowright W \# \Sigma$  in certain cases. To state the classification, we introduce the notion of a *standard action*  $G \curvearrowright W \# \Sigma$  when  $W$  is a smooth manifold.

**Standard actions.** Assume  $W$  is a compact smooth manifold, and fix a group  $G < \text{Diff}^+(W)$  of orientation-preserving diffeomorphisms. For any exotic sphere  $\widehat{\Sigma}$ , there is a smooth action of  $G$  on  $W \# \widehat{\Sigma} \# \cdots \# \widehat{\Sigma}$  obtained by performing the connected sum of  $|G|$  copies of  $\widehat{\Sigma}$  equivariantly along the  $G$ -orbit of a point in  $W$  with trivial stabilizer. See Figure 1. We call an action  $G \curvearrowright W \# \Sigma$  *standard* if it is isomorphic (in the category of smooth oriented  $G$ -manifolds) to this construction.

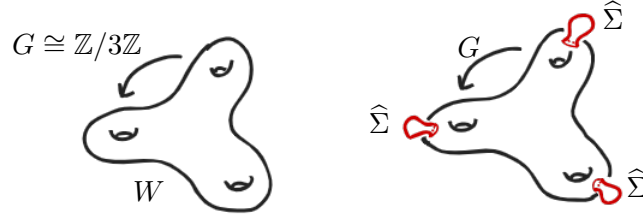


FIGURE 1. An action  $G \cong \mathbb{Z}/3\mathbb{Z} \curvearrowright W$  induces a “standard” action  $G \curvearrowright W \# \hat{\Sigma} \# \hat{\Sigma} \# \hat{\Sigma}$ .

**Theorem A** (Actions on exotic  $n$ -tori). *Let  $T^n = (S^1)^n$  denote the  $n$ -torus ( $n \geq 5$ ), and let  $\Sigma$  be an exotic  $n$ -sphere. Then any free action of a finite cyclic group on  $T^n \# \Sigma$  is standard.*

By a similar approach we can also prove the following result for actions on  $M \# \Sigma$  when  $M$  is a 7-dimensional hyperbolic manifold.

**Theorem 1.2** (Actions on exotic hyperbolic 7-manifolds). *Let  $M^7$  be a 7-dimensional closed hyperbolic manifold, and let  $\Sigma$  be an exotic 7-sphere. Assume that the order of  $G$  is odd. Then every free action of  $G$  on  $M \# \Sigma$  is standard.*

**Remark 1.3.** Combining Theorem A and Theorem 1.2 with rigidity results about flat and hyperbolic manifolds, one finds that (under the assumptions of Theorems A and 1.2) any action  $G \curvearrowright W \# \Sigma$  is “standard with respect to an isometric action”, i.e. the action  $G \curvearrowright W$  in the definition of a standard action is isometric with respect to some locally symmetric metric on  $W$ .

Next we give several remarks that provide more context and motivation for Theorems A and 1.2.

**Remark 1.4** (Connection to Nielsen realization). For  $G < \mathrm{SL}_n(\mathbb{Z})$ , we say that  $G$  is realized by diffeomorphisms of  $T^n \# \Sigma$  if there exists a solution to the following lifting problem

$$(1) \quad \begin{array}{ccc} & \mathrm{Diff}^+(T^n \# \Sigma) & \\ \nearrow & \downarrow \text{action on } \pi_1 & \\ G & \longrightarrow & \mathrm{SL}_n(\mathbb{Z}) \end{array}$$

If there exists a lift where  $G$  acts freely on  $T^n \# \Sigma$ , then we say  $G$  is realized *freely*.

As a corollary to Theorem A, we can completely characterize the finite cyclic subgroups of  $\mathrm{SL}_n(\mathbb{Z})$  that are realized freely on  $T^n \# \Sigma$ . By Theorem A, if  $\Sigma$  is not divisible by  $|G|$  in  $\Theta_n$ , then  $G$  does not act freely on  $T^n \# \Sigma$ . The converse is obvious, so we deduce:

**Corollary 1.5** (Nielsen realization for  $T^n \# \Sigma$ ). *Fix  $d \geq 1$  and a subgroup  $G \cong \mathbb{Z}/d\mathbb{Z}$  of  $\mathrm{SL}_n(\mathbb{Z})$ . Then  $G$  is realized freely on  $T^n \# \Sigma$  if and only if  $\Sigma$  is divisible by  $d$  in  $\Theta_n$ .*

There is a similar corollary of Theorem 1.2, where we consider the lifting problem:

$$(2) \quad \begin{array}{ccc} & \mathrm{Diff}^+(M \# \Sigma) & \\ \nearrow & \downarrow \text{action on } \pi_1 & \\ G & \longrightarrow & \mathrm{Out}^+(\pi_1(M \# \Sigma)) \end{array}$$

We note that the homomorphisms

$$\mathrm{Diff}^+(T^n \# \Sigma) \rightarrow \mathrm{SL}_n(\mathbb{Z}) \quad \text{and} \quad \mathrm{Diff}^+(M \# \Sigma) \rightarrow \mathrm{Out}^+(\pi_1(M \# \Sigma))$$

in (1) and (2) are surjective; this is explained in the authors' previous work [BT19, Thm. 1]. This fact makes the Nielsen realization problem more interesting (if the homomorphism is not surjective, then certain subgroups obviously don't lift). This surjectivity is not true for  $M \# \Sigma$  if we include orientation-reversing diffeomorphisms, as observed by Farrell–Jones [FJ90], and it's also not true for all exotic smooth structures on  $T^n$  or  $M$ . For example,  $\mathrm{Diff}^+(T') \rightarrow \mathrm{SL}_n(\mathbb{Z})$  is not surjective for  $T'$  of the form

$$(3) \quad T' = T^n \setminus (T^k \times \mathrm{int}(D^{n-k})) \cup_f T^k \times D^{n-k},$$

where  $T^k \times D^{n-k} \subset T^n$  is a framed subtorus,  $k \geq 1$ , and the gluing

$$f : T^k \times S^{n-k-1} \rightarrow T^k \times S^{n-k-1}$$

has the form  $f = \mathrm{id}_{T^k} \times \phi$ , where the isotopy class  $[\phi] \in \pi_0 \mathrm{Diff}(S^{n-k-1}) \cong \Theta_{n-k}$  is nontrivial. The non-surjectivity of  $\mathrm{Diff}^+(T') \rightarrow \mathrm{SL}_n(\mathbb{Z})$  in this case can be proved in a similar way to [BT19, Thm. B].

**Remark 1.6** (Connection to the Zimmer program). Another motivation for Theorem A is the following question.

**Question 1.7.** Let  $T'$  be an exotic  $n$ -torus (i.e.  $T'$  is homeomorphic, but not diffeomorphic, to  $T^n$ ). Does there exist a smooth, faithful action of  $\mathrm{SL}_n(\mathbb{Z})$  on  $T'$ ?

In general, the Zimmer program seeks to classify actions of lattices in Lie groups, like  $\mathrm{SL}_n(\mathbb{Z})$ , on smooth manifolds; see [Fis11, Fis19]. There seems to be no “obvious” action of  $\mathrm{SL}_n(\mathbb{Z})$  on any exotic torus  $T'$ , so one might suspect that the answer to Question 1.7 is “no”. As evidence toward this, one can try to prove the following statement.

**Problem 1.8.** For an exotic  $n$ -torus  $T'$ , show that the natural homomorphism

$$\mathrm{Diff}^+(T') \rightarrow \mathrm{SL}_n(\mathbb{Z})$$

is not a split surjection.

See [KRH10] for a result that addresses Problem 1.8 in the real-analytic category.

As mentioned above, it is possible to use the methods of [BT19, Thm. B] to show that if  $\mathrm{Diff}^+(T') \rightarrow \mathrm{SL}_n(\mathbb{Z})$  is surjective, then  $T'$  is diffeomorphic to  $T^n \# \Sigma$ , for some  $\Sigma \in \Theta_n$ . Thus the most interesting case of Problem 1.8 is to show that  $\mathrm{Diff}^+(T^n \# \Sigma) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z})$  is not split when  $\Sigma \neq S^n \in \Theta_n$ . To illustrate how Theorem A could be useful in solving this problem, we note the following sample application of Corollary 1.5, which could be the starting point for further analysis and lead to examples where  $\mathrm{Diff}^+(T^n \# \Sigma) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z})$  does not split.

**Corollary 1.9** (Fixed point theorem). *Let  $\Sigma$  be a generator of  $\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$ . For any splitting  $\mathrm{SL}_7(\mathbb{Z}) \rightarrow \mathrm{Diff}^+(T^7 \# \Sigma)$  of the surjection  $\mathrm{Diff}^+(T^7 \# \Sigma) \rightarrow \mathrm{SL}_7(\mathbb{Z})$ , every finite order element of  $\mathrm{SL}_7(\mathbb{Z})$  acts with a fixed point on  $T^7 \# \Sigma$ .*

**Remark 1.10** (Non-free actions). Understanding non-free actions of  $\mathbb{Z}/d\mathbb{Z}$  on  $T^n \# \Sigma$  is more subtle for a variety of reasons. One difficulty (which is related to our proofs) comes from the fact that topological rigidity (the Borel conjecture) is not generally true for crystallographic groups with torsion [CK91, CDK15].

Furthermore, it seems likely that there exist actions  $\mathbb{Z}/d\mathbb{Z} \curvearrowright T^n \# \Sigma$  where  $\Sigma$  is not divisible by  $d$  in  $\Theta_n$ . For example, a generator  $\Sigma$  of the subgroup  $bP_{4k} < \Theta_{4k-1}$  of homotopy  $4k - 1$ -spheres which bound a parallelizable manifold has an action of  $\mathrm{SO}(2k - 1)$  with

a circle as the fixed set [Hsi67, §1]. If  $G \cong \mathbb{Z}/d\mathbb{Z} \subset \mathrm{SO}(2k-1)$  acts on  $T^{4k-1}$  with a 1-dimensional fixed set and the same normal representation, then one can construct an action of  $G$  on  $T^n \# \Sigma$  by an equivariant connected sum. If  $k = 2$ , then  $bP_8 = \Theta_7$ , so the generator  $\Sigma \in bP_8$  is not divisible by any divisor  $d > 1$  of  $|\Theta_7| = 28$ .

**Remark 1.11** (Non-cyclic free actions). It is possible that Theorem A can be extended to any finite  $G$  (not just cyclic). The assumption  $G = \mathbb{Z}/d\mathbb{Z}$  is used in two places in the proof of Theorem A: (1) If  $G \cong \mathbb{Z}/d\mathbb{Z}$  acts freely on  $T^n$ , then  $T^n/G$  is parallelizable, and (2) for  $k \gg 0$  the suspension  $\Sigma^k T^n \rightarrow \Sigma^k(T^n/G)$  of the quotient map  $T^n \rightarrow T^n/G$  splits (up to homotopy) as a map  $S^{n+k} \vee X \rightarrow S^{n+k} \vee Y$  that is “diagonal” (see Proposition 5.3 for a precise statement). This conclusion is obtained by a carefully constructed equivariant Whitney embedding of  $T^n$  in  $\mathbb{R}^{n+k}$ . This makes special use of the fact that  $G$  is cyclic, but a similar construction could exist more generally.

**Remark 1.12.** The assumption that  $|G|$  is odd in Theorem 1.2 could be avoided if the natural map  $F : [M, \mathrm{Top}/\mathrm{O}] \rightarrow [M, \mathrm{Top}/\mathrm{PL}]$ , which is surjective when  $\dim M = 7$ , admits a splitting. Here the domain/codomain of  $F$  can be interpreted as the group of concordance classes of smooth/PL structures on  $M$ , and the map  $F$  is forgetful. One can show that  $F$  splits if  $H_4(M; \mathbb{Z}/2\mathbb{Z})$  is generated by embedded submanifolds with trivial normal bundle, but we do not know if this is ever the case.

**1.2. Asymmetric manifolds  $W \# \Sigma$ .** Next we state our second main result. We say that an oriented smooth manifold  $X$  is *asymmetric* if  $\mathrm{Diff}^+(X)$  does not contain a nontrivial finite subgroup.

**Theorem B** (Asymmetric smoothings of hyperbolic manifolds). *For every  $n_0 \geq 5$  and  $d \geq 1$ , there exists  $n \geq n_0$ , a closed hyperbolic  $n$ -manifold  $M$ , and an exotic sphere  $\Sigma \in \Theta_n$ , so that  $|\mathrm{Isom}(M)| \geq d$  and  $M \# \Sigma$  is asymmetric.*

Theorem B answers Question 3 of [BT19].

**Remark 1.13** (Other results about asymmetric manifolds). The first examples of asymmetric aspherical manifolds were constructed by Conner–Raymond–Weinberger [CRW72]. These examples (some of which are solvmanifolds) are asymmetric in the topological category (i.e.  $\mathrm{Homeo}(W)$  does not contain any nontrivial finite subgroup), and they are shown to be asymmetric by arranging that  $\mathrm{Out}(\pi_1(W))$  is torsionfree.

In the hyperbolic setting, Long–Reid [LR05] gave examples of asymmetric hyperbolic  $n$ -manifolds for each  $n \geq 2$  (see also [BL05]). These examples are shown to be asymmetric by arranging that  $\mathrm{Isom}(M) \cong \mathrm{Out}(\pi_1(M)) = 1$ .

Theorem B exhibits a different phenomenon: these examples have many topological symmetries (in particular  $\mathrm{Out}(\pi_1)$  may be arbitrarily large), but no smooth ones.

**Remark 1.14** (Simplest instance of Theorem B). The simplest nontrivial instance of Theorem B is as follows. Let  $M$  be a hyperbolic 7-manifold such that  $\mathrm{Isom}(M) = \mathrm{Isom}^+(M) = \mathbb{Z}/7\mathbb{Z}$  acts freely on  $M$  (such examples exist, as we recall below), and let  $\Sigma$  be a generator of  $\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$ . Then  $M \# \Sigma$  is asymmetric. We could also deduce that this example is asymmetric using Theorem 1.2.

**Remark 1.15** (Connection to Nielsen realization). Consider the lifting problem in (2). The statement “ $M \# \Sigma$  is asymmetric” is equivalent to the statement “no nontrivial subgroup of  $\mathrm{Out}(\pi_1(M \# \Sigma))$  is realized by diffeomorphisms of  $M \# \Sigma$ .” Thus Theorem B gives extreme examples of the non-realizability of the outer automorphism group by diffeomorphisms. This strengthens a result of Farrell–Jones [FJ90, Thm. 1] who gave examples where

$$\mathrm{Diff}(M \# \Sigma) \rightarrow \mathrm{Out}(\pi_1(M \# \Sigma))$$

is not surjective. On the other hand, the image of this homomorphism has index at most 2 in  $\text{Out}(\pi_1(M\#\Sigma))$ , so to produce asymmetric examples, one needs a different obstruction than the one found by [FJ90].

**1.3. Questions and problems.** One can view Theorems A and 1.2 as rigidity results for free actions on  $W\#\Sigma$  when  $W$  is locally symmetric and nonpositively curved. It is unclear in what generality such rigidity might hold.

**Problem 1.16.** Give an example of a nonpositively curved locally symmetric  $W$  and a free action  $G \curvearrowright W\#\Sigma$  that is not standard.

**Remark 1.17** (Possible approach to Problem 1.16). Here is one possible source of examples. Let  $M$  be hyperbolic. Fix a geodesic  $\gamma \subset M$ , an isometry  $g \in \text{Isom}^+(M)$  acting freely, and consider the orbit  $\Gamma = \gamma \sqcup g(\gamma) \sqcup \dots$ . Use  $\Gamma$  and  $[f] \in \pi_0(\text{Diff}(S^{n-2})) \cong \Theta_{n-1}$  to define a new smooth structure  $M'$ , similar to (3). If  $\Gamma$  is nullhomologous, then  $M'$  is diffeomorphic to  $M$ , and one can use this to construct an action of  $G = \langle g \rangle$  on  $M'' := M\#(\widehat{\Sigma}^{|G|})$  such that  $M''/G$  is not diffeomorphic to  $M\#\Sigma$ , which implies the action is not standard (c.f. Lemma 2.1). It seems challenging to prove that such a construction exists.

We also propose the following weaker form of rigidity.

**Definition 1.18** (Divisibility property). We say that a smooth manifold  $W$  has the *divisibility property* if  $G$  acts freely on  $W\#\Sigma$  only if  $\Sigma$  is divisible by  $|G|$  in  $\Theta_n/I(W)$ , where  $I(W) = \{\Sigma \in \Theta_n : W\#\Sigma \cong W\}$  is the inertia group of  $W$ .

For example,  $n$ -tori have the divisibility property by Theorem A.

**Problem 1.19.** Give an example of an aspherical  $n$ -manifold  $W$  that does not have the divisibility property.

The approach to Problem 1.16 suggested in Remark 1.17 would not solve Problem 1.19.

**Remark 1.20.** It will be apparent in the course of our arguments that the divisibility property is related to the filtration

$$F_0 \subset F_1 \subset \dots \subset F_r = [W, \text{Top}/\text{O}]$$

arising from the Atiyah-Hirzebruch spectral sequence, and whether or not any of the extensions

$$0 \rightarrow F_0 \rightarrow F_k \rightarrow F_k/F_0 \rightarrow 0$$

do not split. Here  $F_0 \cong \Theta_n/I(W)$ .

**Section Outline.** In §2 we give a characterization of standard actions; in §3 we explain the necessary background from smoothing theory; and in §4 we prove two results about flat manifolds. In §5 – §7 we prove the main theorems.

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## 2. STANDARD ACTIONS

The following lemma gives a characterization of standard actions that we will use to prove that actions are standard.

**Lemma 2.1** (Standard actions). *Let  $W$  be a closed smooth manifold, and fix  $\Sigma \in \Theta_n$ . A smooth, free, finite group action  $G \curvearrowright W \# \Sigma$  is standard if and only if there exists a finite subgroup  $G' < \text{Diff}^+(W)$  that is conjugate to  $G < \text{Diff}^+(W)$  in  $\text{Homeo}^+(W)$ , an exotic sphere  $\widehat{\Sigma} \in \Theta_n$ , and a diffeomorphism*

$$(W \# \Sigma)/G \cong (W/G') \# \widehat{\Sigma}.$$

*Proof of Lemma 2.1.* The forward direction is immediate from the definition of a standard action with  $G = G'$ . Conversely, assume that there exists  $G, \widehat{\Sigma}$  and a diffeomorphism

$$f : (W/G') \# \widehat{\Sigma} \rightarrow (W \# \Sigma)/G.$$

On the one hand, the covering map  $W \# \Sigma \rightarrow (W \# \Sigma)/G$  pulls back along  $f$  to a covering

$$p : X \rightarrow (W/G') \# \widehat{\Sigma}.$$

On the other hand, there is a standard action  $G' \curvearrowright W \# (\widehat{\Sigma}^{|G|})$  which defines a smooth covering space

$$p' : W \# (\widehat{\Sigma}^{|G|}) \rightarrow (W/G') \# \widehat{\Sigma}.$$

The lemma follows by showing that  $X$  and  $W \# (\widehat{\Sigma}^{|G|})$  are (smoothly) isomorphic covering spaces of  $(W/G') \# \widehat{\Sigma}$ . Since the smooth structure on a smooth cover of a smooth manifold is uniquely determined, it suffices to observe that  $p$  and  $p'$  are isomorphic in the topological category. This holds because both are topologically equivalent to the covering  $W \rightarrow W/G'$ . This holds for  $p'$  by construction, and holds for  $p$  by our assumption that  $G \curvearrowright W \# \Sigma$  and  $G' \curvearrowright W$  are topologically conjugate.  $\square$

### 3. BACKGROUND FROM SMOOTHING THEORY

In this section we recall the basic setup of smoothing theory and state the classification theorem for concordance classes of smooth structures, which is used in Sections 5 and 7. We also prove in Lemmas 3.3 and 3.5 a result about concordance classes of the smooth structures  $T^n \# \Sigma$  and  $M \# \Sigma$  ( $M$  hyperbolic) that will be used in the proofs of the main theorems.

**Smooth structures.** A *smooth structure* or *smoothing* of a topological manifold  $W$  is a pair  $(U, h)$  where  $U$  is a smooth manifold and  $h : U \rightarrow W$  is a homeomorphism (this in particular determines a smooth atlas on  $W$  by push-forward). Two smooth structures  $(U_0, h_0)$  and  $(U_1, h_1)$  of  $W$  are *concordant* if there exists a smooth cobordism  $V$  between  $U_0$  and  $U_1$  and a homeomorphism  $V \rightarrow W \times [0, 1]$  which restricts to  $h_i$  on  $U_i$ ,  $i = 0, 1$ .

The set of concordance classes of smooth structures on  $W$  is denoted by  $S(W)$ .

In the following classification theorem [KS77, Essay IV, Theorem 10.1],  $\text{Top/O}$  denotes the homotopy fiber of the natural map  $B\text{O} \rightarrow B\text{Top}$ , where  $\text{O} = \text{colim } \text{O}(n)$  is the infinite orthogonal group, and  $\text{Top} = \text{colim } \text{Homeo}(\mathbb{R}^n, 0)$ , with  $\text{Homeo}(\mathbb{R}^n, 0)$  the topological group of homeomorphisms of  $\mathbb{R}^n$  that fix the origin. We denote by  $[W, \text{Top/O}]$  the set of homotopy classes of based maps; this is equivalent to the set of homotopy classes of unbased maps since  $\text{Top/O}$  is an  $H$ -space under Whitney sum.

**Theorem 3.1.** *Let  $W$  be a closed topological manifold of dimension  $n \geq 5$ . Then a choice of a smooth structure  $U \xrightarrow{h} W$  on  $W$  determines a bijection  $S(W) \xrightarrow{\cong} [W, \text{Top/O}]$  under which  $(U, h)$  is sent to the homotopy class of the constant map.*



To compute  $[W, \text{Top}/\text{O}]$ , we use that  $\text{Top}/\text{O}$  is in fact an infinite loop space [BV73, p.216]. This affords two different approaches:

- (1) For any  $k \geq 1$ , if we write  $\text{Top}/\text{O} = \Omega^k Y$ , then

$$[W, \text{Top}/\text{O}] \cong [W, \Omega^k Y] \cong [\Sigma^k W, Y].$$

Thus information about the homotopy type of  $\Sigma^k W$  (which can be simpler than that of  $W$ ) can allow us to make conclusions about  $[W, \text{Top}/\text{O}]$ .

- (2) We can view  $[W, \text{Top}/\text{O}]$  as the 0-th group of a cohomology theory. Then the Atiyah–Hirzebruch spectral sequence can be used to gain information about  $[W, \text{Top}/\text{O}]$ . We elaborate on this more below.

**Localization of  $\text{Top}/\text{O}$ .** For the proof of Theorem B, we will use the localization  $\text{Top}/\text{O}_{(p)}$  of  $\text{Top}/\text{O}$  at an odd prime  $p$ . The homotopy groups of the infinite loop space  $\text{Top}/\text{O}_{(p)}$  are the  $p$ -torsion subgroups of  $\Theta_n$ :

$$\pi_n(\text{Top}/\text{O}_{(p)}) = \Theta_n \otimes \mathbb{Z}_{(p)}$$

When localized, the short exact sequence  $0 \rightarrow bP_{n+1} \rightarrow \Theta_n \rightarrow \Theta_n/bP_{n+1} \rightarrow 0$  splits. In fact, there is a splitting on the level of spaces:

**Theorem 3.2** (Localization of  $\text{Top}/\text{O}$ ). *Let  $p$  be an odd prime. Then there are infinite loop spaces  $B = B(p)$  and  $C = C(p)$ , and infinite loop maps  $\beta : B \rightarrow \text{Top}/\text{O}_{(p)}$  and  $\alpha : C \rightarrow \text{Top}/\text{O}_{(p)}$  such that the map  $B \times C \xrightarrow{\beta \times \alpha} \text{Top}/\text{O}_{(p)} \times \text{Top}/\text{O}_{(p)} \xrightarrow{\text{multiplication}} \text{Top}/\text{O}_{(p)}$  is an equivalence of infinite loop spaces. Furthermore, the map  $\beta$  induces an isomorphism from  $\pi_n(B)$  onto  $bP_{n+1} \otimes \mathbb{Z}_{(p)}$ .*

This is proved in [Lan88, §5.]. To compare the statement above to what appears in [Lan88], one should note that  $\text{PL}/\text{O}_{(p)} \simeq \text{Top}/\text{O}_{(p)}$  when  $p$  is an odd prime because  $\text{Top}/\text{PL}_{(p)} \simeq K(\mathbb{Z}/2\mathbb{Z}, 3)_{(p)}$  is contractible. (Localization preserves fibrations of simply connected spaces.)

**The Atiyah–Hirzebruch spectral sequence.** The infinite loop space  $B = B(p)$  from Theorem 3.2 defines spectrum and also a cohomology theory  $\mathbb{E}^*$ . In particular, for any space  $W$ ,  $\mathbb{E}^0(W) = [W, B]$ . When  $W$  is a closed manifold, the groups  $\mathbb{E}^*(W)$  can be computed using the Atiyah–Hirzebruch spectral sequence. This is a spectral sequence with  $E_2$  page

$$E_2^{p,q} = H^p(W; \pi_{-q}(B)),$$

that converges to  $\mathbb{E}^{p+q}(W)$ . (For general  $W$ , the spectral sequence converges conditionally, but when  $W$  is a closed manifold, the spectral sequence always converges.)

**Smooth structures  $W \# \Sigma$ .** View  $W \# \Sigma$  as  $(W \setminus \text{int } D^n) \cup_{\phi} D^n$ , where  $D^n$  is glued to  $W \setminus \text{int } D^n$  along the common boundary  $\partial D^n = S^{n-1}$  by  $\phi \in \text{Diff}(S^{n-1})$  whose isotopy class  $[\phi] \in \pi_0(\text{Diff}(S^{n-1})) \cong \Theta_n$  corresponds to  $[\Sigma]$ . From this point-of-view, there is a “standard” homeomorphism  $\iota : W \# \Sigma \rightarrow W$ , which is the identity on  $W \setminus \text{int } D^n$  and  $\iota|_{D^n}$  is the cone of  $\phi$  (which is a homeomorphism, but not generally a diffeomorphism).

The map  $W \rightarrow \text{Top}/\text{O}$  that classifies  $(W \# \Sigma, \iota)$  can be obtained by a composition

$$W \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O},$$

where  $c$  collapses the complement of  $D^n \subset W$  to a point, and  $f$  classifies  $\Sigma \in S(S^n) \cong \pi_n(\text{Top}/\text{O})$ .

The isomorphism  $S(W) \cong [W, \text{Top}/\text{O}]$  is equivariant with respect to the obvious action of  $\text{Homeo}(W)$  on both sets. The following Lemmas 3.3 and 3.5 describe this action on the concordance classes represented by  $W \# \Sigma$  for some  $\Sigma \in \Theta_n$ .

**Lemma 3.3.** *Let  $n \geq 5$  and fix  $\Sigma \in \Theta_n$ . Then, for any  $h \in \text{Homeo}^+(T^n)$ , the smooth structures  $(T^n \# \Sigma, \iota)$  and  $(T^n \# \Sigma, h \circ \iota)$  are concordant.*

**Remark 3.4.** Lemma 3.3 is not generally true if  $T^n \# \Sigma$  is replaced by another smooth structure, such as those in (3); c.f. [BT19, §4].

*Proof of Lemma 3.3.* The map  $T^n \rightarrow \text{Top}/\text{O}$  that classifies  $(T^n \# \Sigma, \iota)$  factors as

$$T^n \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O}$$

where  $c$  is the collapse map associated to an embedding  $D^n \hookrightarrow T^n$ , and the isotopy class  $[f] \in \pi_n(\text{Top}/\text{O}) \cong \Theta_n$  corresponds to  $\Sigma$ . Then to prove the lemma, it suffices to show that the maps

$$T^n \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O} \quad \text{and} \quad T^n \xrightarrow{h} T^n \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O}$$

are homotopic. Since  $\text{Out}(\pi_1(T^n)) \cong \text{GL}_n(\mathbb{Z})$  and maps to a  $K(\mathbb{Z}^n, 1)$  are determined by their effect on the fundamental group,  $h$  is homotopic to a linear automorphism, and by a further isotopy we can obtain a homeomorphism  $h'$  such that  $h'(D^n) = D^n$ . Then  $f \circ c = f \circ c \circ h'$  is homotopic to  $f \circ c \circ h$ , as desired.  $\square$

We have a similar result for hyperbolic manifolds, with only a minor adjustment of the proof.

**Lemma 3.5.** *Let  $M$  be a closed hyperbolic  $n$ -manifold ( $n \geq 5$ ) and  $\Sigma \in \Theta_n$ . Then, for any  $h \in \text{Homeo}^+(M)$ , the smooth structures  $(M \# \Sigma, \iota)$  and  $(M \# \Sigma, h \circ \iota)$  are concordant.*

*Proof.* As in the proof of Lemma 3.3, we consider the collapse map  $c : M \rightarrow S^n$  induced by the inclusion of a disk  $D^n \hookrightarrow M$ , and we take the homotopy class  $[f] \in \pi_n(\text{Top}/\text{O}) \cong \Theta_n$  corresponding to  $\Sigma$ , and we want to show that the maps

$$M \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O} \quad \text{and} \quad M \xrightarrow{h} M \xrightarrow{c} S^n \xrightarrow{f} \text{Top}/\text{O}$$

are homotopic. By Mostow rigidity,  $\text{Out}(\pi_1(M)) \cong \text{Isom}(M)$ , and so  $h$  is homotopic to an (orientation-preserving) isometry, and by a further isotopy we can obtain a homeomorphism  $h'$  so that  $h'(D^n) = D^n$ . Then again we conclude that  $f \circ c = f \circ c \circ h'$  is homotopic to  $f \circ c \circ h$ .  $\square$

#### 4. TWO FACTS ABOUT FLAT MANIFOLDS

Here we prove two structural results (Propositions 4.1 and 4.2) about flat manifolds with cyclic holonomy. These are used for the proof of Theorem A.

**Proposition 4.1** (Cyclic holonomy implies mapping torus). *Fix  $d \geq 2$ . Let  $N$  be a flat manifold with a  $G = \mathbb{Z}/d\mathbb{Z}$  cover  $T^n \rightarrow N$ . Then  $N$  is diffeomorphic to a mapping torus  $T^{n-1} \rightarrow N \rightarrow S^1$ .*

Proposition 4.1 includes the case when  $N$  is a flat manifold with holonomy group  $G = \mathbb{Z}/d\mathbb{Z}$  (but it also includes, the (easier) case that  $N$  is itself the torus). For flat manifolds with holonomy  $G = \mathbb{Z}/d\mathbb{Z}$  with  $d$  prime, the proposition is proved in [Vas70, Thm. 3.6]. The general case is not much harder using an observation of [CW89, Lem. 1].



*Proof.* The group  $\Gamma = \pi_1(N)$  is a torsion-free subgroup of  $\text{Isom}^+(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \text{SO}(n)$ . Let  $\mathbb{Z}^n < \Gamma$  be the subgroup corresponding to the given covering  $T^n \rightarrow N$ . There is a short exact sequence

$$(4) \quad 1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow G \rightarrow 1.$$

Let  $A$  denote  $\mathbb{Z}^n$  with its  $\mathbb{Z}[G]$ -module structure coming from the above extension, and let  $\xi \in H^2(G; A)$  denote the Euler class of the extension. We know that  $\xi \neq 0$  because otherwise  $\Gamma$  would be isomorphic to  $\mathbb{Z}^n \rtimes G$ , which is not torsion-free.

Denoting a generator of  $G$  by  $g$ , consider the homomorphism  $m : A \rightarrow A$  defined by multiplication by  $1 + g + \cdots + g^{d-1} \in \mathbb{Z}[G]$ . Let  $A' = \ker(m)$  and  $A'' = \text{im}(m)$ , so there is a short exact sequence of  $\mathbb{Z}[G]$  modules:

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0.$$

Consider the following portion of the associated long exact sequence in cohomology with coefficients

$$H^2(G; A') \rightarrow H^2(G; A) \rightarrow H^2(G; A'')$$

Recalling that  $H^2(G; A) \cong A^G / \ker(m)$  (and similarly for any  $\mathbb{Z}[G]$  module), we deduce that  $H^2(G; A') = 0$ , so  $H^2(G; A) \rightarrow H^2(G; A'')$  is injective. By construction  $A'' \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  is a trivial module, so  $H^2(G; A'') \cong H^2(G; \mathbb{Z}) \oplus \cdots \oplus H^2(G; \mathbb{Z})$ . Since  $\xi \neq 0$ , there is surjective composition  $q : A \twoheadrightarrow A'' \twoheadrightarrow \mathbb{Z}$  so that the image of  $\xi$  under the induced homomorphism

$$q_* : H^2(G; A) \hookrightarrow H^2(G; A'') \rightarrow H^2(G; \mathbb{Z})$$

is nonzero.

Set  $B = \ker(q)$ . This is a  $\mathbb{Z}[G]$ -submodule of  $A$ , which implies that  $B$  is a normal subgroup of  $\Gamma$ . By construction, the extension

$$1 \rightarrow A/B \rightarrow \Gamma/B \rightarrow \Gamma/A \rightarrow 1$$

does not split (its Euler class is  $q_*(\xi) \neq 0$ ). Here  $\Gamma/A \cong G \cong \mathbb{Z}/d\mathbb{Z}$  and  $A/B \cong \mathbb{Z}$ , so the fact that  $q_*(\xi) \neq 0$  implies that this extension is central and  $\Gamma/B \cong \mathbb{Z}$ .

Thus  $\Gamma \cong B \rtimes \mathbb{Z}$ , and one can see (e.g. from the structure of  $\text{Isom}(\mathbb{R}^n)$ ) that this isomorphism is realized topologically by a fibration  $T^{n-1} \rightarrow N \rightarrow S^1$ .  $\square$

**Proposition 4.2** (Flat mapping torus is parallelizable). *Let  $M_f$  be a flat manifold that has the structure of a mapping torus  $T^{n-1} \rightarrow M_f \rightarrow S^1$  whose monodromy  $f$  is orientation-preserving and has finite order. Then  $M_f$  is parallelizable.*

A version of this is proved by [Tho65] with a different assumption. We give an alternate argument.

*Proof of Proposition 4.2.* To show  $M_f$  is parallelizable, it suffices to construct an  $n$ -frame field on  $\widetilde{M}_f \cong \mathbb{R}^n$  that is  $\Gamma$ -invariant, where  $\Gamma = \pi_1(M_f)$ . Write  $\Gamma = \mathbb{Z}^{n-1} \rtimes_{f_*} \mathbb{Z}$ , and decompose  $\mathbb{R}^n$  accordingly as  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ .

The group  $\Gamma$  has a generating set  $\gamma_1, \dots, \gamma_{n-1}, \eta$ , where  $\langle \gamma_1, \dots, \gamma_{n-1} \rangle \cong \mathbb{Z}^{n-1}$  acts by translations of  $\mathbb{R}^{n-1} \times \mathbb{R}$  that are trivial in the second factor, and  $\eta$  acts on  $(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$  by  $\eta(x, t) = (f_*(x) + \beta, t + \frac{1}{d})$ , where  $(\beta, f_*) \in \mathbb{R}^{n-1} \rtimes \text{SO}(n-1)$ .

Define an orthonormal  $n$ -frame on  $\mathbb{R}^n$  as follows. First define a frame along  $\mathbb{R}^{n-1} \times 0$  by choosing an orthonormal frame at the origin, and moving it along  $\mathbb{R}^{n-1}$  by parallel transport. Choose this frame compatible with the decomposition  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ .

Now let  $\alpha_t$  be a path from the identity  $I$  to  $f_*$  in  $\text{SO}(n-1)$ , defined for  $t \in [0, 1/d]$ . Define a frame on  $\mathbb{R}^{n-1} \times \{t\}$  by acting by  $\alpha_t$  on the framing of  $\mathbb{R}^{n-1} \times \{0\}$ .

This extends in an obvious way to a framing on  $\mathbb{R}^{n-1} \times \mathbb{R}$  that is  $\eta$ -invariant. The resulting framing is  $\gamma_i$ -invariant for each  $i$ , since by construction it is constant on  $\mathbb{R}^{n-1} \times \{t\}$  for each  $t \in \mathbb{R}$ . Since  $\Gamma$  is generated by  $\eta$  and the  $\gamma_i$ , the framing is  $\Gamma$  invariant.  $\square$

## 5. ACTIONS ON EXOTIC TORI (THEOREM A)

In this section we prove Theorem A. We begin with a brief sketch of the argument, and then give the details in the following subsections.

Fix  $\Sigma \in \Theta_n$  and suppose that  $G \cong \mathbb{Z}/d\mathbb{Z}$  acts freely on  $T^n \# \Sigma$  by orientation-preserving diffeomorphisms.

- **Step 1 (rigidity).** Let  $(T^n \# \Sigma, \iota)$  denote the standard concordance class, where  $\iota : T^n \# \Sigma \rightarrow T^n$  is the homeomorphism given by coning. First we show that  $(T^n \# \Sigma, \iota)$  is in the image of a homomorphism

$$\pi^* : [\bar{T}, \text{Top}/O] \rightarrow [T^n, \text{Top}/O],$$

induced by a certain covering map  $\pi : T^n \rightarrow \bar{T}$  (see Proposition 5.1). The main ingredient is the geometric and topological rigidity of flat manifolds.

- **Step 2 (splitting).** We show that there are isomorphisms

$$(5) \quad \begin{array}{ccc} [T^n, \text{Top}/O] & \cong & [S^n, \text{Top}/O] \oplus A \\ \pi^* \uparrow & & \\ [\bar{T}, \text{Top}/O] & \cong & [S^n, \text{Top}/O] \oplus \bar{A} \end{array}$$

where  $A$  and  $\bar{A}$  are finite abelian groups. With respect to this splitting, the homomorphism  $\pi^*$  is diagonal (see Corollary 5.4).

Given Steps 1 and 2, the proof is completed as follows. The concordance class  $(T^n \# \Sigma, \iota)$  belongs to the subgroup  $[S^n, \text{Top}/O] \subset [T^n, \text{Top}/O]$ . By Step 1,  $(T^n \# \Sigma, \iota)$  is in the image of  $\pi^*$ , and by Step 2, this implies that  $(T^n \# \Sigma)/G$  is diffeomorphic to  $\bar{T} \# \hat{\Sigma}$  for some  $\hat{\Sigma} \in \Theta_n$ . Then we apply Lemma 2.1 to deduce that the action  $G \curvearrowright T^n \# \Sigma$  is standard.

### 5.1. Step 1: rigidity.

**Proposition 5.1.** *Fix an exotic  $n$ -sphere  $\Sigma$ , and fix a finite subgroup  $G < \text{Diff}^+(T^n \# \Sigma)$  acting freely on  $T^n \# \Sigma$ .*

- (i) *There exists an action  $G' \curvearrowright T^n$  that is isometric with respect to some flat metric and a homeomorphism  $T^n \# \Sigma \rightarrow T^n$  that is equivariant with respect to some isomorphism  $G \cong G'$ .*
- (ii) *Assume  $n \geq 5$  and let  $\pi : T^n \rightarrow T^n/G'$  denote the quotient map. Then the smooth structure  $(T^n \# \Sigma, \iota)$  is in the image of the homomorphism*

$$\pi^* : [T^n/G', \text{Top}/O] \rightarrow [T^n, \text{Top}/O].$$

*Proof.* Use  $\iota : T^n \# \Sigma \rightarrow T^n$  to view the action of  $G$  on  $T^n \# \Sigma$  as an action of  $G$  on  $T^n$  by homeomorphisms. By assumption, the action  $G \curvearrowright T^n$  is free, so the quotient  $T^n \rightarrow T^n/G$  is a covering map, and there is an exact sequence

$$1 \rightarrow \mathbb{Z}^n \rightarrow \pi_1(T^n/G) \rightarrow G \rightarrow 1.$$

The group  $\pi_1(T^n/G)$  is the fundamental group of a flat manifold  $\bar{T}$  by [AK57, Thm. 1]. Consider the corresponding extension of  $\pi_1(\bar{T})$ .

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \pi_1(T^n/G) & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \uparrow \cong & & \parallel \\ 1 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \pi_1(\bar{T}) & \longrightarrow & G \longrightarrow 1 \end{array}$$

By the Borel conjecture for flat manifolds [FH83], the isomorphism  $\pi_1(T^n/G) \cong \pi_1(\bar{T})$  above is induced by a homeomorphism  $h : T^n/G \cong \bar{T}$ . By construction, this homeomorphism lifts to homeomorphism  $\tilde{h} : T^n \rightarrow T^n$ , i.e. we have the following diagram

$$(6) \quad \begin{array}{ccccc} T^n \# \Sigma & \xrightarrow{\iota} & T^n & \xrightarrow{\tilde{h}} & T^n \\ \downarrow & & \downarrow & & \downarrow \pi \\ U := (T^n \# \Sigma)/G & \xrightarrow{\bar{\iota}} & T^n/G & \xrightarrow{h} & \bar{T} \end{array}$$

The vertical maps are covering maps, and the homeomorphism  $\tilde{h}$  conjugates the given action of  $G$  on  $T^n$  to an isometric action  $G' \curvearrowright T^n$ . This proves (i).

To show (ii), we conclude from Diagram (6) that the induced map

$$\pi^* : S(\bar{T}) \cong [\bar{T}, \text{Top/O}] \rightarrow [T^n, \text{Top/O}] \cong S(T)$$

sends  $(U, h \circ \bar{\iota})$  to  $(T^n \# \Sigma, \tilde{h} \circ \iota)$ . Indeed,  $\pi^*$  sends an arbitrary element  $(W, g) \in S(\bar{T})$  to  $(\widehat{W}, \widehat{g}) \in S(T)$ , where  $\widehat{W}$  is the pullback covering space with the smooth structure such that  $\widehat{W} \rightarrow W$  is smooth; see the following diagram.

$$\begin{array}{ccc} \widehat{W} & \xrightarrow{\widehat{g}} & T^n \\ \downarrow & & \downarrow \pi \\ W & \xrightarrow{g} & \bar{T} \end{array}$$

By Lemma 3.3  $(T^n \# \Sigma, \tilde{h} \circ \iota)$  is concordant to  $(T^n \# \Sigma, \iota)$ . Therefore,  $(T^n \# \Sigma, \iota)$  is in the image of  $\pi^*$ .  $\square$

**5.2. Step 2: compatible splitting of the top cell.** For an open embedding  $e : X \hookrightarrow Y$  of manifolds, we denote  $e' : Y' \rightarrow X'$  the induced map of 1-point compactifications. Recall that a smooth  $n$ -manifold  $X$  is *stably parallelizable* if it can be smoothly embedded in  $\mathbb{R}^{n+k}$ , for some  $k \geq 1$ , with trivial normal bundle.

The following lemma is well known. We give a proof which will be helpful in preparation for the Proposition 5.3.

**Lemma 5.2** (Splitting the top cell). *Let  $W$  be a stably parallelizable closed  $n$ -manifold. Then for  $k \geq n$ , there is a homotopy equivalence  $S^{n+k} \vee Z_W \rightarrow \Sigma^k W$ , where  $Z_W$  is a finite CW-complex of dimension  $< n + k$ .*

*Proof.* Give  $W$  a cell structure with a single  $n$ -cell, and give  $\Sigma^k W$  the induced cell structure, which has a single  $(n + k)$ -cell. Let  $Z = (\Sigma^k W)^{(n+k-1)}$  be the  $(n + k - 1)$ -skeleton. There is a cofiber sequence

$$Z \xrightarrow{i} \Sigma^k W \xrightarrow{q} S^{n+k}$$

Since  $W$  is stably parallelizable, there is a framed Whitney embedding  $j : W \times D^k \hookrightarrow \mathbb{R}^{n+k}$ . The map

$$p : S^{n+k} \xrightarrow{j'} \Sigma^k(W_+) \rightarrow \Sigma^k W$$

is a right inverse to the map  $q$  (up to homotopy) since the composition  $q \circ p : S^{n+k} \rightarrow S^{n+k}$  has degree 1.

*Claim.* The map  $p \vee i : S^{n+k} \vee Z \rightarrow \Sigma^k W$  is a homotopy equivalence.

*Proof of Claim.* Since the domain and codomain are simply connected, it suffices to show that  $p \vee i$  is a homology equivalence by Whitehead's theorem [Hat02, Cor. 4.33]. It is easy to see that  $p \vee i$  induces an isomorphism on  $H_\ell$  for  $\ell \leq n+k-2$  since  $Z$  is the  $(n+k-1)$  skeleton; see e.g. [Hat02, Lem. 2.34(c)]. It remains to treat the cases  $\ell = n+k-1$  and  $\ell = n+k$ .

For  $\ell = n+k$ , the composition  $S^{n+k} \vee Z \xrightarrow{p \vee i} \Sigma^k W \xrightarrow{q} S^{n+k}$  induces an isomorphism on  $H_{n+k}$  (since  $q \circ p$  has degree 1), and since each of these spaces has  $H_{n+k} = \mathbb{Z}$ , it follows that  $(p \vee i)_* : H_{n+k}(S^{n+k} \vee Z) \rightarrow H_{n+k}(\Sigma^k W)$  is an isomorphism (and also that  $q_* : H_{n+k}(\Sigma^k W) \rightarrow H_{n+k}(S^{n+k})$  is an isomorphism).

For  $\ell = n+k-1$ , it suffices to show that  $i_* : H_{n+k-1}(Z) \rightarrow H_{n+k-1}(\Sigma^k W)$  is injective. Considering the long-exact sequence of the pair  $(\Sigma^k W, Z)$ , it is equivalent to show that the homomorphism  $H_{n+k}(\Sigma^k W) \rightarrow H_{n+k}(\Sigma^k W, Z)$  is surjective. This homomorphism can be identified with  $q_* : H_{n+k}(\Sigma^k W) \rightarrow H_{n+k}(\Sigma^k W/Z)$ , which we observed is an isomorphism in the preceding paragraph.

This proves the claim, and finishes the proof of the lemma.  $\square$

**Proposition 5.3** (Compatible splitting). *Let  $\pi : T^n \rightarrow Q$  be the quotient by a free action of  $G = \mathbb{Z}/d\mathbb{Z} \curvearrowright T^n$ . Fix  $k \geq n$  and let  $\Sigma^k T^n \simeq S^{n+k} \vee Z_{T^n}$  and  $\Sigma^k Q \simeq S^{n+k} \vee Z_Q$  be splittings as in Lemma 5.2 (these exist by Proposition 4.2). Then the map*

$$(7) \quad S^{n+k} \hookrightarrow S^{n+k} \vee Z_{T^n} \xrightarrow{\simeq} \Sigma^k T^n \xrightarrow{\Sigma^k(\pi)} \Sigma^k Q \xrightarrow{\simeq} S^{n+k} \vee Z_Q \twoheadrightarrow Z_Q,$$

where the fourth map is a homotopy inverse to the splitting of Lemma 5.2, is homotopically trivial.

We do not know a reason for Proposition 5.3 to be true for general covering spaces. A key property of  $Q$  used in the proof below is that  $Q$  is a mapping torus of a homeomorphism of a stably-parallelizable manifold.

*Proof of Proposition 5.3.* Set  $G = \langle f \rangle$ . Fix an equivariant embedding  $T^{n-1} \hookrightarrow V$  into a  $\mathbb{R}[G]$ -module  $V$ ; c.f. [Pal57, Mos57]. Without loss of generality, we can assume that the action  $f \curvearrowright V$  is orientation-preserving (if not, replace  $V$  by  $V \oplus V$ , for example).

Let  $f_t$  be a homotopy in  $\text{SO}(V)$  from  $f$  to the identity. We assume this homotopy takes place for  $t \in (0, \infty)$ , with  $f_t = \text{id}$  for  $t \leq 1$  and  $f_t = f$  for  $t \geq 2$ .

The map  $F(v, t) = (f_t(v), t)$  is a homeomorphism of  $V \times (0, \infty)$ . Let  $M_F$  be the mapping torus of  $F$ . We can view  $M_F$  as a bundle

$$V \rightarrow M_F \rightarrow \mathbb{R}^2 \setminus \{0\}.$$

Since the bundle is trivial near 0, we can extend  $M_F$  to a bundle  $V \rightarrow \widehat{M}_F \rightarrow \mathbb{R}^2$ . This latter bundle is trivial  $\widehat{M}_F \cong V \times \mathbb{R}^2$  since  $\mathbb{R}^2$  is contractible. By construction,  $Q = M_f$  embeds in  $M_F \subset \widehat{M}_F$ .

Now we lift to an equivariant embedding of  $T^n$ . Identify  $\mathbb{R}^2 \cong \mathbb{C}$ , and consider the  $d$ -fold cover

$$\phi : V \times \mathbb{C} \rightarrow V \times \mathbb{C} \quad \text{given by} \quad (v, z) \mapsto (v, z^d).$$

This is a regular cover, branched over  $V \times \{0\}$ . The pre-image of  $M_f$  under this cover is  $M_{fd} = M_{\text{id}} = T^n$ . Thus we have an embedding  $T^n \hookrightarrow V \times \mathbb{R}^2$  that is equivariant with respect to the deck group actions of the coverings  $\pi : T^n \rightarrow M_f$  and  $\phi : V \times \mathbb{R}^2 \rightarrow V \times \mathbb{R}^2$ , and the quotient by these actions yields an embedding  $Q = M_f \hookrightarrow V \times \mathbb{R}^2$ . Since  $T^n$  and  $Q$  are stably parallelizable by Lemma 4.2, the normal bundles of  $T^n \subset V \times \mathbb{R}^2$  and  $Q \subset V \times \mathbb{R}^2$  are trivial. Thus we have a commutative diagram in the category of  $G$ -spaces.

$$\begin{array}{ccc} T^n \times D^k & \xrightarrow{j_{T^n}} & V \times \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \phi \\ Q \times D^k & \xrightarrow{j_Q} & V \times \mathbb{R}^2 \end{array}$$

We use collapse maps of the embeddings  $j_{T^n}, j_Q$  to get a commutative diagram

$$(8) \quad \begin{array}{ccc} S^{n+k} & \xrightarrow{p_{T^n}} & \Sigma^k T^n \\ \phi' \downarrow & & \downarrow \Sigma^k(\pi) \\ S^{n+k} & \xrightarrow{p_Q} & \Sigma^k Q \end{array}$$

For  $W = T^n$  or  $Q$ , the maps  $p_W : S^{n+k} \rightarrow \Sigma^k W$  are the maps that appear in the proof of Lemma 5.2. In particular, the composition

$$(9) \quad S^{n+k} \xrightarrow{p_Q} \Sigma^k Q \simeq S^{n+k} \vee Z_Q \twoheadrightarrow Z_Q$$

is homotopically trivial. Then the composition (7) is homotopically trivial because it factors through (9) by virtue of diagram (8).  $\square$

We now use Proposition 5.3 to prove that there is a “compatible splitting” of

$$\pi^* : [Q, \text{Top}/O] \rightarrow [T^n, \text{Top}/O].$$

Let  $\pi : T^n \rightarrow Q$  be the quotient by a free action of  $G = \mathbb{Z}/d\mathbb{Z}$ . Let  $u_Q : D^n \hookrightarrow Q$  be an embedded disk, chosen sufficiently small so that it lifts to an embedding  $\tilde{u} : \sqcup_d D^n \hookrightarrow T^n$ . Let  $u_{T^n} : D^n \hookrightarrow T^n$  be an embedded disk that contains the image of  $\tilde{u}$ , so  $\tilde{u}$  factors as  $\sqcup_d D^n \xrightarrow{j} D^n \xrightarrow{u_{T^n}} T^n$ . This leads to the following commutative diagram of spaces.

$$\begin{array}{ccc} T^n & \xrightarrow{u'_{T^n}} & S^n \\ & \searrow \tilde{u}' & \downarrow j' \\ & & V_d S^n \\ \pi \downarrow & & \downarrow \Delta \\ Q & \xrightarrow{u'_Q} & S^n \end{array}$$

Here the map  $\Delta$  is the identity map on each  $S^n$ . The composition  $\Delta \circ j' : S^n \rightarrow S^n$  has degree  $d$ . Suspending this diagram and combining with Diagram (8), we obtain

$$(10) \quad \begin{array}{ccccc} S^{n+k} & \xrightarrow{p_{T^n}} & \Sigma^k T^n_+ & \xrightarrow{u'_{T^n}} & S^{n+k} \\ \text{deg}=d \downarrow & & \downarrow & & \\ S^{n+k} & \xrightarrow{p_Q} & \Sigma^k Q_+ & \xrightarrow{u'_Q} & S^{n+k} \end{array}$$

The composition  $S^{n+k} \rightarrow S^{n+k}$  in each row is a degree-1 map. Recalling that  $\text{Top/O}$  is an infinite loop space, let  $Y$  be a space such that  $\Omega^k Y \simeq \text{Top/O}$ . Apply  $[-, Y]$  to Diagram (10), and use the adjunction  $[A, \Omega B] \cong [\Sigma A, B]$  to arrive at the following diagram.

$$\begin{array}{ccccc} [S^n, \text{Top/O}] & \xrightarrow{(u'_Q)^*} & [Q, \text{Top/O}] & \xrightarrow{p_Q^*} & [S^n, \text{Top/O}] \\ & & \downarrow \pi^* & & \downarrow \mu_d \\ [S^n, \text{Top/O}] & \xrightarrow{(u'_{T^n})^*} & [T^n, \text{Top/O}] & \xrightarrow{p_{T^n}^*} & [S^n, \text{Top/O}] \end{array}$$

Here  $\mu_d$  is multiplication by  $d$ . We have proved the following corollary.

**Corollary 5.4.** *The maps  $(u'_Q)^*$  and  $(u'_{T^n})^*$  are split injections, and there exist splittings  $p_Q^*, p_{T^n}^*$  with the property that*

$$(11) \quad p_{T^n}^* \circ \pi^* = \mu_d \circ p_Q^*.$$

**5.3. Finishing the proof of Theorem A.** In Proposition 5.1(ii), we showed that the concordance class  $((T^n \# \Sigma)/G, h \circ \bar{\iota}) \in S(\bar{T})$  maps to  $(T^n \# \Sigma, \iota)$  under  $\pi^*$ . By Corollary 5.4,  $U$  is concordant, hence diffeomorphic (by [KS77, Essay I, Theorem 4.1]) to  $\bar{T} \# \hat{\Sigma}$  for some  $\hat{\Sigma} \in \Theta_n$ . By Proposition 5.1(i), the action  $G \curvearrowright T^n \# \Sigma$  is topologically conjugate to an isometric action on  $T^n$ . Therefore, we can apply Lemma 2.1 to conclude that the action of  $G$  on  $T^n \# \Sigma$  is standard.

## 6. ACTIONS ON EXOTIC HYPERBOLIC 7-MANIFOLDS (THEOREM 1.2)

In this section we prove Theorem 1.2. The general strategy is similar to the proof of Theorem A. Assuming that  $G$  acts freely on  $M \# \Sigma$ , we find an isometric action  $G' \curvearrowright M$  and show that  $(M \# \Sigma, \iota)$  is in the image of the homomorphism

$$\pi^* : [M/G', \text{Top/O}] \rightarrow [M, \text{Top/O}]$$

induced by the quotient map  $\pi : M \rightarrow M/G'$  (Proposition 6.1). We would like to use this to show that  $(M \# \Sigma)/G$  is diffeomorphic to  $M \# \hat{\Sigma}$  for some  $\hat{\Sigma} \in \Theta_n$ . Here there is some difficulty, as we are not able to prove an analogue of the space-level splitting of Proposition 5.3. Instead we study  $\pi^*$  using the Atiyah–Hirzebruch spectral sequence. Our inability to resolve extension problems in the spectral sequence ultimately forces us to restrict to dimension 7. We argue in two steps.

**Step 1: rigidity.** In this step we show that if  $G$  acts freely on  $M \# \Sigma$ , then there exists a free, isometric action  $G' \curvearrowright M$  such that if  $\pi : M \rightarrow M/G'$  is the quotient map, then the smooth structure  $(M \# \Sigma, \iota)$  is in the image of the homomorphism

$$\pi^* : [M/G', \text{Top/O}] \rightarrow [M, \text{Top/O}].$$



This statement is proved in two steps, which are stated in Proposition 6.1. These statements, which do not use the assumption  $\dim M = 7$ , will be used again for the proof of Theorem B.

**Proposition 6.1.** *Fix a closed hyperbolic  $n$ -manifold  $M$ , an exotic  $n$ -sphere  $\Sigma$ , and a finite subgroup  $G < \text{Diff}^+(M \# \Sigma)$  acting freely on  $M \# \Sigma$ .*

- (i) *There exists a subgroup  $G' < \text{Isom}(M)$  and a homeomorphism  $M \# \Sigma \rightarrow M$  that is equivariant with respect to some isomorphism  $G \cong G'$ .*
- (ii) *Denoting the quotient map  $\pi : M \rightarrow M/G$ , then the smooth structure  $(M \# \Sigma, \iota)$  is in the image of the homomorphism*

$$\pi^* : [M/G', \text{Top/O}] \rightarrow [M, \text{Top/O}].$$

Proposition 6.1(i) is similar to part of [CLW18, Thm. 1.5], which says that if  $\text{Isom}^+(M)$  acts freely on  $M$ , then every finite subgroup of  $\text{Homeo}^+(M)$  is conjugate into  $\text{Isom}^+(M)$ . However, in Proposition 6.1, we do not assume that the full group  $\text{Isom}^+(M)$  acts freely; nevertheless our proof of Proposition 6.1 is similar to the corresponding part of the proof of [CLW18, Thm. 1.5].

*Proof of Proposition 6.1.* Let  $\iota : M \# \Sigma \rightarrow M$  be the standard homeomorphism defined in §3. We use  $\iota$  to view the action of  $G$  on  $M \# \Sigma$  as an action of  $G$  on  $M$  by homeomorphisms. The induced homomorphism  $G \rightarrow \text{Out}(\pi_1(M))$  is injective by [Bor83]; denote the image  $G' < \text{Out}(\pi_1(M))$ . By Mostow rigidity,  $\text{Out}(\pi_1(M)) \cong \text{Isom}(M)$ , so  $G'$  is a group of isometries of  $M$ . To prove the proposition, it suffices to show that the actions  $G \curvearrowright M$  and  $G' \curvearrowright M$  are topologically conjugate.

*Claim.*  $G'$  acts freely on  $M$ .

*Proof of Claim.* The subgroup  $G' < \text{Out}(\pi_1(M))$  determines an extension

$$(12) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \text{Aut}(\pi_1(M)) & \longrightarrow & \text{Out}(\pi_1(M)) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \Gamma & \longrightarrow & G' \longrightarrow 1 \end{array}$$

By Mostow rigidity, the extension in the top row in the Diagram (12) is equivalent to the extension

$$1 \rightarrow \pi_1(M) \rightarrow N(\pi_1(M)) \rightarrow \text{Isom}(M) \rightarrow 1,$$

where  $N(\pi_1(M))$  denotes the normalizer in  $\text{Isom}(\mathbb{H}^n)$ . Consequently, the group  $\Gamma$  can be identified with the group of all lifts of isometries of  $G'$  to the universal cover  $\mathbb{H}^n$ . Then  $G'$  acts freely on  $M$  if and only if  $\Gamma$  is torsion-free. We prove the Claim by showing  $\Gamma$  is torsion free.

To show that  $\Gamma$  is torsion free, we give another description of the extension of  $G'$  in (12). Recall that  $G$  acts freely on  $M$ , so the quotient  $M \rightarrow M/G$  is a covering map, which determines an extension

$$1 \rightarrow \pi_1(M) \rightarrow \pi_1(M/G) \rightarrow G \rightarrow 1.$$

By construction, the homomorphism  $G \xrightarrow{\cong} G' < \text{Out}(\pi_1(M))$  that classifies this extension, induces an isomorphism of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(M/G) & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \cong & & \downarrow \cong \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \Gamma & \longrightarrow & G' \longrightarrow 1 \end{array}$$

The group  $\pi_1(M/G)$  is torsion-free because  $M/G$  is a closed aspherical manifold. Thus  $\Gamma$  is torsion free. This completes the proof of the Claim.

It remains to obtain a homeomorphism  $M \rightarrow M$  that conjugates the actions of  $G$  and  $G'$ . By the Borel conjecture for hyperbolic manifolds [FJ89, Cor. 10.5], the isomorphism  $\pi_1(M/G) \cong \Gamma \cong \pi_1(M/G')$  is induced by a homeomorphism  $h : M/G \cong M/G'$ . By construction, this homeomorphism lifts to an equivariant homeomorphism  $\tilde{h} : M \rightarrow M$ , as desired. This completes the proof of (i).

In order to prove (ii), we note from part (i) that we have the a commutative diagram

$$(13) \quad \begin{array}{ccccc} M \# \Sigma & \xrightarrow{\iota} & M & \xrightarrow{\tilde{h}} & M \\ \downarrow & & \downarrow & & \downarrow \pi \\ (M \# \Sigma)/G & \xrightarrow{\bar{\iota}} & M/G & \xrightarrow{h} & M/G' \end{array}$$

As in the proof of Proposition 5.1, from this diagram, we conclude that the homomorphism

$$\pi^* : S(M/G') \cong [M/G', \text{Top}/O] \rightarrow [M, \text{Top}/O] \cong S(M)$$

sends  $((M \# \Sigma)/G, h \circ \bar{\iota})$  to  $(M \# \Sigma, \tilde{h} \circ \iota)$ . By Lemma 3.5, the smooth structures  $(M \# \Sigma, \tilde{h} \circ \iota)$  and  $(M \# \Sigma, \iota)$  are concordant, so  $(M \# \Sigma, \iota)$  is in the image of  $\pi^*$ .  $\square$

**Step 2: computation with the Atiyah–Hirzebruch spectral sequence.** Consider the Atiyah–Hirzebruch spectral sequence for the cohomology theory  $\mathbb{E}^*$  determined by the infinite loop space  $\text{Top}/O$ , so that  $\mathbb{E}^0(M) = [M, \text{Top}/O]$ . This spectral sequence has  $E_2$ -page

$$E_2^{p,q} = H^p(M; \pi_{-q}(\text{Top}/O)).$$

Recall that  $\pi_k(\text{Top}/O) = \Theta_k$  if  $k \neq 3$  and  $\pi_3(\text{Top}/O) \cong \mathbb{Z}/2\mathbb{Z}$ . In particular  $\pi_k(\text{Top}/O) = 0$  for  $k \in \{0, 1, 2, 4, 5, 6\}$ . Thus the spectral sequence yields an exact sequence

$$H^7(M; \Theta_7) \xrightarrow{c} [M, \text{Top}/O] \xrightarrow{q} H^3(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow 0$$

Identifying  $H^7(M; \Theta_7) \cong \Theta_7$ , the homomorphism  $c$  is the map  $\Sigma \mapsto (M \# \Sigma, \iota)$ . (Aside: We will not need to interpret the homomorphism  $q$ , but it can be identified with the natural map  $[M, \text{Top}/O] \rightarrow [M, \text{Top}/\text{PL}]$  that sends a smooth structure to its corresponding PL structure.)

The map  $\pi : M \rightarrow M/G'$  induces a map of spectral sequences and a commutative diagram

$$\begin{array}{ccccc} \Theta_7 & \xrightarrow{c} & [M/G', \text{Top}/O] & \xrightarrow{q} & H^3(M/G'; \mathbb{Z}/2\mathbb{Z}) \\ \downarrow \cdot |G| & & \downarrow \pi^* & & \downarrow \pi^* \\ \Theta_7 & \xrightarrow{c} & [M, \text{Top}/O] & \xrightarrow{q} & H^3(M; \mathbb{Z}/2\mathbb{Z}) \end{array}$$

The left vertical map is multiplication by  $|G|$ , which is the degree of the cover  $\pi$ . As  $|G|$  is odd, the right vertical map is injective by a transfer argument [Hat02, Prop. 3G.1].

To complete the proof of Theorem 1.2, we note that by Step 1,  $(M \# \Sigma, \iota) = \pi^*(x)$ , where  $x = ((M \# \Sigma)/G, h \circ \bar{\iota})$ ; c.f. Diagram (13). Since  $(M \# \Sigma) = c(\Sigma)$ , we have  $0 = q(\pi^*(x)) = \pi^*(q(x))$ , which implies that  $q(x) = 0$  since the right vertical map is injective. Thus  $x = c(\hat{\Sigma})$  for some  $\Sigma \in \Theta_7$ , i.e.  $(M \# \Sigma)/G$  is concordant, hence diffeomorphic, to  $(M/G') \# \hat{\Sigma}$ . By Proposition 6.1(i), the action  $G \curvearrowright M \# \Sigma$  is topologically conjugate to an isometric action on  $M$ . Therefore, we can apply Lemma 2.1 to conclude that the action of  $G$  on  $M \# \Sigma$  is standard.  $\square$

## 7. ASYMMETRIC SMOOTHINGS (THEOREM B)

Recall from Proposition 6.1 that if  $M$  is a closed hyperbolic manifold and  $G$  acts freely on  $M\#\Sigma$ , then the concordance class of  $(M\#\Sigma, \iota)$  is in the image of a certain homomorphism

$$\pi^* : [M/G', \text{Top}/O] \rightarrow [M, \text{Top}/O].$$

We would like to use this to conclude that  $M$  satisfies the divisibility property (Definition 1.18), similar to Step 2 in the proof of Theorem A. If we could show this, then it would suffice to find  $M$  and  $\Sigma$  with the property that  $\Sigma$  is not divisible by  $|G|$  for any nontrivial group  $G < \text{Isom}^+(M)$ .

Unfortunately, we don't know how to prove that hyperbolic manifolds satisfy the divisibility property in general (it seems difficult to produce a geometric construction that would yield a version of Proposition 5.3 in the hyperbolic case). Instead, as in the proof of Theorem 1.2, we study  $\pi^*$  using the Atiyah–Hirzebruch spectral sequence. Here the difficulty is (as usual) potentially nontrivial differentials and extension problems, but we show these issues can be avoided for a proper choice of  $M, \Sigma$  and by localizing  $\text{Top}/O$  at an odd prime.

*Proof of Theorem B.*

**Step 1: the construction.** Fix  $n_0 \geq 5$ ,  $d \geq 1$ , and choose  $n = 4k - 1 \geq n_0$  and an odd prime  $p$  such that the  $p$ -torsion subgroup of  $bP_{n+1}$  is nontrivial and the  $p$ -torsion subgroup of  $bP_{m+1}$  is trivial for  $m < n$ . This is possible because the set of primes that divide  $|bP_{4k}|$  for some  $k$  is infinite. For example,  $|bP_{4k}|$  is divisible by  $2^{2k-1} - 1$  (see [KM63, §7]), and it is not difficult to show that if  $s, t$  are relatively prime, then  $2^s - 1$  and  $2^t - 1$  are relatively prime.

Let  $\mathbb{Z}_{(p)}$  denote the set of rational numbers with denominator relatively prime to  $p$ . The group  $(bP_{n+1})_{(p)} := bP_{n+1} \otimes \mathbb{Z}_{(p)}$  is the  $p$ -torsion of  $bP_{n+1}$ . Since  $bP_{n+1}$  is cyclic,  $(bP_{n+1})_{(p)} \cong \mathbb{Z}/p^a\mathbb{Z}$  for some  $a \geq 1$ . Choose a generator  $\Sigma \in (bP_{n+1})_{(p)}$ .

Next choose a closed oriented hyperbolic  $n$ -manifold  $M$  such that (i)  $\text{Isom}^+(M) = \text{Isom}(M)$ , (ii)  $\text{Isom}(M)$  is a  $p$ -group where every element has order divisible by  $p^a$ , and (iii)  $\text{Isom}(M)$  acts freely on  $M$ . Such examples exist by the construction of Belolipetsky–Lubotzky [BL05, Thm. 1.1]; see also [BT19, Thm. 6].

**Step 2: the computation.** Take  $M$  and  $\Sigma$  as in Step 1. We claim that  $N = M\#\Sigma$  is asymmetric. Fix a finite order element  $g \in \text{Diff}^+(M\#\Sigma)$  and denote  $G = \langle g \rangle$ . Suppose for a contradiction that  $g \neq \text{id}_N$ . By a result of Borel [Bor83], the induced map  $G \rightarrow \text{Out}(\pi_1(N)) \cong \text{Isom}(M)$  is injective, so the order of  $g$  is  $p^b$  for some  $b \geq a$ .

By Proposition 6.1, there is a degree  $|G|$  covering map  $\pi : M \rightarrow \overline{M}$  and  $x \in [\overline{M}, \text{Top}/O]$  such that  $\pi^*(x) = (M\#\Sigma, \iota)$ , where  $\pi^* : [\overline{M}, \text{Top}/O] \rightarrow [M, \text{Top}/O]$ .

To explain the remainder of the argument, we consider the following commutative diagram.

$$(14) \quad \begin{array}{ccccc} [\overline{M}, \text{Top}/O] & \xrightarrow{\pi^*} & [M, \text{Top}/O] & \xleftarrow{(i')^*} & [S^n, \text{Top}/O] \cong \Theta_n \\ \downarrow & & \downarrow & & \downarrow \\ [\overline{M}, \text{Top}/O_{(p)}] & \longrightarrow & [M, \text{Top}/O_{(p)}] & \xleftarrow{\quad} & [S^n, \text{Top}/O_{(p)}] \cong \Theta_n \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ [\overline{M}, B] & \xrightarrow{0} & [M, B] & \xleftarrow{\cong} & [S^n, B] \cong bP_{n+1} \otimes \mathbb{Z}_{(p)} \end{array}$$

In the top row, the map  $i' : M \rightarrow S^n$  is the collapse map induced by the inclusion of a disk  $i : D^n \hookrightarrow M$ . The vertical maps are induced by maps

$$\text{Top/O} \rightarrow \text{Top/O}_{(p)} \rightarrow B \times C \rightrightarrows B.$$

where the second arrow is a homotopy inverse to the equivalence of Theorem 3.2.

*Claim.*  $[M, B] \cong H^n(M; \pi_n(B)) \cong bP_{n+1} \otimes \mathbb{Z}_{(p)}$  and similarly for  $\overline{M}$ .

*Proof of Claim.* We prove this using the Atiyah–Hirzebruch spectral sequence. As discussed in §3, this spectral sequence has  $E_2$ -page

$$E_2^{i,-j} = H^i(M; \pi_j(B)),$$

and converges to  $\mathbb{E}^{i-j}(M)$ , where  $\mathbb{E}^*$  denotes the cohomology theory associated to the infinite loop space  $B$ . In particular, to determine,  $[M, B] = \mathbb{E}^0(M)$ , we focus on the terms  $E_2^{i,-i} = H^i(M; \pi_i(B))$ . By our choice of  $p$ , and from the fact that  $M$  is a closed, oriented  $n$ -manifold, we have

$$H^i(M; \pi_i(B)) = \begin{cases} \mathbb{Z}/p^a\mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Furthermore, by construction  $\pi_k(B) = bP_{n+1} \otimes \mathbb{Z}_{(p)}$  is 0 for  $k < n$ , so the term  $E_2^{n,-n}$  receives no nontrivial differentials, and the claim follows.

The induced map

$$H^n(\pi) : H^n(\overline{M}; \pi_n(B)) \cong \mathbb{Z}/p^a\mathbb{Z} \rightarrow \mathbb{Z}/p^a\mathbb{Z} \cong H^n(M; \pi_n(B))$$

is multiplication by  $\deg(M \xrightarrow{\pi} \overline{M}) = |G|$ , so  $H^n(\pi)$  is the zero map because  $p^a$  divides  $|G|$  by construction. This explains the arrow labeled “0” in Diagram (14).

Now we conclude. On the one hand, the image of  $(M \# \Sigma, \iota)$  under  $[M, \text{Top/O}] \rightarrow [M, B]$  is nonzero because  $(M \# \Sigma, \iota) = (i')^*(\Sigma)$ , and the “right side” of Diagram (14) commutes. On the other hand,  $(M \# \Sigma, \iota)$  is in the kernel of  $[M, \text{Top/O}] \rightarrow [M, B]$  because  $(M \# \Sigma, \iota) = \pi^*(x)$ , and the “left side” of Diagram (14) commutes. This contradiction implies that our finite order element  $g \in \text{Diff}(N)$  must have been trivial, and this completes the proof of Theorem B.  $\square$

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Mauricio Bustamante

Departamento de Matemáticas, Universidad Católica de Chile

mauricio.bustamante@mat.uc.cl

Bena Tshishiku

Department of Mathematics, Brown University

benatshishiku@brown.edu