Homework 3

Math 106

Due Friday, Sept 29 by 11:59pm

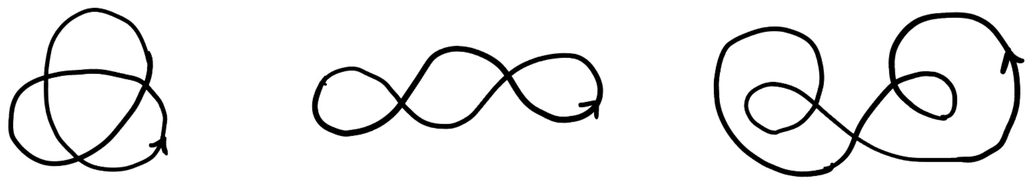
Name:

Topics covered: curvature, Frenet frame, fundamental theorem of space curves, rotation number

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use ed discussions!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Let* $\alpha : [0, L] \to \mathbb{R}^2$ *be a unit-speed plane curve, and assume* $\alpha(0) = \alpha(L)$ *and* $\alpha'(0) = \alpha'(L)$*. Write* $\alpha'(t) = (\cos\theta(t), \sin\theta(t))$*, where* $\theta : [0, L] \to \mathbb{R}$ *is a smooth function. Then* $\frac{1}{2\pi}\big(\theta(L) - \theta(0)\big)$ *is an integer, called the rotation number of* α*. Determine the rotation number of the following curves. Please show some work.*



Solution. 2, 1, 0 □

Problem 2. *Write down a linear system of differential equations in functions $f_1, \ldots, f_6$ that is satisfied by $f_1 = T \cdot N$, $f_2 = T \cdot B$, $f_3 = N \cdot B$, $f_4 = T \cdot T$, $f_5 = N \cdot N$, $f_6 = B \cdot B$ when T, N, B are a Frenet frame.*[1] *Verify that the functions $f_1 = f_2 = f_3 = 0$ and $f_4 = f_5 = f_6 = 1$ is a solution to your system of differential equations.*[2]

Solution. Using the hint we obtain the following system.

$$
\begin{aligned}
f_1' &= \kappa f_5 - \kappa f_4 - \tau f_2 \\
f_2' &= \kappa f_3 + \tau f_1 \\
f_3' &= -\kappa f_2 - \tau f_6 + \tau f_5 \\
f_4' &= 2\kappa f_1 \\
f_5' &= 2\kappa f_1 - 2\tau f_3 \\
f_6' &= 2\tau f_3
\end{aligned}
$$

It is easy to check that when $(f_1, \ldots, f_6) = (0, 0, 0, 1, 1, 1)$ both sides of each equation evaluate to 0. □

[1] Hint: differentiate these dot product functions, and express the answer back in terms of these functions. The final answer should be a system of differential equations involving $f_1, \ldots, f_6$, not T, N, B.

[2] Remark: recall this fact is used to prove the fundamental theorem of space curves.

Problem 3. *Let $F : V \to V$ be a linear operator on a finite-dimensional inner-product space. Prove the following are equivalent.*

(a) F preserves the inner product.

(b) F preserves lengths of vectors.

(c) F preserves orthonormality (i.e. it sends an ONB to another ONB).

(d) F preserves orthonormality of some orthonormal basis (i.e. there exists an ONB that is sent to an ONB under F).

A map satisfying these conditions is called a linear isometry or an orthogonal map.[3]

Solution. (a) implies (b) because the length is given in terms of the dot product.

(b) implies (a) because $|v+w|^2 = (v+w)\cdot(v+w) = |v|^2 + |w|^2 + 2v \cdot w$ implies the dot product can be expressed in terms of the norm.

(a) implies (c) since orthonormal is the relations $v_i \cdot v_i = 1$ and $v_i \cdot v_j = 0$, which is given in terms of the inner product.

(c) implies (b) is obvious (given any vector, extend to an orthonormal basis)

(c) implies (d) is obvious.

(c) implies (b): fix an ONB $u_1, \dots, u_n$ such that $F(u_1), \dots, F(u_n)$ is also an ONB. If $v = \sum c_i u_1$, then

$$|F(v)|^2 = \left|\sum c_i F(u_i)\right| = \sum c_i^2 = \left|\sum c_i u_i\right| = |v|^2.$$

$\square$

[3]Remark: for $\mathbb{R}^2$ with the standard inner product, the orthogonal maps are rotations and reflections.

Problem 4. *Show that the curvature and torsion of a curve are invariant under rigid motions.* [4] [5] [6]

Solution. Summary: The most interesting thing is to show the $\beta = R \circ \alpha$ has the same curvature and torsion as α when R is an orthogonal map with positive determinant. Curvature is easy to show by direct computation: $\kappa_\beta = |\beta''| = |R\alpha''| = |\alpha''| = \kappa_\alpha$ (note orthogonal maps preserve the norm). For the torsion, first show that the tangent and normal vectors differ by R. Then the binormal vectors agree up to a sign since R is orthogonal (sends ONB to ONB). Since the determinant is positive, the binormal vectors agree. □

[4] Recall that (by our definition) a rigid motion is a composition of translations and orthogonal maps.
[5] Remark: this is the converse of the fundamental theorem of space curves.
[6] Hint: You'll need to show something about rigid motions and the cross product...

Problem 5. *The tangent line is the line that best approximates a curve at a point. Similarly, the osculating circle is the circle that best approximates a plane curve at a point. Recall that for points a, b, c in the plane, not on a line, there is a unique circle passing through these points. Write $C(a, b, c)$ for the center of this circle. The osculating circle at $\alpha(t)$ is defined as the circle through $\alpha(t)$ with center*

$$C = \lim_{s \to 0} C(\alpha(t-s), \alpha(t), \alpha(t+s)).$$

(i) Fix $\lambda > 0$ and define $\beta(s) = (s, \lambda s^2)$. For $s \neq 0$, compute the center of the circle that passes through $\beta(s), \beta(0)$, and $\beta(-s)$. [7]

(ii) Assume α satisfies $\alpha(0) = (0, 0)$ and $\alpha'(0) = (1, 0)$. Use the preceding part and the Taylor expansion of $\alpha(t)$ to show the radius of the osculating circle at $\alpha(0)$ is $1/\kappa$, where $\kappa = \kappa(0)$ is the curvature. [8] [9]

Solution. (i) Following the hint, we want to find a, b, r that satisfy the equations

$$\begin{aligned} (s-a)^2 + (\lambda s^2 - b)^2 &= r^2 \\ (-s-a)^2 + (\lambda s^2 - b)^2 &= r^2 \\ a^2 + b^2 &= r^2 \end{aligned}$$

The solution is $a = 0$, $b = \frac{1}{2\lambda}(1 + \lambda^2 s^2)$.

(ii) Write $\kappa = \kappa(0)$. The Taylor expansion $\alpha(t) \approx (t, \frac{\kappa}{2} t^2)$. Using this approximation, the osculating circle has radius $\frac{1}{\kappa}$ by the computation in part (i). $\square$

[7] Hint: You can solve this by finding a, b, r so that $\beta(\pm s), \beta(0)$ satisfy the equation $(x-a)^2 + (y-b)^2 = r^2$.

[8] Hint: use specifically the degree-2 Taylor approximation.

[9] Remark: this problem gives a geometric interpretation for the curvature.