Homework 3

Math 123

Due February 17, 2023 by 5pm

Name:

Topics covered: trees, Prüfer codes, spanning trees, counting graphs

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems. You must type your solutions alone.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!

Problem 1. *Determine which trees have Prüfer codes that*

(a) contain only one value;

(b) contain exactly two values;

(c) have distinct values.

You should explain your answer, but you don't need to give careful proof.

Solution. (a) The Prüfer code $(a, \ldots, a)$ is the star graph with central vertex labeled a.

(b) The tree corresponding to Prüfer code with coordinates a, b are trees that are a union of two stars with centers a, b connected by edge ab.

(c) If all the values of the Prüfer code are distinct, then the tree only has two leaves (the indices not in the code), which implies that the tree is isomorphic to the path P_n. □

Problem 2. *Prove that if $T_1, \ldots, T_k$ are pairwise-intersecting subtrees of a tree T, then T has a vertex that belongs to all of $T_1, \ldots, T_k$.* [1] [2]

Solution. First we prove a lemma.

Lemma: If $A, B \subset T$ are subtrees that intersect nontrivially, then $A \cap B$ is also a tree.

Proof of lemma: Since $A \cap B$ is a subgraph of a tree, it has no cycles, so to show it's a tree, it suffices to show it's connected. Let u, v be vertices of $A \cap B$. Let $P \subset T$ be a path joining u and v. Since A is a tree, there is a path Q in A between u and v. By uniqueness of paths in T, $P = Q$. Similarly, we conclude $P \subset B$. Thus $P \subset A \cap B$.

Now we solve the problem by inducting on k. The base cases $k = 1, 2$ are trivial.

For the induction step, suppose given k subtrees $T_1, \ldots, T_k$ for $k \geq 3$. To show $T_1 \cap \cdots \cap T_k$ is nonempty, we write this as

$$\big(T_1 \cap \cdots \cap T_{k-2}\big) \cap T_{k-1} \cap T_k.$$

Write $S = T_1 \cap \cdots \cap T_{k-2}$. By the induction hypothesis and the lemma, S is a nonempty tree. By assumption $T_{k-1} \cap T_k$ is nonempty; the induction hypothesis also implies that $S \cap T_{k-1}$ and $S \cap T_k$ are nonempty. Then by the induction hypothesis applied to S, T_{k-1}, T_k implies that

$$T_1 \cap \cdots \cap T_k = S \cap T_{k-1} \cap T_k$$

is nonempty. □

Problem 3. *Let G_n be the graph whose vertices are orderings of the elements of $\{1, \ldots, n\}$ with $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ adjacent if they differ by switching a pair of adjacent entries.*[3]

(a) The graph G_3 is isomorphic to a familiar graph. Which one is it?

(b) Show that G_n is connected. [4]

[1] Remark: This is a graph-theoretic analog of Helly's theorem.

[2] Hint: use induction on k.

[3] Note: Here the sequence $(a_1, \ldots, a_n)$ does not have repetition. Each element of $\{1, \ldots, n\}$ appears exactly once.

[4] Tangential remark: here you are proving that a certain collection of permutations generate the symmetric group.

Solution. (a) The graph G_3 is isomorphic to C_5.

(b) Fix a vertex $a = (a_1, \dots, a_n)$. It suffices to construct path (or even just a walk) from a to $v = (1, 2, 3, \dots, n)$. (Then given any two vertices a, b we can obtain a walk from a to b by concatenating walks from a to v and v to b.)

We can show a walk a to v exists by induction on n. The case $n = 1$ is trivial since G_1 is a single point. For the induction step, fix $(a_1, \dots, a_n)$. Let k be the index with $a_k = 1$. By swapping positions $(k, k-1)$, then $(k-1, k-2)$, and so on until $(2, 1)$, we obtain a new vertex $(a'_1, \dots, a'_n)$ with $a'_1 = 1$. Now by the induction hypothesis, there exists a walk from $(a'_2, \dots, a_n)$ to $(2, \dots, n)$. Concatenating these walks gives a walk $(a_1, \dots, a_n)$ to $(1, \dots, n)$ as desired. □

Problem 4. *Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.*

Solution. Let $1, \dots, n$ be the vertices of K_n, and fix edge $e = \{n-1, n\}$. To solve the problem, we will count spanning trees containing e, showing there are $2n^{n-3}$ of them. If we do this, then the number of trees not containing e is $n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$.

Let T be a tree containing e. In the construction of its Prüfer code, where we inductively delete the smallest leaf, the last remaining edge will be e. Thus the last entry in the code is either n or $n-1$. There are exactly $2n^{n-3}$ such sequences.

To finish the proof, we show that if T is a spanning tree of K_n that doesn't contain e, then $P(T)$ does not end in either of $n-1, n$. Consider the unique path P in T between the vertices $n-1$ and n. Let x be the vertex on P closest to n; by assumption P has at least two edges, so $x \neq n-1$. Observe that during the construction of the Prüfer code, at some stage we obtain the tree P. (No vertex of P can be deleted until $n-1$ becomes the smallest leaf, since the algorithm produces a tree at each stage.) Now it follows that $P(T)$ ends in $x \notin \{n-1, n\}$. This completes the proof. □

Problem 5. *Prove that the number of even graphs (i.e. graphs where every vertex has even degree) with vertex set $\{1, \dots, n\}$ is $2^{\binom{n-1}{2}}$.* [5]

Solution. Let H be a graph with vertex set $\{1, \dots, n-1\}$. We define an even graph $\phi(H)$ with vertices $\{1, \dots, n\}$ by taking H and adding an edge $\{i, n\}$ precisely when i has odd degree in H. Then in $\phi(H)$, each vertex $1, \dots, n-1$ has even degree. Note also that the degree of n in $\phi(H)$ is the number of vertices of odd degree in H. By the degree-sum formula this number is even. Thus $\phi(H)$ is an even graph.

Conversely, suppose that G is an even graph with vertex set $\{1, \dots, n\}$. Define $\psi(G)$ as the graph obtained from G by deleting n. This graph has vertices $\{1, \dots, n-1\}$.

We claim that ϕ, ψ are inverses, thus establishing the desired bijection. Clearly $\psi(\phi(H)) = H$ since we add a vertex and then we remove it. Now consider $\phi(\psi(G))$. When we remove vertex n from G, the vertex i in $\psi(G)$ has odd degree if and only if $\{i, n\}$ is an edge in G. Thus by construction $\phi(\psi(G))$ adds this edge back to $\psi(G)$ and nothing more. This shows $\phi(\psi(G)) = G$. □

Problem 6. *Consider an alternative version of Bridg-it, where the player that forms a path connecting their ends loses. Give a strategy that shows that Player 2 can always win.* [6]

[5] Hint: establish a bijection to the set of all graphs with vertex set $\{1, \dots, n-1\}$.

[6] You may use the following fact, which is similar to one we proved: Given spanning trees, $T, T' \subset G$ and an edge

Solution. As in class, we consider an equivalent version of the game, which is played on a graph G that is the union of two spanning trees (blue/green). Here the rules are that Player 1 adds blue/green double edges to G, and Player 2 deletes edges from G. The graph has a "dummy" edge connecting the top to bottom, and this edge is not allowed to be touched by either player (since it does not correspond to any move in Bridg-it).

Strategy: Suppose Player 1 adds a blue edge e parallel to a green edge. By the lemma in the footnote, there is an edge e' of the blue graph so that adding e and removing e' preserves the fact that the blue graph is a spanning tree. Player 2 deletes the edge e'. If Player 1 adds a green edge parallel to a blue edge, the play is similar. After each round in the game, we have a graph that is a union of two (disjoint) spanning trees. At the end, there are only double edges which are blue and green copies of the same spanning tree, which shows that Player 1 has lost.

There is one issue: we need to argue that Player 2 is never forced to remove the dummy edge. Note that in Player 2's move, they can apply the footnote by looking for a cycle in one of the graphs and removing one edge from that cycle. The key observation is that if the cycle contains the dummy edge, then either Player 2 can remove a different edge from this cycle, or Player 1 has already lost. □

e of T but not T', there exists an edge of e' of T' but not T such that $T' - e' + e$ is a spanning tree. (Note the difference between this and the statement we proved in class!)