Veech Fibrations

Sam Freedman (Brown)

Joint with Trent Lucas (Brown)

Gatsby Spring 2024

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The Nielsen realization problem for K3 surfaces

Benson Farb and Eduard Looijenga *

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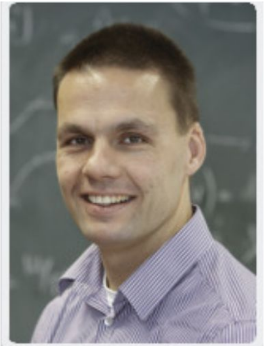

PERIODIC POINTS ON VEECH SURFACES AND THE MORDELL-WEIL GROUP OVER A TEICHMÜLLER CURVE

MARTIN MÖLLER

The Nielsen realization problem for K3 surfaces

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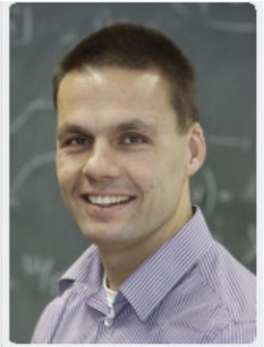

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VEECH FIBRATIONS

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General Veech Fibrations

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(Generically) minimal complex surfaces of general type + smooth 4-manifolds

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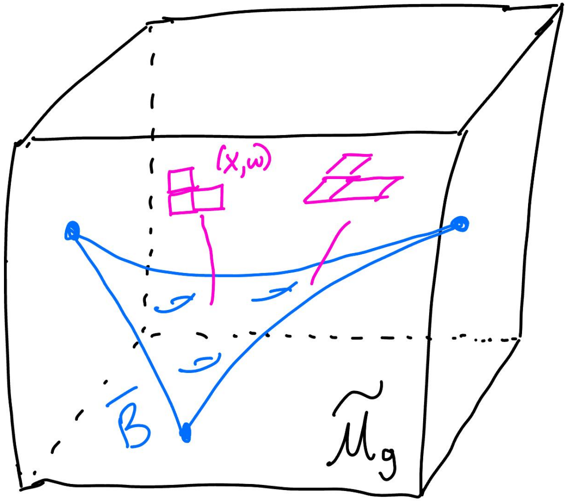

$$
\begin{tikzcd}
\widetilde{\mathbb{X}} \arrow[r] \arrow[dr] & \overline{\mathbb{X}} \arrow[d] & \mathbb{X} \arrow[l, hook'] \arrow[r] \arrow[d] & \mathbb{X}_{\mathrm{univ}} \arrow[d] \\
 & \overline{B} & B \arrow[l, hook'] \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}_g \arrow[d] \\
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C.f. Möller *Variations of Hodge structures of a Teichmüller curve*, 2005

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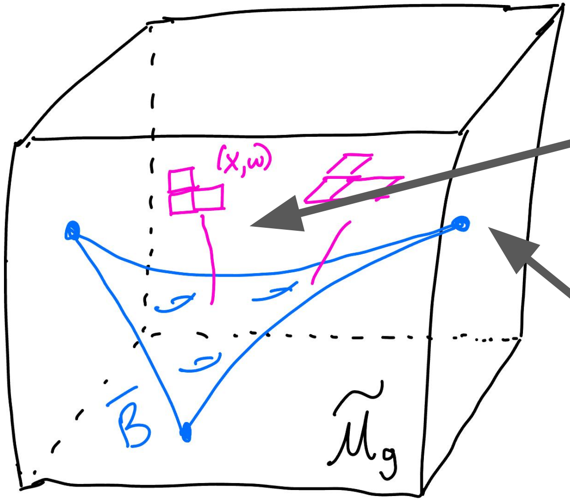

Smooth fibers are Veech surfaces

Singular nodal fibers at cusps

Caution: base is *cover* of Teichmüller curve

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From general fibrations to congruence fibrations

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Problem:
Formulas for Euler characteristic, signature, etc. require explicit computation of cover $B \to C$.

$$e(\widetilde{\mathbb{X}}) = \boxed{\chi(\overline{B})}\chi(X) + \boxed{T}$$

$$\sigma(\widetilde{\mathbb{X}}) = -2\kappa_\mu \boxed{\chi(B)} - \frac{2}{3}\boxed{T}$$

$$T := \sum_{c \in \Delta} \sum_{i=1}^{\ell_c} k_i^c$$

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Solution:
Restrict to *congruence Veech fibrations* built from congruence subgroups of the MCG.

$$\begin{tikzcd}
B = \mathbb{H}^2/\Gamma(n) \arrow[r] \arrow[d] & \mathcal{M}_1^{[n]} \arrow[d] \\
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(Generalizing elliptic modular surfaces)

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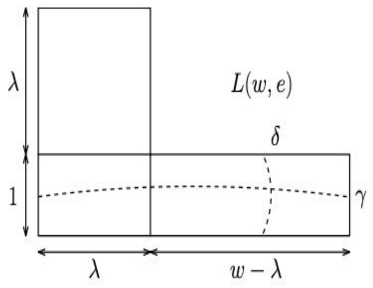
(Generalizing elliptic modular surfaces)

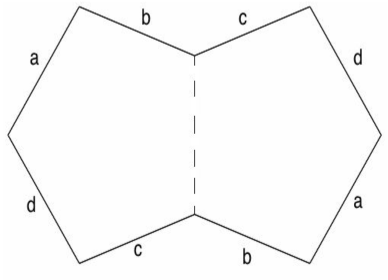
Problem:
Requires describing the representation $\rho_m : \mathrm{Aff}^+(X,\omega) \to \mathrm{Sp}(H_1(X;\mathbb{Z}/m\mathbb{Z}))$.
Difficult for any specific choices of surface and level m!

Case of algebraically primitive examples

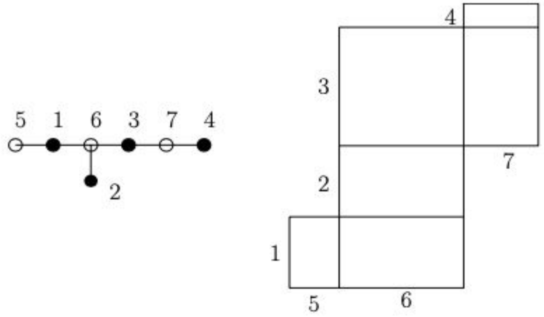
Case of algebraically primitive examples



Genus 2 “L-tables”
(non-square discriminant)

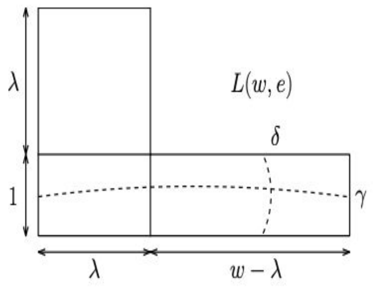


Regular n-gons
(n = p, 2p, 2^k)



Sporadic E_7 and E_8 examples

Case of algebraically primitive examples



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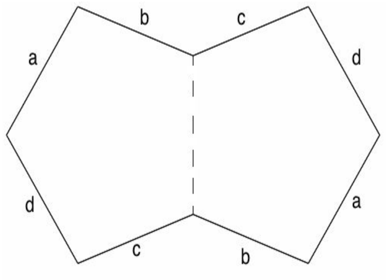

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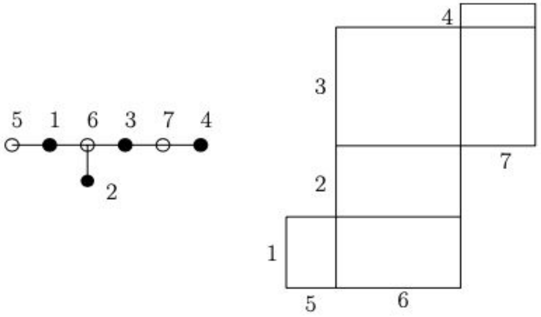

Sporadic E_7 and E_8 examples

Theorem (F. – Lucas):
For surfaces in the above three families, and for primes *p* in a computable infinite set depending on the surface, we have:

1. $\pi_1(\widetilde{\mathbb{X}_p}) \to \pi_1(\overline{B}_p)$ is an isomorphism,
2. $\widetilde{\mathbb{X}_p}$ is a minimal complex surface of general type, and
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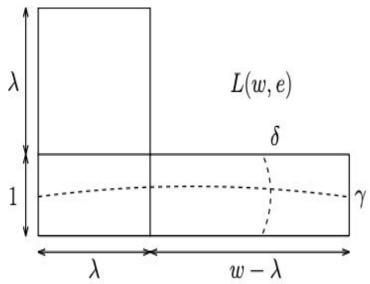
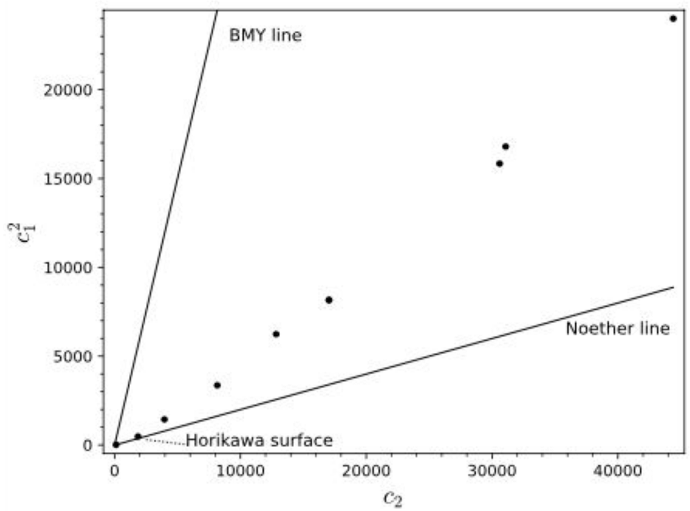


Genus 2 “L-tables”
(non-square discriminant)

$$\boxed{e(\widetilde{\mathbb{X}}_{q,p}) = dg + d(q-3)\left(\frac{1}{2} - \frac{1}{q} - \frac{1}{p}\right)}$$

$$\boxed{\sigma(\widetilde{\mathbb{X}}_{q,p}) = -\frac{d(q^2-1)}{4q}}$$



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arXiv:2310.02325

Nielsen-Thurston types and Teichmüller translation lengths of mapping classes along random walks

Dongryul M. Kim

Dept. of Mathematics, Yale University

GATSBY 2024 Spring

Random walk on MCG

S: closed connected orientable surface of genus ≥ 2.

$$\mu : \text{prob on } \mathrm{MCG}(S)$$

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This gives a prob $\mathbb{P}$ on $\mathrm{MCG}(S)^{\mathbb{N}}$,

"Random walk on MCG"

Always assume that $\langle \operatorname{supp} \mu \rangle < \mathrm{MCG}(S)$ is non-elementary.

Type and translation length along RW

Theorem (Maher-Tiozzo 2018)

Suppose that $\# \operatorname{supp} \mu < \infty$*.*

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Suppose that $\# \operatorname{supp} \mu < \infty$. *Then* $\mathbb{P}$*-a.e.* $(\omega_n) \in \mathrm{MCG}(S)^{\mathbb{N}}$*, there exists* $N > 0$ *such that*

$$\omega_n \text{ is pseudo-Anosov for all } n \geq N.$$

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Suppose that $\# \operatorname{supp} \mu < \infty$*. Then* $\mathbb{P}$*-a.e.* $(\omega_n) \in \mathrm{MCG}(S)^{\mathbb{N}}$*,*

$$\lim_{n \to \infty} \frac{\log \lambda(\omega_n)}{n} = L_\mu$$

where λ *is the stretch factor of p-A and* L_μ *is a const determined by* μ *(called 'drift').*

Note: for p-A $g \in \mathrm{MCG}(S)$, $\log \lambda(g) =$ Teich. tr. length = top. entropy.

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where λ is the stretch factor of p-A and L_μ is a const determined by μ (called 'drift').

This establishes the SLLN for translation lengths.

Finite quotients of hyperbolic 3-manifold groups

Tam Cheetham-West

Yale University

GATSBY Spring 2024

Introduction...

- An infinite group G is residually finite if every non-trivial element of G survives to some finite quotient of G.

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- An infinite group G is residually finite if every non-trivial element of G survives to some finite quotient of G.
- For a group G, let

$$\mathcal{C}(G) = \{Q : G \twoheadrightarrow Q, |Q| < \infty\}$$

  be the set of finite quotients of G. One can ask to what extent $\mathcal{C}(G)$ determines G. This is a well-studied question for several classes of groups.

Profinite rigidity

- A *profinitely rigid* group G is one for which $\mathcal{C}(G) = \mathcal{C}(H)$ $\implies G \cong H$ (G, H f.g., r.f.)

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- A *profinitely rigid* group is a residually finite group that is completely determined (among residually finite groups) up to isomorphism by its collection of finite quotients
- We do not know whether free groups or surface groups are profinitely rigid

Profinite rigidity for Kleinian groups

Theorem (Bridson-McReynolds-Reid-Spitler)

The following Kleinian groups are profinitely rigid

- π_1(*The Weeks manifold*)
- $PSL(2, \mathbb{Z}[\omega])$ *where* $\omega^2 + \omega + 1 = 0$

Theorem (C)

The following fibered 3-manifolds have profinitely rigid π_1

- *0-surgery on* 6_2
- *0-surgery on* 6_3

The knots 6_2 and 6_3

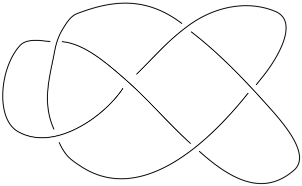


Figure 1: 6_2

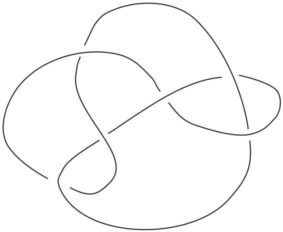


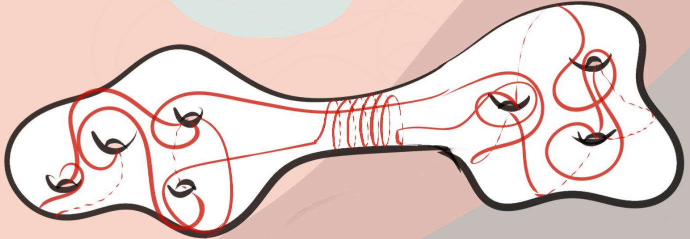
Figure 2: 6_3

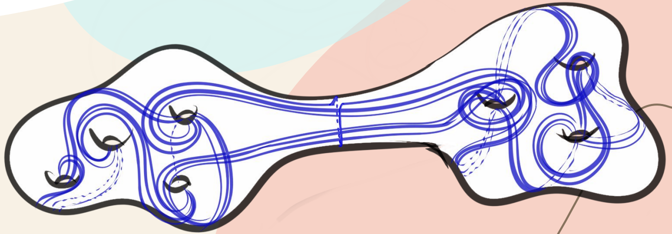
And...

Thanks for listening

Are these curves different?

Sayantika Mondal
Graduate Center, CUNY





Filling Curves

Cut along curve —--> Disks and annuli

Alternate Characterization:
A closed curve on a surface is said to be filling if it intersects every essential simple, non-peripheral closed curve on the surface.

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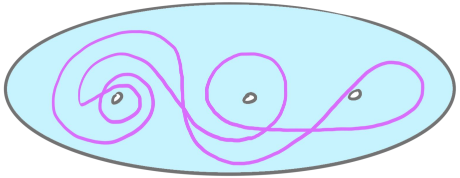
Example:



Topological Types:

Two curves are said to be of the same topological type if there is a mapping class group element (think Dehn twists!) taking one to the other.

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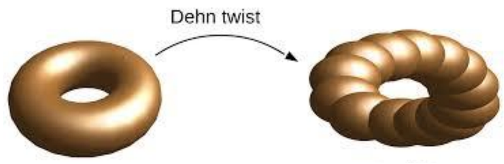

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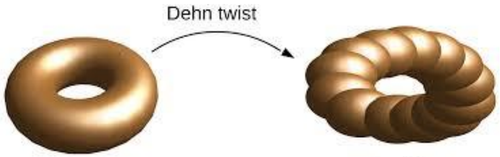

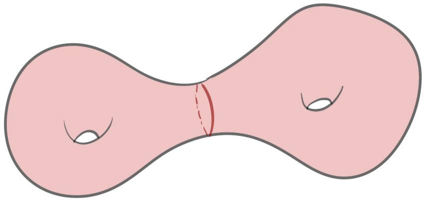

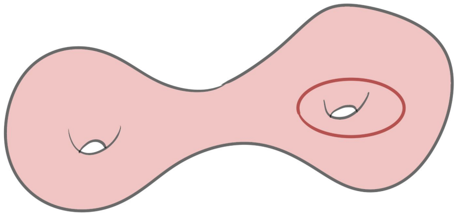

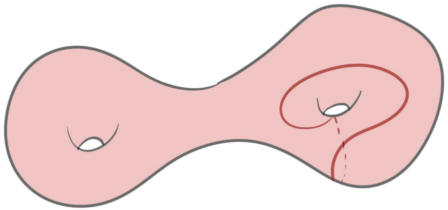

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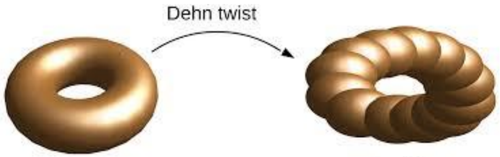

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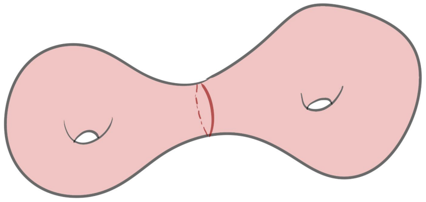
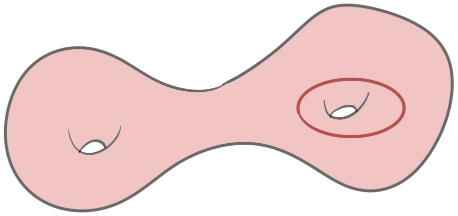
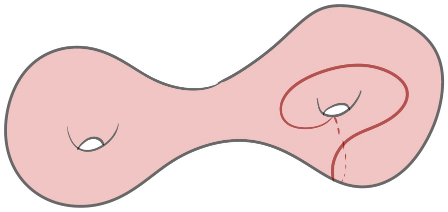
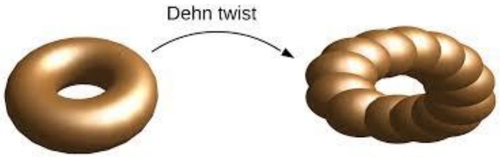






Different

Topological Types:



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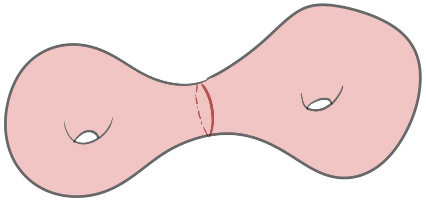
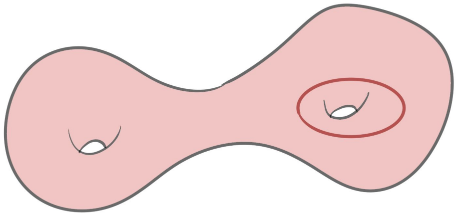
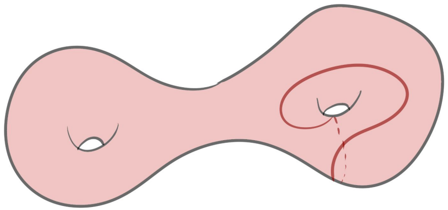






Different **Same**

A first example...

Minimal filling curve

Filling curve with the minimum self intersection number on a given surface.

A first example...

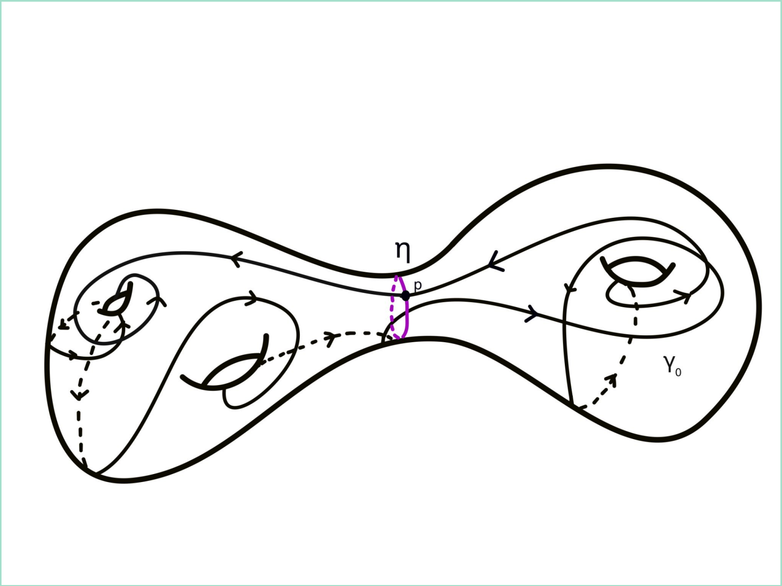
Minimal filling curve

Filling curve with the minimum self intersection number on a given surface.

- Exists!
- This is sort of the simplest example on a surface.

Building filling curves (an infinite family)

Start with a (minimal) filling curve γ_0



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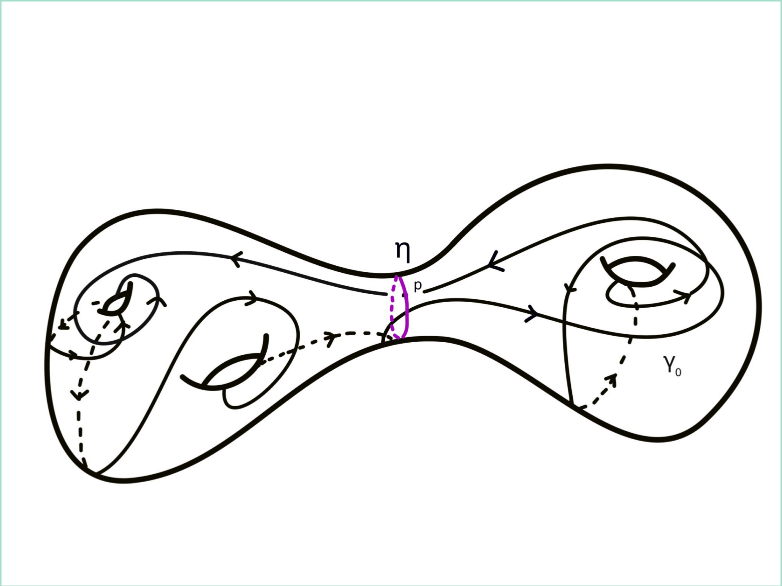

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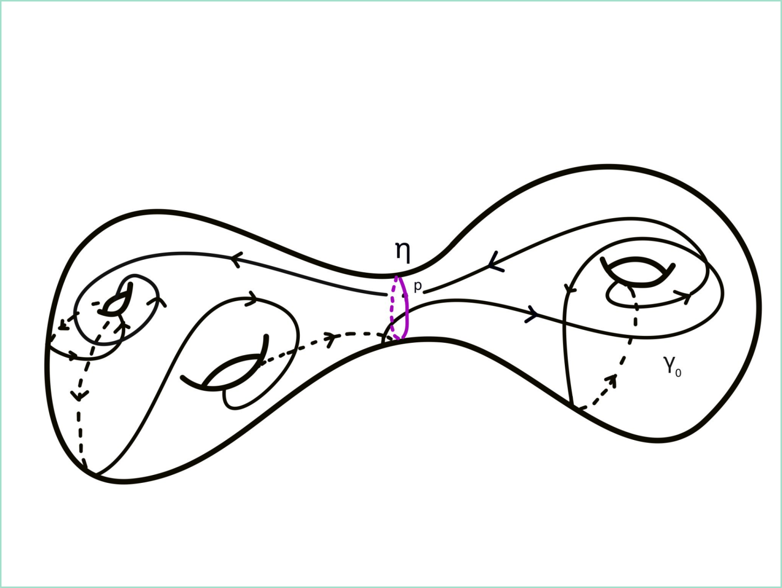
Given a pair of positive integers (m, n) , let γ be the curve $\eta^m * \gamma_0^n$ (based at the intersection point)



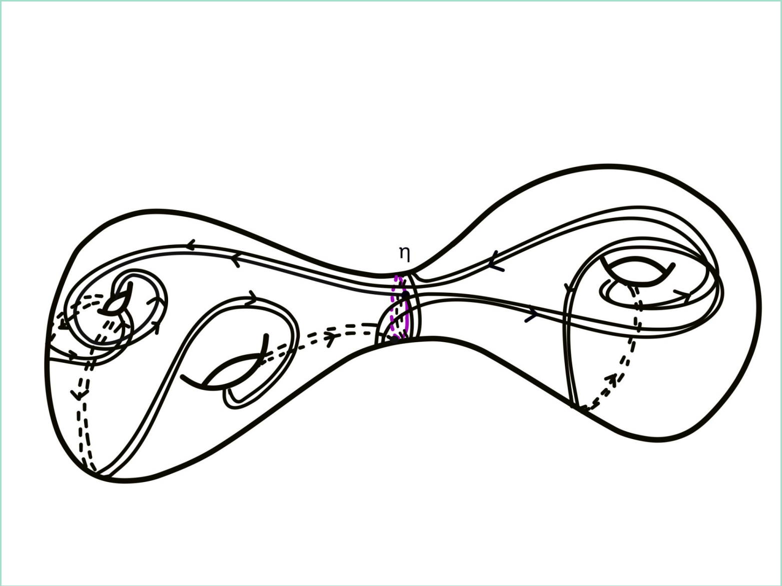
Building filling curves (an infinite family)

Start with a (minimal) filling curve γ_0

Cut at an intersection point with a separating curve.

Given a pair of positive integers (m, n) , let γ be the curve $\eta^m * \gamma_0{}^n$ (based at the intersection point)

A (2,2) curve.



Are these different?

For every n, m we get a curve.

Are the different types?

How can we tell?

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A first guess...

Self -Intersection number

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$$i(\gamma, \gamma) = i(\gamma_0, \gamma_0)n^2 + (i(\gamma_0, \eta)n - 1)m$$

A first guess...

Self -Intersection number

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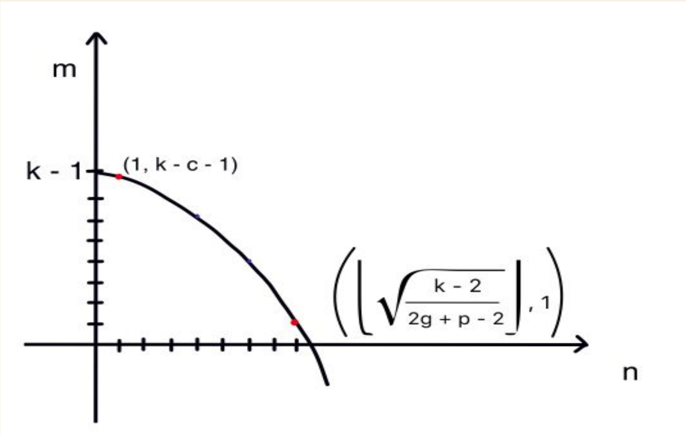

Solve for k

A better invariant...

For (ϕ, X) in Teich(Σ) . Let $\ell_\gamma(X)$ denote the 'X-length' of the geodesic in the free homotopy class of $\phi(\gamma)$.

We define the length infimum of γ as follows:

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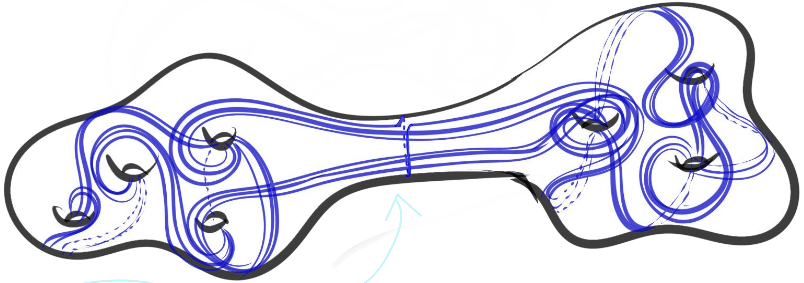
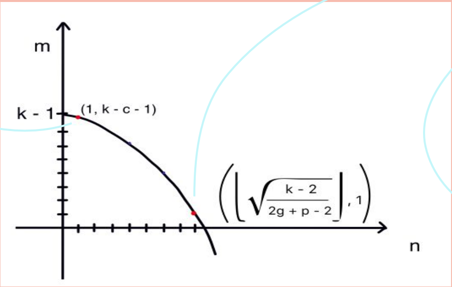
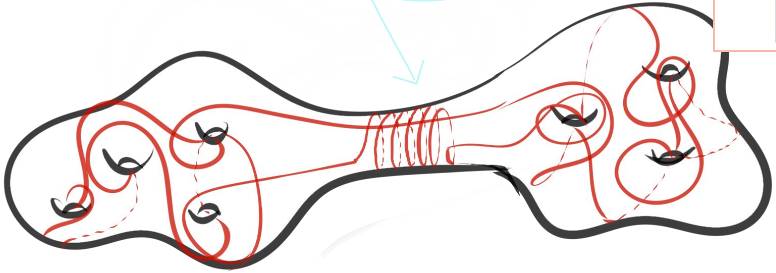
! Realized uniquely and is a MCG invariant

Back to slide 1







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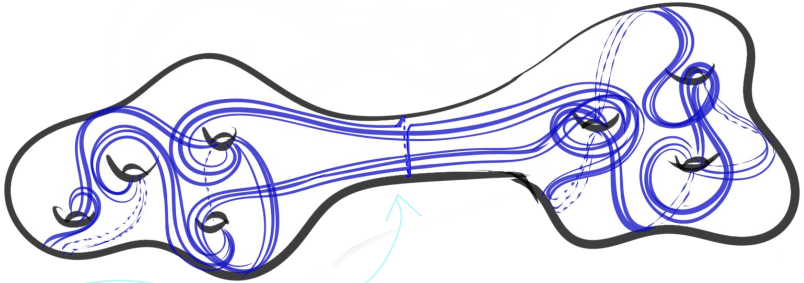

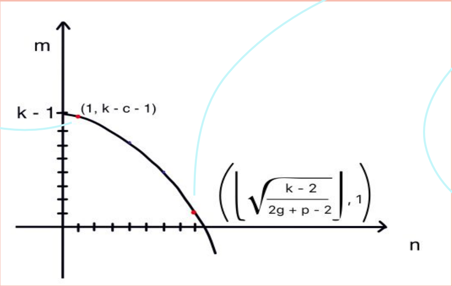

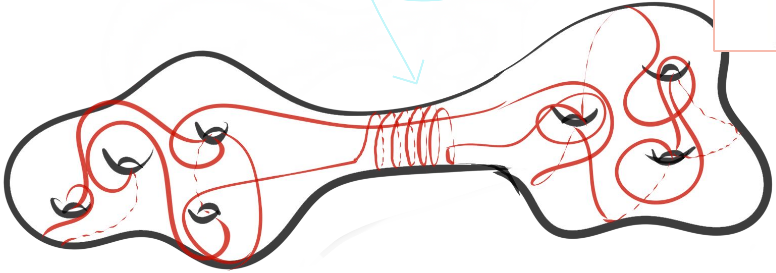

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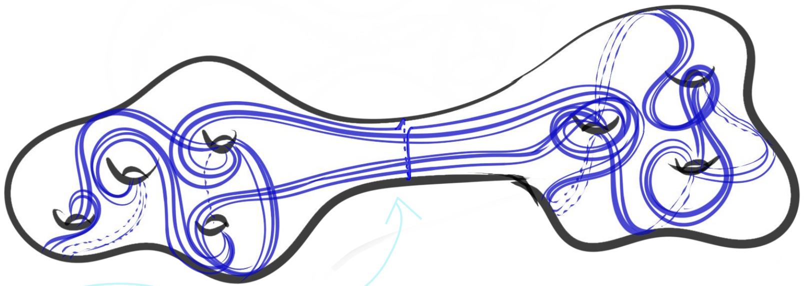
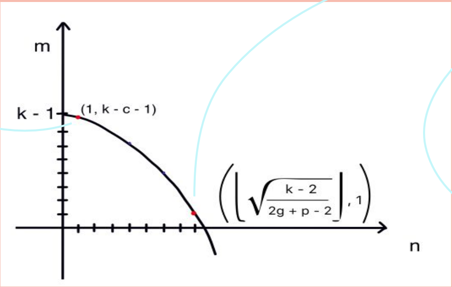
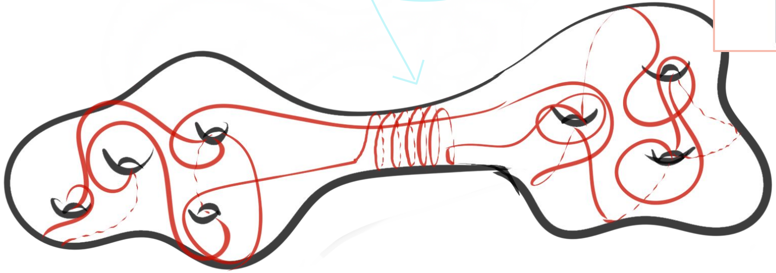




$k \lesssim m_{\beta_k}$



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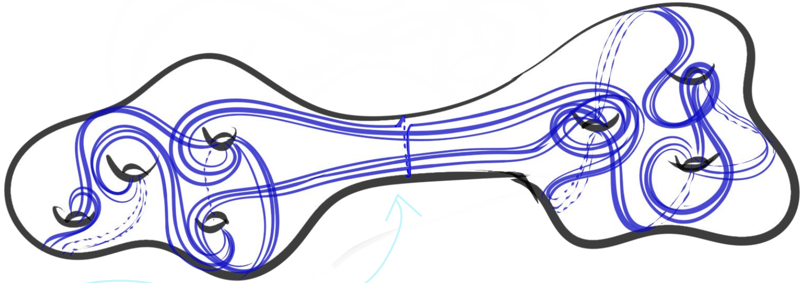

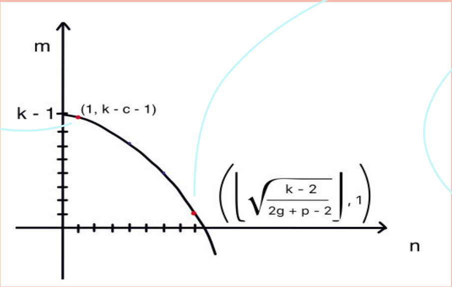

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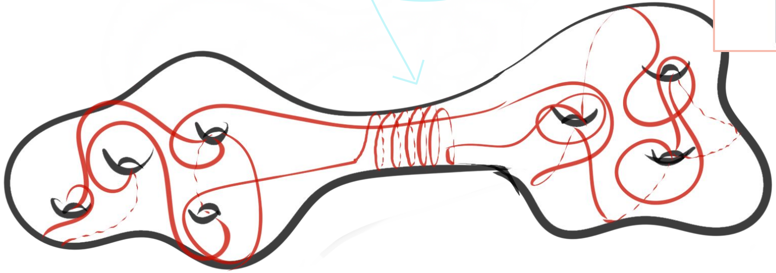

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So these are of different topological types!

Theorem (ish):

For any finite type surface and any choice of natural number k (some minor restrictions),
Can build curves with same intersection numbers (k) but of different infimum length and as a result topological types.
....and some more info on the inf metrics.

Joint work with A. Basmajian

Thanks!

smondal@gradcenter.cuny.edu

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Octonions and exceptional Lie groups G_2 and F_4

Christa Ishimwe

Wesleyan University

GATSBY, April 20, 2024

Introduction

Classical Lie groups can be realized as isometry groups of projective spaces over skew fields $\mathbb{R}, \mathbb{C},$ and $\mathbb{H}$. When they were first discovered, the exceptional Lie groups could not be realized as isometry groups of any known mathematical structure. Now we know that that $\mathrm{Aut}(\mathbb{O}) \cong G_2$ and $\mathrm{Aut}(\mathbb{O}P^2) \cong F_4$.

Constructing $\mathbb{O}$ and $\mathbb{O}P^2$

- We construct $\mathbb{O}$ as an extension of $\mathbb{C}$ through the Cayley-Dickson method.
- Given a division algebra A over the reals , we usually define the n-dimensional projective space as the quotient $A^{n+1}/\sim$ where $x \sim y$ if $\exists \lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. This gives us **homogeneous coordinates**.
- The above method does not work for the octonions because they are not associative. Instead of using **homogeneous coordinates**, we use **Veronese Coordinates**. We can also obtain a space isomorphic to $\mathbb{O}P^2$ from the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$.

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Key steps for proving $\mathrm{Aut}(\mathbb{O}) \cong G_2$

- Prove that if $f \in \mathrm{Aut}(\mathbb{O})$ then f is $\mathbb{R}$-linear and preserves length.
- Prove that $\mathrm{Aut}(\mathbb{O})$ is a compact lie group by showing that it is a closed subgroup of the compact Lie group $O(7)$.
- Prove that $\mathrm{Aut}(\mathbb{O})$ is simple.
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- By the classification of simple compact connected and simply connected Lie groups, conclude that $\mathrm{Aut}(\mathbb{O}) \cong G_2$.

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The Lie group F_4

- The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ consists of 3×3 hermitian matrices with octonion entries.
- $\mathbb{O}P^2$ correspond to trace 1 projections in $\mathfrak{h}_3(\mathbb{O})$.
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Surfaces Subgroups of Cocompact Lattices of Isometries of Hyperbolic Spaces

Zhenghao Rao

Geometry and Topology Seminar at Brown and Yale
April 20, 2024

Department of Mathematics
Brown University

Definitions

- Kleinian group: discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$
- Fuchsian group: finitely generated discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (the limit set is contained in a round circle)
- quasi-Fuchsian group: finitely generated discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with the limit set contained in a topological circle. ($\Leftrightarrow$ quasiconformally conjugate to a Fuchsian group)

Surface Subgroup Theorem

Theorem (Kahn-Marković, 2012)

Let $M^3 = \mathbb{H}^3/\mathcal{G}$ *denote a closed hyperbolic three manifold where* $\mathcal{G}$ *is a Kleinian group, and let* $\epsilon > 0$*. Then there exists a Riemann surface* $S_\epsilon = \mathbb{H}^2/F_\epsilon$ *where* F_ϵ *is a Fuchsian group, and* F_ϵ *is* $(1+\epsilon)$*-quasiconformally conjugate to a subgroup of* $\mathcal{G}$*.*

Main Results

Theorem (Rao, 2023)

Let Γ be a genus-2 quasi-Fuchsian group and G be a cocompact Kleinian group. For any $K > 1$, there is a surface subgroup $H < G$ that is K-quasiconformally conjugate to a finite index subgroup $F < \Gamma$.

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Theorem (Rao, 2023)

For any cocompact Kleinian group G, the set of the Hausdorff dimension of limit sets of all surface subgroups of G is dense in $[1, 2]$.

More Generalization

Theorem (Hamenstädt, 2015)

Let $\mathcal{G}$ be a simple rank one Lie group of non-compact type distinct from $\mathrm{SO}(2m, 1)$ for some $m \geq 1$ and $\Gamma < \mathcal{G}$ be a cocompact lattice. Then Γ contains surface subgroups.

Thank You!

$$S^2 \times S^2 \xrightarrow{/\iota} \mathbb{CP}^2 \xrightarrow{/c} S^4$$

$$\iota(x,y) = (y,x) \qquad c[z_1 : z_2 : z_3] = [\bar{z}_1 : \bar{z}_2 : \bar{z}_3]$$

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alg. geo

Massey '72
Kuiper '73

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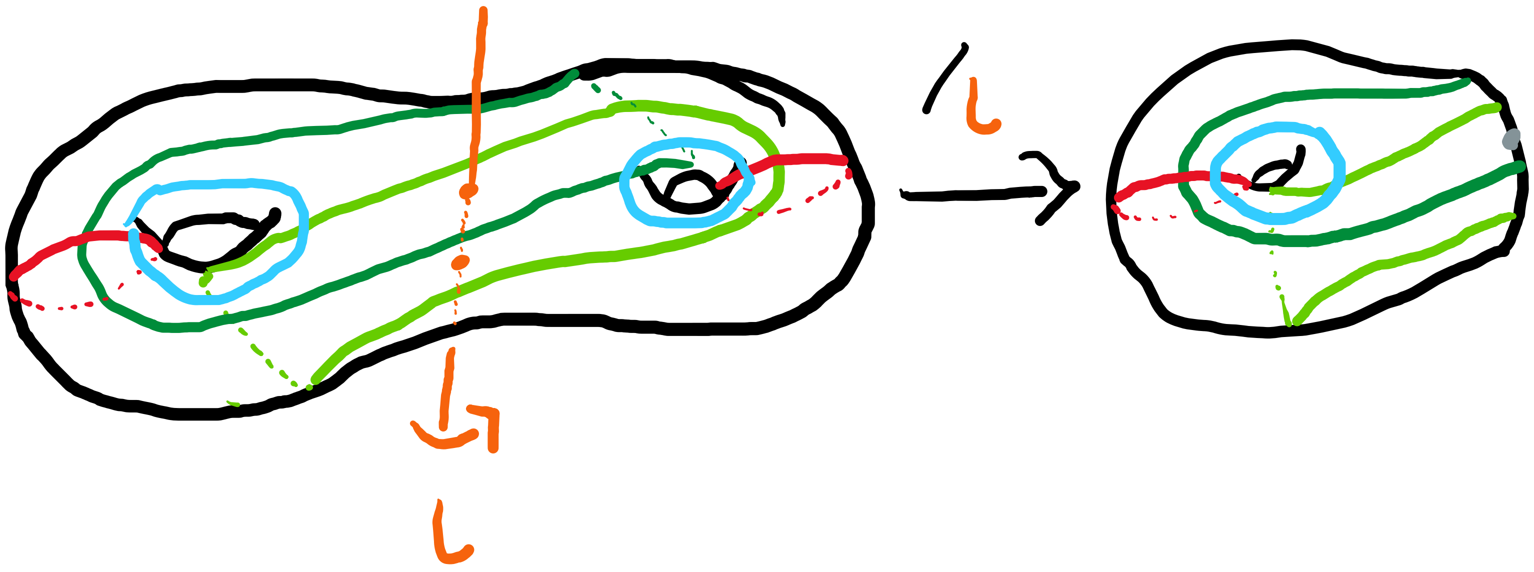

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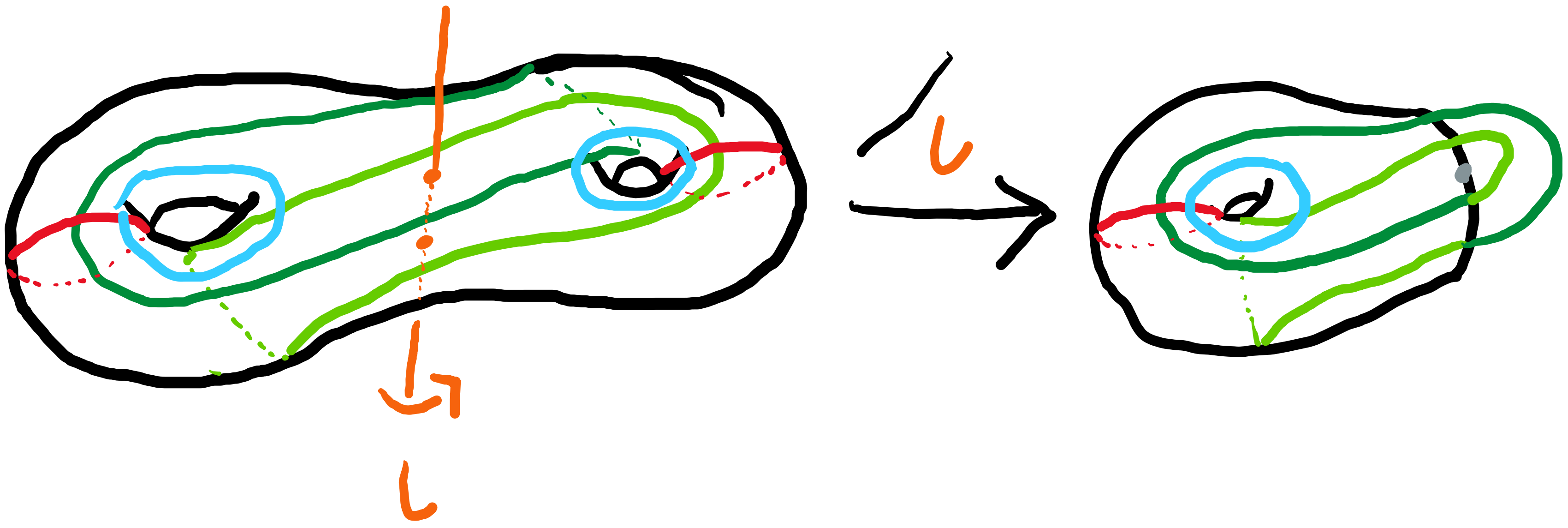

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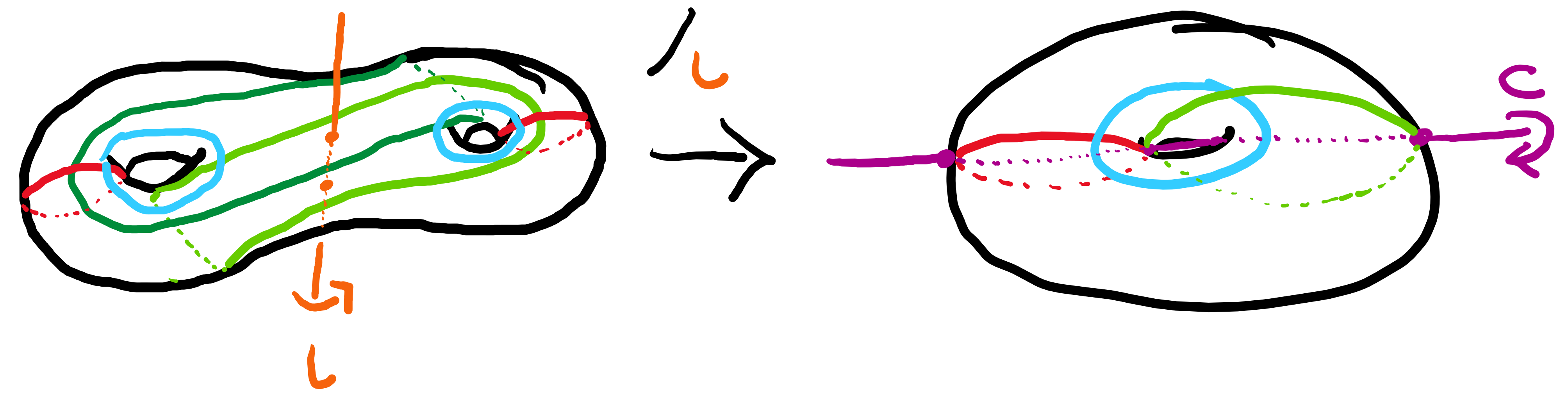

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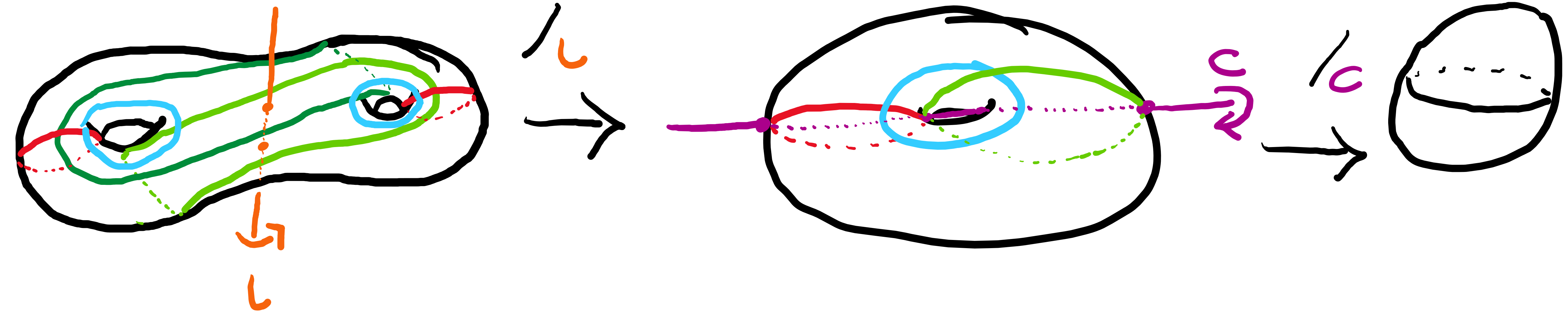

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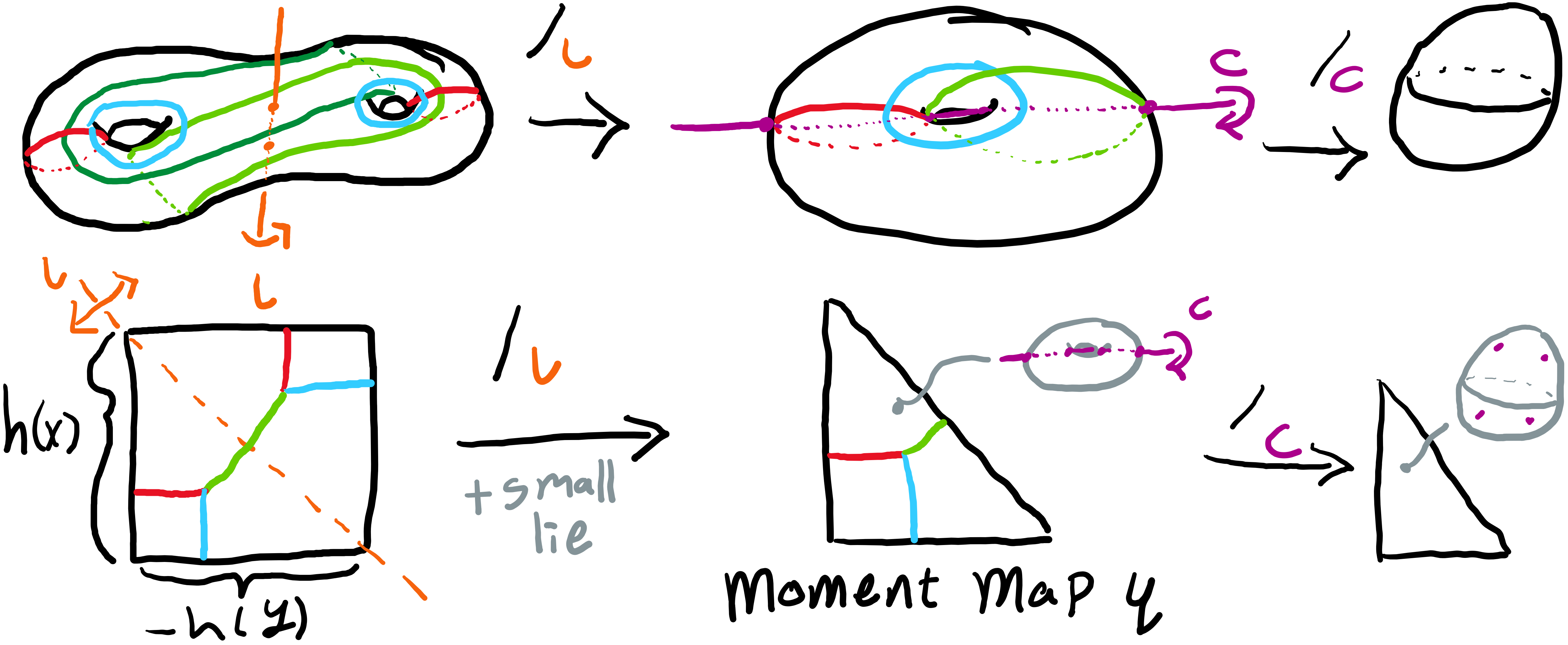

Flows vs foliations: universal circles and ideal boundaries

Junzhi Huang

Yale University

GATSBY, Spring 2024

Depth-1 foliation from spinning

Let M be a fibered closed hyperbolic 3-manifold with fiber F and pseudo-Anosov monodromy $f : F \to F$:

$$M = F \times [0,1]/(x,1) \sim (f(x),0).$$

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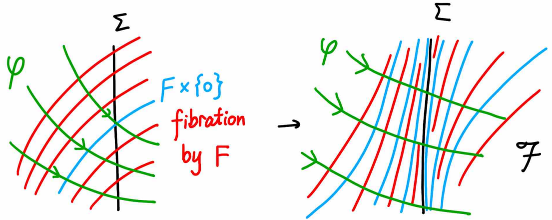
We can "spin" the fibration of M around Σ to obtain a taut depth-1 foliation $\mathcal{F}$ in M transverse to ϕ. The foliation $\mathcal{F}$ has a unique compact leaf Σ, and all the other leaves are surfaces of infinite type spiraling towards Σ.



A circle from the foliation

Let $\widetilde{M}$ be the universal cover of M and let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{M}$.

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The circle $\mathfrak{S}$ admits a faithful $\pi_1(M)$-action. For any leaf λ of $\widetilde{\mathcal{F}}$, there is a monotone map $I_\lambda : \mathfrak{S} \to \partial_\infty \lambda$ (by identifying λ with $\mathbb{H}^2$) so that for any $\gamma \in \pi_1(M)$, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "I_\lambda"] \arrow[d, "\gamma"'] & \partial_\infty \lambda \arrow[d, "\gamma"] \\ \mathfrak{S} \arrow[r, "I_{\gamma\lambda}"] & \partial_\infty(\gamma\lambda) \end{tikzcd}$$

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\end{tikzcd}$$

The universal circle exists and the π_1-action is faithful for any taut foliation with hyperbolic leaves in an atoroidal 3-manifold. In general, the construction is not canonical and not unique.

The pseudo-Anosov suspension flow

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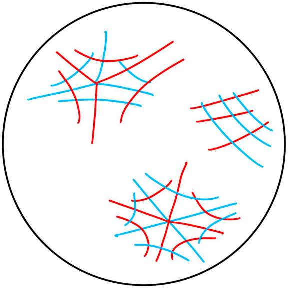
By fibration, we can identify $\mathcal{O}$ with the universal cover of the fiber F, and $\mathcal{F}_{\mathcal{O}}^{s/u}$ with the lift of the stable/unstable foliations of f.



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The flow space $\mathcal{O}$ has a natural ideal boundary $\partial\mathcal{O}$, and every ray of $\mathcal{F}_{\mathcal{O}}^{s/u}$ has a well-defined ideal endpoint. There is a natural $\pi_1(M)$-action on $\partial\mathcal{O}$.

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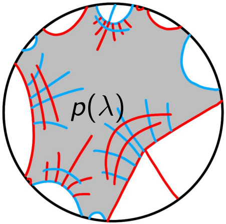
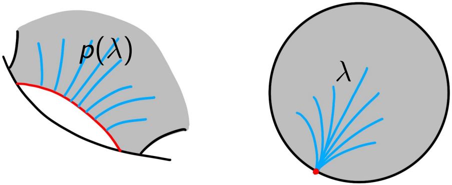
Each boundary leaf correspond to a unique point at infinity of λ.





The identification of the two circles

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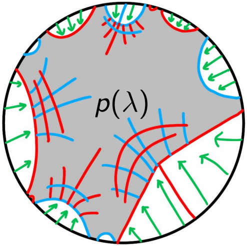
Theorem

There is a $\pi_1(M)$-equivariant homeomorphism $T : \mathfrak{S} \to \partial\mathcal{O}$. The monotone map $I_\lambda \circ T^{-1}$ is given by collapsing each component of $\partial\mathcal{O} - \overline{p(\lambda)}$ to the ideal point corresponding to the boundary leaf of $p(\lambda)$ it is facing.



Thank you!