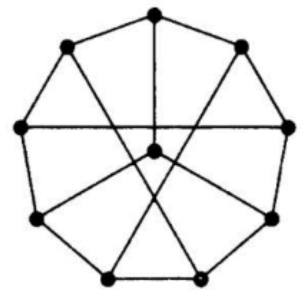
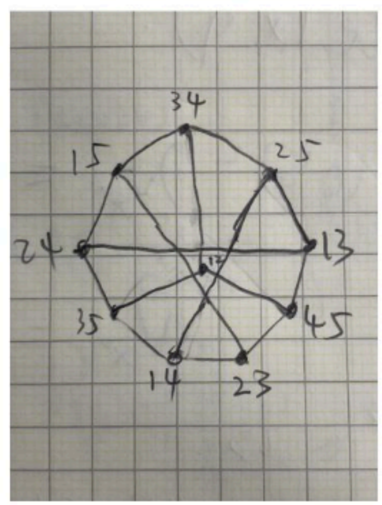
Problem 1. *Prove that the graph below is isomorphic to the Petersen graph.*[1]



Solution. Below is an isomorphism over the two graphs represented by corresponding vertex labels. One can check that with the following labeling of the graph, two vertices are adjacent if and only if their labels are disjoint. Therefore, this graph is an isomorphism to the Petersen graph, with the bijection represented by the labels of the vertices.



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Problem 2. *How many cycles of length n are there in the complete graph K_n?*

Solution. Label the vertices from 1 to n. We know that there are n! orderings of these labels. Each cycle in the graph corresponds to one such ordering (e.g. 1, 2, 3, ...,n is a cycle), but each cycle could be denoted by n different starting points and also in two orderings (forward and reverse) hence for every cycle we have 2n possible permutations in our ordering, implying that the total number of cycles will be $\frac{n!}{2n} = \frac{(n-1)!}{2}$ □

Problem 3. *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \ldots, a_k)$ with coordinates $a_i \in \{0, 1\}$ and with an edge between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if they differ in exactly one coordinate.*[2]

(a) *Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.*

(b) *Let $K_{2,3}$ be the complete bipartite graph with 2 red vertices, 3 blue vertices, and all possible edges between red and blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.*

Solution. 1. Let $v_1, v_2, v_3, v_4 \in Q_k$ be a 4-cycle, in that order.

By the definition of Q_k, the two non-adjacent vertices v_1, v_3 must differ in exactly 2 coordinates. They must be one of the following pairs of coordinates,

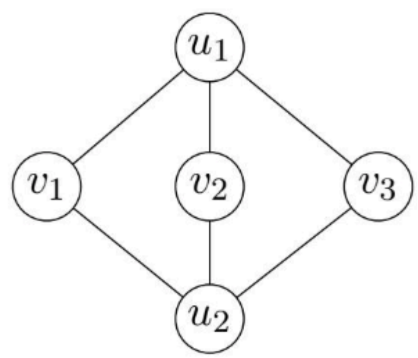
$$(\ldots, 1, \ldots, 0, \ldots), (\ldots, 0, \ldots, 1, \ldots)$$
$$(\ldots, 0, \ldots, 0, \ldots), (\ldots, 1, \ldots, 1, \ldots)$$

There is a unique 4-cycle containing these two points. This is because the only vertices that are adjacent to both v_1, v_3 are the other pair of coordinates listed above.

It suffices to show that any other case of 4-cycles intersecting would share the opposite vertices, and thus be the same cycle. We see that this is true if they intersect at 2 or more edges, and if they share 3 or more vertices. Suppose they intersect at 2 or more vertices. Then they will either share a single edge, or they will intersect at the opposite vertices.

Thus the only cases possible are disjoint cycles, intersecting at 1 vertex, or intersecting at 1 edge.

2. In $K_{2,3}$, we can find two 4-cycles which intersect at 2 edges, thus breaking the above property, preventing $K_{2,3}$ from being a subgraph of Q_k.



$\square$

Problem 4. *For a graph $G = (V, E)$, the complement of G is the graph $\overline{G} = (V, \overline{E})$, where $\{u, v\} \in \overline{E}$ if and only if $\{u, v\} \notin E$. Prove or disprove: If G and H are isomorphic, then the complements $\overline{G}$ and $\overline{H}$ are also isomorphic.*

Solution. Suppose that $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are isomorphic. Then by definition there exists a bijection $f : V_G \to V_H$ such that $\{u, v\} \in E_G$ if and only if $\{f(u), f(v)\} \in E_H$ for $u, v \in V_G$. Note that for $u, v \in V_G$:

$$\begin{aligned}
\{u, v\} \in \overline{E}_G &\iff \{u, v\} \notin E_G \text{ by the definition of the complement} \\
&\iff \{f(u), f(v)\} \notin E_H \text{ by the definition of } f \\
&\iff \{f(u), f(v)\} \in \overline{E}_H \text{ by the definition of the complement}
\end{aligned}$$

Thus f gives an isomorphism between $\overline{G} = (V_G, \overline{E}_G)$ and $\overline{H} = (V_H, \overline{E}_H)$. As such, the complements $\overline{G}$ and $\overline{H}$ are also isomorphic. $\square$

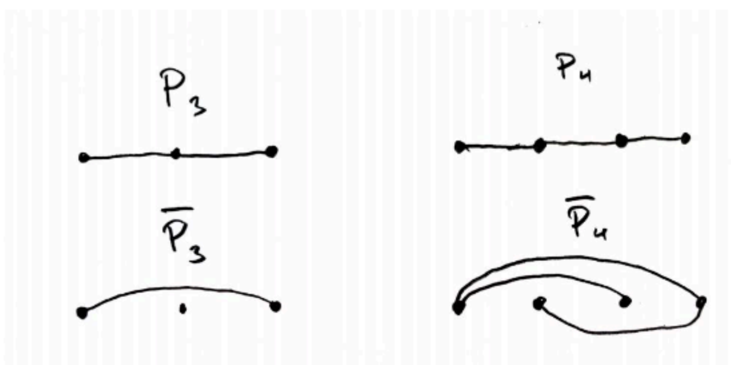
Problem 5.

(a) *Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices. It has $n-1$ edges.)*

(b) *We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n is divisible by 4 or $n-1$ is divisible by 4.* [3]

In fact, whenever n or $n-1$ is divisible by 4, there is a self-complementary graph with n vertices – see the bonus problem below.

Solution. (a) The graphs are as follows



(b) Suppose that G is self-complementary and has n vertices. Suppose that G has e edges. We know that at K_n has $n(n-1)/2$ edges, and that if we combine the edges from G and $\bar{G}$, we will get K_n that, as $G \simeq \bar{G}$, $2e = n(n-1)/2$. Thus, $4e = n(n-1)$. As n and $n-1$ differ only by one, one of them must be odd and the other must be even. Thus as $4|n(n-1)$ and $4 \nmid n$ or $4 \nmid n-1$ as one of them is odd, it must be that $4|n$ or $4|n-1$.

$\square$

Problem 6. *Prove that the Petersen graph has no cycles of length* 3 *or* 4. [5]

Solution. In class, we defined the Petersen graph as the 2-element subsets of $\{1,2,3,4,5\}$ connected by disjointness, i.e., if there is an edge between two vertices (two 2-element subsets), then the intersection of the two sets is the null set $\emptyset$.

(i) Without loss of generality, we choose the first vertex as the subset $\{1,2\}$. Then its possible neighbors are subsets $\{3,4\}, \{3,5\}$, and $\{4,5\}$. To form a cycle, we have to choose two vertices and they have to be distinct. No matter how we choose two subsets of of these three, their intersection is not $\emptyset$, i.e., $\{3,4\} \cap \{3,5\} = \{3\}, \{3,4\} \cap \{4,5\} = \{4\}, \{4,5\} \cap \{3,5\} = \{5\}$. Hence, we cannot have an edge between them, which means that we cannot add the third edge to form a cycle of length 3.

(ii) Following from (i), the two neighbors of $\{1,2\}$ can be any two among $\{3,4\}, \{3,5\}$, and $\{4,5\}$. If we want to have a cycle of length 4, then we need to find one more 2-subset that is disjoint of the union of the two neighbors of $\{1,2\}$. Hence, this new neighbor can only be a subset of $\{1,2,3,4,5\} \setminus \{3,4,5\} = \{1,2\}$, which means that the only possible choice is the first vertex we have chosen. Hence, we cannot form a cycle of length 4.

By (i) and (ii), we conclude that the Petersen graph has no cycles of length 3 or 4. $\square$

Problem 7 (Bonus). *Let G, H be a self-complementary graphs, and assume G has with $4k$ vertices. Construct a self-complementary graph obtained by taking the union of G and H and adding some edges.*[5] *Deduce that if either n or $n-1$ is divisible by 4, then there is a self-complementary graph with n vertices.*

Solution. Note: for a vertex v, $\overline{v}$ denotes the same vertex after taking the complement. Similarly, for a set of vertices V, $\overline{V}$ denotes the same set of vertices after taking the complement.

First, realize that we only need to add edges which connect vertices of G and vertices of H, as $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are already self-complementary. Additionally, each vertex in G can have up to $4k-1$ edges, an odd number. This means that given $v \in V_G$ and the complement $\overline{v} \in V_{\overline{G}}$, then $\deg(v)$ will be of the opposite parity as $\deg(\overline{v})$ since $\deg(v) + \deg(\overline{v}) = 4k-1$. Since G is self-complementary, it follows that we must have the same number of vertices with odd degree as even degree. Let $V_{2G} \subseteq V_G$ be the set of vertices of G with even degree and $V_{1G} \subseteq V_G$ be the set of vertices of G with odd degree. Note that $\overline{V_{2G}} \cong V_{1G}$ and $\overline{V_{1G}} \cong V_{2G}$. This is because after taking the complement, every vertex in V_G switches parity of degree, so vertices with odd degree will have an even degree and vertices with even degree will have odd degree.

To turn $G \cup H$ into a self-complementary graph, we will simply connect every vertex of H to every $v \in V_{2G}$. When we take the complement of $G \cup H$, the edges connecting vertices of H to vertices of V_{2G} will disappear and every vertex will now be connected to $\overline{V_{1G}}$ (the bar tells us that we have taken the complement of this set of vertices). From above, however, we know that $\overline{V_{1G}} \cong V_{2G}$. Thus we get what we started with – every $v \in V_H$ is connected to every vertex of G with an even degree. Note that we did not specify whether $\#H = 4j$ or if $\#H = 4j+1$, so this works as long as H is self-complementary.

We now use induction to prove that we can produce a self-complementary graph with $n = 4k$ vertices. We have shown that the base case where $k = 1$ is true in problem 5a – the base case is simply P_4. Our inductive hypothesis is that there exists a self-complementary graph with $n = 4k$ vertices. We need to show that there exists a self-complementary graph with $n = 4(k+1)$ vertices. Using our previous result, let G be the graph from our induction hypothesis with $4k$ vertices, and let $H = P_4$. We can take the union of G and H to produce a self-complementary graph with $n = 4(k+1)$ vertices this way, completing this case.

We also want to prove that we can produce a self-complementary graph with $n = 4k + 1$ vertices. The base case where $k = 0$ is trivial – this is simply K_1, which is clearly self-complementary. Our inductive hypothesis is that there exists a self-complementary graph with $n = 4k + 1$ vertices. We need to show that there exists a self-complementary graph with $n = 4(k + 1) + 1$ vertices. Using our previous result, let H be the graph from our induction hypothesis with $4k + 1$ vertices, and let $G = P_4$. We can take the union of G and H to produce a self-complementary graph with $n = 4(k + 1) + 1$ vertices this way, completing this case.

$\square$