Problem 1. *Compute the first fundamental forms of the ellipsoid*

$$\phi(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$$

where a, b, c are positive constants.

Solution. We first compute partial derivatives, and see that:

$$\begin{aligned}\phi_u(u,v) &= (a\cos u\cos v, b\cos u\sin v, -c\sin u)\\ \phi_v(u,v) &= (-a\sin u\sin v, b\sin u\cos v, 0).\end{aligned}$$

We can now calculate that:

$$\begin{aligned}E(u,v) &= \langle \phi_u, \phi_u\rangle = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u\\ F(u,v) &= \langle \phi_u, \phi_v\rangle = (b^2 - a^2)\cos u\sin u\cos v\sin v\\ G(u,v) &= \langle \phi_v, \phi_v\rangle = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v.\end{aligned}$$

Written out explicitly, the first fundamental form will be

$$\begin{pmatrix} E & F \\ F & G\end{pmatrix} = \begin{pmatrix} a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u & (b^2-a^2)\cos u\sin u\cos v\sin v \\ (b^2-a^2)\cos u\sin u\cos v\sin v & a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v\end{pmatrix}.$$

□

Problem 2. *Derive a general formula for the area of a surface of revolution (say, of a curve in the xz-plane, revolving about the z-axis) by finding a chart and computing the first fundamental form.*[1] *Use this to compute the area of the torus surface from class.*

Solution. Let α be a curve in the xz plane.

$$\alpha(t) = (x(t), z(t))\ t \in [a, b]$$

Then, we can define the coordinate chart of the surface of revolution, ϕ:

$$\phi(t, \theta) = (x(t)cos(\theta), x(t)sin(\theta), z(t))\ \ \theta \in [0, 2\pi], t \in [a, b]$$

We calculate the first fundamental form of this chart:

$$\begin{aligned}
\phi_t &= (x'(t)cos(\theta), x'(t)sin(\theta), z'(t)) \\
\phi_\theta &= (-x(t)sin(\theta), x(t)cos(\theta), 0 \\
E = \phi_t \cdot \phi_t &= x'(t)^2cos^2(\theta) + x'(t)^2sin^2(\theta) + z'(t)^2 = x'(t)^2 + z'(t)^2 \\
F = \phi_t \cdot \phi_\theta &= -x'(t)x(t)sin(\theta)cos(\theta) + x'(t)x(t)sin(\theta)cos(\theta) = 0 \\
G = \phi_\theta \cdot \phi_\theta &= x^2(t)sin^2(\theta) + x^2(t)cos^2(\theta) = x^2(t)
\end{aligned}$$

Using the first fundamental form, we can derive the general formula for an area of a surface of revolution (of a curve in the xz-plane revolving around the z-axis)

$$Area(\phi(t, \theta)) = \int \sqrt{EG - F^2}\delta t\delta\theta = \int \sqrt{x'^2(t)(x'(t)^2 + z'(t)^2)}\delta t\delta\theta$$

We use this to compute the area of the torus surface. For the torus, we define $\alpha(t)$ as the curve in the xz-plane being rotated around the z-axis:

$$\begin{aligned}
\alpha(t) &= (x(t), z(t)) = (rcos(t) + \theta, rsin(t))\ \ t \in [0, 2\pi] \\
\phi(t, \theta) &= ((rcos(t) + d)cos(\theta), (rcos(t) + d)sin(\theta), rsin(t))\ \ t \in [0, 2\pi]\theta \in [0, 2\pi] \\
Area(\phi(t, \theta)) &= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(rcos(\theta) + d)^2r^2}\delta t\delta\theta \\
&= 2\pi\int_0^{2\pi} \sqrt{(rcos(\theta) + d)^2r^2}\delta\theta \\
&= 2\pi r\int_0^{2\pi} rcos(\theta) + d\delta\theta \\
&= 2\pi r(2\pi d + r\int_0^{2\pi} rcos(\theta)\delta\theta) \\
&= 2\pi r \cdot 2\pi d
\end{aligned}$$

□

[1] Perhaps you did this some other way in MVC.

Problem 3. *Consider the following parameterization of the Möbius band.*[2]

$$\phi(t,\theta) = \big((2 + t\cos(\theta/2))\cos\theta, (2 + t\cos(\theta/2))\sin\theta, t\sin(\theta/2)\big) \quad t \in [-1,1], \theta \in [0, 2\pi]$$

Use this to compute the area of the Möbius band.[3] [4]

Solution.

$$\begin{aligned}
\phi_t &= (\cos\frac{\theta}{2}, \cos\frac{\theta}{2}\sin\theta, \sin\frac{\theta}{2}) \\
\phi_\theta &= (-2\sin\theta - t\sin\theta\cos\frac{\theta}{2} - \frac{t}{2}\sin\frac{\theta}{2}\cos\theta,\ 2\cos\theta - \frac{t}{2}\sin\frac{\theta}{2}\sin\theta + t\cos\frac{\theta}{2}\cos\theta,\ \frac{t}{2}\cos\frac{\theta}{2}) \\
&= (-\frac{t}{2}\sin\frac{\theta}{2}\cos\theta - \sin\theta(2 + t\cos\frac{\theta}{2}), -\frac{t}{2}\sin\frac{\theta}{2}\sin\theta + \cos\theta(2 + t\cos\frac{\theta}{2}), \frac{t}{2}\cos\frac{\theta}{2}) \\
E &= \phi_t \cdot \phi_t = \cos^2\frac{\theta}{2}\cos^2\theta + \cos^2\frac{\theta}{2}\sin^2\theta + \sin^2\frac{\theta}{2} \\
&= \cos^2\frac{\theta}{2}(\cos^2\theta + \sin^2\theta) + \sin^2\frac{\theta}{2} \\
&= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \\
F &= \phi_t \cdot \phi_\theta = 0
\end{aligned}$$

since ϕ_t traces along the length of the band while ϕ_θ traces across the width of the band, $\phi_\perp\phi_\theta$.

$$\begin{aligned}
G &= \phi_\theta \cdot \phi_\theta \\
&= \frac{t^2}{4}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + t\sin\frac{\theta}{2}\cos\theta\sin\theta(2 + t\cos\frac{\theta}{2}) + \sin^2\theta(t\cos\frac{\theta}{2} + 2)^2 \\
&\quad + \frac{t^2}{4}\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2} - t\sin\frac{\theta}{2}\cos\theta\sin\theta(2 + t\cos\frac{\theta}{2}) + \cos^2\theta(t\cos\frac{\theta}{2} + 2)^2 \\
&\quad + \frac{t^2}{4}\cos^2\frac{\theta}{2} \\
&= \frac{t^2}{4}\sin^2\frac{\theta}{2}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}) + (\sin^2\theta + \cos^2\theta)(t\cos\frac{\theta}{2} + 2)^2 + \frac{t^2}{4}\cos^2\frac{\theta}{2} \\
&= \frac{t^2}{4}\sin^2\frac{\theta}{2} + \frac{t^2}{4}\cos^2\frac{\theta}{2} + (t\cos\frac{\theta}{2} + 2)^2 \\
&= \frac{t^2}{4} + (t\cos\frac{\theta}{2} + 2)^2
\end{aligned}$$

$$\begin{aligned}
\text{Area} &= \int_\theta\int_t \sqrt{EG - F^2} = \int_0^{2\pi}\int_{-1}^{1}\sqrt{\frac{t^2}{4} + (t\cos\frac{\theta}{2} + 2)^2} \\
&\approx 25.4131 \text{ (numerically)}
\end{aligned}$$

□

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[2] Remark: you probably found a similar parameterization on last week's homework.

[3] Aside: why does the area make sense?? The Möbius band does not have a unit normal!

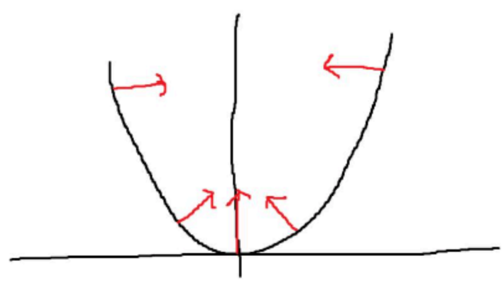
[4] Encouragement: the computation gets complicated, but it should simplify (have faith!).

Problem 4. *Describe the image of the Gauss map of the following surfaces. Do* <u>*not*</u> *compute using a chart – there is an easier way to reason.*

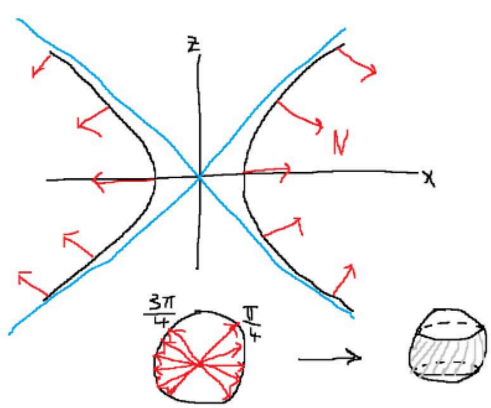
(a) *paraboloid* $z = x^2 + y^2$

(b) *hyperboloid* $x^2 + y^2 - z^2 = 1$

Solution. (a) Any plane containing the z-axis intersects the paraboloid in an upward parabola (e.g. the plane $y = 0$ intersects the paraboloid in $z = y^2$). Also, the union of all such parabolas is the entire surface. Each point p on the surface along a single parabola has normal vector $N(p)$ that consistently points inward to the z-axis and has positive z component (if we fix one of two orientations). The z component of $N(p)$ is maximized at $N(0,0,0) = (0,0,1)$ and approaches 0 as $|p| \to \infty$. Thus the union of all $N(p)$ across all parabolas (i.e. the image of the Gauss map) is the upper unit hemisphere with open boundary.



(b) The intersection of any plane containing the z-axis with the hyperboloid is a hyperbola. If we fix the normal orientation to be radially outward and consider the plane $y = 0$, we can see that the normal vectors are $(\pm 1, 0, 0)$ when $z = 0$ and asymptotically approach $(\pm 1, 0, \pm 1)$. If we center all these normal vectors about the origin, we see than their image is the set of points on the unit circle in the xz-plane with $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$. Now, since the intersecting hyperboloids are radially symmetric, we get the same sub-circle of normals as we rotate the plane. Thus the image of the gauss map is the surface of revolution of $\alpha(\theta) = (\cos\theta, \sin\theta)$ about the z-axis where $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$ (geometrically, this is a ring). Equivalently, it is the set of points on S^2 with angle within $\frac{\pi}{4}$ of the xy-plane.



□

Problem 5. *In this problem you prove the spectral theorem for self-adjoint linear operators of* $\mathbb{R}^2$. [5]

Fix a self-adjoint linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(p) = A(p) \cdot p$ *and let* $\alpha : [0, 2\pi] \to \mathbb{R}^2$ *denote the standard parameterization of the unit circle.*

(a) *Compute* $(f \circ \alpha)'(t)$ *and prove that* t *is a critical point of* $f \circ \alpha$ *if and only if* $\alpha(t)$ *is an eigenvector of* A. *Relate the corresponding eigenvalue to* f. [6]

(b) *Use facts from calculus to deduce that either* A *is a scalar matrix or* A *has two distinct eigenvalues.*

(c) *Prove that there exist a pair of eigenvectors for* A *that form an orthonormal basis for* $\mathbb{R}^2$. [7]

Solution. Part a)

I start by computing $(f \circ \alpha)'(t)$:

$$(f \circ \alpha) = f(\alpha(t)) = A(\alpha(t)) \cdot \alpha(t)$$

From the homework 1, I know that when I take the derivative of the expression, I will get

$$f(\alpha(t))' = A\alpha'(t) \cdot \alpha(t) \ + A\alpha(t) \cdot \alpha'(t)$$

Because A is defined as a self-adjoint map, then I know that:

$$A\alpha'(t) \cdot \alpha(t) \ = \alpha'(t) \cdot A\alpha(t)$$

Meaning I will get

$$f(\alpha(t))' = 2A\alpha(t) \cdot \alpha'(t)$$

I set this equal to 0 now in order to find the critical value t:

$$f(\alpha(t))' = 2A\alpha(t) \cdot \alpha'(t) = 0$$

This then implies that $2A\alpha(t)$ is orthogonal to $\alpha'(t)$ by the definition of the dot product

From homework 1, we know that if $|\alpha(t)|$ is constant, which in this case it is because it is just 1 because $\alpha(t)$ is the unit circle parameterization, then α is orthogonal to α'

Therefore we know that any vector orthogonal to $\alpha'(t)$ can only be in the direction of $\alpha(t)$ because we are in R^2 space, meaning that when $\alpha(t)$ is transformed by A, it, it must be some scalar multiple of $\alpha(t)$, namely $\lambda\alpha(t)$. Therefore, t is a critical value that will solve $2A\alpha(t) \cdot \alpha'(t) = 0$ whenever $2A\alpha(t) = \lambda\alpha(t)$, which is only the case iff $\alpha(t)$ is an eigenvector of A.

To relate the corresponding eigenvalue λ to f, it is simply

$$f(\alpha(t)) = A\alpha(t) \cdot \alpha(t) = \lambda\alpha(t) \cdot \alpha(t) = \lambda * 1 = \lambda$$

[5] Remark: This special case of the theorem is the most important case for us in the course.

[6] Recall: A critical point of a function $g : \mathbb{R} \to \mathbb{R}$ is a value t so that $g'(t) = 0$. An eigenvalue of a linear operator A is a nonzero vector w so that $Aw = cw$ for some scalar c.

[7] Hint: this is just a linear algebra exercise. Make sure to use the hypothesis...

Part b

From calculus, we know that if a function is continuous on a closed interval, then it is bounded, and will always achieve at least one maximum and one minimum value, provided that they are in the interior of the interval.

If we look at our function $f(\alpha(t))$, which is continuous on a closed interval, being 0 and 2π, then we know it must achieve at least one minimum value and one maximum value, whose corresponding critical value would the t would correspond to one eigenvalue of A (proved in part a). From linear algebra, we know that for R^2 space, there can only be two unique eigenvalues, meaning that we can only have one maximum and one minimum value exactly in our interval $(0, 2\pi)$. This would mean that there are two critical points for $f(\alpha(t))$, meaning there are 2 eigenvalues for A.

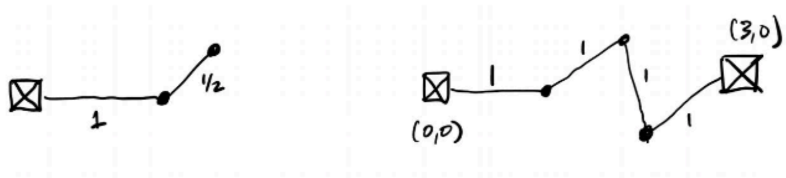
For the scalar matrix, this would be the case where the maximum value equaled the minimum value in our interval, in which case the function is just a flat line, meaning every t in our interval $(0, 2\pi)$ is a critical point. If we go back to $2A\alpha(t) = \lambda\alpha(t)$, it is only possible for every t to be a critical value iff A is a scalar matrix.

Part c

From part a, we showed that there exists some t that $\alpha(t)$ is an eigenvector. However, notice if we go through part a again but with $f(\alpha(t))' = 2A\alpha'(t) \cdot \alpha(t)$ by the fact that A is a self-adjoint linear map, in which case we could go through the exact same reasoning as before to show that there exists some t that would make $\alpha'(t)$ an eigenvector of A. From homework, we know that because $|\alpha(t)| = 1$ is constant, then $\alpha \cdot \alpha' = 0$. Finally, we know that the unit circle is always unit speed meaning $|\alpha'(t)|$ is always 1 and we just stated that $|\alpha(t)| = 1$. With these facts, it is evident that α and α', who are both eigenvectors of A, form an ONB for R^2. □

very nice

Problem 6 (Bonus). *A linkage is a collection of rods in the plane. The following picture shows two different linkages.*



The configuration space of a linkage is the set of all possible positions of the joints of the rods (this is a subset of some Euclidean space). The configuration space of the two linkages above are surfaces. Identify these surfaces.

Solution. The first surface is a torus living in $\mathbb{R}^4$. To see this, notice the space can be parameterized by the angle of the first rod and the angle of the second rod. These parameters each live in S^1. $S^1 \times S^1 = T^2$.

The second surface is a banana. The middle joint lives somewhere in the intersection of the disks of radius 2 around each anchor. For each position of the middle joint, there are 4 possible configurations, depending on whether each of the first and third joints stick up or down. Transitions between different configuration families occur smoothly at the border. Thus, the surface is represented by 4 lenses, glued together as follows:

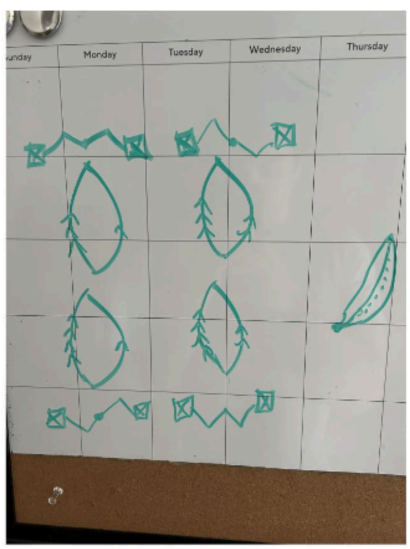


Figure 1: The gluing diagram for the second linkage's surface

It's easy to see that gluing these pieces together results in a banana (homeomorphic to the sphere).

□