Convex cocompact subgroups of the Goeritz group

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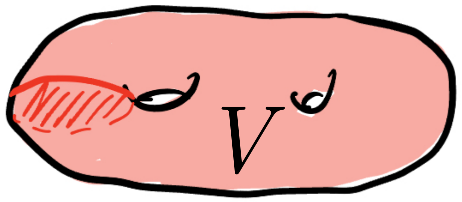
AMS Sectional, GaTech

3/18/2023

I. The Goeritz group

Goeritz group $\mathbb{G}_g$



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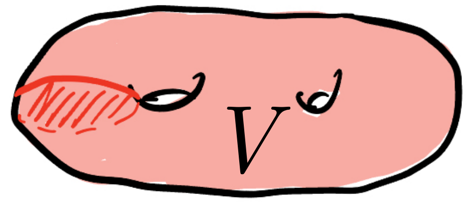

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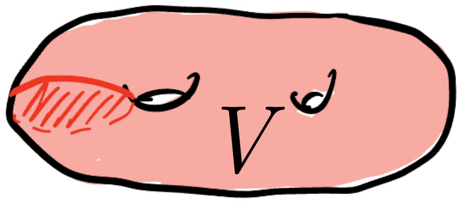

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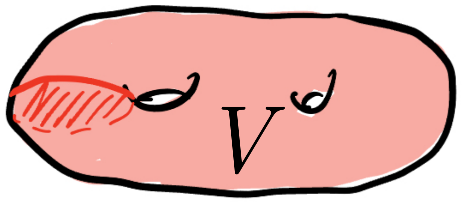

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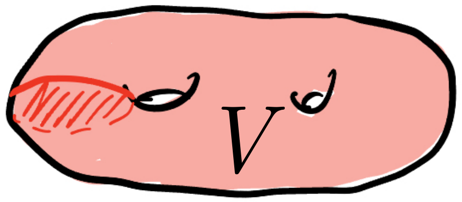

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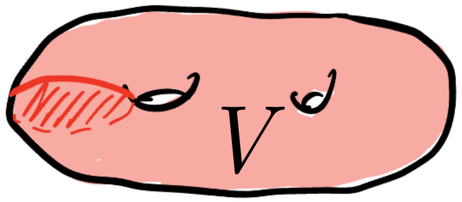

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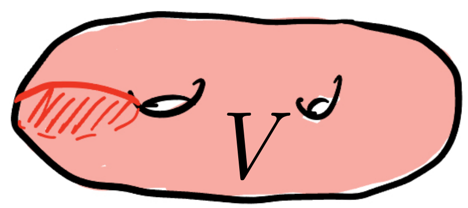

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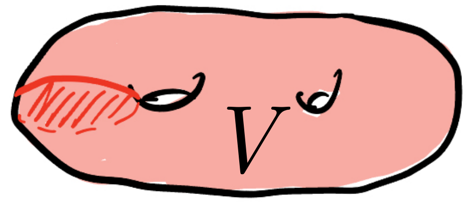

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  $\rightsquigarrow \quad \mathbb{G}_g \hookrightarrow \mathrm{Mod}(S_g)$ intersection of handlebody groups

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Conjecture (Powell 1977) $\mathbb{G}_g$ finitely generated by braid moves.

True for $g = 2,3$ (Goeritz 1933, Scharlemann-Freedman 2018)

Genus 2 Goeritz group

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Theorem (Goeritz 1933, Scharlemann 2004)

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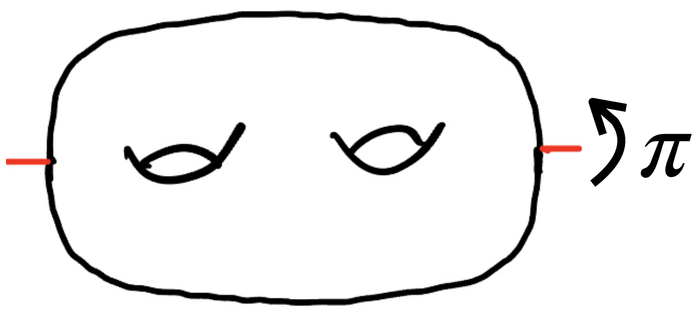
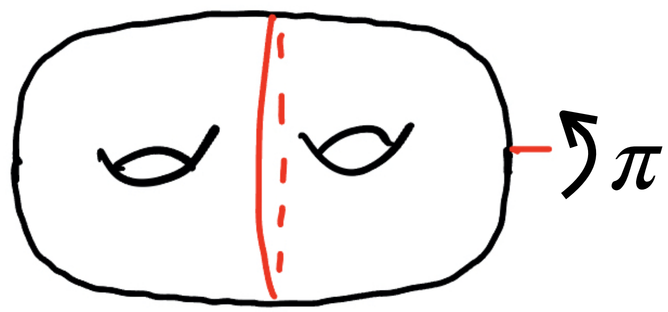
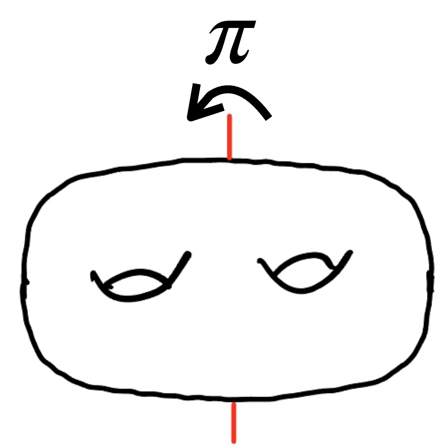


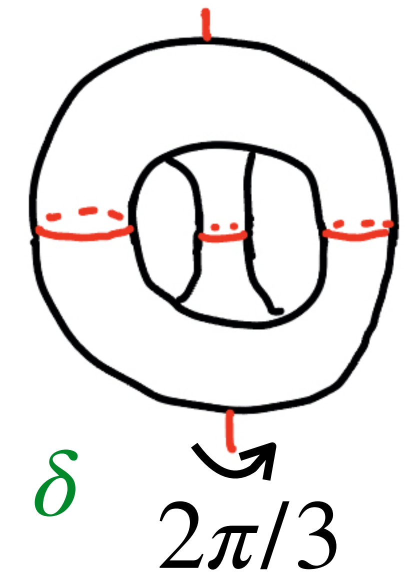
α hyperelliptic



β half-twist



γ rotation



δ

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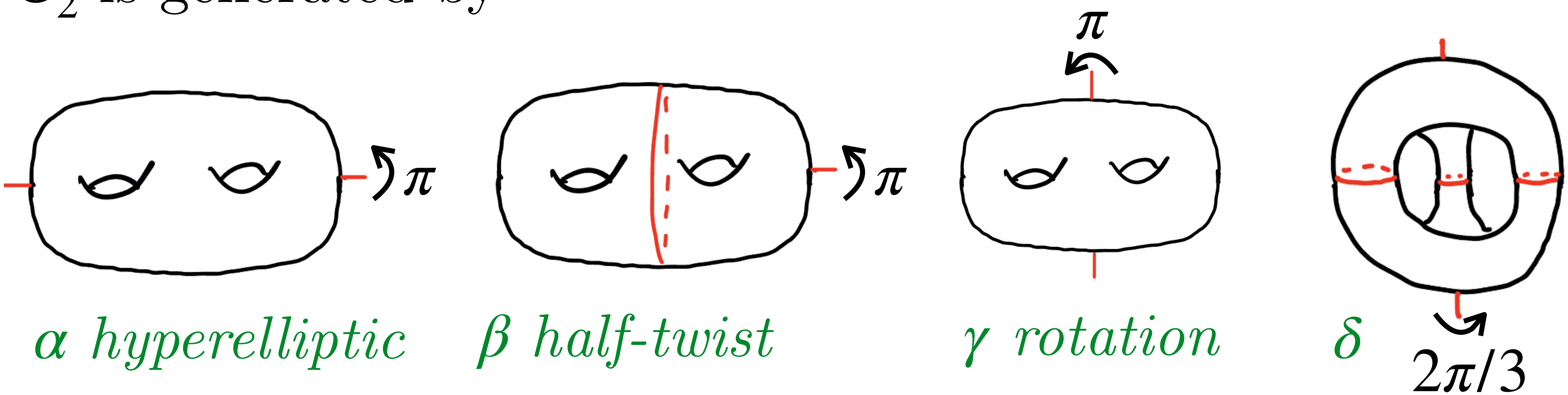

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Theorem (Akbas, Cho 2008) $\mathbb{G}_2$ is finitely presented.

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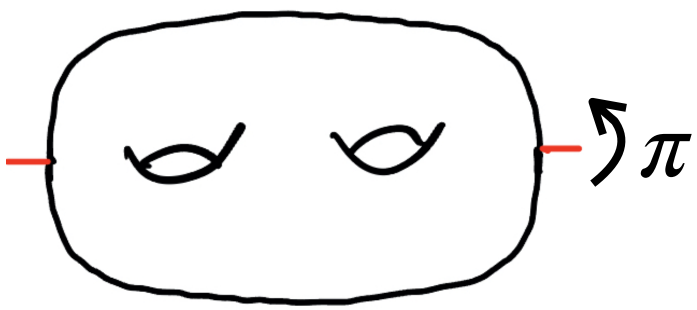

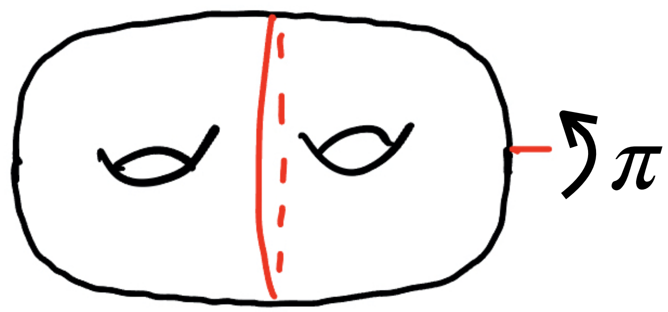

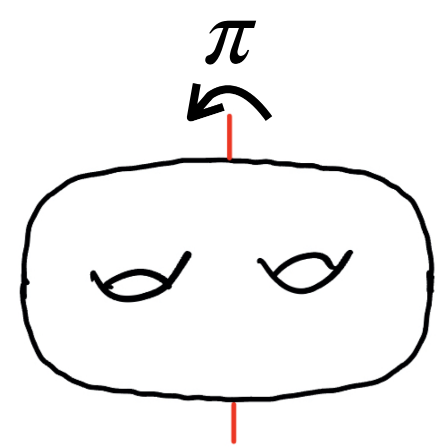

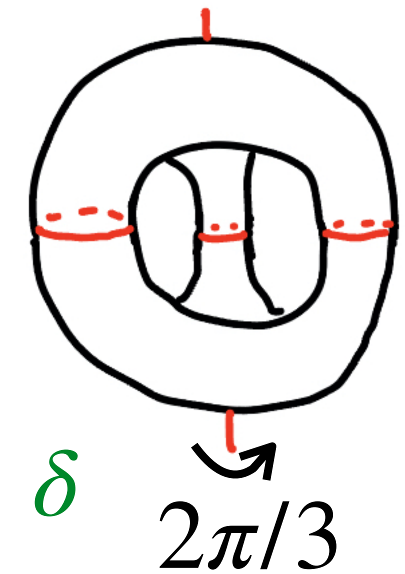

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Theorem (Akbas, Cho 2008) $\mathbb{G}_2$ is finitely presented.

$$\mathbb{G}_2 \cong \left[(\underset{\alpha}{\mathbb{Z}_2} \times \underset{\beta}{\mathbb{Z}}) \rtimes \underset{\gamma}{\mathbb{Z}_2}\right] *_{\underset{\alpha\ \ \beta}{\mathbb{Z}_2\times\mathbb{Z}_2}} (\underset{\gamma,\delta}{S_3} \times \underset{\alpha}{\mathbb{Z}_2})$$

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<u>Focus of this talk</u>: Geometry and topology of $\mathbb{G}_2$

1. Nielsen-Thurston classification for $\mathbb{G}_2$
2. Purely pseudo-Anosov subgroups of $\mathbb{G}_2$

Yes, $\phi \in \mathbb{G}_2$ *is a product of* $\alpha, \beta, \gamma, \delta$*, but how does* ϕ *act on* S_g *(or* S^3*)?*

II. Nielsen-Thurston classification for $\mathbb{G}_2$

Nielsen-Thurston theory for $\mathbb{G}_g$

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Question (Reducible in $\mathbb{G}_g$) Which multicurves are canonical reduction systems for reducible $\phi \in \mathbb{G}_g$? (CRS = intersection of maximal RS.) Which subsurfaces support pseudo-Anosovs?

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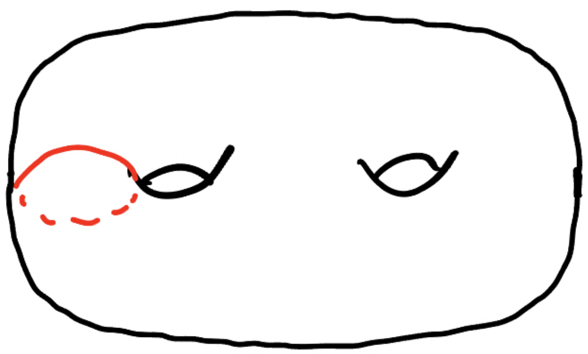

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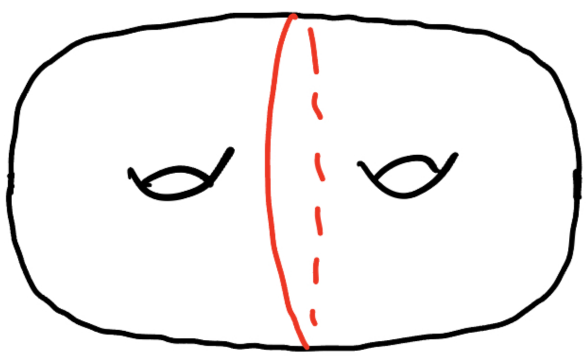

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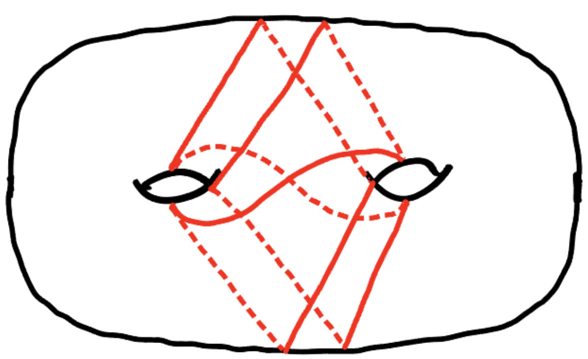
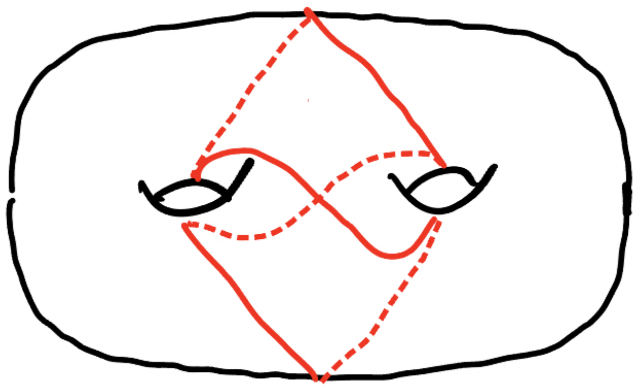


non-separating, but not primitive



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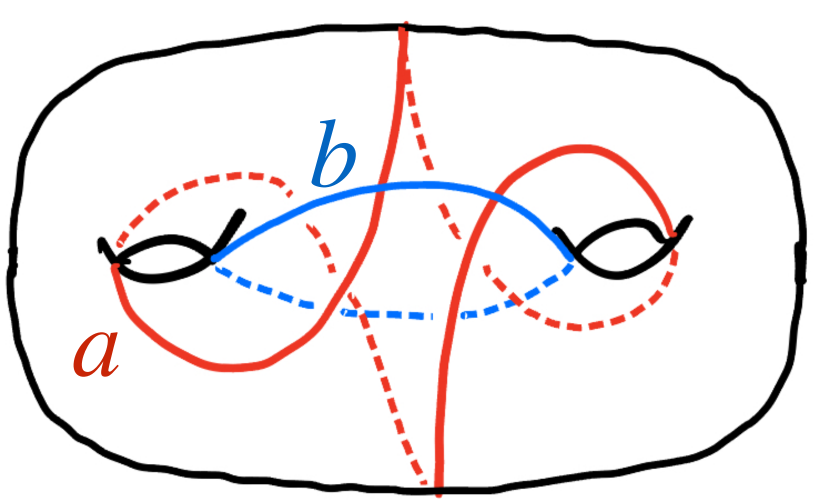

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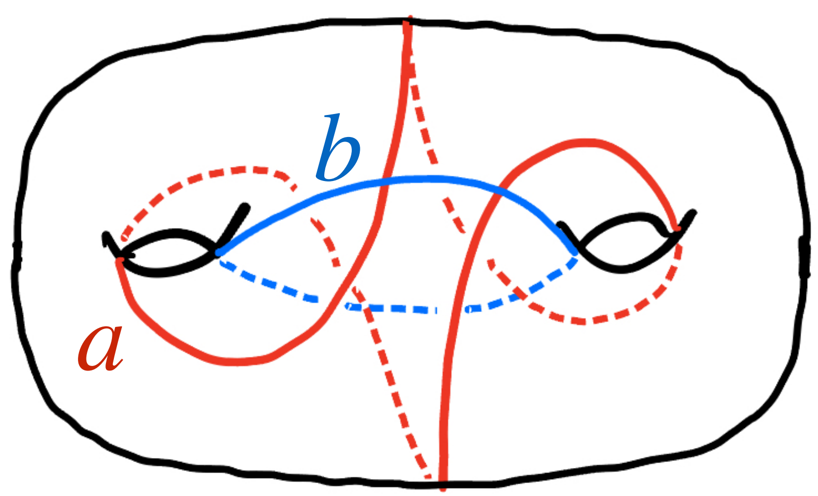

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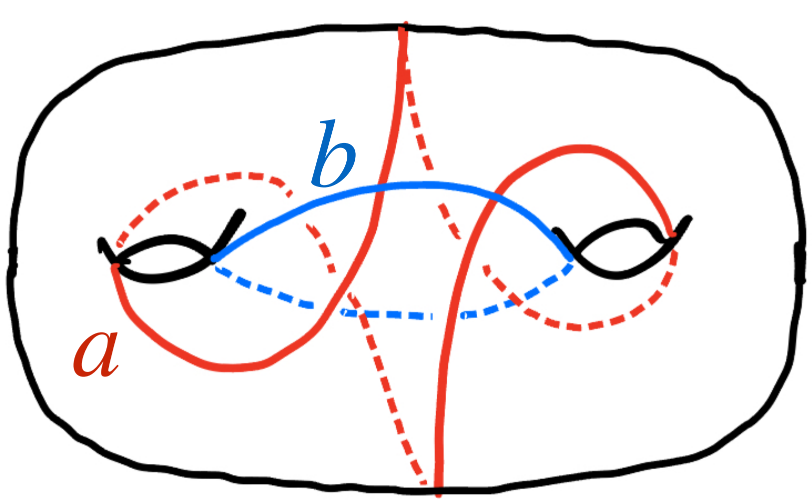

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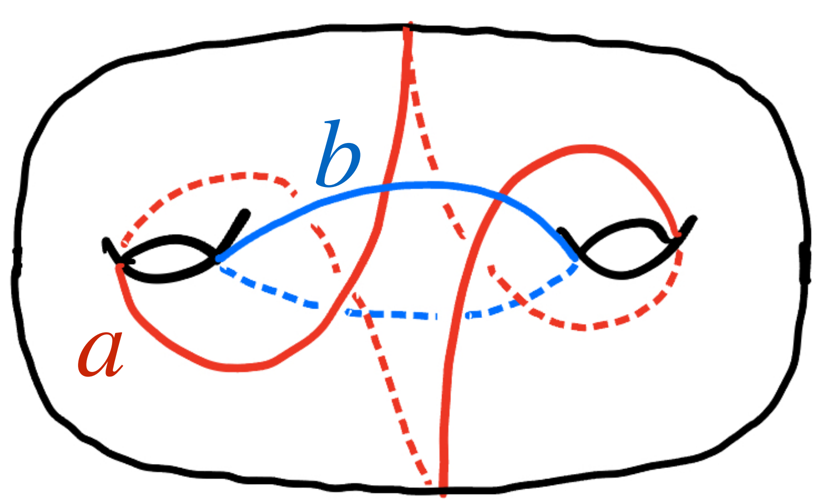
ϕ is pseudo-Anosov $\Leftrightarrow \rho(\phi)$ hyperbolic.



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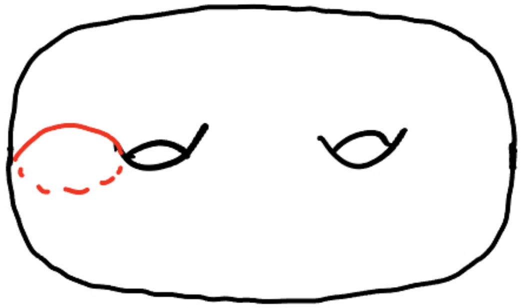

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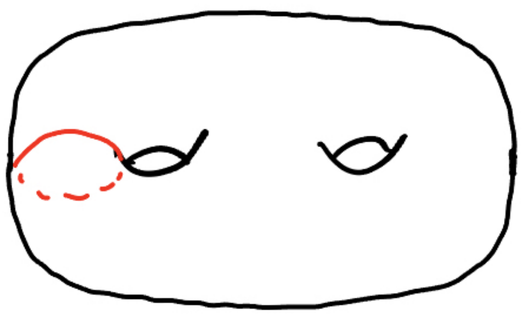

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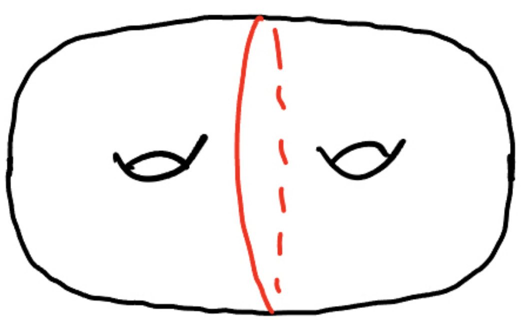

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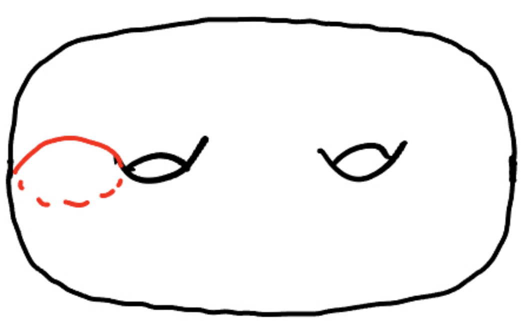

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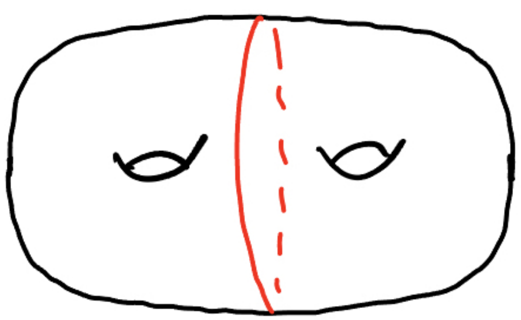

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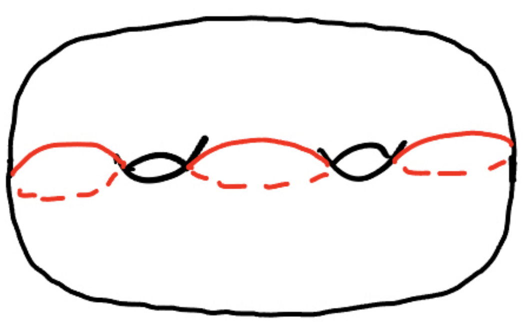

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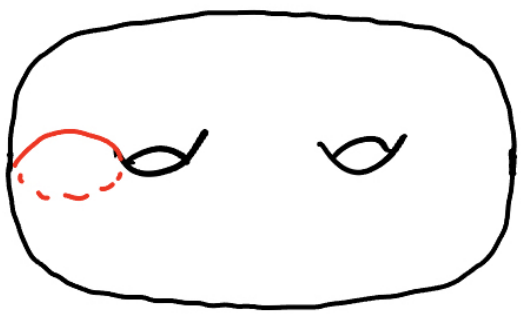

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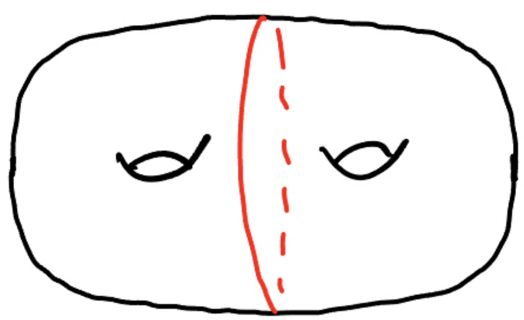

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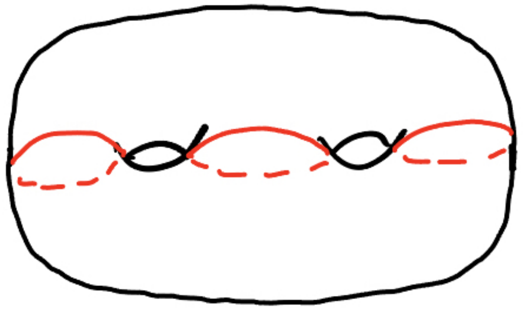

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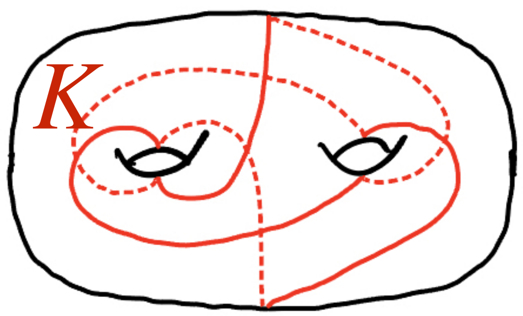

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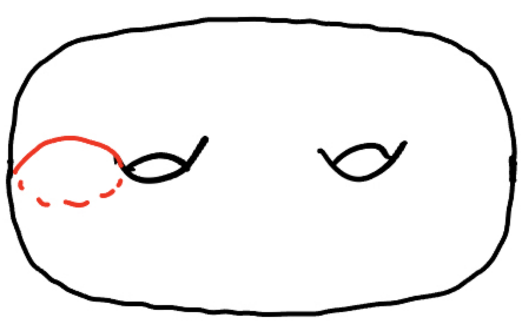

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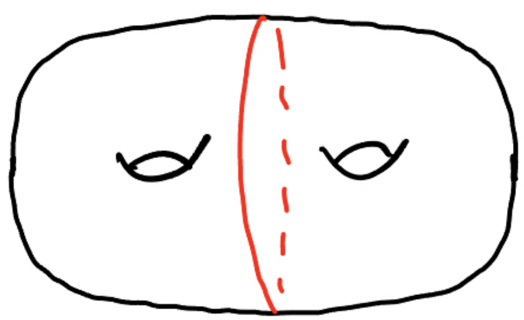

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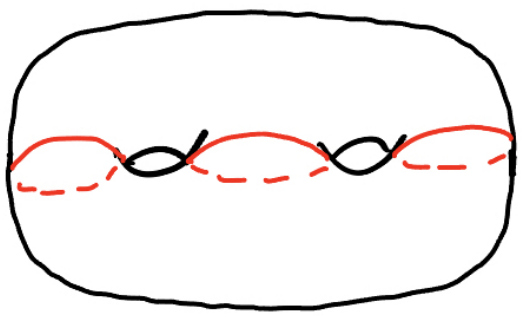

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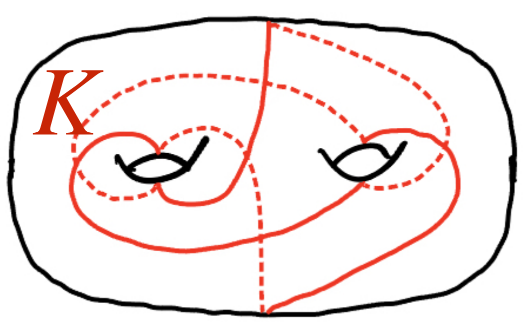

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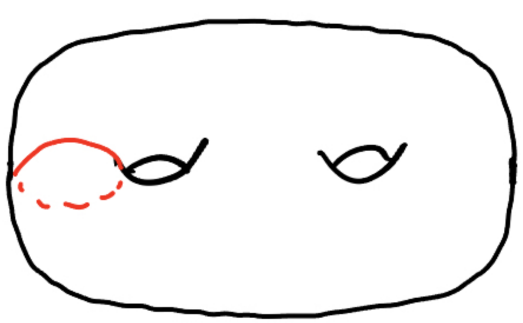

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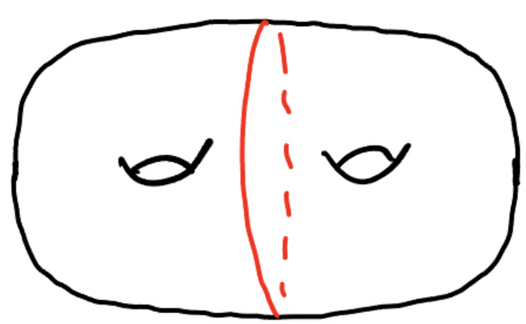

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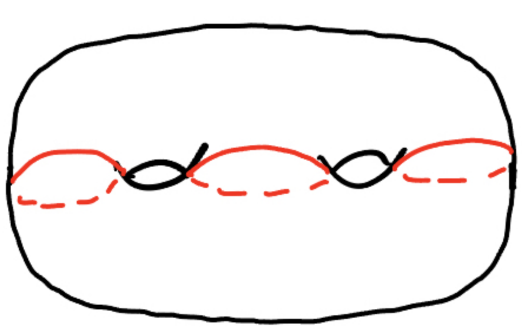

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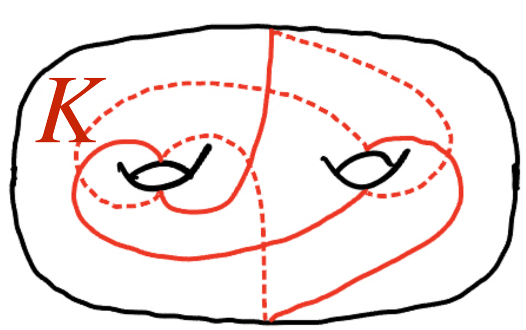

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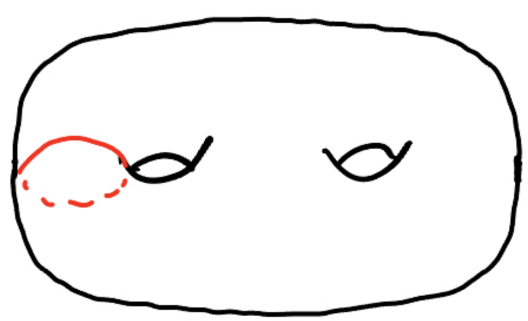

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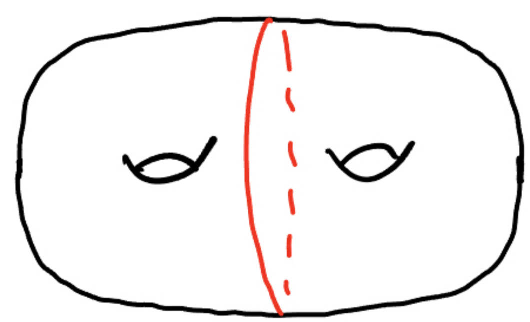

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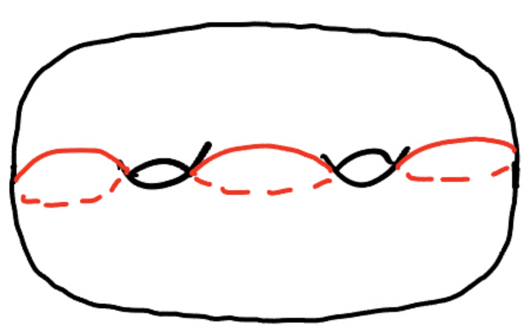

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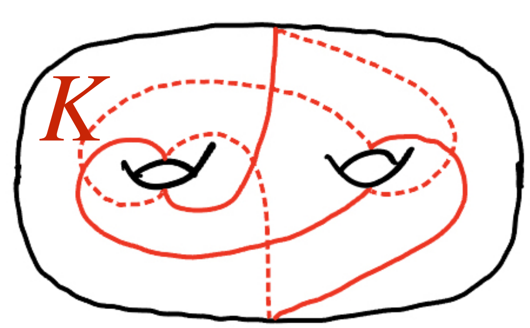

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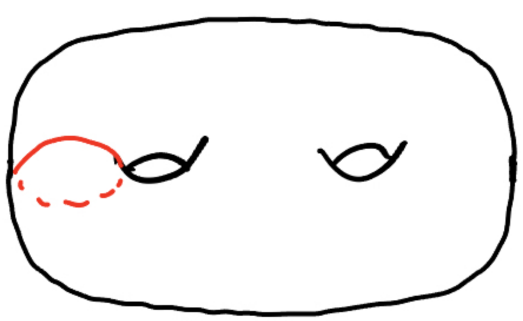

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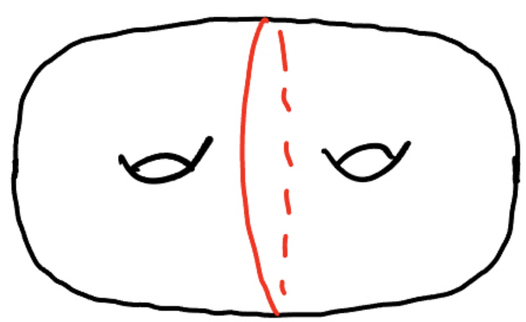

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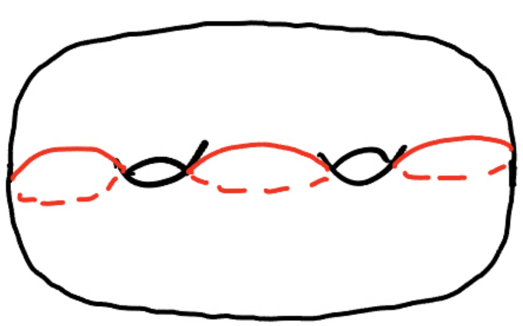

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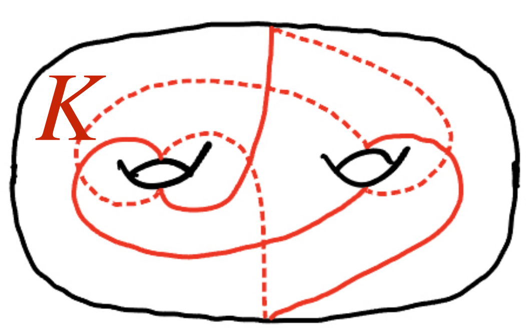

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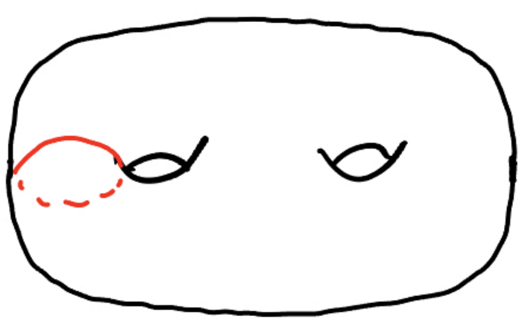

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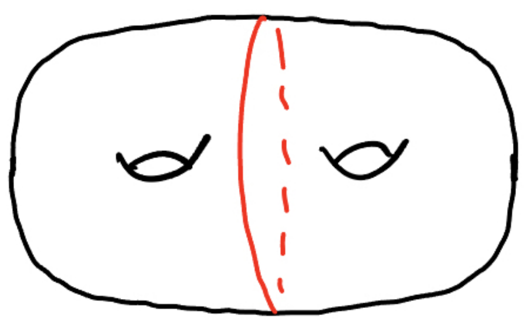

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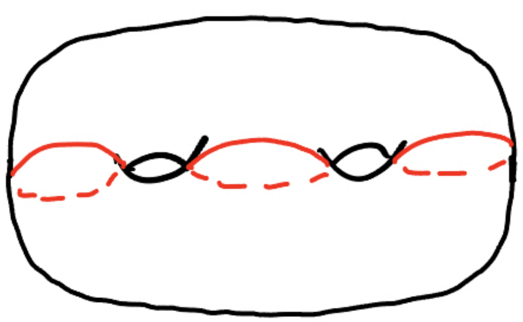

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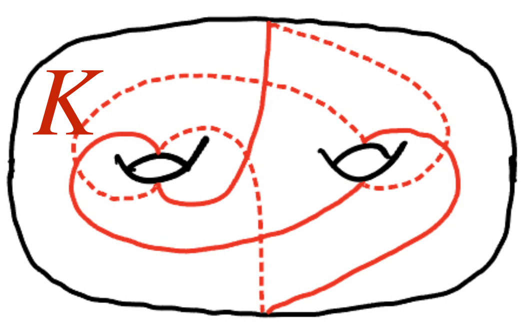

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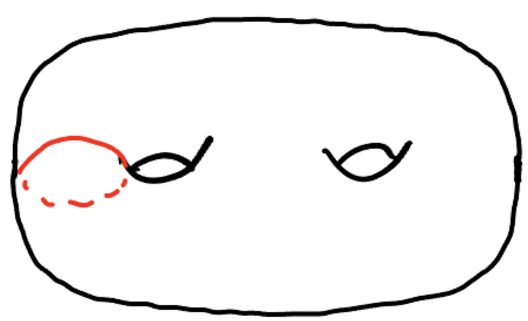

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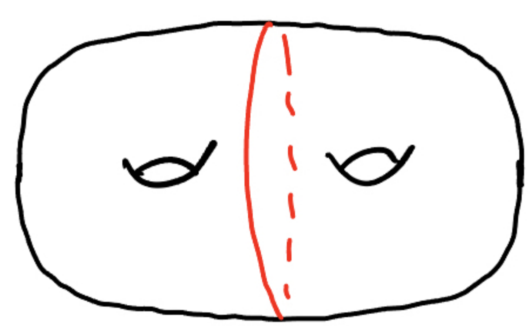

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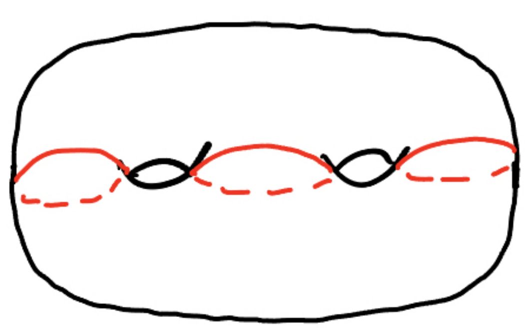

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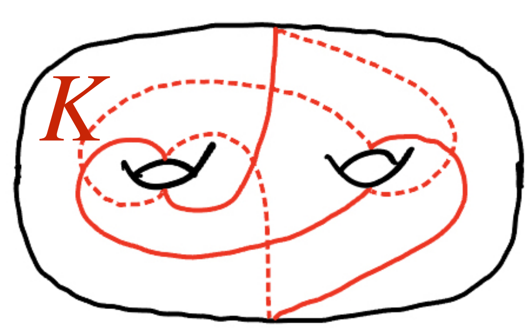

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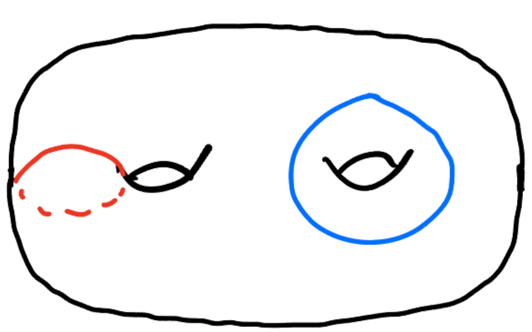

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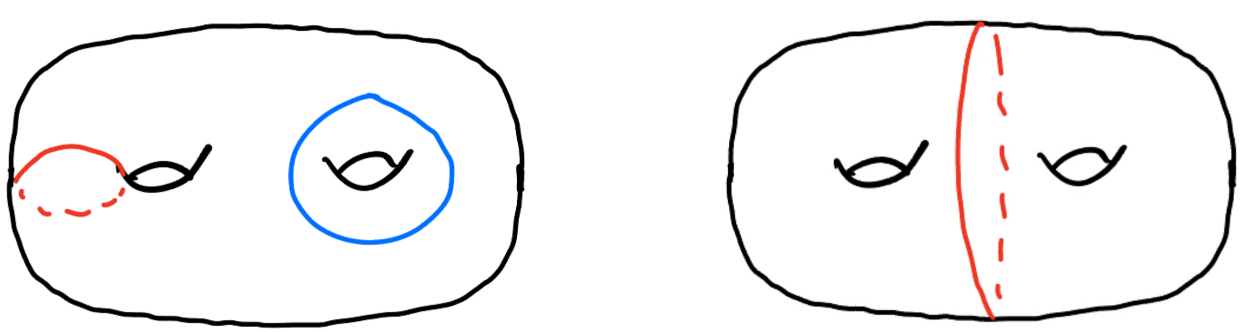

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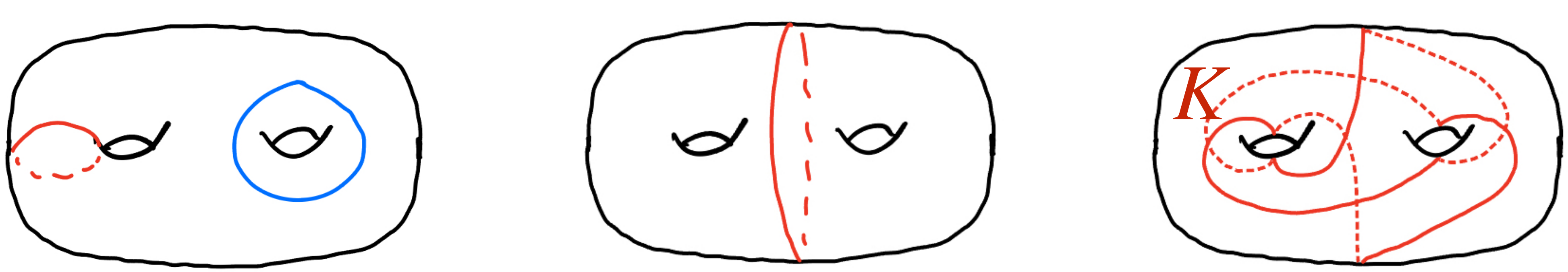

III. Purely pseudo-Anosov subgroups of $\mathbb{G}_2$

Hyperbolic surface-group extensions

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$$
\begin{array}{ccccccccc}
 & & & & & & \mathrm{Mod}(S) & & \\
 & & & & & & \wr\!\| & & \\
1 \to & \pi_1(S) & \to & \mathrm{Aut}\big(\pi_1(S)\big) & \to & & \mathrm{Out}\big(\pi_1(S)\big) & \to & 1 \\
 & \| & & \uparrow & & & \uparrow & & \\
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$$\begin{tikzcd}
 & & & \mathrm{Mod}(S) \arrow[d, phantom, "\cong" rotate=90] & \\
1 \arrow[r] & \pi_1(S) \arrow[r] \arrow[d, equal] & \mathrm{Aut}\big(\pi_1(S)\big) \arrow[r] & \mathrm{Out}\big(\pi_1(S)\big) \arrow[r] & 1 \\
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\end{tikzcd}$$

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For $G < \mathrm{Mod}(S)$, when is Γ_G a hyperbolic group?

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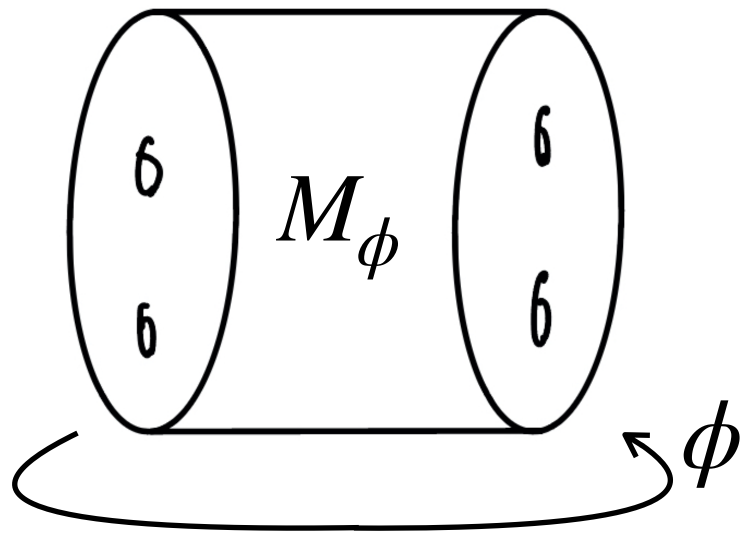

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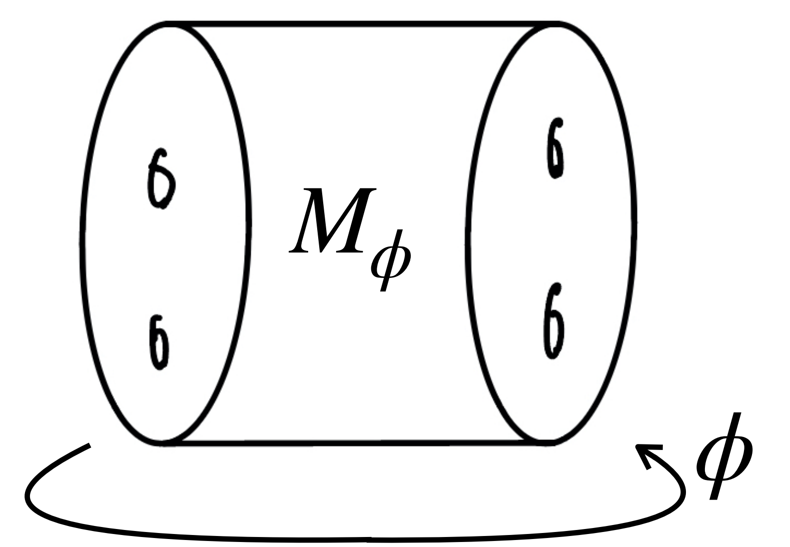

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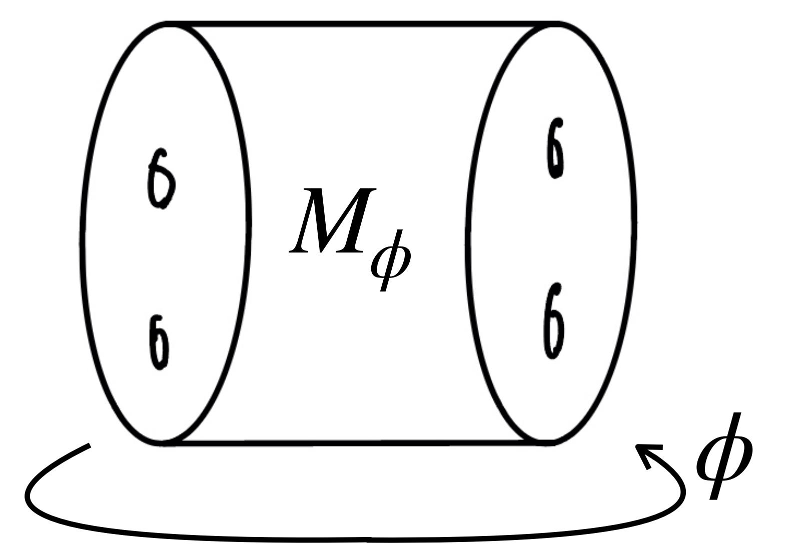

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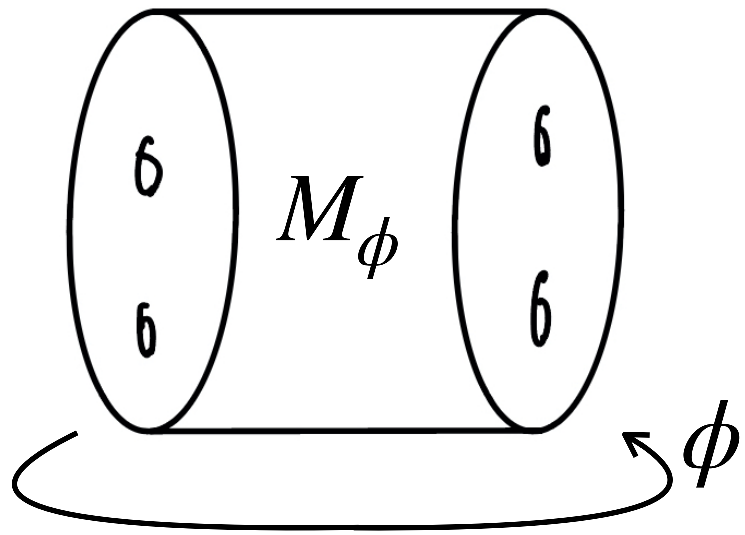

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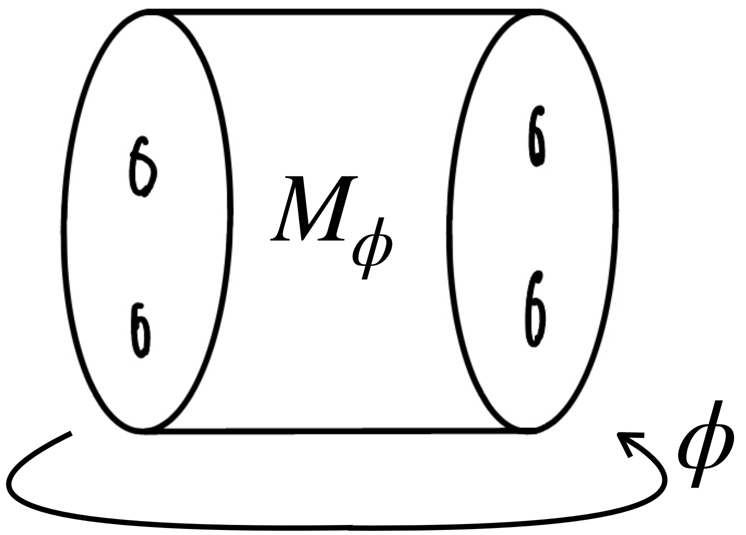

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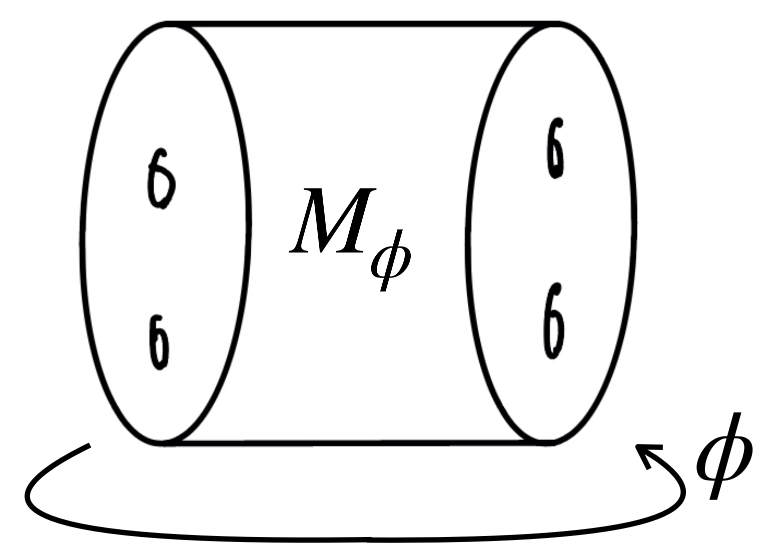
Theorem (Thurston). $\phi \in \mathrm{Mod}(S)$. TFAE

(i) M_ϕ hyperbolic 3-manifold

(ii) $\pi_1(M_\phi)$ hyperbolic group

(iii) ϕ pseudo-Anosov



Example $G = \langle \beta^2\delta, \delta\beta^2 \rangle < \mathrm{Mod}(S_2)$ (purely pA)

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- genus-2 Goeritz group $\mathbb{G}_2$ (T)

IV. Proof techniques of main results

Main results

$\mathbb{G}_2 < \mathrm{Mod}(S_2)$ Goeritz group

Theorem 1 $\phi \in \mathbb{G}_2 < Mod(S_2)$ is pseudo-Anosov $\iff$

ϕ is not conjugate into any of the following subgroups

- $\langle \alpha, \beta, \gamma\delta \rangle$ (primitive curve stabilizer)
- $\langle \alpha, \beta, \gamma \rangle$ (reducing curve stabilizer)
- $\langle \alpha, \gamma, \delta \rangle$ (primitive pant stabilizer)
- $\langle \alpha, \beta\delta\beta^{-1}\delta, \gamma\delta \rangle$ (figure-8 stabilizer)

Theorem 2

$G < \mathbb{G}_2$ f.g. purely pA $\implies G < \mathrm{Mod}(S_2)$ convex cocompact.

Corollary (explicit examples) For $n \geq 2$

$G_n = \langle \beta^n\delta, \delta\beta^n \rangle$ is purely pseudo-Anosov (by Thm1) $\implies$

convex cocompact (by Thm2) $\implies \Gamma_{G_n}$ hyperbolic.

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$\mathscr{P} \sim \mathrm{Cone}(\mathbb{G}_2, H)$, where $H < \mathbb{G}_2$ is stabilizer of $a \in \mathscr{P}$.

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Theorem (T). The only (∞-diameter) witnesses are $X = S$ and $X =$ genus-1 subsurface bounding a fig-8 knot $K \subset S \subset S^3$.

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then G contains reducible element.

Proof sketch

WTS $G < \mathbb{G}_2$ f.g. purely p.A. $\implies$ are convex cocompact.

Fix $G < \mathbb{G}_2$ f.g. purely pseudo-Anosov

$$
\begin{tikzcd}
\mathbb{G}_2 \arrow[rr, mapsto] & & \mathrm{Cone}\big(\mathbb{G}_2, \mathrm{Stab}_{\mathbb{G}_2}(a)\big) \arrow[d, leftrightarrow, "\text{q.i. (Cho } +\varepsilon)"] \\
G \arrow[u, mapsto, "\substack{\text{q.i. emb.} \\ (\mathbb{G}_2 \text{ virtually free})}"'] \arrow[urr, "\substack{\text{q.i. emb.} \\ \text{(Abbott-Manning } +\varepsilon)}"'] \arrow[rr] \arrow[drr] & & \mathscr{P} \arrow[d, mapsto, "\text{not a q.i. emb.}"] \\
& & \mathscr{C}(S)
\end{tikzcd}
$$

WTS: $G \to \mathscr{C}(S)$ is q.i. emb.

suffices to show $G \to \mathscr{P}$ is q.i. emb.

(distance formula + BBKL)

Show if $G \to \mathscr{P}$ is q.i. emb but $G \to \mathscr{C}(S)$ is not,

then G contains reducible element.

$\square$