Problem 1. *Let* $\alpha : I \to \mathbb{R}^3$ *and* $\beta : I \to \mathbb{R}^3$ *be two curves. Let* $\alpha \cdot \beta : I \to \mathbb{R}$ *be the function defined by* $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$ *(dot product). Prove that*

$$(\alpha \cdot \beta)'(t) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t).$$

Solution.

$$\begin{aligned}
(\alpha \cdot \beta)'(t) &= \lim_{h\to 0} \frac{(\alpha \cdot \beta)(t+h) - (\alpha \cdot \beta)(t)}{h} \\
&= \lim_{h\to 0} \frac{\alpha(t+h) \cdot \beta(t+h) - \alpha(t) \cdot \beta(t)}{h} \\
&= \lim_{h\to 0} \frac{\alpha(t+h) \cdot \beta(t+h) - \alpha(t) \cdot \beta(t+h) + \alpha(t) \cdot \beta(t+h) - \alpha(t) \cdot \beta(t)}{h} \\
&= \lim_{h\to 0} \beta(t+h) \cdot \frac{\alpha(t+h) - \alpha(t)}{h} + \lim_{h\to 0} \alpha(t) \cdot \frac{\beta(t+h) - \beta(t)}{h} \\
&= \lim_{h\to 0} \beta(t+h) \cdot \lim_{h\to 0} \frac{\alpha(t+h) - \alpha(t)}{h} + \lim_{h\to 0} \alpha(t) \cdot \lim_{h\to 0} \frac{\beta(t+h) - \beta(t)}{h} \\
&= \beta(t) \cdot \alpha'(t) + \alpha(t) \cdot \beta'(t)
\end{aligned}$$

$\square$

Problem 1. *Let $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ be two curves. Let $\alpha \cdot \beta : I \to \mathbb{R}$ be the function defined by $(\alpha \cdot \beta)(t) = \alpha(t) \cdot \beta(t)$ (dot product). Prove that*

$$(\alpha \cdot \beta)'(t) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t).$$

Solution. We write $(\alpha \cdot \beta)(t) = x_\alpha(t)x_\beta(t) + y_\alpha(t)y_\beta(t) + z_\alpha(t)z_\beta(t)$. Now we can differentiate:

$$\begin{aligned}
(\alpha \cdot \beta)'(t) &= \frac{d}{dt}\left(x_\alpha(t)x_\beta(t) + y_\alpha(t)y_\beta(t) + z_\alpha(t)z_\beta(t)\right) \\
&= x'_\alpha(t)x_\beta(t) + x_\alpha(t)x'_\beta(t) + y'_\alpha(t)y_\beta(t) + y_\alpha(t)y'_\beta(t) + z'_\alpha(t)z_\beta(t) + z_\alpha(t)z'_\beta(t) \\
&= \Big(x'_\alpha(t)x_\beta(t) + y'_\alpha(t)y_\beta(t) + z'_\alpha(t)z_\beta(t)\Big) + \Big(x_\alpha(t)x'_\beta(t) + y_\alpha(t)y'_\beta(t) + z_\alpha(t)z'_\beta(t)\Big) \\
&= \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t)
\end{aligned}$$

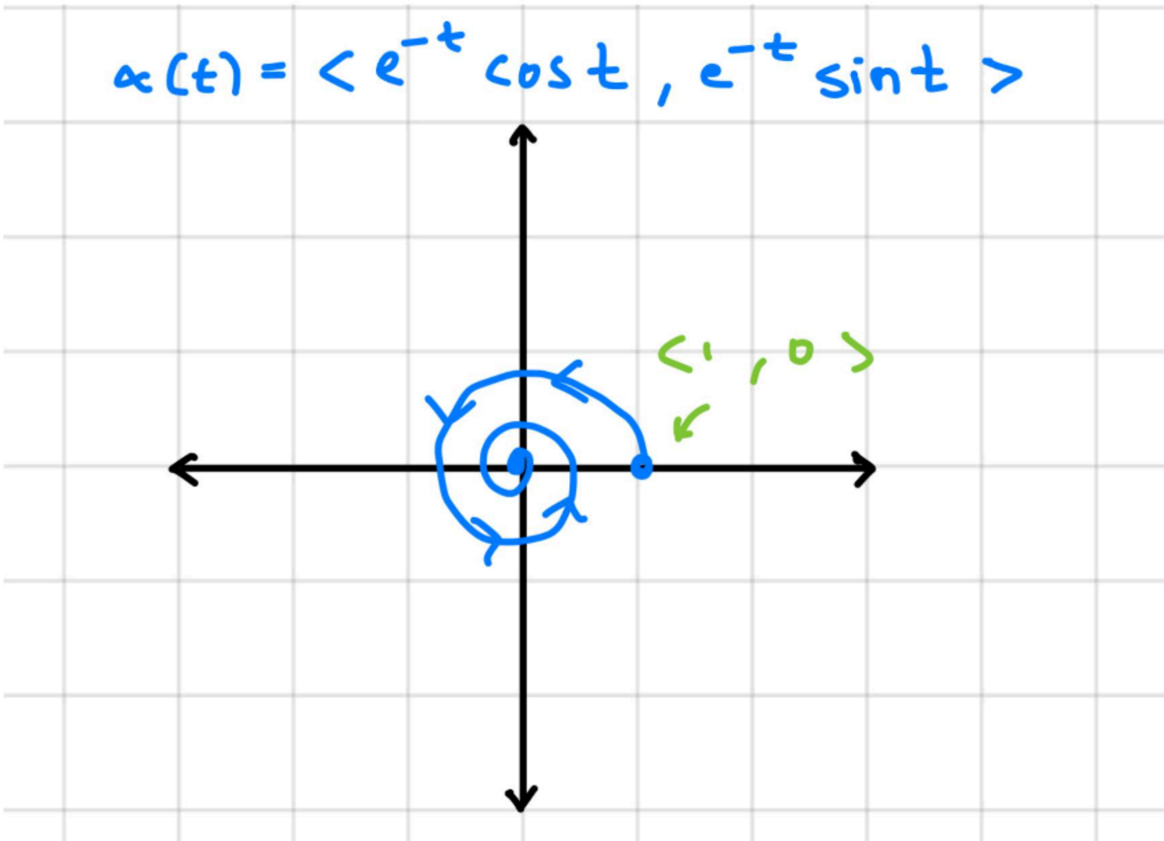
$\square$

Problem 2. *Let $\alpha : I \to \mathbb{R}^3$ be a curve. Prove that $|\alpha(t)|$ is constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.*

Solution. We clearly have that $|\alpha(t)|$ is constant iff $|\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ is constant, and $\alpha(t) \cdot \alpha(t)$ is constant iff $(\alpha(t) \cdot \alpha(t))'$ is zero for all $t \in I$. By Problem 1, we have that $(\alpha(t) \cdot \alpha(t))' = \alpha(t) \cdot \alpha'(t) + \alpha(t)' \cdot \alpha(t) = 2\alpha(t) \cdot \alpha'(t)$. By the definition of the dot product, $2\alpha(t) \cdot \alpha'(t)$ is zero for all $t \in I$ iff $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$. Following the chain of if and only if statements gives the desired statement. $\square$

Problem 4. *The curve* $\alpha(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$ *for* $t \in [0, \infty)$ *is called the logarithmic spiral. Sketch this curve, and compute its length.*

Solution. To sketch the curve, notice that $\alpha(t)$ parametrizes the circle with radius e^{-t} centered at $(0,0)$. As t increases, the radius decreases, so the curve spirals in toward the origin.



To compute the length of the curve, we must compute $\int_0^\infty |\alpha'(t)|\,dt$.

$\alpha'(t) = \langle -e^{-t}\cos(t) - e^{-t}\sin(t), -e^{-t}\sin(t) + e^{-t}\cos(t)\rangle$. Thus, $|\alpha'(t)| = \sqrt{2e^{-2t}(\cos^2(t) + \sin^2(t))} = \sqrt{2e^{-2t}} = \sqrt{2}e^{-t}$.

Thus, $\int_0^\infty |\alpha'(t)|\,dt = \sqrt{2}\int_0^\infty e^{-t}\,dt = \sqrt{2}e^{-t}\big|_0^\infty = \sqrt{2}$. The logarithmic spiral described has length $\sqrt{2}$.

$\square$

Problem 5. *Let* $\alpha : [a,b] \to \mathbb{R}^3$ *be a curve. Let* v *be a unit vector. Prove that*

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

Choose v *appropriately to deduce that the shortest path between any two points is a straight line.*

Solution. As v is independent of t, we have $v' = 0$ and thus $\alpha(t) \cdot v' = 0$. This yields

$$(\alpha(b) - \alpha(a)) \cdot v = \alpha(t) \cdot v \Big|_a^b = \int_a^b (\alpha \cdot v)'(t) \, dt = \int_a^b \left(\alpha'(t) \cdot v + \alpha(t) \cdot v'\right) \, dt = \int_a^b \alpha'(t) \cdot v \, dt,$$

where the second equality comes from the fundamental theorem of calculus and the third equality comes from **Problem 1**. To prove the inequality, note that $|v| = 1$ and therefore

$$\int_a^b \alpha'(t) \cdot v \, dt \leq \left| \int_a^b \alpha'(t) \cdot v \, dt \right| \leq \int_a^b \left|\alpha'(t) \cdot v\right| \, dt \leq \int_a^b \left|\alpha'(t)\right| |v| \, dt = \int_a^b \left|\alpha'(t)\right| \, dt, \qquad (3)$$

where the second inequality comes the triangle inequality and the third comes from the Cauchy-Schwarz inequality.

If $\alpha(b) - \alpha(a) = 0$, then the distance between these two points is zero. Thus suppose $\alpha(b) - \alpha(a) \neq 0$ and let v be given by

$$v = \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}.$$

This gives

$$(\alpha(b) - \alpha(a)) \cdot \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|} = |\alpha(b) - \alpha(a)| \leq \int_a^b \left|\alpha'(t)\right| \, dt,$$

implying that the line segment that connects $\alpha(a)$ to $\alpha(b)$ is the shortest possible path. $\square$

Problem 6. *Fix a curve $\alpha : I \to \mathbb{R}^3$ and fix $[a,b] \subset I$. For a partition*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

we defined $L(\alpha, P) = \sum |\alpha(t_{i+1}) - \alpha(t_i)|$ and $|P| = \max(t_{i+1} - t_i)$. Prove that for each $\epsilon > 0$, there exists $\delta > 0$ so that if $|P| < \delta$, then

$$\left| \int_a^b |\alpha'(t)|\, dt - L(\alpha, P) \right| < \epsilon.$$

[2] [3]

Solution. We want to show that for any partition P with $|P| < \delta$ for some $\delta > 0$, we have that

$$\left| \int_a^b |\alpha'(t)|\, dt - L(\alpha, P) \right| < \epsilon$$

To do so, it suffices to show that we can find a δ such that

$$\left| \int_a^b |\alpha'(t)|\, dt - \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) \right| < \frac{\epsilon}{2}$$

$$\left| \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) - \sum_P |\alpha(t_{n+1}) - \alpha(t_n)| \right| < \frac{\epsilon}{2}$$

By the triangle inequality, this will give us the desired overall bound.
By definition, we know that

$$\lim_{|P| \to 0} \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) = \int_a^b |\alpha'(t)|\, dt$$

By the definition of limit, this means there is some δ_1 such that $|P| < \delta_1$ gives us the first required bound. Now we want to find a δ_2 which gives us the second bound, and then let $\delta = \min(\delta_1, \delta_2)$. Consider the second expression:

$$\left| \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) - \sum_P |\alpha(t_{n+1}) - \alpha(t_n)| \right| = \left| \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) - |\alpha(t_{n+1}) - \alpha(t_n)| \right|$$

$$= \sum_P (t_{n+1} - t_n) \left[|\alpha'(t_n)| - \frac{|\alpha(t_{n+1}) - \alpha(t_n)|}{t_{n+1} - t_n} \right]$$

Using the reverse triangle inequality, we can bound this term:

$$\leq \sum_P (t_{n+1} - t_n) \left| \alpha'(t_n) - \frac{\alpha(t_{n+1}) - \alpha(t_n)}{t_{n+1} - t_n} \right|$$

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[2]Suggestion: First replace $\int_a^b |\alpha'(t)|\, dt$ by a Riemann sum, and compare the Riemann sum to $L(\alpha, P)$. For the latter, it may help to use the mean value theorem for vector-valued functions (you will need to figure out which function to apply it to).

[3]This problem is probably harder than most HW problems for the course. Please ask questions if you are stuck.

$$= \sum_P (t_{n+1} - t_n) \left| \frac{\alpha'(t_n)(t_{n+1} - t_n)}{t_{n+1} - t_n} - \frac{\alpha(t_{n+1}) - \alpha(t_n)}{t_{n+1} - t_n} \right|$$

Let $f_n(t) = \alpha'(t_n)t - \alpha(t)$. We can rewrite the numerator:

$$= \sum_P (t_{n+1} - t_n) \left| \frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n} \right|$$

By the multivariate mean value theorem, we can bound this again, with $t_n^* \in (t_n, t_{n+1})$ given by the mean value theorem:

$$\leq \sum_P (t_{n+1} - t_n) \left| f_n'(t_n^*) \right|$$

Recall that $f(t) = \alpha'(t_n)t - \alpha(t)$, so $f'(t) = \alpha'(t_n) - \alpha'(t)$. Plugging this in, we see that:

$$\leq \sum_P (t_{n+1} - t_n) \left| \alpha'(t_n) - \alpha'(t_n^*) \right|$$

At this point we shift gears for a moment. First, we note that α is smooth so in particular, α' is continuous. Further, $I = [a, b]$ is compact, which means that $\alpha' : I \to \mathbb{R}^3$ is uniformly continuous. That means that there exists some δ_2 such that $|t_n^* - t_n| < \delta_2$ implies that $|\alpha'(t_n) - \alpha'(t_n^*)| < \frac{\epsilon}{2(b-a)}$. Thus, if $|P| < \delta_2$, we can again bound:

$$< \sum_P (t_{n+1} - t_n) \frac{\epsilon}{2(b-a)}$$

$$= \frac{\epsilon}{2(b-a)} \sum_P (t_{n+1} - t_n)$$

Observe that the sum is telescoping and evaluates to $b - a$. Putting all the inequalities together, we get that

$$\left| \sum_P |\alpha'(t_n)|(t_{n+1} - t_n) - \sum_P |\alpha(t_{n+1}) - \alpha(t_n)| \right| < \frac{\epsilon}{2}$$

when $|P| < \delta_2$. As noted before, taking $\delta = \min(\delta_1, \delta_2)$ gives us the desired inequality (via triangle inequality).