RESEARCH SUMMARY

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My research area is geometric topology. I primarily study manifolds, fiber bundles, and group actions. My research also has ties to geometric group theory and arithmetic groups. Below I summarize three major topics in my research: Nielsen realization, arithmetic groups, and aspherical manifolds.

1. Group actions and Nielsen realization

For a manifold M, there is a natural surjection $\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M)$ from the homeomorphism group $\mathrm{Homeo}(M)$ to the mapping class group

$$\mathrm{Mod}(M) := \pi_0 \mathrm{Homeo}(M).$$

The Nielsen realization problem asks, for each subgroup $G < \mathrm{Mod}(M)$, if there is a solution to the following lifting problem

$$\begin{tikzcd} & \mathrm{Homeo}(M) \arrow[d, two heads] \\ G \arrow[r, hook] \arrow[ur, dashed, "L"] & \mathrm{Mod}(M) \end{tikzcd}$$

Some motivations for this problem are as follows.

Topology. If $G < \mathrm{Mod}(M)$ is the monodromy of an M-bundle $E \to B$, then a solution to the lifting problem implies the bundle has a *flat connection* (a foliation with certain properties). It's difficult to find M-bundles with no flat connection, but this problem can be approached via Nielsen realization.

Algebra vs. Topology. If $\dim M \leq 3$ and M is aspherical, then $\mathrm{Mod}(M)$ is isomorphic to $\mathrm{Out}(\pi_1(M))$, and the realization problem asks when a group of symmetries of $\pi_1(M)$ can be promoted to a group of symmetries of M.

Geometry. For a surface S_g, Nielsen originally asked if every finite $G < \mathrm{Mod}(S_g)$ lifts to a group of isometries of S_g with respect to some hyperbolic metric. This was proved by Kerckhoff [Ker83]. A more general conjecture (still open, but possibly naïve) is that whenever $G < \mathrm{Mod}(S_g)$ lifts to $\mathrm{Homeo}(S_g)$, there is a lift by Nielsen–Thurston representatives.

In general, there is no obvious conjectural answer for when the lifting problem can be solved. My research has focused on understanding prominent examples and finding new techniques and phenomena. Below I emphasize three different approaches: cohomological, dynamical, homotopy-theoretic.

Cohomological approaches. Thurston asked if $\mathrm{Homeo}(S_g) \to \mathrm{Mod}(S_g)$ splits for a surface of genus $g \geq 2$ [kir97]. This was solved by Markovic [Mar07]. Earlier work by Morita [Mor87] showed that there is no splitting over finite-index $G < \mathrm{Mod}(S_g)$ valued in diffeomorphisms, when $g \gg 0$. Morita's approach is to show that the induced map on (group) cohomology $H^*(-;\mathbb{Q})$ is not injective, and as part of the proof, he constructs an S_g-bundle $E \to B^6$ over an aspherical 6-manifold with no flat connection.

<u>The K3 manifold</u>. Giansiracusa–Kupers and I [GKT21] prove an analogue of Morita's theorem for the K3 4-manifold: $\mathrm{Diff}(\mathrm{K3}) \to \mathrm{Mod}(\mathrm{K3})$ does not split over finite-index $G < \mathrm{Mod}(\mathrm{K3})$. We also show there exists a K3-bundle $E \to B^8$ over an aspherical 8-manifold with no flat connection. It would be interesting to generalized this argument to other 4-manifolds. This result is complementary to recent work of Farb–Looijenga [FL24] who study Nielsen realization for finite subgroups of $\mathrm{Mod}(\mathrm{K3})$.

<u>Point-pushing</u>. A different (cohomological) approach to Morita's theorem was found by Bestvina–Church–Souto [BCS13], who prove, for a based surface $(S_g, *)$ of genus $g \geq 2$, $\mathrm{Diff}(S_g, *) \to \mathrm{Mod}(S_g, *)$ does not split over the "point-pushing" subgroup $\pi_1(S_g) < \mathrm{Mod}(S_g, *)$. In [Tsh15], I extend their result to many other locally symmetric manifolds, e.g. finite manifold covers of $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ and complex hyperbolic manifolds $\Gamma\backslash\mathbb{C}H^n$.

<u>$\mathrm{Mod}(S_g)$ acting on 3-manifolds</u>. Although $\mathrm{Homeo}(S_g) \twoheadrightarrow \mathrm{Mod}(S_g)$ does not split ($g \geq 2$), there is an action of $\mathrm{Mod}(S_g)$ on the unit tangent bundle T^1S_g by homeomorphisms that splits a natural map $\mathrm{Homeo}(T^1S_g) \twoheadrightarrow \mathrm{Mod}(S_g)$.

The existence of the action $\mathrm{Mod}(S_g) \curvearrowright T^1S_g$ is surprising, and one might guess that it is unique in some sense. For example, Souto [Sou10] proves that there is no splitting of $\mathrm{Homeo}^{\pm}(T^1S_g) \twoheadrightarrow \mathrm{Mod}^{\pm}(S_g)$ valued in diffeomorphisms. More generally, for a circle bundle $M \to S_g$ there are surjections

$$\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g),$$

and, conjecturally, this composition splits only for the unit tangent bundle.

L. Chen and I [CT23] show that $\mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g)$ splits if and only if the Euler characteristic $\chi(S_g)$ divides the Euler number of $e(M \to S_g)$. This is a homological argument. With more dynamical techniques, together with my student Alina al Beaini, we show that $\mathrm{Homeo}(S_g \times S^1) \twoheadrightarrow \mathrm{Mod}(S_g)$ does not split for infinitely many g [BCT23] (and similarly for more general M).

Dynamical approaches. A general strategy to show $G < \mathrm{Mod}(M)$ does not lift to $\mathrm{Homeo}(M)$ is to

(1) find relations in G that
(2) cannot be satisfied by homeomorphisms for some dynamical reason.

Indeed, this is (very roughly) the strategy of Markovic [Mar07] to prove $\mathrm{Homeo}(S_g) \to \mathrm{Mod}(S_g)$ does not split. The following table illustrates some ways my collaborators and I have found to combine algebra/dynamics to prove realization results [ST16, CT22, BCT23].

G	algebraic fact in $\mathrm{Mod}(M)$	dynamical ingredients
surface braid groups $B_n(S_g)$	$[[B_n, B_n], [B_n, B_n]] = [B_n, B_n]$ derived subgroup perfect $n \geq 5$	Thurston stability
Sphere twists for 3-manifolds	$G \cong (\mathbb{Z}/2\mathbb{Z})^i$	equivariant sphere theorem
$\mathrm{Mod}(S_g)$ lifting to $\mathrm{Homeo}(S_g \times S^1)$	$\mathrm{Mod}(S_g)$ generated by centralizers $\mathrm{Mod}(S_g) = \langle C(\alpha^2), C(\alpha^p)\rangle$	rigidity+smoothing finite group actions on 3-manifolds

Homotopy-theoretic approaches. Schoen–Yau [SY79] asked the following version of Nielsen's problem: for negatively curved M, can the finite group $\mathrm{Out}(\pi_1(M))$ be realized as a group of isometries with respect to some negatively curved Riemannian metric? Farrell–Jones [FJ90] showed the answer is "no" by giving examples of negatively curved M for which the natural map $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$ is not surjective(!).

The Farrell–Jones examples are of the form $M = N\#\Sigma$, where N is hyperbolic and Σ is a homotopy sphere. Their method uses homotopy theory (specifically smoothing theory) and topological rigidity (Borel conjecture).

Motivated by the Farrell-Jones examples, one would like to compute the image of $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$ and determine if this map splits over subgroups of the image.

Nielsen realization for homotopy hyperbolic manifolds. For $M = N\#\Sigma$ as above, restricting to orientation-preserving diffeomorphisms gives a surjection $\mathrm{Diff}^+(M) \twoheadrightarrow \mathrm{Out}^+(\pi_1(M))$. In this case, M. Bustamante and I [BT23] completely solve the lifting problem when N is a hyperbolic 7-manifold (with a technical assumption) and Σ is a homotopy 7-sphere.

We also show that, among negatively curved manifolds M, the index of

$$\mathrm{Image}\big[\,\mathrm{Diff}^+(M) \to \mathrm{Out}^+(\pi_1(M))\big] < \mathrm{Out}^+\left(\pi_1(M)\right)$$

can be arbitrarily large, in contrast to the case $M = N\#\Sigma$ [BT22].

Homotopy tori and the Zimmer program. For $\mathcal{T}$ a homotopy n-torus (smooth manifold homotopy equivalent to T^n), there is a natural map

$$\mathrm{Diff}(\mathcal{T}) \to \mathrm{Out}\left(\pi_1(\mathcal{T})\right) \cong \mathrm{GL}_n(\mathbb{Z}).$$

This splits for $\mathcal{T} = T^n$, and conjecturally this is the only example, according to Fisher–Melnick [FM22], motivated by the Zimmer program. Bustamante–Krannich–Kupers and I [BKKT23] give examples $\mathcal{T} = T^n\#\Sigma$ for which the only homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Diff}^+(\mathcal{T})$ is the trivial map. As an application of the techniques, we can also give examples of Anosov actions $\mathbb{Z}^2 \curvearrowright T^n$ that cannot be homotoped to a smooth action on $T^n\#\Sigma$.

2. **Arithmetic groups, monodromy, and cohomology**

The study of arithmetic groups, e.g. $\mathrm{SL}_n(\mathbb{Z})$, is a classical topic connecting to many areas, and these groups appear in a variety of ways in my work.

Monodromy of holomorphic bundles. For a holomorphic S_g-bundle $E \to B$, Griffiths–Schmid [GS75] asked if its monodromy group $\Gamma_E < \mathrm{Sp}_{2g}(\mathbb{Z})$ is an arithmetic group. This is a subtle question, and both answers occur [DM86, Ven14]. Salter and I [ST20] answer this question for certain holomorphic S_g-bundles constructed by Atiyah–Kodaira. As a topological consequence, motivated by Thurston's theory of fibering 3-manifolds, we show that these bundles (whose total spaces are 4-manifolds) fiber in exactly two ways.

For a more general monodromy question, fix a cover $S' \to S$ of surfaces. Then there is a virtual representation $\mathrm{Mod}(S) \dashrightarrow \mathrm{Sp}(H_1(S'))$. When is the image arithmetic? Questions about orbits of this action on $H_1(S')$ are related to a well-known conjecture of Putman–Wieland [PW13]. My student Trent Lucas has completed a case study of the low-genus cases, showing the image is arithmetic whenever $\mathrm{genus}(S') \leq 3$ [Luc24].

Unstable cohomology. In studying bundles with fiber M, a fundamental problem is to compute the ring of characteristic classes $H^*(B\,\mathrm{Diff}(M))$. When $M_g^{2d} = \#_g(S^d \times S^d)$, this ring is known in a range $* \ll g$ (Mumford's conjecture) [GRW14, MW07, Mum83]. Little is known about $H^*(B\,\mathrm{Diff}(M_g))$ when $* \geq g$, although there have been recent breakthroughs [CGP18].

In [Tsh21], I produce new classes in $H^g(B\,\mathrm{Diff}'(M_g^{2d}))$ and for certain finite-index "congruence" subgroups $\mathrm{Diff}'(M_g) < \mathrm{Diff}(M_g)$, when $d \gg g$ is even. This result suggests an approach for finding new cohomology in finite-index subgroups of $\mathrm{Mod}(S_g)$ (i.e. $d = 1$), which I plan to explore in future work.

The cohomology produced above comes from the cohomology of arithmetic groups, specifically tori in locally symmetric spaces. D. Studenmund and I [ST22] compute lower bounds on how the subspace generated by these classes grows in congruence subgroups. For example, for the level-s principal congruence subgroup $\Gamma(s) < \mathrm{SL}_{n+1}(\mathbb{Z})$, we show

$$\dim H_n(\Gamma(p^\ell); \mathbb{Q}) \gtrsim |\,\mathrm{SL}_{n+1}(\mathbb{Z}) : \Gamma(p^\ell)|^{\frac{n+1}{n^2+2n}} \quad \text{for } p \text{ prime, } \ell \gg 0.$$

Arithmetic mapping tori. By a theorem of Margulis [Mar91], a lattice Γ in a semisimple Lie group is arithmetic if and only if Γ has infinite index in its commensurator. In contrast, no general arithmeticity characterization for lattices in *solvable* Lie groups is known. In [Tsh22] for solvable lattices of the form $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ with $A \in \mathrm{GL}_n(\mathbb{Z})$ hyperbolic and semisimple, I provide an arithmeticity criterion in terms of the eigenvalues of A. The case A is irreducible was solved by Grunewald–Platonov [GP98]. It would be interesting to prove an analogous theorem for groups of the form $\Gamma = \pi_1(S_g) \rtimes_\phi \mathbb{Z}$ with pseudo-Anosov monodromy $\phi \in \mathrm{Out}(\pi_1(S_g)) \cong \mathrm{Mod}(S_g)$.

3. Hyperbolic groups and aspherical manifolds

Wall and Cannon conjectures. In the classification of aspherical manifolds, the guiding conjectures are the Wall and Borel conjectures. These predict that for each finitely generated Poincaré duality group G, (Wall) there exists an aspherical manifold M with $\pi_1(M) \cong G$, and (Borel) M is unique up to homeomorphism. These conjectures are known to be true for many groups/manifolds coming from geometry.

For example, Bartels–Lück–Weinberger [BLW10] prove the Wall conjecture for hyperbolic groups with sphere boundary $\partial G \cong S^n$, $n \geq 5$.

Lafont and I [LT19] extend the result [BLW10] by proving a relative version: if a hyperbolic group has boundary a $(n-2)$-dimensional Sierpinski space, $n \geq 7$, then $G = \pi_1(M)$ where M is a compact aspherical manifold with aspherical boundary.

The Cannon conjecture, a version of Wall's conjecture in geometric group theory and low-dimensional topology, predicts that a torsion-free hyperbolic group G with boundary $\partial G \cong S^2$ the 2-sphere is the fundamental group of a closed hyperbolic 3-manifold. Work of Bestvina–Mess [BM91] shows that that $\partial G \cong S^2$ at least implies that G is a Poincaré duality group.

In connection to a relative version of Cannon's conjecture, G. Walsh and I [TW20] show that a relatively hyperbolic group $(G, \mathcal{P})$ is a 3-dimensional Poincaré duality pair if and only if its Bowditch boundary $\partial(G, \mathcal{P})$ is the 2-sphere.

Gromov's geometrization question for groups. Thurston's geometrization implies that a closed aspherical 3-manifold is hyperbolic if its fundamental group does not contain $\mathbb{Z}^2$. Gromov proposed a group-theoretical generalization: a group G (with a finite $K(G,1)$) that contains no Baumslag–Solitar subgroup is necessarily hyperbolic. Recently, a counterexample has been provided by Italiano–Martelli–Migliorini [IMM23] via a construction of hyperbolic 5-manifolds, but Gromov's conjecture might be correct for e.g. surface group extensions $1 \to \pi_1(S_g) \to \widetilde{G} \to G \to 1$.

For extension groups $\widetilde{G}$ as above, Gromov's conjecture specializes to a conjecture of Farb–Mosher that purely pseudo-Anosov subgroups $G < \mathrm{Mod}(S_g)$ are convex cocompact. This is known for many classes of examples G of geometric origin.

In [Tsh24], I prove the Farb–Mosher conjecture for the subgroups of the genus-2 Goeritz group, mapping classes that extend to the genus-2 Heegaard splitting of S^3. As a corollary of the proof, I give a characterization of pseudo-Anosov elements in the Goeritz group, analogous to Thurston's test for pseudo-Anosov's in the subgroup generated by Dehn twists T_a, T_b about a pair of filling curves.

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