Problem 1. *Let $G = (V, E)$ be a bipartite graph with maximum vertex degree Δ.*

(a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.

(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution. (a) Let Q be a vertex cover of G. If we cover any $v \in V$ we will cover a maximum of Δ edges, since Δ is the maximum vertex degree. Thus, at minimum, to cover all edges we need $|E|/\Delta$ vertices:

$$|Q| \geq |E|/\Delta$$

Now since G is bipartite, and M is a maximal matching

$$|M| \geq |E|/\Delta$$

by König's theorem.

(b) Let $G = (V, E)$ denote an arbitrary subgraph with of $K_{n,n}$ with $|E| > (k-1)n$ (Eq 1) and Δ denote the maximum vertex degree of G. Let $|M|$ denote some mathcing. By Eq 1 and (a) we write:

$$|M| \geq \frac{|E|}{\Delta} > \frac{(k-1)n}{\Delta}$$

Multplying by Δ/n:

$$|M|\frac{\Delta}{n} \geq \frac{|E|}{n} > (k-1)$$

Because G is $K_{n,n}$, $\Delta \leq n$ so

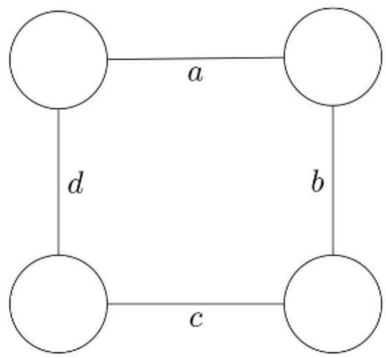
$$|M| \geq |M|\frac{\Delta}{n} \geq \frac{|E|}{n} > (k-1)$$

Now, since $k-1$ is an integer, $\frac{|E|}{n}$ is some fraction greater than or equal to this integer, and $|M|$ is an integer, we can say $|M| \geq k$

$\square$

Problem 2. *Fix $k \geq 2$, and let Q_k denote hypercube graph (from HW1). Prove that Q_k has at least $2^{2^{k-2}}$ perfect matchings.*

Solution. Prove this by induction.
Base case: k=2. We have a square, and it is clear that there are two perfect matchings:$\{a, c\}$ and $\{b, d\}$.



$\square$

Inductive step: suppose there are at least $2^{2^{k-2}}$ perfect matchings for $Q_k \Rightarrow$ there are at least $2^{2^{(k+1)-2}}$ matchings for Q_{k+1}. Recall that each vertex in Q_k is represented by some tuple $(v_1, \ldots, v_k)$ and each vertex in Q_{k+1} is either $(1, v_1, \ldots, v_k)$ or $(0, v_1, \ldots, v_k)$. Therefore, Q_{k+1} can be partitioned to two Q_ks: one consists of all tuples starting with '1' and the other starting with '0'. There are edges between two Q_ks. All vertices in Q_{k+1} are in one of the partition, so if each of Q_k has a perfect matching, then so does Q_{k+1} as all vertices are saturated. By the inductive hypothesis, each Q_k has at least $2^{2^{k-2}}$ perfect matchings, so there are at least $(2^{2^{k-2}})^2 = 2^{2^{k-1}}$ perfect matchings in Q_{k+1}. 'At least' because all those matchings do not involve edges between two Q_ks.
This holds for every k, so the statement holds.

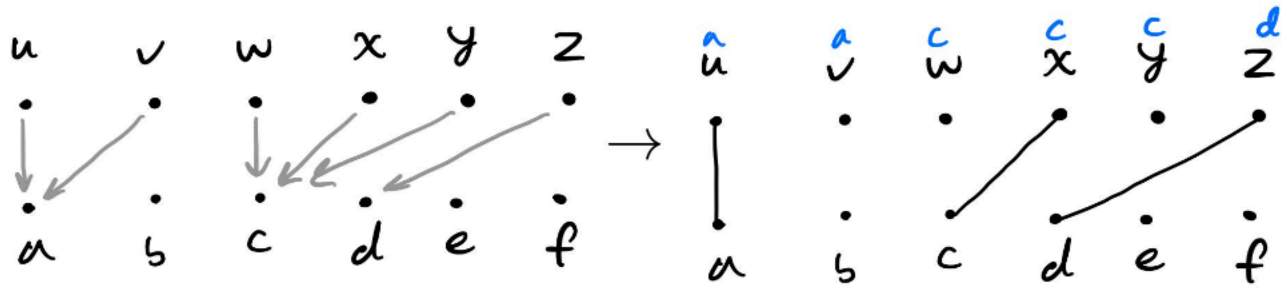
Problem 3. *Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.*

Cats $\{u, v, w, x, y, z\}$	Giraffes $\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

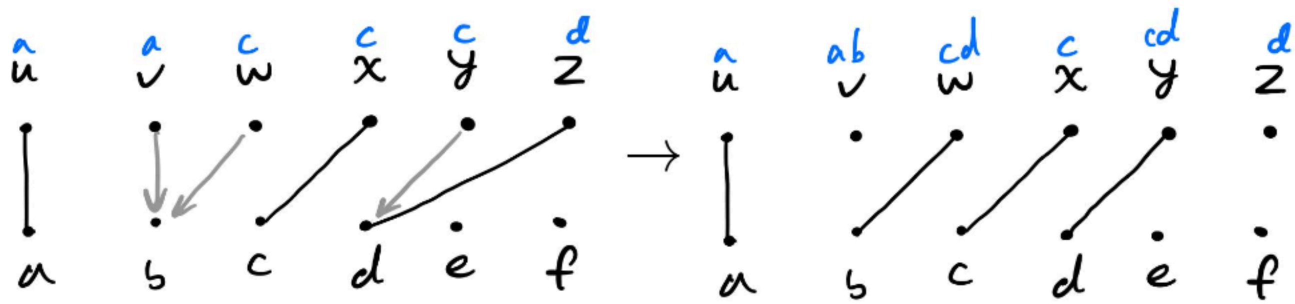
To receive full credit, you should show your work.

Solution. Note: As permitted by a post on Campuswire, my solution to this problem will exceed a page due to the size of the pictures involved. (I think it's still a page's worth of informational content.)

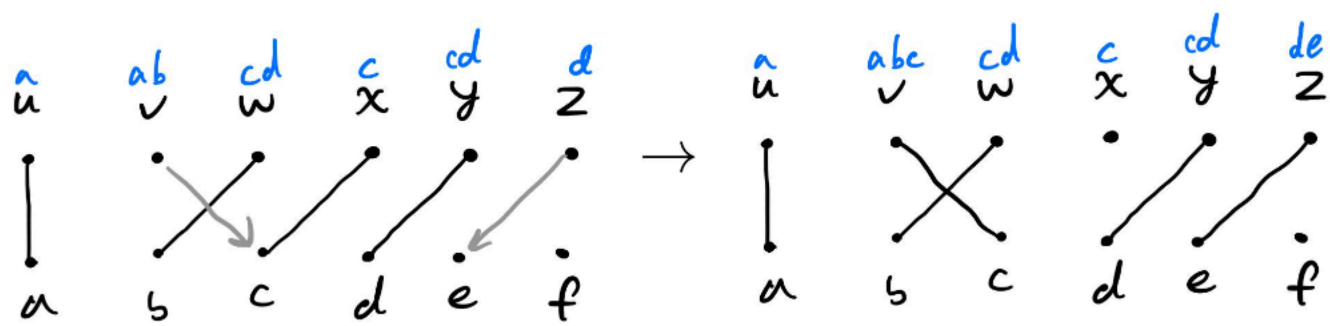
We begin with the cats proposing. The proposals in round 1 are shown on the left, and the resulting matchings on the right:



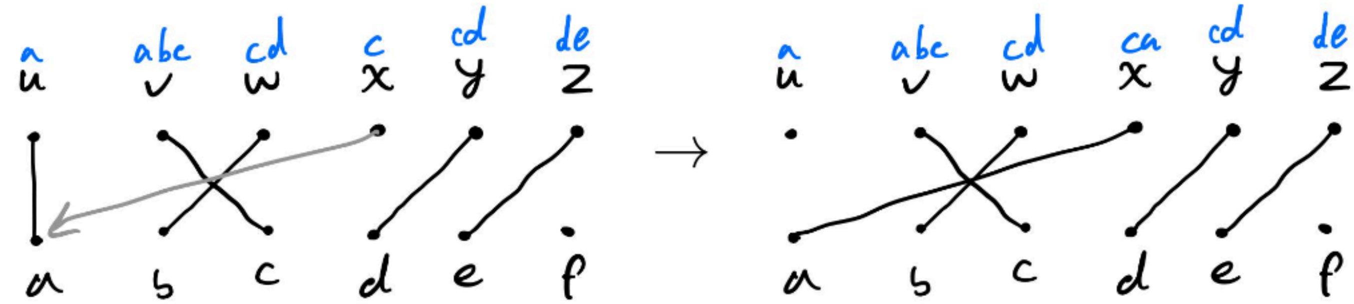
The new proposals for round 2 are shown in gray below (atop the existing matchings from round 1; note that prior proposals are listed above each proposer), with the results to the right:



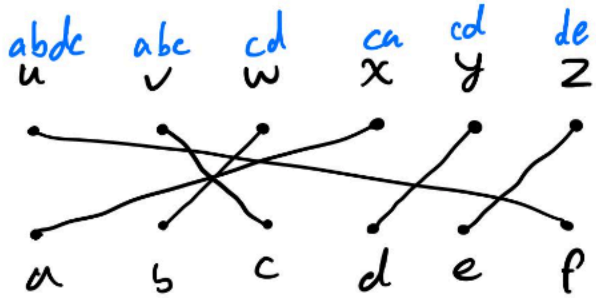
Round 3 goes as follows:



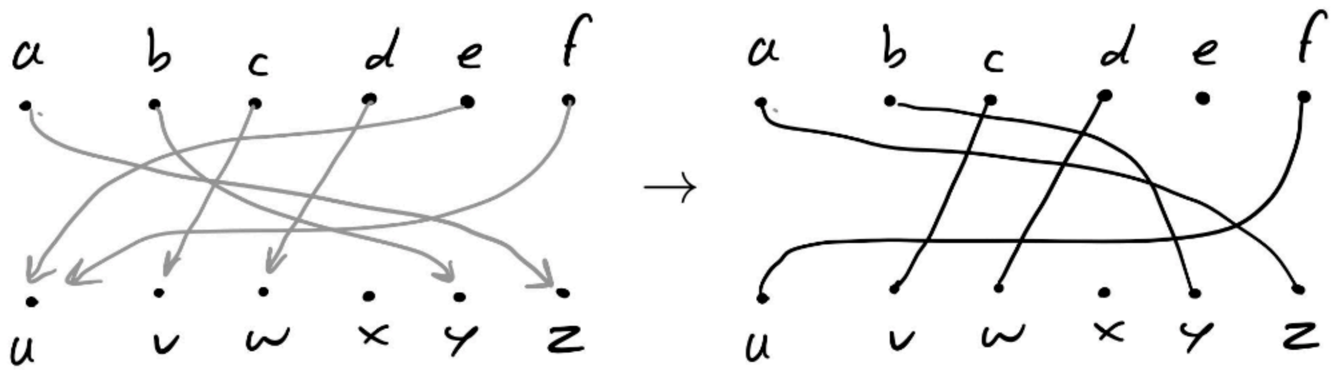
Then round 4:



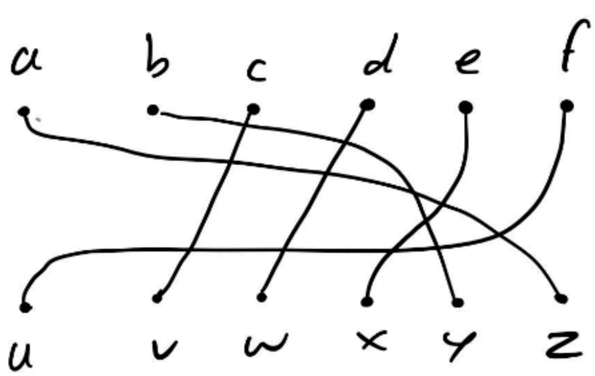
In round 5, u proposes to b and is rejected. In round 6, u proposes to d and is turned down again. In round 7, u proposes to c and is snubbed once more. Finally, in round 8, u proposes to f, who accepts, giving us the following final stable matching:



Now for the giraffes. (I don't bother to write down prior proposals this time because the process is much shorter.) Here's round 1:



In round 2, e proposes to v and is rejected. In round 3, e proposes to x and is accepted, giving the following final stable matching:



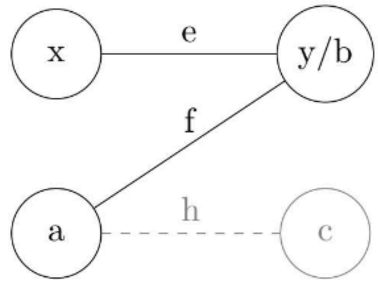
Problem 4. *Let $G = (X \sqcup Y, E)$ be a bipartite graph satisfying $|N(S)| > |S|$ for each nonempty $S \subset X$. Prove that every edge of G belongs to some matching that saturates X.*

Solution. Let $\{x, y\} \in E$. Consider the subgraph of G with x, y, and all edges incident to x and y removed, G'. Let $S \subset G'$ and let $N(S)$ be the neighbors of S in G and $N'(S)$ be the neighbors of S in G'. Note that we have only removed one vertex from Y so we have removed at most one neighbor of S from the set of neighbors in G. Thus, $|N'(S)| + 1 \geq |N(S)|$. Thus, as we know that $|N(S)| > |S|$, we have that $|N'(S)| + 1 > |S|$ so, as these are integers, $|N'(S)| \geq |S|$. Thus, by Hall's theorem, we know that there is a matching of G' that saturates $X \setminus \{x\}$. By the way we have constructed G', we know that this matching will not contain x or y. Thus, we know that we can take this matching in G and add $\{x, y\}$ to it such that we have a matching saturating X in G containing $\{x, y\}$. Thus, as our choice of edge was arbitrary, we know that for any edge in E, we can find a matching that saturates X containing it. $\square$

Problem 5. *Complete the proof of König's theorem that we started in class.*

Solution. We will consider each case in turn:[1]

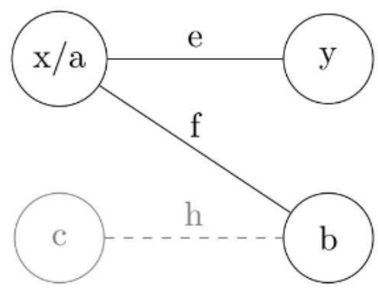
$b = y$:



(i) If there is an M-alternating path starting from an unsaturated vertex of X and passing through e, $y \in Q$ by construction, so $b \in Q$, so f is covered by Q.

(ii) If there is *not* an M-alternating path starting from an unsaturated vertex of X and passing through e, we know that a must be saturated (otherwise we would in fact have such a path). Thus, there is some edge $h = \{a, c\} \in M$. We claim there is not an M-alternating path starting from an unsaturated vertex of X and passing through h. If there were, it would start with an edge not in M going from a vertex in X to a vertex in Y, and every edge in M would be used going from Y to X, so it would go through h by going first to c and then a, so the path could then be extended by adding f and then e to create an alternating path from an unsaturated vertex in X passing through e, which contradicts our assumption for this case. Thus, h is an edge in M that is not part of such a path, so $a \in Q$ by construction, so f is covered by Q.

$a = x$:



(i) If there is an M-alternating path starting from an unsaturated vertex of X and passing through e, we know that b must be saturated by M (otherwise we have an M-augmenting path by adding f to the alternating path, which is guaranteed to end at x/a for reasons similar to the ones discussed above). Thus, there is some edge $h = \{c, b\} \in M$. Note that by adding f and h to the M-alternating path from an unsaturated vertex of X passing through e, we get such a path passing through h, so $b \in Q$ by construction. Thus, f is covered by Q.

(ii) If there is *not* an M-alternating path starting from an unsaturated vertex of X and passing through e, then $x \in Q$ by construction, so $a \in Q$, so f is covered by Q.

We have shown that all edges are covered by Q, so Q is in fact a vertex cover, and we have proven König's theorem. $\square$

[1] See Campuswire post #63 for the beginning of the proof being assumed in this problem.

Problem 6. *A deck with mn cards with m values and n suits consists of one card for each value in each suit. The cards are dealt into an $n \times m$ array. Prove that there is a set of m cards, one in each column, having distinct values.*

Solution. We can create a bipartite graph $G = (X \sqcup Y, E)$, with each vertex in X representing a value, and each vertex in Y representing a column. For every card, create an edge between it's value and the column it's in and allow for multiple edges between vertices. Because there n rows, each column must have n cards and therefore it's corresponding vertex must have a degree of n. Likewise, because there are n cards for every value, each value vertex must have a degree of n. So we constructed a n-regular bipartite graph and by Hall's Theorem, there must be a perfect matching. This perfect matching is a set of m cards each in a unique column. $\square$