The game of cat and mouse

Bena Tshishiku

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Variation. Game begins by cat and mouse each choosing their starting position.

Translation to graph theory

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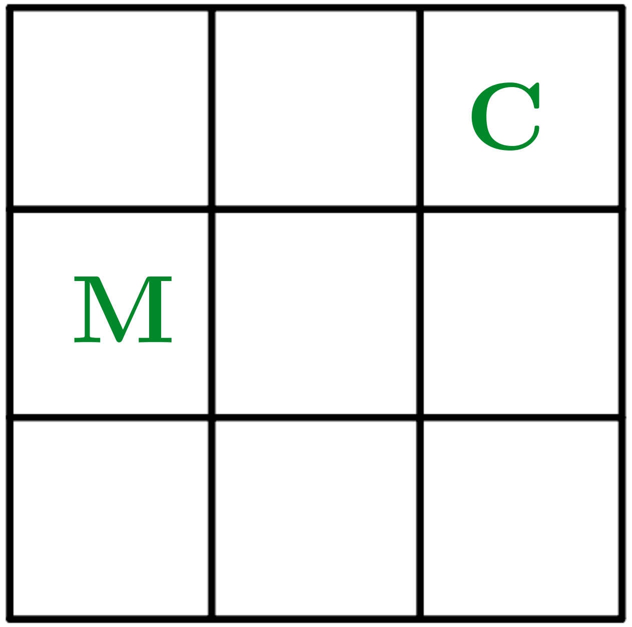

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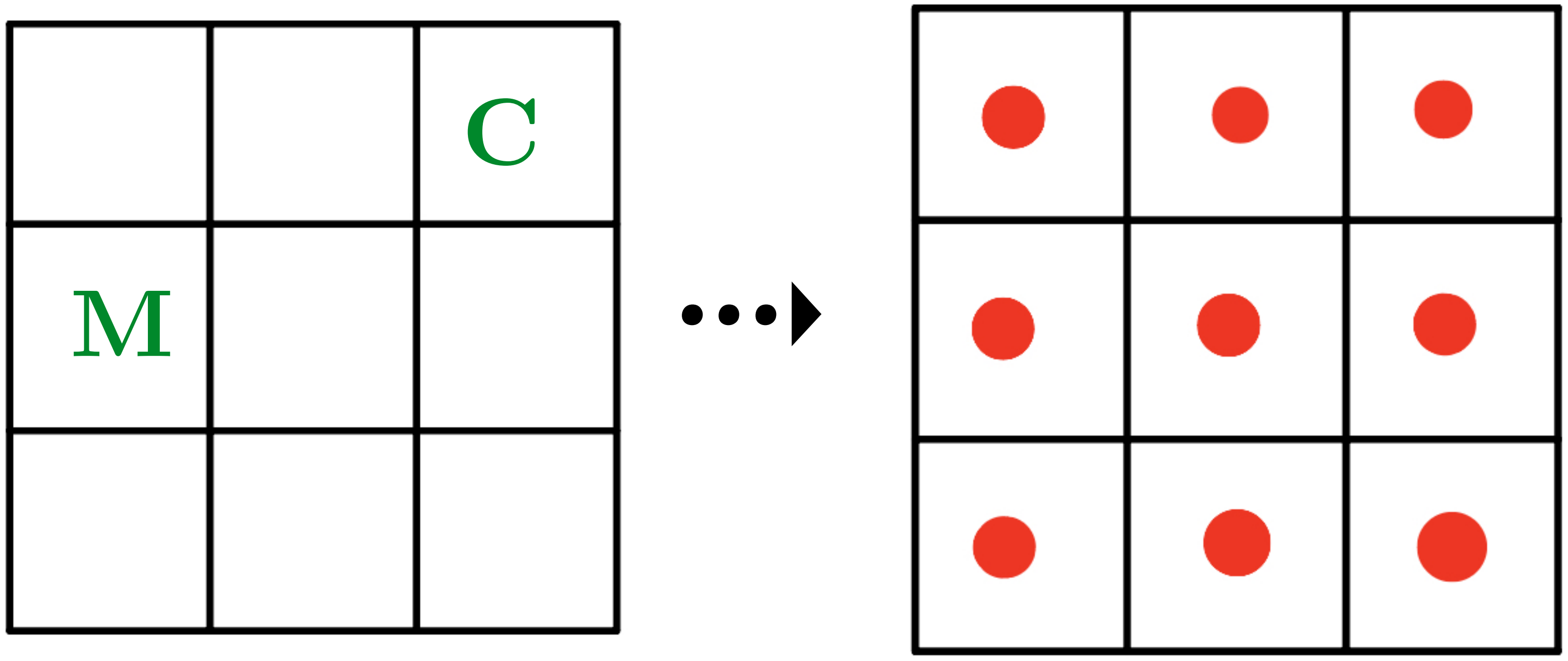

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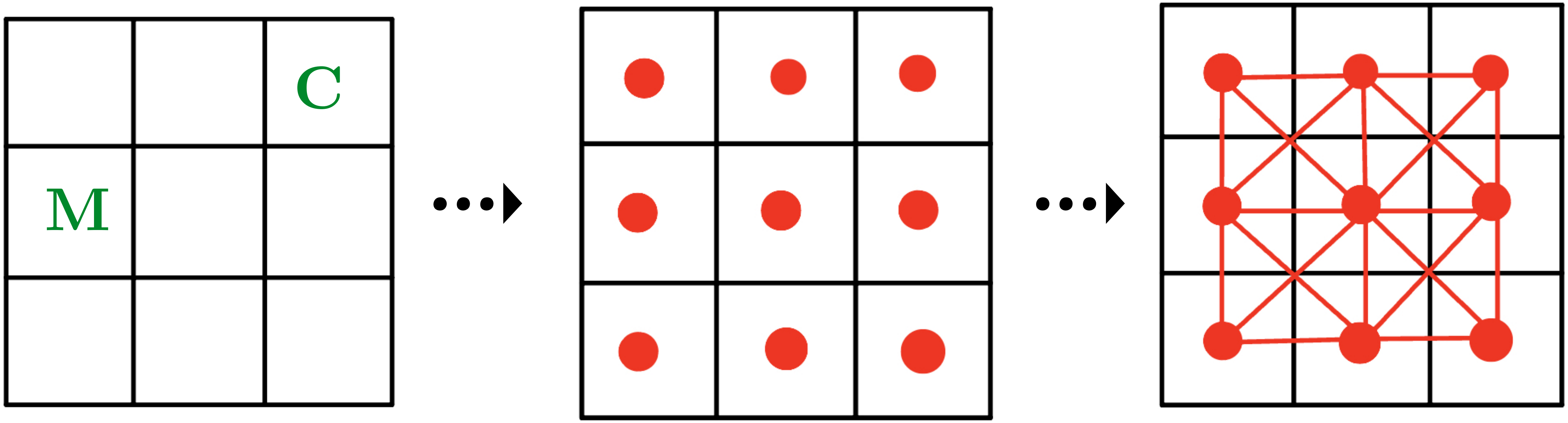

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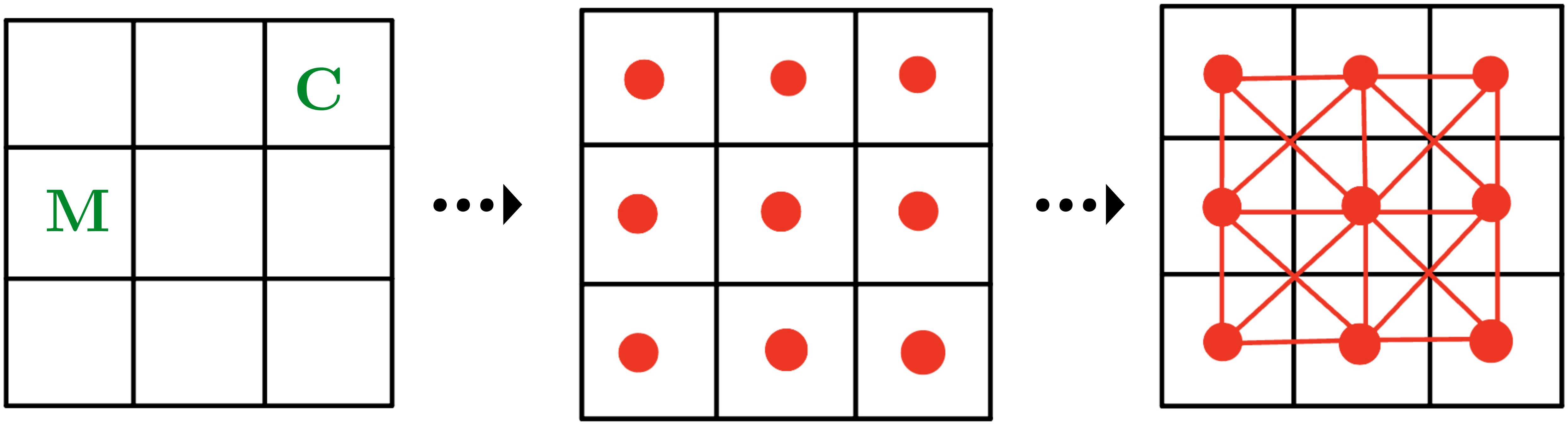
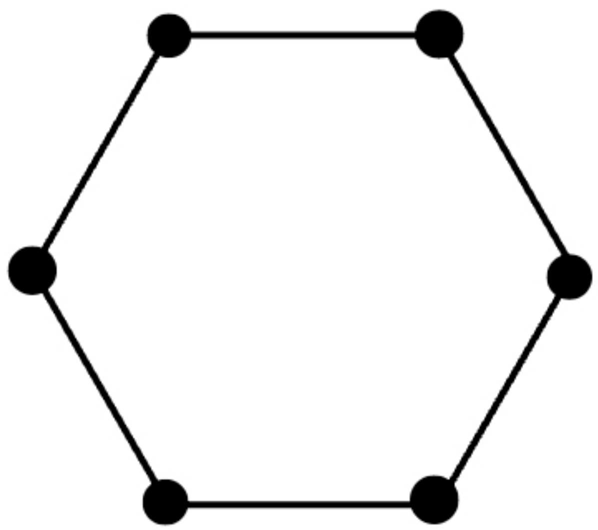


Cat and mouse play on vertices of the graph, moving to adjacent vertices.

Question. For these graphs who has a winning strategy (cat or mouse)?

(A)



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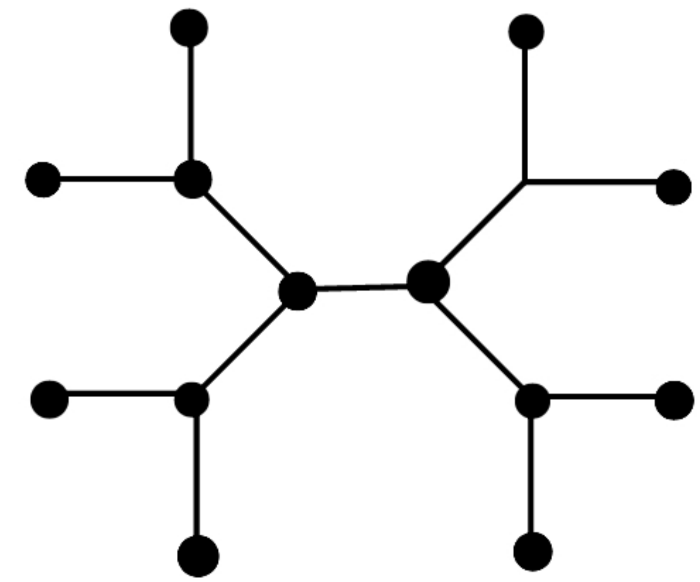

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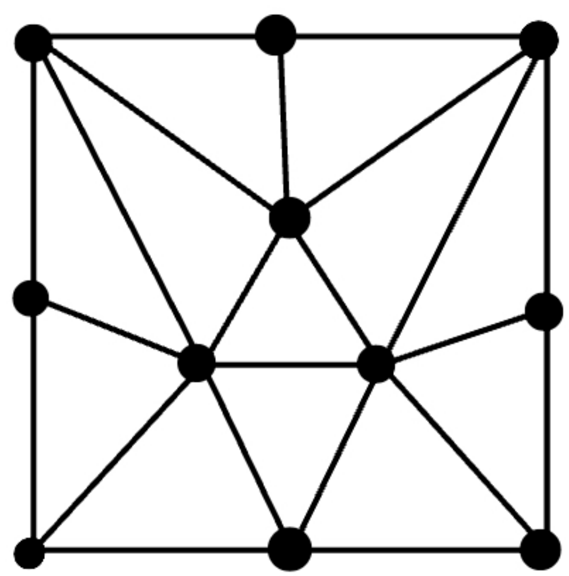

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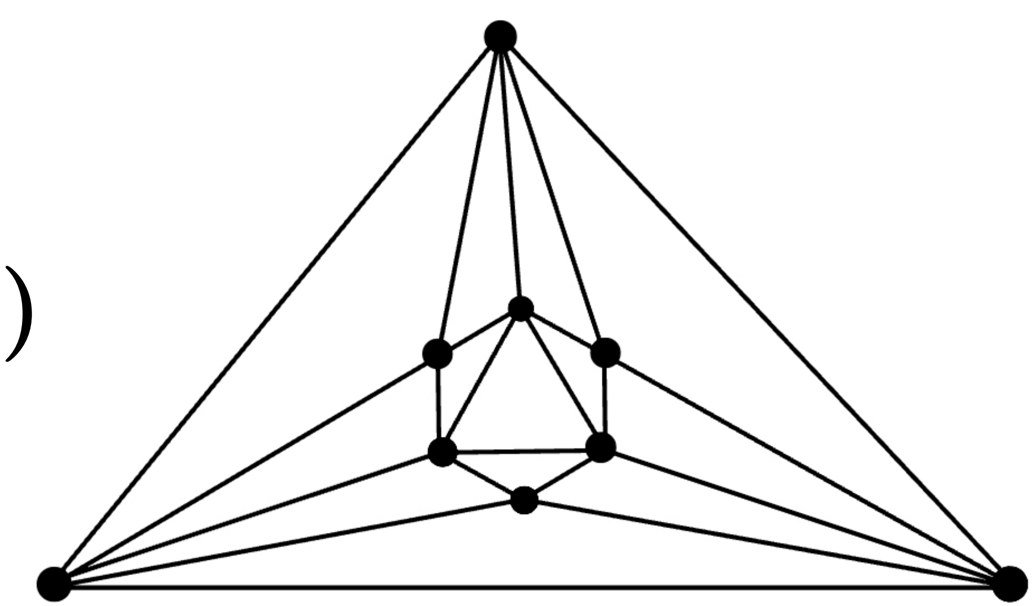

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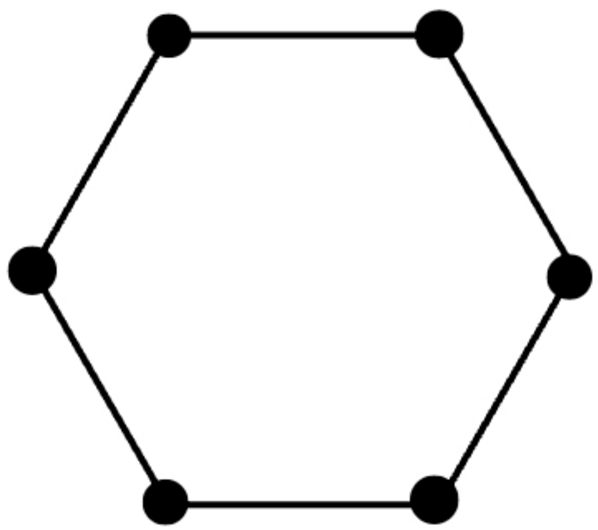

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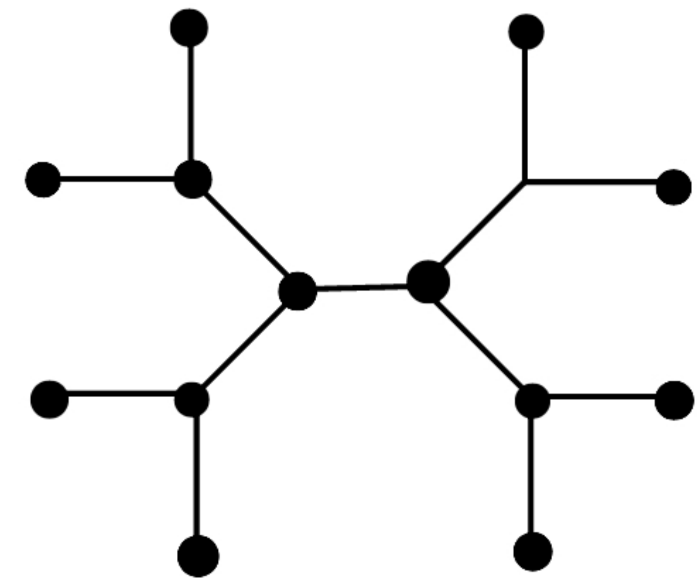

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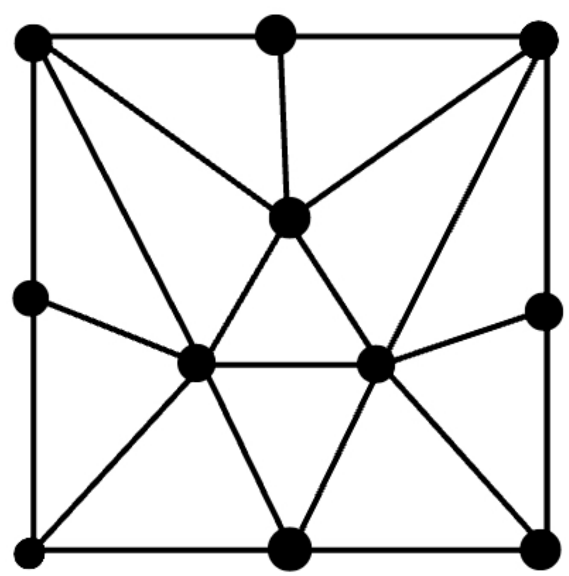

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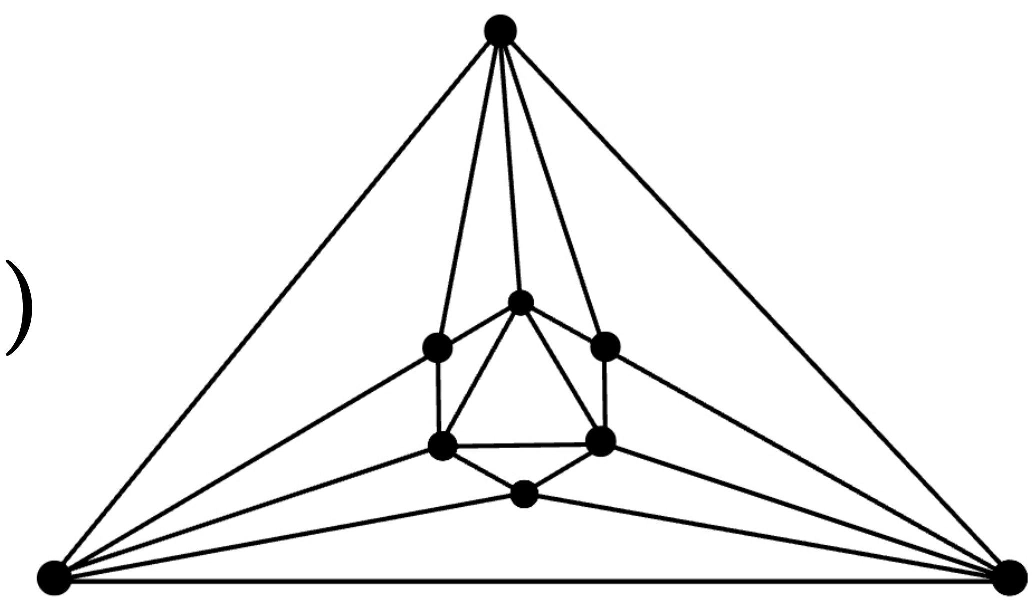

How generally to tell if a graph is cat win?

(Combinatorial) Game Theory

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Game theory: mathematical theory that analyzes strategy and decision making.

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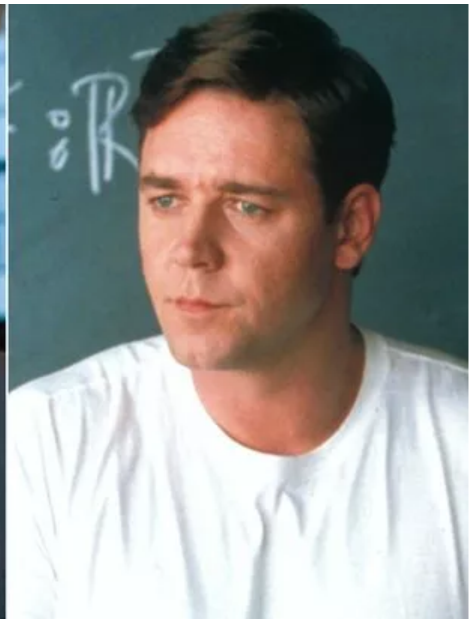

John Nash Russell Crowe

Zermelo's theorem. In any* finite 2-player game without chance (e.g. chess, nim, cat-mouse) one player has a winning strategy.

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Proof is non-constructive!

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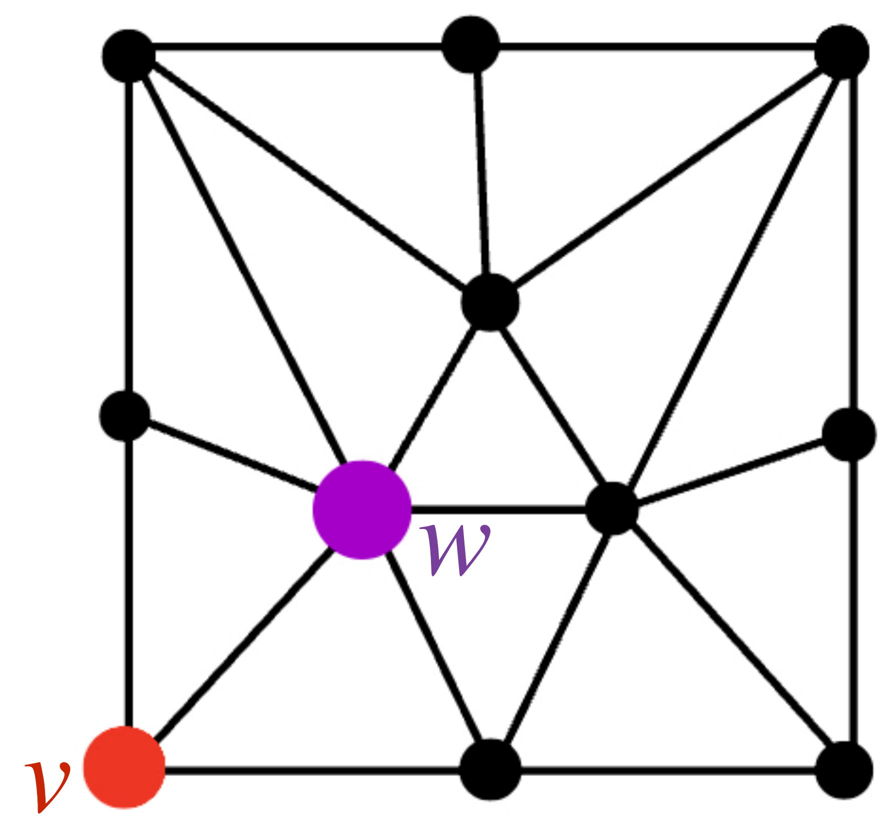

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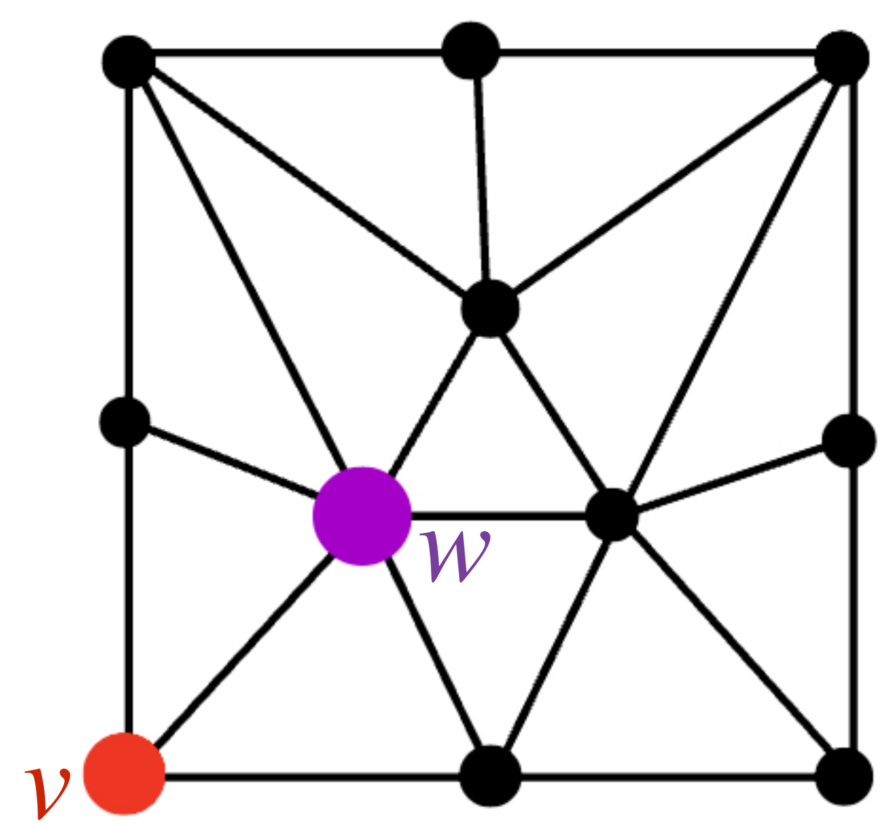
Remark. Having a dominated vertex is a *local property* of a graph.



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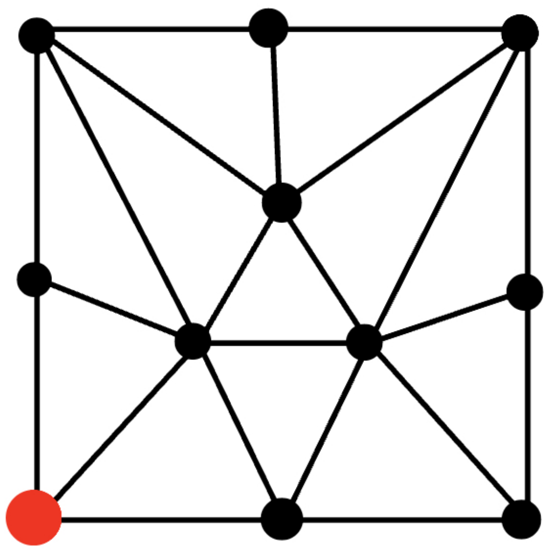

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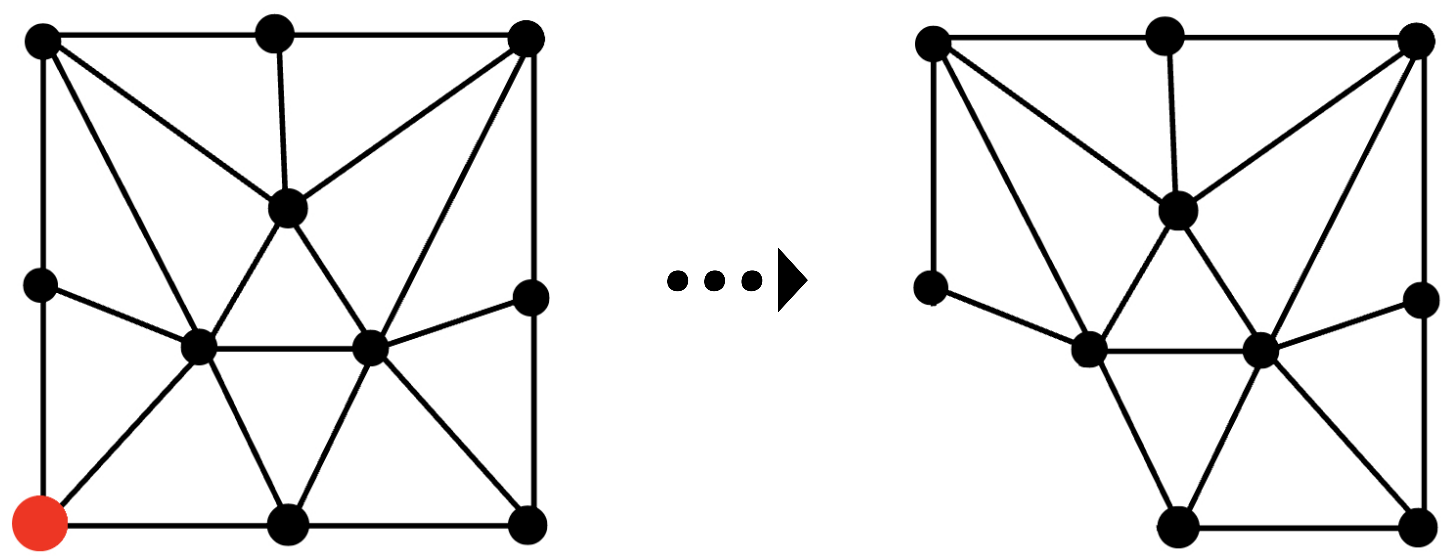

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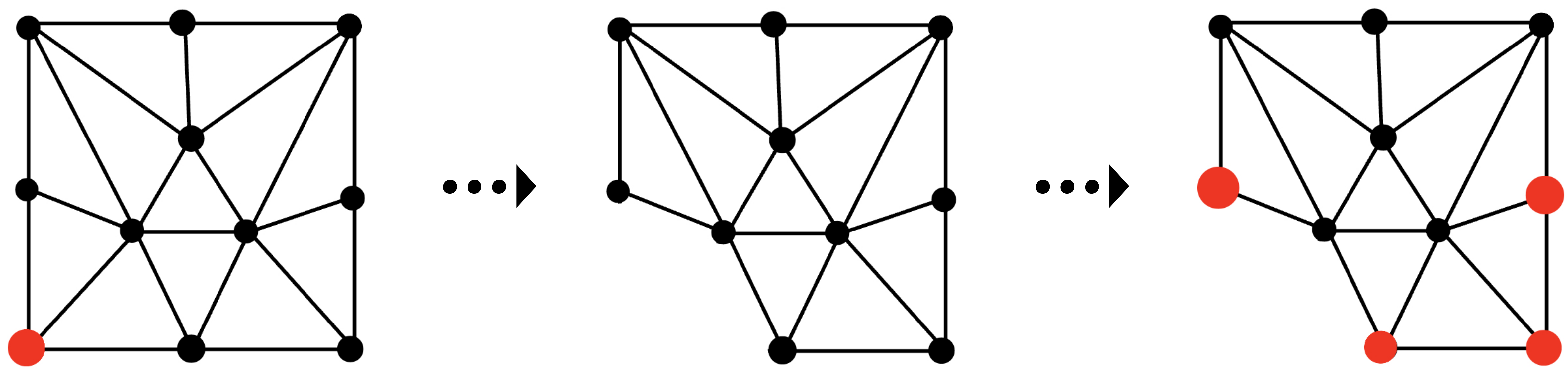

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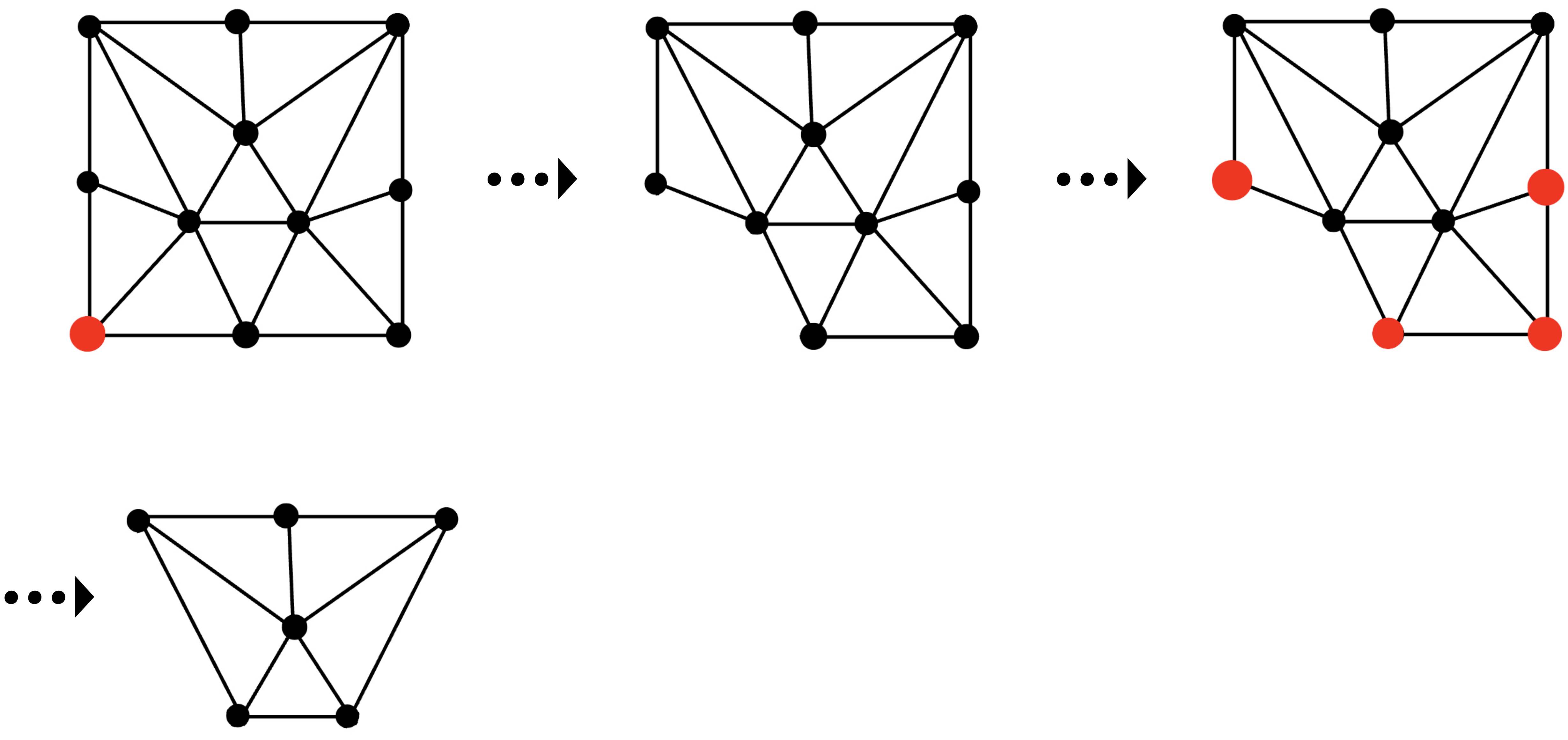

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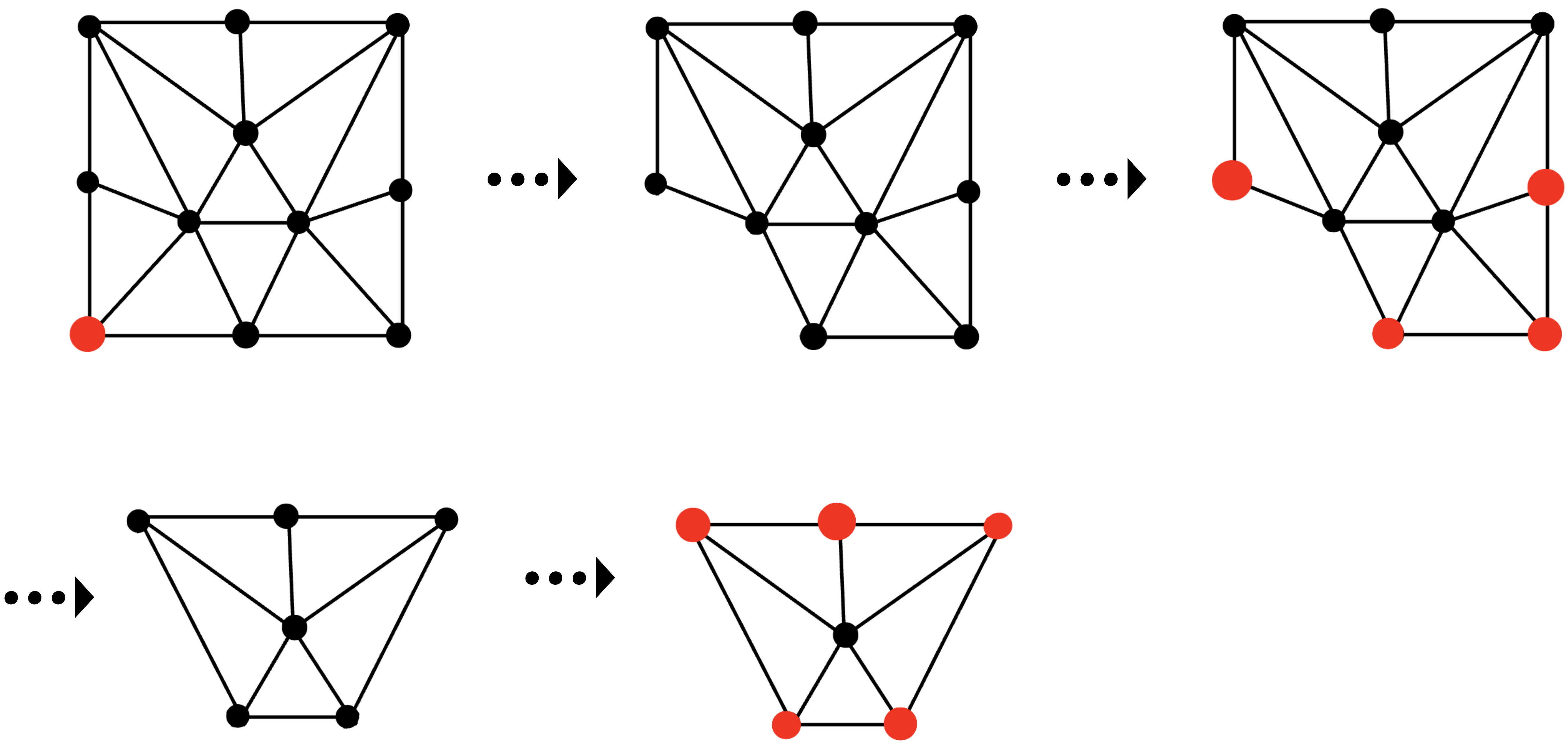

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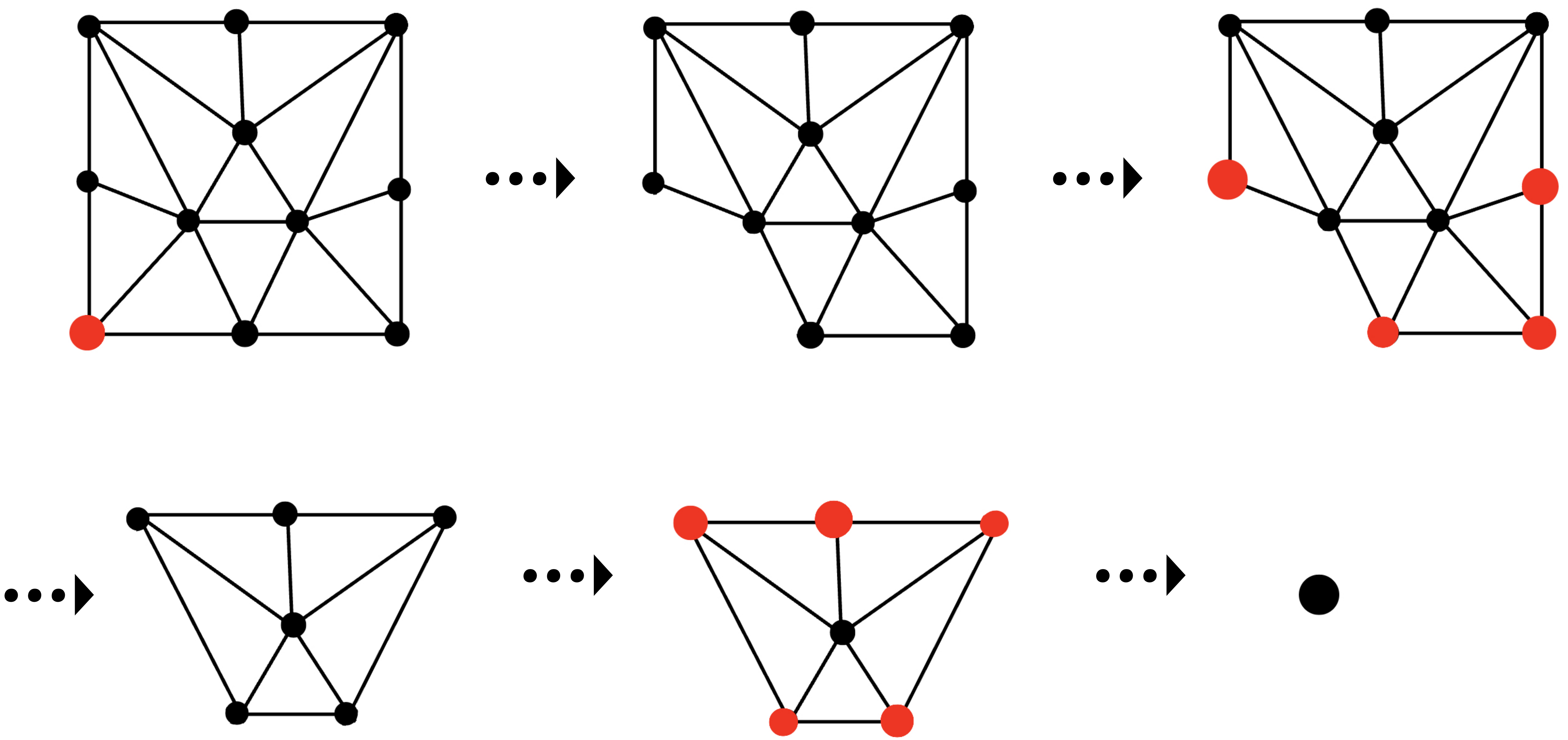

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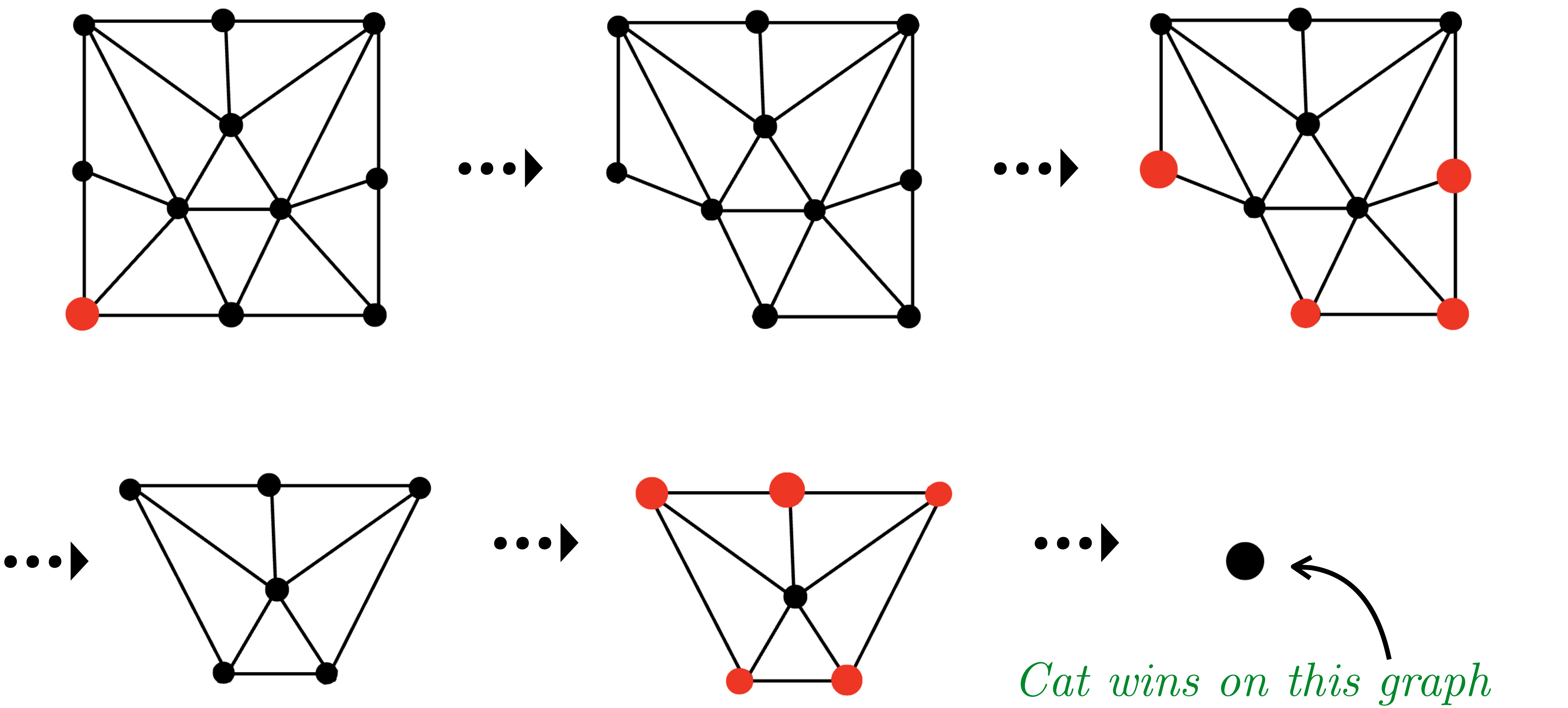

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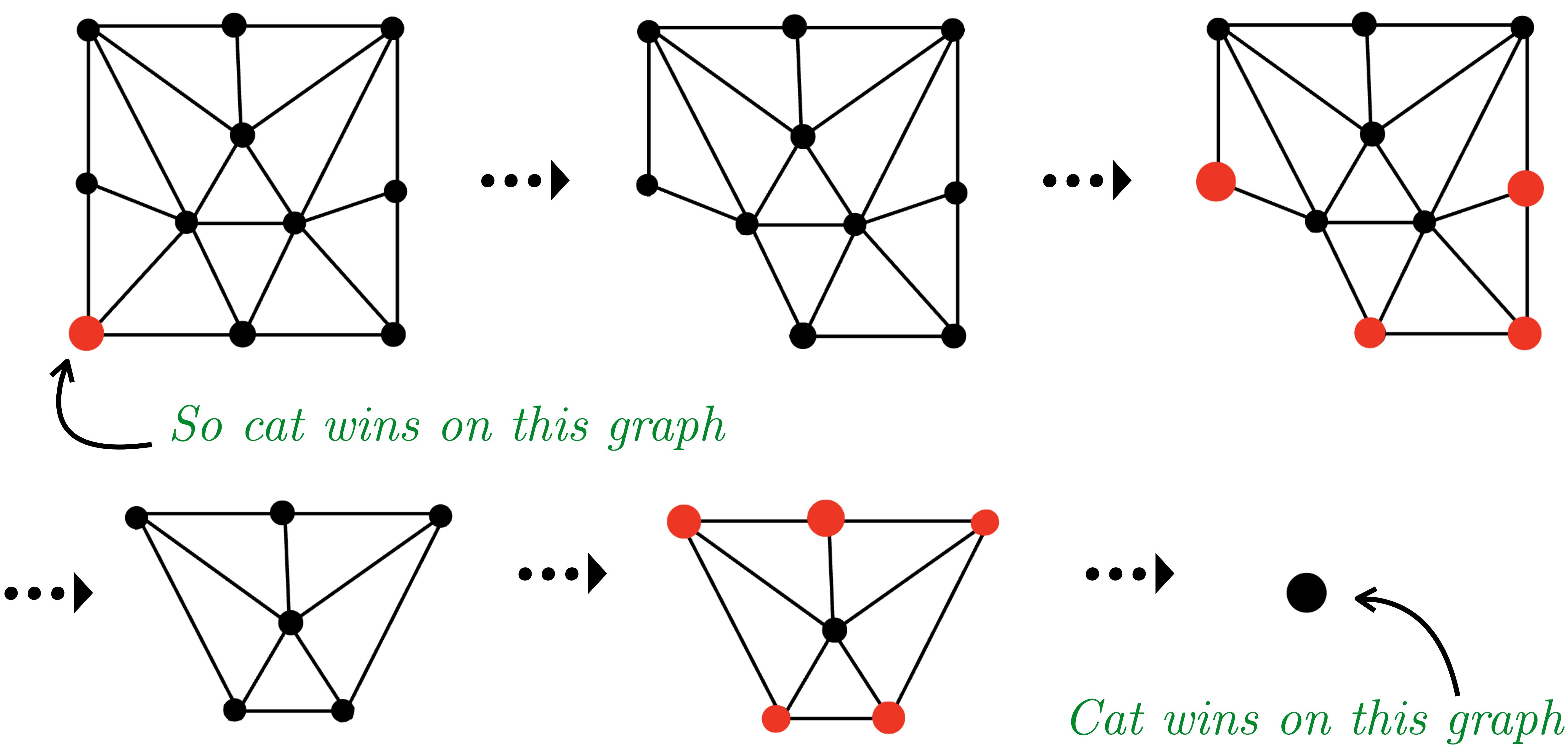

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Cat has winning strategy on G if and only if can order vertices $v_1, \ldots, v_n$ so that v_i is dominated by one of $v_{i+1}, \ldots, v_n$ for each $i = 1, \ldots, n-1$.

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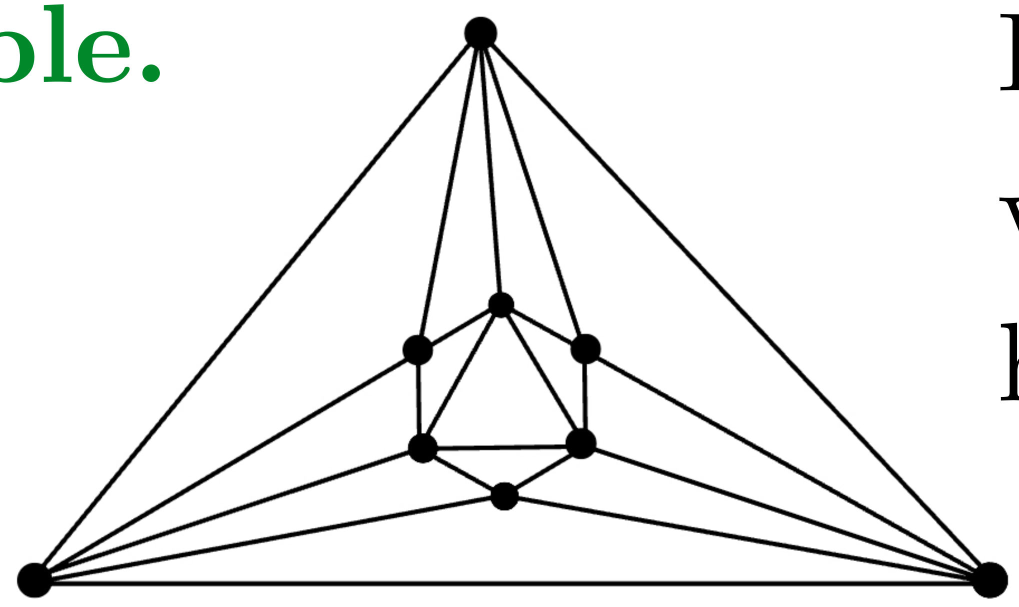
Example.



If G has no dominated vertex, then the mouse has winning strategy.

Cat number and a conjecture

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Variation: More cats!

How many cats are needed to ensure the cats win?

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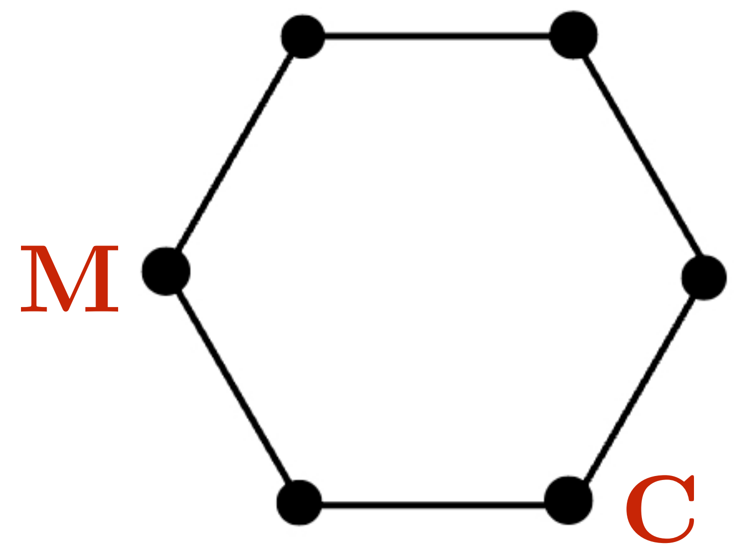

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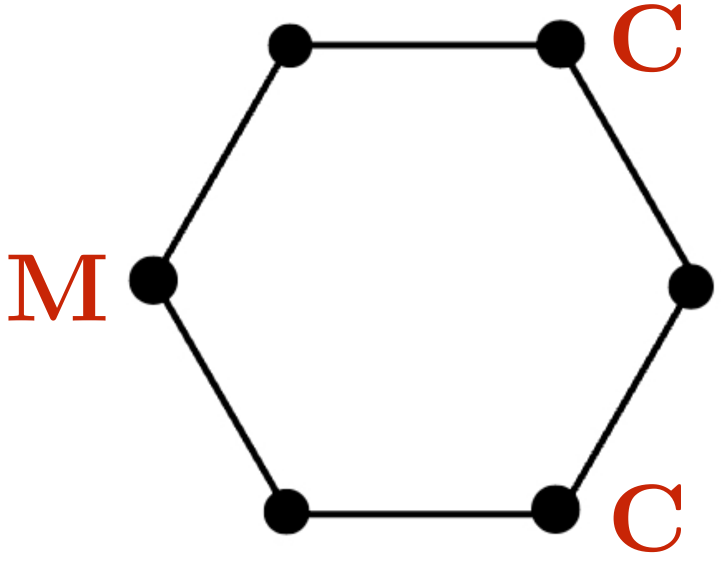

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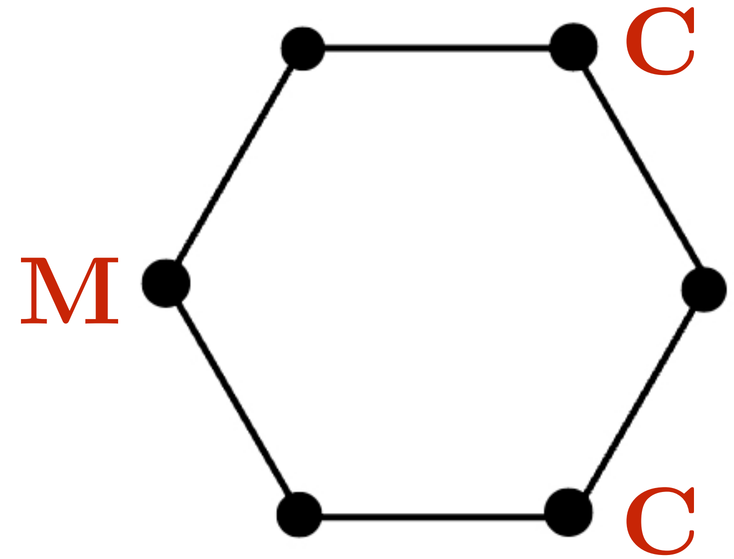

Conjecture (Meyniel, 1985). In a graph with n vertices, don't need more than $\sqrt{n}$ cats to catch a mouse.

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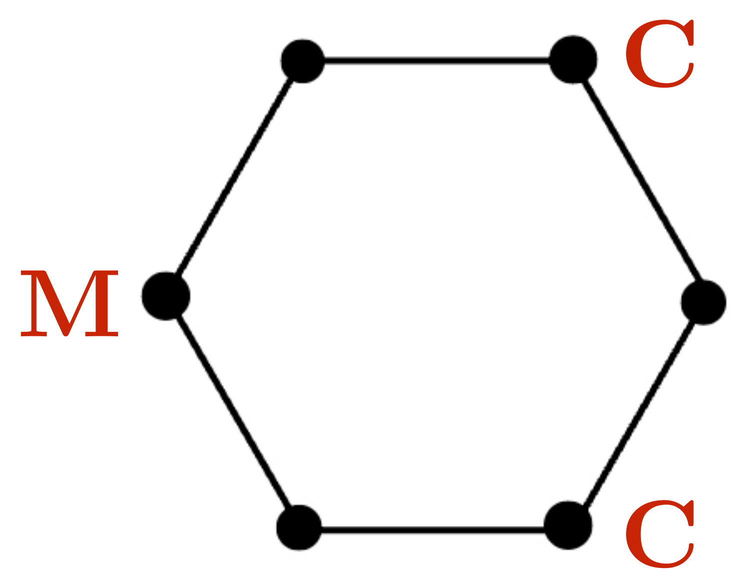

Conjecture (Meyniel, 1985). In a graph with n vertices, don't need more than $\sqrt{n}$ cats to catch a mouse. *More precisely the maximum cat number among n-vertex graphs is $O(\sqrt{n})$.*


Thank you