Homework 7

Math 123

Due March 16, 2023 by midnight

Name:

Topics covered: Graph coloring, chromatic polynomial, Turan graphs

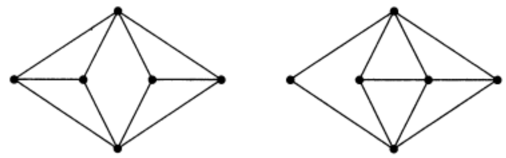
Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Give an example or explain why no example exists: A graph G that is neither complete nor an odd cycle, but for which the greedy coloring uses $\Delta(G) + 1$ colors.*

Solution. We give an example: Let P_4 be the path with vertices (v_1, v_2, v_3, v_4), where v_1, v_4 have degree 1. Consider the greedy coloring where we order the vertices v_1, v_4, v_3, v_2. This uses $3 = \Delta(G) + 1$ colors. $\square$

Problem 2. *Give a very short proof that the following two graphs have the same chromatic number.*[1]



Solution. In fact these graphs have the same chromatic polynomials by the edge deletion-contraction formula. Apply the edge-contraction deletion formula to the left-most horizontal edge in each graph. The graphs obtained by deleting an edge are isomorphic; the same statement holds for the contractions. Thus the two graphs have the same chromatic polynomial. In particular, they have the same chromatic number. □

[1] Note: solutions that construct optimal colorings of these graphs will not receive credit.

Problem 3. *Let $G = M_{n_1,\ldots,n_k}$ be a complete k-partite graph with $n = n_1 + \cdots + n_r$ vertices. Show that if $n_i - n_j \geq 2$ for some i, j, then there exists a k-partite graph with n vertices and more edges than G.*

Solution. Move one vertex v from the i-th group to the j-th group to get a new r-partite graph G'. All the vertex degrees are unchanged, except the degrees in the i-th and j-th group. In G', the i-th group has $n_i - 1$ vertices each with degree $n - n_i + 1$, and the j-th group has $n_j + 1$ vertices each with degree $n - n_j - 1$. Then in total the degree sum for vertices in G' in the i-th and j-th groups is

$$(n_i - 1)(n - n_i + 1) + (n_j + 1)(n - n_j - 1) = \big[n_i(n - n_i) + n_j(n - n_j)\big] + 2(n_i - n_j - 1)$$

The summand in brackets is the total degree sum from the i-th and j-th groups in G. The term $2(n_i - n_j - 1)$ is positive since $n_i - n_j \geq 2$. This shows the degree sum of G' is larger than the degree sum of G, so G' has more edges than G. $\square$

Problem 4. *Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.* [2] [3]

Solution. After perturbing the line arrangement, we can assume that no line is parallel to the x-axis. Order the intersections $v_1, \ldots, v_n$ by decreasing height (two points may have the same height, but we can order vertices of the same height arbitrarily). Now we apply a greedy coloring: color the vertices in order with the first available color.

We claim this is a 3-coloring. The only way we need more than three colors is if there is a vertex v and (at least) three vertices below v that are connected to v by an edge. But this means that there are three lines that meet at v, contradicting our assumption. Therefore, the greedy coloring uses at most 3 colors. □

[2] Suggestion: start by looking at some explicit examples.
[3] Hint: use a greedy coloring with an appropriate vertex ordering.

Problem 5. *Let G be a graph with chromatic number k. Show that for every k-coloring of G and for each color i, there is a vertex of color i that is adjacent to vertices of the other $k-1$ colors.* [4]

Solution. We prove the contrapositive: Suppose there is a k-coloring of G and a color i such that for each vertex v of color i, there is a color $j(v)$ not adjacent to v. We want to show $\chi(G) < k$. Change the color of v from i to $j(v)$. Our assumptions imply that this is still a coloring, but now without using color i, so $\chi(G) < k$ as desired. □

[4] Hint: think back to the proof that a graph with chromatic number k has at least $\binom{k}{2}$ edges.

Problem 6 (West 5.1.38)**.** *Prove that $\chi(G) = \omega(G)$ when the complement $\bar{G}$ is bipartite.* [5] [6] [7]

Solution. Observation 1: If $\bar{G}$ is bipartite, then for any coloring of G, for each color i, there are at most two vertices with color i. Given a coloring of G, we get a labeling of the vertices of $\bar{G}$ with the property that vertices not connected by an edge have different colors. Then if $\bar{G}$ has a bipartition $V = \sqcup Y$, then none of the X vertices have the same color, and same for Y, so there are at most two vertices with each color.

Observation 2: A k-coloring of G gives a matching of $\bar{G}$ of size $|V| - k$. To see this, define a matching by matching an X vertex to a Y vertex of the same color, if it exists. Among the $|V|$ vertices, if there are k colors each with multiplicity 1 or 2, then there are $|V| - k$ pairs of vertices with the same color (pigeonhole principle).

Observation 3: A matching of $\bar{G}$ of size m gives a coloring of G with $|V| - m$ colors. Given a matching of $\bar{G}$, we color the vertices giving each pair of matched vertices a unique color and giving the remaining vertices new colors as well to get a coloring of G (in this coloring vertices have the same color only if they are connected by an edge in $\bar{G}$, so they are not connected by an edge in G). A matching of size m gives a coloring with $|V| - m$ colors.

Combining Observations 2 and 3 (and their proofs) we get a bijection between colorings of G and matchings of $\bar{G}$. From this we conclude that

$$\chi(G) = |V| - \max\{|M| : M \text{ matching of } \bar{G}\}.$$

Observation 4: If $H \cong K_r \subset G$, then $V \setminus V(H)$ is a vertex cover of $\bar{G}$. The fact that H is a clique in G implies that there are no edges between $V(H)$ in $\bar{G}$. This implies that $V \setminus V(H)$ is a vertex cover of $\bar{G}$.

Observation 5: If Q is a vertex cover of $\bar{G}$, then $V \setminus Q$ spans a clique in G. If Q is a vertex cover of $\bar{G}$, then there are no edges between vertices of $V \setminus Q$ in $\bar{G}$, so these vertices span a clique in G.

Combining Observations 4 and 5, we obtain

$$|V| - \min\{|Q| : Q \text{ vertex cover of } \bar{G}\} = \omega(G).$$

We finish by applying König's theorem with the preceding observations:

$$\begin{aligned} \chi(G) &= |V| - \max\{|M| : M \text{ matching of } \bar{G}\} \\ &= |V| - \min\{|Q| : Q \text{ vertex cover}\} \\ &= \omega(G) \end{aligned}$$

□

[5] Here $\omega(G)$ is the clique number: the largest m so that G contains K_m.

[6] Hint: look to apply König's theorem. (!)

[7] This is a pretty challenging problem. If you want more hints, please ask.

Problem 7 (Bonus)**.** *Let $G = (V, E)$ be the unit distance graph in the plane: $V = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. (a) Use the hexagonal tiling to prove $\chi(G) \leq 7$. (b) Prove that $\chi(G) \geq 3$.*

Solution. □