Homework 2

Math 123

Due February 10, 2023 by 5pm

Name:

Topics covered: bipartite graphs, Euler tours, degree sum formula, trees

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems. You must type your solutions alone.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!

Problem 1. *The complete bipartite graph $K_{n,m}$ is the graph with $n+m$ vertices $v_1, \dots, v_n$ and $u_1, \dots, u_m$ and edges $\{v_i, u_j\}$ for each $1 \le i \le n$ and $1 \le j \le m$. Determine the values n, m so that $K_{n,m}$ is Eulerian.*

Solution. In $K_{n,m}$ every vertex has degree either n or m. So $K_{n,m}$ is Eulerian if and only if n, m are both even. □

Problem 2. *Prove or disprove:*

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian graph with an even number of vertices has an even number of edges.

Solution.

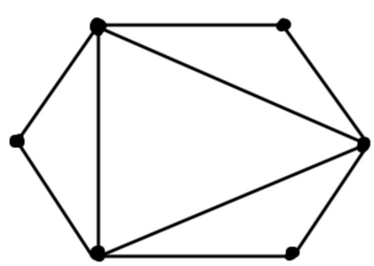
(a) This is true. Suppose $G = (V, E)$ is Eulerian and bipartite. Write $V = A \cup B$ for the bipartition. Observe that

$$\sum_{v \in A} \deg(v) = |E|$$

since every edge is incident to one vertex of A and one vertex of B. Since G is Eulerian, $\deg(v)$ is even for each $v \in A$. This shows $|E|$ is even.

Alternate proof (learned from Sreshtaa): Observe that a closed walk in a bipartite graph visits an even number of edges (on level of vertices, it goes red, blue, red etc). An Eulerian tour is a closed walk that visits $|E|$ edges. Combining this with the observation we conclude that in an Eulerian graph $|E|$ is even.

(b) This is false. The following graph is a counterexample. Each vertex has degree 2 or 4 (so it's Eulerian) and it has an even number of vertices (6), but it has an odd number of edges (9).



□

Problem 3. *Prove that every tree has at least two vertices of degree 1. Classify trees with exactly two vertices of degree 1.* [1]

Solution. From class we know a tree with n vertices has $n-1$ edges. Then by the degree-sum formula

$$\sum_{v \in V} deg(v) = 2|E| = 2(|V| - 1) = 2|V| - 2.$$

[1]Useful fact: a tree with n vertices has $n-1$ edges. We will show this in class next week.

There are $|V|$ terms in the sum, and each term is ≥ 1 (no vertex has degree 0 because trees are connected). From this it follows that there must be at least two vertices of degree 1.

We claim a tree with two vertices of degree 1 is isomorphic to P_n. We prove this by induction on the number of vertices. (It is helpful to note that, by the degree-sum formula, if T has two vertices of degree 1, then all the other vertices have degree 2.)

Base case: For $n = 2$, observe that there is a unique tree with 2 vertices, namely P_2 and it does indeed have 2 vertices of degree 1.

Induction step: Assume T has $n \geq 3$ vertices and let u, w be the vertices of degree 1. Consider the unique edge $\{u, v\}$ incident to u. Note that $v \neq w$ since otherwise T would be P_2. Then v has degree 2 and $T \setminus u$ is a graph with $n - 1$ vertices and exactly two vertices of degree 1 (they are v, w). By the induction hypothesis, $T \setminus u = P_{n-1}$, and T is obtained from P_{n-1} by adding an edge to one of the vertices of degree 1, which shows $T = P_n$. □

Problem 4. *Determine the number of graphs with* 7*-vertices, each of degree* 4 *(up to isomorphism).* [2]

Solution. Let G be a 7-vertex, 4-regular graph. Consider the complement $\bar{G}$. This is a 7-vertex, 2-regular graph.

Claim. A graph H where every vertex has degree 2 is a union of cycles.

Since a cycle has at least 3 vertices, this implies that a 7-vertex, 2-regular graph H is isomorphic to one of C_7 or $C_4 \sqcup C_3$. This shows that there are two 7-vertex, 4-regular graphs.

Proof of Claim: It suffices to show that a connected graph where every vertex has degree 2 is isomorphic to C_n, where n is the number of vertices. To see this, let H be a connected 2-regular graph. Removing any edge e leaves a connected graph with exactly two vertices of degree 1. By the previous problem, $H \setminus e$ is isomorphic to P_n, so H is obtained from P_n by adding an edge between the degree-1 vertices. This shows $H = C_n$.

Alternate proof of claim. Again assume H is connected and 2-regular. Since H has even vertex degrees, H is Eulerian, so there is an Euler tour. This Euler tour (which by definition is a closed walk) is in fact a cycle: since every vertex is degree-2, it is visited exactly once on the Euler tour. This shows that H is a cycle. □

Problem 5. *Use induction on the number of edges to prove that a graph with no odd cycle is bipartite.*

Solution. Base case. A graph with no edges can be 2-colored arbitrarily.

Induction step. Fix G and fix an edge e. Consider $G \setminus e$. If G has no odd cycle, then neither does $G \setminus e$, so by the induction hypothesis, $G \setminus e$ is bipartite. We claim that the 2-coloring of the vertices of $G \setminus e$ also gives a 2-coloring of G. The only thing to check is that the edge e is not monochromatic. There are two cases, depending on whether $G \setminus e$ is connected or not.

If $G \setminus e$ is connected, then there's a path P in $G \setminus e$ between the endpoints of e. Observe that e is monochromatic if and only if P has an even number of edges, which is true if and only if $P \cup e$ forms of a cycle of odd length. Thus, if G has no odd cycles, then e is not monochromatic.

[2]Hint: consider the complement. Your solution should not be long. Use may want to use the previous problem.

If $G \setminus e$ is disconnected, then the endpoints of e lie in different components, and we can change the coloring on one component to get a coloring of G so that e is not monochromatic. □

Problem 6. *Suppose there are two mountain trails, each starting at sea level and ending at the same elevation. Suppose hikers A,B start hiking these two different trails at the same time. The Mountain Climber Problem asks if it is possible for A and B to hike to the top of their individual trails in a way so that they have the same elevation at every time.*[3] *We model the trails by functions* $f, g : [0,1] \to [0,1]$ *with* $f(0) = g(0) = 0$ *and* $f(1) = g(1) = 1$. *In this problem you solve the Mountain Climber Problem in the case when* f *and* g *are piecewise linear continuous functions.*[4]

(a) Consider

$$Z = \{(x, y) \in [0,1] \times [0,1] : f(x) = g(y)\}$$

Assuming f, g *are piecewise linear, determine the local picture near* (x, y) *in* Z*, considering cases based on the local pictures of* f *and* g *near* x *and* y*, respectively.*

(b) Observe that Z *can be given the structure of a graph* G. *Show that* G *has exactly two vertices of odd degree. Deduce that there is a path in* G *from* $(0,0)$ *to* $(1,1)$.

Solution. If f or g have locally constant regions, we can collapse these to get now functions f and g which are either locally increasing, decreasing, or have a local extremum at each point. If we can solve the problem in this simpler case, we can get a solution in the more general case without too much work (details omitted).

(a) Fix $(x, y) \in Z$. There are a few cases to consider.

- The points x, y are not local extrema for f, g. In this case Z looks like a line near (x, y). In order to preserve having the same height, the hikers have only one choice of direction to move.
- The point x is a local extremum for f, but y is not a local extremum for g, or vice versa. Again Z looks like a line near (x, y). In order to preserve having the same height, y has only one choice for which direction to move (this involves backtracking).
- The points x, y are both local minima or both local maxima for f, g. In this case Z looks like two lines crossing near (x, y). Here both hikers have a choice for which direction to move to keep their heights equal.
- The points $(0,0)$ and $(1,1)$. Near $(0,0)$ the set Z looks like a ray (half line), since both hikers only have one choice, which is to start ascending the mountain. The situation at $(1,1)$ is similar.
- The point x is a local minimum of f and the point y is a local maximum of g. In this case (x, y) is an isolated point of Z. There is no way for the hikers to move in a way that maintains that they have the same height.

[3] It is important to note that the hikers are allowed to backtrack.

[4] A function $f : [0,1] \to \mathbb{R}$ is piecewise linear if it's possible to express $[0,1]$ as a union of finitely many intervals, so that f is linear ($x \mapsto ax + b$) on each.

- The point x is a local min of f and g is constant as you approach y from the left and increasing as you approach y from the right (or the obvious variations of this scenario). In this case Z looks like a tripod near (x, y). Here if y increases elevation, then x has two choices, but if y moves staying flat then x has to stay fixed.

(b) Since Z is made up of some straight lines and isolated vertices, we can give G the structure of a graph by adding a vertex for each isolated vertex, at $(0, 0)$ and $(1, 1)$, and wherever two lines cross. With this graph structure, each vertex has degree either 0, 1, 2, or 4, respectively, and there are exactly two vertices with degree 1.

We want to show that $(0, 0)$ to $(1, 1)$ belong the same component of G. The component of G containing $(0, 0)$ has an even number of vertices of odd degree (by the degree-sum formula), so $(1, 1)$ must also belong to this component. A path from $(0, 0)$ to $(1, 1)$ gives a solution to the mountain climber problem. $\square$