Homework 1

Math 123

Due February 3, 2023 by 5pm

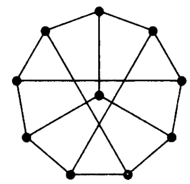
Name:

Topics covered: graph, subgraph, cycle, path, vertex degrees,

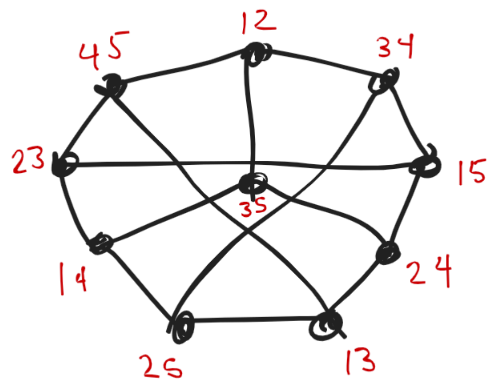
Instructions:

- This assignment must be submitted on Gradescope by the due date. Gradescope Entry Code: RZ277D.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck, please ask for help (from me, a TA, a classmate).

Problem 1. *Prove that the graph below is isomorphic to the Petersen graph.*[1]



Solution. We do this by labeling the vertices of the graph with 2-element subsets of $\{1, \ldots, 5\}$ so that edges correspond to disjointness.



□

Problem 2. *How many cycles of length n are there in the complete graph K_n?*

Solution. Denote $V(K_n) = \{1, \ldots, n\}$. An n-cycle can be expressed as an ordering $(a_1, \ldots, a_n)$ of $1, \ldots, n$ (each vertex appearing once). Note that reversing the order $(a_n, \ldots, a_1)$ specifies the same cycle. Furthermore, changing the cycle by a cyclic permutation does not change the cycle, e.g. for $n = 3$, 123 and 231 and 312 each describe the same cycle. The number of sequences $(a_1, \ldots, a_n)$ is $n!$, and the number of sequences equivalent to a given one is $2 \cdot n$ (coming from cyclic permutations and reversing direction), so the total number of n-cycles is $n!/(2n) = \frac{(n-1)!}{2}$. □

Problem 3. *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \ldots, a_k)$ with coordinates $a_i \in \{0, 1\}$ and with an edge between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if they differ in exactly one coordinate.*[2]

(a) *Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.*

(b) *Let $K_{2,3}$ be the complete bipartite graph with 2 red vertices, 3 blue vertices, and all possible edges between red and blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.*

[1] Hint: label the graph.

[2] Suggestion: Draw Q_k for $k = 2$ and $k = 3$.

Solution.

(a) A 4-cycle containing vertex a is determined by two coordinates i, j and the other vertices b, c, d of the 4-cycle are obtained by either changing i, j or both, respectively. Denote $C_{i,j}(a)$ the corresponding 4-cycle containing a.

Assume two 4-cycles are not disjoint. Choose a common vertex a, so that the two cycles can be denoted $C_{i,j}(a)$ and $C_{k,\ell}(a)$ as above. Then either $\{i,j\}$ and $\{k,\ell\}$ are either disjoint or share a single index. In the first case (disjoint), then $C_{i,j}(a)$ and $C_{k,\ell}(a)$ intersect only at a. In the second case, then $C_{i,j}(a)$ and $C_{k,\ell}(a)$ share a single edge. This completes the proof.

(b) The graph $K_{2,3}$ contains two 4-cycles C, D that have two edges in common. This property is not shared by 4-cycles in Q_k by (a), so $K_{2,3}$ is not a subgraph of Q_k. $\square$

Problem 4. *For a graph $G = (V, E)$, the complement of G is the graph $\bar{G} = (V, \bar{E})$, where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$. Prove or disprove: If G and H are isomorphic, then the complements $\bar{G}$ and $\bar{H}$ are also isomorphic.*

Solution. This statement is true. Suppose $\phi : V(G) \to V(H)$ is gives an isomorphism between G and H. This means that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$. Equivalently, $\{u, v\} \notin E(G)$ if and only if $\{\phi(u), \phi(v)\} \notin E(H)$. This says exactly that ϕ gives an isomorphism between $\bar{G}$ and $\bar{H}$. $\square$

Problem 5.

(a) Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices. It has $n-1$ edges.)

(b) We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n is divisible by 4 or $n-1$ is divisible by 4. [3]

In fact, whenever n or $n-1$ is divisible by 4, there is a self-complementary graph with n vertices – see the bonus problem below.

Solution.

(a) the complement of P_3 is a union of P_2 and P_1. The complement of P_4 is isomorphic to P_4.

(b) Assume G is self-complementary with n vertices. We can write $n = 4k + a$ with k an integer and $a \in \{0, 1, 2, 3\}$. To solve this problem we show $a = 0$ or $a = 1$.

Self-complementary implies in particular that G and $\bar{G}$ have the same number of edges. Since $|E(G)| + |E(\bar{G})| = |E(K_n)|$, this implies that K_n has an even number of edges. Recall that the number of edges of K_n is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

So if $\binom{n}{2}$ is even, then $n(n-1)$ is divisible by 4. Substituting $n = 4k + a$, we have

$$n(n-1) = (4k+a)(4k+a-1) = 16k^2 + 8ka + a^2 - 4k - a = 4(4k^2 + 2ka - k) + a^2 - a$$

[3] Hint: count edges

In order for $n(n-1)$ to be divisible by 4, we need $a^2 - a$ to be divisible by 4. This is true for $a = 0$, $a = 1$, but is not true for $a = 2$ and $a = 3$.

Thus if there exists an n-vertex self-complementary graph, then $n = 4k$ or $n = 4k+1$, which proves the claim. □

Problem 6. *Prove that the Petersen graph has no cycles of length* 3 *or* 4. [4]

Solution. Let's write $X = \{1, 2, 3, 4, 5\}$.

First we argue that there is no 3-cycle. The existence of a 3-cycle would mean there are 2-element subsets $A, B, C \subset X$ that are pairwise disjoint. This would force X to have at least 6 elements, a contradiction.

Next observe that if $A \neq C \subset X$ are 2-element subsets that are not disjoint, then A and C have exactly one element in common, i.e. $A \cup C$ has 3 elements. Then there is a unique 2-element subset $B \subset X$ so that A, C are both disjoint from B.

Suppose for a contradiction that G has a 4-cycle (A, B, C, D, A) with A, B, C, D distinct 2-element subsets of X. Since there are no 3-cycles, then A, C are not disjoint, and since they are distinct, $A \cap C$ has exactly one element and $A \cup C$ has three elements. Since A, C are both adjacent to both B and D, this means $A \cup C$ is disjoint from B and D. But $A \cup C$ has three elements and X has only five elements, so this forces $B = D$, a contradiction. □

Problem 7 (Bonus)**.** *Let G, H be a self-complementary graphs, and assume G has with $4k$ vertices. Construct a self-complementary graph obtained by taking the union of G and H and adding some edges.*[5] *Deduce that if either n or $n-1$ is divisible by* 4*, then there is a self-complementary graph with n vertices.*

Solution. Form a graph $G * H$ by starting with $G \cup H$ and combining each even-degree vertex of G to every vertex of H.

To see that $G * H$ is self-complementary, the key observation is that when we take the complement of G, odd-degree vertices become even-degree vertices and vice versa (this is because since the degree of vertices in K_{4k} is $4k-1$, which is odd). Consequently, when we take complement of $G * H$, then odd-degree vertices of G are connected to every vertex of H. But odd-degree vertices of G are even-degree vertices of $\bar{G}$, so the complement is $\bar{G} \cup \bar{H}$ with even degree vertices of $\bar{G}$ connected to every vertex of H.

Now $P_4 * \cdots * P_4$ is a self-complementary graph with $4k$ vertices and $K_1 * P_4 * \cdots * P_4$ is a self-complementary graph with $4k+1$ vertices. □

[4]Hint: use the definition of Petersen graph given in class.

[5]Hint: How does the degree of even/odd vertices of G change after taking the complement?