Homework 6

Math 123

Due March 10, 2023 by midnight

Name:

Topics covered: vertex cuts, connectivity, Menger's theorem, network flows

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1 (West 4.1.2)**.** *Let G be a graph.*

(a) Give a counterexample to the following statement: If e is a cut-edge of G, then at least one vertex of e is a cut-vertex of G. [1]

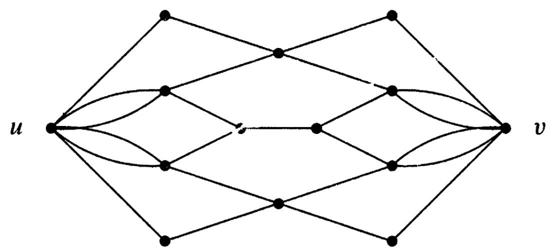
(b) Add a hypothesis to correct the above statement.

Solution. (a) Consider $G = K_2$. Removing a vertex leaves K_1 which is connected.

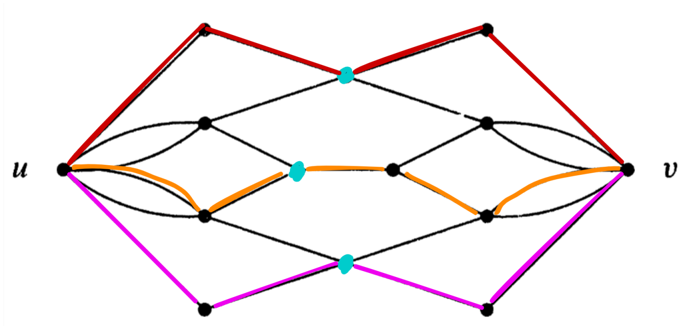
(b) If we add the assumption that G has at least 3 vertices, then the statement is true. If G is already disconnected, we're done. Otherwise, for $e = \{u, v\}$, without loss of generality, v is connected by an edge to some vertex w, and the graph $G \setminus \{v\}$ is disconnected (no path u to w). □

[1] We did not define cut edge in class, but it means what you most likely guess.

Problem 2 (West 4.2.1)**.** *Compute (with proof) $\kappa(u,v)$ for the graph below.*



Solution. We claim that $\kappa(u,v) = 3$. First we observe that there is a vertex cut of size 3, illustrated in teal below.

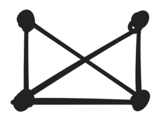


To show we cannot do better, by Menger's theorem, it suffices to find 3 disjoint paths from u to v. See the figure. □

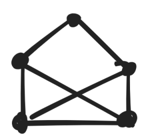
Problem 3 (West 4.1.10)**.** *Find (with proof) the smallest* 3*-regular graph with* $\kappa(G) = 1$. [2]

Solution. Suppose $G = (V, E)$ is 3-regular and $\kappa(G) = 1$. Then there exists $v \in V$ so that $G \setminus \{v\}$ has at least two components. Suppose G has exactly two components G_1, G_2. In one of these components, say G_1, there are two vertices of degree 2 and the remaining vertices have degree 3; in the other component G_2 there is one vertex of degree 2 and the remaining vertices have degree 3.

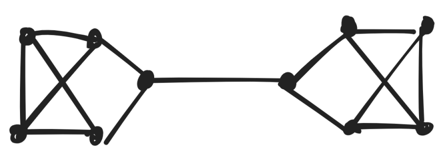
What is the smallest G_1 can be? Its degree sequence is $(2, 2, 3, \ldots, 3)$ and it must have at least one vertex of degree 3, and hence two by the vertex degree formula. We find that an example with exactly two vertices of degree 3 does exists:



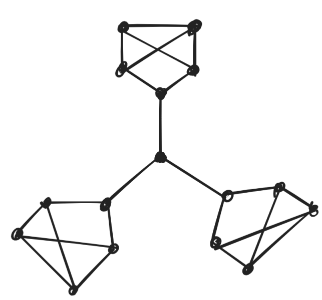
What is the smallest G_2 can be? It's degree sequence is $(2, 3, 3, \ldots, 3)$. By the degree-sum formula, there are an even number of vertices of degree 3. Observe that H can't have only 3 vertices. Then we must have at least 4 vertices of degree 3. This is possible:



Altogether we conclude that G is the graph below.



Note that it's also possible that $G \setminus \{v\}$ has 3 components. In this case, arguing similarly, each component of $G \setminus \{v\}$ has exactly one vertex of degree 2, and the smallest graph we can obtain is pictured below. It has more edges than the graph above.



[2]Hint: consider a 1-element vertex cut S. What does $G \setminus S$ look like?

$\square$

Problem 4 (West 4.2.20)**.** *Fix $k \geq 2$ and let Q_k be the hypercube graph. Prove that for any pair of vertices x, y there exist k pairwise disjoint (x, y)-paths.*

Solution. We prove this by induction. Base case $k = 2$ was treated in class, so we focus on the induction step.

Fix vertices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. We consider two cases.

Case 1: x, y do not have any coordinates in common. Up to symmetry $x = (0, \ldots, 0)$ and $y = (1, \ldots, 1)$.

Note that a path in Q_k from x to y is specified by specifying an ordering of $1, \ldots, k$ which specifies the coordinates are changed from 0 to 1. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vertex with a 1 in position i. Note that a path from x to y is specified by adding the e_i in some order, e.g. one path has vertices

$$x, \;\; x + e_1, \;\; x + e_1 + e_2, \;\; \ldots, \;\; x + e_1 + \cdots + e_k = y.$$

Note then that we can specify a path from x to y by an ordering of $1, \ldots, k$. Observe that the paths

$$(1, 2, \ldots, k), \;\; (2, 3, \ldots, k, 1), \;\; \ldots, \;\; (k, 1, 2, \ldots, k-1)$$

have no interior vertices in common, i.e. they are disjoint.

Case 2: x, y have two vertices in common. Without loss of generality (using symmetry) $x_n = y_n = 0$. Let $H, H' \cong Q_{k-1}$ be the subgraph spanned by vertices u with $u_n = 0$ and $u_n = 1$, respectively. Then $x, y \in H$. By the induction step there are $k - 1$ vertex disjoint paths in H from x to y. Let $x' = (x_1, \ldots, x_{n-1}, 1)$ and $y' = (y_1, \ldots, y_{n-1}, 1)$, and let P be a path between x', y' in H'. Observe that x, x', P, y', y specifies a path from x to y, which is disjoint the paths we found in H. Thus we have found k disjoint paths, and this completes the proof. $\square$

Problem 5 (West 4.2.23). *Use Menger's theorem to prove König's theorem: if $G = (X \sqcup Y, E)$ is bipartite the maximum size of a matching of G is equal to the minimum size of a vertex cover of G.* [3]

Solution. Let $G = (X \sqcup Y, E)$ be a bipartite graph.

Recall the easy direction of König: Given a matching M, a vertex cover must have at least as many vertices as edge of M. This shows

$$\min\{|Q| : \text{ vertex cover}\} \geq \max\{|M| : \text{ matching}\}.$$

We show the reverse inequality. For this, let M be a maximum matching of G. We need to find a vertex cover Q with $|Q| \leq |M|$.

Following the hint, define G' to contain G as a subgraph with two additional vertices $\{a, b\}$ and edges from a to every $x \in X$ and from b to every $y \in Y$.

Observe that a collection of disjoint (a, b)-paths in G' defines a matching of G of the same size (record the first X-Y edge – this is a matching by the disjointness condition for the paths). This shows that $|M| \geq \lambda(a, b)$.

By Menger's theorem, $\lambda(a, b) = \kappa(a, b)$. Note that if Q is an (a, b) vertex cut of G', then Q is also a vertex cover of G. Show the contrapositive, if Q is not a vertex cover, then there is an edge of G with neither endpoint in Q. Then there is a path from (a, b), so Q is not a vertex cut of G'. Thus $\kappa(a, b) \geq |Q|$.

Altogether, we've shown

$$|M| \geq \lambda(a, b) = \kappa(a, b) \geq |Q|,$$

which is the desired inequality. □

[3] Hint: consider graph G' obtained by adding vertices a, b to G and connecting a to every vertex of X and b to every vertex of Y.

Problem 6. *Use the matrix-tree theorem*[4] *to prove Cayley's theorem.*[5]

Solution. Write the matrix M_1 from the matrix-tree theorem applied to K_n, which has the property that $\det(M_1)$ is the number of spanning trees. This is an $(n-1) \times (n-1)$ matrix with $n-1$ on the diagonal and -1 in every other entry.

We want to show $\det(M_1) = n^{n-2}$. Recall that the determinant is the product of the eigenvalues with multiplicity. It suffices to show that the eigenvalues are 1 with multiplicity 1 and n with multiplicity $n-2$. Observe that the matrix $M - nI$ is the matrix where all the entries are -1. This matrix has rank 1 and nullity $n-2$, so M_1 has a subspace of dimension $n-2$ consisting of eigenvectors with eigenvalue n. Note also that $(1, \dots, 1)$ is an eigenvector with eigenvalue 1. This completes the proof. $\square$

[4] From the end of lecture 2/28

[5] Use a connection between the determinant and eigenvalues. It may help to first try to guess the form of the answer. For the love of algebra, do NOT compute any determinants!