RESEARCH STATEMENT

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My research area is geometric topology. I primarily study manifolds, fiber bundles, and group actions. My research also has ties to geometric group theory and arithmetic groups. Below I summarize three major topics in my research: Nielsen realization, arithmetic groups, and aspherical manifolds.

1. Group actions and Nielsen realization

For a manifold M, there is a natural surjection $\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M)$ from the homeomorphism group $\mathrm{Homeo}(M)$ to the mapping class group

$$\mathrm{Mod}(M) := \pi_0 \,\mathrm{Homeo}(M).$$

The Nielsen realization problem asks, for each subgroup $G < \mathrm{Mod}(M)$, if there is a solution to the following lifting problem

$$\begin{tikzcd} & \mathrm{Homeo}(M) \arrow[d, two heads] \\ G \arrow[r, hook] \arrow[ur, dashed, "L"] & \mathrm{Mod}(M) \end{tikzcd}$$

When a lift exists, we say $G < \mathrm{Mod}(M)$ is *realizable.*

The optimist's conjecture.

For a surface S_g, Nielsen originally asked if every finite $G < \mathrm{Mod}(S_g)$ is realized by a group of isometries of S_g with respect to some hyperbolic metric. This was proved by Kerckhoff [Ker83].

A finite-order homeomorphism representing a finite-order mapping class is an example of a *Nielsen–Thurston representative*, which are particularly simple elements in each isotopy class. The following conjecture would generalize Kerckhoff's theorem.

Conjecture 1 (optimist's conjecture)**.** If $G < \mathrm{Mod}(S_g)$ is realizable in $\mathrm{Homeo}(S_g)$, then there is a realization by Nielsen–Thurston representatives.

Conjecture 1 holds for every $G < \mathrm{Mod}(S_g)$ that is known to be realizable. This includes finite groups, free groups, abelian groups, Veech groups, certain right-angled Artin groups, ... In each case, the relations in G are simple enough that they can be satisfied by Nielsen–Thurston representatives.

For $G = \mathrm{Mod}(S_g)$, Conjecture 1 predicts that $\mathrm{Homeo}(S_g) \to \mathrm{Mod}(S_g)$ does not split for $g \geq 2$. This was originally asked by Thurston in Kirby's problem list and proved by Markovic [Mar07].

Problem 2. Find new examples of infinite $G < \mathrm{Mod}(S_g)$ that are not realizable.

For example, Salter and I [ST16] show most surface braid groups are not realizable by diffeomorphisms.

Theorem 3 (Realizing braid groups)**.** *Let $B_n \cong \mathrm{Mod}(\mathbb{D}^2, n)$ denote the n-stranded braid group.*

- *For $n \geq 5$, the braid group B_n is not realizable by diffeomorphisms. More generally, surface braid groups $B_n(S_g) < \mathrm{Mod}(S_{g,n})$ are not realizable when $n \geq 6$.*
- (Corollary) *If $g \geq 2$, then* $\mathrm{Mod}(S_g)$ *is not realizable by diffeomorphisms.*

This has since been improved by L. Chen who shows B_n is not realizable by homeomorphisms and that the pure braid group PB_n is not realizable by area-preserving homeomorphisms. When $g \geq 2$ and $n \geq 6$, Theorem 3 gives an alternative proof of a result of Bestvina–Church–Souto [BCS13].

There are many examples of Problem 2 to consider. Some that I find interesting are (i) braid subgroups generated by a "chain" of Dehn twists, (ii) the handle-dragging subgroup $\pi_1(US_g) < \mathrm{Mod}(S_{g+1})$, (iii) the purely pseudo-Anosov surface subgroups constructed by Kent–Leininger [KL24]. This is an area where more techniques are needed.

3-manifolds.

For general 3-manifolds M^3, it's unclear how to formulate a version of Conjecture 1 even for finite $G < \mathrm{Mod}(M^3)$.

Problem 4. Give a criterion, applying to all 3-manifolds M, that characterizes when finite $G < \mathrm{Mod}(M)$ is realizable.

Currently Problem 4 is only solved for special families of M. It would be interesting to solve this problem for special G (e.g. cyclic) and arbitrary M. L. Chen and I [CT22] solve Problem 4 for G generated by sphere twists (the 3D analogue of Dehn twists for surfaces) for any M.

Theorem 5 (Realizing sphere twists)**.** *Let M be a closed, oriented 3-manifold. A subgroup $G < \mathrm{Mod}(M)$ generated by sphere twists is realizable if and only if G is cyclic and M is diffeomorphic to a connected sum of lens spaces.*

Combining Theorem 5 with older works, it should be possible to solve Problem 4.

Nielsen realization and the topology of bundles.

For $G = \pi_1(B)$, a homomorphism $G \to \mathrm{Diff}(M)$ determines an M-bundle $E \to B$ with a *flat connection* (a foliation with certain properties). Whether or not a given bundle admits a flat connection is a poorly understood problem.

Problem 6. Give new examples of M-bundles that are not flat (i.e. have no flat connection).

This can be approached with Nielsen realization: if $G \to \mathrm{Mod}(M)$ is not realizable in $\mathrm{Diff}(M)$, then no bundle with this monodromy is flat.

Morita [Mor87] gave the first examples of non-flat S_g-bundles. Giansiracusa–Kupers and I [GKT21] apply similar ideas to the K3 4-manifold.

Theorem 7 (Non-flat K3 bundles)**.** *Let M^4 be the* K3 *surface.*

- *Finite-index $G < \mathrm{Mod}(M)$ are not realizable by diffeomorphisms.*
- *The tautological bundle over the moduli space of Einstein metrics on M is not flat.*
- *There exists an aspherical 8-manifold B, and a M-bundle $E \to B^8$ that admits no flat connection.*

It would be interesting to extend Morita's argument to other 4-manifolds.

As another example, the following theorem builds on work of Bestvina–Church–Souto [BCS13] who prove a similar statement in the surface case.

Theorem 8. *Let $M = \Gamma\backslash G/K$ be locally symmetric of noncompact type. If $\mathbb{Q}$-rank$(\Gamma) \geq 1$ or M has a nonzero Pontryagin class, then*

- *The pointpushing group $\pi_1(M,*) < \mathrm{Mod}(M,*)$ is not realizable in* $\mathrm{Diff}(M,*)$.
- *The product bundle $M \times M \to M$ is not flat relative to the diagonal $\Delta \subset M \times M$.*

The theorem applies e.g. to finite manifold covers of $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, $n \geq 3$, and compact complex hyperbolic manifolds $\Gamma\backslash\mathbb{C}H^n$.

Classifying realizations.

There are very few known examples of natural actions of mapping class groups $\mathrm{Mod}(M)$ and outer automorphism groups $\mathrm{Out}(\pi_1(M))$ on manifolds. Here are some examples:

1. (Cheeger [Gro00]) $\mathrm{Mod}(S_g)$ acts on the unit tangent bundle US_g.
2. $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ acts on the n-torus T^n.
3. (Mostow rigidity) For hyperbolic M^n, $n \geq 3$, $\mathrm{Out}(\pi_1 M)$ acts on M.

We view each of these as a rare gem whose properties deserve study.

Problem 9. Show the actions above are unique (this is made precise below).

<u>$\mathrm{Mod}(S_g)$ acting on 3-manifolds.</u> The dearth of examples of interesting actions of $\mathrm{Mod}(S_g)$ on 3-manifolds leads us to the following conjecture.

Conjecture 10. If $\mathrm{Mod}(S_g)$ acts faithfully on a 3-manifold M^3, then $M = US_g$ and the action is conjugate to Cheeger's construction.

Conjecture 10 appears to be out of reach in general, but it contains interesting cases that are more approachable. For example, the action $\mathrm{Mod}(S_g) \curvearrowright US_g$ is not smooth, so Conjecture 10 implies, in particular, that this action is not homotopic to a smooth action. This was proved by Souto [Sou10] (for the extended mapping class group).

As another special case of Conjecture 10, for a circle bundle $M \to S_g$, the natural surjection

$$\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g)$$

should split only for $M = US_g$. Evidence for this is provided by the following theorem, proved in joint works with L. Chen and my student Alina al Beaini [CT23, BCT23].

Theorem 11 (Realizing $\mathrm{Mod}(S_g)$ on circle bundles)**.** *Fix an oriented circle bundle $M \to S_g$ and let $e(M) \in H^2(S_g;\mathbb{Z}) \cong \mathbb{Z}$ be its Euler class/number.*

(i) $\mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g)$ *splits* $\Leftrightarrow 2-2g = \chi(S_g)$ *divides* $e(M)$.
(ii) $\mathrm{Homeo}(S_g \times S^1) \twoheadrightarrow \mathrm{Mod}(S_g)$ *does not split for infinitely many* g.

<u>Actions of $\mathrm{SL}_n(\mathbb{Z})$ on n-manifolds.</u> Motivated by Souto's result, for any smooth structure $\mathfrak{T}$ on T^n, we can ask whether the action $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright T^n$ is homotopic to a smooth action on $\mathfrak{T}$, i.e. whether one can split the map

$$\mathrm{Diff}(\mathfrak{T}) \to \mathrm{Out}(\pi_1\mathfrak{T}) \cong \mathrm{GL}_n(\mathbb{Z}). \tag{1}$$

Conjecture 12. The map (1) splits only for the standard torus $\mathfrak{T} = T^n$.

Conjecture 12 is implied by a conjecture of Fisher–Melnick [FM22] that proposes a classification of actions $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright M^n$ (as part of Zimmer program).

The most interesting case of Conjecture 12 is when $\mathfrak{T} = T^n\#\Sigma$, where Σ is a smooth homotopy n-sphere. Bustamante–Krannich–Kupers and I [BKKT23] prove a partial result. Below $\eta(\Sigma) \in \Theta_{n+1}$ is a group-valued invariant defined by Milnor–Munkres–Novikov.

Theorem 13 (Actions of $\mathrm{SL}_n(\mathbb{Z})$ on homotopy tori)**.** *Let $\mathfrak{T} = T^n\#\Sigma$, $n \geq 5$.*

(i) $\mathrm{Mod}(M) \to \mathrm{SL}_n(\mathbb{Z})$ *splits if and only if* $\eta(\Sigma)$ *is divisible by* 2.
(ii) If $\eta(\Sigma)$ *is not divisible by* 2, *then* $\mathrm{Diff}(\mathfrak{T}) \to \mathrm{GL}_n(\mathbb{Z})$ *does not split. In addition, every homomorphism* $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Diff}(\mathfrak{T})$ *is trivial.*

Actions of $\mathrm{Out}(\pi_1 M)$ on hyperbolic manifolds. For the final instance of Problem 9, let $\mathfrak{M}$ be a smooth structure on a hyperbolic manifold M. For any finite group action $G \curvearrowright M$, we ask whether this action is homotopic to a smooth action on $\mathfrak{M}$. This can be phrased in terms of splitting the map

$$\mathrm{Diff}(\mathfrak{M}) \to \mathrm{Out}(\pi_1 \mathfrak{M}) \cong \mathrm{Out}(\pi_1 M). \tag{2}$$

Whether this map splits when $\mathfrak{M}$ is negatively curved was asked by Schoen–Yau [SY79], as a generalization of Nielsen's question. A negative answer was given by Farrell–Jones [FJ90], who observed that an orientation-reversing involution on M can be homotoped to an involution of $\mathfrak{M} = M\#\Sigma$ only if the homotopy n-sphere Σ is divisible by 2 in the group Θ_n of homotopy n-spheres.

The following result of Bustamante and I [BT23] (building on previous work [BT22]) extends the Farrell–Jones result to arbitrary finite groups and proves a converse, at least in dimension 7.

Theorem 14. *Let M is a hyperbolic 7-manifold, and assume* $\mathrm{Isom}(M)$ *acts freely on M. Let Σ be a homotopy 7-sphere. An action $G \curvearrowright M$ is homotopic to a smooth action on $M\#\Sigma$ if and only if Σ is divisible by $|G|$ in Θ_n.*

2. Arithmetic groups, monodromy, and cohomology

The study of arithmetic groups, e.g. $\mathrm{SL}_n(\mathbb{Z})$, is a classical topic connecting to many areas. My interest in these groups comes primarily from their relation to the study of manifold bundles.

Monodromy of holomorphic bundles.

An surface bundle $S_g \to E \to B$ has a monodromy representation

$$\rho : \pi_1(B) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

In general $\mathrm{Image}(\rho) < \mathrm{Sp}_{2g}(\mathbb{Z})$, called the *monodromy group*, can be any subgroup, but if $E \to B$ is a *holomorphic* fibration, then Deligne [Del87] proved that the Zariski closure of its monodromy group $\Gamma_E < \mathrm{Sp}_{2g}(\mathbb{Z})$ is semi-simple, and Griffiths–Schmid [GS75] asked:

Question 15 (Griffiths–Schmid)**.** When is the monodromy group of a holomorphic S_g-fibration an arithmetic group?

It's known that both answers occur [DM86, Ven14].

There is an instance of Question 15 for every cover $S_g \to S_h$ of surfaces (possibly branched): there is a holomorphic S_g-bundle $E \to \mathcal{M}'_h$, where $\mathcal{M}'_h$ is a finite cover of the moduli space $\mathcal{M}_h$ of genus-h Riemann surfaces. For these examples, the arithmeticity question is related to a conjecture of Putman–Wieland [PW13] that would imply that $\mathrm{Mod}(S_g)$ does not virtually surject to $\mathbb{Z}$.

My student Trent Lucas has studied Problem 15 for all covers $S_g \to S_h$ with $g \leq 3$. His analysis includes 17 previously unstudied cases. He finds that the monodromy group is always arithmetic when $g \leq 3$ [Luc24].

Salter and I [ST20] answer Question 15 for certain holomorphic S_g-bundles constructed by Atiyah–Kodaira. As a topological consequence, we compute the number of fiberings of these examples, a result motivated by Thurston's theory of fibering 3-manifolds.

Theorem 16 (Atiyah–Kodaira bundles)**.** *Let $S_g \to E \to S_h$ be one of the classical holomorphic families constructed by Atiyah and Kodaira. If h is sufficiently large, then*

(i) *the image of the monodromy $\pi_1(S_h) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is an arithmetic group;*
(ii) *the 4-manifold E fibers as a surface bundle in exactly two ways.*

Unstable cohomology.

In studying M-bundles, a fundamental problem is to compute the ring of characteristic classes $H^*(B\,\mathrm{Diff}(M))$. When $M_g^{2d} = \#_g(S^d \times S^d)$, this ring is known in a range $* \ll g$ (Mumford's conjecture) [GRW14, MW07, Mum83]. Little is known about $H^*(B\,\mathrm{Diff}(M_g))$ when $* \geq g$, although there have been recent results [CGP18].

In [Tsh21], I produce new classes in $H^g(B\,\mathrm{Diff}'(M_g^{2d}))$ and for certain finite-index "congruence" subgroups $\mathrm{Diff}'(M_g) < \mathrm{Diff}(M_g)$, when $d \gg g$ is even. This is related to arithmetic groups via the following theorem of [Tsh21].

Theorem 17 (New cohomology for lattices in $\mathrm{SO}(p,q)$)**.** *Fix $1 \leq p \leq q$ with $p+q \geq 3$. Let $\Lambda \subset \mathbb{R}^{p+q}$ be a lattice with an integral bilinear form of signature (p,q). There exists a finite-index subgroup $\Gamma < \mathrm{SO}(\Lambda)$ so that $\dim H^p(\Gamma;\mathbb{Q}) \neq 0$.*

The cohomology in Theorem 17 comes from flat p-dimensional tori in the associated locally symmetric manifolds. D. Studenmund and I [ST22] compute lower bounds on the dimension of the subspace generated by these classes in congruence subgroups. For example, for SL_{n+1} we show:

Theorem 18 (Cohomology growth, congruence subgroups of $\mathrm{SL}_{n+1}(\mathbb{Z})$)**.** *Fix $n \geq 2$, and let $\Gamma(s) < \mathrm{SL}_{n+1}(\mathbb{Z})$ denote the level-s principal congruence subgroup. Then for each prime p,*

$$\dim H_n(\Gamma(p^\ell);\mathbb{Q}) \gtrsim |\,\mathrm{SL}_{n+1}(\mathbb{Z}) : \Gamma(p^\ell)|^{\frac{n+1}{n^2+2n}} \quad \textit{for } \ell \gg 0.$$

The above approach for constructing characteristic classes suggests a way to find new cohomology in finite-index subgroups of $\mathrm{Mod}(S_g)$. I plan to explore this in future work.

Arithmetic mapping tori.

By a theorem of Margulis [Mar91], a lattice Γ in a semisimple Lie group is arithmetic if and only if Γ has infinite index in its commensurator. In contrast, no general arithmeticity characterization for lattices in *solvable* Lie groups is known. In [Tsh22] for solvable lattices of the form $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$, I provide an arithmeticity criterion in terms of the eigenvalues of A.

Theorem 19 (Arithmeticity criterion)**.** *Fix $A \in \mathrm{GL}_n(\mathbb{Z})$ hyperbolic and semisimple. Then $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ is arithmetic if and only if $\log(\mu)$ and $\log(\nu)$ are commensurable for any real monomials μ, ν in the eigenvalues of A and their inverses.*

The case A is hyperbolic and irreducible was solved by Grunewald–Platonov [GP98], albeit with a more complicated criterion in terms of algebraic tori.

It would be interesting to prove an analogous theorem that characterizes arithmeticity for groups $\Gamma = \pi_1(S_g) \rtimes_\phi \mathbb{Z}$ with $\phi \in \mathrm{Out}(\pi_1(S_g)) \cong \mathrm{Mod}(S_g)$ pseudo-Anosov, in terms of some property ϕ.

3. Aspherical manifolds and hyperbolic groups

Wall and Cannon conjectures.

In the classification of aspherical manifolds, the basic existence and uniqueness problems are as follows.

Conjecture 20.

- (Wall) If G is a finitely-generated Poincaré duality group, then there exists a closed aspherical manifold M with $\pi_1(M) \cong G$.
- (Borel) Two closed aspherical manifolds M, M' with $\pi_1(M) \cong \pi_1(M')$ are homeomorphic.

These conjectures are known to be true for many groups/manifolds coming from geometry. For example, Bartels–Lück–Weinberger [BLW10] prove the Wall conjecture for hyperbolic groups with sphere boundary $\partial G \cong S^n$, $n \geq 5$. Lafont and I [LT19] relative version that extends [BLW10].

Theorem 21 (Wall conjecture, special case)**.** *Let G be a hyperbolic group whose Gromov boundary is an $(n-2)$-dimensional Sierpinski space. If $n \geq 7$, then $G \cong \pi_1(M)$ where M is a compact aspherical manifold with aspherical boundary.*

The Cannon conjecture, a version of Wall's conjecture in geometric group theory and low-dimensional topology, predicts that a torsion-free hyperbolic group G with boundary $\partial G \cong S^2$ the 2-sphere is the fundamental group of a closed hyperbolic 3-manifold. Bestvina–Mess [BM91] showed that $\partial G \cong S^2$

implies that G is a Poincaré duality group (a necessary condition for $G \cong \pi_1(M^3)$).

In connection to a relative version of Cannon's conjecture, G. Walsh and I [TW20] show the following result, also proved by Manning–Wang [MW20].

Theorem 22 (Duality for relatively hyperbolic groups)**.** *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Then $(G, \mathcal{P})$ is a 3-dimensional Poincaré duality pair if and only if the Bowditch boundary $\partial(G, \mathcal{P})$ is the 2-sphere.*

Hyperbolization of groups.

Thurston's geometrization implies that a closed aspherical 3-manifold is hyperbolic if its fundamental group does not contain $\mathbb{Z}^2$. Gromov proposed a group-theoretical analogue: a group G (with a finite $K(G, 1)$) that contains no Baumslag–Solitar subgroup is necessarily hyperbolic. Recently, a counterexample has been provided by Italiano–Martelli–Migliorini [IMM23] via a construction of hyperbolic 5-manifolds, but Gromov's conjecture might be correct for e.g. surface group extensions

$$1 \to \pi_1(S_g) \to G \to \Gamma \to 1. \tag{3}$$

For such G, Gromov's conjecture specializes to a conjecture of Farb–Mosher [FM02].

Conjecture 23 (Farb–Mosher)**.** If $\Gamma < \mathrm{Mod}(S_g)$ and every nontrivial element of $\mathrm{Mod}(S_g)$ is pseudo-Anosov, then G is convex cocompact in $\mathrm{Mod}(S_g)$ (and therefore the extension group G in (3) is hyperbolic).

Conjecture 23 is an active area of interest – there are new examples of Kent–Leininger [KL24] that (as of this writing) could be counterexamples. Nevertheless, Conjecture 23 is known for many classes of examples G of geometric origin. In [Tsh24], I verify the conjecture for subgroups of the genus-2 *Goeritz group*, the subgroup of $\mathrm{Mod}(S_2)$ of mapping classes that extend to the genus-2 Heegaard splitting of S^3. As a corollary of the proof, we characterize pseudo-Anosov elements in the Goeritz group.

Theorem 24 (Pseudo-Anosovs in the Goeritz group)**.** *Let $\mathcal{G} < \mathrm{Mod}(S_2)$ be the genus-2 Goeritz group. Then*

(i) *Conjecture 23 is true for subgroups of $\mathcal{G}$.*
(ii) *An element of $\mathcal{G}$ is reducible if and only if it stabilizes one of the following: (a) a primitive mulit-disk, (b) a reducing sphere, or (c) an embedding of the figure-8 knot on $S \subset S^3$.*

In Theorem 24(ii), the possibilities (a)–(c) are the "geometrically obvious" ways for an element of $\mathcal{G}$ to be reducible. By the theorem, these are the only ways to be reducible in $\mathcal{G}$. Combined with the known presentation for $\mathcal{G}$, (ii) gives an effective way to test if an element of $\mathcal{G}$ is pseudo-Anosov and to construct explicit purely pseudo-Anosov subgroups.

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