Problem 1. *Let w be a vector field on a surface S. Given a smooth function $f : S \to \mathbb{R}$, define $w(f) : S \to \mathbb{R}$ by*

$$w(f)(p) = (f \circ \alpha)'(0)$$

where $\alpha : I \to S$ is a curve such that $\alpha(0) = p$ and $\alpha'(0) = w(p)$. For functions f, g and real numbers λ, μ, prove

$$w(\lambda f + \mu g) = \lambda w(f) + \mu w(g) \quad \text{and} \quad w(fg) = w(f)g + fw(g).$$

Explain the significance of (a) from a linear algebra point-of-view.

Solution. Observe that the first equation tells us that w is a linear operator on the set of smooth functions $f : S \to \mathbb{R}$. To see that the first equation holds, we expand definitions and use linearity of differentiation:

$$\begin{aligned} w(\lambda f + \mu g) &= ((\lambda f + \mu g) \circ \alpha)'(0) \\ &= (\lambda f \circ \alpha + \mu g \circ \alpha)'(0) \\ &= (\lambda f \circ \alpha)'(0) + (\mu g \circ \alpha)'(0) \\ &= \lambda (f \circ \alpha)'(0) + \mu (g \circ \alpha)'(0) \\ &= \lambda w(f) + \mu w(g) \end{aligned}$$

To get the second equation, we again expand the definition and use the product rule:

$$\begin{aligned} w(fg) &= (fg \circ \alpha)'(0) \\ &= ((f \circ \alpha)(g \circ \alpha))'(0) \\ &= \Big[(f \circ \alpha)'(g \circ \alpha) + (f \circ \alpha)(g \circ \alpha)'\Big](0) \\ &= (f \circ \alpha)'(0)(g \circ \alpha)(0) + (f \circ \alpha)(0)(g \circ \alpha)'(0) \\ &= w(f)g(p) + w(g)f(p) \\ &= w(f)g + w(g)f \end{aligned}$$

Note that we simplified using the fact that $\alpha(0) = p$. $\square$

Problem 2. *True or false: the Mobius band from HW5 can be made out of a strip of paper by gluing the ends. Explain your answer.*

Solution.

$$\phi(t,\theta) = \big((2 + t\cos(\theta/2))\cos\theta, (2 + t\cos(\theta/2))\sin\theta, t\sin(\theta/2)\big) \quad t \in [-1,1], \theta \in [0, 2\pi]$$

We will calculate the Gaussian curvature along the median circle, i.e. $t = 0$.

$$\phi_t = (\cos(\theta/2)\cos\theta,\ \cos(\theta/2)\sin\theta,\ \sin(\theta/2))$$

$$\phi_\theta = (-2\sin\theta, 2\cos\theta, 0)$$

Thus, $E = 1,\ F = 0,\ G = 4.$

$$\phi_{tt} = (0,0,0)$$

$$\phi_{t\theta} = (-\frac{1}{2}\sin(\theta/2)\cos\theta - \cos(\theta/2)\sin\theta, -\frac{1}{2}\sin(\theta/2)\sin\theta + \cos(\theta/2)\cos\theta, \frac{1}{2}\cos(\theta/2))$$

$$\phi_{\theta\theta} = (-2\cos\theta, -2\sin\theta, 0)$$

$$N = \frac{\phi_t \times \phi_\theta}{|\phi_t \times \phi_\theta|} = (-\sin(\theta/2)\cos(\theta), -\sin(\theta/2)\sin(\theta), \cos(\theta/2))$$

Thus

$$e = N\phi_{tt} = 0$$

$$f = N\phi_{t\theta} = 1/2$$

$$g = N\phi_{\theta\theta} = 2\sin(\theta/2)$$

Thus the Gaussian curvature along the median circle is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{0 - 1/4}{4 - 0} = -\frac{1}{16}$$

Since $K \neq 0$, by Theorem Egregium, this Mobius band is not locally isometric to a plane. Hence, the Mobius band cannot be made out of a strip of paper. The answer is False. $\square$

Problem 3. *Let $\phi : U \to S^2$ be a spherical coordinates chart. Compute in coordinates the functions $\phi_{uu}, \phi_{uv}, \phi_{vv}, N_u, N_v$ as linear combinations of ϕ_u, ϕ_v, N.*

Solution. Recall that we defined $\phi(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$. We note that $N = \phi$ because we are on the sphere, and so naturally $N_u = (1)\phi_u$ and $N_v = (1)\phi_v$. For the other values, we calculate that:

$$\begin{aligned}
\phi_u &= (\cos u \cos v, \cos u \sin v, -\sin u) \\
\phi_v &= (-\sin u \sin v, \sin u \cos v, 0) \\
\phi_{uu} &= (-\sin u \cos v, -\sin u \sin v, -\cos u) = -\phi = (-1)N \\
\phi_{uv} &= (-\cos u \sin v, \cos u \cos v, 0) \\
\phi_{vv} &= (-\sin u \cos v, -\sin u \sin v, 0).
\end{aligned}$$

Thus, we need to find representations of ϕ_{uv} and ϕ_{vv}. There are a few ways to do this, but I'm going to start with a more general result. Suppose the vectors v_1, v_2, v_3 form an orthogonal basis for $\mathbb{R}^3$. Then their normalizations $v_i/|v_i|$ will be an orthonormal basis. To express a vector in terms of an orthonormal basis, the coefficients will equivalently be the dot product, ie we have that:

$$w = \left(w \cdot \frac{v_1}{|v_1|}\right)\frac{v_1}{|v_1|} + \left(w \cdot \frac{v_1}{|v_1|}\right)\frac{v_2}{|v_2|} + \left(w \cdot \frac{v_2}{|v_1|}\right)\frac{v_2}{|v_2|} = \frac{w \cdot v_1}{|v_1|^2}v_1 + \frac{w \cdot v_2}{|v_2|^2}v_2 + \frac{w \cdot v_3}{|v_3|^2}v_3.$$

This gives us a formula for expressing w as a linear combination in our basis. As we've shown before, ϕ_u, ϕ_v, N are an orthogonal basis. We note from here that $|\phi_u| = \cos^2 u \cos^2 u + \cos^2 u \sin^2 u + \sin^2 u = 1 = |N| = |\phi|$ and $|\phi_v| = \sin^2 u$. We now calculate the needed dot products:

$$\begin{aligned}
\phi_{uv} \cdot \phi_u &= -\cos^2 u \cos v \sin v + \cos^2 u \cos v \sin v = 0 \\
\phi_{uv} \cdot \phi_v &= \sin u \cos u \sin^2 u + \sin u \cos u \cos^2 u = \sin u \cos u \\
\phi_{uv} \cdot N &= -\sin u \cos u \sin v \cos v + \sin u \cos u \sin v \cos v = 0 \\
\phi_{vv} \cdot \phi_u &= -\sin u \cos u \cos^2 v - \sin u \cos u \sin^2 v = -\sin u \cos u \\
\phi_{vv} \cdot \phi_v &= \sin^2 u \sin v \cos v - \sin^2 u \sin v \cos v = 0 \\
\phi_{vv} \cdot N &= -\sin^2 u \cos^2 v - \sin^2 u \sin^2 v = -\sin^2 u
\end{aligned}$$

Applying what we saw before, we now observe that:

$$\begin{aligned}
\phi_{uv} &= \frac{\phi_{uv} \cdot \phi_u}{|\phi_u|^2}\phi_u + \frac{\phi_{uv} \cdot \phi_v}{|\phi_v|^2}\phi_v + \frac{\phi_{uv} \cdot N}{|N|^2}N = \frac{\sin u \cos u}{\sin^2 u}\phi_v = (\cot u)\phi_v \\
\phi_{vv} &= \frac{\phi_{vv} \cdot \phi_u}{|\phi_u|^2}\phi_u + \frac{\phi_{vv} \cdot \phi_v}{|\phi_v|^2}\phi_v + \frac{\phi_{vv} \cdot N}{|N|^2}N = (-\sin u \cos u)\phi_u + (-\sin^2 u)N.
\end{aligned}$$

And we are done (jeez). $\square$

Problem 4 (dC, 4.4.3). *Show that the surfaces $\phi(u,v) = (u\cos v, u\sin v, \log u)$ and $\psi(u,v) = (u\cos v, u\sin v, v)$ have the same Gauss curvature, but $\psi \circ \phi^{-1}$ is not an isometry.* [1]

Solution. WTS that ϕ and ψ have the same Gauss curvature but that $\psi \circ \phi^{-1}$ is not an isometry

$$\phi_u = (\cos v, \sin v, \frac{1}{u})$$

$$\phi_v = (-u\sin v, u\cos v, 0)$$

$$E = \phi_u \cdot \phi_u = \cos^2 v + \sin^2 v + \frac{1}{u^2} = 1 + \frac{1}{u^2}$$

$$F = \phi_u \cdot \phi_v = \cos v(-u\sin v) + \sin v(u\cos v) + \frac{1}{u} \cdot 0 = 0$$

$$G = \phi_v \cdot \phi_v = u^2\sin^2 v + u^2\cos^2 v + 0^2 = u^2$$

As $F = 0$ we don't need to go through the process of calculating the second fundamental form. Instead we see

$$\sqrt{EG - F^2} = \sqrt{(1 + \frac{1}{u^2})u^2} = \sqrt{u^2 + 1}$$

$$E_v = 0$$

$$G_u = 2u$$

$$K = -\frac{1}{2\sqrt{u^2+1}}((\frac{0}{\sqrt{u^2+1}}_v + (\frac{2u}{\sqrt{u^2+1}})_u)$$

$$= \frac{-1}{2\sqrt{u^2+1}} \cdot \frac{2\sqrt{u^2+1} - 2u \cdot \frac{1}{2\sqrt{u^2+1}} \cdot 2u}{u^2+1} \cdot \frac{\sqrt{u^2+1}}{\sqrt{u^2+1}}$$

$$= \frac{-2(u^2+1) - 2u^2}{2(u^2+1)^2} = \frac{-1}{(u^2+1)^2} = K$$

$$\psi_u = (\cos v, \sin v, 0)$$

$$\psi_v = (-u\sin v, u\cos v, 1)$$

$$E = \psi_u \cdot \psi_u = \cos^2 v + \sin^2 v + 0^2 = 1$$

$$F = \psi_u \cdot \psi_v = \cos v(-u\sin v) + \sin v u\cos v = 0$$

$$G = \psi_v \cdot \psi_v = u^2\sin^2 v + u^2\cos^2 v + 1^2 = u^2 + 1$$

Thus we see

$$\sqrt{EG - F^2} = \sqrt{u^2 + 1}$$

this is the same as for ϕ so

$$E_v = 0$$

$$G_u = 2u$$

Thus

$$K_\psi = K_\phi$$

But we see that $\psi \circ \phi^{-1}$ cannot be an isometry because the coefficients of the 1st Fundamental Form are not the same. ☐

[1] This shows the converse of the Theorem Egregium is false.

Problem 5. *Let S be a surface, and suppose $\phi : U \to S$ is a coordinate chart whose first fundamental form satisfies $F = 0$ and $E = \lambda = G$ for some function λ.*[2]

(a) Prove that $\phi_{uu} + \phi_{vv}$ is orthogonal to ϕ_u and ϕ_v. [3]

Solution. On the one hand, the partial derivatives of E and G yield

$$\phi_{uu} \cdot \phi_u = \tfrac{1}{2}(\phi_u \cdot \phi_u)_u = \tfrac{1}{2}E_u = \tfrac{1}{2}G_u = \tfrac{1}{2}(\phi_v \cdot \phi_v)_u = \phi_{uv} \cdot \phi_v, \tag{1}$$
$$\phi_{vv} \cdot \phi_v = \tfrac{1}{2}(\phi_v \cdot \phi_v)_v = \tfrac{1}{2}G_v = \tfrac{1}{2}E_v = \tfrac{1}{2}(\phi_u \cdot \phi_u)_v = \phi_{uv} \cdot \phi_u. \tag{2}$$

On the other hand, the partial derivatives of F yield

$$\begin{aligned}
(\phi_{uu} + \phi_{vv}) \cdot \phi_u &= \phi_{uu} \cdot \phi_u + \phi_{vv} \cdot \phi_u \\
&= \phi_{uv} \cdot \phi_v + \phi_{vv} \cdot \phi_u \qquad \text{by (1)} \\
&= (\phi_u \cdot \phi_v)_v = F_v = 0, \\
(\phi_{uu} + \phi_{vv}) \cdot \phi_v &= \phi_{uu} \cdot \phi_v + \phi_{vv} \cdot \phi_v \\
&= \phi_{uu} \cdot \phi_v + \phi_{uv} \cdot \phi_u \qquad \text{by (2)} \\
&= (\phi_u \cdot \phi_v)_u = F_u = 0.
\end{aligned}$$

□

(b) By (a), $\phi_{uu} + \phi_{vv} = \mu N$ for some μ. Compute μ.

Solution. Since N is a unit vector,

$$\mu = (\phi_{uu} + \phi_{vv}) \cdot N = \phi_{uu} \cdot N + \phi_{vv} \cdot N = e + g.$$

□

(c) Show that if S is a minimal surface, then ϕ is harmonic, i.e. $\phi_{uu} + \phi_{vv} = 0$. [4]

Solution. If S is a minimal surface, its mean curvature is zero everywhere. That is,

$$\begin{aligned}
0 &= \frac{1}{2}\operatorname{tr}(-DN_p) = \frac{1}{2}\operatorname{tr}\left(\mathrm{II}_p(\mathrm{I}_p)^{-1}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{bmatrix} e & f \\ f & g \end{bmatrix}\begin{bmatrix} 1/\lambda & 0 \\ 0 & 1/\lambda \end{bmatrix}\right) \\
&= \frac{1}{2}\operatorname{tr}\begin{bmatrix} e/\lambda & f/\lambda \\ f/\lambda & g/\lambda \end{bmatrix} = (e+g)/(2\lambda),
\end{aligned}$$

so $e + g = 0$ and $\phi_{uu} + \phi_{vv} = 0N = 0$. □

[2]This is called an isothermal chart. Such a chart always exists, but this is not so easy to prove.
[3]Hint: use the partial derivatives of the functions $\phi_u \cdot \phi_v$ and $\phi_u \cdot \phi_u = \phi_v \cdot \phi_v$.
[4]Technically, it may be better to say the coordinate functions of ϕ are harmonic.