

IMPA minicourse exercises

Constructions of exotic spheres

- (1) For each of the constructions of exotic 7-sphere given in lecture give a variation of the construction that produces the standard sphere S^7 .
- (2) Show that the inclusion of a disk $D^n \hookrightarrow S^n$ induces an isomorphism $\pi_0 \operatorname{Diff}(D^n, \partial D^n) \cong \pi_0 \operatorname{Diff}(S^n)$.¹³ Use this to show that $\Sigma_{\phi \circ \psi} \cong \Sigma_{\phi} \# \Sigma_{\psi}$, where Σ_{ϕ} denotes $D^n \cup_{\phi} D^n$ and $\phi, \psi \in \operatorname{Diff}^+(S^{n-1})$. (This shows the natural “clutching” map $\pi_0 \operatorname{Diff}^+(S^n) \rightarrow \Theta_n$ is a homomorphism.)
- (3) (Challenge) Use one of the constructions of exotic 7-spheres to give an explicit construction of a diffeomorphism of S^6 that is not isotopic to the identity. (Which construction seems most useful?)
- (4) (Reading project) Check out the paper “Eight faces of the Poincaré homology sphere” by Kirby–Scharlemann, which gives eight constructions of the Poincaré homology 3-sphere and explains why they are equivalent. Some of these are similar to the constructions of exotic 7-spheres from lecture.

Warped product metrics

- (1) Check that the warped product $\mathbb{R} \times_{\cosh t} \mathbb{R}$ is a model for hyperbolic space. Describe geodesics in this model.
- (2) Prove that a surface of revolution is isometric to a warped product and derive the Bishop–O’Neill formula for sectional curvature in this case.¹⁴
- (3) Let \mathbb{D} be the Poincaré disk model of the hyperbolic plane, so the metric is $ds^2 = 4 \frac{dx^2 + dy^2}{(1-r^2)^2}$, where $r^2 = x^2 + y^2$. Consider the exponential map $\mathbb{R}^2 = T_o \mathbb{D} \rightarrow \mathbb{D}$, where o is the origin. The pullback of the hyperbolic metric on \mathbb{D} to \mathbb{R}^2 is a warped product metric. Compute the warping function.¹⁵

Gluing and homeomorphisms

- (1) Fix manifolds A, B and assume ∂A and ∂B are diffeomorphic. For a diffeomorphism $\phi : \partial A \rightarrow \partial B$, write $M_{\phi} = A \cup_{\phi} B$.
 - (a) Prove that if ϕ, ψ are isotopic then M_{ϕ} and M_{ψ} are diffeomorphic.

¹³Hint: there is a fibration $\operatorname{Diff}^+(S^n) \rightarrow \operatorname{Emb}^+(D^n, S^n)$. This embedding space is homotopy equivalent to the frame bundle, but here we only need π_0 and π_1 of the embedding space, which aren’t so hard to compute.

¹⁴Hint: to get the right formulas, use a unit-speed parameterization of the curve that is being revolved. This should work better than the parameterization of the curve viewed as a graph of a function..

¹⁵Hint: it may help to determine parameterizations for the unit speed geodesics through the origin in \mathbb{D} . Being unit speed gives a differential equation that is solved by a hyperbolic trig function.

- (b) Prove that if $\psi = f_B \circ \phi \circ f_A$, where $f_A : \partial A \rightarrow \partial A$ is a diffeomorphism that extends to A and similarly for f_B , then M_ϕ is diffeomorphic to M_ψ .
- (2) Prove that $M \# \Sigma$ is homeomorphic to M for any manifold M and homotopy sphere Σ .

The group Θ_n

- (1) Let X be a simply connected n -manifold. Show that X is h -cobordant to S^n if and only if X bounds a contractible manifold.^{16 17}
- (2) Let Σ be a homotopy sphere. Show that $\Sigma \# \bar{\Sigma}$ bounds a contractible manifold. Deduce that $\bar{\Sigma}$ is the inverse of Σ in Θ_n .^{18 19}
- (3) Use the h -cobordism theorem to show that the following are equivalent.
- (a) Σ is smoothly concordant to S^n .
 - (b) Σ is (smoothly) h -cobordant to S^n .
 - (c) Σ is orientation-preservingly diffeomorphic to S^n .
- Deduce that there is a bijection between Θ_n and $\mathbb{S}(S^n)$.

Smoothing theory

- (1) The following statements were made during lecture.
- $\Theta_7 = \mathbb{Z}/28\mathbb{Z}$
 - S^7 has 15 smooth structures
- How can these both be true?
- (2) For based spaces A, B , there is a homotopy equivalence

$$\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$$

where $\Sigma(-)$ denote the suspension and $(-) \vee (-)$ denotes the wedge and $(-) \wedge (-)$ is the smashed product. Use this to deduce the homotopy type of $\Sigma(S^n \times S^n)$ and to compute $\mathbb{S}(S^n \times S^n) \cong [S^n \times S^n, \text{Top/O}]$.²⁰

- (3) Using the same strategy as the previous exercise, compute $[T^n, \text{Top/O}]$. (I stated this computation in lecture, but didn't prove it.)
- (4) Look up a statement of the s -cobordism theorem, and use it to deduce that concordance implies diffeomorphism.²¹

¹⁶Hint: use the h -cobordism theorem.

¹⁷Hint: This exercise has a short solution where you judiciously add or subtract disks.

¹⁸Consider $(\Sigma \setminus D^n) \times I$.

¹⁹For this and the previous exercise, the details are in Kervaire–Milnor “Groups of homotopy spheres I,” Section 2.

²⁰You will want to use that Top/O is an infinite loop space and the adjunction $[X, \Omega Y] = [\Sigma X, Y]$. This was also used in lecture.

²¹Remark: I believe I said h -cobordism theorem in class, but our manifolds are not necessarily simply connected...

- (5) Assume M is a noncompact n -manifold. Prove that $M\#\Sigma$ is concordant (hence diffeomorphic) to M for every homotopy n -sphere Σ .²²
23
- (6) Give an example of two smooth structures that are diffeomorphic but not concordant.²⁴
- (7) (Reading project) Check out the article “Minicourse on smoothing theory” by Jim Davis (you can google it). It contains a short exposition of smoothing theory and might help you better understand the discussion of smoothing theory from last lecture (and a bit for the next lecture).
- (8) Prove that if $T^n\#\Sigma$ is diffeomorphic to T^n , then they are also concordant.²⁵
- (9) Prove that if $T^n\#\Sigma$ is concordant to T^n , then Σ is concordant to S^n , using the following outline. Embed T^n in the interior of D^{n+1} . Cutting D^{n+1} along T^n yields two manifolds with boundary; let W denote the component whose boundary has two components (one is T^n and the other is S^n). Alter W to get a manifold W' whose boundary components are $T^n\#\Sigma$ and \bar{S}^n . Assuming $T^n\#\Sigma$ is concordant to T^n , construct a contractible manifold whose boundary is Σ . Now use an exercise from the previous section (on Θ_n).
- (10) Complete the details in the proof of the statement: Let M be a closed hyperbolic manifold of dimension ≥ 5 . If $M\#\Sigma$ is diffeomorphic to M , then $M\#\Sigma$ is concordant to M .

Hyperbolic manifolds

- (1) Show that orientable surfaces are stably parallelizable.
- (2) Let M a closed hyperbolic manifold with residually finite fundamental group.²⁶ Prove that for any R , there is a finite cover of M that has injectivity radius $> R$. (In the process of finding a proof, figure out the correct assumptions – you shouldn’t need much hyperbolic geometry.)
- (3) Prove that exotic negatively curved manifolds do not admit any locally symmetric metric.
- (4) (Reading project) Check out the paper “Virtually spinning hyperbolic manifolds” by Long–Reid, where they prove that a closed hyperbolic

²²Use the fact (mentioned in lecture) that the map $\mathbb{S}(S^n) \rightarrow \mathbb{S}(M)$ defined by $\Sigma \mapsto M\#\Sigma$ is induced by a map $M \rightarrow S^n$ that collapses the complement of a ball in M to a point.

²³Hint: recall (or look up) the Hopf degree theorem.

²⁴This can be done with either S^n or $S^n \times S^n$ or T^n . We will also discuss this in the next lecture.

²⁵Hint: follow the proof of the analogous fact from lecture. Topological rigidity in this case was proved by Farrell–Hsiang.

²⁶This means that the intersection of finite index subgroups is trivial. Equivalently, for any element of the group, there is a surjection to a finite group where that element survives.

MINICOURSE: EXOTIC HYPERBOLIC MANIFOLDS

manifold has a finite cover that is spin (i.e. the second Stiefel–Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is zero).

- (5) (Challenge) Find a proof that closed hyperbolic manifolds are virtually stably parallelizable. Less ambitious would be to show that characteristic classes (Stiefel–Whitney, Pontryagin) vanish in a finite cover. Or show that M is virtually almost complex (this is also implied by virtually stably parallelizable).