NIELSEN REALIZATION FOR SPHERE TWISTS ON 3-MANIFOLDS

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Abstract. For a 3-manifold M, the twist group $\mathrm{Twist}(M)$ is the subgroup of the mapping class group $\mathrm{Mod}(M)$ generated by twists about embedded 2-spheres. We study the Nielsen realization problem for subgroups of $\mathrm{Twist}(M)$. We prove that a nontrivial subgroup $G < \mathrm{Twist}(M)$ is realized by diffeomorphisms if and only if G is cyclic and M is a connected sum of lens spaces, including $S^1 \times S^2$. We also apply our methods to the Burnside problem for 3-manifolds and show that $\mathrm{Diff}(M)$ does not contain an infinite torsion group when M is reducible and not a connected sum of lens spaces.

1. Introduction

The mapping class group $\mathrm{Mod}(M)$ of a smooth, closed oriented manifold M, is the group of isotopy classes of orientation-preserving diffeomorphisms of M. Denoting $\mathrm{Diff}^+(M)$ the group of orientation-preserving diffeomorphisms, there is a natural projection map

$$\pi : \mathrm{Diff}^+(M) \to \mathrm{Mod}(M)$$

sending a diffeomorphism to its isotopy class. A subgroup $i : G \hookrightarrow \mathrm{Mod}(M)$ is called *realizable* if there is a homomorphism of $\rho : G \to \mathrm{Diff}^+(M)$ such that $\pi \circ \rho = i$.

$$\begin{tikzcd} & \mathrm{Diff}^+(M) \arrow[d, "\pi"] \\ G \arrow[ur, dashed, "\rho"] \arrow[r, "i"'] & \mathrm{Mod}(M) \end{tikzcd}$$

The Nielsen realization problem asks which finite groups $G < \mathrm{Mod}(M)$ are realizable. When M is a surface, every finite subgroup of $\mathrm{Mod}(M)$ is realizable by work of Kerkchoff [Ker83]. For other manifolds, only sporadic results have been obtained; see [Par21, §2] for a summary of results in dimension 3, [FL21, BK19, Lee22, Kon22] for results in dimension 4, and [FJ90, BW08, BT22] for higher dimensions.

When M is a (closed, oriented) 3-manifold, by Hong–McCullough [HM13, Thm. 4.1] its mapping class group fits into an exact sequence

$$1 \to \mathrm{Twist}(M) \to \mathrm{Mod}(M) \to \mathrm{Out}(\pi_1(M)),$$

where the *twist group* $\mathrm{Twist}(M) < \mathrm{Mod}(M)$ is the subgroup generated by twists about embedded 2-spheres (see §2 for the precise definition). McCullough [McC90] proved that $\mathrm{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^d$ for some $d \geq 0$.

We address the Nielsen realization problem for subgroups of $\mathrm{Twist}(M)$. This problem was studied by Zimmermann [Zim21] for $M = \#_k(S^1 \times S^2)$, but there is an error in his argument; see [Zim22]. Nevertheless, using some of the same ideas, we correct the argument, and we generalize from $\#_k(S^1 \times S^2)$ to all 3-manifolds, giving a precise condition for which subgroups can be realized or not.

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To state the main result, recall that for any pair of coprime integers p, q, there is a lens space $L(p, q)$. Every lens space is covered by S^3 with the exception of $L(0, 1) \cong S^1 \times S^2$.

Main Theorem. *Fix a closed, oriented* 3*-manifold M, and fix a nontrivial subgroup* $1 \neq G < \mathrm{Twist}(M)$*. Then G is realizable if and only if G is cyclic and M is diffeomorphic to a connected sum of lens spaces.*

In particular, the subgroup of $\mathrm{Twist}(M)$ generated by a single twist is *sometimes* realizable (depending on the topology of M), but the subgroup generated by distinct commuting twists is *never* realizable.

Application: the Burnside property for diffeomorphism groups. Recall that a *torsion group* is a group where every element has finite order. The existence of finitely-generated, *infinite* torsion groups was proved by Golod–Shafarevich [Gol64, Gv64] and Adian-Novikov [AN68]. A group H is said to have the *Burnside property* if every finitely-generated torsion subgroup of H is finite. Burnside and Schur proved that linear groups have the Burnside property [Bur02, Sch12] . E. Ghys and B. Farb asked homeomorphism groups of a compact manifolds have the Burnside property; see [Fis11, Question 13.2] and [Fis17, §5]. As an application of the tools used to prove the Main Theorem, we prove that certain diffeomorphism groups have the Burnside property.

Theorem 1.1. *Let M be a compact oriented* 3*-manifold. Assume that M is reducible and not a connected sum of lens spaces. Then* $\mathrm{Diff}(M)$ *has the Burnside property, i.e. every finitely-generated torsion subgroup of* $\mathrm{Diff}(M)$ *is finite.*

Smooth vs. topological Nielsen realization. The topological mapping class group $\mathrm{Mod}_H(M)$ is defined as the group of isotopy classes of orientation-preserving homeomorphisms of M. There is a natural projection map

$$\mathrm{Homeo}^+(M) \to \mathrm{Mod}_H(M),$$

and the Nielsen realization problem can also be asked for subgroups of $\mathrm{Mod}_H(M)$. For 3-manifolds, the smooth and topological mapping class groups coincide $\mathrm{Mod}(M) \cong \mathrm{Mod}_H(M)$ by Cerf [Cer59], who proved that $\mathrm{Diff}^+(M)$ and $\mathrm{Homeo}^+(M)$ are homotopy equivalent. Surprisingly, the realization problem for finite groups is also the same in the topological and smooth categories in dimension 3.

Theorem 1.2 (Pardon, Kirby–Edwards)**.** *Let M be a closed oriented 3-manifold. A finite subgroup $G <$* $\mathrm{Mod}(M)$ *is realizable by homeomorphisms if and only if it is realizable by diffeomorphisms.*

In particular, this allows us to strengthen the conclusion of the Main Theorem. We emphasize that the group G in Theorem 1.2 is finite; however, we do not know an example of a 3-manifold M and an infinite group $G < \mathrm{Mod}(M)$ that is realizable by homeomorphisms but not diffeomorphisms.

Proof of Theorem 1.2. Let $\rho : G \to \mathrm{Homeo}^+(M)$ be a realization of $G < \mathrm{Mod}(M)$. By Pardon [Par21], ρ can be approximated uniformly by a smooth action $\rho' : G \to \mathrm{Diff}^+(M)$. Since $\mathrm{Homeo}^+(M)$ is locally path-connected by Kirby–Edwards [EK71], we know that $\rho'(g)$ and $\rho(g)$ are isotopic in $\mathrm{Homeo}^+(M)$ for each $g \in G$, and hence also isotopic in $\mathrm{Diff}^+(M)$ since $\mathrm{Mod}_H(M) = \mathrm{Mod}(M)$. □

In dimension 4, the situation is different. Even the groups $\mathrm{Mod}(M)$ and $\mathrm{Mod}_H(M)$ may differ. This is true when M is a K3 surface by work of Donaldson [Don87] (c.f. [FL21, §1.1]) and Quinn [Qui86, Thm. 1.1]. See also Ruberman [Rub98]. Regarding Nielsen realization, Baraglia–Konno [BK19] give a simple example of an order-2 mapping class of a K3 surface that can be realized by homeomorphisms but not by diffeomorphisms.

Related work. The following remarks connect the Main Theorem to other previous work.

Remark 1.3 (Twist group for $S^1 \times S^2$)**.** The group $\mathrm{Twist}(S^1 \times S^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We construct a realization of this group in §5. This example seems to be overlooked in some of the literature on finite group actions on geometric 3-manifolds. It is a folklore conjecture of Thurston that any finite group action on a geometric 3-manifold is geometric (i.e. acts isometrically on some geometric structure). It is easy to see that our realization of the twist group, which also appears in work of Tollefson [Tol73], does not preserve any geometric structure on $S^1 \times S^2$, so it is a simple counterexample to Thurston's conjecture.

According to Meeks–Scott [MS86], Thurston proved some cases of his conjecture, but these results were not published. Meeks–Scott [MS86] proved Thurston's conjecture for manifolds modeled on $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil, $\mathbb{E}^3$, and Sol. In [MS86, Thm. 8.4] it is asserted (incorrectly) that Thurston's conjecture also holds for 3-manifolds modeled on $S^2 \times \mathbb{R}$ (in particular $S^2 \times S^1$); they give an argument, but in the case when some $g \in G$ has positive-dimensional fixed set (as is the case for our realization of $\mathrm{Twist}(S^1 \times S^2)$), they cite a preprint of Thurston that seems to have never appeared.

Remark 1.4 (Sphere twists in dimension 4)**.** For a 4-manifold W, for each embedded 2-sphere $S \subset W$ with self-intersection $S \cdot S = -2$, there is a sphere twists $\tau_S \in \mathrm{Mod}(W)$, which has order 2. There are several results known about realizing the subgroup generated by a sphere twist, both positive and negative; see Farb–Looijenga [FL21, Cor. 1.10], Konno [Kon22, Thm. 1.1], and Lee [Lee22, Rmk. 1.7]. It would be interesting to determine precisely when a sphere twist is realizable in dimension 4.

About the proof of the Main Theorem. The proof is divided into two parts: construction and obstruction (corresponding to the "if" and "only if" directions in the theorem statement). For the obstruction part of the argument, we prove the following constraint on group actions on reducible 3-manifolds (see Theorems 4.1 and 4.2).

Theorem 1.5. *Let M be a closed, oriented, reducible 3-manifold. Assume M is not a connected sum of copies of $\mathbb{R}P^3$. Let $G < \mathrm{Diff}^+(M)$ be a finite subgroup that acts trivially on $\pi_1(M)$. Then G is cyclic. If G is nontrivial, then M is a connected sum of lens spaces.*

Section outline. In §2, we recall results about sphere twists and the twist group. In §3 we explain results from minimal surface theory that allow us to decompose a given action into actions on irreducible 3-manifolds. In §4 and 5 we prove the "obstruction" and "construction" parts of the Main Theorem, respectively. In §6, we prove Theorem 1.1.

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2. Sphere twists and the twist subgroup

In this section we collect some facts about sphere twists and the group they generate. In §2.1 we recall the definition of sphere twists and recall that they act trivially on $\pi_1(M)$ and $\pi_2(M)$. In §2.2 we give a computation for the twist group of any closed, oriented 3-manifold (Theorem 2.4). The computation can be deduced by combining different results from the literature and gives a precise generating set.

2.1. **Sphere twists and their action on homotopy groups.** Fix a closed oriented 3-manifold M. We recall the definition of a sphere twist. Fix an embedded 2-sphere $S \subset M$ with a tubular neighborhood $U \cong S \times [0,1] \subset M$, and fix a closed path $\phi : [0,1] \to \mathrm{SO}(3)$ based at the identity that generates $\pi_1(\mathrm{SO}(3))$. Define a diffeomorphism of U by

$$T_S(x,t) = \big(\phi(t)(x), t\big) \tag{1}$$

and extend by the identity to obtain a diffeomorphism T_S of M. The isotopy class $\tau_S \in \mathrm{Mod}(M)$ of T_S is called a *sphere twist*. The *twist subgroup* of $\mathrm{Mod}(M)$, denoted $\mathrm{Twist}(M)$, is the subgroup generated by all sphere twists.

Lemma 2.1 (Action of sphere twists on π_1)**.** *Let M be a closed, oriented 3-manifold with a 2-sided embedded sphere $S \subset M$. Then τ_S acts trivially on $\pi_1(M)$.*

Remark 2.2 (Action on $\pi_1(M)$ vs. $\pi_1(M,*)$)**.** When we say $f \in \mathrm{Diff}(M)$ acts trivially on $\pi_1(M)$, we mean the induced outer automorphism is trivial (recall that after choosing a path from $f(*)$ to $*$, there is an induced automorphism

$$\pi_1(M,*) \xrightarrow{f_*} \pi_1(M, f(*)) \cong \pi_1(M,*),$$

which is well-defined up to inner automorphisms). If f is a diffeomorphism fixing $* \in M$, then we say f acts trivially on $\pi_1(M,*)$ if the induced map $f_* : \pi_1(M,*) \to \pi_1(M,*)$ is the identity. The distinction between f being trivial on $\pi_1(M)$ or $\pi_1(M,*)$ will be important in later sections. In general, if f acts trivially on $\pi_1(M)$ and fixes $* \in M$, then we can only conclude that f acts on $\pi_1(M,*)$ by conjugation.

Lemma 2.1 is well-known, e.g. it is implicit in [McC90]. It can by proved as follows. Let $* \in M$ be a fixed point of the diffeomorphism T_S defined in (1). After choosing a prime decomposition of M, one can show that each element of $\pi_1(M,*)$ is represented by a loop contained entirely in the fixed set of T_S.

Remark 2.3 (Action on $\pi_2(M)$)**.** Laudenbach [Lau74, Appendix III] showed that a diffeomorphism of $(M,*)$ that acts trivially on $\pi_1(M,*)$ also acts trivially on $\pi_2(M,*)$. See [BBP21, Thm. 2.4] for a short proof. As a generalization of the argument given there, if f is a diffeomorphism of $(M,*)$ that acts on $\pi_1(M,*)$ by conjugation, then the action of f on $\pi_2(M,*)$ agrees with the action of an element of $\pi_1(M,*)$. Furthermore, if f acts trivially on $\pi_1(M)$, then f can be isotoped to a diffeomorphism that acts on $\pi_1(M,*)$ by conjugation, hence on $\pi_2(M,*)$ by $\pi_1(M,*)$. Later we will use the fact that if two elements of $\pi_2(M,*)$ differ by the action of $\pi_1(M,*)$, then maps representing the homotopy classes are freely homotopic.

2.2. **Generators and relations in the twist group.** In this section we compute $\mathrm{Twist}(M)$ for every closed, oriented 3-manifold.

Theorem 2.4. *Let M be a closed, orientable 3-manifold with prime decomposition $M = \#_k(S^1 \times S^2)\#P_1\#\cdots\#P_\ell$, where the P_i are irreducible. Let $\ell' \leq \ell$ be the number of the P_i that are lens spaces. Then*

$$\mathrm{Twist}(M) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^k & \text{if } \ell - \ell' \leq 1 \\ (\mathbb{Z}/2\mathbb{Z})^{k+\ell-\ell'-1} & \text{otherwise.} \end{cases}$$

We were not able to find Theorem 2.4 in the literature, although it can be deduced by combining various old results.

For the proof of Theorem 2.4, we will use the following explicit construction of M. Let X be the complement of $\ell + 2k$ open disks in S^3. For each P_i choose a closed embedded disk $D_i \subset P_i$, and let Y be the compact manifold

$$Y := \left[\coprod_{i=1}^{\ell} P_i \setminus \mathrm{int}(D_i)\right] \sqcup \left[\coprod_k S^2 \times [-1,1]\right]. \tag{2}$$

We form M by gluing X and Y along their boundary $\partial X \cong \partial Y$. See Figure 1.

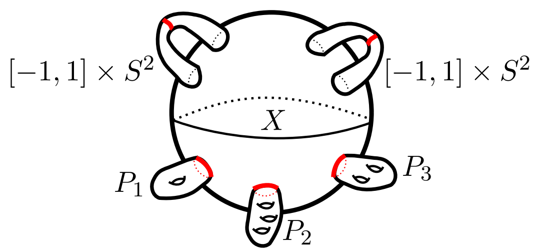

FIGURE 1. Construction of $M = \#_k(S^1 \times S^2)\#P_1\#\cdots\#P_\ell$. The red set represents spheres that generate $\mathrm{Twist}(M)$.

To state the following lemma, recall that the mapping class group $\mathrm{Mod}(Z)$ of a manifold with boundary is the group $\mathrm{Diff}_\partial(Z)$ of diffeomorphisms that restrict to the identity on ∂Z, modulo isotopies that are the identity on ∂Z.

Lemma 2.5 (Pants relation)**.** *Let Z be the complement in S^3 of three disjoint 3-balls. Let S_1, S_2, S_3 be the three boundary components of Z. Then $\tau_{S_1}\tau_{S_2}\tau_{S_3} = 1$ in $\mathrm{Mod}(Z)$.*

Lemma 2.5 can be proved in an elementary fashion by constructing an explicit isotopy. We explain the idea briefly. Represent τ_{S_i} by a diffeomorphism T_{S_i}, supported on a neighborhood $N(S_i) \cong S_i \times [0,1]$ of $S_i = S_i \times \{0\}$, where the path $\phi : [0,1] \to \mathrm{SO}(3)$ used to define T_{S_i} is a family of rotations with a fixed axis. In particular the fixed set of T_{S_i} in $N(S_i)$ has the form $\big(S_i \times \{0,1\}\big) \cup \big(\{p_i, q_i\} \times [0,1]\big)$ for a pair of (antipodal) points $p_i, q_i \in S_i$. Observe that there is an isotopy of T_{S_i} on $N(S_i)$ to a diffeomorphism T_i that is the identity on a tubular neighborhood of $\{p_i\} \times [0,1]$ and such that the restriction of T_i to $S_i \times \{1\}$ has support equal to an annulus and is a Dehn twist on that annulus (perform the isotopy level-wise on $N(S_i) = S_i \times [0,1]$ and supported near $\{p_i\} \times [0,1]$). Since T_i is not the identity on $S_i \times \{1\}$ it does not extend in an obvious way to an isotopy of Z. However, if we choose an arc $\alpha \subset Z$ joining $p_1 \in S_1$ to $p_2 \in S_2$ (and whose intersection with $N(S_i)$ is $\{p_i\} \times [0,1]$), then we can perform a similar isotopy of $T_{S_1} \circ T_{S_2}$ on a regular neighborhood N of $S_1 \cup \alpha \cup S_2$ to a diffeomorphism that is the identity on a neighborhood of α. By a further isotopy, we obtain a diffeomorphism T supported on a collar neighborhood of S_3', the boundary component of N in the interior of Z, and T represents the sphere twist $\tau_{S_3'}$. Finally, we observe that S_3' is parallel to S_3, so this proves the relation $\tau_{S_1}\tau_{S_2}\tau_{S_3} = 1$ in $\mathrm{Mod}(Z)$.

For the proof of Theorem 2.4 we also use the following theorem.

Theorem 2.6 (Hendriks, Friedman-Witt)**.** *Let P be an irreducible 3-manifold. Fix an embedded ball $D \subset P$, and let $\tau_S \in \mathrm{Mod}(P, D)$ be the sphere twist about a sphere S parallel to ∂D. Then $\tau_S = 1$ if and only if P is a lens space.*

Proof. By work of Hendriks [Hen77] (see [FW86, Cor. 2.1]), the twist $T_S \in \mathrm{Diff}_\partial(P\backslash\mathrm{int}(D))$ is not homotopic to the identity (rel boundary), unless P is either a lens space or a prism manifold (the latter are manifolds covered by S^3 whose fundamental group is an extension of a dihedral group).

First consider the case when P is a lens space. Since we assume P is irreducible, $P \neq S^1 \times S^2$, so P is covered by S^3. In this case T_S is isotopic to the identity (rel boundary) by [FW86, Lem. 3.5]. (Aside: T_S is also isotopic to the identity when $P = S^1 \times S^2$, which can be seen using Lemma 2.5.)

When P is a prism manifold, T_S is not isotopic to the identity (rel boundary). See [FW86, Thm. 2.2]. The statement there does not include one family of prism manifolds S^3/D^*_{4m}. This is because the argument uses the (generalized) Smale conjecture, which was not proved for S^3/D^*_{4m} at the time the paper was written. See [FW86, Remark after Corollary 2.2]. Fortunately, the generalized Smale conjecture has now been confirmed for all prism manifolds (in fact for all elliptic 3-manifold, with the exception of $\mathbb{R}P^3$). See [HKMR12] and [BK17]. $\square$

Proof of Theorem 2.4. First observe that $\mathrm{Twist}(M) = 1$ when M is irreducible. When $M = S^2 \times S^1$, then $\mathrm{Twist}(M) \cong \mathbb{Z}/2\mathbb{Z}$ was first computed by Gluck [Glu62, Thm. 5.1]. Given this, it remains to consider the case when M is reducible and not prime.

Step 1: a generating set for Twist(M). For $i = 1, \ldots, \ell$, fix an embedded sphere $S_i \subset Y$ that is parallel to the boundary component of $P_i \setminus \mathrm{int}(D_i)$. For $j = 1, \ldots, k$, let $S_j' \subset Y$ be the embedded sphere $S^2 \times \{0\}$ in the j-th copy of $S^2 \times [0, 1]$. Let $\mathrm{Twist}(Y)$ be the subgroup of $\mathrm{Mod}(Y)$ generated by the sphere twists $\{\tau_{S_1}, \ldots, \tau_{S_\ell}\} \cup \{\tau_{S_1'}, \ldots, \tau_{S_k'}\}$. Consider the composition

$$(\mathbb{Z}/2\mathbb{Z})^{\ell+k} \xrightarrow{\rho} \mathrm{Twist}(Y) \xrightarrow{\pi} \mathrm{Twist}(M), \tag{3}$$

where

$$\rho(a_1, \ldots, a_\ell, b_1, \ldots, b_k) = \tau_{S_1}^{a_1} \cdots \tau_{S_\ell}^{a_\ell} \tau_{S_1'}^{b_1} \cdots \tau_{S_k'}^{b_k},$$

and π is the restriction of the homomorphism $\mathrm{Mod}(Y) \to \mathrm{Mod}(M)$. The composition $\pi \circ \rho$ is surjective by [McC90, Prop. 1.2]. In the rest of the proof we compute the kernels of π and ρ.

Step 2: global relation among sphere twists. In this step we compute the kernel of $\pi : \mathrm{Twist}(Y) \to \mathrm{Twist}(M)$. We do this by first identifying $\mathrm{Mod}(Y)$ with a subgroup of $\mathrm{Mod}(M, D)$, where $D \subset M$ is an embedded ball, and then we examine how the kernel of the forgetful homomorphism $\mathrm{Mod}(M, D) \to \mathrm{Mod}(M)$ intersects $\mathrm{Twist}(Y)$.

Let $D \subset X \subset M$ be an embedded ball. Let $\mathrm{Emb}(D, M)$ be the space of embeddings that respect the orientation. Let $\mathrm{Emb}^e_D(X, M)$ be the space of embeddings $X \to M$ that

(i) restrict to the inclusion on D and (ii) that extend to a diffeomorphism of M. Consider the following diagram, which consists of two fiber sequences (c.f. [Pal60]).

$$\begin{tikzcd} \mathrm{Diff}_\partial(Y) \arrow[r] & \mathrm{Diff}(M,D) \arrow[r] \arrow[d] & \mathrm{Emb}^e_D(X,M) \\ & \mathrm{Diff}^+(M) \arrow[d] & \\ & \mathrm{Emb}(D,M) & \end{tikzcd}$$

The "horizontal" fiber bundle in the diagram splits as a product by [HM87, Thm. 1]. Consequently, $\mathrm{Mod}(Y) \to \mathrm{Mod}(M,D)$ is injective. Then from the preceding diagram, we obtain

$$\begin{tikzcd} & \pi_1\big(\mathrm{Emb}(D,M)\big) \arrow[d, "\delta"] \\ \mathrm{Mod}(Y) \arrow[r, hook] & \mathrm{Mod}(M,D) \arrow[d] \\ & \mathrm{Mod}(M) \end{tikzcd}$$

As is well-known, the space $\mathrm{Emb}(D,M)$ is homotopy equivalent to the (oriented) frame bundle of M, which is diffeomorphic to $M \times \mathrm{SO}(3)$ since closed oriented 3-manifolds are parallelizable. Then $\pi_1\big(\mathrm{Emb}(D,M)\big) \cong \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}$. A generator of the $\mathbb{Z}/2\mathbb{Z}$ factor maps under δ to a sphere twist about $S := \partial D$. From Lemma 2.5 we deduce that τ_S is equal to the product of sphere twists about each boundary component of X. For each $i = 1, \ldots, \ell$ (resp. $j = 1, \ldots, k$) the number of boundary components of X that are isotopic to S_i (resp. S'_j) is one (resp. two); since sphere twists have order 2, we deduce the relation $\tau_S = \tau_{S_1} \cdots \tau_{S_\ell}$ in $\mathrm{Mod}(M,D)$. This proves $\tau_{S_1} \cdots \tau_{S_\ell}$ belongs to $\ker(\pi)$.

We claim that $\ker(\pi) = \langle \tau_{S_1} \cdots \tau_{S_\ell} \rangle$. To see that $\ker(\pi)$ is not larger, it suffices to show that if $\gamma \in \pi_1(M) < \pi_1\big(\mathrm{Emb}(D,M)\big)$ is nontrivial, then $\delta(\gamma)$ is not in the image of $\mathrm{Twist}(Y) \to \mathrm{Mod}(M,D)$. This is easy to see because each element of $\mathrm{Twist}(Y) < \mathrm{Mod}(M,D)$ acts trivially on $\pi_1(M,*)$ (where the basepoint $*$ belongs to D), whereas $\delta(\gamma)$ acts by a nontrivial conjugation on $\pi_1(M,*)$ (note that the fundamental group of a reducible, non-prime 3-manifold has trivial center).

Step 3: local triviality of sphere twists. Here we compute the kernel of the map $\rho : (\mathbb{Z}/2\mathbb{Z})^{\ell+k} \to \mathrm{Twist}(Y)$ defined in (3). Since the spheres $S_1, \ldots, S_\ell, S'_1, \ldots, S'_k$ belong to distinct components of Y, it suffices to determine which of the given generators for $\mathrm{Twist}(Y)$ is trivial in $\mathrm{Twist}(Y)$. Sphere twists in $S^2 \times [-1,1]$ components are nontrivial by [Lau73]. See also [BBP21] who prove this by considering the action on framings. It remains then to consider when the twists τ_{S_i} (about the boundary of $P_i \setminus \mathrm{int}(D_i)$) are nontrivial. By Theorem 2.6, $\tau_{S_i} \in \mathrm{Mod}(P_i, D_i) < \mathrm{Mod}(Y)$ is trivial if and only if P_i is a lens space.

Conclusion. Combining this with the proceeding steps gives a full list of relations among the twists $\tau_{S_1}, \ldots, \tau_{S_\ell}$ and $\tau_{S'_1}, \ldots, \tau_{S'_k}$, i.e. a generating set for the kernel of the homomorphism $\pi \circ \rho : (\mathbb{Z}/2\mathbb{Z})^{\ell+k} \to \mathrm{Twist}(M)$ defined in (3). For example, if $\ell = 1$, then $\tau_{S_1} = 1$ is the only relation. If $\ell \geq 2$ and $\ell > \ell'$, then we have the relation $\tau_{S_1} \cdots \tau_{S_\ell} = 1$ and one additional relation for each of the S_i that bounds a lens space. If $\ell \geq 2$ and $\ell = \ell'$, then one of these relations is redundant. This completes the proof. □

For later use, we record the following corollary of Theorem 2.4. For the manifold $S^1 \times S^2$, we call a sphere of the form $* \times S^2$ a *belt sphere* (we use this terminology because this sphere can be viewed as the belt sphere of a handle attachment).

Corollary 2.7. *Let $M = \#_k(S^1 \times S^2) \# P_1 \# \cdots \# P_\ell$, and assume each P_i is a lens space. Then $\mathrm{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^k$ is generated by twists about the belt spheres of the $S^1 \times S^2$ summands.*

3. Decomposing finite group actions on 3-manifolds

In this section we explain some general structural results for certain finite group actions on 3-manifolds, which will allow us to decompose a G-manifold M^3 into simpler G-invariant pieces. For our application to the Main Theorem we are particularly interested in actions that are trivial on $\pi_i(M)$ for $i = 1, 2$.

3.1. **Equivariant sphere theorem.** The main result of this section is Theorem 3.1. In order to state it, we introduce some notation. Let $\mathbb{S}$ be a collection of disjoint embedded spheres in a 3-manifold M. Define $M_{\mathbb{S}}$ as the result of removing an open regular neighborhood of each $S \in \mathbb{S}$ and capping each boundary component with a 3-ball. The 3-manifold $M_{\mathbb{S}}$ is a closed, but usually not connected. This process is illustrated in Figure 2.

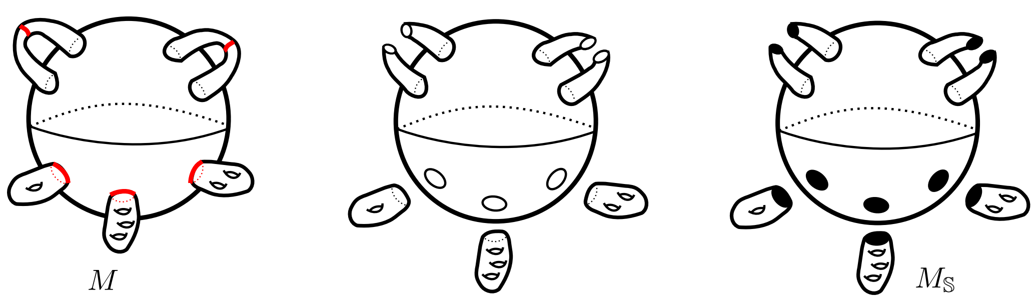

Figure 2. Cutting and capping along spheres $M \rightsquigarrow M_{\mathbb{S}}$.

Theorem 3.1. *Let M be a closed oriented 3-manifold and let G be a finite subgroup of $\mathrm{Diff}^+(M)$.*

(1) There exists a G-invariant collection $\mathbb{S}$ of disjoint embedded spheres in M such that the components of $M_{\mathbb{S}}$ are irreducible.

(2) If $M \neq S^1 \times S^2$ and G acts trivially on $\pi_1(M)$, then G preserves every element in $\mathbb{S}$. Furthermore, for each $S \in \mathbb{S}$, G preserves each of the two boundary components of a G-equivariant regular neighborhood of S.

We call a collection of spheres as in the statement of Theorem 3.1 a *sphere system* for G.

Remark 3.2. Without loss of generality, one can assume that no $S \in \mathbb{S}$ bounds a ball in M by removing any sphere that bounds a ball from $\mathbb{S}$. Similarly, if G preserves every element of $\mathbb{S}$, then we can also assume that no pair $S \neq S' \in \mathbb{S}$ bound an embedded $S^2 \times [0,1]$ in M. We call a sphere system with these additional properties an *essential sphere system* for G.

Part (1) of Theorem 3.1 is due to Meeks–Yau; see [MY80, c.f. Thm. 7]. An alternate approach was given by Dunwoody [Dun85, Thm. 4.1]. The main tools used in these works are minimal surface theory (in the smooth and PL categories). For the proof of Theorem 3.1(2), we use the following lemmas.

Lemma 3.3. *Let $S_0, S_1 \subset M$ be disjoint embedded spheres. If S_0 and S_1 are ambiently isotopic, then they bound an embedded $S^2 \times [0,1]$ in M.*

Lemma 3.3 follows from [Lau73, Lem. 1.2] and the Poincaré conjecture (Laudenbach proves that homotopic spheres bound an h-cobordism, and every h-cobordism is trivial by Perelman's resolution of the Poincaré conjecture).

Lemma 3.4. *Let h be an orientation-preserving homeomorphism of $S^2 \times [0,1]$. If h that interchanges the two boundary components, then h acts on $H_2(S^2 \times [0,1]) \cong \mathbb{Z}$ by -1.*

Proof of Lemma 3.4. Set $A = S^2 \times [0,1]$. Consider the arc $\alpha = * \times [0,1]$ and the sphere $\beta = S^2 \times 0$. After orienting α and β, we view them as homology classes $\alpha \in H_1(A, \partial A)$ and $\beta \in H_2(A)$, which generate these groups. Since h interchanges the components of ∂A, $h(\alpha) = -\alpha$. Since h is orientation-preserving,

$$\alpha \cdot \beta = h(\alpha) \cdot h(\beta) = -\alpha \cdot h(\beta).$$

This implies $h(\beta) = -\beta$ because the intersection pairing $H_1(A, \partial A) \times H_2(A) \to \mathbb{Z}$ is a perfect pairing by Poincaré–Lefschetz duality. $\square$

Proof of Theorem 3.1(2). Let $\mathbb{S}$ be a G-invariant collection of embedded spheres as in Theorem 3.1(1). Fix $S \in \mathbb{S}$ and $g \in G$. We want to show that $g(S) = S$. Suppose for a contradiction that $g(S)$ is disjoint from S. Fix an embedding $f : S^2 \to M$ with $f(S^2) = S$. Since g acts trivially on $\pi_1(M)$, the maps f and $g \circ f$ are homotopic (see Remark 2.3), hence isotopic by a result of Laudenbach and the Poincaré conjecture; c.f. [Lau73, Thm. 1].

By Lemma 3.3, the spheres S and $g(S)$ bound a submanifold $A \cong S^2 \times [0,1]$. Let $k \geq 2$ be the smallest power of g so that $g^k(S) = S$.

Suppose that $k = 2$. First we show that $g(A) = A$. By assumption $S \cup g(S)$ is g-invariant; hence so to is its complement. Then $g(A)$ is a component of $M \setminus \big(S \cup g(S)\big)$. Then if $g(A) \neq A$, we conclude that M is the union of A and $g(A)$, glued along their common boundary, so $M \cong S^2 \times S^1$ (this is the only S^2-bundle over S^1 with orientable total space). This contradicts our assumption, so $g(A) = A$. By Lemma 3.4, $[g(S)] = -[S]$ in $H_2(M)$. By assumption that G acts trivially on $\pi_1(M)$ together with Remark 2.3, we deduce that $[g(S)] = [S]$. Therefore, $2[S] = 0$ in $H_2(M)$. If $S \subset M$ is non-separating, then there is a closed curve $\gamma \subset M$ so that $[S] \cdot [\gamma] = 1$; this implies that $[S]$ has infinite order in $H_2(M)$, which is a contradiction. If $S \subset M$ is separating, then $M \setminus A$ is a union of two components $M_1 \sqcup M_2$ that are interchanged by g. This contradicts the fact that G acts trivially on $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$.

For the case $k \geq 3$, we prove the following.

Claim. If $k \geq 3$, then

$$A \cup gA \cup \cdots \cup g^i A \cong \begin{cases} S^2 \times [0,1] & \text{if } i \leq k-2 \\ S^2 \times S^1 & \text{if } i = k-1 \end{cases}$$

From the Claim, we deduce $M \cong S^2 \times S^1$, which contradicts our assumption and implies $k = 1$, as desired. Therefore, it only remains to prove the claim.

The claim can be proved inductively. We explain the case $i = 1$; the general case is very similar. We want to show that A and $g(A)$ have disjoint interiors, for this implies that $A \cup g(A) \cong S^2 \times [0,1]$. First observe that $g^2(S)$ cannot be in the interior of A. If it were, then $g^2(S)$ and $g(S)$ bound an annulus $A' \subset A$. Note that $g(A) \neq A'$ because we can average a metric so that g acts isometrically (and isometries are volume preserving). But if $g(A) \neq A'$, then M is the union of A' and $g(A)$, glued along their common boundary, so $M \cong S^2 \times S^1$, which again is a contradiction.

Similarly, we can show $g^{-1}(S)$ is disjoint from A. We use this to deduce that A and $g(A)$ have disjoint interiors. If not, then we can find a path in $g(A)$ that is disjoint from $g(S)$ and connects a point of A to a point of the complement A^c. Such a path necessarily intersects S, which implies $S \subset g(A)$. Equivalently, $g^{-1}(S) \subset A$, which contradicts the fact that $g^{-1}(S)$ is disjoint from A. This proves the case $i = 1$ of the claim, which can be used as the base case in an induction by a similar argument. □

3.2. **Decomposing an action along invariant spheres.** Here we explain how we use Theorem 3.1 to decompose an action $G \curvearrowright M$ into smaller pieces. We also prove a result about the action on the fundamental group of the pieces under the assumption that G acts trivially on $\pi_1(M)$.

Fix a finite subgroup $G < \mathrm{Diff}^+(M)$ and assume G acts trivially on $\pi_1(M)$. Let $\mathbb{S}$ be an essential sphere system for G (Theorem 3.1 and Remark 3.2).

Observe that there is an induced action of G on $M_{\mathbb{S}}$. To construct it, recall a classical result of Brouwer, Eilenberg, and de Kerékjártó [Bro19, dK19, Eil34] that every finite subgroup of $\mathrm{Homeo}^+(S^2)$ is conjugate to a finite subgroup of SO(3), hence extends from the unit sphere $S^2 \subset \mathbb{R}^3$ to the unit ball $D^3 \subset \mathbb{R}^3$. In this way the action of G on $M \setminus \bigcup_{S \in \mathbb{S}} S$ extends to an action on $M_{\mathbb{S}}$, which can be made smooth as well.

Remark 3.5 (global fixed points)**.** Since G acts trivially on $\mathbb{S}$ and preserves the boundary components of a regular neighborhood of each $S \in \mathbb{S}$, the center of each of the added 3-balls in $M_{\mathbb{S}}$ contains a global fixed point for the G-action; we call these *canonical fixed points*. Each component of $M_{\mathbb{S}}$ contains at least one canonical fixed point.

The following proposition will be important for our proof of the Main Theorem.

Proposition 3.6. *Fix $G < \mathrm{Diff}^+(M)$ acting trivially on $\pi_1(M)$ and fix an essential sphere system $\mathbb{S}$ for G. Let N be a component of $M_{\mathbb{S}}$, and let $p \in N$ be a canonical fixed point, as defined in Remark 3.5. Then G acts trivially on $\pi_1(N, p)$.*

For the proof, it may be helpful to remember that a nonseparating 2-sphere in an oriented 3-manifold is the belt sphere of a $S^2 \times S^1$ summand.

Proof. Fix $g \in G$. We show that the action of g on $\pi_1(N, p)$ is trivial. The statement is only interesting when $\pi_1(N)$ is nontrivial, so we assume this.

Let $S \in \mathbb{S}$ be the sphere associated with p. Fix an equivariant regular neighborhood $\nu(S) \cong S \times [-1, 1]$ of S in M, and denote $S_\pm = S \times \{\pm 1\}$. The spheres S_+, S_- have canonical embeddings in $M_{\mathbb{S}}$, which bound 3-balls B_+, B_-, respectively. For definiteness, we assume that $p \in B_+$. Let $q \in S$ be a fixed point of g, and let $q_\pm$ be the corresponding point of $S_\pm$.

Let $N_0 \subset M$ be the component of $M \setminus \bigcup_{S \in \mathbb{S}} \nu(S)$ corresponding to N. It suffices to show that g acts trivially on $\pi_1(N_0, q_+)$. Indeed, there is an obvious equivariant inclusion $N_0 \subset N$ which induces an isomorphism $\pi_1(N_0, q_+) \cong \pi_1(N, q_+)$ (because $N \setminus N_0$ is a union of 3-balls). Furthermore, g acts trivially on $\pi_1(N, q_+)$ if and only if g acts trivially on $\pi_1(N, p)$. This is because p and q_+ are connected by a path contained in the fixed set N^g.[1] Our goal will be to show g acts trivially on $\pi_1(N_0, q_+)$.

By Remark 2.2, the action of g on $\pi_1(M, q)$ is by conjugation by some element $\alpha \in \pi_1(M, q)$. Our basic strategy is as follows. We consider cases based on whether S is separating or nonseparating in M, and in the latter case, whether B_+ and B_- belong to the same or different components of $M_{\mathbb{S}}$. These cases are pictured in Figure 3. In each of these cases, we give a free product decomposition of $\pi_1(M, q)$ that contains $\pi_1(N_0, q_+)$ as a g-invariant free factor. Using additional structure from the specific case, we conclude $\alpha = 1$, which implies that g acts trivially on $\pi_1(M, q)$ and also on $\pi_1(N_0, q_+) < \pi_1(M, q)$.

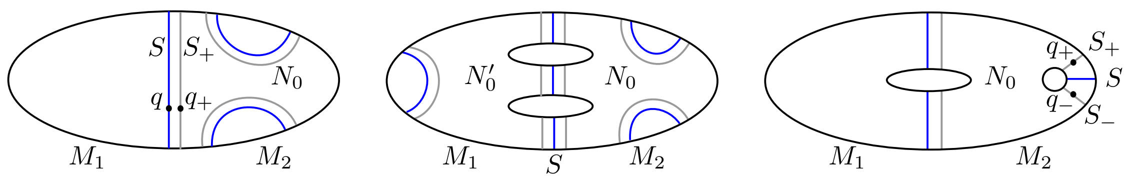

FIGURE 3. Different cases in the proof of Proposition 3.6.

Case 1. Suppose that S is separating in M. Let M_1, M_2 be the closures of the components of $M \setminus S$, and assume that M_2 contains N_0. See Figure 3 (left). Then $\pi_1(M_2, q) \cong \pi_1(N_0, q_+) * \Gamma$ for some group Γ, and there is a decomposition

$$\pi_1(M, q) \cong \pi_1(M_1, q) * \pi_1(N_0, q_+) * \Gamma. \tag{4}$$

The group $\pi_1(M_1, q)$ is nontrivial because otherwise M_1 is a 3-ball, which contradicts our assumption that $\mathbb{S}$ is an essential sphere system. Since the g action preserves each of M_1 and N_0, the free factors $\pi_1(M_1, q)$ and $\pi_1(N_0, q_+)$ are also preserved by g. Since g acts by conjugation by $\alpha \in \pi_1(M, q)$ this implies that α must belong to both $\pi_1(M_1, q)$ and $\pi_1(N_0, q_+)$. Therefore $\alpha = 1$, as desired.

Case 2. Suppose that S is nonseparating and that $B_\pm$ belong to different components of $M_{\mathbb{S}}$. Let N' be the component of $M_{\mathbb{S}}$ that contains B_-. Let $S = S_1, S_2, \ldots, S_{r+1}$ be the spheres in $\mathbb{S}$ along which N and N' are glued in M. Here $r \geq 1$ by the assumption that B_- is disjoint from N. Let M_1, M_2 be the closure of the components of $M \setminus (S_1 \cup \cdots \cup S_{r+1})$, and assume that $N' \subset M_1$ and $N \subset M_2$. See Figure 3 (middle).

First assume $r = 1$. Then $\pi_1(M_1, q)$ is nontrivial since otherwise S_1 and S_2 are parallel, which contradicts the fact that $\mathbb{S}$ is essential. Again we have a decomposition as in (4), where the first two factors are invariant by g. Then we conclude that g acts trivially on $\pi_1(N_0, q_+)$ by the same argument as in Case 1.

[1] Note that in general, changing the basepoint can change the automorphism to a nontrivial conjugation.

Now assume $r \geq 2$. If $\pi_1(M_1, q)$ is nontrivial, we can repeat the preceding argument. Assume then that $\pi_1(M_1, q) = 1$. Now we have a decomposition

$$\pi_1(M, q) \cong F_r * \pi_1(N_0, q_+) * \Gamma.$$

Here $\pi_1(M_2, q) \cong \pi_1(N_0, q_+) * \Gamma$ as before. Next we describe generators of the free group F_r. For each $i = 1, \ldots, r$, choose a point $q_i \in (S_i)^g$, choose[2] a path η_i in N_0 from q to q_i, and choose a path η_i' in M_1 from q_i to q. Then F_r is generated by the loops $\gamma_i = \eta_i * \eta_i'$ for $i = 1, \ldots, r$.

Now we compute the action of g. On the one hand,[3]

$$g(\gamma_i) \sim g(\eta_i) * \eta_i' \sim g(\eta_i) * \overline{\eta_i} * \eta_i * \eta_i' = (g(\eta_i) * \overline{\eta_i}) * \gamma_i,$$

so g acts on γ_i by left multiplication by the element $\beta_i = g(\eta_i) * \overline{\eta_i} \in \pi_1(N_0, q_+)$.

On the other hand, g acts on γ_i by conjugation by an element $\alpha \in \pi_1(M, q_0)$. The only way $\alpha\gamma_i\alpha^{-1} = \beta_i\gamma_i$ for each i is if α and the β_i are trivial. To see this, consider the word length on $\pi_1(M, q)$ given by the generating set $\{s : s \in \pi_1(N_0, q_+) \text{ or } s \in F_r\}$, then the word length of $\alpha\gamma_i\alpha^{-1}$ is odd, but the word length for $\beta_i\gamma_i$ is 2 unless $\beta_i = 1$. This implies that $\beta_i = 1$. Then $\gamma_i = \alpha\gamma_i\alpha^{-1}$ for every i, which implies that $\alpha = 1$.

Case 3. Suppose that S is nonseparating and that $B_\pm$ both belong to N. Here

$$\pi_1(M, q) \cong \mathbb{Z} * \pi_1(N_0, q_+) * \Gamma$$

for some group Γ. To describe a generator for the $\mathbb{Z}$ factor, fix a path η in N_0 from q_+ to q_- and otherwise disjoint from the spheres $S_\pm$. This path corresponds in an obvious way to an element $\hat{\eta} \in \pi_1(M, q)$, which generates the $\mathbb{Z}$ factor. Observe that $g(\eta)$ is another path in N_0 from q_+ to q_- and that the concatenation $g(\eta) * \bar{\eta}$ is an element of $\pi_1(N_0, q_+)$. The action of g on $\pi_1(M, q)$ sends $\hat{\eta}$ to $\widehat{g(\eta)}$ (the loop corresponding to the path $g(\eta)$ in N_0). Since $g(\eta)$ and $g(\eta) * \bar{\eta} * \eta$ are homotopic paths from q_+ to q_-, we conclude that g acts on $\hat{\eta}$ by multiplication on the left by

$$\beta := g(\eta) * \bar{\eta} \in \pi_1(N_0, q_+) < \pi_1(M, q).$$

On the other hand, g acts on $\pi_1(M, q_+)$ by conjugation by some element α, and this conjugation preserves $\pi_1(N, q_+) < \pi_1(M, q_+)$, so $\alpha \in \pi_1(N, q_+)$. Consequently, $\beta\hat{\eta} = \alpha\hat{\eta}\alpha^{-1}$ in $\pi_1(M, q_+)$. Arguing as in Case 2 (using a word length), we conclude that $\alpha = 1$. $\square$

4. Obstructing realizations

In this section we prove the "only if" direction of the Main Theorem. This can be deduced quickly from the following more general statements.

Theorem 4.1 (π_1-trivial action on irreducible 3-manifold)**.** *Let N be a closed, oriented, irreducible 3-manifold with basepoint $p \in N$. Suppose there exists a nontrivial, finite-order element $f \in \mathrm{Diff}^+(N, p)$ that acts trivially on $\pi_1(N, p)$. Then N is a lens space.*

[2]Technically, we should use $q_{i,\pm}$ and $q_\pm$ since q, q_i are not points of N_0, but will ignore this minor issue to avoid making the notation more cumbersome.

[3]Here the symbol $\sim$ indicates homotopic loops based at q. For the first homotopy, note that the paths $g(\eta_i')$ and η_i' are homotopic rel endpoints because M_1 is simply connected.

Theorem 4.2 (π_1-trivial action on reducible 3-manifold)**.** *Let M be a closed, oriented, reducible 3-manifold. Let $G < \mathrm{Diff}^+(M)$ be a finite subgroup that acts trivially on $\pi_1(M)$. Then G is cyclic unless M is either $S^2 \times S^1$ or a connected sum of projective spaces $\mathbb{R}P^3 \# \cdots \# \mathbb{R}P^3$.*

We comment on the excluded cases in Theorem 4.2. First observe that any finite subgroup of SO(3) acts π_1-trivially on $S^2 \times S^1$. The same is true for finite subgroups of $\mathrm{SO}(4)/\{\pm 1\}$ acting on $\mathbb{R}P^3$. Furthermore, by equivariant connected sum, one finds that any finite subgroup of SO(3) acts π_1-trivially on $\mathbb{R}P^3 \# \mathbb{R}P^3$, and every finite dihedral group has a π_1-trivial action on each manifold of the form $\mathbb{R}P^3 \# \cdots \# \mathbb{R}P^3$.

Proof of Main Theorem: obstruction. Suppose $1 \neq G < \mathrm{Twist}(M)$ is realizable. The fact that $\mathrm{Twist}(M) \neq 1$ implies that either $M = S^2 \times S^1$ or M is reducible and not prime. In the former case, there is nothing to prove, so we assume M is reducible and has a nontrivial prime decomposition. This assumption together with Lemma 2.1 allow us to apply Theorem 4.2 and conclude that either G is cyclic or M is a connected sum of copies of $\mathbb{R}P^3$. The latter case is excluded because $\mathrm{Twist}(\mathbb{R}P^3 \# \cdots \# \mathbb{R}P^3)$ is the trivial group (Corollary 2.7).

To show that M is a connected sum of lens spaces, fix a sphere system $\mathbb{S}$ for G (Theorem 3.1). It suffices to show that each component of $M_{\mathbb{S}}$ is a lens space, and this is implied directly by Proposition 3.6 and Theorem 4.1. □

We proceed to prove Theorems 4.1 and 4.2. Our argument for Theorem 4.1 is inspired by an argument of Borel [Bor83] that shows that a finite group G acting faithfully on a closed aspherical manifold N and $\pi_1(N)$ has trivial center, then G also acts faithfully on $\pi_1(N)$ (by outer automorphisms).

Remark 4.3 (Lifting actions to universal covers)**.** In this remark we recall some facts that are useful for Theorem 4.1. Let N be a closed manifold. Recall that $\widetilde{N}$ can be defined as the set of paths $\alpha : [0,1] \to N$ with $\alpha(0) = *$, up to homotopy rel endpoints. Using this description, there is a left action $\pi_1(N,*) \times \widetilde{N} \to \widetilde{N}$ given by pre-concatenation of paths $[\gamma].[\alpha] = [\gamma * \alpha]$, and there is a left action

$$\mathrm{Diff}(N,*) \times \widetilde{N} \to \widetilde{N}$$

given by post-composition $f.[\alpha] = [f \circ \alpha]$, and this action lifts the action of $\mathrm{Diff}(N,*)$ on N. Furthermore, if $f \in \mathrm{Diff}(N,*)$ acts trivially on $\pi_1(N,*)$, then the lift $[\alpha] \mapsto [f \circ \alpha]$ commutes with the deck group action and fixes the homotopy class of the constant path, as well as every other homotopy class corresponding to an element of $\pi_1(N,*)$.

Proof of Theorem 4.1. As observed in Remark 4.3, we can lift f to a finite-order diffeomorphism F that commutes with the deck group $\pi_1(N,*)$ and has a global fixed point.

First we show that $\pi_1(N)$ is finite. Suppose for a contradiction that $\pi_1(N)$ is infinite. This implies $\widetilde{N}$ is contractible.[4] By Smith theory [Smi34], the fixed set $(\widetilde{N})^F$ is connected, and simply connected. Since F acts smoothly, $(\widetilde{N})^F$ is a smooth 1-dimensional manifold, hence it is homeomorphic to $\mathbb{R}$. Since $\pi_1(N)$ commutes with F, it acts on $(\widetilde{N})^F \cong \mathbb{R}$, and this action is free and properly discontinuous since the action of $\pi_1(N,*)$ on $\widetilde{N}$ has these properties. This implies that $\pi_1(N,*) \cong \mathbb{Z}$, which contradicts the fact that N is a closed, aspherical 3-manifold ($\mathbb{Z}$ is not a 3-dimensional Poincaré duality group).

[4]By Hurewicz, $\pi_3(\widetilde{N}) \cong H_3(\widetilde{N})$. Since $\pi_1(N)$ is infinite, $\widetilde{N}$ is noncompact, so $H_3(\widetilde{N}) = 0$. Similarly, all higher homotopy groups vanish by Hurewicz's theorem.

Since $\pi_1(N)$ is finite, its universal cover is diffeomorphic to S^3 by the Poincare conjecture. As in the preceding paragraph, consider the action of F on $\widetilde{N} \cong S^3$. By Smith theory and smoothness of the action, the fixed set is a smooth, connected 1-dimensional manifold with nontrivial fundamental group. Hence $(\widetilde{N})^F \cong S^1$. Since $\pi_1(N)$ acts freely on $(\widetilde{N})^F$ this implies $\pi_1(N)$ is cyclic, which implies that N is a lens space. $\square$

Proof of Theorem 4.2. By assumption, G acts trivially on $\pi_1(M)$. Let $\mathbb{S}$ be an essential sphere system for G (Theorem 3.1 and Remark 3.2). Proposition 3.6 and Theorem 4.1 combine to show that each component of $M_{\mathbb{S}}$ is a lens space $L(p, q)$ with $p \neq 0$. We consider three cases.

Case 1. Suppose that there exists a component N of $M_{\mathbb{S}}$ that is diffeomorphic to $L(1, 0) \cong S^3$. Let k be the number of elements of $\mathbb{S}$ that meet N. Then $k \geq 3$ because $\mathbb{S}$ is essential. This implies that N^G has at least 3 points. By the Smith conjecture [Mor84], the action of G on $N \cong S^3$ is conjugate into $\mathrm{SO}(4)$, and the fact that $|N^G| \geq 3$ implies that G is conjugate into $\mathrm{SO}(2)$. Therefore G is cyclic.

Case 2. Suppose there exists a component N of $M_{\mathbb{S}}$ that is diffeomorphic to a lens space $L(p, q)$ with $p \geq 3$. Fix a canonical fixed point $* \in N$. By Remark 4.3, the action of G on $(N, *)$ lifts to an action on $S^3 = \widetilde{N}$ that fixes at least $p = |\pi_1(N)|$ points. By the Smith conjecture, the action of G on S^3 is conjugate into $\mathrm{SO}(4)$; furthermore, since $G \curvearrowright S^3$ has at least $p \geq 3$ fixed points, G is conjugate into $\mathrm{SO}(2)$, so G is cyclic.

Case 3. In the remaining case, every component of $M_{\mathbb{S}}$ is $L(2, 1) \cong \mathbb{R}P^3$. Since M is obtained from $M_{\mathbb{S}}$ by removing balls and tubing components to each other (or themselves), this implies that
$$M \cong \#_k(S^1 \times S^2) \# \mathbb{R}P^3 \# \cdots \# \mathbb{R}P^3.$$
If $M = \mathbb{R}P^3 \# \cdots \# \mathbb{R}P^3$ we are done (as this is a possible conclusion of the theorem), so we assume $k \geq 1$. This implies that there is a component $N \cong \mathbb{R}P^3$ of $M_{\mathbb{S}}$ such that some non-separating sphere $S \in \mathbb{S}$ meets N. This sphere contributes two canonical fixed points $x, y \in N$ to the G action on N. Using this, if we lift the action of G on (N, x) to $\widetilde{N} \cong S^3$, we find that G is conjugate into $O(2) < \mathrm{SO}(4)$. (The action of $G < \mathrm{SO}(4)$ on S^3 fixes two (antipodal) points, so G is conjugate into $\mathrm{SO}(3)$. Furthermore, there is another pair of antipodal points that are permuted by G, so G is conjugate into $O(2)$, which is the stabilizer of a line in $\mathrm{SO}(3)$.)

Since G is conjugate into $O(2)$, the group G is either cyclic or dihedral. Supposing that G is a noncyclic dihedral group, we will obtain a contradiction by showing that G acts non-trivially on $\pi_1(M)$.

First, observe that there is $g \in G$ so that x, y belong to different components of the fixed set N^g (which is a 1-dimensional manifold). For example, when the rotation subgroup of G has order > 2, the action of G on N has exactly two fixed points, and any reflection in $g \in G$ has the desired property (by reflection we mean an element not in the rotation subgroup). The case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is similar, but in that case, there is a unique choice for g.

We will show g acts nontrivially on $\pi_1(M)$. Here we use an analysis similar to Case 3 in the proof of Proposition 3.6, and we will use the same notation (e.g. N_0, $S_\pm$, and $q_\pm$) from that argument. Here N_0 is diffeomorphic to $\mathbb{R}P^3$ minus a finite collection of disjoint open 3-balls, and $\pi_1(M, q) \cong \mathbb{Z} * \pi_1(N_0, q_+) * \Gamma$ for some group Γ.

Fix a path η in $N_0 \subset N$ between q_+ and q_-. This arc determines a loop $\hat{\eta}$ in $\pi_1(M, q)$ that generates the $\mathbb{Z}$ factor in the decomposition above. One computes that the concatenation $\beta := g(\eta) * \bar{\eta}$ is a loop that generates $\pi_1(N_0, q_+) \cong \mathbb{Z}/2\mathbb{Z}$ (this can be written down explicitly – it is helpful to choose η to be a semi-circular arc). Therefore, $g(\hat{\eta}) = \beta\hat{\eta}$, where $\beta \neq 1$. By the argument from the proof of Proposition 3.6 we conclude that g does not act by conjugation on $\pi_1(M, q)$, so g acts nontrivially on $\pi_1(M)$. □

5. Constructing realizations

In this section we prove the "if" direction of the Main Theorem. We state this as the following theorem.

Theorem 5.1. *Let M be a connected sum of lens spaces. Then every cyclic subgroup of* $\mathrm{Twist}(M)$ *is realizable.*

Fix M as in Theorem 5.1, and write the prime decomposition

$$M = \#_k(S^1 \times S^2) \# P_1 \# \cdots \# P_\ell,$$

where each P_i is a lens space different from $L(0, 1) \cong S^1 \times S^2$.

To prove Theorem 5.1, given a nontrivial element $g \in \mathrm{Twist}(M)$ we define $\gamma \in \mathrm{Diff}(M)$ such that $\gamma^2 = id$ and $[\gamma] = g$ in $\mathrm{Mod}(M)$. The basic approach is to define an order-2 diffeomorphism of

$$\sqcup_k(S^1 \times S^2) \sqcup P_1 \sqcup \cdots \sqcup P_\ell \tag{5}$$

in such a way that the diffeomorphisms on the components can be glued to give an order-2 diffeomorphism of M. On each component of (5) we perform one of the following diffeomorphisms.

- (constant π rotation) Define
$$R_0 : S^1 \times S^2 \to S^1 \times S^2$$
by $id \times r$, where $r : S^2 \to S^2$ is any π rotation (choose one – the particular axis is not important).
- (nonconstant π rotation) Let $c : [0, 1] \to \mathbb{R}P^2$ be a closed path that generates $\pi_1(\mathbb{R}P^2)$, and let $\alpha : \mathbb{R}P^2 \to \mathrm{SO}(3)$ be the map that sends $\ell \in \mathbb{R}P^2$ to the π-rotation whose axis is ℓ. Now define
$$R_1 : S^1 \times S^2 \to S^1 \times S^2$$
by $(t, x) \mapsto (t, \alpha(c(t))(x))$.

  Since $\alpha \circ c : [0, 1] \to \mathrm{SO}(3)$ defines a nontrivial element of $\pi_1(\mathrm{SO}(3))$, the diffeomorphism R_1 represents the generator of $\mathrm{Twist}(S^1 \times S^2) \cong \mathbb{Z}/2\mathbb{Z}$. This shows that $\mathrm{Twist}(S^1 \times S^2)$ is realized. This involution appears in [Tol73, §1] in a slightly different form.
- (lens space rotation) Fix p, q relatively prime and with $p \geq 2$. View $L(p, q)$ as the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$ action generated by $(z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi iq/p}w)$. Define
$$R_{p,q} : L(p, q) \to L(p, q)$$
as the involution induced by $(z, w) \mapsto (z, -w)$ on S^3 (which descends to $L(p, q)$ since it commutes with the $\mathbb{Z}/p\mathbb{Z}$ action).

Each of the diffeomorphisms R_0, R_1, and $R_{p,q}$ has 1-dimensional fixed set. The representation in the normal direction at a fixed point is the antipodal map on $\mathbb{R}^2$ (there is no other option since these diffeomorphisms are involutions). Lemma 5.2 below allows us to glue these actions along their fixed sets.

Lemma 5.2. *Suppose M, M' are oriented manifolds, each with a smooth action of a finite group G. Assume that $x \in M$ and $x' \in M'$ are fixed points of G, and that the representations T_xM and $T_{x'}M'$ are isomorphic by an orientation reversing map. Then M and M' can be glued along regular neighborhoods B and B' of x and x' so that there is a smooth action of G on $M\#M'$ that restricts to the given action on $M \setminus B$ and $M' \setminus B'$.* $\square$

Remark 5.3. The condition that the isomorphism $T_xM \cong T_{x'}M'$ be orientation-reversing appears because the connected sum of two oriented manifolds is defined by deleting an open ball from each and identifying the boundaries of these balls by an orientation-reversing diffeomorphism. This condition is always satisfied if each tangent space contains a copy of the trivial representation (choose an appropriate reflection).

Remark 5.4 (Useful isotopies)**.** To prove that $\gamma \in \mathrm{Diff}(M)$ is in the isotopy class of $g \in \mathrm{Twist}(M)$, the following observation will be useful. The fixed set of $R_{p,q}$ acting on $L(p,q)$ contains[5] the image C of the circle $\{(z,0) : |z| = 1\} \subset S^3$. The isotopy $h_t(z,w) = (z, e^{\pi i(1-t)}w)$, $0 \le t \le 1$, descends to $L(p,q)$ to give an isotopy between $R_{p,q}$ and the identity, and h_t fixes C for each t.

Similarly, it's possible to isotope R_1 to R_1', which is a constant π-rotation (say about the z-axis) on a neighborhood of $* \times S^2$, for some fixed basepoint $* \in S^1$ (observe that R_1' is still an involution). Furthermore, we can isotope R_1' to a diffeomorphism that is the identity near $* \times S^2$ and in such a way that the isotopy at time $t \in [0,1]$ is a rotation by angle $\pi(1-t)$ (about the z-axis) on each sphere in a regular neighborhood of $* \times S^2$. The fixed set restricted to a neighborhood of $* \times S^2$ remains constant during this isotopy.

Finally, we can isotope R_0 to the identity so that at time t the diffeomorphism is a constant rotation by angle $\pi(1-t)$ (about the fixed axis).

On a neighborhood of a fixed point, the local picture of the isotopies of $R_{p,q}$, R_1', and R_0 looks the same. This will allow us to perform these isotopies equivariantly on connected sums.

We proceed now to the proof of Theorem 5.1. First we warm up with the case $M = \#_k(S^1 \times S^2)$ and then we do the general case.

5.1. Realizations for connected sums of $S^1 \times S^2$.

Fix $k \ge 1$ and consider
$$M_k := \#_k(S^1 \times S^2).$$
Let S_i be a belt sphere in the i-th connect summand, and denote the sphere twist about S_i by τ_i. The twists $\tau_1, \ldots, \tau_k$ form a basis for $\mathrm{Twist}(M_k) \cong (\mathbb{Z}/2\mathbb{Z})^k$, c.f. Corollary 2.7.

Fix a nonzero element
$$g = a_1\tau_1 + \cdots + a_k\tau_k$$
in $\mathrm{Twist}(M_k)$. We start by defining an involution $\hat{\gamma}$ of $\sqcup_k(S^1 \times S^2)$. For ease of exposition, let $W_i = S^1 \times S^2$ denote the i-th component of $\sqcup_k(S^1 \times S^2)$. Define $\hat{\gamma}$ on W_i to be R_0 or R_1, depending on whether the coefficient a_i is 0 or 1, respectively. Next we glue using Lemma 5.2 to obtain an involution γ of $M_k = W_1 \# \cdots \# W_k \cong \#_k(S^1 \times S^2)$. There are multiple ways to describe the gluing; for example, choose $k-1$ distinct fixed points

[5] It's possible that the fixed set is larger (this is true for $L(2,1) \cong \mathbb{R}P^3$), but this is not important.

$x_1, \ldots, x_{k-1} \in W_k$, and for $1 \leq i \leq k-1$, glue W_i to W_k along regular (equivariant) neighborhoods of x_i and an arbitrary fixed point $y_i \in W_i$ (the neighborhoods of $x_1, \ldots, x_k$ should be chosen to be small enough so that they are disjoint).

To see that $\gamma \in \mathrm{Diff}(M_k)$ is in the isotopy class g, recall the short exact sequence of Laudenbach

$$1 \to \mathrm{Twist}(M_k) \to \mathrm{Mod}(M_k) \to \mathrm{Out}(\pi_1(M_k)) \to 1.$$

It's easy to check that γ acts trivially on $\pi_1(M_k)$, so γ represents a mapping class in $\mathrm{Twist}(M_k)$. The particular isotopy class is determined by the action on trivializations of the tangent bundle of M_k, and in this way one can check that $[\gamma] = g$ in $\mathrm{Twist}(M_k)$. We do not spell out the details of this because we give an alternate argument in the next section in the general case.

Remark 5.5. We cannot realize a non-cyclic subgroup of $\mathrm{Twist}(M_n)$ using this construction because it is not possible to choose the axis for R_0 so that (1) R_0 and R_1 have a common fixed point and (2) R_0 and R_1 commute. Indeed, §4 proves no non-cyclic subgroup of $\mathrm{Twist}(M_n)$ is realized.

5.2. **Realizations for connected sum of lens spaces.** Now we treat the general case

$$M = \#_k(S^1 \times S^2)\#L(p_1, q_1)\# \cdots \#L(p_\ell, q_\ell),$$

where each $L(p_j, q_j)$ is a lens space different from $L(0, 1) \cong S^1 \times S^2$. Our approach is similar to the preceding section.

Recall from Corollary 2.7 that $\mathrm{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^k$ is generated by twists $\tau_1, \ldots, \tau_k$ in the belt spheres of the $S^1 \times S^2$ summands.

Fix a nonzero element

$$g = a_1\tau_1 + \cdots + a_k\tau_k$$

in $\mathrm{Twist}(M)$. We start by defining an involution $\hat{\gamma}$ of

$$\sqcup_k(S^1 \times S^2) \sqcup L(p_1, q_1) \sqcup \cdots \sqcup L(p_\ell, q_\ell).$$

Let W_i denote the i-th component diffeomorphic to $S^1 \times S^2$. Define $\hat{\gamma}$ on $L(p_j, q_j)$ to be R_{p_j,q_j}, and on W_i to be R_0 or R_1', depending on whether the coefficient a_i is 0 or 1, respectively. (Recall that R_1' is similar to R_1, but it has a product region.)

Next we glue using Lemma 5.2 to obtain an involution γ of M. We glue by the following pattern. First we glue $W_1, \ldots, W_k$. Choose $k-1$ distinct fixed points $x_1, \ldots, x_{k-1} \in W_k$, and for $1 \leq i \leq k-1$, glue W_i to W_k along regular (equivariant) neighborhoods of x_i and an arbitrary fixed point $y_i \in W_i$ (as was done in the preceding section). Next glue $L(p_j, q_j)$ to W_k in a region where $\hat{\gamma}$ acts as a product (we can choose $x_1, \ldots, x_{k-1}$ and the regular neighborhoods of these points to ensure that there is room to do this). In this way we obtain an involution $\gamma \in \mathrm{Diff}(M)$.

We need to check that γ is in the isotopy class of g. Using the isotopies defined in Remark 5.4, we can isotope $\hat{\gamma}$ to a map that is the identity on each $L(p_j, q_j)$ component and each component W_i such that $a_i = 0$, and is a sphere twist on each component W_i such that $a_i = 1$. By construction these isotopies glue to give an isotopy of γ to a product of sphere twists representing g.

This completes the proof of Theorem 5.1. □

Question 5.6. Are any two realizations of $g \in \mathrm{Twist}(M)$ conjugate in $\mathrm{Diff}(M)$?

6. Burnside Problem for $\mathrm{Diff}(M)$ for reducible 3-manifold M

In this section, we prove Theorem 1.1, which follows quickly from Lemma 6.1.

Lemma 6.1. *Fix a closed, oriented 3-manifold M, and consider the group*

$$K := \ker \left[\mathrm{Diff}(M) \xrightarrow{\Phi} \mathrm{Out}(\pi_1(M)) \right].$$

If M is reducible and that M is not a connected sum of lens spaces, then K is torsion free.

Remark 6.2 (A strong converse to Lemma 6.1)**.** If M is a connected sum of lens spaces, then M has a faithful S^1-action, so K contains S^1 as a subgroup. To see this, observe that each lens space has an S^1 action with global fixed points, so by performing the connected sum equivariantly along fixed points (similar to the construction in Section 5) we obtain an S^1 action on M.

Proof of Theorem 1.1. If $\mathrm{Diff}(M)$ has an infinite torsion group G, then either the image of Φ has an infinite torsion subgroup or the kernel K of Φ has an infinite torsion subgroup. By [McC90, Cor. 2.2], $\mathrm{Out}(\pi_1(M))$ contains a torsion-free finite-index subgroup. This implies that $\Phi(G)$ is finite, since every element of $\Phi(G)$ has finite order. On the other hand, by Lemma 6.1 that K is torsion free, so $G \cong \Phi(G)$ is finite. □

Proof of Lemma 6.1. Fix a nontrivial subgroup $G = \mathbb{Z}/\ell\mathbb{Z} < K$, and fix a sphere system $\mathbb{S}$ for G (Theorem 3.1). By Proposition 3.6, the action of G on each component N of $M_{\mathbb{S}}$ is trivial on $\pi_1(N, p)$ as an automorphism. This implies that each component N of $M_{\mathbb{S}}$ is a lens space by Theorem 4.1. □

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