Problem 1. *The helicoid is the surface given by the chart*

$$\phi(u,v) = (v\cos u, v\sin u, u), \quad u, v \in \mathbb{R}.$$

Use a mathematica `ParametricPlot3D` *(or similar) to plot this surface. Compute (by hand) the first and second fundamental forms* I, II *and mean curvature* H *of this surface.*[1]

Solution. $\phi_u = (-v\sin u, v\cos u, 1), \qquad \phi_v = (\cos u, \sin u, 0),$

$$\phi_{uu} = (-v\cos u, -v\sin u, 0), \quad \phi_{uv} = (-\sin u, \cos u, 0), \quad \phi_{vv} = (0,0,0),$$

$$\phi_u \times \phi_v = (-\sin u, \cos u, -v\cos^2 u - v\sin^2 u) = (-\sin u, \cos u, v)$$

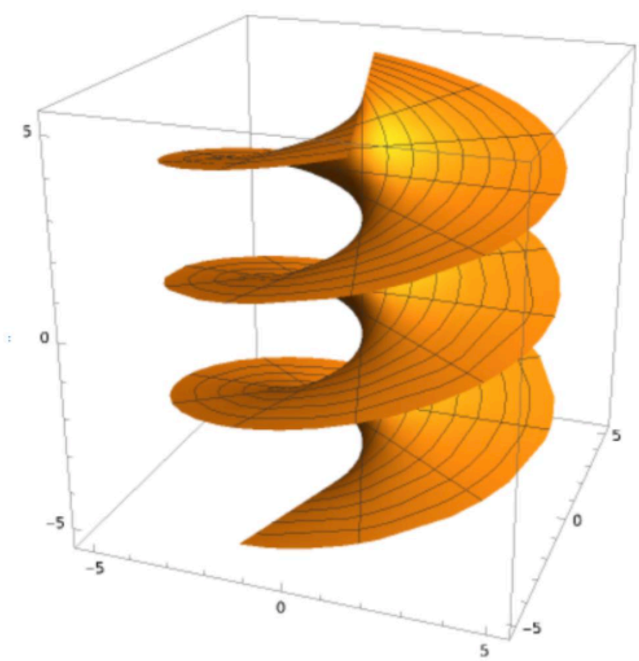
$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-\sin u, \cos u, v)}{\sqrt{1+v^2}}$$

$$\begin{aligned} E &= \phi_u \cdot \phi_u = v^2\sin^2 u + v^2\cos^2 u + 1^2 &&= 1+v^2, \\ F &= \phi_u \cdot \phi_v = -v\sin u\cos u + v\cos u\sin u = 0, \\ G &= \phi_v \cdot \phi_v = \cos^2 u + \sin^2 u &&= 1, \end{aligned}$$

$$\begin{aligned} e &= N \cdot \phi_{uu} = (1+v^2)^{-1/2}(v\cos u\sin u - v\sin u\cos u) = 0, \\ f &= N \cdot \phi_{uv} = (1+v^2)^{-1/2}(\sin^2 u + \cos^2 u) &&= (1+v^2)^{-1/2}, \\ g &= N \cdot \phi_{vv} = (1+v^2)^{-1/2}(0) &&= 0, \end{aligned}$$

$$\mathrm{I} = \begin{bmatrix} 1+v^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathrm{II} = \begin{bmatrix} 0 & (1+v^2)^{-1/2} \\ (1+v^2)^{-1/2} & 0 \end{bmatrix},$$

$$H = \left(-\frac{1}{2}\right)\frac{eG + gE - 2fF}{EG - F^2} = \left(-\frac{1}{2}\right)\frac{0+0-2\cdot 0}{EG-F^2} = 0. \qquad \square$$



[1] Hint: Your answer for the mean curvature, if correct, will be exceedingly simple.

Problem 2. *Consider the curve*[2]

$$\alpha(t) = (t - \tanh t, \operatorname{sech} t, 0), \quad t > 0.$$

Let S be the surface obtained by revolving α about the x-axis. Use a mathematica `ParametricPlot3D` *(or similar) to plot this surface. Compute (by hand) the first and second fundamental forms* I, II *and Gauss curvature K of this surface.*[3]

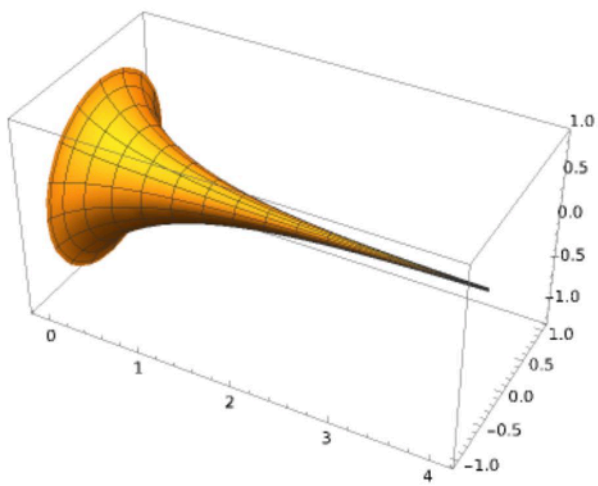
Solution. We begin by noting that we can obtain the parametrization of the surface by multiplying the parametrization of the curve by the rotation matrix $R_x(\theta)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Taking the product $\alpha(t)R_x(\theta)$, we obtain a coordinate map for the surface:

$$\phi(t,\theta) = (t - \tanh t, \operatorname{sech} t \cos\theta, \operatorname{sech} t \sin\theta)$$

for $t > 0$ and $0 < \theta < 2\pi$. The result from plotting (with $0 < t < 5$ for computational efficiency) is:



We now compute the first fundamental form $\mathrm{I} = \left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$, where $E = \phi_t \cdot \phi_t$, $F = \phi_t \cdot \phi_\theta$, and $G = \phi_\theta \cdot \phi_\theta$. We have:

$$\begin{aligned} \phi_t &= (1 - \operatorname{sech}^2(t), -\operatorname{sech}(t)\tanh(t)\cos(\theta), -\operatorname{sech}(t)\tanh(t)\sin(\theta)), \\ \phi_\theta &= (0, -\operatorname{sech}(t)\sin(\theta), \operatorname{sech}(t)\cos(\theta)). \end{aligned}$$

Applying the identity $\tanh^2(t) + \operatorname{sech}^2(t) = 1$, we obtain:

$$\begin{aligned} \phi_t &= (\tanh^2(t), -\operatorname{sech}(t)\tanh(t)\cos(\theta), -\operatorname{sech}(t)\tanh(t)\sin(\theta)), \\ \phi_\theta &= (0, -\operatorname{sech}(t)\sin(\theta), \operatorname{sech}(t)\cos(\theta)). \end{aligned}$$

[2] Recall the hyperbolic trig functions are defined by $\cosh(t) = \frac{e^t + e^{-t}}{2}$, $\sinh(t) = \frac{e^t - e^{-t}}{2}$, etc. I suggest you look up formulas for the derivatives and identities satisfied by these functions.

[3] Hint: Your answer for the Gauss curvature, if correct, will be exceedingly simple.

Applying the dot product, we obtain:

$$\begin{aligned}
E &= \phi_t \cdot \phi_t = \tanh^4(t) + \operatorname{sech}^2(t)\tanh^2(t) = \tanh^2(t)(\tanh^2(t) + \operatorname{sech}^2(t)) = \tanh^2(t), \\
F &= \phi_t \cdot \phi_\theta = \operatorname{sech}^2(t)\tanh(t)\cos(\theta)\sin(\theta) - \operatorname{sech}^2(t)\tanh(t)\sin(\theta)\cos(\theta) = 0, \text{ and} \\
G &= \phi_\theta \cdot \phi_\theta = \operatorname{sech}^2(t)\sin^2(\theta) + \operatorname{sech}^2(t)\cos^2(\theta) = \operatorname{sech}^2(t).
\end{aligned}$$

Thus, we have:

$$\mathrm{I} = \begin{pmatrix} \tanh^2(t) & 0 \\ 0 & \operatorname{sech}^2(t) \end{pmatrix}.$$

To compute the second fundamental form, $\mathrm{II} = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)$, where $e = -\phi_t \cdot \hat{n}_t$, $f = -\phi_t \cdot \hat{n}_\theta$, and $g = -\phi_\theta \cdot \hat{n}_\theta$, we first compute the unit normal $\hat{n}$. We have:

$$\begin{aligned}
\hat{n} &= \frac{\phi_t \times \phi_\theta}{|\phi_t \times \phi_\theta|} \\
&= \frac{(-\tanh(t)\operatorname{sech}^2(t), -\tanh^2(t)\operatorname{sech}(t)\cos(\theta), -\tanh^2(t)\operatorname{sech}(t)\sin(\theta))}{\sqrt{\det \mathrm{I}}} \\
&= \frac{(-\tanh(t)\operatorname{sech}^2(t), -\tanh^2(t)\operatorname{sech}(t)cos(\theta), -\tanh^2(t)\operatorname{sech}(t)\sin(\theta))}{\operatorname{sech}(t)\tanh(t)} \\
&= (-\operatorname{sech}(t), -\tanh(t)\cos(\theta), -\tanh(t)\sin(\theta)).
\end{aligned}$$

Toward computing e, f, and g, we also calculate $\hat{n}_t$ and $\hat{n}_\theta$:

$$\begin{aligned}
\hat{n}_t &= (\operatorname{sech}(t)\tanh(t), -\operatorname{sech}^2(t)\cos(\theta), -\operatorname{sech}^2(t)\sin(\theta)) \\
\hat{n}_\theta &= (0, \tanh(t)\sin(\theta), -\tanh(t)\cos(\theta))
\end{aligned}$$

Now, we can compute e, f, and g:

$$\begin{aligned}
e &= -\phi_t \cdot \hat{n}_t = -\tanh(t)\operatorname{sech}(t)(\operatorname{sech}^2(t)(\cos^2(\theta) + \sin^2(\theta)) + \tanh^2(t)) = -\tanh(t)sech(t). \\
f &= -\phi_t \cdot \hat{n}_\theta = 0. \\
g &= -\phi_\theta \cdot \hat{n}_\theta = \operatorname{sech}(t)\tanh(t)\sin^2(\theta) + \operatorname{sech}(t)\tanh(t)\cos^2(\theta) = \operatorname{sech}(t)\tanh(t).
\end{aligned}$$

Finally, we can write the second fundamental form:

$$\mathrm{II} = \begin{pmatrix} -\tanh(t)\operatorname{sech}(t) & 0 \\ 0 & \operatorname{sech}(t)\tanh(t) \end{pmatrix}.$$

The Gauss curvature is given by:

$$\begin{aligned}
K &= \frac{\det \mathrm{II}}{\det \mathrm{I}} \\
&= \frac{-\operatorname{sech}^2(t)\tanh^2(t)}{\operatorname{sech}^2(t)\tanh^2(t)} \\
&= -1
\end{aligned}$$

Yet again, we've achieved an exceedingly simple result! Whoop whoop!

$\square$

Problem 3. *Let S be a surface with a unit normal $N : S \to S^2$. Let $\alpha : I \to S$ be a unit speed curve. Assume that $\alpha'(t)$ is a principal direction for each t.*[4] *Show that the curvature $\kappa = \kappa_\alpha$ of α satisfies $\kappa = |k_n k_N|$, where k_n is the normal curvature and k_N is the curvature of $N \circ \alpha$.* [5]

Solution. The key here is to make liberal use of the fact that because α is a line of curvature, $(N \circ \alpha)' = DN_p(\alpha') = \lambda\alpha'$ for some real-valued function $\lambda(t)$. Now we just compute k_n and k_N.

$$0 = ((N \circ \alpha) \cdot \alpha')' = DN_p(\alpha') \cdot \alpha' + N \cdot \alpha'' = \lambda + N \cdot \alpha''.$$

Thus, $k_n = N \cdot \alpha'' = -\lambda$. Now we compute k_N using the formula from homework 2:

$$k_N = \frac{|(N \circ \alpha)' \times (N \circ \alpha)''|}{|(N \circ \alpha)'|^3} = \frac{|\lambda\alpha' \times \lambda\alpha''|}{|\lambda\alpha'|^3} = \frac{\lambda^2|\alpha''|}{|\lambda^3|} = \frac{\kappa_\alpha}{|\lambda|}.$$

For the second equality, we are using the fact that α' is an eigenvector of DN_p. For the third inequality, we are using the fact that α is unit speed, which gives that $|\alpha'| = 1$ and that α' is perpendicular to α'', so the magnitude of their cross product is the product of their magnitudes.

Now it is clear that $|k_n k_N| = |\lambda \cdot \frac{\kappa_\alpha}{|\lambda|}| = |\kappa_\alpha| = \kappa_\alpha$.

□

[4] Remark: In this case, α is a called a *line of curvature*.

[5] Hint: note that $N \circ \alpha$ is not necessarily unit speed. Use a formula for curvature from a previous problem.

Problem 4. *Let S be a surface, and fix $q \in \mathbb{R}^3$. Define $f : S \to \mathbb{R}$ by $f(p) = |p - q|^2$. Give a formula for $Df_p(w)$ directly using the way we defined the derivative of a function on a surface in class. When is p a critical point*[6] *of f?*

Solution. Let $\alpha : I \to S$ be a curve on S, such that $\alpha(0) = p$ and $\alpha'(0) = w$. Then, we can express $Df_p(w)$ as $Df_p(\alpha'(0))$:

$$\begin{aligned} Df_p(\alpha'(t)) &= (f \circ \alpha)'(t)|_{t=0} \\ &= \frac{d}{dt}((\alpha(t) - q)(\alpha(t) - q))|_{t=0} \\ &= \alpha'(t)(\alpha(t) - q) + (\alpha(t) - q)\alpha'(t)|_{t=0} \\ &= 2\alpha'(t)(\alpha(t) - q)|_{t=0} \\ &= 2w \cdot (p - q) \end{aligned}$$

The point p is a critical point of f when for all w at a point p, $Df_p = 2w \cdot (p - q) = 0$. Thus, p is a critical point when w and $p - q$ are orthogonal. Since w can be any vector in the tangent plane, $p - q$ must then be in the orthogonal space of the tangent plane, so $p - q$ must be parallel to the normal line at p. □

[6] We say that $p \in S$ is a critical point of $f : S \to \mathbb{R}$ if $Df_p = 0$.

Problem 5. *Let $S \subset \mathbb{R}^3$ be a surface. Suppose that there exists a point $q \in \mathbb{R}^3$ such that the normal line through $p \in S$ passes through q for each $p \in S$. Prove that S is contained in a sphere.* [7]

Solution. Consider the function $f(p) = |p-q|^2$ where q is the point contained in the normal line through every $p \in S$. We know by problem 4 that because q is contained in the normal line through every p, $Df_p \equiv 0$ for every $p \in S$. This means that f is a constant function and that $|p-q|^2 = r^2$ for all $p \in S$ for some constant r^2. By defintion, this means that S is a subset of the sphere of radius r around q.
(Technically this only proves that $|p-q|^2$ is constant across any given coordinate chart, meaning that S could contain subsets of several spheres of different radii around q, as remarked in Ed Discussion post #43, but that's still pretty cool, and its equivalent to the intended claim if we assume S is connected.) $\square$

[7]Hint: Define an appropriate function on S whose derivative is constant...