Problem 1. *The complete bipartite graph $K_{n,m}$ is the graph with $n+m$ vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$ and edges $\{v_i, u_j\}$ for each $1 \le i \le n$ and $1 \le j \le m$. Determine the values n, m so that $K_{n,m}$ is Eulerian.*

Solution. We know that a graph is Eulerian iff all vertices have even degree. For $K_{n,m}$, every vertex u_i has edges to all the v_j and therefore has degree n, and similarly, every vertex v_j has edges to all the u_i and therefore has degree m. Therefore, all the vertices of $K_{n,m}$ have even degree iff n and m are even. Therefore, $K_{n,m}$ is Eulerian iff n, m are even. $\square$

[redacted]

[redacted]

Problem 2. *Prove or disprove:*

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian graph with an even number of vertices has an even number of edges.

Solution. (a) We will show that this proposition is true. Since a bipartite graph contains two disjoint sets of vertices, with each vertex adjacent only to vertices from the opposite set, the number of edges is equal to the sum of degrees of the vertices in one partition (which, by definition, must be equal to the sum of degrees of the vertices in the other partition). If there is an Euler tour, then all degrees are even, and the sum of even numbers is even.

(b) We will show that this proposition is false. Consider a graph G with a cycle of length 3 and a cycle of length 4, joined at a single vertex. G has 6 vertices and 7 edges, and it contains an Euler tour, so this is a counterexample.

□

[redacted]

Problem 4. *Determine the number of graphs with* 7*-vertices, each of degree* 4 *(up to isomorphism).*
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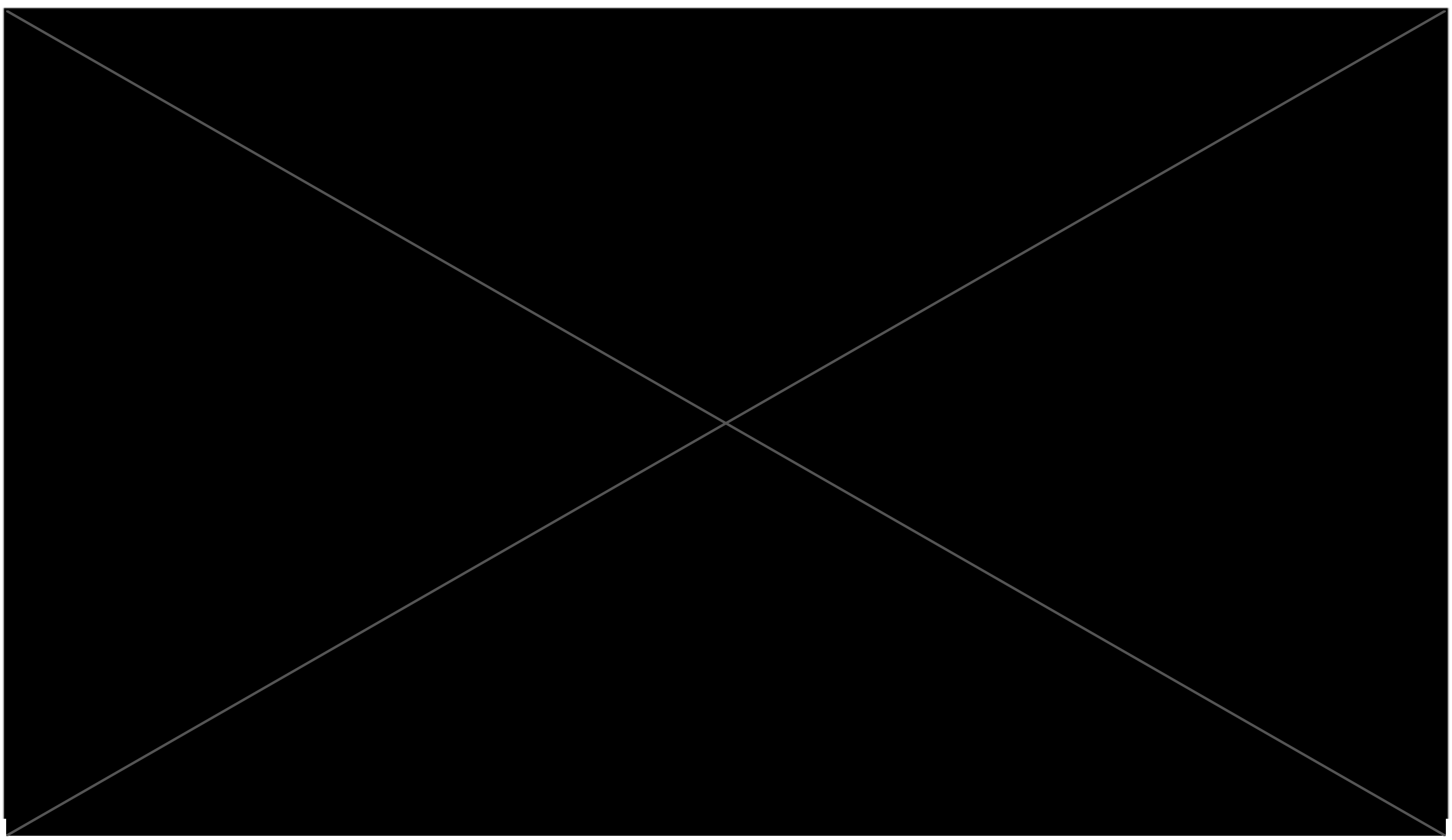
Solution. Instead look at the complements of these graphs. In a complement graph, each of the 7 vertices would have degree 2 $(6-4)$, and there would be 7 edges in total. To ensure that the number of edges is equal to the number of vertices, and each vertex has degree exactly 2, we need our graph to be made up of one or more non-overlapping cycles. There are only two ways to do this: either our complement graph is a heptagon, or it consists of a disjoint square and triangle. Since iff the complements of two graphs are isomorphic, the graphs are isomorphic (proven on the last homework), we conclude that there are only two graphs with 7 vertices, each of degree 4 (up to isomorphism). □



Problem 5. *Use induction on the number of edges to prove that a graph with no odd cycle is bipartite.*

This is clearly true for 0 edges, which is our base case.

Now suppose that graphs with $\leq n$ edges with no odd cycle are bipartite. Let G be a graph with $n+1$ edges with no odd cycle.

Delete an edge e from G. This gives a graph H with n edges, which cannot have an odd cycle because then G would have an odd cycle. By our assumption, H must be bipartite. Thus, we can two-color H.

If in this two-coloring, the two vertices v, w incident to e are colored opposite colors, then the coloring remains valid if we re-add e, hence G is bipartite.

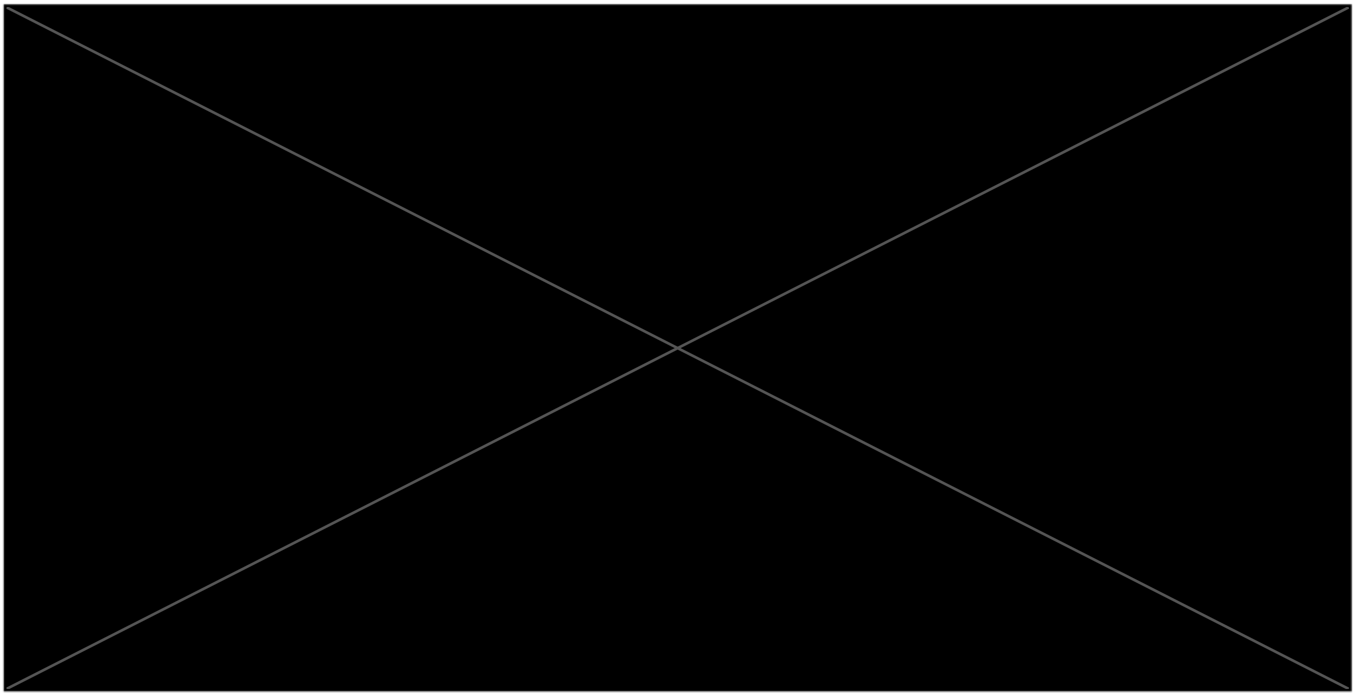
If v and w are colored the same and there exists a path P from v to w in H, then P must have even length (because with each step along the path the color of the reached vertex flips, hence for v and w to be the same color an even number of steps must be taken between them). But this even path from v to w in H induces one in G, which yields an odd cycle by moving along e after completion of the path, a contradiction.

Finally, we consider the case where v and w are colored the same and where there does not exist a path from v to w in H. Since such a path does not exist, v and w are in different connected components. Since there is no edge between these components, flipping the color of every vertex in the connected component C of w also yields a valid coloring of H, as any two vertices of opposite color in C will remain opposite colors after the flip. Thus this yields a coloring of H where v and w are of opposite colors, meaning e can be safely re-added to the graph yielding a 2-coloring of G.

This completes the induction, hence any graph with no odd cycle is bipartite.

Solution. Collaborated with [redacted]

□



Problem 6. *Suppose there are two mountain trails, each starting at sea level and ending at the same elevation. Suppose hikers A,B start hiking these two different trails at the same time. The Mountain Climber Problem asks if it is possible for A and B to hike to the top of their individual trails in a way so that they have the same elevation at every time.*[3] *We model the trails by functions* $f, g : [0,1] \to [0,1]$ *with* $f(0) = g(0) = 0$ *and* $f(1) = g(1) = 1$. *In this problem you solve the Mountain Climber Problem in the case when* f *and* g *are piecewise linear continuous functions.*[4]

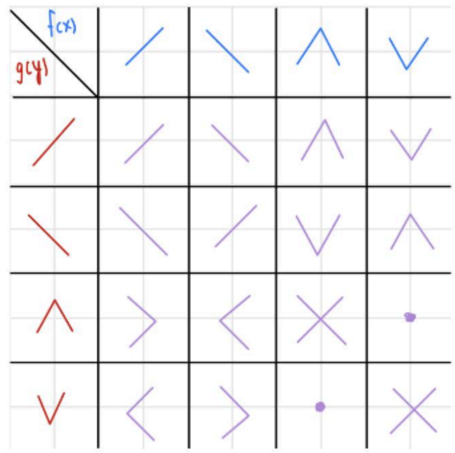
(a) *Consider*

$$Z = \{(x, y) \in [0,1] \times [0,1] : f(x) = g(y)\}$$

Assuming f, g *are piecewise linear, determine the local picture near* (x, y) *in* Z, *considering cases based on the local pictures of* f *and* g *near* x *and* y, *respectively.*

(b) *Observe that* Z *can be given the structure of a graph* G. *Show that* G *has exactly two vertices of odd degree. Deduce that there is a path in* G *from* $(0,0)$ *to* $(1,1)$.

Solution. The local picture for f and/or g is either locally linear (sloping upward or downward), a kink (ie a cusp where f/g do not change sign on the slope), or a cusp that is a local extrema. Note that having a kink in f or g does not produce a qualitatively different result in Z compared to simply being locally linear, as the important piece is knowing the sign of the slope of f and g locally around x and y. If the kink does not change the sign of the slope, then there will be no qualitative difference. As such, we will not include kinks separately. The combinations of local images in f and g and the resulting local image in Z is shown in the table below:



To illustrate how the table is constructed, consider the case where both f and g have a local maximum at the same elevation. If both hikers are at the local maxima in both f and g, then they only place they can go is down. For hiker A on path f, this could involve going left or right, thus either increasing or decreasing x. Similarly, hiker B on path g can also decrease their elevation by going left or right, either increasing or decreasing y. This means that we have four options in Z: either both hikers can decrease their elevation by going left (decreasing both x and y), both hikers can decrease elevation by going right (increasing both x and y), or by going in opposite directions (ie x increases and y decreases or y increases and x decreases). This gives us the intersection of 4 lines that we see in Z which corresponds to two local maxima in f and g. The other cells in the table are filled out in a similar manner.

We note that if we have a local maximum in f and a local minimum in g where the local extrema in each function are at the exact same elevation, we get a point where $f(x) = g(y)$ and therefore have a point in Z. However, this point is not connected to any other point in Z because there is no way for hikers A and B to change elevation in the same direction to leave this point. Suppose that hiker A and hiker B are at that local max and local min respectively. Now suppose that hiker A starts descending by either going left or right. Hiker B cannot match hiker A's elevation since hiker B is at a local minimum and can only go up. As such, there is no way for both hikers to move from the point while maintaining the same elevation, giving us an isolated point in Z.

For part (b), we can turn Z into a graph by defining vertices as $z \in Z$ such that:

1. z is an endpoint, ie $(0, 0)$ or $(1, 1)$
2. z is an isolated point unconnected to any other elements of Z, ie where f and g are at different local extrema at the same elevation
3. z is the intersection of four line segments, ie where f and g are both local maxima or local minima

The edges will simply be the line segment(s) which connect these vertices. Note that edges may consist of multiple line segments and may include kinks.

By definition, we can see that we only have two vertices of odd degree: $(0, 0)$ and $(1, 1)$. Further, we know that these vertices must be connected. To see why, assume to the contrary that $(0, 0)$ and $(1, 1)$ are not connected so that there is no path from $(0, 0)$ to $(1, 1)$. This implies that $(0, 0)$ and $(1, 1)$ are in separate components of G. Examine the component containing $(0, 0)$. We note that there is at least one other vertex, as $(0, 0)$ is incident to an edge. Further, we know that the sum of degrees must be even. This implies that we have another vertex of odd degree, v, in the component containing $(0,0)$ in order to make the sum of degrees even. We know $v \neq (1,1)$ as $(0,0)$ and $(1,1)$ are in separate components. However, this implies that we have more than two vertices of odd degree, a contradiction.

$\square$