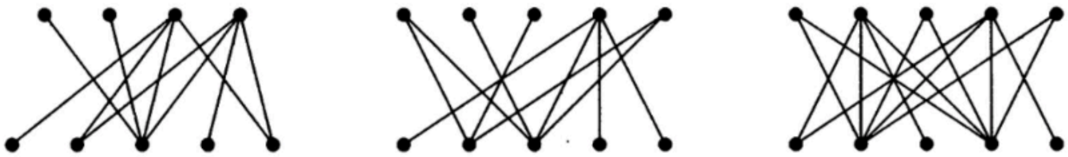
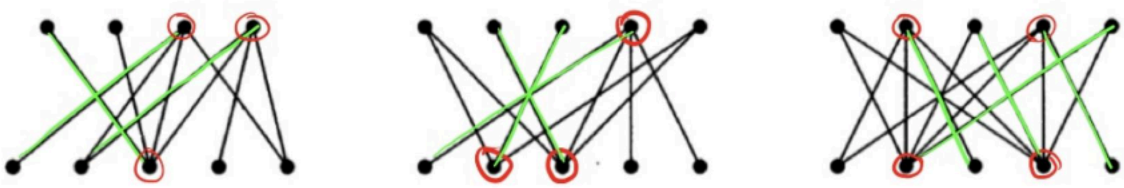
Problem 1. *Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem.*



Solution. König's Theorem states that for a bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover. Since the above graphs are all bipartite, we can just find the minimum vertex covers.



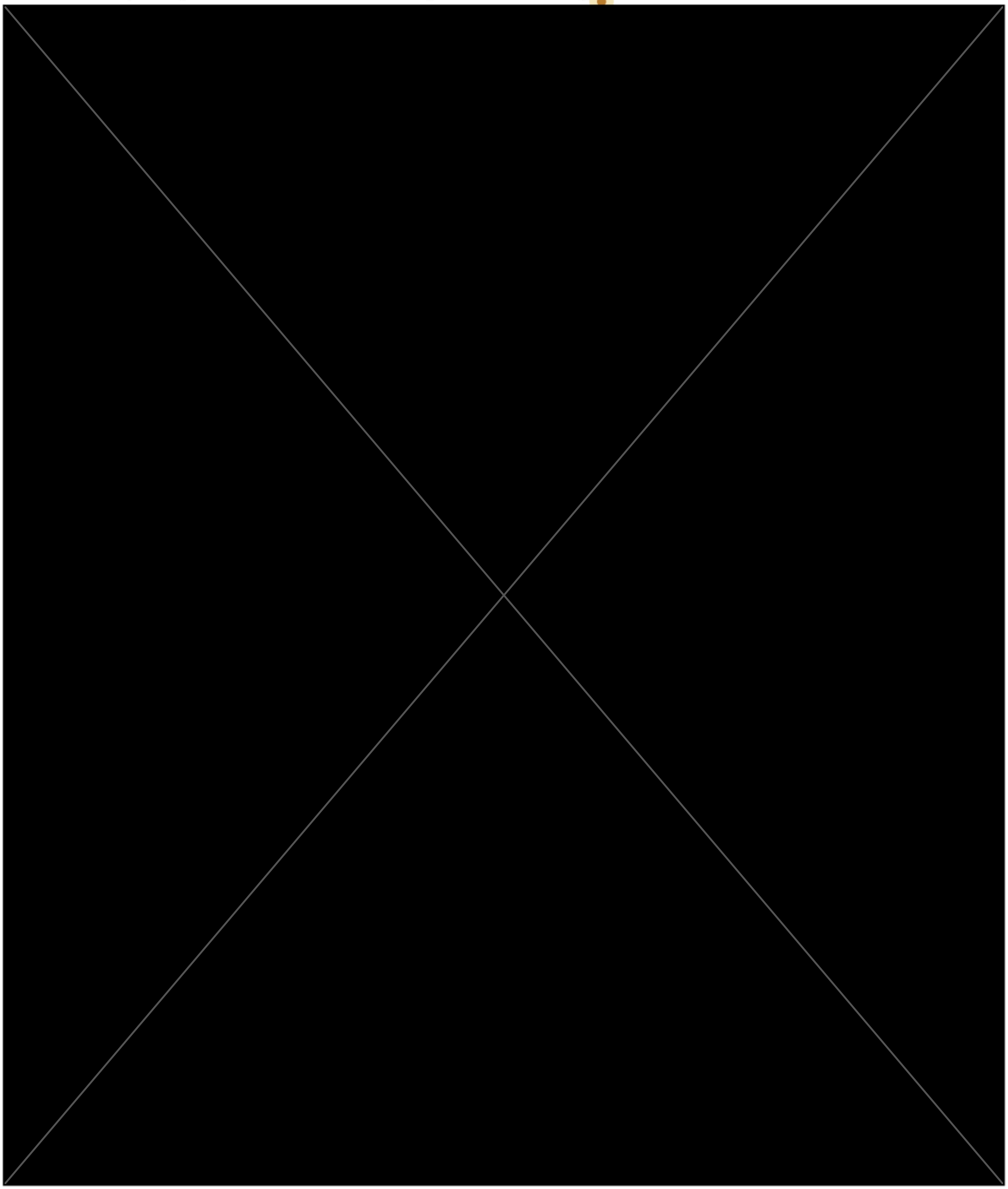
For each of the above vertex covers, shown in red, we can see that they are minimal because removing any single one would disconnect edges. We also can see that each vertex in each vertex cover has a higher degree than each vertex not in the vertex cover, so there will be no combination □

[redacted]

[redacted]

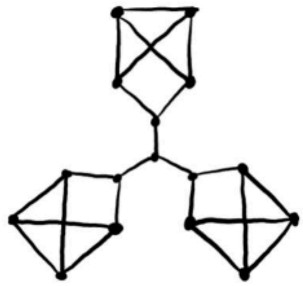
Problem 2. *Prove or disprove: every tree T has at most one perfect matching.*

Solution. Let M and M' be perfect matchings of tree T. Because the matchings are perfect, we know that in $M \triangle M'$, every vertex either has degree 0 or degree 2. Therefore, each component of $M \triangle M'$ must be either a cycle or isolated vertex. Because a tree has no cycle, $M \triangle M'$ cannot have cycles either. Thus, we see that $M \triangle M'$ only contains isolated points. By the definition of symmetric difference, we know that M and M' must either both share each edge or both not contain each edge, showing that $M = M'$. Therefore, if a tree T has a perfect matching, it is unique. In other words, every tree T has at most one perfect matching. $\square$

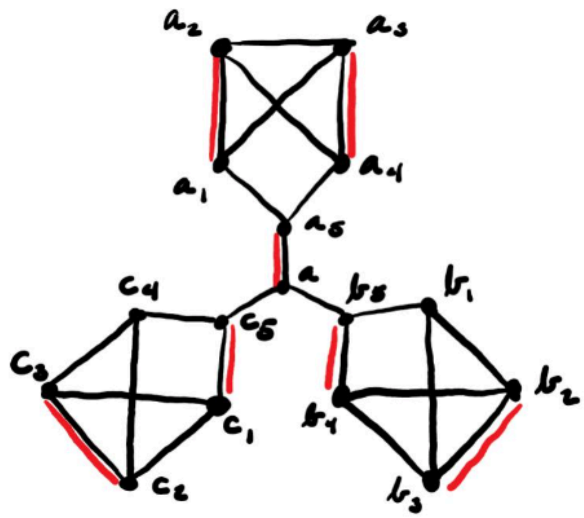


Problem 3. *Construct a 3-regular graph with an even number of vertices and no perfect matching. Give proof that your graph has the desired property.*

Solution. The graph below is an example of a 3-regular graph with 16 vertices no perfect matching.



To prove that this graph has no perfect matching, consider the matching M below. We will prove that there is no M-augmenting path, and thus that M is maximal.



If this graph were to have an M-augmenting path, it would have to go from b_1 to c_4 since these are the only two unsaturated vertices. It is clear that to make a path from the b_i vertices to the c_j vertices in the above labeling, this path must include the two-edge path $\{b_5, a\}$ to $\{a, c_5\}$. Both of these edges are not in M, so this path portion cannot be in an M-augmenting path. Since this is the only way to get between b_1 and c_4, it is not possible to construct an M-augmenting path. Thus, M must be maximal, so this graph has no perfect matching.

$\square$

[redacted]

Problem 4. *Two people play a game on a graph G, alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins. Prove: (i) if G has a perfect matching, then the second player has a winning strategy; (ii) if G has no perfect matching, then the first player has a winning strategy.*[1]

Solution. (i) If there exists a perfect matching M, then for any vertex v Player 1 selects, Player 2 can select the vertex adjacent to v on M. This is a unique choice and always exists because after Player 2 selects the vertex v' adjacent to v on M, Player 1 must select a vertex u adjacent to v' but not over an edge in M because two edges in a matching cannot be connected to the same vertex. By nature of perfect matching, there must a vertex connected to u by matching: this is Player 2's next selection. If following this strategy, for any move Player 1 makes, Player 2 has a move. Therefore, Player 2 wins.

(ii) If there is no perfect matching, let us consider the maximum matching M. Player 1 should select a vertex not saturated by M. Player 2 will then be forced to select a vertex v saturated by M, because if two adjacent unsaturated vertices exist then the matching is not maximum (from class). Player 1 can then choose the match of v. Player 2 must always select a saturated vertex because if they select an unsaturated vertex then the game has revealed an M-augmenting path between two unsaturated vertices, contradiction with M being a maximum matching. Player 2 always selects a saturated vertex and Player 1 can always respond with its match.

□

[redacted]

Problem 5. *A permutation matrix P has exactly one 1 in each row and column and the remaining entries are 0. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all the row sums and column sums equal k.* [2]

Solution. ($\Leftarrow$)

If a matrix M is a sum of k permutation matrices, since the sum of permutation matrix in each row and column is 1, the sum on each row and column of M is k.

($\Rightarrow$)

We induct on k.

We know that for $k = 0$, the matrix is the sum of 0 permutation matrices, thus the base case is true.

For the inductive case, for each $M \in \mathbb{R}^{n\times n}$, we consider the following multi-edge bipartite graph:

Let the vertices be $V = X \sqcup Y$, where $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_n\}$. For each vertex pair x_i and y_j, there are M_{ij} edges connecting them. We notice that the degrees of each vertex is the sum of the values in a row for x_i, and the sum of values in a column in y_j. This means that this bipartite graph is k-regular.

By a Corollary of Hall's Theorem, we know this graph contains a perfect matching. A perfect matching in this case corresponds to one and only one 1 on each row and column of the matrix, i.e. a permutation matrix. Removing these edges is equivalent to subtracting a permutation matrix from M. By the above, we know that this new $M' = M - P$ has row and column sums $k-1$, thus its corresponding graph is $(k-1)$-regular. Hence we have inductive case. $\square$

[redacted]

Problem 6 (Bonus)**.** *Use Problem 5 construct your own original 4×4 magic square.*[3] *Make sure the diagonals also have the same value (you will need to think about how to ensure this). Also your example should not be boring.*

Solution. $k = 168$

$$
1\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
+2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
+27\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}
+48\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
+56\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
+15\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
+8\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
+11\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
=\begin{bmatrix} 1+2 & 15+8 & 27+11 & 48+56 \\ 56+11 & 27+48 & 1+8 & 2+15 \\ 48+8 & 2+11 & 56+15 & 1+27 \\ 27+15 & 1+56 & 2+48 & 8+11 \end{bmatrix}
$$

$$
=\begin{bmatrix} 3 & 23 & 38 & 104 \\ 67 & 75 & 9 & 17 \\ 56 & 13 & 71 & 28 \\ 42 & 57 & 50 & 19 \end{bmatrix}
$$

□

[2] Hint: it may help to consider graphs with multiple edges.

[3] First look up what a magic square is...