

Regression analysis (in R)

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Outline for the course

- ▶ Introduction to statistics
- ▶ Hypothesis testing (in R)
- ▶ Regression analysis (in R)
- ▶ Multiple testing (in R)

Regression analysis

In statistics, **regression analysis** is a tool to quantify relationships between 2 or more variables

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In regression, variables are classified as

- ▶ **Response:** outcome variable of interest (dependent)
- ▶ **Explanatory:** covariate(s) to explain the response (independent)

Examples?

Regression analysis

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- ▶ For a continuous response (e.g. blood sugar level), a typical choice is **linear regression**

Today, we will focus on linear regression

Linear regression

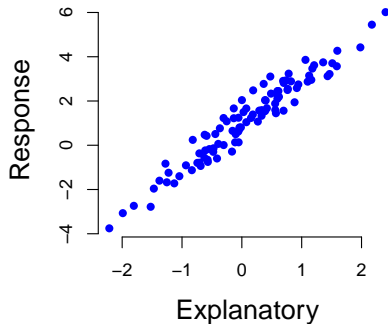
Linear regression

Linear regression is a statistical tool to capture **linear** relationships between a single **continuous** response variable and one or more explanatory variables (may be discrete or continuous)

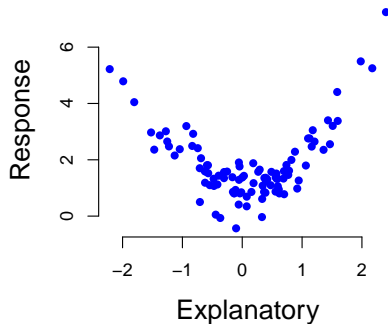
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Linear relationship



Non linear relationship

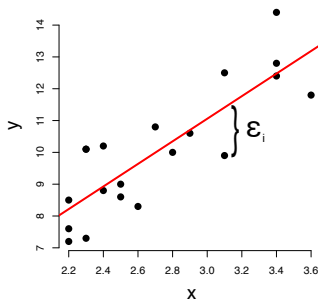


Simple linear regression

In **simple linear regression**, the aim is to capture a linear relationship between a response (y) and a **single** covariate (x)

The simple linear regression model is written as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, \dots, n$$

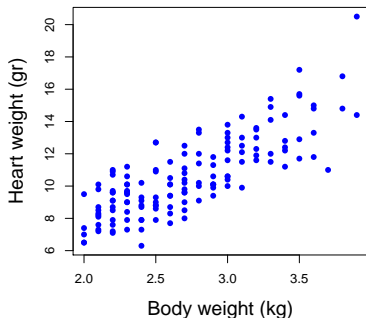


- ▶ β_0 is the intercept
- ▶ β_1 is the slope (gradient)
- ▶ ϵ_i is the error term

(*) Plot the response on the vertical axis and the covariate on the horizontal axis

Simple linear regression: an example

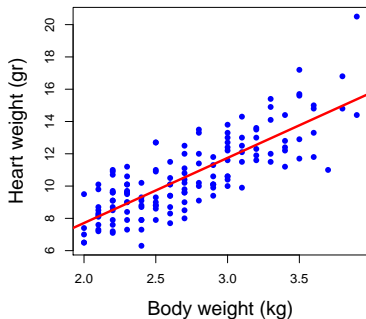
The data in Fisher (1947) shows heart and body weights for 144 cats



Is there a relationship between heart
and body weight?

Simple linear regression: an example

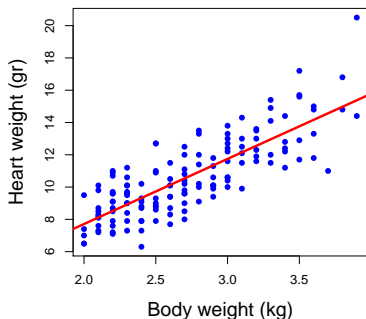
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Simple linear regression: an example

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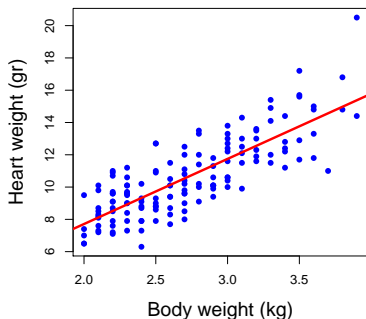
$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where

- ▶ y_i = heart weight of cat i
- ▶ x_i = body weight of cat i

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How to estimate β_0 and β_1 ?

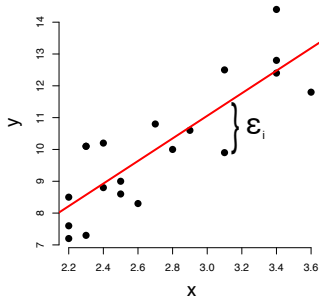
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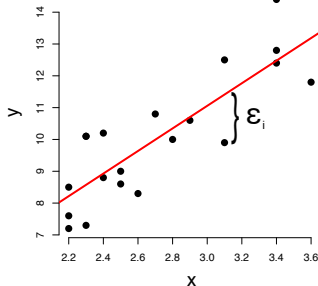
For example, by minimizing the **residuals** of the regression



$$\epsilon_i = Y_i - (\beta_0 + \beta_1 x_i) \equiv Y_i - \hat{Y}_i$$

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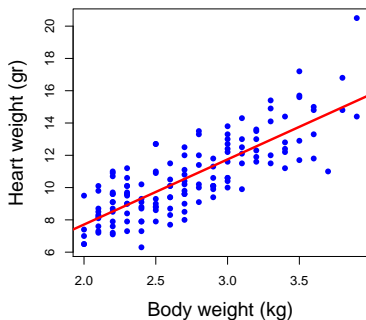
$$\epsilon_i = Y_i - (\beta_0 + \beta_1 x_i) \equiv Y_i - \hat{Y}_i$$

Typically, this is done by minimizing the **Sum of Square Errors (SSE)**

$$\text{SSE} = \sum_{i=1}^n \epsilon_i^2$$

Simple linear regression: an example

The data in Fisher (1947) shows heart and body weights for 144 cats



In this example we have

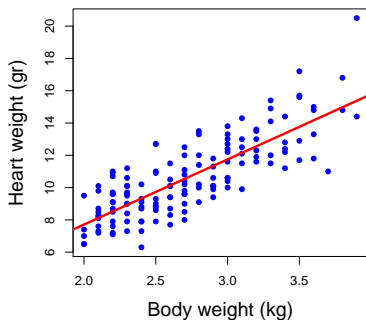
► $\beta_0 = -0.36$

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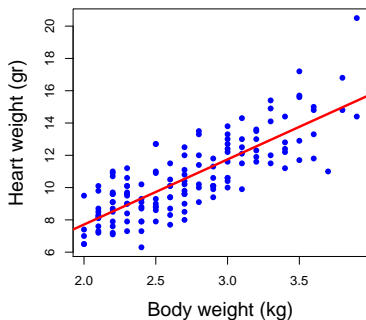
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For a 1kg increase in body weight,
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Is this increase statistically
significant?

Simple linear regression

We can answer this using hypothesis testing:

$$H_0 : \beta_1 = 0 \quad \text{vs} \quad H_1 : \beta_1 \neq 0$$

Recall: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

→ if $\beta_1 = 0$, y_i and x_i are not (linearly) dependent

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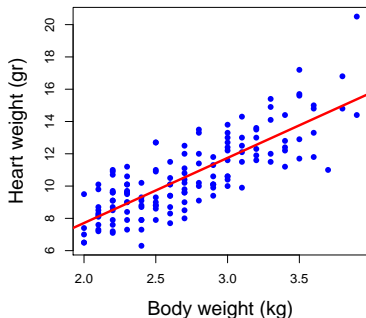
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Assuming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we can derive a t -test for β_1

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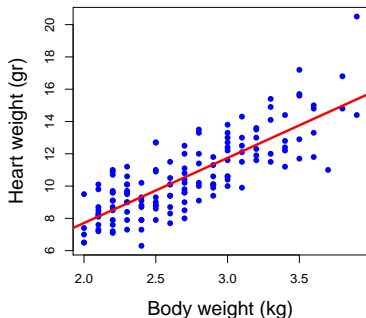
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The p -value is $< 2 \times 10^{-16}$

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We reject H_0 , i.e. we conclude that the linear relationship between heart and body weight is statistically significant ($\alpha = 0.05$)

Simple linear regression: Analysis of Variance (ANOVA)

ANOVA aims to decompose the total variance of y

Recall: $\text{Var}(y) = \sum_{i=1}^n (y_i - \bar{y})^2 / (n - 1)$, with $\bar{y} = (\sum_{i=1}^n y_i) / n$

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$$\underbrace{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}_{\text{SSR (Regression)}} = \underbrace{\sum_{i=1}^n (y_i - \bar{y})^2}_{\text{SST (Total)}} - \underbrace{\sum_{i=1}^n (y_i - \hat{y}_i)^2}_{\text{SSE (Error)}}$$

- ▶ SST quantifies the total variability of y
- ▶ SSE quantifies residual variability of y (unexplained by x)
- ▶ SSR quantifies how much of the variability of y is explained by x

Simple linear regression: Analysis of Variance (ANOVA)

The **coefficient of determination** R^2 is defined as the proportion of total variability of y that is explained by x

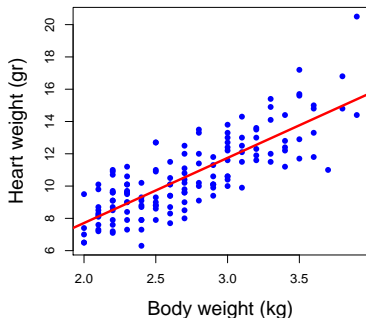
$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

- ▶ $0 \leq R^2 \leq 1$
- ▶ If $R^2 = 0$, x explains none of the variability of y
- ▶ If $R^2 = 1$, x explains all of the variability of y (perfect fit!)

Warning: this assumes the relationship between x and y is linear

Simple linear regression: an example

The data in Fisher (1947) shows heart and body weights for 144 cats



In this example we have

► $R^2 = 0.65$

Body weight explains 65% of the variability of a cat's heart weight

Simple linear regression versus correlation

Pearson's correlation ($\rho_{x,y}$) quantifies linear dependency between variables x and y

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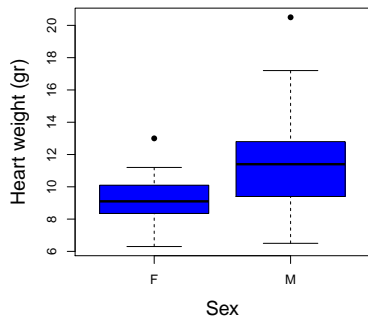
Let's play: <http://guessthecorrelation.com>

Questions + practical

Simple linear regression: categorical covariates

Simple linear regression: an example

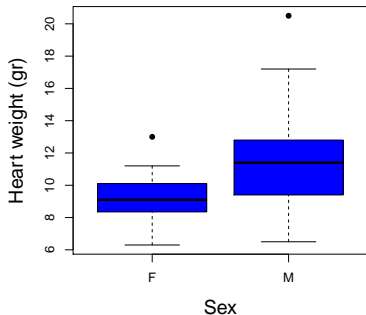
The data in Fisher (1947) also contains sex information



Is there a relationship between sex and a cat's heart weight?

Simple linear regression: an example

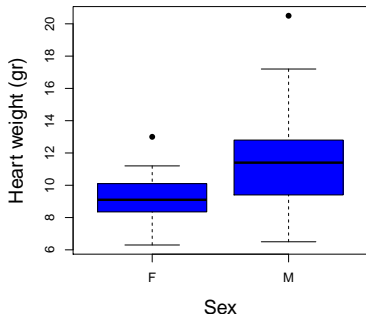
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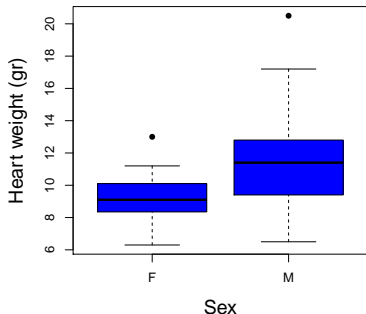
In this example we have:

$$t = -5.35$$

$$p\text{-value} = 3.38 \times 10^{-7}$$

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Can we answer the same question using linear regression?

Simple linear regression: categorical covariates

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$$\text{Recall: } y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Instead of defining a single regression effect β_1 , we need to estimate an effect for **each level** of the categorical covariate

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This can be done using **dummy variables**

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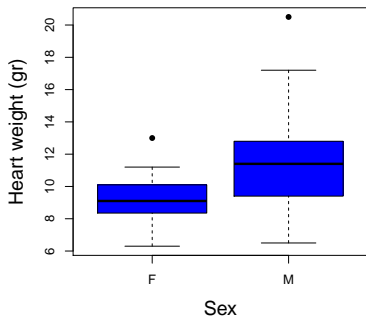
Note: sex has **two** levels, but we only define **one** dummy variable (here we left *female* as a **reference category**)

Simple linear regression: an example

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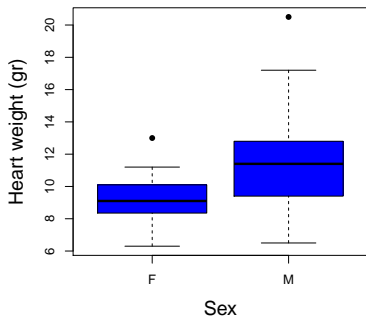
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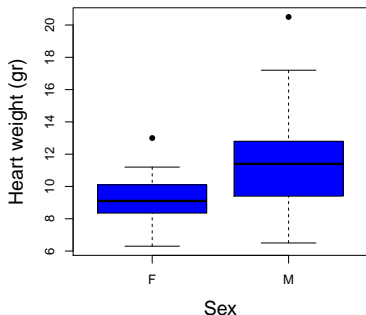
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$$y_i = \beta_0 + \beta_1 D_i + \epsilon_i$$

- ▶ For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
- ▶ For a female cat: $\hat{y}_i = \beta_0$

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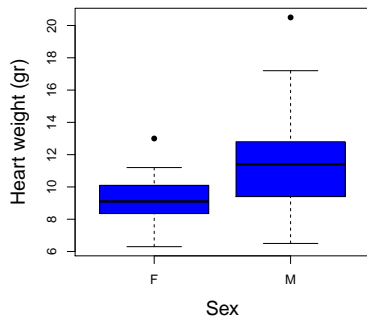
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- For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
- For a female cat: $\hat{y}_i = \beta_0$

⇒ β_1 quantifies the difference between female and male cats

Simple linear regression: an example

Is there a relationship between sex and a cat's heart weight?

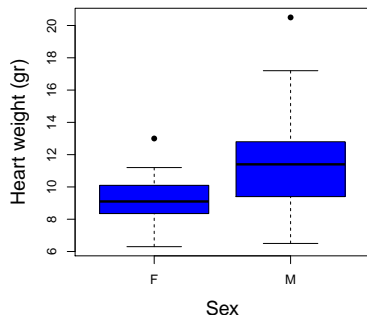


We can answer this using the test

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$$H_0 : \beta_1 = 0 \quad \text{vs} \quad H_1 : \beta_1 \neq 0$$

Assuming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we derive a t -test:

$$\beta_1 = 2.12$$

$$t = 5.35$$

$$p\text{-value} = 3.38 \times 10^{-7}$$

Multiple linear regression

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$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

Multiple linear regression: an example

For example, with 2 continuous covariates:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- ▶ y_i : heart weight for cat i
- ▶ x_{i1} : body weight for cat i
- ▶ x_{i2} : age for cat i

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- ▶ **Global:** to assess the joint effect of all covariates

$$H_0 : \beta_1 = \cdots = \beta_p = 0 \text{ vs } H_1 : \text{at least one } \beta_j \neq 0$$

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Assuming $\epsilon_i \sim N(0, \sigma^2)$, we can use an *F*-test (from ANOVA table)

Multiple linear regression: an example

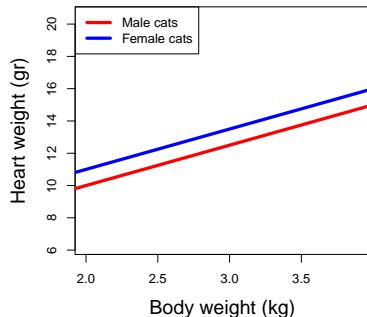
... or mixing a continuous and a categorical covariate

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- ▶ y_i : heart weight for cat i
- ▶ x_{i1} : body weight for cat i
- ▶ $x_{i2} = \begin{cases} 1, & \text{if cat } i \text{ is male;} \\ 0, & \text{if cat } i \text{ is female.} \end{cases}$

Multiple linear regression: an example



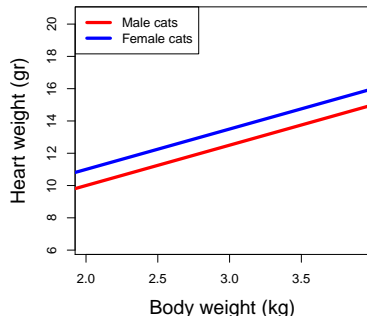
For a male cat:

$$\hat{y}_i = \beta_0 + \beta_1 x_{i1} + \beta_2$$

For a female cat:

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Multiple linear regression: an example



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For a female cat:

$$\hat{y}_i = \beta_0 + \beta_1 x_{i1}$$

⇒ parallel lines, effect of body weight is independent of sex

⇒ β_2 quantifies a global difference between female and male cats

Multiple linear regression: an example

What if we think the effect of body weight depends on sex?

Multiple linear regression: an example

What if we think the effect of body weight depends on sex?

We can define an **interaction effect**

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \epsilon_i,$$

Recall:

- ▶ y_i : heart weight for cat i
- ▶ x_{i1} : body weight for cat i
- ▶ $x_{i2} = \begin{cases} 1, & \text{if cat } i \text{ is male;} \\ 0, & \text{if cat } i \text{ is female.} \end{cases}$

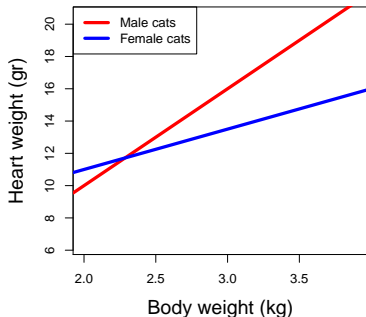
Multiple linear regression: an example

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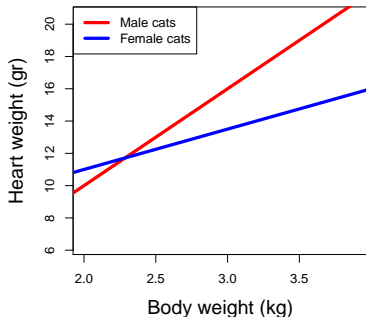
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Multiple linear regression: an example



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⇒ crossing lines, effect of body weight depends on sex

⇒ β_2 quantifies a global difference between female and male cats

⇒ β_3 quantifies the differential effect of body weight

Questions + practical

Assumptions in linear regression

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Recall: the multiple linear regression model is written as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

What assumptions underlie this model?

Assumptions in linear regression

- ▶ Firstly, we need more observations than variables ($n > p$)

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- ▶ Secondly, we assume that errors ϵ_i are **independent** and **identically distributed** with

$$\epsilon_i \sim N(0, \sigma^2)$$

Assumptions in linear regression

- ▶ Firstly, we need more observations than variables ($n > p$)

This might not hold in some genomics applications ...
... but that's outside the scope of this course!

- ▶ Secondly, we assume that errors ϵ_i are **independent** and **identically distributed** with

$$\epsilon_i \sim N(0, \sigma^2)$$

How can we assess this?

Assumptions in linear regression

Idea: after estimating the regression coefficients, define residuals as

$$\hat{\epsilon}_i = y_i - \hat{y}_i$$

Use these estimated residuals to diagnose model quality

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More about this in the practical ...

Assumptions in linear regression

What if things go wrong?

Assumptions in linear regression

What if things go wrong?

If the residuals are not good enough to pass the diagnostic criteria, we need to revisit the model. For example

- ▶ We can transform some of the covariates or
- ▶ We can transform the response variable

Assumptions in linear regression: covariate transformation

Suppose a simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Covariate transformation can be useful in situations where the relationship between y and x is **not linear**

Assumptions in linear regression: covariate transformation

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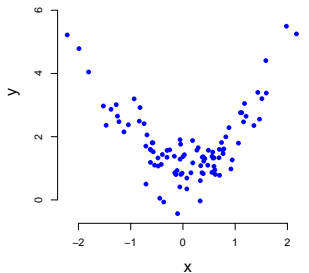
Covariate transformation can be useful in situations where the relationship between y and x is **not linear**

Idea: replace x_i by a transformed version of x_i (e.g. $x_i^* = x_i^2$)

Assumptions in linear regression: covariate transformation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

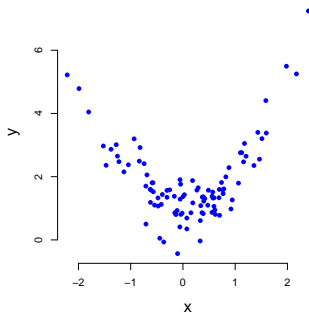
Non linear relationship



Assumptions in linear regression: covariate transformation

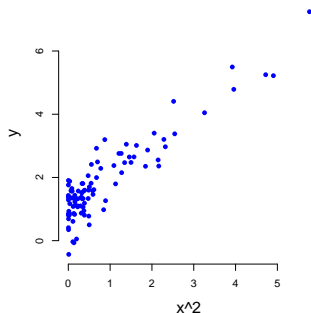
$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Non linear relationship



$$y_i = \beta_0 + \beta_1 x_i^2 + \epsilon_i$$

Linear relationship



Assumptions in linear regression: response transformation

Suppose a simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Response transformation can be useful in situations where the variance of y is not constant as a function of x

We refer to this as **heteroskedastic** errors

Assumptions in linear regression: response transformation

Suppose a simple linear regression model

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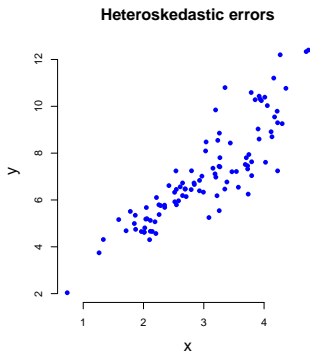
Response transformation can be useful in situations where the variance of y is not constant as a function of x

We refer to this as **heteroskedastic** errors

Idea: replace y_i by a transformed version of y_i (e.g. $y_i^* = \log(y_i)$)

Assumptions in linear regression: response transformation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

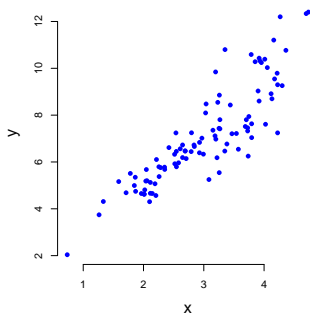


Assumptions in linear regression: response transformation

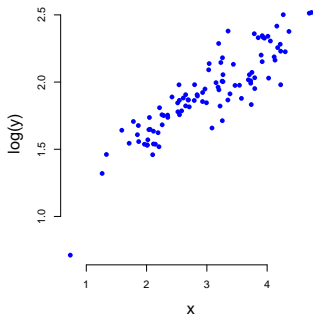
$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$\log(y_i) = \beta_0 + \beta_1 x_i + \epsilon_i$$

Heteroskedastic errors



Homoskedastic errors



Questions + practical