Regression analysis (in R)

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Outline for the course

- **Introduction to statistics**
- **Hypothesis testing (in R)**
- **Regression analysis (in R)**

Regression analysis

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In regression, variables are clasified as

- **Response**: outcome variable of interest (dependent)
- **Explanatory**: covariate(s) to explain the response (independent)

Examples?

Regression analysis

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- For a continuous response (e.g. blood sugar level), a typical choice is **linear regression**

Today, we will focus on linear regression

Linear regression

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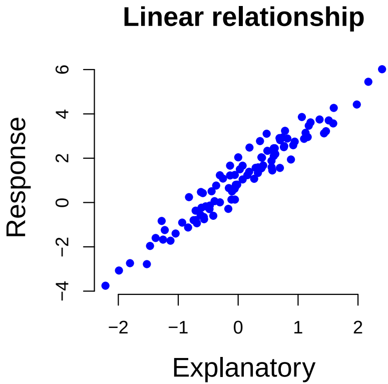
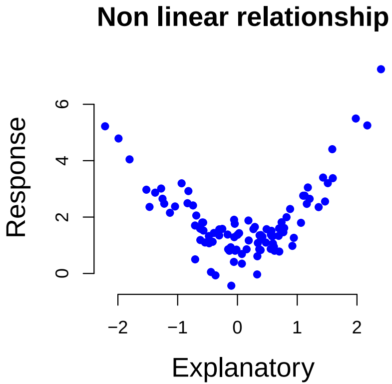


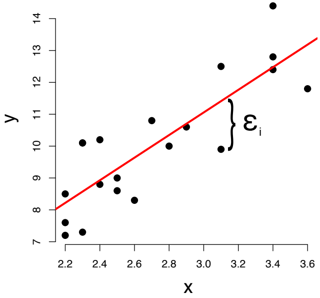


Simple linear regression

In **simple linear regression**, the aim is to capture a linear relationship between a response (y) and a **single** covariate (x)

The simple linear regression model is writen as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad i = 1, \ldots, n$$

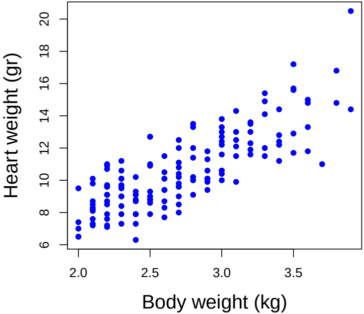


- β_0 is the intercept
- β_1 is the slope (gradient)
- ϵ_i is the error term

(*) Plot the response on the vertical axis and the covariate on the horizontal axis

Simple linear regression: an example

The data in Fisher (1947) shows heart and body weights for 144 cats



Is there a relationship between heart and body weight?

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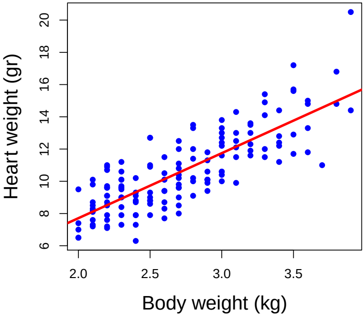

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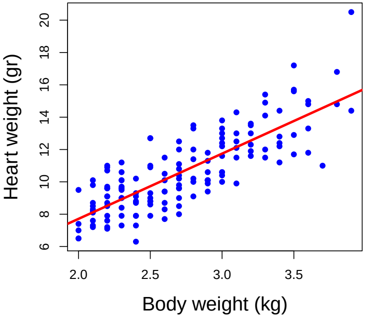

We can fit a simple linear regression

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where

- y_i = heart weight of cat i
- x_i = body weight of cat i

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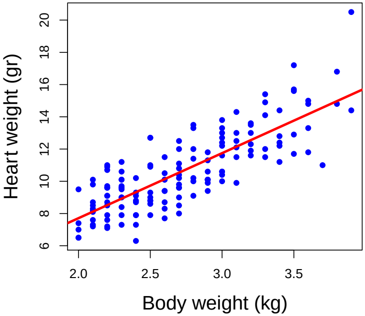

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How to estimate β_0 and β_1?

Simple linear regression

Idea: find β_0 and β_1 such that the line is a good fit for the data

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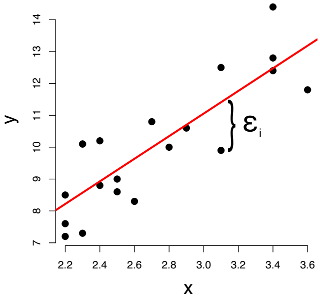

For example, by minimizing the **residuals** of the regression

$$\epsilon_i = Y_i - (\beta_0 + \beta_1 x_i) \equiv Y_i - \hat{Y}_i$$

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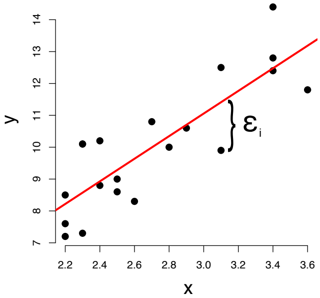

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Typically, this is done by minimizing the **Sum of Square Errors (SSE)**

$$\text{SSE} = \sum_{i=1}^{n} \epsilon_i^2$$

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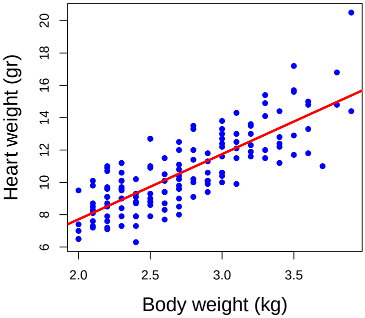

In this example we have

- $\beta_0 =$ -0.36
- $\beta_1 = 4.03$

How to interpret this?

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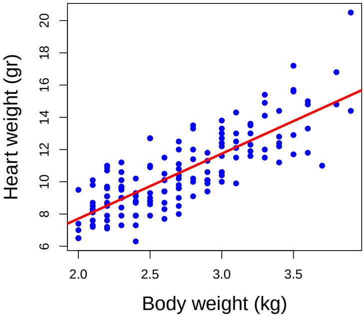

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For a 1kg increase in body weight, we expect heart weight to increase by 4.03gr

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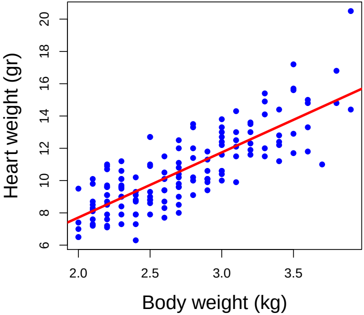

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How to interpret this?
For a 1kg increase in body weight, we expect heart weight to increase by 4.03gr

Is this increase statistically significant?

Simple linear regression

We can answer this using hypothesis testing:

$$H_0 : \beta_1 = 0 \quad \text{vs} \quad H_1 : \beta_1 \neq 0$$

Recall: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

$\rightarrow$ if $\beta_1 = 0$, y_i and x_i are not (linearly) dependent

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Assumming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we can derive a *t*-test for β_1

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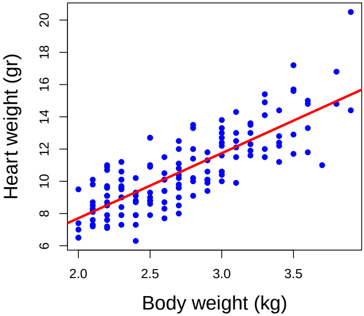

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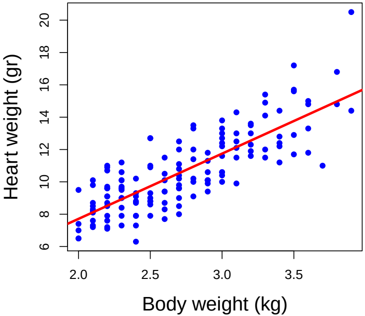

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We reject H_0, i.e. we conclude that the linear relationship between heart and body weight is statistically significant ($\alpha = 0.05$)

Simple linear regression: Analysis of Variance (ANOVA)

ANOVA aims to decompose the total variance of y

Recall: $\mathrm{Var}(y) = \sum_{i=1}^{n}(y_i - \bar{y})^2/(n-1)$, with $\bar{y} = (\sum_{i=1}^{n} y_i)/n$

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$$\underbrace{\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2}_{\text{SSR (Regression)}} = \underbrace{\sum_{i=1}^{n} (y_i - \bar{y})^2}_{\text{SST (Total)}} - \underbrace{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}_{\text{SSE (Error)}}$$

- SST quantifies the total variability of y
- SSE quantifies residual variability of y (unexplained by x)
- SSR quantifies how much of the variability of y is explained by x

Simple linear regression: Analysis of Variance (ANOVA)

The **coefficient of determination** R^2 is defined as the proportion of total variability of y that is explained by x

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}$$

- $0 \leq R^2 \leq 1$
- If $R^2 = 0$, x explains none of the variability of y
- If $R^2 = 1$, x explains all of the variability of y (perfect fit!)

Warning: this assumes the relationship between x and y is linear

Simple linear regression: an example

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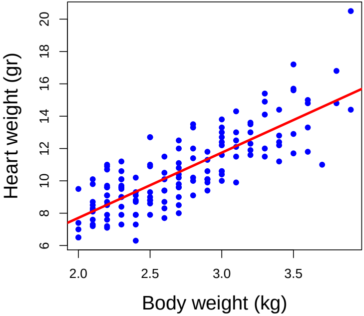

In this example we have

- $R^2 = 0.65$

Body weight explains 65% of the variability of a cat's heart weight

Simple linear regression versus correlation

Pearson's correlation ($\rho_{x,y}$) quantifies linear dependency between variables x and y

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Examples?

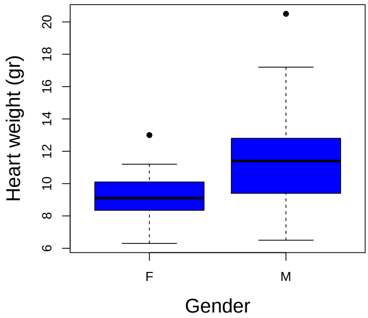
Let's play: `http://guessthecorrelation.com`

Questions $+$ practical

Simple linear regression: categorical covariates

Simple linear regression: an example

The data in Fisher (1947) also contains gender information



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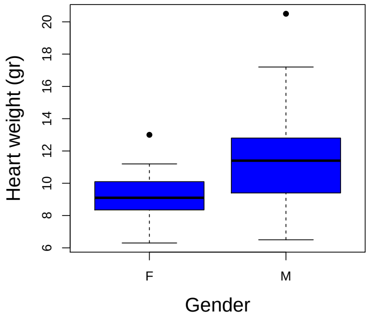

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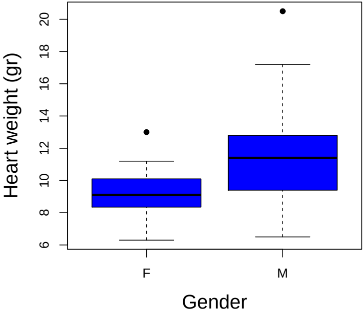

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$$t = \text{-}5.35$$
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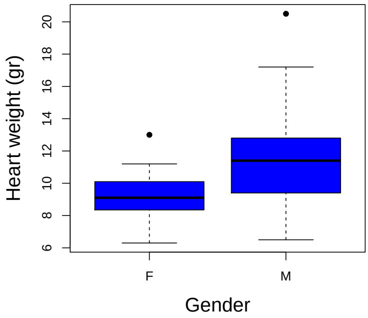

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Can we answer the same question using linear regression?

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Instead of defining a single regression effect β_1, we need to estimate an effect for **each level** of the categorical covariate

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This can be done using **dummy variables**

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Note: gender has **two** levels, but we only define **one** dummy variable (here we left *female* as a **reference category**)

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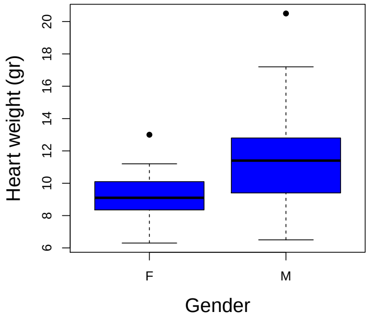

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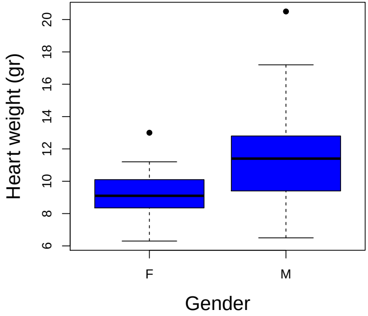

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- For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
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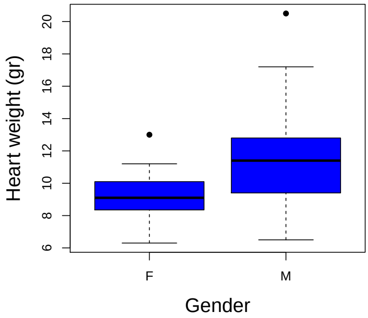

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- For a male cat: $\hat{y}_i = \beta_0 + \beta_1$
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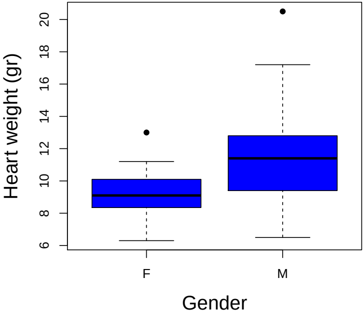
$\Rightarrow$ β_1 quantifies the difference between female and male cats

Simple linear regression: an example

Is there a relationship between gender and a cat's heart weight?



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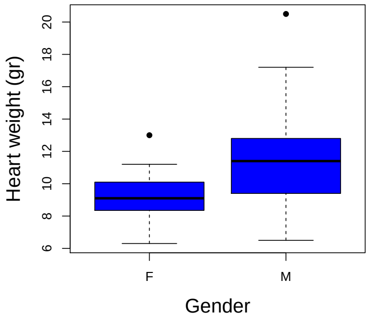

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Assuming $\epsilon_i \sim \text{Normal}(0, \sigma^2)$, we derive a t-test:

$$\beta_1 = 2.12$$
$$t = 5.35$$
$$p\text{-value} = 3.38 \times 10^{-7}$$

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$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

Multiple linear regression: an example

For example, with 2 continuous covariates:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- y_i: heart weight for cat i
- x_{i1}: body weight for cat i
- x_{i2}: age for cat i

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- **Global**: to assess the joint effect of all covariates

$$H_0 : \beta_1 = \cdots = \beta_p = 0 \ \text{ vs } \ H_1 : \text{at least one } \beta_j \neq 0$$

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Multiple linear regression: an example

... or mixing a continuous and a categorical covariate

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where

- y_i: heart weight for cat i
- x_{i1}: body weight for cat i
- $x_{i2} = \begin{cases} 1, \text{if cat } i \text{ is male;} \\ 0, \text{if cat } i \text{ is female.} \end{cases}$

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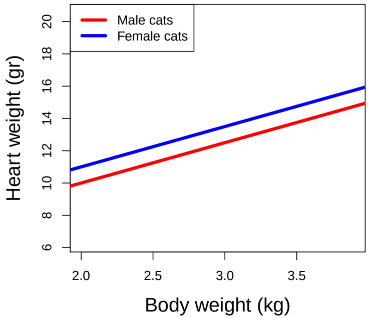

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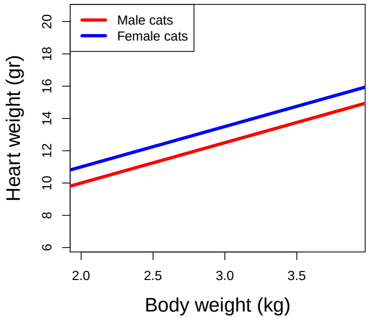

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$\Rightarrow$ parallel lines, effect of body weight is independent of gender

$\Rightarrow$ β_2 quantifies a global difference between female and male cats

Multiple linear regression: an example

What if we think the effect of body weight depends on gender?

Multiple linear regression: an example

What if we think the effect of body weight depends on gender?

We can define an **interaction effect**

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i12} + \epsilon_i,$$

Recall:

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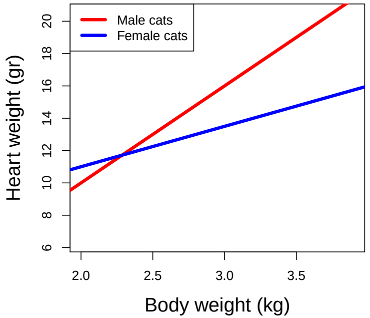

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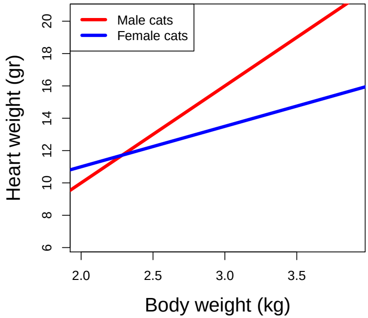
Multiple linear regression: an example



For a male cat:

$$\hat{y}_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \beta_3 x_{i1}$$

For a female cat:

$$\hat{y}_i = \beta_0 + \beta_1 x_{i1}$$

$\Rightarrow$ crossing lines, effect of body weight depends on gender

$\Rightarrow$ β_2 quantifies a global difference between female and male cats

$\Rightarrow$ β_3 quantifies the differential effect of body weight

Questions + practical

Assumptions in linear regression

Assumptions in linear regression

Recall: the multiple linear regression model is written as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i$$

What assumptions underlie this model?

Assumptions in linear regression

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$$\epsilon_i \sim \mathrm{N}(0, \sigma^2)$$

How can we assess this?

Assumptions in linear regression

Idea: after estimating the regression coefficients, define residuals as

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Use these estimated residuals to diagnose model quality

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More about this in the practical ...

Assumptions in linear regression

What if things go wrong?

Assumptions in linear regression

What if things go wrong?

If the residuals are not good enough to pass the diagnostic criteria, we need to revisit the model. For example

- We can transform some of the covariates or
- We can transform the response variable

Assumptions in linear regression: covariate transformation

Suppose a simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Covariate transformation can be useful in situations where the relationship between y and x is **not linear**

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Idea: replace x_i by a transformed version of x_i (e.g. $x_i^* = x_i^2$)

Assumptions in linear regression: covariate transformation

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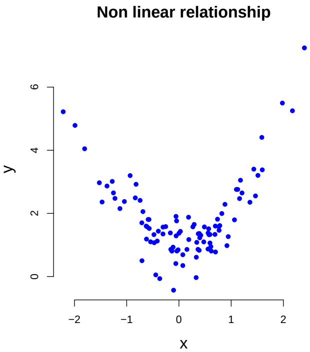
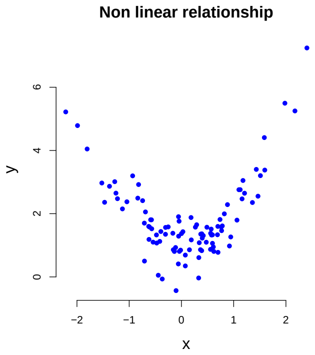
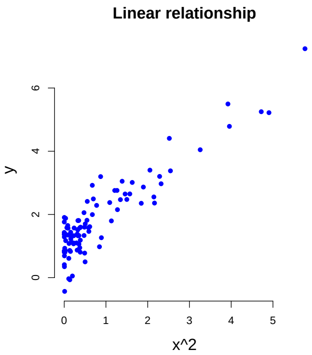


Assumptions in linear regression: covariate transformation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$y_i = \beta_0 + \beta_1 x_i^2 + \epsilon_i$$





Assumptions in linear regression: response transformation

Suppose a simple linear regression model

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Response transformation can be useful in situations where the variance of y is not constant as a function of x

We refer to this as **heteroskedastic** errors

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Idea: replace y_i by a transformed version of y_i (e.g. $y_i^* = \log(y_i)$)

Assumptions in linear regression: response transformation

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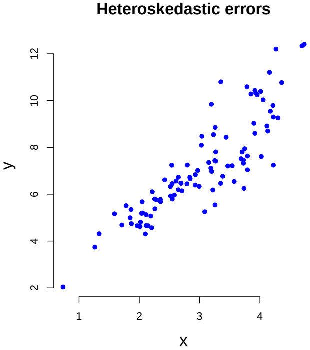
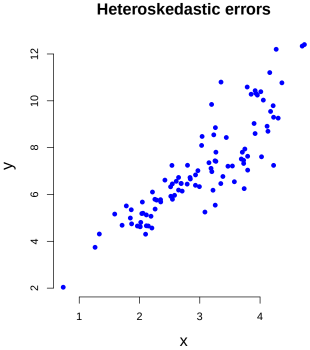
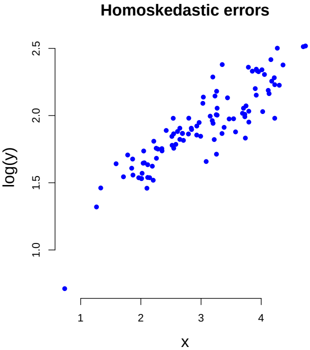


Assumptions in linear regression: response transformation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$\log(y_i) = \beta_0 + \beta_1 x_i + \epsilon_i$$





Questions + practical