Algebraic Topology

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2020-2021

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1 Cohomology

1.1 Homomorphisms of groups

Definition 1.1. Let A be a group and G an abelian group. We define

$$\operatorname{Hom}(A, G) = \{\text{group homomorphism } \varphi : A \to G\}$$

as an abelian group with structure given by $(f + g)(a) = f(a) + g(a)$ and $0(a) = 0 \in G$.

Example 1.2. $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ with the isomorphism $\varphi \mapsto \varphi(1)$, as for any homomorphism $\varphi : \mathbb{Z} \to G$, $\varphi(1)$ fully describes the map.

Example 1.3. $\mathrm{Hom}(\mathbb{Z}/n, G) \cong \ker f$ where $f : G \to G$ such that $x \mapsto n \cdot x$. This can be seen as any homomorphism $\varphi : \mathbb{Z}/n \to G$ must preserve the property that $n\overline{x} = \overline{0}$. We also see that $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \cong 0$.

Example 1.4. $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(n, m)$ is just an application of the first example.

Lemma 1.5. *Let $\{A_i\}_{i\in\mathcal{I}}$ be a sequence of abelian groups and G an abelian group. Then*

$$\mathrm{Hom}\left(\bigoplus_{i\in\mathcal{I}} A_i, G\right) \cong \prod_{i\in\mathcal{I}} \mathrm{Hom}(A_i, G)$$

via the map $f \mapsto \prod_{i\in\mathcal{I}} f\big|_{A_i}$.

Corollary 1.6. $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$.

Let $f : A \to B$ be a group homomorphism. Then there is an induced group homomorphism

$$\begin{aligned} f^* &: \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G), \\ &(\varphi : B \to G) \mapsto (\varphi \circ f : A \to B \to G). \end{aligned}$$

Similarly, if $g : G \to H$ be a group homomorphism between two abelian groups. Then we have the induced group homomorphism

$$\begin{aligned} g^* &: \mathrm{Hom}(A, G) \to \mathrm{Hom}(A, H), \\ &(\varphi : A \to G) \mapsto (g \circ \varphi : A \to G \to H). \end{aligned}$$

1.2 Cochain complexes

Definition 1.7. A *cochain complex* $D = (D^*, \delta_*)$ is a sequence of abelian groups and group homomorphisms

$$\dots \xrightarrow{\delta_{-1}} D^0 \xrightarrow{\delta_0} D^1 \xrightarrow{\delta_1} D^2 \xrightarrow{\delta_2} \dots \xrightarrow{\delta_{n-1}} D^n \xrightarrow{\delta_n} \dots$$

such that $\delta_{i+1} \circ \delta_i = 0$. We call the δ_i the *coboundary maps*. As with chain complexes, we will use non-negative chain complexes (so $C_i = 0$ for all $i < 0$). The *cohomology groups* of D are defined as

$$H^n(D) = \frac{\ker(\delta_n : D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} : D^{n-1} \to D^n)}.$$

Elements of $\ker \delta_n$ and $\mathrm{im}\, \delta_{n-1}$ are called *n-cocycles* and *n-coboundaries* respectively.

For any chain complex, there is a *dual* cochain complex. Let $C = (C_*, \partial_*)$ be a chain complex

$$\xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0.$$

For all $n \in \mathbb{N}_0$, we define

$$\delta_n = \partial_{n+1}^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G).$$

Thus we have the cochain complex $D = (\mathrm{Hom}(C_*, G), \delta_*)$

$$0 \to \mathrm{Hom}(C_0, G) \xrightarrow{\delta_0} \mathrm{Hom}(C_1, G) \xrightarrow{\delta_1} \mathrm{Hom}(C_2, G) \xrightarrow{\delta_2} \dots .$$

We call D the cochain complex *dual* to the chain complex C. The *cohomology* of C with coefficients in G is defined as

$$H^n(C) = \frac{\ker(\delta_n : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G))}{\mathrm{im}(\delta_{n-1} : \mathrm{Hom}(C_{n-1}, G) \to \mathrm{Hom}(C_n, G))}.$$

1.3 Cohomology of a space

We define the singular cohomology of a space as one may expect. Let X be a space, $A \subset X$, and G an abelian group. Then our singular chain group is

$$C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G),$$
$$\delta_n = \partial_{n+1}^*.$$

Then our singular cohomology groups are defined by

$$H^n(X, A; G) = \frac{\ker \delta_n}{\mathrm{im}\, \delta_{n-1}}.$$

We can also do the same for spaces that admit a cellular structure:

$$C^n_{\mathrm{CW}}(X; G) = \mathrm{Hom}(C_n^{\mathrm{CW}}(X), G),$$
$$\delta_n^{\mathrm{CW}} = \left(\partial_{n+1}^{\mathrm{CW}}\right)^*,$$
$$H^n_{\mathrm{CW}}(X; G) = \frac{\ker \delta_n^{\mathrm{CW}}}{\mathrm{im}\, \delta_{n-1}^{\mathrm{CW}}}.$$

Theorem 1.8. *Let X be a CW complex, G be an abelian group, and $n \in \mathbb{N}_0$. Then*

$$H^n(X; G) \cong H^n_{CW}(X; G).$$

Example 1.9. We consider S^1. We give S^1 the cellular structure $e_0 \cup e_1$ where we glue the endpoints of e_1 to e_0. Now $C_0^{\mathrm{CW}} \cong C_1^{\mathrm{CW}} \cong \mathbb{Z}$ and $C_i^{\mathrm{CW}} \cong 0$ for $i > 1$. Thus we get the CW chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

(reread notes on cellular structure to convince yourself of the map 0). Lets now examine the dual cochain complex:

$$0 \to \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{0^*} \mathrm{Hom}(\mathbb{Z}, G) \to 0$$

(note that the ordering of this sequence has flipped). But we have seen $\mathrm{Hom}(\mathbb{Z}, G) \cong G$, and also $0^* = 0$, so we get

$$0 \to G \xrightarrow{0} G \to 0.$$

Thus

$$H^i_{\mathrm{CW}}(S^1; G) \cong \begin{cases} G & i \in \{0, 1\}, \\ 0 & \text{else.} \end{cases}$$

In fact,

$$H^i_{\mathrm{CW}}(S^n; G) \cong \begin{cases} G & i \in \{0, n\}, \\ 0 & \text{else.} \end{cases}$$

Example 1.10. We now consider $\mathbb{RP}^2$. We can consider this space as D^2 with antipodal boundary points identified. This realises a cellular structure $e_0 \cup e_1 \cup e_2$, giving us the CW chain complex

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

(reread notes on cellular structure if you are not convinced by the maps). Thus our standard homology groups are

$$H_i(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/2 & i = 1, \\ 0 & \text{else.} \end{cases}$$

We now examine the CW cochain complex

$$0 \to \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{0^*} \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{2^*} \mathrm{Hom}(\mathbb{Z}, G) \to 0.$$

That is,

$$0 \to G \xrightarrow{0} G \xrightarrow{g \mapsto 2 \cdot g} G \to 0.$$

This is not easy to characterise generally, so we take $G = \mathbb{Z}$. Then we get the cohomology groups

$$H^i(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/2 & i = 2, \\ 0 & \text{else.} \end{cases}$$

1.4 Cochain maps

Definition 1.11. A *cochain map* $f : C^* \to D^*$ between two cochain complexes $C = (C^*, \delta_*^C)$ and $D = (D^*, \delta_*^D)$ is a sequence f_* of homomorphisms $f_n : C_i \to D_i$ such that the diagram

$$\begin{tikzcd} C^n \arrow[r, "\delta_n^C"] \arrow[d, "f_n"] & C^{n+1} \arrow[d, "f_{n+1}"] \\ D^n \arrow[r, "\delta_n^D"] & D^{n+1} \end{tikzcd}$$

commutes for all $n \in \mathbb{N}_0$.

Definition 1.12. A *cochain homotopy* is a sequence of homomorphisms $h_n : C_n \to D_{n-1}$ such that

$$f_n - g_n = h_{n+1} \circ \delta_n^C + \delta_{n-1}^D \circ h_n$$

for all $n \in \mathbb{N}_0$. That is, the diagram

$$\begin{tikzcd} C^{n-1} \arrow[r, "\delta_{n-1}^C"] \arrow[d, "f_{n-1}-g_{n-1}"'] & C^n \arrow[r, "\delta_n^C"] \arrow[d, "f_n-g_n"] \arrow[dl, "h_n"] & C^{n+1} \arrow[d, "f_{n+1}-g_{n+1}"] \arrow[dl, "h_{n+1}"] \\ D^{n-1} \arrow[r, "\delta_{n-1}^D"] & D^n \arrow[r, "\delta_n^D"] & D^{n+1} \end{tikzcd}$$

commutes for all $n \in \mathbb{N}_0$.

Lemma 1.13.

1. *Let $f : C^* \to D^*$ be a cochain map between two cochain complexes $C = (C^*, \delta_*^C)$ and $D = (D^*, \delta_*^D)$. Then the induced map*

$$\begin{aligned} f_* : H^n(C) &\to H^n(D), \\ [c] &\mapsto [f_n(c)], \end{aligned}$$

   is well-defined.

2. *If cochain maps $f, g : C^* \to D^*$ are cochain homotopic then $f_* = g_*$.*

Lemma 1.14.

1. *Let $f : C_* \to D_*$ be a chain map. Then the induced map*

$$\begin{aligned} f^* : \mathrm{Hom}(D_*, G) &\to \mathrm{Hom}(C_*, G), \\ (\varphi : D_* \to G) &\mapsto (\varphi \circ f : C_* \to D_* \to G), \end{aligned}$$

   is a cochain map.

2. *If $f, g : C_* \to D_*$ are homotopic chain maps, then f^* and g^* are homotopic cochain maps.*

1.5 Properties of cohomology

1. Let $f : (X, A) \to (Y, B)$ be a map of pairs of spaces. Then f_* is a chain map which induces a map $f^* : H^n(Y, B; G) \to H^n(X, A; G)$ on cohomology.

2. Let $f, g : (X, A) \to (Y, A)$ be homotopic maps of pairs of spaces. Then $f^* = g^*$.

3. If $f : X \to Y$ is a homotopy equivalence, f^* defines an isomorphism.

4. Let (X, A) be a pair of spaces. Then there is a long exact sequence on cohomology

$$0 \to H^0(X, A; G) \to H^0(X; G) \to H^0(A; G) \xrightarrow{\delta^*} H^1(X, A; G) \to \dots$$

using the inclusion map $i : A \to X$, quotient map $q_* : C_*(X) \to C_*(X, A) = C_*(X)/C_*(A)$ and the connecting homomorphism δ^*.

Theorem 1.15. *Let X be a space and G be an abelian group. Let $Z \subset A \subset X$ such that $\overline{Z} \subset \mathring{A}$. Then the inclusion*

$$i : (X \setminus Z, A \setminus Z) \overset{(}{\hookrightarrow} X, A)$$

induces an isomorphism

$$i^* : H^n(X, A; G) \xrightarrow{\cong} H^n(X \setminus Z, A \setminus Z; G)$$

on cohomology.

Theorem 1.16. *Let G be an abelian group and X a space such that $X = \mathring{A} \cup \mathring{B}$ for $A, B \subset X$. Then there is a long exact sequence*

$$\dots \to H^n(X; G) \xrightarrow{i^* \oplus i^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B; G)$$
$$\xrightarrow{d} H^{n+1}(X; G) \to \dots$$

where d is the connecting homomorphism.

1.6 Cup products

The main benefit of working with cohomology is that we can give it ring structure, by defining the *cup product*. Our aim is to construct a map

$$\smile : H^i(X; R) \times H^j(X; R) \to H^{i+j}(X; R)$$

for a commutative ring R which satisfies the multiplication axioms for ring structure. $x \smile b$.

We first define $\smile$ on cochain groups, and show that it extends to a well-defined map on cohomology. We define

$$\begin{aligned}\smile : C^i(X;R) \times C^j(X;R) &\to C^{i+j}(X;R),\\ (\varphi(\sigma), \psi(\sigma)) &\mapsto \varphi(\sigma\big|_{[0,\dots,i]})\psi(\sigma\big|_{[i,\dots,i+j]}).\end{aligned}$$

Intuitively, we are evaluating φ on the *front* i-face and ψ on the *back* j-face.

Theorem 1.17.

1. *If φ and ψ are cocycles, then $\varphi \smile \psi$ is a cocycle.*
2. *$\smile$ is well-defined on homology classes.*

To prove this, we need the following.

Lemma 1.18. *Let X be a space, R be a commutative ring, $\varphi \in C^i(X;R)$, and $\psi \in C^j(X;R)$ for some $i, j \in \mathbb{N}_0$. Then*

$$\delta(\varphi \smile \psi) = (\delta\varphi \smile \psi) + (-1)^i(\varphi \smile \delta\psi).$$

Proof. Let X be a space and $\sigma : \Delta^{i+j+1} \to X$ be a singular simplex. Then

$$\begin{aligned}(\delta\varphi \smile \psi)(\sigma) &= \sum_{k=0}^{i+1}(-1)^k\varphi\left(\sigma\big|_{[0,\dots,\hat{k},\dots,i+1]}\right)\cdot\psi\left(\sigma\big|_{[i+1,\dots,i+j+1]}\right)\\ (-1)^i(\varphi \smile \psi) &= \sum_{k=i}^{i+j+1}(-1)^k\varphi\left(\sigma\big|_{[0,\dots,i]}\right)\cdot\psi\left(\sigma\big|_{[i,\dots,\hat{k},\dots,i+j+1]}\right)\end{aligned}$$

and so by adding the RHS we get

$$\delta(\varphi \smile \psi) = (\delta\varphi \smile \psi) + (-1)^i(\varphi \smile \delta\psi).$$ ☐

Now we prove the theorem.

Proof.

1. Let $\varphi \in C^i(X;R)$ and $\psi \in C^j(X;R)$ be cocycles. Then

$$\begin{aligned}\delta(\varphi \smile \psi) &= (\delta\varphi \smile \psi) + (-1)^i(\varphi \smile \delta\psi)\\ &= (0 \smile \psi) + (-1)^i(\varphi \smile 0)\\ &= 0 + 0 = 0.\end{aligned}$$

2. Let $\varphi \in C^i(X;R)$, $\psi \in C^j(X;R)$, $\theta \in C^{i-1}(X;R)$, and $\chi \in C^{j-1}(X;R)$ be cocycles. We aim to show that

$$[(\varphi + \delta\theta) \smile (\psi + \delta\chi)] = [\varphi \smile \psi].$$

We recall that $\delta(\varphi \smile \psi) = (\delta\varphi \smile \psi) + (-1)^i(\varphi \smile \delta\psi)$, and so we get

$$\begin{aligned}\varphi \smile \delta\chi &= (-1)^i \delta(\varphi \smile \chi),\\ \delta\theta \smile \psi &= \delta(\theta \smile \psi),\\ \delta\theta \smile \delta\chi &= \delta(\theta \smile \delta\chi).\end{aligned}$$

Thus

$$\begin{aligned}&((\varphi + \delta\theta) \smile (\psi + \delta\chi)) - (\varphi \smile \psi)\\ = \delta &\left((-1)^i(\varphi \smile \chi) + (\theta \smile \psi) + (\theta \smile \delta\chi)\right)\end{aligned}$$

which is a coboundary. □

1.7 The cohomology ring

Proposition 1.19. *Let R be a commutative ring and let X be a space.*

1. *The cup product is R-bilinear and associative.*
2. *The conglomerate $H^*(X;R) = \bigoplus_{n\in\mathbb{N}_0} H^n(X;R)$, with operations $+$ and $\smile$, forms a ring. The unit of the ring is 1_X, the cohomology class in $H^0(X;R)$ is represented by the constant map $X \to R$ that sends $x \mapsto 1 \in R$ for every $x \in X$.*

Proof. 1. First we prove associativity. Let $\varphi \in C^i(X;R)$, $\psi \in C^j(X;R)$, $\chi \in C^k(X;R)$, and $(\sigma : \Delta^{i+j+k} \to X) \in C_{i+j+k}(X;R)$. Then

$$\begin{aligned}(\varphi \smile \psi) \smile \chi(\sigma) &= (\varphi \smile \psi)(\sigma\big|_{[0,\dots,i+j]}) \cdot \chi(\sigma\big|_{[i+j,\dots,i+j+k]})\\ &= \varphi(\sigma\big|_{[0,\dots,i]}) \cdot \psi(\sigma\big|_{[i,\dots,i+j]}) \cdot \chi(\sigma\big|_{[i+j,\dots,i+j+k]})\\ &= \varphi \smile (\psi \smile \chi)(\sigma).\end{aligned}$$

Showing that φ is R-bilinear is trivial.

2. We have already shown that the cup product is R-bilinear and associative, and we know that cohomology is an abelian group with $+$. Thus, we have left to show that 1_X is the multiplicative identity, but this is clear. So we are done.

□

Theorem 1.20. *Let $f : X \to Y$ be a map of spaces. Then*

$$f^* : H^*(Y;R) \to H^*(X;R)$$

is a ring homomorphism.

Proof. Let $\varphi \in C^i(Y;R)$, $\psi \in C^j(X;R)$, and $(\sigma : \Delta^{i+j} \to X) \in C_{i+j}(X)$. Then

$$\begin{aligned}
(f^*(\varphi) \smile f^*(\psi))(\sigma) &= f^*(\varphi)(\sigma\big|_{[0,\dots,i]}) \cdot f^*(\psi)(\sigma\big|_{[i,\dots,i+j]}) \\
&= (\varphi \circ f)(\sigma\big|_{[0,\dots,i]}) \cdot (\psi \circ f)(\sigma\big|_{i,\dots,i+j]}) \\
&= (\varphi \smile \psi)(f \circ \sigma) \\
&= f^*(\varphi \smile \psi)(\sigma).
\end{aligned}$$

We already know f^* is a homomorphism for the abelian group, so we are done. □

Corollary 1.21. *Let X and Y be spaces such that $X \simeq Y$. Then*

$$H^*(X;R) \cong H^(Y;R)$$

are isomorphic rings.

Proof. As $X \simeq Y$, there is $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. That is, $(f \circ g)^* = f^* \circ g^* = \mathrm{id}$ and $(g \circ f)^* = g^* \circ f^* = \mathrm{id}$. Thus f^* and g^* are inverse ring homomorphisms of each other. □

Theorem 1.22 (Graded commutativity)**.** *Let X be a space, $[\varphi] \in H^i(X;R)$, and $[\psi] \in H^j(X;R)$. Then*

$$[\varphi \smile \psi] = (-1)^{ij}[\psi \smile \varphi] \in H^{i+j}(X;R).$$

The proof for this can be found in Hatcher.

1.8 Cap products

In this subsection we define *cap products*, another R-bilinear product:

$$\frown : H^i(X;R) \times H_j(X;R) \to H_{j-i}(X;R).$$

As with the cup product, we define the cap product on the chain and cochain level, and then show that they induce well-defined maps on homology and cohomology.

Definition 1.23 (Cap product)**.** Let $\varphi \in C^i(X;R)$ be a singular cochain and $\sigma : \Delta^j \to X$ with $i \leq j$. We define

$$\varphi \frown \sigma = \varphi(\sigma\big|_{[0,\dots,i]}) \otimes \sigma\big|_{[i,\dots,j]} \in R \otimes C_{j-i} = C_{j-i}(X;R).$$

We extend to all of $C_j(X;R)$ by linearity. When $i \geq j$, we take $\frown$ to be the zero map (by definition).

Lemma 1.24. *Let X be a space, R a commutative ring, $\varphi \in C^i(X;R)$, and $\sigma \in C_j(X;R)$. Then*

$$\partial(\varphi \frown \sigma) = (-1)^i(-\delta\varphi \frown \sigma + \varphi \frown \partial\sigma) \in C_{j-i-1}(X;R).$$

Proof. □

Lemma 1.25. *The cap product induces a well-defined map*

$$\begin{aligned} H^j(X;R) \times H_j(X;R) &\to H_{j-i}(X;R), \\ ([\varphi],[\sigma]) &\mapsto [\varphi \frown \sigma]. \end{aligned}$$

We have two key lemmas on the cap product: the identity of cohomology acts as identity on homology, and a special case wher cap product is the same as evaluation.

Lemma 1.26. *Let X be a space, R a commutative ring, $\sigma \in H_i(X;R)$, and $1_X \in H^0(X;R)$ be the identity in the cohomology ring. Then*

$$1_X \frown \sigma = \sigma.$$

Lemma 1.27. *Suppose that X is a path connected space and R is a commutative ring. Then*

$$\begin{aligned} \frown : H^i(X;R) \times H_i(X;R) &\to H_0(X;R) \cong R \\ ([\varphi],[\sigma]) &\mapsto \langle \varphi, \sigma \rangle. \end{aligned}$$

Theorem 1.28 (Cup-cap formula)**.** *Let X be a space, R a commutative ring, $\varphi \in C^i(X;R)$, $\psi \in C^j(X;R)$, and $\sigma \in C_n(X;R)$. Then*

$$\varphi \frown (\psi \frown \sigma) = (\psi \smile \varphi) \frown \sigma \in C_{n-i-j}(X;R).$$

Theorem 1.29. *Let $f : X \to Y$ be a map of spaces. Let $\varphi \in C^i(Y;\mathbb{Z})$ and $\sigma \in C_n(X;\mathbb{Z})$. Then*

$$f_*(f^*(\varphi) \frown \sigma) = \varphi \frown f_*(\sigma).$$

2 Relationship between homology and cohomology

2.1 Ext groups

Lemma 2.1. *Let G be an abelian group and let $0 \to A \to B \to C \to 0$ be a short exact sequeunce of abelian groups. Then*

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$$

is exact.

Example 2.2. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

for $p \in \mathbb{Z}_{>1}$. We apply the functor $\mathrm{Hom}(-, \mathbb{Z})$ to get

$$0 \to \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}\, \mathbb{Z}, \mathbb{Z} \to 0$$

which is equivalent to

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0.$$

But this is *not* exact, since multiplication by p is not onto.

We aim to measure the failure of $\mathrm{Hom}(-, G)$ to be right exact, and to do this we define Ext groups.

Definition 2.3. Let H and G be abelian group. An exact sequence

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_2} H \to 0$$

with F_0 and F_1 free abelian group is called a *free resolution* of H. Then we define

$$\begin{aligned}
\mathrm{Ext}^0(H, G) &= \ker(f_1^* : \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)), \\
\mathrm{Ext}^1(H, G) &= \mathrm{coker}(f_1^* : \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)).
\end{aligned}$$

From a categorical approach, we take the chain complex above with $H = 0$, apply $\mathrm{Hom}(-, G)$ and then take cohomology to get $H^n(\mathrm{Hom}(F_*, G))$ for $n \in \{0, 1\}$. Note that we could continue doing this to get Ext^2, Ext^3, etc. but we note that these are all zero for abelian groups.

Let H be a finitely generated abelian group. Then by the classification of finitely generated abelian groups,

$$H \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$$

for $n \in \mathbb{N}_0$, primes $p_1, \ldots, p_k$, and positive integers $n_1, \ldots, n_k$. We can construct a resolution of length one:

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{(0, \bigoplus_{i=1}^{k} p_i^{n_i})} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

Lemma 2.4. *Given two resolutions*

$$0 \to F_1 \to F_0 \to H \to 0$$

and

$$0 \to F_1' \to F_0' \to H \to 0$$

of H and a homomorphism $\varphi : H \to H$, we have the following.

1. *There is a chain map between the resolutions inducing φ.*

2. *Any two such chain maps are chain homotopic.*

Proposition 2.5. *The grouips $\mathrm{Ext}^i(H,G)$ are independent of the choice of free resoltuion of H.*

Example 2.6. We compute $\mathrm{Ext}^n(\mathbb{Z},\mathbb{Z})$. There is a free resolution

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

So $F_* = 0 \to \mathbb{Z} \to 0$. Taking $\mathrm{Hom}(-,\mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0.$$

Thus

$$\mathrm{Ext}^i(\mathbb{Z},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & \text{else.} \end{cases}$$

Example 2.7. We compute $\mathrm{Ext}^n(\mathbb{Z}/p,\mathbb{Z})$. There is a free resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p \to 0.$$

So $F_* = 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0$. Taking $\mathrm{Hom}(-,G)$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0$$

and so

$$\mathrm{Ext}^i(H,G) \cong \begin{cases} \mathbb{Z}/p & i = 1, \\ 0 & \text{else.} \end{cases}$$

Proposition 2.8. *Let G and H be abelian groups.*

1. $\mathrm{Ext}^0(H,G) \cong \mathrm{Hom}(H,G)$.

2. *If H is a free abelian group, then* $\mathrm{Ext}^1(H,G) = 0$.

3. *If H is a finitely generated abelian group, then* $\mathrm{Ext}^1(H,\mathbb{Z})$ *is precisely the torsion subgroup of H.*

4. $\mathrm{Ext}^1(H,\mathbb{Q}) = 0$ *for any H.*

5. *Let $H_1,\dots,H_k$ and $G_1,\dots,G_k$ be abelian groups. Then*

$$\mathrm{Ext}^1\left(\bigoplus_{i=1}^k H_i, G\right) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H_i,G)$$

and

$$\mathrm{Ext}^1\left(H, \bigoplus_{i=1}^k G_i\right) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H,G_i).$$

2.2 Universal coefficient theorem

Definition 2.9. Let (C_*, ∂) be a chain complex of free abelian groups and G be an abelian group. The *evaluation map* is

$$\begin{aligned} \mathrm{ev} : H^n(C;G) &\to \mathrm{Hom}(H_n(C), G) \\ [\varphi : C_n \to G] &\mapsto ([c] \mapsto \varphi(c)). \end{aligned}$$

We may denote this map $\langle [\varphi], [c] \rangle = \varphi(c)$ (called the Kronecker pairing).

Lemma 2.10. ev *is a well-defined group homomorphism.*

Theorem 2.11. *Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. For each $n \in \mathbb{N}_0$, there is a natural (in C_*) short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(C), G) \to 0$$

that splits; thus,

$$H^n(C;G) \cong \mathrm{Ext}^1(H_{n-1}, G) \oplus \mathrm{Hom}(H_n(C), G).$$

Theorem 2.12 (Universal coefficient theorem)**.** *Let X be a space and G an abelian group. For each $n \in \mathbb{N}_0$, ther eis a natural (in X) short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(X), G) \to H^n(X;G) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(X), G) \to 0$$

that splits; thus,

$$H^n(X;G) \cong \mathrm{Ext}^1(H_{n-1}(X), G) \oplus \mathrm{Hom}(H_n(X), G).$$

Theorem 2.13 (UCT for field coefficients)**.** *Let X be a topological space and $\mathbb{F}$ be a field. For each $n \in \mathbb{N}_0$, evaluation gives rise to a vector space isomorphism*

$$H^n(X;\mathbb{F}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}}(H_n(X;\mathbb{F}), \mathbb{F}).$$

2.3 Manifolds

Definition 2.14. An *n-dimensional manifold M* is a topological space that is Hausdorff, second countable, and locally n-Euclidean.

Definition 2.15. A manifold is said to be *closed* if it is compact and has empty boundary.

Example 2.16.
1. The sphere S^n is an n-dimensional manifold.
2. A product of spheres $S^{n_1} \times S^{n_2} \times \ldots \times S^{n_k}$ is a manifold of dimension $n_1 + n_2 + \ldots + n_k$.

3. The surface of genus g, Σ_g, is a manifold.
4. The real project space $\mathbb{RP}^n$ is an n-dimensional manifold.
5. The complex projective space

$$\mathbb{CP}^n = \mathbb{C}^{n+1}/\mathbb{C}^\times$$

   is a $2n$-dimensional manifold.
6. The orthogonal group $O(n)$ is a manifold of dimension $n(n-1)/2$.
7. The group $\mathbb{Z}/p$ acts on S^3 as follows: let p and q be coprime positive integers. Consider $S^3 \subset \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. Write $\mathbb{Z}/p \cong C_p$ where C_p is the cyclic group generated by $\eta = e^{2\pi i/p} \in S^1$. Then

$$\eta^j \cdot (z, w) = (\eta \cdot z, \eta^q \cdot w).$$

   The quotient is $S^3/C_p = L(p, q)$, the lens space.

2.4 Orientations

Throughout this subset, M is an n-dimensional manifold with $\partial M = \varnothing$ and R is a commutative ring. Suppose that $n \geq 1$. Note that $\partial M = \varnothing$ is not essential, but it simplifies exposition.

Lemma 2.17. *For $x \in M$, $H_n(M, M \setminus \{x\}; R) \cong R$.*

Definition 2.18. A local R-orientation of M at x is a generator of $H_n(M, M \setminus \{x\}; R)$.

Lemma 2.19. *Let $\alpha_x \in H_n(M; M\setminus\{x\}; R)$. There is an open neighbourhood U of x at $\alpha \in H_n(M, M \setminus U; R)$ with $\alpha_x = j_x^U(\alpha)$, where $j_x^U : H_n(M : M \setminus U; R) \to H_n(M, M \setminus \{x\}; R)$ is the inclusion induced map.*

Lemma 2.20 (Coherence lemma)**.** *If α_x generated $H_n(M, M \setminus \{x\}; R)$, then $U \subset M$ and α can be chosen so that $\alpha_y = j_y^U(\alpha)$ generated $H_n(M, M\setminus\{y\}; R)$ for all $y \in U$.*

To prove this we need the following lemma.

Lemma 2.21. *Every neighbourhood $W \ni x$ contains $u \ni x$ open such that for every $y \in U$,*

$$j_y^U : H_n(M, M \setminus U; R) \to H_n(M, M \setminus \{y\}; R)$$

is an isomorphism.

Definition 2.22. Let $U \subset M$. An element $\alpha \in H_n(M, M \setminus U; R)$ such that $j_y^U(\alpha)$ generated $H_n(M, M \setminus \{y\}; R)$ for every $y \in U$ is called an *R-local orientation at U.*

Definition 2.23. Let $\{U_i\}_{i \in \mathcal{I}}$ be an open cover of M by open subsets. For each i, let $\alpha_i \in H_n(M, M \setminus U_i; R)$ be a local R-orientation at U_i. This is called an *R-orientation system* if for every pair $i, k \in \mathcal{I}$, if $x \in U_i \cap U_k$ then $j_x^{U_i}(\alpha_i) = j_x^{U_k}(\alpha_k)$.

We say that two R-orientation systems $(M, \{U_i, \alpha_i\}_{i \in \mathcal{I}})$ and $(M, \{V_i, \beta_i\}_{i \in \mathcal{I}})$ if and only if $\alpha_x = \beta_x$ for all $x \in M$. A *global R-orientation of M* is an equivalence class of R-orientation systems on M. If a global R-orientation of M exists, M is said to be *R-orientable*.

Proposition 2.24. *Suppose that M is connected and R-orientable. Then two R orientations that agree at a point are equal.*

Proposition 2.25. *Every manifold M has a unique $\mathbb{Z}/2$ orientation.*

Proposition 2.26. *If M is orientable then it is R-orientable for every R.*

Proposition 2.27. *Every simply connected, connected manifold M is orientable.*

2.5 The fundamental class

Theorem 2.28. *Let M be a compact, connected n-manifold.*

1. *If M is R-orientable, then $H_n(M, \partial M; R) \cong R$.*
2. *If M is not R-orientable, then $H_n(M, \partial M; R) = 0$.*

We also note if ∂M is non-empty, or if M is non-compact, then $H_n(M; R) = 0$.

Definition 2.29. A choice of generator

$$[M]_R \in H_n(M, \partial M; R)$$

is called an *R-fundamental class* of M.

We make some remarks.

1. If $R = \mathbb{Z}$, then we simply write $[M]$ for the fundamental class.
2. Note the special case where $\partial M = \varnothing$.
3. If $\partial M = \varnothing$, then the fundamental class has the property that it maps to a local R-orientation at x, for every $x \in M$, under the map $H_n(M; R) \to H_n(M, M \setminus \{x\}; R)$.
4. Given a triangulation of M, one can think of $[M]_R$ (roughly) as a formal sum of all the top dimensional simplices of M, with coefficients a unit of R. Whether or not an orientation exists is essentially the same as asking whether there is a choice of units such that this sum of simplices has trivial boundary, and so gives rise to a cycle.

2.6 Poincaré duality

Theorem 2.30 (Poincaré duality). *Let M be a compact, connected, R-orientable n-manifold. Then capping with an R-fundamental class induces isomorphisms*

$$- \frown [M]_R : H^{n-r}(M;R) \xrightarrow{\cong} H_r(M, \partial M; R),$$
$$- \frown [M]_R : H^{n-r}(M, \partial M; R) \xrightarrow{\cong} H_r(M;R)$$

for every $r \in \mathbb{N}_0$.

We have a variant for closed manifolds.

Theorem 2.31. *Let M be a closed, connected, R-orientable n-manifold. Then capping with an R-fundamental class induces an isomorphism*

$$- \frown [M]_R : H^{n-r}(M;R) \xrightarrow{\cong} H_r(M;R)$$

for every $r \in \mathbb{N}_0$.

Corollary 2.32. *Let M be a closed and connected n-manifold. Then $H_r(M;\mathbb{Z}/2) \cong H_{n-r}(M;\mathbb{Z}/2)$.*

This is immediate from the Poincaré duality and the universal coefficient theorem with field coefficients.

Corollary 2.33. *Let M be a closed, connected, orientable n-manifold. Then*

$$H_r(M;R) \cong \operatorname{Hom}(H_{n-r}(M;\mathbb{Z}), R) \oplus \operatorname{Ext}^1(H_{n-r-1}(M;\mathbb{Z}), R)).$$

Again, this is immediate from the Poincaré duality and the universal coefficient theorem with field coefficients.

2.7 Applications of Poincaré duality

Example 2.34. We now apply Poincaré duality to compute some cup products. Recall tthe complex projective plane

$$\mathbb{CP}^2 = \frac{\mathbb{C}^3 \setminus \{(0,0,0)\}}{(z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2); \lambda \in \mathbb{C} \setminus \{0\}}.$$

This has a CW decomposition with three cells: one cell of dimension 0, 2, and 4. That is,

$$\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4 = S^2 \cup_\eta D^4.$$

This is a 4-dimensional, oriented manifold. We let $\varphi = [\mathbb{CP}^1]^* \in H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ be a generator and $[\mathbb{CP}^2] \in H_4(\mathbb{CP}^2;\mathbb{Z})$ be a fundamental class. Let $\psi \in H^2(\mathbb{CP}^2;\mathbb{Z})$. Then

$$\langle \varphi \smile \psi, [\mathbb{CP}^2] \rangle = (\varphi \smile \psi) \frown [\mathbb{CP}^2] = \psi \cup (\varphi \cap [\mathbb{CP}^2]).$$

By the Poincaré duality, the map $- \frown [\mathbb{CP}^2] : H^2(\mathbb{CP}^2;\mathbb{Z}) \to H_2(\mathbb{CP}^2;\mathbb{Z})$ is an isomorphism; thus, $\varphi \frown [\mathbb{CP}^2]$ is a generator of $H_2(\mathbb{CP}^2)$, so $\varphi \frown [\mathbb{CP}^2] = \pm[\mathbb{CP}^1]$. We now note that $\mathrm{Ext}^1(H_1(\mathbb{CP}^2;\mathbb{Z});\mathbb{Z}) = \mathrm{Ext}^1(0,\mathbb{Z}) = 0$, so

$$\mathrm{ev} : H^2(\mathbb{CP}^2;\mathbb{Z}) \to \mathrm{Hom}(H_2(\mathbb{CP}^2;\mathbb{Z}),\mathbb{Z})$$

is an isomorphism by UCT. We have that

$$\langle \varphi \smile \psi, [\mathbb{CP}^2]\rangle = \psi \frown (\varphi \cap [\mathbb{CP}^2]) = \mathrm{ev}(\psi)(\varphi \frown [\mathbb{CP}^2]).$$

Taking $\psi = \varphi$, we have is an isomorphism by UCT. We have that

$$\langle \varphi \smile \varphi, [\mathbb{CP}^2]\rangle = \mathrm{ev}(\varphi)(\varphi \frown [\mathbb{CP}^2]) = [\mathbb{CP}^1]^*(\pm[\mathbb{CP}^1]) = \pm 1.$$

It follows that $\varphi \frown \varphi = \pm[\mathbb{CP}^2]^*$ is a dual fundamental class. In particular, the cup product

$$\smile : H^2(\mathbb{CP}^2;\mathbb{Z}) \times H^2(\mathbb{CP}^2;\mathbb{Z}) \to H^4(\mathbb{CP}^2;\mathbb{Z})$$

is non-trivial. In general, $\psi = n\varphi$ for some $n \in \mathbb{Z}$ and so

$$\langle \varphi \smile \psi, [\mathbb{CP}^2]\rangle = \mathrm{ev}(n\varphi)(\varphi \frown [\mathbb{CP}^2]) = n \cdot [\mathbb{CP}^1]^*(\pm[\mathbb{CP}^1]) = \pm n.$$

Proposition 2.35. *The spaces $S^2 \wedge S^4$ and $\mathbb{CP}^2$ are not homotopy equivalent.*

Proof. We note that the homology (and cohomology) of these spaces coincide, ($\mathbb{Z}$ in dimensions 0, 2, and 4) so we cannot argue on this. Isntead, we use cup products to distinguish the homotopy types. Suppose that there is a homotopy equivalence $f : S^2 \wedge S^4 \to \mathbb{CP}^2$. Then we have the following commutative diagram.

$$\begin{tikzcd}
H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) \arrow[r, "\smile"] \arrow[d, "f^* \times f^*"] & H^4(\mathbb{CP}^2) \arrow[d, "f^*"] \\
H^2(S^2 \wedge S^4) \times H^2(S^2 \wedge S^4) \arrow[r, "\smile"] & H^4(S^2 \wedge S^4)
\end{tikzcd}$$

This commutes by the functoriality of the cup product. Since f is a homotopy equivalence, the vertical maps are isomorphisms. We let $(1,1) \in \mathbb{Z} \times \mathbb{Z} \cong H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2)$. This maps to $\pm 1 \in H^4(\mathbb{CP}^2)$ and then to $\pm 1 \in H^4(S^2 \wedge S^4)$. If we instead take the other direction, $(1,1) \in \mathbb{Z} \times \mathbb{Z} \cong H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2)$ maps to $\pm(1,1) \in H^2(S^2 \wedge S^4) \times H^2(S^2 \wedge S^4)$. But the cup product on the product of wedges of spheres vanishes, thus this maps to $0 \in H^4(S^2 \wedge S^4)$. Thus the diagram is not commutative and so no such f may exist. $\square$

We recall that $\mathbb{CP}^2 = S^2 \cup_\eta D^4$. The 4-cell ($D^4$) is attached by a map $\eta : S^3 \to S^2$, which is a famous map called the Hopf map. If η was null homotopic (that is, homotopic to a constant map) then $S^2 \cup_\eta D^4 \simeq S^2 \wedge S^4$. Since we showed that this is not the case, η must be non-trivial. The set of based homotopy classes of maps from S^3 to S^2 form a group, a higher dimensional analogue of the fundamental group, called $\pi_3(S^2)$. We have just shown that $\pi_3(S^2)$ is non-trivial. in fact, $\pi_3(S^2) \cong \mathbb{Z}$ and η is a generator.

Example 2.36. We compute the cup products of $S^2 \times S^2$:

$$\smile: H^2(S^2 \times S^2) \times H^2(S^2 \times S^2) \to H^4(S^2 \times S^2)$$

but note $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$ and $H^4(S^2 \times S^2) \cong \mathbb{Z}$. We can represent this by a 2×2 matrix. Let $p_i : S^2 \times S^2 \to S^2$ be the projection of the ith factor and let $\theta \in H^2(S^2)$ be a generator. Then $p_1^*(\theta)$ and $p_2^*(\theta)$ are generators of $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$. We have that

$$p_i^*(\theta) \smile p_i^*(\theta) = p_i^*(\theta \smile \theta) = p_i^*(0) = 0.$$

Thus the matrix looks like

$$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}.$$

We know that the off-diagonal entries are equal by the symmetry of the cup product. Let $\varphi = (1, 0)$ and $\psi = (0, 1)$ in $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$. Then

$$(\varphi \smile \psi) \frown [S^2 \times S^2] = \psi \frown (\varphi \frown [S^2 \times S^2]) = \mathrm{ev}(\psi)(\varphi \frown [S^2 \times S^2]) = m$$

for some $m \in \mathbb{Z}$. This m is such that

$$\varphi \frown [S^2 \times S^2] = n \cdot [S^2 \times \mathrm{pt}] + m \cdot [\mathrm{pt} \times S^2]$$

$$0 = (\varphi \smile \varphi) \frown [S^2 \times S^2] = \mathrm{ev}(\varphi)(\varphi \frown [S^2 \times S^2]) = n.$$

Proposition 2.37. *The spaces $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent.*

Proof. Both spaces are closed, orientable 4-manifolds with homology $\mathbb{Z}, 0, \mathbb{Z}^2, \mathbb{Z}$. The cup product on $S^2 \times S^2$ is represented by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the cup product on $\mathbb{CP}^2$ is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose there is a homotopy equivlance $f : S^2 \times S^2 \to \mathbb{CP}^2 \# \mathbb{CP}^2$. Let $\beta = (1, 0) \in H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ be a generator. Then $\beta \smile \beta = \pm[\mathbb{CP}^2 \# \mathbb{CP}^2]^*$. Therefore

$$f^*(\beta) \smile f^*(\beta) = \pm[\mathbb{CP}^2 \# \mathbb{CP}^2]^* = \pm[S^2 \times S^2]^*.$$

Suppose $f^*(\beta) = (a, b)$. Then

$$f^*(\beta) \smile f^*(\beta) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2ab$$

which is even, which contradicts the above. Thus, not such f can exist. $\square$