Representation Theory IV, Michaelmas Term, Assignment 1
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1. (a) Find the matrices of all the elements of S_3 in the permutation representation, with respect to the basis e_1, e_2, e_3.

Solution:

$$(1\quad 2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1\quad 2\quad 3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(2\quad 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1\quad 3\quad 2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(1\quad 3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad e = I.$$

(b) Find another basis such that the matrices all take the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & ? & ? \\ 0 & ? & ? \end{pmatrix}$$

and determine the unknown entries for your basis.

Solution: We fix the basis $(e_1+e_2+e_3, e_1-e_2, e_2-e_3)$. Then we get the following matrices.

$$(1\quad 2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1\quad 2\quad 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$(2\quad 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad (1\quad 3\quad 2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(1\quad 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad e = I.$$

2. Suppose that V is a representation of G and that W_1 and W_2 are irreducible subrepresentations. Show that either $W_1 = W_2$ or $W_1 \cap W_2 = \{0\}$.

Solution: Let ρ be the homomorphism for the representation V, and ρ_{W_1} and ρ_{W_2} for W_1 and W_2 respectively. We observe that as W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V. Thus we have another subrepresentation $(\rho_{W_1 \cap W_2}, W_1 \cap W_2)$. Note that this is a subrepresentation of V, W_1, and W_2. But, W_1 and W_2 are irreducible. Thus $W_1 \cap W_2$ must be $W_1 = W_2$ or $\{0\}$.

3. Consider $G = S_n$ with its permutation representation action on $\mathbb{C}^n$ characterized by

$$\pi(g)e_i = e_{g(i)},$$

and let $V \subset \mathbb{C}^n$ be the subspace

$$\left\{(a_1, \ldots, a_n) : \sum_{i=1}^{n} a_i = 0\right\}.$$

Mimic the last part of Example 1.16 to show that V is irreducible.

Solution: Let $U \subset V$ be a non-zero subrepresentation. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in U$ be non-zero. We see that

$$\begin{aligned}
\boldsymbol{b}_1 &= \boldsymbol{a} - \pi\left(\begin{pmatrix}1 & 2\end{pmatrix}\right)\boldsymbol{a} = (a_1 - a_2, a_1 - a_2, 0, \ldots, 0) \\
\boldsymbol{b}_2 &= \boldsymbol{a} - \pi\left(\begin{pmatrix}2 & 3\end{pmatrix}\right)\boldsymbol{a} = (0, a_2 - a_3, a_3 - a_2, 0 \ldots, 0) \\
&\vdots \\
\boldsymbol{b}_{n-1} &= \boldsymbol{a} - \pi\left(\begin{pmatrix}n & n-1\end{pmatrix}\right)\boldsymbol{a} = (0, \ldots, 0, a_{n-1} - a_n, a_n - a_{n-1}).
\end{aligned}$$

Clearly, all of these $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-1}\}$ are linearly independent and are in V. As

$$\dim V = n - 1 = \dim\left(\langle \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-1}\rangle\right),$$

we must have that $U = V$.

6. Classify the irreducible representations of D_n when $n \geq 4$ is even. Write out the list explicitly when $n = 4$.

Solution: Let (ρ, V) be a irreducible complex representation of D_n where $n \mid 4$ and $n \geq 4$. Let $v \in V$ be an eigenvector for $\rho(r)$ with eigenvalue λ. So $\rho(r)v = \lambda v$. We observe that as $(\rho(r))^n = 1$, $(\rho(r))^n v = \lambda^n v$ so λ must be an nth root of unity. We now set $w = \rho(s)v$ and we claim that w is also an eigenvector of $\rho(r)$ we eigenvalue λ^{-1}. Indeed,

$$\begin{aligned}
\rho(r)w &= \rho(r)\rho(s)v \\
&= \rho(rs)v \\
&= \rho(sr^{-1})v \\
&= \rho(s)\rho(r^{-1})v \\
&= \rho(s)(\lambda^{-1}v) \\
&= \lambda^{-1}w.
\end{aligned}$$

As v and w are eigenvectors of $\rho(r)$, we have that $\rho(r)v, \rho(r)w \in \langle v, w\rangle$. We further see that $\rho(s)w = v \in \langle v, w\rangle$ and $\rho(s)(v) = w \in \langle v, w\rangle$ and thus $\langle v, w\rangle$ is a subrepresentation V. But V is irreducible, so $V = \langle v, w\rangle$.

(i) Suppose $\lambda \neq \lambda^{-1}$. Thus v and w are distinct eigenvectors of $\rho(r)$ that are linearly independent. Thus $\dim V = 2$. Fixing the basis (v, w), we have the representation

$$\begin{aligned}
\rho(r) &= \begin{pmatrix}\lambda & 0 \\ 0 & -\lambda\end{pmatrix}, \\
\rho(s) &= \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}.
\end{aligned}$$

As stated before, λ is a nth root of unity. We get a unique representation ρ_k for $\lambda = e^{2\pi ik/n}$, $k \in \{1, 2, \ldots, (n-2)/2\}$. For $e^{2\pi ik/n}$ with $k \in \{1, 2, \ldots, (n-2)/2\}$ we get the same representations as before (taking basis (w, v)).

(ii) We now suppose $\lambda = \lambda^{-1}$, that is $\lambda \in \{1, -1\}$.

(a) Suppose $\lambda = 1$. We consider $v+w$ and $v-w$, both of which cannot be 0. If $v+w \neq 0$, then we see that

$$\rho(s)(v+w) = w + v \in \langle v+w \rangle$$
$$\rho(r)(v+w) = v + w \in \langle v+w \rangle$$

Thus $\langle v+w \rangle$ is a subrepresentation of V, but as V is irreducible we get $V = \langle v+w \rangle$. We see that in this scenario, we have the trivial representation ($\rho(s) = \rho(r) = \begin{pmatrix}1\end{pmatrix}$). Now we suppose that $v - w \neq 0$, then we see that

$$\rho(r)(v-w) = v - w$$
$$\rho(s)(v-w) = -(v-w)$$

and thus by a similar argument to before $\langle v - w \rangle = V$. Here, we have the sign representation: $\rho(r) = \begin{pmatrix}1\end{pmatrix}$ and $\rho(s) = \begin{pmatrix}-1\end{pmatrix}$.

(b) Now suppose $\lambda = -1$. We again consider the vectors $v+w$ and $v-w$. If $v+w \neq 0$,

$$\rho(r)(v+w) = -(v+w)$$
$$\rho(s)(v+w) = v+w,$$

thus $\langle v+w \rangle$ is a subrepresentation of V and thus $V = \langle v+w \rangle$ since V is irreducible. Here we have the representation $\rho(r) = \begin{pmatrix}-1\end{pmatrix}$, $\rho(s) = \begin{pmatrix}1\end{pmatrix}$. Finally, for $v - w \neq 0$, we have

$$\rho(r)(v-w) = -(v-w)$$
$$\rho(s)(v-w) = -(v-w)$$

thus we get the representation $\rho(r) = \rho(s) \begin{pmatrix}-1\end{pmatrix}$.

Dimension	$\rho(r)$	$\rho(s)$
2	$\begin{pmatrix} e^{\pi i/2} & 0 \\ 0 & e^{-\pi i/2} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
1	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
1	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$
1	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
1	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}-1\end{pmatrix}$