Representation Theory

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1 Representation theory of finite groups

1.1 Representations

First, we try to introduce some motivation and intuition on representation theory, to make sense of why are defining them. Describing the full behaviour of a group can be challenging, so to assist our analysis we may look to see how our group *acts* (this language will be formalised) on objects we understand better; for example, vector spaces. Thus effectively reducing abstract algebra to linear algebra.

We restrict our analysis here to finite groups and finite dimensional complex vector spaces (e.g. $\mathbb{C}^n$), unless otherwise stated.

Definition 1.1 (Representation)**.** Let K be a field and G be a group. A *representation* of G over K is a pair (ρ, V) such that V is a K-vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

The dimension of a given representation (ρ, V) is the dimension of the vector space V.

Here, $\mathrm{GL}(V)$ denotes the general linear group of V: the set of all bijective linear transformations $V \to V$ with the functional composition as group operation. A slight abuse of notation: we may often refer to a representation as solely a vector space V or a homomorphism ρ, depending on the relevance of either. Further, we may use the notation gv in place of $\rho(g)v$ given the alternative perspective of a representation: a linear action of G on a vector space.

Note that if we fix a basis of our vector space, then we see that our representation is equivalent to a homomorphism $G \to \mathrm{GL}_n(K)$ where $\mathrm{GL}_n(K)$ denotes the group of invertible $n \times n$ matrices with entries in K (with the operation of ordinary matrix multiplication). In particular, for $n = 1$ we have $G \to K^\times$.

Example 1.2. For a introductory example, we consider S_n. We claim (without proof) that the sign function $\epsilon : S_n \to \{1, -1\}$ (that is, the number of inversions for a permutation) is a homomorphism. This induces a one-dimensional representation of S_n, $(\rho, \mathbb{C})$, where $(\rho(\sigma))(w) = \mathrm{sign}(\sigma) \cdot w$, which is bijective as $\mathrm{im}\,\mathrm{sign} \subset \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Here we think of $\rho(\sigma) : \mathbb{C} \to \mathbb{C}$ as the bijective automorphism that multiplies a complex number by the sign of σ.

We of course have a trivial example too.

Example 1.3. Let G be a group and V a K-vector space for some field K. Let $\rho : G \to \mathrm{GL}(V)$ be the homomorphism sending every element to the identity, that is, $\rho(g) = \mathrm{id}_V$ for all $g \in G$. We call this the *trivial represention* *on V.*

We now consider representation of cyclic groups.

Example 1.4. Let G be a cyclic group with generator g and (ρ, V) be some representation of G. ρ may be completely determined by V and $\rho(g)$. Indeed, let $h \in G$. Then $h = g^k$ for some $k \in \mathbb{Z}$. So $\rho(h) = \rho(g^k) = \rho(g)^k$. If G has infinite order (such as $(\mathbb{Z}, +)$), then $\rho(g)$ may be any invertible linear map, but if G has order $n \in \mathbb{N}$ then $\rho(g)$ must also satify $\rho(g)^n = \mathrm{id}_V$.

Lets move onto a less trivial example, we recall the notion of a group G *acting* on a set X: a group action of G on X is a function $\alpha : G \times X \to X$ with the identity property $\alpha(e_G, x) = x$ and the compatibility property $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$.

Definition 1.5 (Permutation representation)**.** Suppose a group G acts on a set X, and V is a vector space over a field K with basis $\{e_x : x \in X\}$. Then we define the *permutation representation* of G over K associated to X with $ge_x = e_{gx}$.

We have introduced a brand new notion and immediately connected it to existing theory, so lets do an example.

Example 1.6. Consider S_n with its permutation representation over $\mathbb{R}$, $(\pi, \mathbb{R}^n)$, where $(\pi(g))(e_i) = \pi(g)e_i = e_{g(i)}$, where $\{e_1, \ldots, e_n\}$ is the standard basis over $\mathbb{R}^n$. With this standard basis, we realise $\pi(g)$ to be some permutation matrix. To clarify, S_n is acting on $\{1, \ldots, n\}$.

1.2 Subrepresentations and irreducibility

We continue our theory with the notion of a *subrepresentation.*

Definition 1.7. A *subrepresentation* of a representation (ρ, V) of G is a subspace $W \subset V$ such that $\rho(g)w \in W$ for all $g \in G$ and $w \in W$.

We notice that (ρ_W, W) is also a representation of G, where $\rho_W(g)w = \rho(g)w$ for all $w \in W$.

Example 1.8. Lets approach the permutation representation of S_3 over $\mathbb{C}^3$ for which $\sigma(e_i) = e_{\sigma(i)}$ (this is the notation we discussed earlier!). Lets look at some possible subrepresentations.

- Let $W_1 = \langle e_1 + e_2 + e_3 \rangle$. We claim that this is a subrepresentation. Indeed, let $\sigma \in S_3$ and $w = k(e_1 + e_2 + e_3) \in W_1$ where $k \in \mathbb{C}$. Then $\sigma w = k(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)}) = k(e_1 + e_2 + e_3) = w$ (that is, the action of S_n is trivial) so W_1 is a subrepresentation.
- Let $W_2 = \{(x, y, z) \in \mathbb{C}^3 : x + y + z = \lambda\}$ for some $\lambda \in \mathbb{C}$. σ permutes the entries of a vector, and thus the sum of the entries does not change.

Thus $\sigma w = w$ for $w \in W_2$, and so W_2 is a subrepresentation.

- Let $W_3 = \{0\} \subset \mathbb{C}^3$. Indeed, no matter how we permute the entries of the zero vector, we still have the zero vector.

These subrepresentations show us the existence of trivial subrepresentations, as well as infinite families of subrepresentations.

Definition 1.9. A representation is said to be *irreducible* if it is non-zero and has no subrepresentations other than itself and $\{0\}$.

Let V be a representation with subrepresentations W_1 and W_2 such that every element can be uniquely expressed as the sum of an element of W_1 and an element of W_2. Then we say that V is the *internal direct sum* of W_1 and W_2, denoted $V = W_1 \oplus W_2$. We can further generalise this to any finite number of subrepresentations.

Definition 1.10 (Decomposable)**.** A representation that is the direct sum of irreducible subrepresentations is said to be *decomposable.*

We will move to see that all complex representations of finite groups are decomposable.

1.3 Morphisms between representations

Now we define a morphism between representations.

Definition 1.11. Let (ρ, V) and (σ, W) be representations of a group G over a field K. A *G-homomorphism* $V \to W$ is a K-linear map $\phi : V \to W$ such that for all $g \in G$: $\phi \circ \rho(g) = \sigma(g) \circ \phi$. We write $\mathrm{Hom}_G(V, W)$ for the vector space of G-homomorphism from V to W.

In other words, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho(g)"] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

We define a *G-isomorphism* as a bijective G-homomorphism, as one would expect. If such an isomorphism exists between two vectors space V and W, we denote this $V \cong W$ and say the representation are isomorphic.

Lemma 1.12. *Suppose V and W are representations of G.*

1. *If $T \in \mathrm{Hom}_G(V, W)$ is an isomorphism, then $T^{-1} \in \mathrm{Hom}_G(W, V)$.*
2. *Suppose $\dim V = \dim W$ and we fix a bases for both. Then $\rho_V \cong \rho_W$ if and only if there is $T \in \mathrm{GL}_{\dim V}(\mathbb{C})$ such that $T\rho_v(g)T^{-1} = \rho_w(g)$ for all $g \in G$.*

This lemma is a simple exercise on the definitions. Note by $\rho_V \cong \rho_W$, we mean that the representations are isomorphic.

Lemma 1.13. *Let V and W be two representations and $\phi \in \mathrm{Hom}_G(V, W)$. Then $\ker \phi \subset V$ and $\operatorname{im} \phi \subset W$ are subrepresentations.*

Proof. $\ker \phi$ and $\operatorname{im} \phi$ are subspaces, so we have left to show that they are perserved under the action of g. Let $g \in G$ and $v \in \ker \phi$. Then $\phi(gv) = g\phi(v) = 0$, so $gv \in \ker \phi$ thus V is a subrepresentation. Now let $g \in G$ and $w \in \operatorname{im} \phi$. Then there is $v \in V$ such that $\phi(v) = w$. Then $gw = g\phi(v) = \phi(gv) \in \operatorname{im} \phi$ and so W is a subrepresentation. $\square$

Note above we are using a lot of shorthands, although working with two representations (ρ_V, V) and (ρ_W, W), no mention to ρ_V or ρ_W was made. Instead, we opt to talk about the action of the group upon the subspaces.

Example 1.14. We consider two representations of D_3 over $\mathbb{C}$. First, the permutation representation with the standard basis $\{e_1, e_2, e_3\}$. So how may D_3 act on $\{1, 2, 3\}$? Well, observe that $S_3 \cong D_3$ with the group isomorphism $r \mapsto (1\ 2\ 3)$ and $s \mapsto (2\ 3)$. We already know how S_3 acts on $\{1, 2, 3\}$, thus we have

$$\rho_1(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho_1(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we consider a different representation, one may be introduced to D_n as the set of symmetries (rotations and reflections) of a regular n-gon. Thus, letting D_3 on the equilateral triangle centered at 0 with a vertex at $(0, 1)$, we get the following representation of $\mathbb{C}^2$:

$$\rho_2(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \quad \rho_2(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We now define a G-homomorphism $T : \mathbb{C}^3 \to \mathbb{C}^2$ by $e_i \mapsto v_i$, where $\{v_1, v_2, v_3\}$ denote the vertices of the equilateral triangle labelled anticlockwise ($v_1 = (0, 1)$). To calculate the matrix of T, we observe that $v_2 = (-\sqrt{3}/2, -1/2)$ and $v_3 = (\sqrt{3}/2, -1/2)$ and so

$$\begin{pmatrix} 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}.$$

One may check that for all $g \in G$, $T\rho_1(g) = \rho_2(g)T$ and so T is indeed a G-homomorphism. We see that $\ker T = \langle e_1 + e_2 + e_3 \rangle$, and $\operatorname{im} T = \mathbb{C}^2$, both subrepresentations. Finally, if we restrict T to $\{(a, b, c) \in \mathbb{C}^3 : a+b+c = 0\}$, then it defines an isomorphism from this subrepresentation to $(\rho_2, \mathbb{C}^2)$.

If V and W are representations of G, then we may form their *external direct sum* as the representation with underlying vector space $V \oplus W$ and such that $g(v, w) = (gv, gw)$ for $g \in G$, $v \in V$, and $w \in W$. If V and W are subrepresentations of some other representation X, then we may call X the direct sum of V and W if the map $V \oplus W \to X$, $(v, w) \mapsto v + w$ is an isomorphism.

1.4 Example: the dihedral group

We aim to understand all irreducible complex representation of the finite dihedral group D_n. We list the elements of D_n as

$$\{r^k, sr^k : k \in \{0, \dots, n-1\}\}.$$

Let (ρ, V) be a irreducible complex representation of D_n, $v \in V$ be a eigenvector for $\rho(r)$ with eigenvalue λ, and $w = \rho(s)v$. We claim that $\langle v, w\rangle$ is a subrepresentation of V. Indeed, it can be shown that $\rho(r)w = \lambda^{-1}w$, $\rho(r)v = \lambda v$, $\rho(s)w = v$, and $\rho(s)v = w$ (by examining the group structure of D_n). As V is irreducible, we have either $V = \langle v, w\rangle$ or $\langle v, w\rangle = 0$; but eigenvectors cannot the zero vector so we have the former. As one may have noticed, v and w are both eigenvectors of $\rho(r)$. We now split into cases.

- Suppose that $\lambda \neq \lambda^{-1}$. Thus v and w are eigenvectors of $\rho(r)$ with distinct eigenvalues, so they are linearly independent. But v and w also span V; thus, $\dim V = 2$. From the above calculations, we see that (with basis v, w)

  $$\rho(r) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  We are in the dihedral group, so we must have $r^n = e$. Thus $\rho(r)^n = I$; that is, $\lambda^n = 1$. As $\lambda \neq \lambda^{-1}$, $\lambda \neq 1$ and $\lambda \neq e^{\pi i}$. The other roots of unity are $\lambda = e^{2\pi i k/n}$ for $k \in \{1, \dots, n-1\}$ (excluding $k = n/2$ if n is even). Observe that $e^{2\pi i k/n} = e^{-2\pi i (n-k)/n}$ which we can obtain by using the base w, v to obtain (as opposed to v, w), thus we have unique representation above with $\lambda = e^{2\pi i k/n}$ for $k \in \{1, \dots, \lceil n/2\rceil - 1\}$.

- Now we assume $\lambda = \lambda^{-1}$.

  - Assume n is odd, then $\lambda = 1$. Observe that $\rho(r)(v+w) = \rho(s)(v+w) = v + w$, so $\langle v + w\rangle$ is a subrepresentation of V. If $v + w \neq 0$, then $\rho(r) = \rho(s) = \mathrm{id}_V$ (the trivial representation). Otherwise, $\rho(r) = \mathrm{id}_V$ and $\rho(s)v = w = -v$ (the sign representation).

  - Assume n is even, then the above case still holds for $\lambda = 1$, but we also consider $\lambda = e^{\pi i} = -1$. By similar argument to the last point, $\langle v+w\rangle$ is a subrepresentation of V; thus, $\langle v+w\rangle \in \{V, \{0\}\}$.

If $\langle v+w \rangle = \{0\}$, then $v+w=0$ and so we get $\rho(r) = (-1)$ and $\rho(s) = (-1)$. Otherwise, we get $\rho(r) = (-1)$ and $\rho(s) = (1)$.

This completes our list of representations, shown in the table below, where the bottom two representations are omitted if n is odd.

Label	Dimension	$\rho(r)$	$\rho(s)$
ρ_k, $k \in \{1, \ldots, \lceil n/2 \rceil - 1\}$	2	$\begin{pmatrix} e^{2\pi i k/n} & 0 \\ 0 & e^{-2\pi i k/n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{1}$	1	1	1
ϵ	1	1	-1
ρ_1	1	-1	-1
ρ_2	1	-1	1

1.5 Schur's lemma

Theorem 1.15 (Schur's lemma). *Let V and W be irreducible finite-dimensional complex representation of some group G and $T : V \to W$ be a G-homomorphism.*

1. *Either T is an isomorphism or $T = 0$.*
2. *If $V = W$, then $T = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$.*
3. $\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{else.} \end{cases}$

Proof. 1: we see this by recalling that $\ker T \subset V$ and $\operatorname{im} T \subset W$ are subrepresentations, then we argue on the irreducibility of V. 2: we claim that $T - \lambda \operatorname{id}_V$ is also a G-homomorphism with non-zero kernel, where λ is an eigenvalue of T (the existence of λ is as V is a complex vector space, and the kernel is precisely the span of the eigenvector(s) for λ). We conclude by arguing on the irreducibility of V. 3: if we suppose that $V \ncong W$, then then by (1) we have $T = 0$, so $\operatorname{Hom}_G(V, W) = \{0\}$. Now suppose $V \cong W$ with isomorphism S. Let $T \in \operatorname{Hom}_G(V, W)$. Observe that $S^{-1} \circ T \in \operatorname{Hom}_G(V, V)$, so by (2) we have that $S^{-1} \circ T = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$. Thus $T = \lambda S$, and so $\operatorname{Hom}_G(V, W) = \langle S \rangle$. □

Schur's lemma is elementary and has some more graspable corollaries.

Corollary 1.16. *Every finite-dimensional irreducible complex representation of an abelian group is one-dimensional.*

Proof. Let G be an abelian group and (ρ, V) a representation as in the statement of the corollary. Fix $h \in G$, and notice that $\rho(h)$ is a G-homomorphism. Thus, $\rho(h) \in \operatorname{Hom}_G(V, V)$ and by Schur's lemma $\rho(h) = \lambda \operatorname{id}_V$ for some

$\lambda \in \mathbb{C}$ (we ignore the case in which $\lambda = 0$, as this leads to a zero-dimensional representation which is not considered to be irreducible). From this, we see that $\langle v \rangle$ forms a subrepresentation of V for all $v \in V \setminus \{0\}$. Following a similar argument on the irreducibility of V, we see that $V = \langle v \rangle$; that is, $\dim V = 1$. $\square$

In the statement on Schur's lemma, G was not assumed to be finite but we did assume that our representation was finite-dimensional. If G was finite, then it is necessary that any irreducible representation of G must also be finite-dimensional.

Proposition 1.17. *Any irreducibile representation of a finite group is finite-dimensional.*

Proof. A proof to this may seem hard to approach, but in fact it requires very little work. Let V be an irreducible representation of a finite group and let $v \in V \setminus \{0\}$. We construct the subspace $V' = \{gv : g \in G\}$ and observe that it is preserved by G (that is, $\rho(g)v \in V'$ for all $v \in V'$ and $g \in G$), this can be easily checked. Thus V' is a subrepresentation of V. We argue on the irreducibility of V that $V = V'$, and V' is finite-dimensional by construction. $\square$

A group homomorphism $\chi : G \to \mathbb{C}^\times$ (or to the multiplicative set of any field) for a group G is called a *multiplicative character* (or *character*, but this causes a clash of notation). If G is abelian, we may define the group

$$\hat{G} = \{\chi : G \to \mathbb{C}^\times : \chi \text{ is a group homomorphism}\}$$

called the *character group* (or *dual group*) of G, which is closed under multiplication.

Example 1.18. We claim $\hat{C}_n \cong C_n$. Let g be a generator of C_n, then χ is uniquely determined by $\chi(g)$. But we observe that $\chi(g)^n = \chi(g^n) = \chi(e) = 1$, thus $\chi(g)$ must be an nth root of unity. That is, $\chi(g) = e^{2\pi ik/n}$ for some $k \in \{0, \ldots, n-1\}$. In fact, we claim the map $k \mapsto e^{2\pi ik/n}$ is a group isomorphism $C_n \cong \mathbb{Z}/n \to \hat{C}_n$. We can extend this further: by the fundamental theorem of finite abelian groups, any finite abelian group is isomorphic to its dual group.

We recall the center of group is the set of elements that commute with every other element.

Proposition 1.19. *Let (ρ, V) be an irreducibility finite-dimensional representation of a group G. The center of G, $Z(G)$, acts on V as a character: there is $\chi : Z \to \mathbb{C}^\times$ such that $\rho(z)v = \chi(z)v$ for all $z \in Z(G)$ and $v \in V$.*

Proof. An example of Schur's lemma, we consider $z \in Z(G)$ and observe that $\rho(z) \in \mathrm{Hom}_G(V, V)$. Thus $\rho(z) = \lambda_z \,\mathrm{id}_V$ for some $\lambda_z \in \mathbb{C}^\times$. Thus $\chi(z) = \lambda_z$ and we are done. We call χ the *central character* of ρ. $\square$

Proposition 1.20. *Let G be a finite group and A an abelian subgroup. Let (ρ, V) be an irreducible representation of G. Then*

$$\dim V \leq |G|/|A| = [G : A].$$

Proof. We restrict our represnetation of G to A to find an irreducible A-subrepresentation W of V. As A is abelian, W is one-dimesional, say spanned by the vector $v \in W$. Thus there is a character χ of A such that $\rho(h)v = \chi(h)v$ for all $h \in A$. We see that $\{\rho(g)v : g \in G\}$ is a subrepresentation of V and thus equal to V (by irreducibility). We now write $g_1A, \ldots, g_rA$ for the left cosets of A where $r = [G : A]$. For $h \in A$ we have $\rho(g_ih)v = \rho(g_i)\rho(h)v = \rho(g_i)\chi(h)v = \chi(h)(\rho(g_i)v)$. Thus $V = \langle \rho(g_i)v : i \in \{1, \ldots, r\}\rangle$; and so must have dimension at most r. $\square$

Example 1.21. Consider D_n with the abelian subgroup C_n and $[D_n : C_n] = 2$. Thus every irreducible representation of D_n must have dimension at most 2.

1.6 Maschke's theorem

Definition 1.22 (Projection)**.** Let V be a vector space and $W \subset V$ a subspace. A linear map $\pi : V \to W$ is a *projection* if $\pi(w) = w$ for all $w \in W$.

Lemma 1.23. *If $\pi : V \to W$ is some projection, then $V = W \oplus \ker \pi$.*

Proof. Let $v \in V$. Then $v = (v - \pi(v)) + \pi(v)$. It follows that $\pi(v - \pi(v)) = 0$ and $\pi(v) \in W$. $\square$

Recall that the characteristic of a field is the smallest number of times one must add the multiplicative identity to get the additive identity. If the sum never reaches the additive identity then the field is said to have characteristic zero.

Theorem 1.24 (Maschke's)**.** *If G is a finite group then every finite-dimensional representation of G over a field whose characteristic does not divide $|G|$ is decomposable.*

Proof. Let V be our representation. If V is irreducible, then we are done. Otherwise, we take a irreducible subrepresentation W and construct a projection $\pi : V \to W$ which is a G-homomorphism. Then $V = W \oplus \ker \pi$, and $\ker \pi$ is a subrepresentation (as $\pi \in \mathrm{Hom}_G(V, W)$). We repeat this line of reasoning starting with $\ker \pi$ as our representation. So, we have left to construct such a π. First, define $\pi_0 : V \to W$ to be a linear map such that $\pi_0\big|_W = \mathrm{id}_W$ (we can construct this by choosing a basis for W and extending it to V, then setting π_0 to be the identity on the basis of W and anything else for V). This itself may not be a G-homomorphism, but we define $\pi(v) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi_0(gv)$ which is a G-homomorphism. Indeed, for $h \in G$

$$\begin{aligned}
\pi(hv) &= hh^{-1}\pi(hv) \\
&= h\frac{1}{|G|}\sum_{g\in G}(gh)^{-1}\pi_0((gh)v) \\
&= h\frac{1}{|G|}\sum_{k\in G}(k)^{-1}\pi_0((k)v) \\
&= h\pi(v).
\end{aligned}$$

Here $k = gh$, and we justify the step in which it was introduced with the following: h is fixed and g iterates over every element of the group once, thus $k = gh$ also iterates over each element exactly once. Indeed, if we suppose otherwise then we would have $g_1 \neq g_2$ such that $g_1h = g_2h$. Multiplying both sides by h^{-1} we get $g_1 = g_2$; a contradiction. We have left to check that π is a projection, for $w \in W$

$$\begin{aligned}
\pi(w) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi_0(gw) \\
&= \frac{1}{|G|}\sum_{g\in G} g^{-1}gw \\
&= \frac{1}{|G|}\sum_{g\in G} w = w.
\end{aligned}$$

Note above we used that W is a subrepresentation of V, so $gw \in W$. □

Both Maschke's theorem and Schur's lemma hold for representation of finite groups over the complex field; we assume this from now on, as well as all representations being finite-dimensional.

Corollary 1.25. *Let V be a representation of G. Then there is the decomposition of V*

$$V \cong W_1 \oplus \ldots \oplus W_r$$

for some irreducible representations $W_1, \ldots, W_r$. Moreover, the number of times each isomorphism class of irreducible representations shows up in the decomposition above is independent of the choice of decomposition.

Proof. Existence has been established. For uniqueness, consider a irreducible representation W and V with decomposition $V \cong W_1 \oplus \ldots \oplus W_r$. Then

$$\dim \operatorname{Hom}_G(W, V) = \sum_i \dim \operatorname{Hom}_G(W, W_i) = |\{i : W \cong W_i\}|$$

by Schur's lemma. This is dependent on V and W, not the choice of decomposition. $\square$

In the proof above, we used a property of Hom that we have maybe not formalised, but would expect to be true.

Lemma 1.26. *If V, V', W, W' are representations of some group G then*

$$\operatorname{Hom}_G(V, W \oplus W') \cong \operatorname{Hom}_G(V, W) \oplus \operatorname{Hom}_G(V, W'),$$
$$\operatorname{Hom}_G(V \oplus V', W) \cong \operatorname{Hom}_G(V, W) \oplus \operatorname{Hom}_G(V', W).$$

Proof. We prove just the first, as the proof for the second follows the same logic. Let $\phi \in \operatorname{Hom}_G(V, W \oplus W')$. Then $\phi(v) = (\phi_W(v), \phi_{W'}(v))$ for linear maps ϕ_W and $\phi_{W'}$. We have left to prove that these are G-homomorphisms. For $v \in V$ and $g \in G$,

$$\begin{aligned} \phi(gv) &= g\phi(v) \\ (\phi_W(gv), \phi_{W'}(gv)) &= g(\phi_W(v), \phi_{W'}(v)) \\ &= (g\phi_W(v), g\phi_{W'}(v)), \end{aligned}$$

thus the action of g commutes with ϕ_W and $\phi_{W'}$ $\square$

We used this fact in an earlier proof, but it is worth a note of its own.

Lemma 1.27. *If ρ is a irreducible representation of G and σ is some other representation of V, then the number of times ρ appears in the decomposition of σ is* $\dim \operatorname{Hom}_G(\rho, \sigma)$.

Example 1.28. In effort to gain intuition behind the technique used to prove Maschke's theorem, we will examine the actual construction of the projection for an example. We consider a representation of S_3 on $\mathbb{C}^3$: the permutation representation. We have the irreducible subrepresentation $V_0 = \langle (1,1,1) \rangle$. We build our G-projection (that is, a projection that is also a G-homomorphism) $\pi : V \to V_0$ is such that

$$\pi(x, y, z) = \frac{1}{|S_3|} \sum_{\sigma \in S_3} \sigma^{-1} \pi_0(\sigma(x, y, z)))$$

where we pick $\pi_0(x, y, z) = \frac{1}{3}(x + y + z)(1, 1, 1)$. Thus

$$\pi(x, y, z) = \frac{1}{3|S_3|} \sum_{\sigma \in S_3} (x + y + z)(1, 1, 1) = \frac{1}{3}(x + y + z)(1, 1, 1).$$

In turns out that, in this case, our *averaging trick* did nothing to change our map, and π_0 was already a G-homomorphism. Following the proof further, we see that $V = V_0 \oplus V_1$ where $V_1 = \ker \pi = \{(x, y, z) : x + y + z = 0\}$ which is irreducible (we can argue this a couple of ways, but note $S_3 \cong D_3$, which is a group who's irreducible representations we have classified). We may have picked another G-equivariant projection and maybe obtained V_1 first. For example, we have that precise scenario when we consider the projection $V \to V_1$, $(x, y, z) \mapsto \frac{1}{3}(2x - y - z, 2y - x - z, 2z - x - y)$, we see that the kernel of this map is V_0 (as we expected).

1.7 The group ring

Definition 1.29 (Group ring)**.** Let G be a finite group. The *group ring* $\mathbb{C}[G]$ has elements as formal linear combinations $\sum_{g \in G} a_g[g]$ with $a_g \in \mathbb{C}$, which are multiplied according to $[g][h] = [gh]$; that is,

$$\sum_{g \in G} a_g[g] \sum_{g \in G} b_g[g] = \sum_{g \in G} a_g b_g [gh].$$

The set $\{[g] : g \in G\}$ form a basis for the group ring vector space, which has dimension $\dim G$.

Example 1.30. Let $x = [e] - [(1\ 2)]$ and $y = 2[(2\ 3)] + [(1\ 2\ 3)]$ be elements on $\mathbb{C}[S_3]$. Then

$$\begin{aligned}
xy &= ([e] - [(1\ 2)])\,(2[(2\ 3)] + [(1\ 2\ 3)]) \\
&= 2[e][(2\ 3)] + [e][(1\ 2\ 3)] - 2[(1\ 2)][(2\ 3)] - [(1\ 2)][(1\ 2\ 3)] \\
&= 2[e(2\ 3)] + [e(1\ 2\ 3)] - 2[(1\ 2)(2\ 3)] - [(1\ 2)(1\ 2\ 3)] \\
&= 2[(2\ 3)] + [(1\ 2\ 3)] - 2[(1\ 2)(2\ 3)] - [(1\ 2)(1\ 2\ 3)] \\
&= 2[(2\ 3)] + [(1\ 2\ 3)] - 2[(1\ 2\ 3)] - [(2\ 3)] \\
&= [(2\ 3)] - [(1\ 2\ 3)]
\end{aligned}$$

Definition 1.31 (Regular representation)**.** Let G be a finite group. Then the *(left) regular representation* of G is $(\rho, \mathbb{C}[G])$ where $\rho(g)\left(\sum_{h \in G} a_g[h]\right) = \sum_{h \in G} a_h[gh]$.

Proposition 1.32. *The (left) regular representation of a finite group indeed defines a representation.*

Proof. Let G be a finite group. Firstly, $\mathbb{C}[G]$ is a $\mathbb{C}$-vector space by construction. We have left to show that ρ is a group homomorphism. Let $g_1, g_2 \in G$

and $\sum_{h\in G} a_g[h] \in \mathbb{C}[G]$. Then

$$\begin{aligned}
\rho(g_1g_2)\left(\sum_{h\in G} a_g[h]\right) &= \sum_{h\in G} a_g[(g_1g_2)h] \\
&= \sum_{h\in G} a_g[g_1(g_2h)] \\
&= \rho(g_1)\left(\sum_{h\in G} a_g[g_2h]\right) \\
&= \rho(g_1)\rho(g_2)\left(\sum_{h\in G} a_g[h]\right)
\end{aligned}$$

□

If (p, V) is some representation of G, then we can *multiply* any element of V by any element of $\mathbb{C}[G]$ by $(\sum a_g[g])v = \sum a_g\rho(g)v$.

We can view the regular representation of a finite group as the permutation representation for the action of G *on itself* (by left multiplication). We may also look at a dual point of view using functions.

Definition 1.33 (Regular representation, functional)**.** Let $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$. We define a representation ρ of G on $\mathbb{C}^G$ by $\rho(g)(f(h)) = f(g^{-1}h)$.

Lemma 1.34. *Let G be a finite group. Then the representations $\mathbb{C}^G$ and $\mathbb{C}[G]$ are isomorphic.*

So when we refer to the regular representation, we may use either of these definitions. Whichever is more convenient.

Theorem 1.35. *Let V be any representation of G. Then there is an isomorphism of vector spaces $\mathrm{Hom}_G(\mathbb{C}[G], V) \to V$. Equivalently,*

$$\dim \mathrm{Hom}_G(\mathbb{C}[G], V) = \dim V.$$

Proof. Let ρ be the regular representation of G. For $f \in \mathrm{Hom}_G(\mathbb{C}[G], V)$, we claim that it is uniquely determined by $f([e])$ as $f([g]) = f(\rho(g)[e]) = \rho(g)f([e])$. Thus we define $\phi : \mathrm{Hom}_G(\mathbb{C}[G], V) \to V$ such that $\phi(f) = f([e])$. Conversely, we recall that we can *multiply* elements of any representation by elements in the regular representation, so we define $\psi : V \to \mathrm{Hom}_G(\mathbb{C}[G], V)$ by $\psi(v)(\sum a_g[g]) = \sum a_g[g]v$. We claim that ϕ and ψ are linear maps that are two-sided inverses of each other, thus proving the theorem. □

This has quite a significant consequence: the sum of the squares of the dimensions of the irreducible representations is equal to the order of the group. We write $\mathrm{Irr}(G)$ for the set of isomorphism classes of irreducible representations of G.

Theorem 1.36. *Let G be a finite group and $(\rho, \mathbb{C}[G])$ be the regular representation. Then*

$$\mathbb{C}[G] \cong \bigoplus_{\rho \in \mathrm{Irr}(G)} \rho^{\dim \rho}$$

and by equating the dimensions of both sides, we get

$$\sum_{\rho \in \mathrm{Irr}(G)} \dim(\rho)^2 = |G|.$$

Proof. By Maschke's theorem, we decompose $\mathbb{C}[G]$ into isomorphism classes of irreducible representation in which each class ρ appears

$$\dim \mathrm{Hom}_G(\mathbb{C}[G], \rho) = \dim \rho$$

times (by the last Theorem). □

The formula at the end of the last theorem is called the *sum of squares formula* and can be quite useful in determining all irreducible representations of a given group. For example, in classifying the irreducible representations of the dihedral group we may just write down enough non-isomorphic representations to satisfy the sum of squares formula, then we are done.

Example 1.37. Consider the dihedral group D_n. We will verify the sum of squares formula. We recall our representations for the dihedral group (note in the table below, the last two representations are omitted if n is odd).

Label	Dimension	$\rho(r)$	$\rho(s)$
ρ_k, $k \in \{1, \dots, \lceil n/2 \rceil - 1\}$	2	$\begin{pmatrix} e^{2\pi ik/n} & 0 \\ 0 & e^{-2\pi ik/n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{1}$	1	1	1
ϵ	1	1	-1
ρ_1	1	-1	-1
ρ_2	1	-1	1

First, we have $|D_n| = 2n$. Lets cover for n odd first. So there are $(n-1)/2$ irreducible representations of dimension 2. Then there are 2 more representations of dimension 1, giving us

$$\sum_{\rho \in \mathrm{Irr}(G)} \rho^{\dim \rho} = \left(\frac{n-1}{2}\right)(2^2) + 2(1^1) = 2n$$

as expected. Similarly, for n even there are $(n/2) - 1$ irreducible represen-taitons of dimension 2, and 4 more of dimension 1. This gives us

$$\sum_{\rho \in \operatorname{Irr}(G)} \rho^{\dim \rho} = \left(\frac{n}{2} - 1\right)(2^2) + 2(1^1) = 2n$$

again as expected.

2 Character theory

2.1 Characters

Definition 2.1 (Character)**.** Let (ρ, V) be a finite-dimensional complex representation of G. The *character* of (ρ, V) is the function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_p(g) = \operatorname{tr}(\rho(g)).$$

We note that trace is a function defined on matrices, but to represent $\rho(g) : V \to V$ as a matrix we must pick a basis for V. The definition above assumes that the trace of any such representation is the same, which is not immediately obvious. But we can see that by applying a change of basis matrix P to a matrix M yields $\operatorname{tr} PMP^{-1} = \operatorname{tr} M$. We can apply a similar argument to get the following lemma.

Lemma 2.2. *Isomorphic representations have the same character.*

It may seem that the character only encodes certain information about the representation; however, the character completely determines the represent-ation. There is also additional structure that allow us to find all of the characters of a group, even if we cannot construct the representations.

Example 2.3. Let $\chi : G \to \mathbb{C}^\times$ be a one-dimensional representation. Then it is its own character.

Example 2.4. Let ρ be the irreducible two-dimensional representation of S_3 and let χ be its character. We observe that $S_3 \cong D_3$, and we have already seen what this representation is; that is

$$\rho(r) = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so

$$\begin{aligned} \chi(e) &= 2 \\ \chi((1\ 2)) = \chi((1\ 3)) = \chi((2\ 3)) &= 0 \\ \chi((1\ 2\ 3) = \chi(1\ 3\ 2) &= e^{2\pi i/3} + e^{-2\pi i/3} = -1. \end{aligned}$$

Lemma 2.5. *If (ρ, V) is a representation of G, then $\chi_\rho(e) = \dim \rho$.*

Proof. We pick some basis of V. Since e is the identity element of G, it will be mapped to the identity matrix. Thus

$$\chi_\rho(e) = \operatorname{tr} I = \sum_{i=1}^{\dim V} 1 = \dim \rho.$$

$\square$

We call $\dim \rho$ the *degree* (or *dimension* of the character χ_ρ).

Lemma 2.6. *If V and W are representations with characters χ and ψ, then $V \oplus W$ has character $\chi + \psi$.*

Proof. This can be shown by examining the block matrices for these representations. $\square$

Lemma 2.7. *If χ is the character of a representation V of G, then*

$$\chi(g) = \overline{\chi(g)}$$

for $g \in G$.

Proof. Let $g \in G$ with order m. We can find a basis such that $\rho(g)$ is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that λ_i is the ith root of unity. Then $\rho(g)^{-1}$ is diagonal with eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$; that it $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$. $\square$

It immediately follows that if g is conjugate to g^{-1}, then $\chi(g)$ is real for every character χ.

Lemma 2.8. *Let χ be the character of a representation ρ of G. If g and h are in the same conjugacy class of G, then*

$$\chi(g) = \chi(h).$$

Proof. Let $h = xgx^{-1}$ for some $x \in G$. Then

$$\chi(h) = \operatorname{tr} \rho(xgx^{-1}) = \operatorname{tr} \left(\rho(x)\rho(g)\rho(x^{-1})\right) = \operatorname{tr} \rho(g) = \chi(g). \qquad \square$$

Definition 2.9 (Class function)**.** Let G be a group. A *class function* on G is a function $G \to \mathbb{C}$ that is constant on conjugacy classes.

The previous lemma can be rephrased as: the character of a representation is a class function. We can organise the character information of a group into a *character table*. This columns are the conjugacy classes of the group, and the rows are the irreducible representations. The entries are the values of the characters of the irreducible representations on elements on the conjugacy class.

Example 2.10. Consider S_3. The following is the character table.

	Class		
Representation	e	(1 2)	(1 2 3)
$\mathbb{1}$	1	1	1
ε	1	-1	1
ρ	2	0	-1

Example 2.11. Consider C_5 and $\omega = e^{2\pi i/n}$. Since G is abelian, all conjugacy classes are singletons. The following is the character table.

	Class				
Representation	e	g	g^2	g^3	g^4
$\mathbb{1}$	1	1	1	1	1
χ	1	ω	ω^2	ω^3	ω^4
χ^2	1	ω^2	ω^4	ω	ω^3
χ^3	1	ω^3	ω	ω^4	ω^2
χ^4	1	ω^4	ω^3	ω^2	ω

Example 2.12. Let G act on a finite set X and let ρ be the permutation representation. Then the character χ is given by

$$\chi(g) = |\mathrm{Fix}(g)|$$

for $g \in G$, where $\mathrm{Fix}(g)$ is the number of fixed points of g. This can be seen by observing that $\rho(g)$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 1)$ and each fixed point of g has a one-to-one correspondence to a column C_i of $\rho(g)$ such that $C_i[j] = 0$ for $j \neq i$ and $C_i[i] = 1$.

Example 2.13. Let G be a finite group and consider the regular representation $V = \bigoplus_{g \in G} \mathbb{C}[g]$. This representation associates to each $g \in G$ the matrix M_g that permutes the basis according to the multiplication of G. For all $x \in G$ and $g \neq e$, $gx \neq X$. Thus, M_g have zero trace for all $g \neq 1$. Thus $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$.

2.2 Orthogonality of characters

Definition 2.14. If χ and φ are two class functions on G, we define their *inner product* to be

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g).$$

This is in fact a Hermitian inner product on the space of class functions of G, whose dimension is the number of conjugacy classes of G.

Theorem 2.15. *Let G be a finite group with conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$. If χ is a class function, we write $\chi(\mathcal{C}_i) = \chi(g)$ for any $g \in \mathcal{C}_i$.*

1. *The irreducible characters are orthonormal with respect to their inner product. We can also see this inner product as the standard inner product of the rows of the character table with entries weighted by $\sqrt{|\mathcal{C}_i|/|G|}$.*
2. *The number of irreducible representations is equal to the number of conjugacy classes.*
3. *The columns of the character table are orthonormal with respect to the weighted inner product from 1.*

Theorem 2.16. *Two irreducible representations of G are isomorphic if and only if they have the same character.*

Theorem 2.17.

1. *A representation is determined up to isomorphism by its character.*
2. *If V is a representation with character χ, then V is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

2.3 Example, S_4

Example 2.18. We will determine the character table of S_4. We first have the trivial representation $\mathbb{1}$ and the sign character ε, which give us the following partial character table.

	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	1	-1

Next we consider the permutation representation V of S_4 on $\{1, 2, 3, 4\}$. This contains a copy of the trivial representation, so we write $V = \mathbb{1} \oplus W$ for some representation W, where

$$\chi(g) = |\mathrm{Fix}(g)| - 1.$$

We notice that we may perform an operation on representation (known as *twisting*), if (ρ, V) is an irreducible representation of G with character χ and ψ is a one-dimensional character of G, then we can define a new representation $\rho\psi$ of G on V by the formula

$$(\rho\psi)(g) = \rho(g)\psi(g).$$

It has character $\chi\psi$. In this case, we look at $\chi\varepsilon$ and see that it must be a character of an irreducible representation.

	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
Class size	1	6	3	8	6
χ	3	1	-1	0	-1
χ^{ε}	3	-1	-1	0	1

We see that

$$\langle\chi\rangle = \frac{1}{24}\left(1(3^2) + 6(1^2) + 3(-1)^2 + 8(0^2) + 6(-1)^2\right) = 1$$

and so χ is indeed irreducible. We note that $\chi \oplus \varepsilon$ differs to χ, and is also irreducible. So we have four of the five representations and have one left to find, ψ. We can find it using the theorems we built in the last section. Firstly, using the sum of squares formula we assert than $\psi(e) = 2$. Then, using the orthonormality of the first column we get the following character table.

	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	1	-1
χ	3	1	-1	0	-1
χ^{ε}	3	-1	-1	0	1
ψ	2	0	2	-1	0

We notice that the character of the final representation was computed without constructing the representation itself.

2.3.1 Character of lifts

Definition 2.19 (Restriction)**.** Let G be a group, H a subgroup of G, and (ρ, V) a representation of G. Then we have a representation $(\rho\big|_H, V)$ called the *restriction* of ρ to G.

Definition 2.20 (Lift)**.** Let K be a normal subgroup of a group G and (ρ, V) be a representation of G/K. We define the *lift* (or *inflation*) $\tilde{\rho}$ or ρ to be the homomorphism $G \to \mathrm{GL}(V)$ defined by $\tilde{\rho}(g) = \rho(gK)$.

We note that $(\tilde{\rho}, V)$ is also a representation of G. We can draw the following diagram.

$$\begin{tikzcd} G \arrow[d, "q"'] \arrow[dr, "\tilde{\rho}"] & \\ G/K \arrow[r, "\rho"] & \mathrm{GL}(V) \end{tikzcd}$$

Example 2.21. Let $G = S_4$ and

$$K = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

Then we claim that

$$G/K \cong S_3.$$

We construct such an isomorphism, we label the three non-identity elements of K as a, b, and c. Then G acts by conjugation on the set $\{a, b, c\}$ (that is, $g \cdot x = gxg^{-1} \in \{a, b, c\}$ for all $g \in G$ and $x \in \{a, b, c\}$). This action induces a homomorphism $f : G = S_4 \to \{\text{permutations of } \{a, b, c\}\} \cong S_3$. We consider $f((1\ 2))$, by considering:

$$\begin{aligned} (1\ 2)a(1\ 2)^{-1} &= a, \\ (1\ 2)b(1\ 2)^{-1} &= c, \\ (1\ 2)c(1\ 2)^{-1} &= b. \end{aligned}$$

Thus $(1\ 2)$ fixes a, and switches b and c; that is, $f((1\ 2)) = (b\ c)$. We can examine other values of f to get that it is indeed surjective and that the kernel is K. Hence $f : S_4/K \xrightarrow{\cong} S_3$.

Lemma 2.22. *There is a bijection between*

1. *the representations of G whose kernel contains K; and*
2. *the representations of G/K.*

Proof. todo □

Corollary 2.23. *Let K be a normal subgroup of a group G. If ρ is a representation of G/K, $\tilde{\rho}$ is irreducible if and only if ρ is.*

Proof. □

Proposition 2.24. *Let ρ be a representation of G with character χ and dimension d. Then*

$$\ker \rho = \{g \in G : \chi(g) = d\}.$$

2.4 Inner products and homomorphisms

Let (ρ, V) and (σ, W) be two complex representation of a finite group G. Then we can define a representation $\mathrm{Hom}(\rho, \sigma)$ on the vector space

$$\mathrm{Hom}(V, W) = \{\text{linear maps } T : V \to W\}$$

with G-action by *'conjugation'*:

$$(g \cdot T)(v) = \sigma(g)T(\rho(g)^{-1}v).$$

Lemma 2.25.

1. *If ρ and σ have characters χ and ψ, then* $\mathrm{Hom}(\rho, \sigma)$ *has character $\overline{\chi}\sigma$.*
2. *We have*
$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G;$$
that is, the G-homomorphisms are the G-fixed points of $\mathrm{Hom}(V, W)$.

Proof. todo □

Theorem 2.26. *If V and W are two representations of G with characters χ and ψ respectively, then*

$$\langle \chi, \psi \rangle = \dim \mathrm{Hom}_G(V, W).$$

Corollary 2.27. *Suppose that V and W are irreducible representations with characters χ and ψ respectively. Then*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. □

2.5 Universal projections

Let G be a group and $\alpha : G \to \mathbb{C}$ be any class function. Then define

$$\pi_\alpha = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha}(g)[g] \in \mathbb{C}[G].$$

For every representation (ρ, V), π_α acts on G, and we also call this π_α:

$$\pi_\alpha(v) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha}(g)\rho(g)v.$$

If we take α to be the constant function with value 1, then π_α is simply the projection $\pi : V \to V^G$.

Lemma 2.28. *If V is any representation, then $\pi_\alpha : V \to V$ is a G-homomorphism.*

Since π_α is a G-homomorphism $V \to V$ for every V, we could call it a *universal* G-homomorphism. We now look at its behaviour on irreducible representations.

Proposition 2.29. *Let α be a class function and let V be an irreducible representation with character χ. Then π_α acts as the scalar*

$$\frac{1}{\dim V}\langle \alpha, \chi\rangle$$

on V. In particular, if ψ is the character of an irreducible representation W, then π_ψ acts as $\frac{1}{\dim W}$ and as 0 on all other irreducible representations.

Let ρ be an irreducible representation of G and V another representation of G. We denote the subrepresentation of V generated by all the subrepresentation of V isomorphic to ρ by $V(\rho)$.

Corollary 2.30. *Let ρ be an irreducible representation of G with character χ and dimension d. Then the operation $d\pi_\psi$ acts, on any G-representation V, as the G equivariant projection $V \to V(\rho)$.*

Corollary 2.31. *The irreducible characters are a basis for the space of class functions.*

We now have enough to prove the rest of Theorem 2.15.

Corollary 2.32. *Part 2 and 3 of Theorem 2.15 are true.*

2.6 Linear algebra constructions

2.6.1 The dual representation

We recall that for two G-representations V and W, we have a G-representation on the space $\mathrm{Hom}(V, W)$ with character $\overline{\chi}_V \chi_W$.

Definition 2.33. Let V be a vector space. Then the *dual space* of V is

$$V^* = \mathrm{Hom}(V, \mathbb{C}).$$

We have $\dim V^* = \dim V$. To see this, let $v_1, \ldots, v_n$ be a basis of V. Then we have the *dual basis* $v_1^*, \ldots, v_n^*$ of V^*, given by

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This is a bilinear map $V^* \times V \to \mathbb{C}$, $(\varphi, v) \mapsto \varphi(v)$. The choice of basis identifies V with $\mathbb{C}^n$, and then the dual basis realises V^* as $1 \times n$ matrices

If V has a G-representation, then we take $\mathbb{C}$ to have the trivial representation to get an action ρ^* of G on V^* defined by

$$\rho^*(g)(\varphi) = \varphi(\rho(g)^{-1}v).$$

We have already seen a formula for the character of $\mathrm{Hom}(V, W)$, thus $\chi_{V^*} = \overline{\chi}_V$.

If the matrix of $\rho(g)$ with respect to some basis is A, then the matrix of $\rho(g)$ with respect to the dual basis is $(A^T)^{-1}$.

2.6.2 Tensor products

Let V and W be two vector spaces. Then the tensor product of V and W, denoted $V \otimes W$ is the $\mathbb{C}$-vector space generated by the symbols $v \otimes w$ for $v \in V$ and $w \in W$ with the *bilinear relations*

$$\begin{aligned} \lambda(v \otimes w) &= (\lambda v) \otimes w = v \otimes (\lambda w), \\ (v + v') \otimes w &= v \otimes w + v' \otimes w, \\ v \otimes (w + w') &= v \otimes w + v \otimes w'. \end{aligned}$$

We can rigorously define this space using quotients, but it is assured that this is understood.

Proposition 2.34. *Let $e_1, \dots, e_n$ be a basis of V and $f_1, \dots, f_m$ be a basis of W. Then*

$$\{e_i \otimes f_i : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$$

is a basis of $V \otimes W$.

In particular, we see that

$$\dim(V \otimes W) = \dim V \dim W.$$

We also note that not every vector of $V \otimes W$ is of the farm $v \otimes w$; for example, let $V = W$ with basis (e, f) then the vector

$$e \otimes e = f \otimes f \in V \otimes V$$

cannot be written in this form.

We claim that a linear map on a tensor product of spaces corresponds to a bilinear map on the cartesian product. This correspondence is defined in the way you would expect, and it is trivial to check the linearity and bilinearity.

If V and W are both representation of G, then $V \otimes W$ is a representation by

$$g(v \otimes w) = gv \otimes gw.$$

We may write $\rho_V \otimes \rho_W$ for this representation. Let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots, f_m$ be a basis for W. If we order the basis of $V \otimes W$ as

$$e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_1 \otimes f_m, e_2 \otimes f_1, \dots$$

then the matrix of g on $V \otimes W$ is $A \otimes B$, the block matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 2.35. *If ρ_V and ρ_W have characters χ_V and χ_W, then*

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

The tensor product is a generalisation of the *twisting* construction we saw earlier. If V is any vector space, then $V \otimes \mathbb{C} \cong V$ via the map $v \otimes \lambda \mapsto \lambda v$. If $(\chi, \mathbb{C})$ is a 1-dimensional representation and (ρ, V) is any representation of G, then $\rho \otimes \chi$ is a representation acting on $V \otimes \mathbb{C}$. We have

$$(\rho \otimes \chi)(g)(v \otimes \lambda) = (\rho(g)v) \otimes (\chi(g)\lambda) \mapsto \chi(g)\rho(g)\lambda v.$$

Thus

$$\rho \otimes \chi \cong \chi\rho.$$

We note that if ρ is irreducible, so is $\chi \otimes \rho$. Furthermore, $\chi \otimes \rho$ may or may not be isomorphic to ρ.

Lemma 2.36. *Let V and W be two finite-dimensional representation of a group. Then*

$$V^* \otimes W \cong \mathrm{Hom}(V, W)$$

as G-modules.

2.6.3 Symmetric and alternating powers

The *symmetric square* $\mathrm{Sym}^2(V)$ of V is the vector space spanned by the symbols vv' with the bilinear relations and $vv' = v'v$ for all $v, v' \in V$. Formally, $\mathrm{Sym}^2(V)$ is the quotient

$$(V \otimes V)/\sim$$

where $v \otimes v' \sim v' \otimes v$.

Proposition 2.37. *Given a basis $e_1, \dots, e_n$ of a vector space V, then e_ie_j with $i \leq j$ are a basis of* $\mathrm{Sym}^2(V)$.

Thus

$$\dim \mathrm{Sym}^2(V) = \frac{n(n+1)}{2}.$$

We also define the *alternating square* $\bigwedge^2(V)$ of V is spanned by elements of the form $v \wedge v'$ subject to the bilinear relations and $v \wedge v' = -v' \wedge v$ for all $v, v' \in V$. We can similarly define this by a quotient.

If V and W are representations of G, then we can define actions of G on these spaces.

$$\begin{aligned} \mathrm{Sym}^2(V) &\to V \otimes V, \\ vv' &\mapsto v \otimes v' + v' \otimes v; \\ \bigwedge^2(V) &\to V \otimes V, \\ vv' &\mapsto v \otimes v' - v' \otimes v. \end{aligned}$$

Any $v \otimes w \in V \otimes W$ can be written in the form

$$v \otimes w = \frac{1}{2}((v \otimes w + w \otimes v) + (v \otimes w - w \otimes v))$$

and so

$$V \otimes V \cong \mathrm{Sym}^2(V) \otimes \bigwedge^2(V).$$

We note that the space $V \otimes V$ has the involution

$$\sigma : v \otimes w \mapsto w \otimes v.$$

As $\sigma^2 = I$, it has eigenvalues ± 1. The decomposition above is the eigenspace decomposition for σ, where $\mathrm{Sym}^2(V)$ is the 1-eigenspace and $\bigwedge^2(V)$ is the (-1)-eigenspace.

Proposition 2.38. *If (ρ, V) has character χ, then*

$$\begin{aligned} \chi_{\mathrm{Sym}^2(V)}(g) &= \frac{1}{2}\left(\chi(g)^2 + \chi\left(g^2\right)\right), \\ \chi_{\bigwedge^2(V)}(g) &= \frac{1}{2}\left(\chi(g)^2 - \chi\left(g^2\right)\right). \end{aligned}$$

2.6.4 Matrices in dimension two

Suppose that (ρ, V) is a representation of G and $\dim V = 2$ with basis e_1, e_2. Let $g \in G$ and

$$\rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the space. We will compute the matrices of the symmetric square and alternating square.

Example 2.39. Let (p, V) be a representation of G such that $\dim V = 2$ with basis e_1, e_2. For $\bigwedge^2(\rho)$, we get the matrix

$$\begin{pmatrix} ad - bc \end{pmatrix}$$

and for $\mathrm{Sym}^2(\rho)$ we get the matrix

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

with the assumed bases.

2.7 The character table of S_5

Example 2.40. Let $G = S_5$. We have the trivial representation $\mathbb{1}$, the sign representation ε, and the permutation representation $V \cong \mathbb{1} \oplus W$ and its twist, as we have seen. So we start our character table.

	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3)(4\,5)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$
Class size	1	10	15	20	20	30	24
$\mathbb{1}$	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
χ	4	2	0	1	-1	0	-1
χ^ε	4	-2	0	1	1	0	1

We now look at $\mathrm{Sym}^2(W)$ and $\bigwedge^2(W)$, which have the following characters.

	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3)(4\,5)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$
Class size	1	10	15	20	20	30	24
$\bigwedge^2\chi$	6	0	-2	0	0	0	1
$\mathrm{Sym}^2\chi$	10	4	2	1	1	0	0

From this, we see that $\bigwedge^2\chi$ is irreducible (but $\mathrm{Sym}^2\chi$ is not) and unfortunately the twist of $\bigwedge^2\chi$ is equal to itself. Now

$$\langle \mathrm{Sym}^2\chi, \mathrm{Sym}^2\chi \rangle = 3,$$

and as it must be the sum of squares, the only way is $1^2 + 1^2 + 1^2$. So it must decompose as the sum of three irreducible representations: ψ_0, ψ_1, and ψ_2. We note that

$$\langle \mathrm{Sym}^2\chi, \mathbb{1} \rangle = 1,$$

so we take $\psi_0 = 1$. Again,

$$\langle \mathrm{Sym}^2\chi, \chi \rangle = 1,$$

so we take $\psi_1 = 1$. No other known representation can make up ψ_2. We know $\operatorname{Sym}^2 \chi - \not\Bbbk - \chi = \psi_2 = \psi$ must be a irreducible representation, and we can compute this directly to get the following.

	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3)(4\,5)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$
Class size	1	10	15	20	20	30	24
$\mathbb{1}$	1	1	1	1	1	1	1
χ	4	2	0	1	-1	0	-1
ψ	5	1	1	-1	1	-1	0
$\operatorname{Sym}^2 \chi$	10	4	2	1	1	0	0

And we finally twist ψ to get the final character table.

	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3)(4\,5)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$
Class size	1	10	15	20	20	30	24
$\mathbb{1}$	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
χ	4	2	0	1	-1	0	-1
χ^ε	4	-2	0	1	1	0	1
$\bigwedge^2 \chi$	6	0	-2	0	0	0	1
ψ	5	1	1	-1	1	-1	0
ψ^ε	5	-1	1	-1	-1	1	0

3 Induced representations

3.1 Definition

Given H a subgroup of some group G, we can restrict a representation of G to get a representation of H. We now move to see how, given a representation of H, extend it to a representation of G, a *induced representation*.

Precisely, let (σ, W) be a representation of H. We want to construct a representation (ρ, V) which contains W as an H-subrepresentation (that is, $\rho\big|_H$ contains σ). Suppose such a representation exists. Then V would have a H-subrepresentation W_0 such that $W_0 \cong W$. Also, for $g \in G$ we must have that $\rho(g)W_0 \subset V$ is a subspace, and this only depends of gH as if $g_1 = g_2 h$ for some $h \in H$, then

$$\begin{aligned} \rho(g_1)W_0 &= \rho(g_2)\rho(h)W_0 \\ &= \rho(g_2)W_0. \end{aligned}$$

Definition 3.1 (Induced representation)**.** Let G be a finite group and H be a subgroup. If (σ, W) is a representation of H, then a representation (ρ, V) is *induced* from (σ, W) if

1. V has a H-subrepresentation W_0 with $W_0 \cong W$ as a H-representation; and
2. if $g_1H, \ldots, g_rH$ are the left cosets of H in G, then

$$V = \rho(g_1)W_0 \oplus \ldots \oplus \rho(g_r)W_0.$$

Example 3.2. Consider the two-dimensional representation $(\rho, \mathbb{C}^2)$ of $G = D_n$ such that

$$\rho(r)\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \qquad \rho(s)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is irreducible for $w \neq \pm 1$. Let $(\chi, \mathbb{C})$ be the one-dimensional representation of $H = \langle r \rangle \cong C_n$, with $\chi(r) = w$. The cosets of C_n in D_n are

$$\{C_n, rC_n\}.$$

If $V_0\langle e_1 \rangle \subset \mathbb{C}^2$, then V_0 is indeed a C_n-subrepresentation isomorphic to χ and $sV_0 = \langle e_2 \rangle$. We clearly have $\mathbb{C}^2 = V_0 \oplus sV_0$, and so the ρ is induced from σ.

Proposition 3.3.

1. *If (σ, W) is a representation of H, then there is a representation (ρ, V) of G induced from (σ, W).*
2. *Any two representations of G induced from the same representation of H are isomorphic.*

We write $\left(\operatorname{Ind}_H^G \sigma, \operatorname{Ind}_H^G W\right)$ for the representation of G induced from (σ, W).

3.2 Frobenius reciprocity

Theorem 3.4. *Let $H \subset G$ be finite groups, V be a representation of H, and W be a representation of G induced from W. Then for any representation U of G, there is an isomorphism of vector spaces*

$$\operatorname{Hom}_G(W, U) \xrightarrow{\cong} \operatorname{Hom}_H(V, U).$$

Corollary 3.5. *Any two representations induced from isomorphic representations of H are isomorphic.*

Corollary 3.6. *Let (p, V) be a representation of H with character χ, and let ψ be any class function on G. Then*

$$\left\langle \operatorname{Ind}_H^G \chi, \psi \right\rangle_G = \left\langle \chi, \operatorname{Res}_H^G \psi \right\rangle_H.$$

Here $\operatorname{Res}_H^G \psi$ denotes the restriction of ψ from G to H.

3.2.1 Example, S_3 to S_4

Example 3.7. We first list the character tables for S_4 and S_3.

	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3\,4)$
ψ_0	1	1	1	1	1
ψ_1	1	-1	1	1	-1
ψ_2	2	0	2	-1	-1
ψ_3	3	1	-1	0	-1
ψ_4	3	-1	-1	0	1
χ_0	1	1		1	
χ_1	1	-1		1	
χ_2	2	0		-1	

We view

$$S_3 = \{\tau \in S_4 : \tau \text{ fixes } 4\} \subset S_4.$$

Frobenius reciprocity implies that

$$\left\langle \operatorname{Ind}_{S_3}^{S_4} \chi_2, \psi_i \right\rangle_{S_4} = \left\langle \chi_2, \operatorname{Res}_{S_3}^{S_4} \psi_i \right\rangle_{S_3}$$

for each $i \in \{0, \ldots, 4\}$. For $i \in \{0,1\}$, $\operatorname{Res}_{S_3}^{S_4} \psi_i = \chi_i$, and so the RHS is 0 these i. For $i \in \{2,3,4\}$, we see that the RHS is 1. Thus, we have

$$\operatorname{Ind}_{S_3}^{S_4} \chi_2 = \psi_2 + \psi_3 + \psi_4.$$

3.3 Characters

We can also use Frobenius reciprocity to calculate the character of an induced representation.

Theorem 3.8. *Let $H \subset G$ be finite groups and let (ρ, V) be a representation of H with character χ. Suppose that C is a conjugacy class of G. Then*

$$\operatorname{Ind}_H^G(\chi)(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi(D_i)$$

where each D_i is a conjugacy class of H.

Example 3.9. Consider $D_4 \subset S_4$ with one-dimension character φ given below.

	Class				
	e	r	r^2	s	rs
Size	1	2	1	2	2
φ	1	-1	1	1	-1