Graph homology to motivate the generalisation of the Tutte polynomial to simplicial complexes

PROJECT IV: COMPUTATIONAL TOPOLOGY,
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Theorem 1. *Let $G = (V, E)$ be a directed graph. Then*

$$H_0(G) \cong \mathbb{Z}^{|C|}, \qquad H_1(G) \cong \mathbb{Z}^{|E|-|V|+|C|}$$

where $C \subset G$ denotes the set of components of G.

Proof. Recall that $H_0(G) = \ker \partial_0 / \operatorname{im} \partial_1$. Now we consider a component of G. We see that any vertex can be obtained by another vertex by adding elements of $\operatorname{im} \partial_1$. Thus all vertices belong to the same equivalence class. It is clear that we cannot do the same for two vertices within different components, thus $H_0(G) \cong \mathbb{Z}^{|C|}$. Now we let $T \subset E'$ be a spanning tree of a component $G' = (V', E')$. We observe that if we add another edge (not in T) to T, we get a cycle. Thus each edge in $E' \setminus T$ corresponds to a cycle in G'. $|E' \setminus T| = |E'| - (|V'| - 1)$. Considering every component, we get $H_1(G) \cong \mathbb{Z}^{|E|-|V|+|C|}$. $\square$