1. Let $G = (V, E)$ be a graph. Recall that a function $c : V \to \{1, \dots, k\}$ is a k-colouring if $c(u) \neq c(v)$ for every pair of adjacent vertices u and v. For $1 \leq i \leq k$, the set $C_i = \{i \in V : c(u) = i\}$ is called a *colour set* of c. For a pair i, j with $1 \leq i < j \leq k$, let $G[i, j]$ denote the subgraph of G induced by the set $C_i \cup C_j$.

   We say that a k-colouring c of G is *safe* if for every pair i, j with $i \neq j$, the subgraph $G[i, j]$ has maximum degree at most 3. The *safety number* of G is the smallest number k such that G has a safe k-colouring.

   Let Safety Number be the problem of deciding if a given graph G has safety number at most k for some given integer k. If k is fixed (so if k is not part of the input), then we denote the problem as k-Safety Number.

   (a) (2 marks) Prove that the safety number of a graph G is at most $|V(G)|$.

   Solution: Consider the trivial colouring in which each vertex is given a distinct colour. Then the induced subgraph by every colour set is either $2P_1$ or P_2 (assuming G is simple), which has degree at most 1.

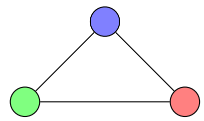
   (b) (8 marks) For every integer $k \geq 1$, give a graph of safety number k that is not a tree. For every integer $k \geq 1$, give also a tree of safety number k. Justify your answers.

   Solution: We denote $\Delta(G)$ as the maximum degree of a graph G.

   For a non-tree and $k = 1$, we claim that $2P_1$ has safety number 1. $2P_1$ is not a tree as it is disconnected, and we have the trivial 1-colouring where both vertices are given the same colour. Then $\Delta(G) = 0 \leq 3$, so the maximum degree of any induced subgraph is 0. Thus the colouring is safe. $2P_1$ is not empty, thus there is no 0-colouring, proving our claim. The graph is drawn below.

   For a non-tree and $k = 2$, we consider $2P_2 = uv + wx$. As before, $2P_2$ is not a tree as it is disconnected, and we have a 2-colouring $c : V(2P_2) \to \{1, 2\}$ where $c(u) = c(w) = 1$ and $c(v) = c(x) = 2$. It is clear that this is a minimal colouring, the existence of edges prohibits a 1-colouring. This colouring is safe as $\Delta(G) = 1 \leq 3$, and so any induced subgraph has maximum degree 1. Thus $2P_2$ has safety number 2. The graph is drawn below.

For a non-tree and $k \geq 3$, we may consider the complete graph K_k. We have $\chi(K_k) = k$, with each vertex having a distinct colour. For $x, y \in V(K_k)$, the induced subgraph $K_k[\{x, y\}]$ is P_2, thus $\Delta(K_k[\{x, y\}]) = 1$. Thus this colouring is safe, and so the safety number k. The graph is drawn below for $k = 3$.



For a tree and $k = 1$, we consider P_1 and the trivial colouring, which is a safe and minimal colouring. Thus the safety number of P_1 is 1.

For a tree and $k \geq 2$, we consider the star graph $G = S_{3(k-1)}$ and let $v \in V(G)$ be the internal node. This is always 2-colourable, where the internal node has one colour and the leaves all share another colour. For $k = 2$ this is safe, as $\Delta(S_{k-1}) = 3$ (and so any induced subgraph must have degree at most 3). So we now consider $k \geq 3$. We first claim that a safe k-colouring exists for G. If we partition the leaves into $k-1$ sets of size 3, then colouring each partition a distinct colour and the internal node a distinct colour, we have us a k-colouring ($k-1$ colour used for the leaves, 1 used for the internal node). This is indeed safe, the subgraph induced from any two distinct partitions is $6P_1$, which has maximum degree 0. Similarly, the subgraph induced from any partition and the internal node is the claw graph, which has maximum degree 3. We finally claim that there is no safe $k-1$-colouring which is safe, and thus the safe k-colouring above must be minimal (and thus the safety number is k). Suppose we have a safe $(k-1)$-colouring $c : V(G) \to \{1, \ldots, k-1\}$. $c(v)$ must have a unique colour, as it is a dominating vertex. So we let $c(v) = 1$ (without loss of generality). We now claim that there is a colour set C with size at least 4. Indeed, we have used 1 colour for the internal node, thus we have $k-2$ colours left to colour the remaining $3(k-1)$ leaves. Observe

$$\frac{3(k-1)}{k-2} \geq \frac{3(k-1)}{k-1} = 3,$$

and so at least one of the colour sets has size 4 or more. We now consider the subgraph induced by $C \cup \{v\}$, which is $K_{1,|C|}$ (as C is a independent set and v is a dominating vertex). But $\Delta(K_{1,|C|}) = |C| \geq 4$, so thus the colouring is not safe. Therefore, the safety number of G is k.

(c) (15 marks) Prove that the class $\mathcal{G}$ of graphs with safety number at most 2 is hereditary. Determine also the corresponding minimal set $\mathcal{F}_{\mathcal{G}}$ of forbidden induced subgraphs for $\mathcal{G}$. Justify your answer.

Solution: We use the following notation: for a function $f : X \to Y$ and $y \in Y$, $f^{-1}(\{y\}) = \{x \in X : f(x) = y\}$. We denote $\Delta(G)$ as the maximum degree of a graph G.

Let G be a graph with safety number at most 2 and $c : V(G) \to \{1, 2\}$ be a safe 2-colouring of G. Then c is also a safe 2-colouring on any induced subgraph G' of G, thus the safety number of G' is at most 2.

Let G be a graph with safety number at most 2 with safe 2-colouring $c : V(G) \to \{1, 2\}$. $\{c^{-1}(\{0\}), c^{-1}(\{1\})\}$ is a partition of $V(G)$, and as c is safe, this means that $\Delta(G) \leq 3$. Also, as G has a safe 2-colouring, it is 2-colourable. Thus $\mathcal{G}$ is precisely the set of bipartite graphs with maximum degree 3. Thus we get the obstruction set

$$\mathcal{F} = \{K_{1,4}, C_3, C_5, C_7, \ldots\}.$$

(d) (15 marks) Prove that 5-Safety Number is constant-time solvable for the class of graphs of diameter at most 4.

Solution: We denote $\Delta(G)$ as the maximum degree of a graph G. We first have some required results.

Lemma. *If a graph G contains K_6 as an induced subgraph, it is not 5-colourable.*

Proof. Let G be a graph and let $A \subset V[G]$ be such that $G[A] = K_6$ (that is, G contains K_6 as an induced subgraph). For a contradiction, suppose we have a 5-colouring $c : V(G) \to \{1, 2, 3, 4, 5\}$ of G. We restrict our 5-colouring to the induced subgraph $G[A]$, but we see that c cannot exist, as $\Delta(G[A]) = 5$. Thus G is not 5-colourable, as required. $\square$

Lemma. *If a graph G contains S_{13} as an induced graph, it does not admit a safe 5-colouring.*

Proof. Let G be a graph and let $A \subset V[G]$ be such that $G[A] = S_{13}$ (that is, G contains S_{13} as an induced subgraph). For a contradiction, suppose we have a safe 5-colouring $c : V(G) \to \{1,2,3,4,5\}$ of G. Then c is also a safe 5-colouring of $G[A] = S_{13}$. As the internal node $v \in G[A]$ is a dominating vertex in this induced subgraph, it must have a unique colour; that is, $P = \{v\}$ is a colour set. Thus we have 4 colours left to colour the 13 leaves. By the pigeonhole principle, at least one of the colour sets has size 4, say $Q \subset A$. As c is a safe 5-colouring, we have $\Delta(G[Q \cup A]) \leq 3$. But $G[Q \cup A] = S_4$, which has maximum degree 4 (precisely $\deg(v) = 4$). Thus we have reached contradiction, and G does not have a safe 5-colouring. □

We also recall the following theorem seen in lectures.

Theorem (Ramsey's theorem)**.** *For every pair of integers $r, s \in \mathbb{N}$, there exists a constant $R(r,s)$ such that every graph on at least $R(r,s)$ vertices has either a clique of size r or an independent set of size s.*

We can now proceed with the proof. Let G be a graph with diameter at most 4.

Claim. If $|V(G)| > \beta$ for some constant $\beta \in \mathbb{N}$ (which we will determine), then G does not admit a safe 5-colouring.

Under the assumption that this claim is true, we see that we have a constant number of possible potential safe 5-colourings we need to check. We can do this in constant time.

Proof of claim. For every vertex $u \in V(G)$ of degree at least $R(5,13)$, by Ramsey's theorem, the induced subgraph $G[N(u)]$ *must* contain either

(1) a clique of size 5; or

(2) an independent set of size 13.

First consider (1). $N(u)$ must contain a clique of size 5, so let $A \subset N(u)$ be such that $G[A]$ is this clique. By definition, u dominates all of its neighbours, thus $G[A \cup \{u\}]$ is a clique of size 6, and so G is not 5-colourable (by our first lemma), and thus does not admit a safe 5-colouring.

Now we consider (2). $N(u)$ must contain a independent set of size 13, so let $A \subset N(u)$ be such $G[A]$ is this independent set. By definition, u dominates all of its neighbours, thus $G[A \cup \{u\}] = S_{13}$. Therefore, G does not admit a safe 5-colouring (by our second lemma).

Thus, for G to admit a safe 5-colouring, every neighbourhood must have size less than $R(5,13)$. We note that $N_i(u) = \varnothing$ for all $i \in \{5,6,7,\dots\}$, and thus we have that

$$|V(G)| \le \beta = 1 + R(5,13) + R(5,13)^2 + R(5,13)^3 + R(5,13)^4$$

where

$$R(5,13) \le \binom{16}{4} = 1820. \qquad \square$$

(e) (15 marks) Prove that 3-Safety Number is polynomial-time solvable for every graph class of bounded treewidth.

Solution: We will first describe the property of have a safe 3-colouring in MSO_2.

Let $G = (V, E)$ be a graph. We first define some formulas to make our writing more economical.

1. For $u, v \in V$, $\mathrm{adj}(u,v)$ is true if and only if u and v are adjacent, so

$$\mathrm{adj}(u,v) = \neg(u = v) \wedge \exists_{e \in E}(i(u,e) \wedge i(v,e)).$$

2. Given $u \in V$ and $A \subset V \setminus \{u\}$, $\mathrm{adj}_0(u,A)$ is true if and only if u is not adjacent to any vertex in A. Thus

$$\mathrm{adj}_0(u,A) = \forall_{w \in A}(\neg\,\mathrm{adj}(u,w)).$$

3. Given $u \in V$ and $A \subset V \setminus \{u\}$, $\mathrm{adj}_1(u,A)$ is true if and only if u is adjacent to exactly one vertex in A. Thus

$$\mathrm{adj}_1(u,A) = \exists_{v_1 \in A}\forall_{w \in A}(\mathrm{adj}(u,w) \iff (w = v_1)).$$

We similarly define $\mathrm{adj}_2(u,A)$ and $\mathrm{adj}_3(u,A)$ as true if and only if u is adjacent to two and three vertices of A respectively, so

$$\begin{aligned}
\mathrm{adj}_2(u,A) = \exists_{v_1,v_2 \in A}&((v_1 \neq v_2) \\
&\wedge \forall_{w \in A}(\mathrm{adj}(u,w) \iff (w = v_1 \vee w = v_2))), \\
\mathrm{adj}_3(u,A) = \exists_{v_1,v_2,v_3 \in A}&((v_1 \neq v_2 \wedge v_1 \neq v_3 \wedge v_2 \neq v_3) \\
&\wedge \forall_{w \in A}(\mathrm{adj}(u,w) \iff (w = v_1 \vee w = v_2 \vee w = v_3))).
\end{aligned}$$

4. Given $u \in V$ and $A \subset V \setminus \{u\}$, $\mathrm{adj}_{\leq 3}(u, A)$ is true if and only if u is adjacent to at most 3 vertices in A, so combining the last point:

$$\mathrm{adj}_{\leq 3}(u, A) = \mathrm{adj}_0(u, A) \wedge \mathrm{adj}_1(u, A) \wedge \mathrm{adj}_2(u, A) \wedge \mathrm{adj}_3(u, A).$$

5. Given $A \subset V(A)$, $\mathrm{I}(A)$ is true if and only if A is an independent set in G, thus

$$\mathrm{I}(A) = \forall_{u,v \in A}(\neg\, \mathrm{adj}(u, v)).$$

6. Given $V_1, V_2, V_3 \subset V$, $P(V_1, V_2, V_3)$ is true if and only if $\{V_1, V_2, V_3\}$ forms a partition of G; that is,

$$\begin{aligned} P(V_1, V_2, V_3) = \forall_{v \in V}(&(v \in V_1 \wedge v \notin V_2 \wedge v \notin V_3) \\ &\vee (v \notin V_1 \wedge v \in V_2 \wedge v \notin V_3) \\ &\vee (v \notin V_1 \wedge v \notin V_2 \wedge v \in V_3)). \end{aligned}$$

We conclude with the MSO_2 formula for a graph having a safe 3-colouring.

$$\begin{aligned} \exists_{V_1,V_2,V_3 \subset V}(&I(V_1) \wedge I(V_2) \wedge I(V_3) \wedge P(V_1, V_2, V_3) \\ &\wedge \forall_{v \in V_1}(\mathrm{adj}_{\leq 3}(v, V_2) \wedge \mathrm{adj}_{\leq 3}(v, V_3) \\ &\wedge \forall_{v \in V_2}(\mathrm{adj}_{\leq 3}(v, V_1) \wedge \mathrm{adj}_{\leq 3}(v, V_3) \\ &\wedge \forall_{v \in V_3}(\mathrm{adj}_{\leq 3}(v, V_1) \wedge \mathrm{adj}_{\leq 3}(v, V_2)). \end{aligned}$$

As the bound for this formula is constant (does not depend on the size of G), we may use Courcelle's theorem to conclude that 3-Safety Number is polynomial-time solvable for every class of bounded treewidth.

2. (a) (5 marks) Let Π be some NP-complete graph problem. Suppose that we can solve Π in polynomial time for P_6-free graphs. Does this mean that we can solve Π in polynomial time for $2P_3$-free graphs? Justify your answer.

> **Solution:** If we found that the class of $2P_3$-free graphs $\mathcal{G}_1$ are a subclass of the class of P_6-free graphs $\mathcal{G}_2$ (that is, $\mathcal{G}_1 \subset \mathcal{G}_2$), then we could solve Π in polynomial restricted to $\mathcal{G}_1$. But we do not have this: P_6 is a graph that is $2P_3$-free, but it is certainly not P_6-free. Thus, the answer is no.

(b) (5 marks) Let Π be some NP-complete graph problem. Let F be the *paw*, that is, F has vertices a, b, c, d and edges ab, bc, ca, cd. Suppose that we can solve Π in polynomial time for paw-free graphs. Does this mean that we can solve Π in polynomial time for bipartite graphs? Justify your answer.

> **Solution:**
>
> **Proposition.** *Every bipartite graph is paw-free.*
>
> *Proof.* Let G be a bipartite graph and for suppose there is $A \subset V(G)$ such that $G[A]$ is the paw (that is, G contains the paw as an induced subgraph). As G is bipartite, it is 2-colourable (give a colour to each vertex partition). Let $c : V(G) \to \{1, 2\}$ be such a colouring. Then c is also a colouring for $G[A]$, the paw. But the paw is not 2-colourable. Namely, it has a clique of size 3; a contradiction. Thus every bipartite graph is paw-free. $\square$
>
> It is immediate that the class of bipartite graphs $\mathcal{G}_1$ is a subclass of the class of paw-free graphs $\mathcal{G}_2$ (that is, $\mathcal{G}_1 \subset \mathcal{G}_2$). Thus, if we can solve Π in polynomial time for paw-free graphs, then we can also solve it for bipartite graphs (as every bipartite graph is paw-free).

(c) (5 marks) Is the class of interval graphs closed under edge contraction?

> **Solution:** The usual definition of an interval graph is as the intersection graph of a set of intervals on the real line, but we restrict this to the intersection graph of a set of *closed* intervals on the real line, as defined by Diner et al., 2015.
>
> Let $\{I_i\}_{i=1}^{n}$ be a finite set of closed intervals where $I_i = [a_i, b_i]$ for $a, b \in \mathbb{R}$ with $a \leq b$, and let $G = (V, E)$ be the corresponding intersection graph;

that is,

$$V = \{v_i\}_{i=1}^n,$$
$$E = \{v_i v_j : I_i \cap I_j \neq \varnothing\}.$$

Pick $v_i v_j \in E$. We claim that $G' = G/v_i v_j$ is an interval graph; that is, there is a finite set of intervals $\mathcal{J}$ such that G' is the interval graph of $\mathcal{J}$. We prove this by construction. For all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, let $I_k \in \mathcal{J}$. Recall that $G/v_i v_j$ may produced from G with the following operations:

- add a new vertex labelled $v_{i\sim j}$;
- add an edge between $v_{i\sim j}$ and every neighbour of v_i and v_j; then
- delete v_i and v_j.

Thus we add the final interval, $I_{i\sim j}$, defined by

$$I_{i\sim j} = [\min\{a_i, a_j\}, \max\{b_i, b_j\}]$$

and let $I_{i\sim j} \in \mathcal{J}$. Thus we have

$$\mathcal{J} = \{I\}_{k\in\{1,\ldots,n\}\setminus\{i,j\}} \cup \{I_{i\sim j}\}.$$

We claim that the interval graph of $\mathcal{J}$ is G'. Indeed, for all $k, l \in \{1, \ldots, n\} \setminus \{i, j\}$, $v_k v_l \in E(G')$ if and only if $I_k \cap I_l \neq \varnothing$ (that is, $v_k v_l \in E(G)$). We have left to consider i and j. But note, $I_i, I_j \subset I_{i\sim j}$ by definition. So if, for some $l \in \{1, \ldots, n\} \setminus \{i, j\}$, $I_l \cap I_{i\sim j}$ if and only if $I_l \cap I_i$ or $I_l \cap I_j$.

Intuitively, we can consider a contraction of two connected intervals on a interval graph as taking the union of the interval, which is also an interval as the existence of an edge ensures that they are connected.

(d) (15 marks) Recall that a graph $G = (V, E)$ is a split graph if there exists two disjoint sets C and I such that $V = C \cup I$, the set C is a clique and the set I is an independent set. Prove that Feedback Vertex Set is polynomial-time solvable for split graphs. Justify your answer.

Solution:

Definition (Split pair)**.** Let G be a graph. If G is split, then we say that (C, I) is a *split pair* if $\{C, I\}$ is a partition of $V(G)$, C is a clique in G, and I is a independent set in G. If C is a maximum clique of G, we call (C, I) a *maximum split pair.*

Definition (Feedback set vertex number)**.** The *feedback vertex set number* of a graph is the size of the smallest feedback vertex set.

Lemma. *Given a split graph, finding a maximum split pair can be done in polynomial-time.*

Proof. Let G be a graph split graph. First, we note that for a maximum split pair (C, I) of G, every vertex of I has degree at most $i-1$ (otherwise it could be brought into the clique), and similarly every vertex of C has degree at least $i-1$. Moreover, if I contains at least one vertex of degree $|C|-1$, then C contains at most one vertex of degree $|C|-1$. We also comment on the obvious existence of a maximum split pair.
For $i \in \{1, \ldots, |V(G)|\}$, we will check if G has a maximum split pair (C, I) with $|C| = i$. We construct the following sets in $O(n^2)$ time:

1. the set I' of vertices of degree at most $i-2$;
2. the set U of vertices of degree $i-1$; and
3. the set C' of vertices of degree at least i.

From the above, if (C, I) is a maximum split pair with $|C_i| = i$, then either

1. $C = C' \cup U$ and $I = I'$;
2. $C = C'$ and $I = I' \cup U$; or
3. there exists a vertex $u \in U$ such that $C = C' \cup \{u\}$ and $I = I' \cup (U \setminus \{u\})$.

Thus we have $|U| + 2 = O(n)$ options to consider, and we can check every option in time $O(n^2)$.
This gives us a total running time of $O(n^4)$, which is polynomial. $\square$

Lemma. *The feedback vertex set number of a graph G with a maximum split pair (C, I) is either $|C| - 2$ or $|C| - 1$.*

Proof. Suppose that $S \subset V(G)$ is a feedback vertex set of G with $|S| \leq |C| - 3$. Then there is at least three vertices of C that aren't deleted, but these form a triangle, and so $G[C]$ cannot be a forest. Thus a feedback vertex set must have size at least $|C| - 2$. Now we recognise a $|C| - 1$ feedback vertex set that always exist: delete $|C| - 1$ vertices of C from G. Then we are left with the disjoint union of a star graph and nP_1 (which is a forest), as otherwise there would be an edge between two vertices of I; a contradiction as I is an independent set. □

Lemma. *Given a graph G and a maximal split pair (C, I), the feedback vertex set number can be computed in polynomial time.*

Proof. Following the reasoning of the previous proof, if S is a minimum feedback vertex set then $|C| - 2$ of the vertices of S must be from C (as otherwise we would have a cycle left over). Thus we permute over every disjoint pair of vertices $\{u, v\} \subset C$ and consider if they share a neighbour in I. If there is a pair which does not share any neighbour of I, then we may take the feedback vertex set number as $|C| - 2$. Otherwise, the feedback vertex set number is $|C| - 1$. This can be done in $O\left(|I|\binom{n}{2}\right) = O(n^3)$ time. □

Combining the above, given a split graph G, we can identity a maximum split pair (C, I) in $O(n^4)$ time. Using this, we can compute the feedback vertex set number in polynomial time.

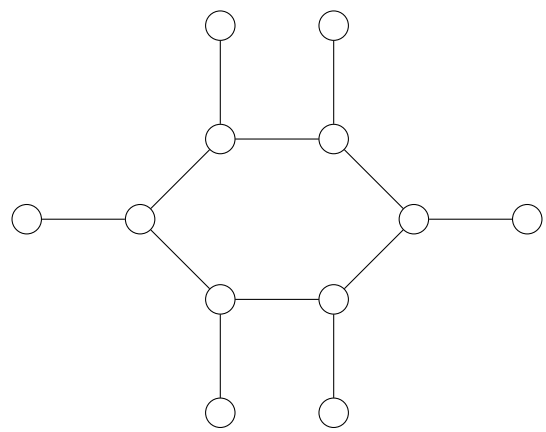
Problem (Feedback Vertex Set).
Instance: let G be a graph and $k \in \mathbb{N}$
Question: does G have a feedback vertex set of size at most k?

Corollary. Feedback Vertex Set *is polynomial-time solvable for split graphs.*

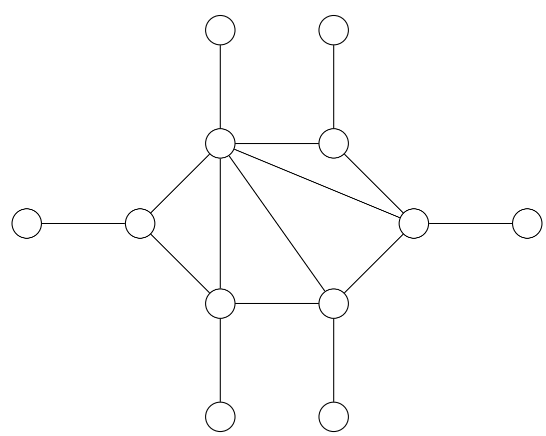
Proof. Let G be a split graph. As G is split, we can compute the feedback vertex set number in polynomial time. Let f be the feedback vertex set number. Then if $k < f$, we output `no`. If $k \geq f$, we output `yes`. □

(e) (20 marks) Let G be the graph that is obtained from the 6-vertex cycle on vertices $u_1, \dots, u_6$ after adding six new vertices $v_1, \dots v_6$ with edges $u_i v_i$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Determine the treewidth of G and prove also that G has clique-width at most 4.

Solution: Consider the graph sunlet_6 (shown below) as defined in the question.



A connected graph with at least two vertices has treewidth 1 if and only if it is a tree, thus $\text{tw}(\text{sunlet}_6) > 1$. We may define the treewidth of G, $\text{tw}(G)$, as one less than smallest clique number of the chordal graphs containing G as a spanning subgraph. Consider the following triangulation of sunlet_6.



This is graph is chordal, contains G as a spanning subgraph, and has clique number 3. Thus, $\text{tw}(\text{sunlet}_6) \leq 2$. But we have seen that $\text{tw}(\text{sunlet}_6) > 1$, so $\text{tw}(\text{sunlet}_6) = 2$.

The bound $\mathrm{cw}(\mathrm{sunlet}_6) \leq 3 \cdot 2^{\mathrm{tw}(\mathrm{sunlet}_6)-1} = 6$ is too high for this question, so we move to a constructive proof of the bound 4 given. This diagram is to be read left to right.

$$\begin{array}{ccccccccc}
1(a) - & \oplus & - & \oplus & - & \oplus & - \eta_{1,2} - \eta_{2,3} - \eta_{3,4} - \rho_{3,2} - \rho_{4,3} - & \oplus & - \eta_{3,4} - \rho_{4,2} - \cdots \\
 & | & & | & & | & & | & \\
 & 2(b) & & 3(b') & & 4(c) & & 4(c') &
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots - & \oplus & - \eta_{3,4} - \rho_{3,2} - \rho_{4,3} - & \oplus & - \eta_{3,4} - \rho_{4,2} - & \oplus & - \eta_{3,4} - \rho_{3,2} - \rho_{4,3} - & \oplus & - \cdots \\
 & | & & | & & | & & | & \\
 & 4(d) & & 4(d') & & 4(d) & & 4(e) &
\end{array}$$

$$\begin{array}{ccccccc}
\cdots - \eta_{3,4} - \rho_{4,2} - & \oplus & - \eta_{3,4} - \rho_{3,2} - \rho_{4,3} - & \oplus & - \eta_{3,4} - \rho_{4,2} - \eta_{3,1} - & \oplus & - \eta_{4,1} \\
 & | & & | & & | & \\
 & 4(f) & & 4(f') & & 4(a') &
\end{array}$$

Thus, in this k-expression we use at most 4 labels, and it constructs sunlet_6. Thus $\mathrm{cw}(\mathrm{sunlet}_6) \leq 4$ as required.