2016

1. Let G be the symmetric group S_n.

   (a) Define the permutation representation (π, V) of G.

   Solution: We take $V = \mathbb{C}^n$ and basis $e_1, \ldots, e_n$. Then

   $$\rho(g)e_i = e_{gi}.$$

   (b) In the special case $n = 3$, i.e., $G = S_3$, write down the matrices $\pi(g)$ for $g \in \{(1), (1\,2), (1\,2\,3)\} \subset G$ with respect to the standard basis of V. Use this to compute the character $\chi_\pi(g)$ for all $g \in G$.

   Solution:

   $$\rho((1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
   $$\rho((1\,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
   $$\rho((1\,2\,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

   χ_π is a class function, and thus fixed on conjugacy classes. Thus

   $$\begin{aligned} \chi_\pi(e) &= 3, \\ \chi_\pi(\text{2-cycles}) &= 1, \\ \chi_\pi(\text{3-cycles}) &= 0. \end{aligned}$$

2. Let G be a finite group with conjugacy classes $C_1, \ldots, C_k$ and let $g_i \in C_i$ be an element for each $i \in \{1, \ldots, k\}$.

(a) Give the definition of a class function of G and define an inner product $\langle -, - \rangle$ between two class functions.

Solution: A class function $\chi : G \to \mathbb{C}$ is a function that is constant on conjugacy classes. Standard inner product:

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi}(g)\psi(g).$$

(b) Let $w = 1/2(-1 + i\sqrt{3})$ be a cube root of unity. The following is the character table for the alternating group A_4:

g_i	(1)	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,3\,2)$
$\#(C_i)$	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	w	w^2
χ_3	1	1	w^2	w
χ_4	3	-1	0	0

Using this table write the following class function f as a sum of irreducible characters. Does f correspond to a character of a finite dimensional representation of A_4?

	(1)	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,3\,2)$
f	2	2	-1	-1

Solution: We combine the class function to the character table.

g_i	(1)	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,3\,2)$
$\#(C_i)$	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	w	w^2
χ_3	1	1	w^2	w
χ_4	3	-1	0	0
f	2	2	-1	-1

Note $w + w^2 = -1$. By inspection, we see that $f = \chi_2 + \chi_3$. f is the character of the direct sum of two representations, so f is indeed the character of a representation.

3. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C})$.

   (a) Compute $\exp(tA)$.

Solution:

$$\begin{aligned} \exp(tA) &= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}^n \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

   (b) Consider the action of $\mathrm{GL}_2(\mathbb{C})$ on the space of polynomials in two variables x_1, x_2 given by

$$(g\varphi)(\boldsymbol{x}) = \varphi(g(\boldsymbol{x})^\intercal),$$

where $\boldsymbol{x} = (x_1, x_2)^\intercal$. Compute the derived action of A.

Solution:

$$\begin{aligned}
A\varphi(x_1, x_2)^\intercal &= \frac{d}{dt}\varphi \exp(tA)(x_1, x_2)^\intercal \Big|_{t=0} \\
&= \frac{d}{dt}\varphi(x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t)^\intercal \Big|_{t=0} \\
&= \frac{\partial \varphi}{\partial x}(x_1, x_2)\frac{d}{dt}(x_1 \cos t - x_2 \sin t) \\
&\qquad + \frac{\partial \varphi}{\partial y}(x_1, x_2)\frac{d}{dt}(x_1 \sin t + x_2 \cos t)\Big|_{t=0} \\
&= -x_2 \frac{\partial \varphi}{\partial x}(x_1, x_2) + x_1 \frac{\partial \varphi}{\partial y}(x_1, x_2).
\end{aligned}$$

Thus

$$A\varphi = -x_2 \frac{\partial \varphi}{\partial x} + x_1 \frac{\partial \varphi}{\partial y}.$$

4. Let $V = \mathbb{C}^2$ be the standard representation of $\mathrm{SL}_2(\mathbb{C})$. Decompose $\Lambda^2(\mathrm{Sym}^3(V))$ into irreducible representations of $\mathrm{SL}_2(\mathbb{C})$.

Solution: Since $\mathrm{SL}_2(\mathbb{C})$ is connected, we can do this at the level of $\mathfrak{sl}_{2,\mathbb{C}}$. We see that, for a representation W

$$\begin{aligned}
\{\text{weights of } \mathrm{Sym}^k(W)\} &= \{\text{sum of unordered } k\text{-tuplets of weights of } W\}, \\
\{\text{weights of } \Lambda^k(W)\} &= \{\text{sum of unordered } k\text{-tuplets of distinct weights of } W\}.
\end{aligned}$$

First, we compute the weights of V:

$$\{\text{weights of } V\} = \{-1, 1\}.$$

The corresponding weight vectors are e_1 and e_{-1}. For $\mathrm{Sym}^3(V)$:

$$\{\text{weights of } \mathrm{Sym}^3(V)\} = \{-3, -1, 1, 3\}.$$

Finally for $\Lambda^2(\mathrm{Sym}^3(V))$:

$$\{\text{weights of } \Lambda^2(\mathrm{Sym}^3(V))\} = \{-4, -2, 0, 0, 2, 4\}.$$

But this is just $\mathrm{Sym}^4(\mathbb{C}^2) \oplus \mathbb{C}$.

5. (a) State the character theoretic formulations of the Frobenius reciprocity.

Solution: Let $H \subset G$ be a subgroup, χ a character of G, and ψ a character of H. Then

$$\langle \mathrm{Ind}_H^G \chi, \psi \rangle = \langle \chi, \mathrm{Res}_H^G \psi \rangle.$$

You may assume that the symmetric group $G = S_4$ has five conjugacy classes, uniquely determined by the cycle type of its elements. The following table contains three of the columns of its character table, where $x, y, z \in \mathbb{C}$.

g_i $\#(C_i)$	(1) 1	$(1\,2)(3\,4)$ 3	$(1\,2\,3)$ 8
χ_1'	1	1	1
χ_2'	1	1	1
χ_3'	2	2	y
χ_4'	3	z	0
χ_5'	x	-1	0

(b) Compute the values of x, y, and z.

Solution: Using column orthogonality, we see that $y = -1$. By the sum of squares formula,

$$x^2 = 4! - 1^2 - 1^2 - 2^2 - 3^2 = 24 - 15 = 9$$

so $x = 3$. Finally, by column orthogonality again, $z = -1$.

(c) Let ρ_i denote the irreducible representations of G corresponding to the characters χ_i' in the character table above, for all $i = 1, \ldots, 5$. Let $H = A_4$ be the subgroup of G. The character table for H is given in Question 2[(b)]. Compute characters for the restriction representations $\mathrm{Res}_H^G \rho_i$ for each $i = 1, \ldots, 5$. (Note that these are representations of the subgroup H).

Solution: The conjugacy classes of A_4 have representatives e, $(1\,2)(3\,4)$, $(1\,2\,3)$, and $(1\,3\,2)$.

g_i $\#(C_i)$	(1) 1	$(1\,2)(3\,4)$ 3	$(1\,2\,3)$ 4	$(1\,3\,2)$ 4
$\mathrm{Res}_H^G \chi_1' = \mathrm{Res}_H^G \chi_2'$	1	1	1	1
$\mathrm{Res}_H^G \chi_3'$	2	2	-1	-1
$\mathrm{Res}_H^G \chi_4' = \mathrm{Res}_H^G \chi_4'$	3	-1	0	0

(d) Let π be the 3-dimensional irreducible representation of H, which corresponds to the character χ_4 in the character table in Question 2[(b)] of H. Write $\mathrm{Ind}_H^G(\pi)$ as a direct sum of irreducible representations of G.

Solution: The solution here comes from examining the character table from Question 2[(b)] and seeing which of the restrictions are irreducible, and if there aren't what their irreducible decomposition is. Then checking the inner product of the induced character against each of them, if it is a 1, then it is in the decomposition.

6. Let G be a group of order 55 explicitly defined by

$$G = \langle x, y : x^{11} = y^5 = 1, yxy^{-1} = x^4 \rangle.$$

This group has 7 conjugacy classes containing $1, 5, 5, 11, 11, 11, 11$ elements, representatives of which can be given explicitly by elements $1, x, x^2, y, y^2, y^3, y^4$ respectively. Moreover, there are two irreducible characters χ_6 and χ_7 of G of dimension strictly bigger than 1, whose character values on the conjugacy classes for x and x^2 are explicitly given by

g_i	x	x^2
χ_6	u	v
χ_7	v	u

.

Here $u, v \in \mathbb{C}$.

(a) Prove that $|G/C(G)| = 5$, and use it to compute characters of all the 1-dimensional representations of G.

(You may take $C(G)$ to be the subgroup generated by x.)

Solution: $G/C(G)$ is generated by $yC(G)$, thus its order is 5. Thus, $G/C(G) \cong C_5$. We have five 1-dimensional representations here, corresponding to multiplication by a root of unity w^i, $i \in \{0, 1, 2, 3, 4\}$ for $w = e^{2\pi/5}$. For each representation, it has character χ_i sending y to w^i. χ_i lifts to G such that $\chi_i(1) = \chi_i(x) = \chi_i(x^2) = 1$ and $\chi_i(y^i) = w^{ij}$ for $i \in \{0, 1, 2, 3, 4\}$. We claim that there are no other characters. Indeed, if $\chi : G \to \mathbb{C}^\times$ is such a character then

$$\chi(x) = \chi(yxy^{-1}) = \chi(x^4) = \chi(x)^4$$

so $\chi(x)$ is a third root of unity. But $\chi(x)$ (by definition of G) is an eleventh root of unity. Thus $\chi(x) = 1$ as 3 and 11 are coprime. But if χ is trivial on x, it must be one of the representations above.

(b) Find dimensions of all the irreducible representations of G.

Solution: We have found all five one-dimensional representation, note these are irreducible as they are one-dimensional and are pairwise-non-isomorphic. Thus, there are two more representations that are not one-dimensional. Let ϕ and ψ be the last two characters of irreducible representations:

$$|G| = 55 = 5 + \phi(e) + \psi(e).$$

The only solution to this is $\phi(e) = \psi(e) = 5$ (as neither can be one-dimensional). Thus our dimensions are: $1, 1, 1, 1, 1, 5, 5$.

(c) Complete the character table of G.

(Hint: You may use here that the tensor product of any irreducible representation with a one-dimensional representation is also irreducible.)

Solution: The following is what we have so far.

g_i	1	x	x^2	y	y^2	y^3	y^4
$\#(C_i)$	1	5	5	11	11	11	11
$\mathbb{1}$	1	1	1	1	1	1	1
χ_1	1	1	1	w	w^2	w^3	w^4
χ_2	1	1	1	w^2	w^4	w	w^3
χ_3	1	1	1	w^3	w	w^4	w^2
χ_4	1	1	1	w^4	w^3	w^2	w
ϕ	5	u	v	?	?	?	?
ψ	5	v	u	?	?	?	?

Here $u, v \in \mathbb{C}$. By the hint, we must have that ϕ and ψ are 0 on y^i. This was a hard question...

7. Let X, Y, and H be the standard basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

(a) Compute the matrix ad_X with respect to the basis X, Y, H.

Solution: We have

$$\begin{aligned}\mathrm{ad}_X(X) &= [X, X] = 0\\ \mathrm{ad}_X(Y) &= [X, Y] = H\\ \mathrm{ad}_X(H) &= [X, H] = -2X.\end{aligned}$$

Thus our matrix with the basis above is

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) Let (π, V) be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Consider the so-called Casimir element

$$\mathcal{C} = \pi(X)\pi(Y) + \pi(Y)\pi(X) + \frac{1}{2}\pi(H)^2.$$

Show that $\mathcal{C}$ commutes with $\pi(X)$.

Solution:

$$\pi(X)\mathcal{C} = \pi(X)^2\pi(Y) + \pi(X)\pi(Y)\pi(X) + \frac{1}{2}\pi(X)\pi(H)^2,$$
$$\mathcal{C}\pi(X) = \pi(X)\pi(Y)\pi(X) + \pi(Y)\pi(X)^2 + \frac{1}{2}\pi(H)^2\pi(X).$$

Some properties of π may come in handy here. In particular, as π is a Lie algebra homomorphism it must commute with the Lie bracket $[-,-]$. Thus

$$[\pi(X), \pi(Y)] = \pi[X, Y] = \pi(H),$$
$$[\pi(X), \pi(H)] = \pi[X, H] = -2\pi(X).$$

These identities can be used to get the answer.

(c) You can assume that $\mathcal{C}$ commutes with the action of all elements in $\mathfrak{sl}_2(\mathbb{C})$. Prove that if V is an irreducible representation, then $\mathcal{C}$ acts as a scalar.

Solution: If $\mathcal{C}$ commutes for all $\pi(A)$, then it is a $\mathfrak{sl}_2(\mathbb{C})$-homomorphism, and thus by Schur's Lemma it acts as a scalar.

(d) What is the scalar in (c) for $V = \mathrm{Sym}^n(\mathbb{C}^2)$, the irreducible representation of highest weight n?

Solution: $\mathcal{C}$ acts as a scalar, so we check its action on e_1^n.

$$\begin{aligned}
\mathcal{C}e_1^n &= XYe_1^n + YXe_1^n + \frac{1}{2}H^2e_1^n \\
&= X(ne_2e_1^{n-1}) + Y(0) + \frac{1}{2}H(ne_1^n) \\
&= nX(e_2e_1^{n-1}) + \frac{n}{2}H(e_1^n) \\
&= ne_1^n + \frac{n^2}{2}e_1^n \\
&= \left(\frac{1}{2}n^2 + n\right)e_1^n.
\end{aligned}$$

Thus $\mathcal{C}$ acts as $\left(\frac{1}{2}n^2 + n\right)$.

8. Consider the orthogonal group O_2 which is generated by the matrices
$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
for $\theta \in \mathbb{R}$. You may assume we have the relation $sr_\theta s = r_{-\theta}$ so that every element in O_2 can be uniquely written as r_θ or $r_\theta s$. Let (π, V) be a finite dimensional irreducible representation of O_2.

   (a) Suppose that $v \in V$ is an eigenvector under SO_2-action. Show that V is spanned by v and sv. In particular, $\dim V \leq 2$.

Solution: As v is a eigenvector under SO_2-action,
$$r_\theta v = \lambda r_\theta v$$
for some $\lambda : \mathrm{SO}(2) \to \mathbb{C}^\times$, which must be a group homomorphism. Then
$$r_\theta sv = sr_{-\theta}v = \lambda r_{-\theta}v$$
and $ssv = v$, so both v and sv are preserved under all r_θ and s. That is, $\langle v, vs\rangle$ is preserved. But V is irreducible, so we have $V = \langle v, sv\rangle$.
For later sections, we note that any irreducible representation of $\mathrm{SO}(2)$ is one-dimensional and is λ_n for some $\lambda_n(r_\theta) = e^{in\theta}$ where $n \in \mathbb{Z}$.

(b) Find all possible π when $\dim V = 1$.

Solution: If $\dim V = 0$, then v and sv are proportional. If $\lambda = \lambda_n$, we have $\lambda_n(r_\theta) = \lambda_n(r_{-\theta})$ for all θ; that is, $n = 0$. Thus SO(2) acts trivially. Since $s^2 = 1$, we can either have $s = 1$ or $s = -1$. We name this $\mathbb{1}$ and ε respectively.

(c) Find all possible π when $\dim V = 2$ by writing down the matrices of $\pi(r_\theta)$ and $\pi(s)$ with respect to the basis $\{v, sv\}$.

Solution: As $\dim V = 2$, $n \neq 0$. Thus

$$\begin{aligned} \pi(r_\theta)v &= e^{in\pi} & \pi(s)v &= sv, \\ \pi(r_\theta)sv &= e^{-in\pi} & \pi(s)sv &= v. \end{aligned}$$

Thus we get the matrices

$$\pi(r_\theta) = \begin{pmatrix} e^{in\pi} & 0 \\ 0 & e^{-in\pi} \end{pmatrix}, \qquad \pi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the representations for $n < 0$ are isomorphic to the ones $n > 0$, so we assume $n > 0$. We label these representations π_n.

(d) Show that the assignments

$$r_\theta \mapsto \hat{r}_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s \mapsto \hat{s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

define an injective group homomorphism $\Phi : \mathrm{O}_2 \to \mathrm{SO}_3$.

Solution: This is clearly a group homomorphism (make sure to check $\hat{s}\hat{r}_\theta\hat{s} = \hat{r}_{-\theta}$, which is true by matrix multiplication). We have left to show that this is injective. It is clear that $\hat{r}_\theta = I$ if and only if $r_\theta = I$. Similarly, $\hat{r}_\theta\hat{s}$ always has bottom right entry -1, so is never the identity.

(e) Let $V^{(3)}$ be the standard representation of SO_3. Decompose $V^{(3)}$

into irreducible representations of O_2 (viewed as a subgroup of SO_3 via Φ in (d)).

Solution: First, we see that $\langle e_3 \rangle$ is preserved by $\Phi(r_\theta)$ and $\Phi(s)$. As $\Phi(s)(e_3) = -e_3$, this is the sign representation. Next, we see that

$$\begin{aligned}
\Phi(r_\theta(e_1 + ie_2)) &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\
&= e_1(\cos\theta + i\sin\theta) + e_2(-\sin\theta + i\cos\theta) \\
&= e_1(\cos\theta + i\sin\theta) - ie_2(\cos\theta + i\sin\theta) \\
&= e^{i\theta}(e_1 - ie_2).
\end{aligned}$$