1 An introduction, with lots of examples

We will recall some elementary results in complexity theory.

First, recall the graph colouring problems Col and k-Col and the following core result.

Theorem 1. 3-Col $\in$ NPC.

There are many ways of dealing with NP-completeness, we will focus on algorithms for restricted inputs. We aim to exploit properties of our input to develop faster algorithms, and in doing so we develop a framework. In the way of justification, problems typically have more structure in their inputs then the most abstract statement of itself, and by exploring the input structure we may learn *why* such a problem is hard.

Example 2. Every planar graph is 4-colourable, thus 4-Col is trivial for planar graphs, but this is certainly not the case for non-planar graphs.

Example 3. Now we examine 3-Col. If we restrict our input to planar graphs as we did for 4-Col, it is still NP-complete; however, every *triangle-free* (this means what you may think, but we will formalise this idea) planar graph is 3-colourable. Hence 3-Col is trivial for triangle-free planar graphs.

Example 4. Consider Col for graphs on diameter 1; that is, the complete graphs. This problem is trivial, and requires n different colours.

Example 5. Consider 4-Col for graphs of diameter at most 2. We claim this this problem is NP-complete by reduction from 3-Col. Let G be a graph of diameter at most 2. We then construct the graph G' as a copy of G, then we add a new vertex adjacent to every vertex of G. We now claim that G is 3-colourable if and only if G' is 4-colourable, which is easy to confirm. It can also be shown that G' has diameter at most 2, and we are done.

Example 6. Consider k-Col for graphs of diameter at most 2, where $k \geq 5$. Again, we claim that this is NP-complete by reduction from 3-Col. We can show this a similar to the last example, but instead of adding a single vertex we add a disjoint clique of $k - 3$ new vertices and attached every vertex of the clique to every vertex of G. After checking that the original graph being 3-colourable if and only if the new graph is k-colourable and that the new graphs has a diameter at most 2, we are done.

Example 7. Consider 3-Col for graphs of diameter at most 4. We solve this by reduction again. We build G' as a copy of G. Let $v_1, \ldots, v_n$ be the vertices of G, then we add the vertices $u_1, \ldots, u_n$ to G' such that v_i is adjacent to u_i for all i. Now add a vertex w which is adjacent to all of u_i. The remaining checking is clear, and thus we are done.

We infact have a near-complete classification of complexity of k-Col.

Theorem 8. *Let $d, k \geq 1$. Then k-*Col *for graphs of diameter at most d is: in* P *if $k \leq 2$ or $d = 1$; or in* NPC *if $k = 3, d \geq 4$ or $k \geq 4, d \geq 2$.*

With the missing cases $(k, d) \in \{(3, 2), (3, 3)\}$, one of these cases has been solved.

Theorem 9. 3-Col *for graphs of diameter 3 is* NP*-complete.*

Leaving the last open question: what is the complexity of 3-Col for graphs of diameter 2?

2 Graph classes

2.1 Initial theory

We build some theory around graph classes (that is, a collection of graphs that can be defined by a property all the graphs share). We first informally introduce *graph operations*: a graph operation π produces a new graph from initial one(s). We have quite simple (elementary) graph operations, such as vertex addition or deletion, and we also have more advanced operations such as graph complement.

Definition 10 (Closed)**.** A graph class $\mathcal{G}$ is *closed under* a graph operation π if for every $G \in \mathcal{G}$, the graph obtained from applying π to G also belongs to $\mathcal{G}$.

To make things clear, we will formally define a (simple undirected) graph. A *graph* is the 2-tuple $G = (V, E)$ where V is a set of *vertices* and $E \subset \{\{x, y\} : x, y \in V, x \neq y\}$ is a set of *edges*. We may be lazy and denote an edge as xy. We will denote $\mathbb{G}$ as the set of all graphs up to graph isomorphisms; that is,

$$\mathbb{G} = \{(V, E)\}/\simeq$$

where V and E are as above and $\simeq$ is the graph isomorphism equivalence relation. Notation will be abused here! For example, let $G = \{\{a, b\}, \varnothing\}$ and $H = \{\{x, y\}, \varnothing\}$. It is correct to write $G \simeq H$, or $[G] = [H]$, but you may find $G = H$ written.

We are particularly vague about graph operations, but we can define some operations more formally. For example, consider the vertex deletion graph operation $\pi : \{((V, E), v) : (V, E) \in \mathbb{G}, v \in V\} \to \mathbb{G}$, where

$$((V, E), v) \mapsto (V \setminus \{v\}, \{\{u_1, u_2\} \in E : u_1 \neq v, u_2 \neq v\}) \,.$$

For this operation, we may denote $G - v = \pi(G, v)$ and

$$G - \{v_1, \ldots, v_n\} = \pi(\ldots(\pi(G, v_1), \ldots), v_n).$$

Also, for graphs G and H, we denote $G + H$ as the disjoint union of graphs G and H (in the way you would expect) and kG (for $k \in \mathbb{Z}_{\geq 0}$) as the disjoint union of k copies of G.

Definition 11 (Hereditary)**.** A graph class $\mathcal{G}$ is said to be *hereditary* if it is closed under vertex deletion.

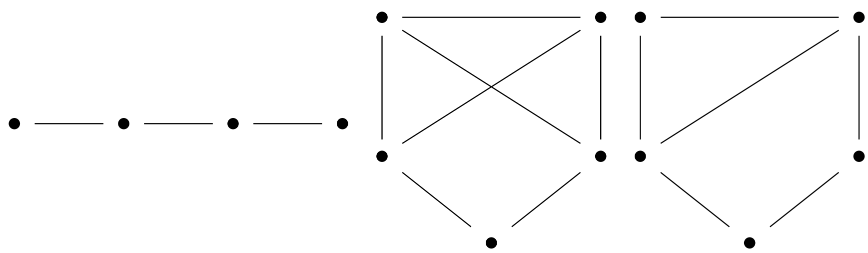
Example 12. The following are hereditary graph classes: bipartite graphs, chordal graphs, interval graphs, perfect graphs, and planar graphs.

Example 13. Consider the class of graphs with diameter 1. One may expect this to be the graph of complete graphs, but there are trivialities here. Consider K_2. If we delete either vertex, we get K_1, which has diameter 0, and is in fact the only non-null graph with such diameter. Thus, the class of graphs with diameter at most 1 is hereditary.

Example 14. Consider the class of graphs of diameter 2. This class is not hereditary, consider the path graph $P_3 = (\{a, b, c\}, \{\{a, b\}, \{b, c\}\})$. Observe that $G - b$ is disconnected, thus the diameter is certainly not 2.

Definition 15 (Induced subgraph)**.** A graph H is an *induced subgraph* of a graph G if G can be modified into H by a sequence of vertex deletions. Furthermore, a graph is said to be *H-free* if H is not an induced subgraph of G. For $G = (V, E)$ and $S \subset V$, we denote $G[S]$ as the *subgraph induced in G by S*.

Example 16. Consider P_4, on the left below. The graph in the middle is P_4-free, while the graph on the right is not P_4-free as P_4 can be obtained by deleting the top right vertex.



For a graph H, we see that the class of H-free graphs is hereditary (this is immediate from the definition).

Definition 17 ($\mathcal{F}$-free)**.** Let $\mathcal{F} = \{H_i\}_{i=1}^{p}$ be a set of graphs. A graph G is $\mathcal{F}$-free if G is H-free for every $H \in \mathcal{F}$.

Example 18. Let $\mathcal{F} = \{C_3, C_5, C_7, \ldots\}$, the set of odd cycles. Then any bipartite graph is $\mathcal{F}$-free.

Definition 19 (Minimal)**.** A family of graphs $\mathcal{F}$ is *minimal* if no graph in $\mathcal{F}$ is an induced subgraph of another graph in $\mathcal{F}$.

Example 20. From a previous example, we see that $\mathcal{F} = \{C_3, C_5, C_7, \ldots\}$ is minimal; deleting any edge from an odd cycle leaves a path graph, which certainly cannot be another odd cycle. However, consider the set of complete graphs $\{K_i\}_{i=0}^{\infty}$. Applying vertex deletion to any K_i results in K_{i-1}, and thus is not minimal.

Theorem 21. *Let $\mathcal{G}$ be a graph class. Then $\mathcal{G}$ is hereditary if and only if there is a unique minimal family of graphs $\mathcal{F}$ such that $\mathcal{G}$ consists of exactly those graphs that are $\mathcal{F}$-free.*

Proof. $\Longrightarrow$: first suppose that $\mathcal{G}$ is hereditary. Let $\mathcal{F}' = \mathbb{G} \setminus \mathcal{G}$. Now let $\mathcal{F} \subset \mathcal{F}'$ such that $H \in \mathcal{F}$ if and only if H is $(\mathcal{F}' \setminus \{H\})$-free. This set is minimal and unique (uniqueness requires a non-trivial argument). Now let $G \in \mathcal{G}$ and suppose that it contains some $H \in \mathcal{F} \subset \mathbb{G} \setminus \mathcal{G}$ as an induced subgraph. As $\mathcal{G}$ is hereditary, $H \in \mathcal{G}$; a contradiction. So $\mathcal{G} \subset \{G \in \mathbb{G} : G \text{ is } \mathcal{F}\text{-free}\}$. Now let $G \notin \mathcal{G}$; that is, $G \in \mathcal{F}'$. Then either $G \in \mathcal{F}$ or G is contains an induced subgraph $H \in \mathcal{F}$. In either case, $\mathcal{G}$ is not $\mathcal{F}$-free and thus $\mathcal{G} = \{G \in \mathbb{G} : G \text{ is } \mathcal{F}\text{-free}\}$. $\Longleftarrow$: suppose $\mathcal{F}$ is as in the statement of the theorem. Let $G = (V, E) \in \mathcal{G}$ and $v \in V$. G is $\mathcal{F}$-free, thus $G - v$ is also $\mathcal{F}$-free, and so $G - v \in \mathcal{G}$; hence, $\mathcal{G}$ is hereditary. $\square$

Definition 22 (Obstruction set)**.** The *obstruction set* $\mathcal{F}_{\mathcal{G}}$ of a hereditary graph class $\mathcal{G}$ is that as defined in Theorem 21.

Example 23. Consider the class of complete bipartite graphs $\mathcal{G}$ (that is, every vertex in one partition is adjacent to every vertex in the other partition). $\mathcal{G}$ is indeed hereditary, we move to understand $\mathcal{F}_{\mathcal{G}}$. Let $G \in \mathcal{G}$. It is clear that G is free from odd cycles, just like for the class of bipartite graphs, but this does not uniquely describe $\mathcal{G}$ as otherwise all bipartite graphs would be complete bipartite! We claim that $\mathcal{F}_{\mathcal{G}} = \{C_3, P_1 + P_2\}$. Indeed, let $G = (V, E)$ be a complete bipartite graph with partitions $A, B \subset V$. As G is bipartite, G is C_3-free. Now suppose that G has $P_1 + P_2 = (\{x, y, z\}, \{\{x, y\}\})$ as an induced subgraph. Without loss of generality, assume $x, z \in A$. Then $y \in B$. Then there no edge $\{z, y\}$; a contradiction. Thus all graphs in $\mathcal{G}$ are $\mathcal{F}_{\mathcal{G}}$-free. Now suppose $G = (V, E)$ is $\{C_3, P_1 + P_2\}$-free. Then G is also $\{C_3, C_5, C_7, \ldots\}$-free, so G is bipartite. Now we assume that G is disconnected. As G is $(P_1 + P_2)$-free, each component must be a single vertex; that is, G is a set of isolated vertices, which is complete bipartite. Assume G is not complete bipartite for a contradiction. Now if G is connected we pick $u \in A$ and $v \in B$ such that u and v are not adjacent. Pick w as the neighbour after u in a path from u to v. As $w, v \in B$, they cannot be connected. But now observe that u, v, and w induce $P_1 + P_2$; a contradiction. Thus all graphs which are $\mathcal{F}_{\mathcal{G}}$-free are in $\mathcal{G}$, and so $\mathcal{F}_{\mathcal{G}}$ is the obstruction set of $\mathcal{G}$.

Obstruction sets are helpful when we want to recognise whether a graph belongs to a specific hereditary graph class. Consider a graph $G = (V, E)$ and a graph class $\mathcal{G}$, and suppose that we want to check whether $G \in \mathcal{G}$. If we know the obstruction set of G and it is finite, say $\mathcal{F}_{\mathcal{G}} = \{H_1, \ldots, H_r\}$, then we have the following polynomial time algorithm for checking membership.

Consider each $H_i = (V_i, E_i) \in \mathcal{F}_{\mathcal{G}}$. By brute force, we check in $O\left(\binom{n}{|V_i|}\right) = O(n^{|V_i|})$ time whether G contains H_i as an induced subgraph. If so, then G is not in $\mathcal{G}$. Otherwise, after considering all H_i, we conclude $G \in \mathcal{G}$.

Of course, this is assuming $\mathcal{F}_{\mathcal{G}}$ is finite, and we have already seen a graph class for which this isn't the case (bipartite graphs). When $\mathcal{F}_{\mathcal{G}}$ is infinite, this certainly gets harder, but in some cases still possible; for example, there is a polynomial time algorithm for checking if G is *odd-hole-free*; that is, $\{C_5, C_7, C_9, \ldots\}$-free.

Definition 24 (Restriction)**.** For a problem Π, denote $\Pi(\mathcal{G})$ as the restriction of Π to graphs in $\mathcal{G}$.

Example 25. Let $\Pi = \textsc{Col}$ and $\mathcal{G}$ the class of planar graphs. Then $\Pi(\mathcal{G})$ is the colouring problem for planar graphs.

A *dichotomy theorem* may state the complexity for certain classes of problems, either easy (P) or hard (NPC). But forming a dichotomy theorem is usually too difficult, so we may strict our problems to hereditary graph classes. But, even then it may be too difficult, so we may restrict to a hereditary graph class with a finite obstruction set, and use algorithms similar to the one prosed above to make some progress. In doing so, we understand more of the complexity of our problem.

Theorem 26. *Let H be a graph and $\mathcal{G}$ be the class of H-free graphs.*

1. *If H is an induced subgraph of P_4 or $P_1 + P_3$, then* $\textsc{Col}(\mathcal{G}) \in \mathsf{P}$.
2. *Otherwise,* $\textsc{Col}(\mathcal{G}) \in \mathsf{NPC}$.

2.2 More operations

Now we develop some more theory by introducing two new graph operations: *edge contraction* and *edge deletion*.

Let $G = (V, E)$ and $e \in E$. The *deletion* of the edge e results in the graph $(V, E \setminus \{e\})$, which we may denote $G - e$. We may formally define this: edge deletion is a graph operation $\pi : \{((V, E), e) : (V, E) \in \mathbb{G}, e \in E\} \to \mathbb{G}$ where

$$((V, E), e) \mapsto (V, E \setminus \{e\}).$$

Example 27. The class of bipartite graphs and the class of planar graphs are both closed under edge deletion, but not all hereditary graph classes are closed under edge deletion. The complete bipartite graphs are clearly not. We also claim that the class of P_4-free graphs are not closed under edge deletion. Indeed, C_4 is P_4 free, but the removal of any edge results in P_4; a graph that is not P_4-free.

Definition 28 (Spanning subgraph)**.** A graph G contains a graph H as a *spanning subgraph* if G can be modified into H by a sequence of edge deletions. Otherwise, G is *H-spanning-subgraph-free.*

We can reformulate problems in terms of these operations; for example, consider the following reformulation of HAMILTON PATH problem.

Problem 29 (HAMILTON PATH).
Instance: a graph $G = (V, E)$.
Question: does G contain $P_{|V|}$ as a spanning subgraph?

Now we move to our other operation: let $G = (V, E)$ be a graph and $e = uv \in E$. The *contraction* of e removes u and v from G but adds a new vertex which is adjacent to all other neighbours of u and v. That is, $\pi : \{((V, E), uv) : (V, E) \in \mathbb{G}, uv \in E\} \to \mathbb{G}$ such that

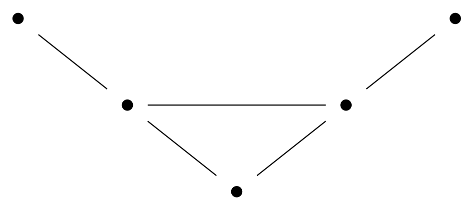
$$\begin{aligned}((V, E), uv) \mapsto ((V \setminus \{u, v\}) \cup \{w\},& \\ \{e \in E : u, v \notin e\} \cup \{\{w, x\} &: \{u, x\} \in E \text{ or } \{v, x\} \in E\}).\end{aligned}$$

We denote this operation by $G/e = \pi(G, e)$ (as opposed to edge deletion, which was $G \setminus e$).

We do note that we are considering simple graphs here, so if you contract an edge whose endpoints share edges with the same vertices then we remove multiple edges such that the graph is simple, as the formalisation above establishes (that is, we are using sets and not multisets).

Definition 30 (Contraction)**.** A graph G contains a graph H as a *contraction* if G can be modified into H by a sequence of edge contractions. Otherwise G is *H-contraction-free.*

Example 31. Consider the class of claw-free graphs (the claw graph is the graph $K_{1,3}$). This class is not closed under edge contraction. This can be shown by looking at the bull graph.



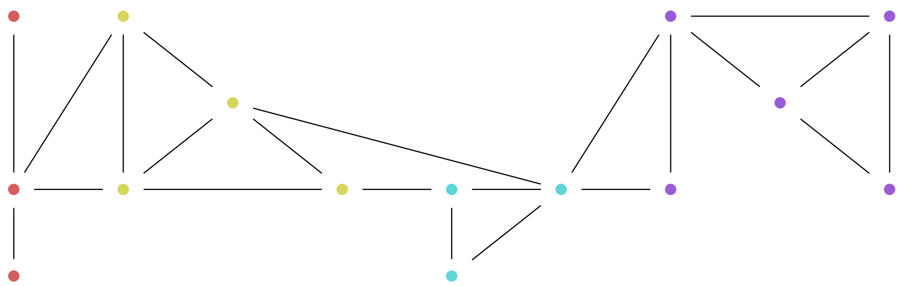
The bull graph can be transformed into the claw graph through edge contraction, but not by vertex deletion.

Theorem 32. *A graph $G = (V, E)$ contains a graph $H = (V', E')$ as a contraction if and only if for every $x \in V$ there exists a non-empty subset $W_x \subset V$ such that*

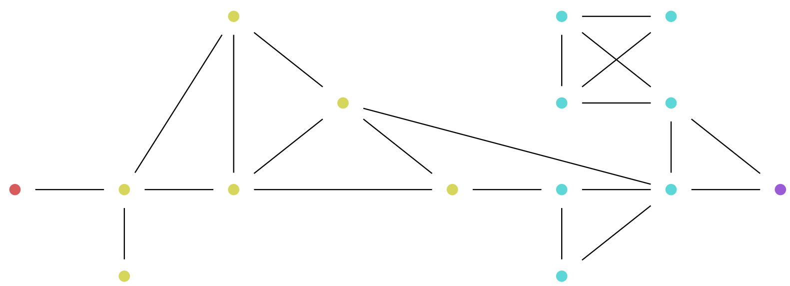
1. *W_x is connected;*
2. *$\mathcal{W} = \{W_x\}_{x \in V}$ is a partition of V; and*
3. *for all $x, y \in V'$, there is $x' \in W_x$ and $y' \in W_y$ with $\{x', y'\} \in E$ if and only if $\{x, y\} \in E$.*

By contracting each W_x to a single vertex, we obtain H. The set $\mathcal{W}$ is called an *H-witness structure* of G, and $W_x \in \mathcal{W}$ is called an *H-witness bag* of G for $x \in V$.

Example 33. Consider the following graph.



Here the colours of the vertices define the partition $\mathcal{W}$, and it is clear to see that this is a P_4-witness structure. In fact, this graph has two different possible partitions. See below another P_4-witness structure.



So $\mathcal{W}$ may not be unique.

We may use combinations of vertex deletions, edge deletions, and edge deletions to obtain smaller graphs, and we have terminology to describe such.

Definition 34. 1. A graph G contains a graph H as a *subgraph* if G can be modified into H by a sequence of vertex deletions and edge deletions. Otherwise, G is *H-subgraph-free.*

2. A graph G contains a graph H as a *minor* if G can be modified into H by a sequence of vertex deletions, edge deletions, and edge contractions. Otherwise, G is *H-minor-free.*

3. A graph G contains a graph H as an *induced minor* if G can be modified into H by a sequence of vertex deletions and edge contractions. Otherwise, G is *H-induced-minor-free.*

Let us consider some general containment problems.

Example 35. Consider the problem of deciding whether a graph H is a spanning subgraph of another $G = (V, E)$. This is NP-complete as for $H = P_{|V|}$, this problem specialises to HAMILTON PATH.

Example 36. Now we consider the problem of deciding whether a graph $G = (V, E)$ contains a fixed graph $H = (V', E')$ as a spanning subgraph. This now becomes polynomial time, as $|V'| = |V|$, so G has constant size too. Thus we check all options by brute force in $O\left(n^{|V|}\right)$ time. Similarly, if we can check in the same time whether H is an induced subgraph.

Theorem 37. *Deciding whether a graph G contains a fixed graph H as a* minor *is solvable in $O(n^3)$ time.*

As we may define a hereditary graph class with an obstruction set, we can do the same for closure under taking minors.

Theorem 38. *Every class of graphs closed under taking minors can be defined by a* finite *set of forbidden minors.*

By forbidden minors, we mean that every graph on the graph class does not contain any of the forbidden minors as minors.

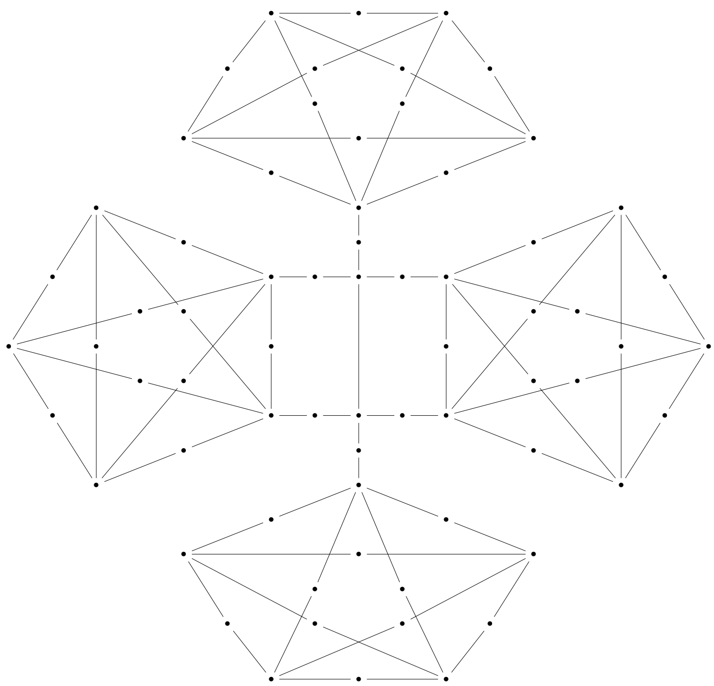
Theorem 39 (Wagner's)**.** *A graph is planar if and only if it is $(K_5, K_{3,3})$-minor-free.*

We have established that for a fixed graph H, deciding whether a graph G contains H as a spanning subgraph, subgraph, induced subgraph, or minor are all in P, but what about for contractions only and for induced minors?

Theorem 40. *Deciding if a graph G contains P_4 as a contraction is* NP-*complete.*

Theorem 41. *Deciding if G contains the H^* as an induced minor is* NP-*complete.*

In the above theorem, H^* is the following graph.



Theorem 42. *There exists graph problems that are* PSPACE*-complete for* $\mathbb{G}$*, but for all proper hereditary subsets* $\mathcal{G} \subset \mathbb{G}$ *can be solved in* $O(1)$ *time.*

3 Graph recognition and modification

We introduce graph recognition and graph modification problems for a graph class $\mathcal{G}$.

Problem 43 ($\mathcal{G}$-Recognition)**.**
Instance: a graph G.
Question: is $x \in \mathcal{G}$?

Example 44. Let $\mathcal{G}$ be the class of triangle-free graphs. Then $\mathcal{G}$-Recognition $\in O(n^3)$ by brute force.

Now let S be a set of graph operations.

Problem 45 ($\mathcal{G}$-Modification(S))**.**
Instance: a graph G and $k \in \mathbb{Z}_{\geq 0}$.
Question: can G be modified into a graph $G' \in \mathcal{G}$ using at most k operations from S?

Example 46.