1. Suppose that V is a representation of G and that W_1 and W_2 are irreducible subrepresentations. Show that either $W_1 = W_2$ or $W_1 \cap W_2 = \{0\}$.

Solution: Let $\boldsymbol{v} \in W_1 \cap W_2$. Then, as W_1 and W_2 are subrepresentations we have $\rho(g)\boldsymbol{v} \in W_1$ and $\rho(g)\boldsymbol{v} \in W_2$ for all $g \in G$.Thus $\rho(g)\boldsymbol{v} \in W_1 \cap W_2$ for all $g \in G$; thus, $W_1 \cap W_2$ is a subrepresentation of W_1 and W_2. As W_1 and W_2 are irreducible, then either $W_1 \cap W_2 = \{0\}$ or $W_1 = W_2 = W_1 \cap W_2$.

2. Consider $G = S_n$ with its permutation representation action on $\mathbb{C}^n$ characterized by

$$\pi(g)e_i = e_{g(i)},$$

and let $V \subset \mathbb{C}^n$ be the subspace

$$\left\{ (a_1, \ldots, a_n) \in \mathbb{C}^n : \sum_{i=1}^{n} a_i = 0 \right\}.$$

Show that V is irreducible.

Solution: Let $W \subset V$ be a non-zero representation, and let $\boldsymbol{a} = (a_1, \ldots, a_n) \in W$. Without loss of generality, we may assume $a_1 \neq a_2$ (if this is not the case, then we can apply a permutation such that it is, since membership in V implies that for all non-zero elements, the entries cannot be equal). Thus

$$\boldsymbol{a} - \pi((1\,2))\boldsymbol{a} = (a_1 - a_2, a_2 - a_1, 0, \ldots, 0) \in W.$$

As $a_1 \neq a_2$, we divide through by $a_1 - a_2$ to get

$$\boldsymbol{e}_1 - \boldsymbol{e}_2 = (1, -1, 0, \ldots, 0) \in W.$$

We see that $\boldsymbol{e}_1 - \boldsymbol{e}_i \in W$ by applying the permutation $(2\,i)$ to $\boldsymbol{e}_1 - \boldsymbol{e}_2$ for all $i \in \{2, \ldots, n\}$. We see that $\{\boldsymbol{e}_1 - \boldsymbol{e}_i\}_{i \in \{2,\ldots,n\}}$ are linearly independent, thus $\dim W \geq n - 1 = \dim V$; thus $W = V$. It is also not hard to see that $\{\boldsymbol{e}_1 - \boldsymbol{e}_i\}_{i \in \{2,\ldots,n\}}$ is a basis of V by inspection.

3. Let (π_V, V) and (π_W, W) be two representations of a finite group G.

   (a) Show that if $T \in \mathrm{Hom}_G(V, W)$ is a G-homomorphism and an isomorphism of vector spaces, then T^{-1} is also a G-homomorphism.

   > **Solution:** As T is an isomorphism, T^{-1} is well defined. Let $w \in W$ and $v \in V$ such that $T(v) = w$. Using that T is a G-homomorphism, we get
   >
   > $$\begin{aligned} T\pi_V v &= \pi_W T v, \\ T\pi_V T^{-1} w &= \pi_W w, \\ \pi_V T^{-1} w &= T^{-1}\pi_W w, \end{aligned}$$
   >
   > as required.

   (b) Assume $\dim V = \dim W = n$ and identify V and W with $\mathbb{C}^n$ by choosing bases. Show that $V \cong W$ as representations of G if and only if there exists a $T \in \mathrm{GL}_n(\mathbb{C})$ such that

   $$T\pi_V(g)T^{-1} = \pi_W(g)$$

   for all $g \in G$.

   > **Solution:** T is a G-homomorphism if and only if the matrix of T is invertible.

4. Prove the following. Let (π, V) be an irreducible representation of a finite group G and $Z = Z(G)$ be the center of G. Then Z acts on V as a character. That is, there exists a homomorphism $\chi : Z \to \mathbb{C}^\times$ such that

   $$\pi(z)v = \chi(z)v$$

   for all $v \in V$.

   > **Solution:** $\pi(z)$ is a G-homomorphism for all $z \in Z$, as it commutes with $\rho(g)$ for all $g \in G$. Thus, by Schur's Lemma, $\rho(z)$ acts by a non-zero scalar, $\chi(z)$, on V. As π is a homomorphism, χ must be too.

5. Find a two-dimensional irreducible representation of C_n over $\mathbb{R}$ (for $n \geq 3$), and prove that it is irreducible. Why does this mean that Schur's lemma doesn't hold with real coefficients?

> **Solution:** We can consider $\rho : C_n \to \mathrm{GL}_2(\mathbb{R})$ the *rotation* representation, given by
>
> $$\rho(g) = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$$
>
> where g is the generator of C_n. Since $\rho(g^k) = \rho(g)^k$, this defines a representation. Let $W \subset \mathbb{R}^2$ be a non-zero proper subrepresentation of this representation. Then W must be a one-dimensional representation which is invariant under $\rho(g)$, but it is clear that this cannot happen. Thus the rotational representation is irreducible.
>
> Schur's Lemma does not hold here, as if it did, every irreducible representation of an abelian group would be one-dimensional.

6. Show that, if V is a vector space, $W \subset V$ is a subspace, and $\pi : V \to W$ is a projection, then

   (a) $V = W \oplus \ker \pi$, and

   > **Solution:** Note that for all $v \in V$, $\pi^2(v) = \pi(v)$. Thus
   >
   > $$\pi(v - \pi(v)) = 0$$
   >
   > and so $v = \pi(v) + (v - \pi(v))$. We have left to show that $W \cap \ker \pi = \{0\}$, but this is immediate.

   (b) $\operatorname{tr} \pi = \dim W$.

   > **Solution:** We consider $\pi : V \to V$. We can write π as the block matrix
   >
   > $$\left(\begin{array}{c|c} I_W & * \\ \hline 0 & 0 \end{array}\right).$$

From this, the result is clear.

7. Suppose that G is a group and $V = \mathbb{C}[G]$, and that $\chi : G \to \mathbb{C}^\times$ is a one-dimensional character. Show that

$$v_X = \sum_{g \in G} \chi^{-1}(g)[g]$$

spans a one-dimensional subrepresentation on which G acts via χ.

Solution: We have to show that $gv_x = \chi(g)v_x$ for all $g \in G$.

$$\begin{aligned} gv_X &= g \sum_{h \in G} \chi^{-1}(h)[h] \\ &= \sum_{h \in G} \chi^{-1}(h)[gh] \\ &= \sum_{g' \in G} \chi^{-1}(g^{-1}g')[g'] \\ &= \chi^{-1}(g^{-1}) \sum_{g' \in G} \chi^{-1}(g')[g'] \\ &= \chi(g)v_X \end{aligned}$$

as required.

8. Decompose the group ring $\mathbb{C}[S_3]$ as the direct sum of irreducible representations of S_3. That is, find explicit irreducible subrepresentations of $\mathbb{C}[S_3]$ such that it is the direct sum of those subrepresentations.

Solution: We have

$$\mathbb{C}[G] \cong \bigoplus_{\rho \in \mathrm{Irr}(G)} \rho^{\dim \rho}$$

and so

$$\mathbb{C}[S_3] \cong \mathbb{1} \oplus \varepsilon \oplus \rho \oplus \rho$$

where ρ is the regular representation (that is, the permutation representation for the action of G on itself). We have left to find the subspaces $V_0, V_1, V_2, V_3 \subset \mathbb{C}[S_3]$ such that V_0 is one-dimensional and S_3 acts trivially on it, V_1 is one-dimensional and S_3 acts as the sign representation on it, and V_2 and V_3 are two-dimensional and distinct, each isomorphic to ρ.

We first note that

$$\alpha = \sum_{h \in G} [h]$$

is preserved under multiplication by g, thus we take $V_0 = \langle \alpha \rangle$. Next consider

$$\beta = \sum_{h \in G} \varepsilon(h)[h].$$

Observe

$$\begin{aligned}
g\beta &= g \sum_{h \in G} \varepsilon(h)[h] \\
&= \sum_{h \in G} \varepsilon(h)[gh] \\
&= \sum_{g' \in G} \varepsilon(g^{-1}g')[g'] \\
&= \varepsilon(g^{-1}) \sum_{g' \in G} \varepsilon(g')[g'] \\
&= \varepsilon(g)\beta
\end{aligned}$$

and so $V_1 = \langle \beta \rangle$. By examining the eigenvectors of $(1\,2\,3)$, we get our last two representations.

9. Find the character tables of the following groups (you shouldn't need to use orthogonality for these). Note that you will have to find conjugacy classes.

   (a) C_4

   Solution: C_4 is abelian, thus all characters have degree 1. For a generator $g \in C_4$, then a degree 1 character $\chi : g \to \mathbb{C}^\times$ is determined by $\chi(g)$, which can be any of $\{1, i, -1, -i\}$. Thus we get the following character table.

	e	g	g^2	g^3
	1	1	1	1
$\mathbb{1}$	1	1	1	1
χ	1	i	-1	$-i$
χ^2	1	-1	1	-1
χ^3	1	$-i$	-1	i

   (b) $C_3 \times C_3$

   Solution: $C_3 \times C_3$ is abelian, so all characters have degree one. This table is just the cartesian product of the table above with itself.

   (c) D_4

   Solution: We have the following conjugacy classes:

   $$\{e\}, \{r, r^{-1}\}, \{r^2\}, \{s, r^2 s\}, \{rs, r^3 s\}.$$

   We have already classified D_4 into four one-dimensional irreducible representations and one two-dimensional irreducible representation.

	e	r	r^2	s	rs
	1	2	1	2	2
$\mathbb{1}$	1	1	1	1	1
ε	1	1	1	-1	-1
ψ_+	1	-1	1	1	-1
ψ_-	1	-1	1	-1	1
χ	2	$w+w^{-1}$	w^2+w^{-2}	0	0

and we observe that $w+w^{-1}=0$ and $w^2+w^{-2}=-2$.

(d) D_5

Solution: We have the following conjugacy classes:

$$\{e\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \{s, rs, r^2s, r^3s, r^4s\}.$$

We have classified D_5 previously too, so we get the following character table.

	e	r	r^2	s
	1	2	2	5
$\mathbb{1}$	1	1	1	1
ε	1	1	1	-1
χ_1	1	$w+w^{-1}$	w^2+w^{-2}	0
χ_2	1	w^2+w^{-2}	$w+w^{-1}$	0

Note that $w=e^{2\pi i/5}$.

10. Let G be a finite group acting on a finite set X. Let χ be the character of the permutation representation. Prove that

$$\chi(g) = |\{x \in X : gx = x\}|.$$

Find the character of the regular representation.

Solution: $\rho(g)$ is a permutation matrix, and the ith column has a 1 in the ith row if and only if it is a fixed point. The regular representation is the permutation representation where $X = G$; that is, G (left) acts on itself. For elements $g, h \in G$ where $g \neq e$, $h \neq gh$. Thus $\chi(g) = 0$. If $g = e$, we get the identity, so $\chi(e) = |G|$.

11. Show that $\sum_{g \in G} a_g[g] \in \mathbb{C}[G]$ is in $Z(\mathbb{C}[G])$ if and only if the function $g \mapsto a_g$ is a class function.

Solution: If an element of $\mathbb{C}[G]$ with all elements of $\mathbb{C}[G]$, then it clearly commutes for $[g]$ for all $g \in G$. The converse can be easily shown by observing

$$Z(\mathbb{C}[G]) = \{z \in \mathbb{C}[G] : z[g] = [g]z \text{ for all } g \in G.$$

12. Let (ρ, V) be a representation of G with character χ and dimension d. Show that

$$|\chi(g)| \leq d$$

for all $g \in G$ with equality if and only if $\rho(g)$ is a scalar matrix. Deduce that

$$\ker \rho = \{g \in G : \chi(g) = d\}.$$

Solution: $\rho(g)$ has finite order, thus its eigenvalues are roots of unity and have absolute value 1. Thus, using the triangle equality, $|\chi(g)| \leq d$.

Equality holds if and only if the eigenvalues are proportional with positive real constants of proportionality. As the eigenvalues are roots of unity, they must be equal. Since $\rho(g)$ is diagonalisable, this means it is a scalar matrix.

If $g \in \ker \rho$, the $\chi(g) = d$. Conversely, if $\chi(g) = d$, then $\rho(g) = \lambda I$.