Algebraic Topology

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1 Simplicial homology

1.1 Introduction and prerequisite algebra

Throughout Michaelmas term, we will develop the notion of homology: a way of associating sequences of algebraic objects (in our case, abelian groups) to other mathematical objects (in our case, topological spaces). Specifically, we look at calculating the nth homology group of a topological space, denoted $H_n(X)$. This group has particular properties which is of interest in topology, namely its invariance over homeomorphisms and its functoriality (that is, for a map $f : X \to Y$, we have a induced map $f_* : H_n(X) \to H_n(Y)$).

We have some prerequisite algebra to help with our homology.

Definition 1.1. The *free abelian group* on generators $\{e_\alpha\}_{\alpha\in A}$ is the group

$$\mathbb{Z}\langle\{e_\alpha\}\rangle = \left\{\sum_{\alpha\in A} n_\alpha e_\alpha : n_\alpha \in \mathbb{Z},\ \text{finitely many } n_\alpha \neq 0\right\}.$$

An element of $\mathbb{Z}\langle\{e_\alpha\}\rangle$ may be called a *formal sum*.

We have a notion of the direct product of groups, but for free abelian groups we have the restriction that formal sums must have finitely many non-zero elements. Thus, the direct product of infinite families of free abelian groups may not be free abelian. From this, we introduce the notion of *direct sum*: let $\{A_i\}_{i\in I}$ be a family of abelian groups, then

$$\bigoplus_{i\in I} A_i = \left\{(a_i)_{i\in I} \in \prod_{i\in I} A_i : \text{ finitely many } a_i \neq e\right\}.$$

This construction allows us to ensure a free abelian group from any family of abelian groups.

1.2 Graph homology

Graph homology is first introduced as a motivating case of the generalised *simplicial homology*. We may describe a directed graph as a 4-tuple

$$G = (V, E, \alpha, \tau)$$

where V is are the vertices, $E \subset \mathcal{P}(V)$ is are the edges, and $\alpha, \tau : E \to V$ describes the inital and terminal vertices of an edge respectively. For example, the directed cycle graph of length 3 may be described as follows:

$$DC_3 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}, \alpha, \tau)$$

with $\alpha(\{v_1, v_2\}) = v_1$, $\alpha(\{v_2, v_3\}) = v_2$, $\alpha(\{v_1, v_3\}) = v_3$, $\tau(\{v_1, v_2\}) = v_2$, $\tau(\{v_2, v_3\}) = v_3$, and $\tau(\{v_1, v_3\}) = v_1$. To visualise a graph, we define the *geometric realisation* $|G|$ of a graph G as the quotient space

$$|G| = \frac{V \cup (E \times [0,1])}{\langle (e,0) \sim \alpha(e), (e,1) \sim \tau(e) : e \in E \rangle}.$$

DC_3, as described above, has the following geometric realisation.

$$\begin{tikzcd}
 & v_1 \arrow[dr] & \\
v_3 \arrow[ur] & & v_2 \arrow[ll]
\end{tikzcd}$$

We now move to the homology groups of a graph G. First, we define the *chain groups* of G.

Definition 1.2. Let $G = (V, E, \alpha, \tau)$ be a graph as defined above. We define

$$C_0(G) = \mathbb{Z}\langle V \rangle, \qquad C_1(G) = \mathbb{Z}\langle E \rangle$$

to be the 0th and 1st *chain group* of G, respectively.

Next, we move to define a map between the chain groups.

Definition 1.3. Let $G = (V, E, \alpha, \tau)$. The *boundary map* of G is a group homomorphism $\partial_G : C_1(G) \to C_0(G)$ defined by $e \mapsto \tau(e) - \alpha(e)$, extending linearly.

By extending linearly, we mean $\partial_G(\sum_i n_i e_i) = \sum_i n_i \partial(e_i)$. Finally, we define the homology of G.

Definition 1.4. Let $G = (V, E, \alpha, \tau)$. Then

$$H_0(G) = \operatorname{coker} \partial_G, \qquad H_1(G) = \ker \partial_G$$

are the 0th and 1st *homology group* of G, respectively.

Note, for a function $f : A \to B$, $\operatorname{coker} f = B / \operatorname{im} f$.

Proposition 1.5. *Let $G = (V, E, \alpha, \tau)$ be a graph. Then*

$$H_0(G) \cong \mathbb{Z}^{c(G)}, \qquad H_1(G) \cong \mathbb{Z}^{|E|-|V|+c(G)}$$

where $c(G)$ denotes the number of connected components in $|G|$.

Proof. Here we will introduce some notions that become more concrete in the following section. We have $H_0(G) \cong C_0(G)/\operatorname{im} \partial_G$. That is, vertices $v_1, v_2 \in V$ lay within the same equivalence class if $v_1 - v_2 \in \operatorname{im} \partial_G$. By the definition of ∂_G, we see this to be true if v_1 and v_2 are connected. Thus we assert that the rank of $H_0(G)$ must equal the number of connected components within G. Now, $H_1(G) \cong \ker \partial_G$: the cycles of G ($e_1, \dots, e_n$ is a cycle in G if and only if $\partial_G(e_1 + e_2 + \ldots + e_n) = 0$). We first assume that G is connected and let T be a spanning tree. Observe that there exists a bijection between $E \setminus T$ and the set of linearly independent cycles of G. Thus $H_1(G) \cong \mathbb{Z}^{|E \setminus T|} = \mathbb{Z}^{|E|-(|V|-1)} = \mathbb{Z}^{|E|-|V|+1}$. A similar argument may extend this to a disconnceted graph to show $H_1(G) \cong \mathbb{Z}^{|E|-|V|+c(G)}$. $\square$

1.3 Simplicial homology

We move to generalise graph homology to n-dimensions. We may recall the n-simplex, the generalisation of a point, line segment, triangle, etc. to arbitrary dimensions.

Definition 1.6. We define the *standard n-simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ as

$$\Delta^n = \left\{ \boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \right\}$$

with the subspace topology from $\mathbb{R}^{n+1}$.

We move to a more abstract notion of a simplex, a family of sets that is closed under taking subsets but we weaken the notion of being tied to a specific space (although this notion is still there). We may denote such a simplex by an ordered list of vertices, note this induces an orientation on the faces. For example, $[v_0, v_1, v_2]$ may denote a 2-simplex (a triangle).

Definition 1.7. Let $\Delta^n = [v_0, \dots, v_n]$ be a n-simplex. A *face* of Δ^n is the $(n-1)$ simplex obtained from Δ^n by removing a vertex.

Now we introduce a class of topological spaces called Δ-complexes.

Definition 1.8. A Δ-structure on a topological space X is a collection of maps $\{\sigma_\alpha\}_{\alpha \in I}$ where for each $\alpha \in I$, $\sigma_\alpha : \Delta^n \to X$ for some $n \in \mathbb{N}_0$. The collection must satisfy the following.

1. For each $\alpha \in I$, $\sigma_\alpha\big|_{\mathring{\Delta}^n}$ is injective.
2. For all $x \in X$, there is $\alpha \in I$ and $y \in \Delta^n$ such that $x = \sigma_\alpha(y)$.
3. For each $\alpha \in I$, the restriction of σ_α to a face of Δ^n is σ_β for some $\beta \in I$.
4. $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open for all $\alpha \in I$.
5. Let $(\rho, \mathbb{R}^{n+1})$ be the permutation representation of S_{n+1}. Then for all $\alpha \in I$, there exists $\beta \in I$ such that $\sigma_\alpha = \sigma_\beta \circ \rho(\tau)$.

A Δ-complex is similar to a simplicial complex in the sense that we construct it by *gluing* simplices together, but the rules of gluing are more lax for a

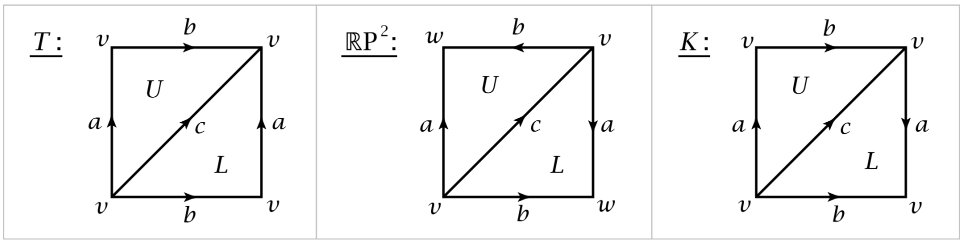

Figure 1: Some examples of Δ-structures for common topological spaces.

Δ-complex. Figure 1 shows some examples of Δ-structures on the torus, projective plane, and klein bottle.

We now move to define the chain groups and homology groups on a Δ-complex much like we did for graphs.

Definition 1.9. Let X be a Δ-complex. For $n \in \mathbb{N}_0$ where $|\{\sigma_\alpha^n \in X\}| \neq 0$, we define the nth chain group of X as

$$C_n^{\text{simp}}(X) = \frac{\mathbb{Z}\langle \sigma_\alpha^n \in X\rangle}{\langle \sigma_\alpha^n \sim \operatorname{sign}(\tau)\sigma_\alpha^n \circ \rho(\tau) : \tau \in S_{n+1}, \sigma_\alpha \in X\rangle}$$

where $(\rho, \mathbb{R}^{n+1})$ is the permutation representation of S_{n+1}. When $|\{\sigma_\alpha^n \in X\}| = 0$ (that is, there are no n-simplices) we define $C_n^{\text{simp}}(X) = 0$.

Here we are identifying all simplices with the ordering of their vertices permuted (with an appropriate sign) as it is clear that $[v_0, v_1, v_2, \ldots, v_n]$ and $[v_1, v_0, v_2, \ldots, v_n]$ represent the same simplex (the sign of the permutation accounts for the orientation).

We now move to define a boundary map between adjacent chain groups, as we did between the 1st and 0th chain groups in graph homology.

Definition 1.10. Let X be a Δ-complex and $n \in \mathbb{N}$. If $C_n^{\text{simp}}, C_{n-1}^{\text{simp}} \neq 0$, we define the nth boundary map as the group homomorphism

$$\begin{aligned} \partial_n^{\text{simp}} : C_n^{\text{simp}} &\to C_{n-1}^{\text{simp}}, \\ [v_0, \ldots, v_n] &\mapsto \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] \end{aligned}$$

and extending linearly. When C_n^{simp} or C_{n-1}^{simp} are empty, then we define ∂_n to be the zero homomorphism. We define ∂_0 to be the zero homomorphism too.

For brevity, we use the notation

$$[v_0, \ldots, \hat{v}_i, \ldots, v_n] = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n].$$

Note that C_n^{simp} is the free abelian group on generators being the n-simplices (where two simplices with the same vertex set permuted are identified graded by the sign of the permutation), so we must extended linearly for ∂_n^{simp} to be well defined. Formally, we mean:

$$\partial_n^{\text{simp}}\left(\sum_\alpha n_\alpha \sigma_\alpha\right) = \sum_\alpha n_\alpha \partial_n^{\text{simp}}(\sigma_\alpha).$$

To motivate the alternating sum: we want to preserve orientation of our vertices (as given to our chain group in the definition of $C_n^{\text{simp}}(X)$).

The natural next step, after following the special case of graph homology, would be to move onto defining the homology groups as the quotients $\ker\delta_n/\operatorname{im}\delta_{n+1}$, but first we need to ensure this quotient is well-defined. Indeed, all subgroups of abelian groups are normal, but we must ensure that $\operatorname{im}\delta_{n+1}\subset\ker\delta_n$. This motivates the following lemma.

Lemma 1.11. *Let $\{\partial_n^{simp}\}_{n\in\mathbb{N}_0}$ be the boundary maps of a Δ-complex. Then for all $n\in\mathbb{N}$,*

$$\partial_{n-1}^{simp}\circ\partial_n^{simp} = 0.$$

Proof. Trivially, $\partial_0^{\text{simp}}\circ\partial_1^{\text{simp}} = 0\circ\partial_1^{\text{simp}} = 0$. We let $n\in\{2,3,\ldots\}$. Let $\sigma = [v_0, v_1, \ldots, v_n]$ a n-simplex. Then we have

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

Thus

$$\begin{aligned}
(\partial_{n-1}\circ\partial_n)(\sigma) &= \partial_{n-1}\left(\sum_{i=0}^n (-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_n]\right)\\
&= \sum_{i=0}^n (-1)^i \partial_{n-1}([v_0,\ldots,\hat{v}_i,\ldots,v_n])\\
&= \sum_{i<j} (-1)^i(-1)^{j-1}\partial_{n-1}([v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]) + {}\\
&\qquad\qquad \sum_{j<i}(-1)^i(-1)^j\partial_{n-1}([v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n])\\
&= 0.
\end{aligned}$$

We can see that the two summations cancel after switching i and j in either summation, it becomes the negative of the first. $\square$

We are now comfortable defining our homology groups.

Definition 1.12. Let X be a Δ-complex with boundary maps $\{\partial_n^{\text{simp}} : C_n^{\text{simp}} \to C_{n-1}^{\text{simp}}\}_{n\in\mathbb{N}_0}$. For $n \in \mathbb{N}_0$, we define the nth *simplicial homology group* of X as

$$H_n^{\text{simp}}(X) = \frac{\ker \partial_n^{\text{simp}}}{\operatorname{im} \partial_{n+1}^{\text{simp}}}.$$

We will look to calculate the simplicial homology groups on the three Δ-complexes introduced in Figure 1.

Example 1.13. For the Δ-complex on the torus, we get the following simplicial chain complex.

$$\begin{tikzcd}
C_2^{\text{simp}}(X) \arrow[r, "\partial_2"'] \arrow[d, "\cong"] & C_1^{\text{simp}}(X) \arrow[r, "\partial_1"'] \arrow[d, "\cong"] & C_0^{\text{simp}}(X) \arrow[r, "\partial_0"'] \arrow[d, "\cong"] & 0 \\
\mathbb{Z}^2 \arrow[r, "M_2"] & \mathbb{Z}^3 \arrow[r, "M_1"] & \mathbb{Z} &
\end{tikzcd}$$

where $M_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. Thus we get

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ \mathbb{Z}^2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.14. For the Δ-complex on the real projective plane, we have the following simplicial chain complex.

$$\begin{tikzcd}
C_2^{\text{simp}}(X) \arrow[r, "\partial_2"'] \arrow[d, "\cong"] & C_1^{\text{simp}}(X) \arrow[r, "\partial_1"'] \arrow[d, "\cong"] & C_0^{\text{simp}}(X) \arrow[r, "\partial_0"'] \arrow[d, "\cong"] & 0 \\
\mathbb{Z}^2 \arrow[r, "M_2"] & \mathbb{Z}^3 \arrow[r, "M_1"] & \mathbb{Z}^2 &
\end{tikzcd}$$

where $M_2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $M_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Thus we get

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.15. For the Δ-complex on the klein bottle, we have the following simplicial chain complex.

$$\begin{tikzcd}
C_2^{\text{simp}}(X) \arrow[r, "\partial_2"'] \arrow[d, "\cong"] & C_1^{\text{simp}}(X) \arrow[r, "\partial_1"'] \arrow[d, "\cong"] & C_0^{\text{simp}}(X) \arrow[r, "\partial_0"'] \arrow[d, "\cong"] & 0 \\
\mathbb{Z}^2 \arrow[r, "M_2"] & \mathbb{Z}^3 \arrow[r, "M_1"] & \mathbb{Z} &
\end{tikzcd}$$

where $M_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. Thus we get

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2 Singular homology

2.1 Definition

We now move to generalise the simplicial homology to a much more powerful version: singular homology.

Definition 2.1 (Chain complex)**.** A *chain complex* is a sequence of abelian groups $\{C_i\}_{i \in \mathbb{Z}}$, and homomorphism $\partial_i : C_i \to C_{i-1}$ such that $\partial_i \circ \partial_{i-1} = 0$, called the *boundary maps*.

Note we may denote a chain complex by the pair (C_*, ∂_*), or just C_*. We mainly consider non-negative chain complexes, that is $C_i = 0$ for all $i < 0$. With just a chain complex, we can define its homology.

Definition 2.2 (Homology)**.** The *homology* $H_*(C)$ of a chain complex $C = (C_*, \partial_*)$ is

$$H_i(C) = \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}.$$

Some language:

1. An element $c \in C_n$ is a *n-chain*.
2. If $\partial_n(c) = 0$, then c is a *n-cycle*.
3. If $c \in \operatorname{im} \partial_{i+1}$, then c is a *n-boundary*.

Informally, we may that $H_i(C)$ is the n-cycles modulo boundaries.

Now we are ready to introduce singular homology.

Definition 2.3 (Singular homology)**.** Let X be a topological space. A *singular n-simplex* of X is a continuous map $\sigma : \Delta^n \to X$. For $n \in \mathbb{Z}_{\geq 0}$, we define the *singular n-chains* $C_n(X)$ as the free abelian groups generated by the singular n-simplices. Let $\sigma \in C_n$ be an n-chain. We define the boundary maps as

$$\partial(\sigma) = \sum_{j=0}^{n} (-1)^j \sigma \circ \iota_j$$

where $\iota_j : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is the inclusion map

$$\begin{cases} v_i \mapsto v_i & i < j, \\ v_i \mapsto v_{i+1} & i \geq j \end{cases}$$

which we may just denote $[v_1, \ldots, \hat{v}_i, \ldots, v_n]$ as before. We extend ∂ linearly as we did before.

This is quite a small a definition, but note here that C_n (when non-trivial) is an uncountable set. A first sight, singular theory seems harder to compute, but it enjoys some formal properties.

Lemma 2.4. *(C_*, ∂) is a chain complex.*

Proof. It is enough to observe that $\partial^2 = 0$. □

Definition 2.5. Let X be a topological space. The nth homology of X is

$$H_n(X) = H_n(C_*(X)).$$

We may denote the n-cycles of X as $Z_n(X)$ and similarly the n-boundaries of X as $B_n(X)$, thus $H_n(X) = Z_n(X)/B_n(X)$. It may seem that we do not have many tools to work with singular homology, but maybe we can something with a trivial space.

Example 2.6. Let X be the empty space. Then $C_n(X) = \{0\}$ for all $n \in \mathbb{Z}_{n \geq 0}$. Thus $H_n(X) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$.

Example 2.7. Let X be the one-point space. Let us reason about the singular homology. First, lets consider $C_0(X)$: the free abelian group generated by the singular 0-simplices. But there is only 1 singular 0-simplex for X: that is, $\sigma : \Delta^0 = \{1\} \to \{\text{pt}\}$. Thus $C_0(X) = \mathbb{Z}\langle\sigma\rangle \cong \mathbb{Z}$. Similarly, for $n \geq 0$, $C_n(X) \cong \mathbb{Z}$ as there is only one n-simplex (a map from Δ^n to a $\{\text{pt}\}$). So we understand $C_n(X)$, what does our boundary maps look like? Let $n \in \mathbb{Z}_{\geq 0}$ and σ_n be the n-simplex. The domain of σ_n is Δ^n, and ∂_n takes σ_n to the sum of the function defined over the $n+1$ faces Δ^{n+1} with the inclusion map. But, there is only one function that this may be, so we get

$$\partial(\sigma_n) = \sum_{j=0}^{n} (-1)^j \sigma_{n-1} \cong \begin{cases} \text{id} & 2 \mid n, \\ 0 & \text{else} \end{cases}$$

(extending linearly). Thus, for $n > 0$.

$$Z_n(X) = B_n(X) = \begin{cases} 0 & 2 \mid n, \\ \mathbb{Z} & \text{else.} \end{cases}$$

We also have $Z_0(X) = \mathbb{Z}$ and $B_0(X) = 0$ as $\partial_1 = \partial_0 = 0$. So

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.8. *Let X be a space with n path components. Then $H_0(X) = \mathbb{Z}^n$.*

Proof. The 0-cycles of X correspond to points in X. Thus, let $c, c' \in Z_0(X)$. c and c' are *homologous* (that is, lie in the same homology class) if $c - c' \in B_0(X)$. That is, if there is a path $\sigma : \Delta^1 \to X$ such that $\sigma(0,1) = c$ and $\sigma(1,0) = c'$. Let $X_1, \ldots, X_n$ be a labelling of the path components of X and let x_i be a point in X_i. We then consider a homomorphism $\mathbb{Z}^n \to H_0(X)$ sending $e_i \mapsto [\Delta^0 \to \{x_i\} \to X]$. We claim such a map is surjective since every 0-simplex in a 0-cycle must be homologous to one of the x_i. We observe the injectivity by the fact that there exists no paths from x_i to x_j, $i \neq j$. $\square$

We will (for now) take for granted the next statement, but we will return to it.

Theorem 2.9. *Let X be a Δ-complex. Then $H_n^{simp}(X) \cong H_n(X)$ for all $n \in \mathbb{Z}$.*

Succeeding this is techniques for calculating singular homology, for now we present some homologies of spaces for intuition.

Example 2.10.

- $H_i(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & \text{else.} \end{cases}$

- $H_i(S^n) = \begin{cases} \mathbb{Z} & i \in \{0, n\}, \\ 0 & \text{else.} \end{cases}$

- $H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i \in \{0, 2n\}, \\ \mathbb{Z}^2 & i = n, \\ 0 & \text{else.} \end{cases}$

- $H_i(S^n \times S^m) = \begin{cases} \mathbb{Z} & i \in \{0, n, m, n+m\}, \\ 0 & \text{else.} \end{cases}$

2.2 Chain maps

One of the main advantages of defining homology of X by the (usually infinite rank) abelian groups generated by all possible continuous maps of an n-simplex into X is that it is easy to prove that it behaves well with respect to maps between spaces.

Definition 2.11 (Chain map)**.** A *chain map* $F : C_* \to D_*$ between chain complexes C_* and D_* is a collection of homomorphisms $F_n : C_n \to D_n$ such that

$$\partial_{n+1}^D \circ F_{n+1} = F_n \circ \partial_{n+1}^C.$$

The above definition can be understand pictorially, F_n is a chain map if the following diagram commutes.

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^C"] \arrow[d, "F_{n+1}"] & C_n \arrow[r] \arrow[d, "F_n"] & \dots \\ \dots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^D"] & D_n \arrow[r] & \dots \end{tikzcd}$$

Lemma 2.12. *A chain map $F : C_* \to D_*$ induces a map on homology*

$$\begin{aligned} F_* : H_n(C_*) &\to H_n(D_*), \\ [c] &\mapsto [F(c)], \end{aligned}$$

for every $n \in \mathbb{Z}_{\geq 0}$.

Proof. We recall that $H_n(C_*) = \ker \partial_n^C / \operatorname{im}_{n+1}^C$. We have to show that this map is well defined (cycles are mapped to cycles, boundaries are mapped to boundaries) and that F_* is a homomorphism.

First, we will show that if c is an n-cycle in C, $F_n(c)$ is a n-cycle in D. Indeed,

$$\partial_n^D(F_n(c)) = F_{n-1}(\partial_n^C(c)) = F_{n-1}(0) = 0.$$

Now let d be a n-boundary. Then there is $e \in C_{n+1}$ such that $\partial_{n+1}^C(e) = d$. Then

$$\begin{aligned} [F_n(c+d)] &= [F_n(c) + F_n(d)] \\ &= [F_n(c) + F_n(\partial_{n+1}^C(e))] \\ &= [F_n(c) + \partial_{n+1}^D(F_{n+1}(e))] \\ &= [F_n(c)]. \end{aligned}$$

We have left to show that F_* is a homomorphism. Let c_1 and c_2 be n-chains in C. Then

$$\begin{aligned}
F_*([c_1]+[c_2]) &= F_*([c_1+c_2]) \\
&= [F_n(c_1+c_2)] \\
&= [F_n(c_1)+F_n(c_2)] \\
&= [F_n(c_1)]+[F_n(c_2)] \\
&= F_*([c_1])+F_*([c_2])
\end{aligned}$$

as required. $\square$

Proposition 2.13. *Let $f : X \to Y$ be a continuous map between topological spaces. Then f induces a chain map $f_* : C_*(X) \to C_*(Y)$ defined by sending each singular simplex $i : \Delta^n \to X$ to $f \circ i : \Delta^n \to Y$.*

Proof. Let $\sigma : \Delta^n \to X$ be a singular $n+1$-simplex. Then

$$\begin{aligned}
(f_n \circ \partial^X_{n+1})(\sigma) &= f_n\left(\sum_{j=0}^{n+1}(-1)^j \sigma \circ \iota_j\right) \\
&= \sum_{j=0}^{n+1}(-1)^j f_n(\sigma \circ \iota_j) \\
&= \sum_{j=0}^{n+1}(-1)^j f \circ \sigma \circ \iota_j \\
&= \sum_{j=0}^{n+1}(-1)^j (f_{n+1} \circ \sigma) \circ \iota_j \\
&= (\partial^Y_{n+1} \circ f_{n+1})(\sigma)
\end{aligned}$$

as required. $\square$

Corollary 2.14. *Let $f : X \to Y$ be a continuous map between topological spaces. Then f induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for every $n \in \mathbb{Z}_{\geq 0}$.*

Proposition 2.15. *If $f = \mathrm{id} : X \to X$ then $f_* = \mathrm{id} : H_n(X) \to H_n(X)$ for every $n \in \mathbb{Z}_{\geq 0}$. Let $f : Y \to Z$ and $g : X \to Y$ be continuous maps. Then $f_* \circ g_* = (f \circ g)_* : H_n(X) \to H_n(Z)$ for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Then

$$f_n([\sigma]) = [f_n(\sigma)] = [f \circ \sigma] = [\sigma]$$

and similarly

$$\begin{aligned}(f_* \circ g_*)([\sigma]) &= [(f_* \circ g_*)(\sigma)] \\ &= [f_*(g \circ \sigma)] \\ &= [f \circ g \circ \sigma] \\ &= [(f \circ g)_*(\sigma)] \\ &= (f \circ g)_*(\sigma)\end{aligned}$$

as expected. □

Proposition 2.16. *Let $f : X \to Y$ be a homeomorphism of topological spaces. Then the induced map $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ is an isomorphism for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof. As f is a homeomorphism, it has a continuous inverse f^{-1}. By the previous proposition, $f_* \circ (f^{-1})_* = (f \circ f^{-1})_* = (\mathrm{id})_* = \mathrm{id}$ and similarly $(f^{-1})_* \circ f_* = \mathrm{id}$, thus $(f^{-1})_*$ is a left and right inverse and so f_* is an isomorphism. □

Thus homology can be used to prove that a pair of topological spaces are not homeomorphic. Specifically, for spaces X and Y, if $H_n(X) \ncong H_n(Y)$, then X and Y are not homeomorphic (as one may expect).

2.3 Exact sequences

Exact sequences is a tool that we will use to compute singular homology.

Definition 2.17 (Exact sequence)**.** A sequence of abelian groups and homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *exact at B* if $\operatorname{im} f = \ker g$. A sequence

$$\dots \to A_{i+1} \to A_i \to A_{i-1} \to \dots$$

is *exact* if $A_{i+1} \to A_i \to A_{i-1}$ is exact at A_i for every i.

Consider the chain complex

$$\dots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots.$$

We have that $\partial_n \circ \partial_{n+1} = 0$, thus $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$. If C_* is exact, then $H_n(C_*) = 0$ for every n.

Definition 2.18 (Short exact sequence)**.** A *short exact sequence* is a five-term exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Consider a short exact sequence as above. As this sequence is exact at A, $\ker f = \operatorname{im} 0 = 0$. Similarly, as it is exact at C, $\operatorname{im} g = \ker 0 = C$. Thus, we conclude that f is injective and g is surjective. As the sequence is exact at B, $\operatorname{im} f = \ker g$. Using this alongside the first isomorphism theorem for groups, we get

$$C \cong B/\ker g = B/\operatorname{im} f \cong B/f(A).$$

Definition 2.19 (Short exact sequence of chain complexes)**.** A *short exact sequence of chain complexes* is a short exact sequence of chain complexes and chain maps

$$0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$$

with $0 \to C_n \to D_n \to E_n \to 0$ a short exact sequence of abelian groups for every n.

Theorem 2.20. *A short exact sequence of chain complexes* $0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$ *determines a long exact sequence in homology groups*

$$\ldots \to H_{n+1}(E) \xrightarrow{\delta} H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\delta} \ldots$$

for some δ*.*

The following diagram above will prove useful for reference in the proof of this theorem.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \ldots \\
\ldots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] \arrow[d, "g_{n+1}"] & D_n \arrow[r, "\partial^D_n"] \arrow[d, "g_n"] & D_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \ldots \\
\ldots \arrow[r] & E_{n+1} \arrow[r, "\partial^E_{n+1}"] \arrow[d] & E_n \arrow[r, "\partial^E_n"] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. We will begin by defining δ, which is effectively done by diagram chasing (a seemingly important skill in algebraic topology!). δ is infact a collection of maps, so we consider $\delta_n : H_n(E) \to H_{n-1}(C)$. We now let $[e] \in H_n(E)$. So $e \in E_n$ with $\partial^E_n(e) = 0$. We have that g is surjective, thus there is $d \in D_n$ such that $g_n(d) = e$. Thus

$$g_{n-1}(\partial^D_n(d)) = \partial^E_n(g_n(d)) = \partial^E_n(e) = 0,$$

thus $\partial_n^D(d) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$. Thus there is $c \in C_{n-1}$ such that $f_{n-1}(c) = \partial_n^D(d)$. Thus define $\delta_n([e]) = [c]$. We now check that this actually defines a homology class (that is, c is a n-cycle in C_{n-1}), but indeed

$$f_{n-2}(\partial_{n-1}^C(c)) = \partial_{n-1}^D(f_{n-1}(c)) = \partial_{n-1}^D(\partial_n^D(d)) = 0$$

and as f is injective, $\partial_{n-1}^C(c) = 0$. We now show that this map is well-defined. First, we show δ_n is independent on the choice of d, so let $d' \in D_n$ be another element such that $g_n(d') = e$. Then $g_n(d - d') = 0$, and so $d - d' \in \ker g_n = \operatorname{im} f_n$. So there is $x \in C_n$ such that $d - d' = f_n(x)$. Thus

$$f_{n-1}(\partial_n^C(x)) = \partial_n^D(f_n(x)) = \partial_n^D(d - d') = \partial_n^D(d) - \partial_n^D(d').$$

Now let $c' \in C_{n-1}$ such that $f_{n-1}(c') = \partial_n^D(d')$. Then

$$f_{n-1}(c - c') = \partial_n^D(d) - \partial_n^D(d') = f_{n-1}(\partial_n^C(x))$$

and as f_{n-1} is injective, we have $c - c' = \partial_n^C(x)$; that is, $c - c'$ is a $(n-1)$-boundary, and so belong to the same homology class. More precisely,

$$[c] = [c' + \partial_n^D(x)] = [c'].$$

Now we show that δ_n is independent on changing e to $e + \partial_{n+1}^E(y)$ for some $y \in E_{n+1}$. We first note that, as g_{n+1} is surjective, there is $w \in D_{n+1}$ such that $g_{n+1}(w) = y$. Thus

$$g_n(\partial_{n+1}^D(w)) = \partial_{n+1}^E(g_{n+1}(w)) = \partial_{n+1}^E(y)$$

and so

$$g_n(d + \partial_{n+1}^D(w)) = g_n(d) + g_n(\partial_{n+1}^D(w)) = e + \partial_{n+1}^E(y).$$

Thus $d + \partial_{n+1}^D(w)$ is the respective d pick for $e + \partial_{n+1}^E(y)$ (as opposed to just e). But observe

$$\partial_n^D(d + \partial_{n+1}^D(w)) = \partial_n^D(d) + \partial_n^D(\partial_{n+1}^D(w)) = \partial_n^D(d)$$

and so $\delta(e + \partial_{n+1}^E(y)) = \delta(e)$. We conclude that $\delta : H_n(E) \to H_{n-1}(C)$ is well-defined.

We now have to show that exactness of the sequence; that is, proving that the long sequence is exact at $H_n(D)$, $H_n(E)$, and $H_n(C)$.

First, we show that the sequence is exact at $H_n(D)$; that is, $\operatorname{im} f_* = \ker g_*$. We first show that $\operatorname{im} f_* \subset \ker g_*$. Let $[d] \in \operatorname{im} f_* \subset H_n(D)$, so there is $c \in C_n$ and $x \in D_{n+1}$ such that $d + \partial_{n+1}^D(x) = f_n(c)$. Thus

$$\begin{aligned} g_*([d]) = [g_n(d)] &= [g_n(d) + \partial_{n+1}^E(g_{n+1}(x))] = [g_n(d) + g_n(\partial_{n+1}^D(x))] \\ &= [g_n(d + \partial_{n+1}^D(x))] = [g_n(f_n(c))] = [0]. \end{aligned}$$

Now we show that $\ker g_* \subset \operatorname{im} f_*$. Let $[d] \in \ker g_* \subset H_n(D)$. That is, there is $x \in E_{n+1}$ such that $g_n(d) = \partial^E_{n+1}(x)$. As g is surjective, there is $d' \in D_{n+1}$ such that $g_{n+1}(d') = x$. Thus

$$g_n(d) = \partial^E_{n+1}(g_{n+1}(d')) = g_n(\partial^D_{n+1}(d'))$$

and so

$$g_n(d - \partial^D_{n+1}(d')) = g_n(d) - g_n(\partial^D_{n+1}(d')) = 0,$$

therefore $d - \partial^D_{n+1}(d') \in \ker g_n = \operatorname{im} f_n$ and so $d - \partial^D_{n+1}(d') = f(c)$ for some $c \in C_n$. Thus

$$[d] = [f(c) + \partial^D_{n+1}(d')] = [f(c)] = f_*([c]) \in \operatorname{im} f_*$$

as required.

Next, we show that the sequence is exact at $H_n(E)$. We start with $\operatorname{im} g_* \subset \ker \delta$. Let $[e] \in \operatorname{im} g_*$; that is, there is $x \in E_{n+1}$ and $d \in D_n$ such that $e + \partial^E_{n+1}(x) = g_n(d)$ with $\partial^D_n(d) = 0$. As δ is well-defined, there is a unique $c \in C_{n-1}$ such that $f(c) = \partial^D_n(d) = 0$ (as defined above), and as f is injective $c = 0$. Thus $\delta_n([e]) = c = 0$; that is, $[e] \in \ker \delta$. Now we show that $\ker \delta \subset \operatorname{im} g_*$. Let $[e] \in \ker \delta \subset H_n(E)$. Let $c \in C_{n-1}$ and $d \in D_n$ such that $\delta([e]) = [c]$, $f_{n-1}(c) = \partial^D_n(d)$, and $g_n(d) = e$ (as above). Then there is $x \in C_n$ such that $c = \partial^C_n(x)$. One may see that $[e] = [g_n(d)] = g_*([d])$ and conclude, but δ is defined on homology classes, that is cycles modulo boundaries, and we cannot say that d is a boundary. Note that

$$\begin{aligned}\partial^D_n(d - f_n(x)) &= \partial^D_n(d) - \partial^D_n(f_n(x)) = \partial^D_n(d) - f_{n-1}(\partial^C_n(x)) \\ &= \partial^D_n(d) - f_{n-1}(c) = 0.\end{aligned}$$

Thus, we have

$$[e] = [g_n(d)] = [g_n(d - f_n(x))] = g_*([d - f_n(x)]).$$

Finally, we show that the sequence is exact at $H_n(C)$, starting with showing $\operatorname{im} \delta \subset \ker f_*$. Let $[c] \in \operatorname{im} \delta \in H_{n-1}(C)$ and $d \in D_n$ and $e \in E_n$ as in the definition of δ. Then

$$f_*([c]) = [f_{n-1}(c)] = [\partial^D_n(d)] = 0$$

as needed. Now $\ker f_* \subset \operatorname{im} \delta$: let $[c] \in \ker f_* \subset H_{n-1}(C)$. That is, there is $x \in D_n$ such that $f_{n-1}(c) = \partial^D_n(x)$. Observe that $\partial^E_n(g_n(x)) = g_{n-1}(\partial^D_n(x)) = g_{n-1}(f_{n-1}(c)) = 0$. Thus $g(x)$ is a n-cycle, and infact $\delta([g_n(x)]) = [c]$ as δ is well-defined.

$$[c] = \delta([g_n(x)]) \in \operatorname{im} \delta$$

and we are done. $\square$

2.4 Homotopy equivalence

Homotopy theory studies objects which may be *continuously deformed* into each other, this deformation is precisely what a *homotopy* is. As always with topology, it is good to have a good intuition about what is going on.

Definition 2.21 (Homotopy)**.** Let $f, g : X \to Y$ be continuous maps between spaces. A *homotopy* from f to g is a map $h : X \times I \to Y$ with $h_(x, 0) = f(x)$ and $h(x, 1) = g(x)$. If such a homotopy exists, we may write $f \sim_h g$ or $f \sim g$.

Example 2.22. Two maps $f, g : \{\text{pt}\} \to X$ are homotopic if and only if $f(\text{pt})$ and $g(\text{pt})$ lay on the same path component of X. We see that any path from the points would define a valid homotopy.

Example 2.23. For any space X and $n \in \mathbb{N}$, any two maps $f, g : X \to \mathbb{R}^n$ are homotopic with the straight line homotopy, define as

$$\begin{aligned} h : X \times I &\to \mathbb{R}^n \\ (x, t) &\mapsto (1 - t)f(x) + tg(x). \end{aligned}$$

Lemma 2.24. *Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ such that $f \sim f'$ and $g \sim g'$. Then $g \circ f \sim g' \circ f'$.*

Homotopy determines an equivalence relation on maps between spaces, and the above lemma shows transitivity (symmetry and reflexivity are clear from the definition). We can also use homotopy to derive an equivalent relation on spaces.

Definition 2.25 (Homotopy equivalent)**.** A map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \sim \text{id}_Y$ and $g \circ f \sim \text{id}_X$. g may be called the [homotopy inverse] of f, and if such functions exist between spaces X and Y, we say that they are *homotopy equivalent*, denoted $X \simeq Y$.

Lemma 2.26. *Let $g, h : Y \to X$ be homotopy inverses of some map $f : X \to Y$. Then $g \sim h$.*

Definition 2.27 (Contractible)**.** A space X is *contactible* if $X \simeq \{\text{pt}\}$.

Example 2.28. $\mathbb{R}^n$ is contractible. Let $f : \mathbb{R}^n \to \{\text{pt}\}$ be defined the only way it can, and $g : \{\text{pt}\} \to \mathbb{R}^n$ with $\text{pt} \to \mathbf{0}$. Then $f \circ g = \text{id}_{\text{pt}}$ and $g \circ f = 0$. Homotopy is an equivalence relation, thus by reflexivity $f \circ g \simeq \text{id}_{\text{pt}}$. For $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, we define the homotopy

$$\begin{aligned} h : \mathbb{R}^n \times I &\to \mathbb{R}^n, \\ (\boldsymbol{x}, t) &\mapsto t\boldsymbol{x}. \end{aligned}$$

Example 2.29. D^n is contractible, and this can be shown in the same way as in (i).

Example 2.30. For all $n \in \mathbb{N}$, $\mathbb{R}^n \setminus \{\mathbf{0}\} \simeq S^{n-1}$. Intuitively, this can be obtained with the inclusion map from S^{n-1} to $\mathbb{R}^n \setminus \{0\}$ and the map $\boldsymbol{x} \mapsto \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ from $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}. The required homotopies can be easily constructed.

Theorem 2.31. *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : H_n(X) \to H_n(Y)$ is an isomorphim for all $n \in \mathbb{Z}_{\geq 0}$.*

So, if spaces X and Y have differing homology, then they are not homotopy equivalent. Using the examples above, we see that

$$H_k(D^n) \cong H_k(\mathbb{R}^n) \cong H_k(\{\mathrm{pt}\}), \qquad H_k(\mathbb{R}^n \setminus \{\mathrm{pt}\}) \cong H_k(S^{n-1})$$

for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geq 0}$. We have the following similar result for fundamental groups.

Theorem 2.32. *Let $f : (X, x) \to (Y, y)$ be a based homotopy equivalence with $f(x) = y$. Then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.*

2.4.1 Mapping cylinders and mapping cones

We now introduce some important homotopy equivalent spaces.

Definition 2.33 (Mapping cylinder)**.** Let $f : X \to Y$ be a map. The *mapping cylinder* of f is

$$M_f = ((I \times X) \sqcup Y)/\!\sim$$

where $\sim$ is the equivalence relation generated by $(0, x) \sim f(x)$ for each $x \in X$, and $\sqcup$ denotes the disjoint union.

Lemma 2.34. *For every map $f : X \to Y$, $M_f \simeq Y$.*

Definition 2.35 (Cone)**.** Let $f : X \to Y$ be a map. The *cone* on a map f is

$$\mathrm{Cone}(f) = C_f = M_f/(X \times \{0\}).$$

One may also view this as the mapping cylinder where the equivalence relation $\sim$ also has $(x, , 0) \simeq (x', 0)$ for all $x \in X$.

Lemma 2.36. *For any space, the cone on the identity map is contractible.*

2.4.2 Retracts

Definition 2.37 (Retract)**.** Let A be a subspace of some space X. Then a map $r : X \to A$ is a *retraction* if $r\big|_A = \mathrm{id}_A$.

Definition 2.38 (Deformation retract)**.** A *deformation retraction* of a space X onto a subspace A is a homotopy $H : X \times I \to X$ between a retraction of X onto A and the identity map on X with $H\big|_{A\times I} = \mathrm{id}_A$.

A retraction is just a map from a space to a subspace that preserves the positions of all points in that subspace, while a deformation retraction is a mapping that captures the concept of continuously shrinking a space into a subspace.

It is clear that a deformation retract defines a retract, but it is not always true that a retract is homotopic to the identity on the space.

Remark. A deformation retract may be defined without the need of leaving points in A fixed throughout the homotopy, and to call the above a *strong deformation retract*.

2.5 Chain homotopy

Chain homotopies are maps between chains that act in a particular nice way; modelling the behaviour of homotopies.

Definition 2.39 (Chain homotopy)**.** Two chain maps $f, g : C_* \to D_*$ are said to be *chain homotopic* if there exists a homomorphism $P_n : C_n \to D_{n+1}$ for each $n \in \mathbb{Z}$ such that

$$f_n - g_n = \partial^D_{n+1} \circ P_n + P_{n-1}\partial^C_n.$$

If f and g are chain homotopic, we write $f \sim g$.

Note that in the following diagrams, we make no statement of commutation.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"] \arrow[dl] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "f_n-g_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}-g_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] & D_n \arrow[r, "\partial^D_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 2.40. *If chain maps $f \sim g : C_* \to D_*$, then $f_* = g_* : H_n(C_*) \to H_n(D_*)$ for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let c be a cycle of C_n. Then $\partial c = 0$. Thus

$$f_*([c]) = [f(c)] = [g(c) + \partial Pc - P\partial c] = [g(c) + P(0)] = [g(c)] = g_*([c])$$

and so $f_* = g_*$. $\square$

Theorem 2.41. *If two maps between spaces are homotopic, then their induced maps on the chain groups are chain homotopic.*

Proof. ☐

Corollary 2.42. *If two maps between spaces are homotopic, then their induced maps on the homology classes are chain homotopic.*

A *chain homotopy equivalence* between two chain complexes is defined as one may expect, and similarly C_* is said to be *chain contractible* if $C_* \simeq 0$.

Lemma 2.43. *If two spaces are homotopy equivalent, then their chain groups are chain homotopy equivalent.*

Proof. This is an immediate consequence of the above theorem: let f and g be witnesses to the homotopy equivalence. Then f_* and g_* witness the chain homotopy equivalence between the corresponding chain groups. ☐

Lemma 2.44. *If two chain groups and chain homotopy equivalent, then their homology groups are isomorphic.*

Proof. If f_* and g_* are witnesses to the chain homotopy equivalences then $f_* \circ g_*, g_* \circ f_* \sim \mathrm{id}_*$. But by an earlier proposition, these maps must be equal to the identity map. Hence they are inverses of each other and we have established the isomorphism. ☐

Corollary 2.45. *If two spaces are homotopy equivalent, then their homology groups are isomorphic.*

2.6 Mayer-Vietoris sequence

We have already seen that a short exact sequence of chain complexes induces a long exact sequence in homology, so now we need a choice of short exact sequence on a topological space.

Theorem 2.46. *Let X be a space and $U, V \subset X$ such that $X = \mathring{U} \cup \mathring{V}$ and*

$$C_*^{\{U,V\}}(X) = \left\{ \sum_i n_i \sigma_i : \sigma_i(\Delta^n) \subset U \text{ or } \sigma_i(\Delta^n) \subset V \right\}.$$

Then $C_*^{\{U,V\}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence.

Theorem 2.47 (Mayer-Vietoris)**.** *Let X be a space and $U, V \subset X$ such that $X = \mathring{U} \cup \mathring{V}$ and let $\mathcal{U} = \{U, V\}$. Then there is a short exact sequence of chain complexes*

$$0 \to C_*(U \cap V) \xrightarrow{\varphi} C_*(U) \oplus C_*(V) \xrightarrow{\psi} C_*^{\{U,V\}}(X) \to 0$$

where $\varphi(x) = (x, -x)$ *and* $\psi(u, v) = u + v$. *Such a sequence induces the long exact sequence in homology:*

$$\ldots \to H_{n+1}(X) \xrightarrow{\delta} H_n(U \cap V) \xrightarrow{\varphi_*} H_n(U) \oplus H_n(V) \xrightarrow{\psi_*} H_n(X) \to \ldots$$

Proof. We have exactness at $C_*(U \cap V)$ as a chain of $U \cap V$ that is the zero chain in U must be the zero chain in $U \cap V$. Let x be a chain of $U \cap V$. Then $(\psi \circ \varphi)(x) = \psi(x, -x) = 0$, thus $\operatorname{im} \varphi \subset \ker \psi$. Now let $u, v \in \ker \psi$. Then $u = -v$, and so $\varphi(u) = (u, -u) = (u, v) \in \operatorname{im} \varphi$. Thus the sequence is exact at $C_*(U) \oplus C_*(V)$. Now, finally we observe that $\operatorname{im} \psi = C_*^{\{U,V\}}$ by definition, and so the sequence is exact. We have already seen how a long exact sequence in homology can be induced from a short exact sequence, and the last piece we need is to see that $H_n^{\{U,V\}}(X) \cong H_n(X)$ by the earlier theorem. □

2.7 Examples

Example 2.48. Consider S^1 as the unit circle in $\mathbb{C}$ parametrised via $f : [0, 1] \to S^1$, $t \mapsto e^{2\pi i t}$. We let $U = f([0, 1/2) \cup (1/2, 1])$ and $V = f((0, 1))$. U and V meet the stipulation of the Mayer-Vietoris sequence, we also note that U and V are contractible and thus

$$H_k(U) \cong H_k(V) \cong \begin{cases} \mathbb{Z} & k = 0, \\ 0 & \text{else.} \end{cases}$$

We observe that $U \cap V$ is disconnected, with two components $f(0, 1/2)$ and $f(1/2, 1)$, both of which are contractible. Thus $U \cap V \simeq \{\text{pt}\} \sqcup \{\text{pt}\}$ and hence

$$H_k(U \cap V) \cong \begin{cases} \mathbb{Z}^2 & k = 0, \\ 0 & \text{else.} \end{cases}$$

The Mayer-Vietoris sequence yields

$$0 \to H_1(S^1) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(S^1) \to 0$$

which is isomorphic to

$$0 \to H_1(S^1) \to \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \to H_0(S^1) \to 0.$$

We wish to understand the map A, which records how the inclusion induced maps send the connected components of $U \cap V$ to the connected components of U and V. U and V are both connected, thus $\iota_U : H_0(U \cap V) \to H_0(U)$ must send $(a, b) \mapsto a + b$ and similarly $\iota_V : H_0(U \cap V) \to H_0(V)$ must do the same. A is recording the map $(\iota_U, -\iota_V)$, thus (with the canonical bases) we represent A with the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

It is clear to see that $H_1(S^1) \cong \ker A \cong \mathbb{Z}$ and similarly $H_0(S^1) \cong \operatorname{coker} A \cong \mathbb{Z}$.

Example 2.49.

Consider $S^n \subset \mathbb{R}^{n+1}$ as the unit sphere and take $U = S^n \setminus \{1, 0, \dots, 0\}$ and $V = S^n \setminus \{(-1, 0, \dots, 0)\}$, clearly these set us up for Mayer-Vietoris and we also see that $U \cong V \cong \mathbb{R}^n \simeq \{pt\}$. Also observe that $U \cap V \cong \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$. Looking at the Mayer-Vietoris sequence we get

$$\dots \to 0 \to H_k(S^n) \to H_{k-1}(S^{n-1}) \to 0 \to \dots$$

for $k > 1$ and so $H_k(S^n) \cong H_{k-1}(S^{n-1})$. By induction, we find that

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & k \in \{0, n\}, \\ 0 & else \end{cases}$$

although this result is not complete without a bit of sniffing around the end of the Mayer-Vietoris sequence.

3 Applications

3.1 Applications of homology

As we have seen through our development of homology theory, it allows us to make statements about general topology.

Theorem 3.1. $\mathbb{R}^n \cong \mathbb{R}^m$ *if and only if* $n = m$.

Proof. If $n = m$, then $\mathbb{R}^n \cong \mathbb{R}^m$. For the converse, suppose $n \neq m$. If $n = 0$ or $m = 0$, then it is clear that $\mathbb{R}^n \ncong \mathbb{R}^m$. So we suppose that $n, m \geq 1$. For a contradiction, suppose $\mathbb{R}^n \cong \mathbb{R}^m$. Then

$$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$$

and it follows that $H_k(S^{n-1}) \cong H_k(S^{m-1})$ for all k; but we have seen this is not true for $n, m \geq 2$. But we also note that $H_0(S^0) \cong \mathbb{Z}^2$, and thus we have reached a contradiction. □

Theorem 3.2. *Let* $n \in \mathbb{Z}_{\geq 0}$ *and* $f : D^n \to D^n$ *be a continuous map. Then* f *has a fixed point; that is, there is* $x \in D^n$ *such that* $f(x) = x$.

Proof. If $n = 0$, we are done. Suppose $n > 0$ and suppose for a contradiction that f has no fixed point. For each $x \in D^n$, we consider the unique line through the points x and $f(x)$. This line intersects S^{n-1} at two points, let $r(x)$ be the point closest to x. It is clear that $r(x) = x$ if $x \in S^{n-1}$, so $r : D^n \to S^{n-1}$ defines a retraction. Now, we observe that D^n is contractible, so for $n \geq 2$, $H_{n-1}(D^n) \cong 0$ but $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, thus this cannot be the case. If $n = 1$, then $H_0(S^0) \cong \mathbb{Z}^2$ while $H_0(D^1) \cong \mathbb{Z}$, which also cannot be the case as there is no injective homomorphism $\mathbb{Z} \to \mathbb{Z}^2$. □

3.2 Split short exact sequences

Definition 3.3 (Split short exact sequence)**.** A short exact sequence of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *split* if there is a homomorphism $s : C \to B$ with $g \circ s = \mathrm{id}_C$.

Proposition 3.4. *For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following are equivalent.*

1. *There is a homomorphism $p : B \to A$ with $p \circ f = \mathrm{id}_A$.*
2. *There is a homomorphism $s : C \to B$ with $g \circ s = \mathrm{id}_C$.*
3. *There is an isomorphism $\theta : A \oplus C \to B$ such that the diagram*

$$\begin{tikzcd}
 & & A \oplus C \arrow[dd, "\cong"', "\theta"] \arrow[dr, "p_2"] & & \\
0 \arrow[r] & A \arrow[ur, "i_1"] \arrow[dr, "f"'] & & C \arrow[r] & 0 \\
 & & B \arrow[ur, "g"'] & &
\end{tikzcd}$$

commutes, where $i_1 : a \mapsto (a, 0)$ and $p_2 : (a, c) \mapsto c$.

Proof. □

Lemma 3.5. *Consider the short exact sequence $0 \to A \to B \to \mathbb{Z}^m \to 0$ for some $m > 0$. Then this short exact sequence splits, so $B \cong A \oplus \mathbb{Z}^m$.*

Proof. For each $e_i \in \mathbb{Z}^m$, choose $b_i \in B$ mapping to e_i. Then we simply map $\sum_i n_i e_i \mapsto \sum_i n_i b_i$. □

4 Variations of homology

4.1 Reduced homology

Reduced homology is a minor modification to homology theory, and it allows us to make much cleaner statements. It is perhaps a more intuition form, in which a point has trivial reduced homology groups.

Definition 4.1 (Augmented chain complex)**.** Define the *augmented chain complex* of a space X, denoted $\tilde{C}_*(X)$, to be

$$\ldots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \ldots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon : C_0(X) \to \mathbb{Z}$, $\sum_i \sigma_i \mapsto \sum_i n_i$.

Definition 4.2 (Reduced homology)**.** The *reduced homology* of a space is $\tilde{H}_n(X) = H_n(\tilde{C}_*(X))$ for each $n \in \mathbb{Z}_{\geq 0}$.

We now just confirm that this is indeed a minor modification, and reduced homology theory acts much in the same way as regular homology theory.

Proposition 4.3.

1. *A map $f : X \to Y$ between spaces induces a chain map between augmented chain groups, and thus homomorphisms between reduced homology classes.*
2. *If two maps between spaces are homologous, then so are their induced chain maps between augmented chain groups, and thus the homomorphisms between reduced homology classes are equal.*
3. $\tilde{H}_k(\varnothing) = \begin{cases} \mathbb{Z} & k = -1, \\ 0 & \textit{else.} \end{cases}$
4. *If X is non-empty, then*

$$H_k(X) = \begin{cases} \tilde{H}_k(X) & k > 0, \\ \tilde{H}_0(X) \oplus \mathbb{Z} & k = 0. \end{cases}$$

4.2 Tensor products

Definition 4.4 (Tensor product)**.** Let A and B be abelian group. The *tensor product* is a quotient of the free abelian group generated by symbols of the form $a_i \otimes b_i$, with $a_i \in A$ and $b_i \in B$

$$A \otimes B = \left\{ \sum_i a_i \otimes b_i \right\} / \sim$$

where the relation $\sim$ is generated by $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$.

We observe that $a \otimes 0 = 0 \otimes a = 0$. We also have the following properties.

1. $A \otimes B \cong B \otimes A$.
2. $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.
3. $\mathbb{Z} \otimes A \cong A$.
4. $\mathbb{Z}/k \otimes A \cong A/kA$.

5. $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong (\mathbb{Z}/n)/(m\mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n)$.
6. If A is a finitely generated abelian group and $A \cong \mathbb{Z}^r \oplus TA$, then $\mathbb{Q} \otimes A \cong \mathbb{Q}^r$.

4.3 Homology with coefficients

Let A, B, and C be abelian groups. For a homomorphism $f : A \to B$, we observe that there is an inuced homomorphism $f \otimes \mathrm{id} : A \otimes C \to B \otimes C$ such that $\sum_i a_i \otimes c_i \mapsto \sum_i f(a_i) \otimes c_i$. We simialrly have a map $\mathrm{id} \otimes f$.

Now let $C = (C_*, \partial_*)$ be a chain complex of abelian groups, and let G be some abelian group. Then we construct the chain complex

$$\ldots \to C_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}} C_{n-1} \otimes G \to \ldots \to 0$$

which is easy to check that it is a chain complex.

Definition 4.5 (Homology with coefficients)**.** The nth homology of a chain complex C_* with coefficients in an abelian group G is $H_n(C_*; G) = H_n(C_* \otimes G)$.

Note that $H_n(C_* \otimes G)$ may not equal $H_n(C_*) \otimes G$, unless G is a field.

Definition 4.6. For a space X, we define $H_n(X; G) = H_n(C_*(X); G)$.

The homology theory we have developed coincides with homology with coefficients in $\mathbb{Z}$.

Lemma 4.7. *Let X be a space and $n \in \mathbb{Z}_{\geq 0}$. The map $C_n(X) \to C_n(X) \otimes \mathbb{Z}$ such that $\sigma \mapsto \sigma \otimes 1$ induces a chain isomorphism and therefore an isomorphism on homology; that is, $H_n(X) \cong H_n(X; \mathbb{Z})$.*

As one may expect, homology with coefficients is invariant under homotopy equivalence, and the Mayer-Vietoris sequence is exact. Commonly used coefficients are $\mathbb{Q}$, $\mathbb{Z}/m$, and $\mathbb{R}$.

Example 4.8. For an abelian group A, $H_k(S^n; A) = \begin{cases} A & k \in \{0, n\}, \\ 0 & \text{else.} \end{cases}$

Example 4.9. We now look at the homology groups for $\mathbb{RP}^2$ and see how they change over commonly chosen coefficients. We can compute the homology of $\mathbb{RP}^2$ by considering the union of a Mobius band and a 2-disk, where the intersection is homotopy equivalent to the common S^1 boundary.

$$H_k(\mathbb{RP}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0, \\ \mathbb{Z}/2 & k=1, \\ 0 & \text{else.} \end{cases}$$

$$H_k(\mathbb{RP}^2;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=0, \\ 0 & \text{else.} \end{cases}$$

$$H_k(\mathbb{RP}^2;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k \in \{0,1,2\}, \\ 0 & \text{else.} \end{cases}$$

$$H_k(\mathbb{RP}^2;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & k=0, \\ 0 & \text{else.} \end{cases}$$

4.4 Relative homology

We eventually move to cellular homology, but to do this we need the tool of *relative homology.*

Definition 4.10 (Relative homology)**.** For a space X and subspace $A \subset X$, define

$$C_n(X,A) = \frac{C_n(X)}{C_n(A)}.$$

The boundary map on $C(X)$ induces a well-defined boundary map on $C(X,A)$ (as you may expect). We then define

$$H_n(X,A) = H_n(C(X,A)).$$

This is another generalisation (as with coefficients) as $H_n(X,\varnothing) \cong H_n(X)$ for every $n \in \mathbb{Z}_{\geq 0}$. Interestingly, let $X \neq \varnothing$ and let $x_0 \in X$. Then $H_n(X,\{x_0\}) \cong \tilde{H}_n(X)$ for every $n \in \mathbb{Z}_{\geq 0}$. The map $\sum_i n_i\sigma_i \mapsto \sum_i n_i\sigma_i - (\sum_i n_i)[x_0]$ induces the isomorphism on the zeroth homology group.

The relative homology $H_n(X,A)$ may be used to help us understand the homology of A or X.

Theorem 4.11. *Let X be a space and $A \subset X$ a subspace.*

1. *There is a short exact sequence of chain complexes*

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{q} C_*(X,A) \to 0$$

with associated long exact sequence in homology

$$\ldots \to H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \to \ldots$$

2. *If $f : (X, A) \to (Y, B)$ is a map of pairs (that is, $f(A) \subset B$), then there is an induced map $f_* : H_n(X, A) \to H_n(Y, B)$ for all $n \in \mathbb{Z}_{\geq 0}$ such that the diagram*

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & \dots \\
\dots \arrow[r] & H_{n+1}(Y,A) \arrow[r, "\delta"] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "q_*"] & H_n(Y,B) \arrow[r] & \dots
\end{tikzcd}$$

commutes.

The property of a map of pairs outlined in (ii) in the above theorem is known as *naturality*, and this property may become useful in later reasoning.

Example 4.12. Let $A \subset X$ and suppose that the inclusion $A \hookrightarrow X$ is a homotopy equivalence. Then every $H_n(X, A)$ is trivial.

Example 4.13. Let $a, b \in \mathbb{R}^2$ be distinct. First, we see that $H_n(\mathbb{R}^2, \{a\})$ is trivial for every n. In contrast, $H_1(\mathbb{R}^2, \{a, b\}) \cong \mathbb{Z}$ but $H_k(\mathbb{R}^2, \{a, b\})$ is trivial for $k \neq 1$. Interesting, in both cases the zeroth homology group vanishes.

Example 4.14. $H_k(D^n, S^{n-1})$ is trivial unless $k = n$, in which case it is isomorphic to $\mathbb{Z}$.

4.5 Excision

Relative homology has a useful trick in we can remove a subsets from a pair without changing their relative homology.

Theorem 4.15. *Let $Z \subset A \subset X$ be subsets such $\overline{Z} \subset \mathring{A}$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ for every $n \in \mathbb{Z}_{\geq 0}$.*

This is also equivalent to the following theorem.

Theorem 4.16. *Let $U, V \subset X$ be subsets of X such that $\mathring{U} \cup \mathring{V} = X$. Then the inclusion $(V, U \cap V) \hookrightarrow (X, U)$ induces an isomorphism $H_n(V, U \cap V) \to H_n(X, U)$ for every $n \in \mathbb{Z}_{\geq 0}$.*

The first theorem is the on to use, but we will prove the second.

Proof. □

We recall that a manifold is a second countable Hausdourff space that is locally homeomorphic to Euclidean space. By second countable, we mean the base (of open sets) is countable. By locally homeomorphic, we mean that every point has a neighbourhood homeomorphic to the open Euclidean n-ball. A surface is a 2-dimensional manifold, and a closed surface is a surface that is compact and without boundary.

Example 4.17. Let Σ be a closed surface and let $p \in \Sigma$. By excision, taking $X = \Sigma$ and $A = \Sigma \setminus \{p\}$, and $Z = \Sigma \setminus U$ (where U is a neighbourhood of p homeomorphic to $\mathbb{R}^2$) we get that $H_2(\Sigma, \Sigma \setminus \{p\}) \cong H_2(U, U \setminus \{p\}) \cong H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$. Then, using our tools from relative homology we have

$$\ldots \to H_2(\mathbb{R}^2) \to H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}) \to H_1(\mathbb{R}^2 \setminus \{0\}) \to H_1(\mathbb{R}^2) \to \ldots$$

which is equivalent to

$$0 \to H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}) \to H_1(\mathbb{R}^2 \setminus \{0\}) \to 0.$$

So $H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}) \cong H_1(S^1) \cong \mathbb{Z}$. This is called the *local homology* of the surface at p.

5 Cellular homology

5.1 Good pairs

Definition 5.1 (Good pair)**.** Let $A \neq \varnothing$ be a closed subset of some space X. Suppose A is a deformation retract of some neighbourhood $V \subset X$ of A, so $A \subset V$.

Example 5.2. Consider $X = \mathbb{R}^2$ and $Y = \{(x, y) \in x : y \leq 0\}$. We present a retract $r : V \to Y$ of V onto Y, where $V = \{(x, y) \in X : y \leq 1\}$:

$$r(x, y) = \begin{cases} (x, y) & y \leq 0, \\ (x, 0) & y \in (0, 1]. \end{cases}$$

There is a clear homotopy $H : V \times I \to V$ from r to id_X,

$$H((x, y), t) = \begin{cases} (x, y) & y \leq 0, \\ (x, ty) & y \in (0, 1]. \end{cases}$$

We recall the Hawaiian earring space $\mathbb{H}$, given by

$$\mathbb{H} = \bigcup_{n=1}^{\infty} \left\{(x, y) \in \mathbb{R}^2 : (x - 1/n)^2 + y^2 = (1/n)^2\right\}$$

and endowed with the subspace topology from the standard topology on $\mathbb{R}^2$.

Example 5.3. We claim the pair $(\mathbb{H}, \{(0, 0)\})$ is *not* a good pair. If we look at any neighbourhood of $(0, 0)$, we see that it would contain infinitely many circles, and thus it must not be homotopy equivalent to a point.

Theorem 5.4. *Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, \{\mathrm{pt}\})$ induces an isomorphism*

$$q_* : H_n(X, A) \xrightarrow{\cong} H_n(X/A, \mathrm{pt}) \cong \tilde{H}_n(X/A).$$

Proof. We consider the following diagram.

$$\begin{tikzcd}
H_n(X/A) \arrow[r, "\cong"] \arrow[d, "q_*"] & H_n(X, V) \arrow[d, "q_*"] & H_n(X \setminus A, V \setminus A) \arrow[l, "\cong"'] \arrow[d, "q_*", "\cong"'] \\
H_n(X/A, \{\mathrm{pt}\}) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n((X/A) \setminus \{\mathrm{pt}\}, (V/A) \setminus \{\mathrm{pt}\}) \arrow[l, "\cong"']
\end{tikzcd}$$

Since (X, A) is a good pair, there is a neighbourhood V of A with $A \simeq V$ (convince yourself that such a homotopy exists). So $H_n(X, A) \cong H_n(X, V)$ and $H_n(X/A, \{\mathrm{pt}\}\}) \cong H_n(X/A, V/A)$ (again, convince yourself of this). Thus the left two horizontal maps are indeed isomorphisms. We can conclude the same for the right two maps by excision. We have left to show that $q_*\big|_{X\setminus A} : H_n(X \setminus A, V \setminus A) \to H_n((X \setminus A)/A, (V \setminus A)/A) = H_n((X/A) \setminus \{\mathrm{pt}\}, (V/A) \setminus \{\mathrm{pt}\})$ is an isomorphism. But $q : (X, A) \to (X/A, \{\mathrm{pt}\})$ is an isomorphism on $X \setminus A$ and so $V \setminus A$ too, as so $q_*\big|_{X\setminus A}$ is an isomorphism. The above diagram commutes as for each square the two routes around the square are induced by the same map of pairs. We conclude that q_* is an isomorphism. □

5.2 CW complex

Definition 5.5 (CW complex)**.** A *CW complex* is a space X with a filtration

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \dots$$

and a collection of maps $\{\varphi_i : S^{n-1} \to X_{n-1}\}_{i \in \mathcal{I}_n}$ called the *attaching maps* that satisfy

1. for all $n \in \mathbb{N}$,
$$X_n = \left(X_{n-1} \sqcup \bigsqcup_{i \in \mathcal{I}_n} D_i^n \right) /\!\sim$$
where $\varphi_i(x) \sim x$ for all $i \in \mathcal{I}_n$ and $x \in \partial D_i^n \cong S^{n-1}$; and
2. $X = \operatorname{colim}_n X_n = \bigcup_{n \in \mathbb{Z}_{\geq 0}} X_n$.

We call X_n the *n-skeleton* of X_n.

Definition 5.6.

1. A CW structure on a space Y is a CW complex X and a homeomorphism $f : X \xrightarrow{\cong} Y$.

2. The dimension of X is -1 if $X = \varnothing$, and is n if this is the largest n such that $X_{n-1} \neq X$. If no such n exists, then the dimension is ∞.
3. A CW complex X is called finite dimensional if $\dim X < \infty$.
4. A CW complex of dimension $\leq n$ is called an n-complex.
5. A CW complex is finite if it has finitely many cells.
6. The maps $\overline{\varphi}_i : D^n \to X$ extending the attaching maps are the characteristic maps.
7. The image of each copy of D^n under $\overline{\varphi}_i$ is called an n-cell, and the image $\overline{\varphi}_i(\mathring{D}^n)$ is called an open n-cell.
8. A subcomplex of a CW complex X is a subset $Y \subset X$ that is the union of cells of X.

Example 5.7 (0-complexes)**.** Every collection of discrete points is a 0-complex.

Example 5.8 (1-complexes)**.** The following are some examples of 1-complexes.

- A CW structure on I can be constructed in two ways.
  - Take two 0-cells and one 1-cell. We glue one endpoint of the 1-cell to one of the 0-cells, and the other endpoint to the other 0-cell.
  - We can also construct I as just one 1-cell, with no 0-cells.
- We can also construct a CW structure on S^1 in two ways.
  - Take one 0-cell and one 1-cell, and glue both endpoints of the 1-cell to the 0-cell.
  - Take two 0-cells and two 1-cells. We glue one endpoint of each 1-cell to one 0-cell, and the other endpoints to the other 0-cell.
- We can build a CW structure over any graph, where the 0-cells are the vertices and the 1-cells are the edges. We identify the endpoints of edges to the vertices adjacent to it.

We introduce the notion of *wedge sum.* If X and Y are *pointed spaces*, the wedge sum of X and Y, denoted $X \vee Y$, is the quotient space of the disjoint union of X and Y with the basepoints of X and Y identified. For example, the wedge sum of two circles $S_1 \vee S_1$ is the appropriately named *figure-eight space.*

Example 5.9 (2-complexes)**.** The following are some examples of 2-complexes.

- A CW structure on S^n can be constructed with two cells, one 0-cell and one n-cell. We identify the boundary of the n-cell to the single 0-cell. For intuition, consider what is happening on S^2. We take a disk and identify the boundary S^1 with a point, wrapping the disk around into S^2.
- We describe a CW structure on $\mathbb{T}$. It can be built as a union of one 0-cell, two 1-cells, and one 2-cell. The 1-skeleton is a wedge $S^1 \vee S^1$, which we may build by identifying all the endpoints of the 1-cells to the 0-cell. To attach the 2-cell, we glue the boundary to a loop in $S^1 \vee S^1$ representing the commutator $aba^{-1}b^{-1} \in \pi_1(S^1 \vee S^1)$, which is a free group on two generators a and b corresponding to generators of the fundamental groups of the two individual circles in the wedge.

Definition 5.10.

1. A space X is Hausdorff if for all $x, y \in X$ there exists open subsets $U, V \subset X$ with $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$.
2. A space X is normal if for every pair of disjoint closed sets $S, T \subset X$, there exists open subsets $U \supset S$ and $V \supset T$ with $U \cap V = \varnothing$.

Theorem 5.11. *Let X be a CW complex and let A be a non-zero subcomplex.*

1. *Every point of X is closed.*
2. *A CW complex X is normal.*
3. *Every cell of X is closed and compact.*
4. *A CW complex is compact if and only if it is finite.*
5. *A CW complex is connected if and only if it is path connected.*
6. *(X, A) is a good pair.*

We note that since points are closed, normal implies Hausdorff.

Example 5.12 (Non-examples)**.**

- The space $\mathbb{E}$ is not a CW complex. We will not prove this, but assume there is a CW complex X consisting of one 0-cell and countably many 1-cells. Consider the 0-cell as a subcomplex A. Then (X, A) must be a good pair; a contradiction (we have already shown that this cannot be a good pair).
- There are many spaces for which we cannot build a CW structure; for example, any non-Hausdorff space or any space in which a point is not closed.

5.3 Degree of maps

Definition 5.13 (Degree of a map)**.** Let $f : S^n \to S^n$ be a map, which induces a homomorphism on the nth homology groups $f_* : H_n(S^n) \to H_n(S^n)$. As $H_n(S^n) \cong \mathbb{Z}$, this determines a homomorphism from $\mathbb{Z} \to \mathbb{Z}$, hence $f_*(a) = da$ for some $d \in \mathbb{Z}$. This d is called the degree of f.

Let us characterise the behaviour of such a notion. We say that a map f *factors* over a map g if there exists a map h such that $f = h \circ g$. We may also that f factors if there exists g and h such that $f = h \circ g$.

1. $\deg(\mathrm{id}_{S^n}) = 1$.
2. If f is not surjective, then $\deg f = 0$. To see this, suppose $x \in S^n$ such that there is no $y \in S^n$ such that $f(y) = x$. Then we see that f factors as $S^n \to S^n \setminus \{x\} \hookrightarrow S^n$, and so f_* factors as $H_n(S^n) \to H_n(S^n \setminus \{x\}) \hookrightarrow H_n(S_n)$. But we see $S^n \setminus \{0\} \cong \mathbb{R}^n$ and so, if $n > 0$, $H_n(S^n \setminus \{x\}) \cong 0$. Thus f_* must be the zero map, and so $\deg f = 0$.
3. If $f \sim g$ (homotopic), then $f_* = g_*$ so $\deg f = \deg g$.
4. $(f \circ g)_* = f_* \circ g_*$ so $\deg(f \circ g) = \deg(f) \deg(g)$.
5. If f is a homotopy equivalence with homotopy inverse g, then $\deg f = \pm 1$ (as $\deg(f) \deg(g) = 1$).

Definition 5.14 (Suspension)**.** The suspension SX of a space X is the quotient space

$$SX = (X \times I)/{\sim}$$

where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. Given a map $f : X \to Y$ between two spaces, the suspension map $Sf : SX \to SY$ is defined by

$$Sf([x, t]) = [f(x), t].$$

Intuitively, the suspension of a space is obtained by stretching X out and collapsing both end faces to points. We recall that the *cone* of a space X is the quotient space $CX = (X \times I)/{\sim}$ where $(x, 0) \sim (y, 0)$ for all $x, y \in X$.

Example 5.15. We show that $SS^n \cong S^{n+1}$ (the cone of S^n is homeomorphic to S^{n+1}). We let $I = [-1, 1]$, $S^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|x\| = 1\}$, and we first define a map $f : S^n \times I$ such that $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}\sqrt{1 - t^2}, t)$. We see that is surjective on $S^{n+1} \setminus (\mathbb{R}^{n+1} \times \{-1, 1\})$, as

$$f\left(\frac{\boldsymbol{y}}{\sqrt{1 - y_{n+2}^2}}, y_{n+2}\right) = \boldsymbol{y}$$

for all $\boldsymbol{y} \in S^{n+1} \setminus (\mathbb{R}^{n+1} \times \{-1, 1\})$. We now define the induced map $\overline{f} : SS^n \to S^{n+1}$ as $[\boldsymbol{x}, t] \mapsto f(\boldsymbol{x}, t)$ and claim that this factors as the composition of the quotient map $q : S^n \times I \to SS^n$ such that $(\boldsymbol{x}, t) \mapsto [\boldsymbol{x}, t]$ and $\overline{f}$. Indeed, $f(\boldsymbol{x}, 1) = f(\boldsymbol{y}, 1)$ and $f(\boldsymbol{x}, -1) = f(\boldsymbol{y}, -1)$ for all $\boldsymbol{x}, \boldsymbol{y} \in S^n$. If $f(\boldsymbol{x}, t) = f(\boldsymbol{y}, s)$, then $\boldsymbol{x} = \boldsymbol{y}$ and $s = t$, or $s = t \in \{-1, 1\}$. So, although f is not injective, $\overline{f}$ is. f is surjective on $S^{n+1} \setminus (\mathbb{R}^{n+1} \times \{-1, 1\})$, but we note that all points in $S^{n+1} \cap (\mathbb{R}^{n+1} \times \{1\})$ belong to the same equivalence class of SS^n. Namely, $[\boldsymbol{x}, 1] = [\boldsymbol{0}, 1]$ and $\overline{f}(\boldsymbol{0}, 1) = (\boldsymbol{0}, 1)$. Similarly, $f(\boldsymbol{0}, -1) = (\boldsymbol{0}, -1)$, and so $\overline{f}$ is surjective.

Proposition 5.16. *Let X and Y be spaces. For $i \in \mathbb{Z}_{\geq 0}$, there is a natural isomorphism $\tilde{H}_{i+1}(SX) \xrightarrow{\cong} \tilde{H}_i(X)$ in the sense that for any map $f : X \to Y$, the following diagram commutes.*

$$\begin{tikzcd}
\tilde{H}_{i+1}(SX) \arrow[r, "\cong"] \arrow[d, "Sf_*"] & \tilde{H}_i(X) \arrow[d, "f_*"] \\
\tilde{H}_{i+1}(SY) \arrow[r, "\cong"] & \tilde{H}_i(Y)
\end{tikzcd}$$

Proof. □

Corollary 5.17. *Let $f : S^n \to S^n$ be a map. Then $\deg Sf = \deg f$*

Example 5.18. We consider the reflection $R_n : S^n \to S^n$ through S^{n-1} living on the equator. We note that $SR_n = R_{n+1}$ for $n \in \mathbb{N}_0$. From this, we see that

$$\deg(R_n) = \deg(SR_{n-1}) = \deg(R_{n-1}) = \ldots = \deg(R_0) = -1.$$

The base case is something for you to verify.

Example 5.19. The antipodal map $a : S^n \to S^n$, $x \mapsto -x$ has degree $(-1)^{n+1}$. This can be seen by noting that this map can be factored as a series of reflections, one for each hyperplane.

Lemma 5.20. *If $f : S^n \to S^n$ has no fixed points then $f \cong a$, so $\deg(f) = (-1)^{n+1}$.*

Proof. A homotopy $h : S^n \times I \to S^n$ can be constructed by

$$h(x, t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

with addition in $\mathbb{R}^{n+1}$. It is an exercise to prove that this homotopy is well-defined. □

Theorem 5.21 (Hairy Ball theorem)**.** *If n is even, then for every continuous vector field v on S^n, there is a point of S^n at which v vanishes.*

Proof. We prove this by considering the contrapositive of the statement. We form $w(x)$ as the associated unit vector of $v(x)$. We construct a homotopy from the the identity on S^n to the antipodal map. We then compare the degree of the antipodal map and the identity map and conclude that n must be odd. $\square$

5.4 Local degree

We need a technique for computing the degrees of map. We suppose $n > 0$ and $f : S^n \to S^n$ is a continuous map, and that for some point $y \in S^n$, $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$. We let $U_1, \dots, U_m$ be disjoint n-disc neighbourhoods of the x_i, and let V be an n-disc neighbourhood of y such that $f(U_i) \subset V$ for each i. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \setminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"] \arrow[ul, "\cong"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

All maps are given by inclusions, quotients, or f. The top isomorphisms come from excision, and the bottom isomorphisms come from the long exact sequence of pairs. We observe that $H_n(U_i, U_i \setminus \{x_i\})$ and $H_n(V_i, V_i \setminus \{x_i\})$ are both canonically identified with $H_n(S^n) \cong \mathbb{Z}$, and this motivates the following definition.

Definition 5.22 (Local degree)**.** Let f be as above. The *local degree* of f at x_i, denoted $\deg(f|x_i)$ is the degree of the map $H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{f_*} H_n(V_i, V_i \setminus \{x_i\})$.

Proposition 5.23. $\deg f = \sum_i \deg(f|x_i)$.

Proof. This can be shown by identifying certain homology groups with $\mathbb{Z}$, and then morphisms become identities. We then argue on the commutativity of the squares. $\square$

Example 5.24. If f is a homeomorphism, every arrow on the diagram becomes an homeomorphism and so $\deg f = \deg(f|x)$ for all $x \in X$.

Example 5.25. We consider $S^1 \subset \mathbb{C}$, and define $f : S^1 \to S^1$ by $f_k(z) = z^k$ for $k > 0$. By setting $y = 0$, we have the pre-image points $x_1, \dots, x_n$. For each i, the restriction $f\big|_{U_i} : U_i \to V$ is homotopic to $r_\theta\big|_{U_i} L U_i \to V$, where

r_θ is a rotation of S^1 through angle θ. Thus $\deg(f_k|x_i) = \deg(r_\theta|x_i) = 1$, since r_θ is a homeomorphism (we note that these homotopies are not global, just their local restrictions). By our proposition, $\deg(f_k) = k$.

Example 5.26. By taking repeated suspensions of the above map, we can construct a map of any degree from $S^m \to S^m$ for every m.