On the Tutte polynomial

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Definition 1 (Tutte polynomial)**.** Let $G = (V, E)$ be an undirected graph. For $A \subset E$, denote $k(A)$ as the number of connected components on the subgraph induced by A. We define the *Tutte polynomial* as

$$T_G(x, y) = \sum_{A \subset E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|}. \tag{1}$$

This polynomial has some useful properties, namely it is indeed a graph property (preserved under isomorphisms), multiplicative over disjoint unions, and the Tutte polynomial of a dual graph G^* of a undirected planar graph G is given by

$$T_{G^*}(x, y) = T_G(y, x).$$

We introduce another invariant of graphs.

Definition 2 (Chromatic polynomial)**.** Let $G = (V, E)$ be a directed graph. We define the *chromatic polynomial* of G, χ_G, as the number of proper k-colourings that exist on G.

The Tutte polynomial is infact a generalisation of the chromatic polynomial. Indeed, for a undirected graph $G = (V, E)$ the Tutte polynomial specialises at $y = 0$:

$$\chi_G(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1-\lambda, 0). \tag{2}$$

We see that if $\chi_G(\lambda)(3) \geq 1$, then there is a three colouring on G. Thus determining the Tutte polynomial is NP-hard. Infact, we have an upper bound on our complexity: it has been shown that it is $\#\mathsf{P}$-complete **annan1995complexities** (while some of the coefficients can be computed in polynomial time).

Moving forward, we can generalised the notion of a Tutte polynomial to simplicial complexes, as outlined by **krushkal2014polynomial**. Within the definition of the Tutte polynomial (**??**), we sum over all spanning subgraphs of G. Using more *higher-dimensional friendly language*, the sum over all subcomplexes of dimension 1 such that the entire 0-skeleton is included in each complex. Translating this to a simplicial complex, we may say that $L \subset K$ is a spanning n-dimensional subcomplex of a simplicial complex K of dimension $\geq n$ if their $(n-1)$-skeletons coincide. We define

$$T_K(x, y) = \sum_{L \subset K^{(n)}} x^{|H_{n-1}(L)|-|H_{n-1}(K)|}y^{|H_n(L)|} \tag{3}$$

where the summation is taken over the *spanning* n-subcomplexes L of K. We can see that (1) and (3) are equivalent for graphs by observing that $n(H) = k(A)+|A|-|V|$ (where $H = (V, A)$) defines the nullity of H: the rank of $H_1(H)$.