1 Lie groups

In this section, when we refer the *Lie groups* we actually mean *linear Lie groups*.

1.1 Definition

Definition 1.1 (Lie group)**.** A *Lie group* is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some n.

We mean *closed* in the topological sense, a more correct definition would include details on *smooth manifolds*, which we do not focus on here.

We denote $\mathfrak{gl}_{n,\mathbb{K}}$ as the set of $n \times n$ matrices with entries in $\mathbb{K}$.

We now generalise the exponential map exp to matrices, using the familiar power series.

Definition 1.2 (Exponential map)**.** Let $X \in \mathfrak{gl}_{n,\mathbb{K}}$. Then define $\exp : \mathfrak{gl}_{n,\mathbb{K}} \to \mathfrak{gl}_{n,\mathbb{K}}$ by

$$\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}.$$

We will omit details on the convergence of exp, but it is convergent for all matrices and can be proved using the Cauchy-Schwartz. In particular, by considering the entry-wise norm.

We have some properties of the exponential map, and comment that it behaves similarly to the normal exp. For all $X, Y, g \in \mathfrak{gl}_{n,\mathbb{K}}$ where g is invertible, and $s, t \in \mathbb{K}$, we have the following.

- $\exp(0) = I$
- $\exp(X + Y) = \exp(X)\exp(Y)$
- $(\exp(X))^{-1} = \exp(-X)$
- $\exp(sX)\exp(tX) = \exp((s + t)X)$
- $g \exp X g^{-1} = \exp(gXg^{-1})$

Proposition 1.3. $\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathfrak{gl}_{n,\mathbb{C}}$ *is differentiable at zero (the zero matrix), and its derivative at the origin is* I*.*

Corollary 1.4. $\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathfrak{gl}_{n,\mathbb{C}}$ *is a local diffeomorphism at zero.*

By this, we mean that exp has an inverse near zero.

We note that $\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathfrak{gl}_{n,\mathbb{C}}$ is *not* injective. In particular, it coincides with our regular exponential map at $n = 1$, so $\exp(2\pi ik) = 1$ for all $k \in \mathbb{Z}$.

Lemma 1.5. $\exp : \mathfrak{gl}_{n,\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ *is surjective.*

We not that $\exp : \mathfrak{gl}_{n,\mathbb{R}} \to \mathrm{GL}_n(\mathbb{R})$ is *not* surjective. Again, for $n = 1$ we see that exp is strictly positive.

Proposition 1.6. $\det \exp = \exp \operatorname{tr}$.

Proof. Let $X \in \mathfrak{gl}_{n,\mathbb{K}}$. We can conjugate X so that it is upper triangular. The result is then immediate. $\square$

1.2 One-parameter subgroups

We have seen that $\exp((s+t)X) = \exp(sX)\exp(tX)$, thus for all $X \in \mathfrak{gl}_{n,\mathbb{C}}$ we can define a group homomorphism $f : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ such that $t \mapsto \exp(tX)$ (here $\mathbb{R}$ is given standard addition $+$).

Similarly, if we consider $G = \mathrm{SO}(2)$ then we can define a group homomorphism $\mathbb{R} \to \mathrm{SO}(2)$ by $t \mapsto$ rotation by t. We note that

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}.$$

Definition 1.7 (One-parameter subgroup)**.** Let G be a Lie group. A *one-parameter subgroup* is a differentiable group homomorphism $\gamma : (\mathbb{R}, +) \to G$. The matrix $\gamma'(0)$ is the *infinitesimal generator*.

Theorem 1.8. *Let $f : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ be a one-parameter subgroup with infinitesimal generator X. Then*

$$f(t) = \exp(tX).$$

1.3 Lie algebras

Definition 1.9 (Lie algebra of a Lie group)**.** Let G be a Lie group. Its *Lie algebra* is

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}_{n,\mathbb{C}} : \exp(\mathbb{R}X) \subset G \right\}.$$

We can alternatively define $\mathfrak{g}$ as the set of infinitesimal generators of all one-parameter subgroups.

Proposition 1.10. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then*

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}_{n,\mathbb{C}} : X = \gamma'(0) \text{ for some map } \gamma : [-a, a] \to G \text{ where } a > 0 \right\}.$$

We may denote the Lie algebra of a Lie group G by $\mathrm{Lie}(G)$.

Example 1.11 (Some Lie algebras)**.**

- $\mathrm{Lie}(\mathrm{GL}_n(\mathbb{K})) = \mathfrak{gl}_{n,\mathbb{K}}$
- $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{K})) = \mathfrak{sl}_{n,\mathbb{K}} = \{X \in \mathfrak{gl}_{n,\mathbb{K}} : \mathrm{tr}(X) = 0\}$
- $\mathrm{Lie}(\mathrm{O}(n)) = \mathfrak{o}_n = \mathrm{Lie}(\mathrm{SO}(n)) = \mathfrak{so}_n = \{X \in \mathfrak{gl}_{n,\mathbb{R}} : X + X^\intercal = 0\}$
- $\mathrm{Lie}(\mathrm{U}(n)) = \mathfrak{u}_n = \{X \in \mathfrak{gl}_{n,\mathbb{C}} : X + X^\dagger = 0\}$
- $\mathrm{Lie}(\mathrm{SU}(n)) = \mathfrak{su}_n = \{X \in \mathfrak{u}_n : \mathrm{tr}(X) = 0\}$

Proposition 1.12. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Then*

1. *$\mathfrak{g} \subset \mathfrak{gl}_{n,\mathbb{C}}$ is a real vector space;*
2. *if $X \in \mathfrak{g}$ and $g \in G$, then $gXg^{-1} \in \mathfrak{g}$; and*
3. *if $X, Y \in \mathfrak{g}$, then*
$$[X, Y] := XY - YX \in \mathfrak{g}.$$

We now define a Lie algebra separate from a Lie group.

Definition 1.13 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ is an $\mathbb{R}$-vector space with a bilinear map (called the *Lie bracket*) $[-,-] : \mathfrak{g}^2 \to \mathfrak{g}$ such that

1. for all $X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$; and
2. the *Jacobi identity* holds: for all $X, Y, Z \in \mathfrak{g}$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

A *Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a subspace which is closed under the Lie bracket.

Example 1.14. Consider $\mathfrak{g} = \mathbb{R}^3$ with $[\boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{v} \times \boldsymbol{w}$. Then $\mathfrak{g}$ is a Lie algebra, and in fact $\mathfrak{g} \cong \mathfrak{so}_3$.

The center of a Lie group is an abelian subgroup of $\mathfrak{g}$.

Definition 1.15. A Lie algebra $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. The *center* of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{Z \in \mathfrak{g} : [Z, X] = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Definition 1.16 (Complex Lie group)**.** A complex Lie group is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ whose Lie algebra is a complex subspace of $\mathfrak{gl}_{n,\mathbb{C}}$.

1.4 Morphisms

Definition 1.17. A *Lie group homomorphism* $\phi : G \to G'$ between two Lie groups is a continuous group homomorphism.

As usual, a Lie group isomorphism is a homomorphism which is bijective with continuous inverse.

Definition 1.18. A *Lie algebra homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$ is an $\mathbb{R}$-linear map such that for all $X, Y \in \mathfrak{g}$,

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)].$$

A Lie algebra isomorphism is an invertible homomorphism.

Definition 1.19 (Derivative)**.** Let $\phi : G \to H$ be a Lie group homomorphism. Define the *derivative* of ϕ as

$$\begin{aligned} &D\phi : \mathfrak{g} \to \mathfrak{h}, \\ &D\phi(X) = \frac{d}{dt}\phi(\exp(tX))\bigg|_{t=0}. \end{aligned}$$

Theorem 1.20. *Let $\phi : G \to H$ be a Lie group homomorphism. Then*

1. *the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D\phi"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

   commutes;

2. *for $g \in G$ and $X \in \mathfrak{g}$, we have*

$$D\phi(gXg^{-1}) = \phi(g)D\phi(X)\phi(g)^{-1};$$

3. *$D\phi$ is a Lie group homomorphism.*

Definition 1.21. Let $\phi : G \to H$ be a Lie group homomorphism. Then ϕ is *holomorphic* if $D\phi$ is $\mathbb{C}$-linear.

Example 1.22. For an non-example, $\det : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$ is *not* holomorphic.

1.5 Representations of Lie groups

We omit the definition of a representation of a Lie group (ρ, V). The only difference to our normal definition is that ρ is a Lie group homomorphism. Similarly, a representation of a Lie algebra (σ, W) is the same but σ is a Lie algebra homomorphism.

We highlight a key difference to our traditional representations: a representation (ρ, V) of a Lie algebra $\mathfrak{g}$ need not satisfy $\rho(XY) = \rho(X)\rho(Y)$. In fact, it is not even certain that $XY \in \mathfrak{g}$. Our definition required only that ρ is $\mathbb{R}$-linear and ρ commutes with the Lie bracket $[-,-]$.

Our notions of G-homomorphisms, isomorphisms, subrepresentations, and irreducibility still hold as normal for representations of Lie groups and Lie algebras.

If G is a complex Lie group, then a holomorphic representation of G is a complex representation whose derivate is $\mathbb{C}$-linear.

Theorem 1.23. *Let A be a Lie group or a Lie algebra.*

1. *If V_1 and V_2 are irreducible finite-dimensional representations of A, then*
$$\dim_A(V_1, V_2) = \begin{cases} 1 & V_1 \cong V_2, \\ 0 & \textit{else.} \end{cases}$$
2. *Any irreducible finite-dimensional representation of an abelian A is 1-dimensional.*
3. *Let (ρ, V). Then ρ has a central character, defined on the center of A.*

Proposition 1.24. *Let (ρ, V) be a finite-dimensional representation of a Lie group G.*

1. *If $W \subset V$ is invariant under $\rho(G)$, then it is invariant under $D\rho(\mathfrak{g})$.*
2. *If $D\rho$ is irreducible, then ρ is irreducible.*
3. *If ρ is unitary, then $D\rho$ is skew-Hermitian.*
4. *Let (ρ', V') be another finite-dimensional representation of G. Then if $\rho \cong \rho'$, then $D\rho \cong D\rho'$.*

If G is connected, the converse hold.

Thus for connected Lie groups, we can test for irreducibility and isomorphisms at the level of Lie algebras.

1.6 Standard constructions for representations of Lie groups

Here we will present some standard constructions for representations of Lie groups. Derivatives are given and not proved, but this is not a difficult task (usually).

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a Lie group.

- We have the obvious action of $g \in G$ on $\mathbb{C}^n$:
$$\rho(g) = g, \qquad D\rho(X) = X.$$

- Let (ρ, V) and (σ, W) be two representations of G. The direct sum $\rho \oplus \sigma$ has derivative
$$D(\rho \oplus \sigma) = D\rho \oplus D\sigma.$$
- We have the determinant representation $\det : G \to \mathbb{C}$, with $D \det = \operatorname{tr}$.
- For a representation (ρ, V) of G, the dual representation (ρ^*, V^*) is defined by
$$(\rho^(g)(\lambda))(v) = \lambda(\rho(g^{-1})(v))$$
for $\lambda \in V^*$. We have
$$D\rho^*(X)(\lambda)(v) = -\lambda(D\rho(X)v).$$
- For two representations (ρ, V) and (σ, W) of G, then the tensor product representation $(\rho \otimes \sigma, V \otimes W)$ is a representation where
$$(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g)$$
and
$$D(\rho \otimes \sigma)(g) = D\rho(g) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes D\sigma(g).$$
- We also have the symmetric powers and alternating powers, which we consider as quotients as the tensor product representation and thus we will omit here.

1.7 The adjoint representation

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We have seen that $\mathfrak{g}$ is closed under conjugation by G; that is, for all $X \in \mathfrak{g}$ and $g \in G$, we have $gXg^{-1} \in \mathfrak{g}$. Thus we have an action on $\mathfrak{g}$ by conjugation, called the *adjoint representation*.

Definition 1.25 (Adjoint representation)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The *adjoint representation* $(\mathrm{Ad}, \mathfrak{g})$ of G is defined by
$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{GL}(\mathfrak{g}), \\ \mathrm{Ad}(X)g &= gXg^{-1}. \end{aligned}$$
We similarly have the *adjoint representation* $(\mathrm{ad}, \mathfrak{g})$ of $\mathfrak{g}$ where
$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}), \\ \mathrm{ad} &= D \, \mathrm{Ad}\,. \end{aligned}$$

We may write $\mathrm{Ad}_g(X)$ instead of $\mathrm{Ad}(g)(X)$, a similarly $\mathrm{ad}_X(Y)$ instead of $\mathrm{ad}(X)(Y)$.

We have seen that for Lie group homomorphisms, the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "D\phi"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"] \\
G \arrow[r, "\phi"] & H
\end{tikzcd}$$

Thus

$$\mathrm{Ad}_{\exp tX} = \exp_{t\,\mathrm{ad}\,X}.$$

Theorem 1.26. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $X, Y \in \mathfrak{g}$.*

1. $\mathrm{ad}_X(Y) = [X, Y] = XY - YX$
2. ad *is a Lie algebra homomorphism, so*

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$$

and the Jacobi identity holds.

Proposition 1.27. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If G is abelian, so is $\mathfrak{g}$. If G is connected, the converse holds.*

1.8 Maschke's theorem

The main corollary of Maschke's theorem was that we can decompose a representation into the directed sum of irreducible representations. But this does *not* hold for infinite groups.

For example, we consider $G = (\mathbb{R}, +)$ and a representation $(\rho, \mathbb{C}^2)$ given by

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

This is reducible, in particular, $\langle e_1 \rangle$ is invariant under $\rho(g)$ for all $g \in G$. But we claim this is not decomposable. Indeed, it can be shown that e_1 is the only eigenvector of $\rho(g)$ up to scalar.

Theorem 1.28. *Every finite-dimensional representation of a compact Lie group is decomposable.*

Some compact Lie groups:

- $U(n)$;
- $\mathrm{SU}(n)$; and

- $\mathrm{SO}(n)$.

Some non-compact Lie groups:

- SL; and
- GL.

Theorem 1.29. *Let (p, V) be an irreducible finite-dimensional representation of $U(1)$ over $\mathbb{C}$. Then*

1. $\dim V = 1$*; and*
2. $\rho : U(1) \to \mathbb{C}^\times$ *has form* $p(z) = z^n$ *for some* $n \in \mathbb{Z}$.

2 $\mathfrak{sl}_{2,\mathbb{C}}$

The aims of this section are as follows.

- Classify the irreducible, finite-dimensional, $\mathbb{C}$-linear representations of $\mathfrak{sl}_{2,\mathbb{C}}$.
- Form methods for decomposing reducible representations of $\mathfrak{sl}_{2,\mathbb{C}}$.

We recap below.

- Denote $\mathfrak{gl}_{n,\mathbb{K}}$ as the set of $n \times n$ matrices with entries in $\mathbb{K}$.
- A *linear Lie group* is a closed (in the topological sense) subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.
- Let $X \in \mathfrak{gl}_{n,\mathbb{C}}$. Then
$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$
- The *Lie algebra* $\mathfrak{g}$ of a linear Lie group G is
$$\mathfrak{g} = \{X \in \mathfrak{gl}_{n,\mathbb{C}} : \exp(\mathbb{R}X) \subset G\}.$$
We may consider the *Lie functor*, $\mathrm{Lie}(-) : \mathrm{GL}_2(\mathbb{C}) \to \mathfrak{gl}_{n,\mathbb{C}}$.
- $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{K})) = \mathfrak{sl}_{n,\mathbb{K}} = \{X \in \mathfrak{gl}_{n,\mathbb{K}} : \mathrm{tr}(X) = 0\}$.

The *standard basis* for $\mathfrak{sl}_{2,\mathbb{C}}$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The idea here is to study representations (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$ by looking at the eigenvectors and eigenvalues of $\rho(G)$.

2.1 Weights

Definition 2.1 (Weight vector)**.** Let (ρ, V) be a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Then a *weight vector* in V is an eigenvector of $\rho(H)$. The eigenvalue is called the *weight*.

Example 2.2.

1. Consider the trivial representation $(\rho, \mathbb{C})$ where $\rho(A) = 0$ for all $A \in \mathfrak{sl}_{2,\mathbb{C}}$. Pick basis $e \in \mathbb{C}^\times$ for $\mathbb{C}$. We have $\rho(H) = 0$ and so the (sole) weight vector is e, and its weight is 0.

2. Consider the standard representation $(\rho, \mathbb{C}^2)$, where $\rho(A) = A$ for all $A \in \mathfrak{sl}_{2,\mathbb{C}}$. So $\rho(H) = H$, so our weight vectors are e_1, e_2 with eigenvalues $1, -1$. Given that e_1 has weight 1 and e_2 has weight -1, we may relabel $e_1 = e_1$ and $e_2 = e_{-1}$.

3. Consider the representation $(\mathrm{ad}, \mathfrak{sl}_{2,\mathbb{C}})$. We examine how ad acts on the standard basis:

$$\begin{aligned} \mathrm{ad}_H(X) &= [H, X] = 2X, \\ \mathrm{ad}_H(Y) &= [H, Y] = -2Y, \\ \mathrm{ad}_H(H) &= [H, H] = 0. \end{aligned}$$

   Thus we have weight vectors X, Y, and H with weights 2, -2, and 0 respectively.

4. Consider the representation $(\rho, \mathbb{C}^2 \otimes \mathbb{C}^2)$ of $\mathfrak{sl}_{2,\mathbb{C}}$ which is the tensor of two standard representations. We pick the standard basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2.$$

   See that

$$(\rho)(H)(e_1 \otimes e_1) = \rho(H)e_1 \otimes e_1 + e_1 \otimes \rho(H)e_1 = 2e_1 \otimes e_1.$$

   In fact, we have the following lemma.

   Lemma 2.3. *If v is a weight vector of weight α and w is a weight vector of weight β, then $v \otimes w$ is a weight vector of weight $\alpha + \beta$.*

   This can be seen by working through the above example. Thus our weights are 2, 0, 0, and -2.

5. Consider the standard representation $(\rho, \mathrm{Sym}^k(\mathbb{C}^2))$ of $\mathfrak{sl}_{2,\mathbb{C}}$. We pick the basis $\{e_1^a e_{-1}^{k-a} : 0 \le a \le k\}$. Then

$$\begin{aligned} \rho(H)(e_1^a e_{-1}^{k-a}) &= \left(\rho(H)e_1^a\right) e_{-1}^{k-a} + e_1^a \left(\rho(H)e_{-1}^{k-a}\right) \\ &= a e_1^a e_{-1}^{k-a} - (k-a) e_1^a e_{-1}^{k-a} \\ &= (2a - k) e_1^a e_{-1}^{k-a}. \end{aligned}$$

So $e_1^a e_{-1}^{k-a}$ is a weight vector with weight $2a - k$. Thus our weights are $\{-k, 2-k, 4-k, \dots, k-4, k-2, k\}$. For example, when $k = 5$ we get $\{-5, -3, -1, 1, 3, 5\}$ as our weights.

We now look at how $\rho(X)$ and $\rho(Y)$ act on weight vectors. For a representation (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$, let

$$V_\alpha = \{v \in V : \rho(H)v = \alpha v\}$$

be the *eigenspace* for $\rho(H)$ with eigenvalue $\alpha \in \mathbb{C}$. We note that we have the decomposition

$$V = \bigoplus_\alpha V_\alpha$$

where we direct sum over the weights of V.

Proposition 2.4. *Let (ρ, V) be a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Let α be a weight of V. Then*

$$\begin{aligned}\rho(X)V_\alpha &\subset V_{\alpha+2},\\ \rho(Y)V_\alpha &\subset V_{\alpha-2}.\end{aligned}$$

We view $\rho(X)$ as a *raising operator*, and $\rho(Y)$ as a *lowering operator*.

Definition 2.5 (Highest weight vector)**.** Let (ρ, V) be a representation of $\mathfrak{sl}_{2,\mathbb{C}}$. A *highest weight vector* $v \in V$ is a weight vector such that $\rho(X)v = 0$. The weight of v is the *highest weight*.

Example 2.6.

1. Consider the standard representation $(\rho, \mathrm{Sym}^5(\mathbb{C}^2))$ of $\mathfrak{sl}_{2,\mathbb{C}}$. Here $v_5 = e_1^5$ is the highest weight, it can easily be checked that $\rho(X)v_5 = 0$.
2. Consider the tensor product of two standard representations $(\rho, \mathbb{C}^2 \otimes \mathbb{C}^2)$. We see that $e_1 \otimes e_1$ is a highest weight vector, but we also have another. We claim that $e_1 \otimes e_{-1} - e_{-1} \otimes e_1$ is a highest weight vector. Both $e_1 \otimes e_{-1}$ and $e_{-1} \otimes e_1$ have weight 0, so $e_1 \otimes e_{-1} - e_{-1} \otimes e_1$ has weight 0. But it can be checked that $\rho(X)$ kills $e_1 \otimes e_{-1} - e_{-1} \otimes e_1$.

2.2 Classification of representations of $\mathfrak{sl}_{2,\mathbb{C}}$

Corollary 2.7. *Any finite-dimensional representation of $\mathfrak{sl}_{2,\mathbb{C}}$ has a highest weight vector.*

Theorem 2.8.

1. *For every $k \in \mathbb{N}_0$, there is a unique $\mathbb{C}$-linear and finite-dimensional representation (up to isomorphism) of $\mathfrak{sl}_{2,\mathbb{C}}$ such that it has a highest weight vector of weight k.*
2. *Every finite-dimensional $\mathbb{C}$-linear irreducible representation of $\mathfrak{sl}_{2,\mathbb{C}}$ is isomorphic to one of the above representations.*

Proof. 1. We can just consider $e_1^k \in \mathrm{Sym}^k(\mathbb{C}^2)$.

2. For this, we just show that $\mathrm{Sym}^k(\mathbb{C}^2)$ is irreducible. Let $W \subset \mathrm{Sym}^k(\mathbb{C}^2)$ be a subrepresentation. W has a highest weight vector of the form $e_1^a e_{-1}^b$ with $a + b = k$, $a \geq 0$, and $b \leq k$. We also have $\rho(X)(e_1^a e_{-1}^b) = be_1^{a+1}e_{-1}^{b-1} \neq 0$ for $b > 0$. Thus e_1^k is the unique highest weight vector in $\mathrm{Sym}^k(\mathbb{C}^2)$, so $e_1^k \in W$. We see that W is invariant under $\rho(Y)$, and by repeated applications of $\rho(Y)$ we see that the entire basis of $\mathrm{Sym}^k(\mathbb{C}^2)$ is in W, and thus $W = \mathrm{Sym}^k(\mathbb{C}^2)$. Thus $\mathrm{Sym}^k(\mathbb{C}^2)$ is irreducible.

□

Lemma 2.9. *Let (ρ, V) be a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. If $v \in V$ is a highest weight vector with weight k. Then*

$$XY^m v = m(k - m + 1)Y^{m-1}v$$

for all $m \in \mathbb{N}_0$.

Lemma 2.10. *If (ρ, V) is a finite-dimensional irreducible representation of $\mathfrak{sl}_{2,\mathbb{C}}$ and $v \in V$ is a highest weight vector with weight $k \in \mathbb{N}_0$, then*

$$V = \langle v, Yv, \ldots, Y^k v\rangle.$$

Corollary 2.11. *If (ρ, V) is a irreducible representation of $\mathfrak{sl}_{2,\mathbb{C}}$, then $\rho(H)$ is diagonalisable.*

2.3 Decomposing $\mathfrak{sl}_{2,\mathbb{C}}$

Lemma 2.12. *Every $A \in \mathfrak{sl}_{2,\mathbb{C}}$ can be written uniquely as $X + iY$ for $X, Y \in \mathfrak{su}_n$.*

Proof. We have $\mathfrak{su}_n = \{X \in \mathfrak{sl}_{n,\mathbb{C}} : X + X^\dagger = 0\}$. Write

$$A = \frac{1}{2}(A - A^\dagger) - \frac{i}{2}(A + A^\dagger).$$

Both components here are in $\mathfrak{su}_n$, and by arguing on the dimensions of $\mathfrak{sl}_{n,\mathbb{C}}$ and $\mathfrak{su}_n$ we see that this must be unique. □

Lemma 2.13. *There is a bijection between the $\mathbb{C}$-linear representations of $\mathfrak{sl}_{n,\mathbb{C}}$ and the complex representations of $\mathfrak{su}_n$.*

Proof. $\tilde{\rho}(X + iY) = \rho(X) + i\rho(Y)$. □

Theorem 2.14 (Complete reducibility for $\mathfrak{sl}_{2,\mathbb{C}}$). *Let V be a finite-dimensional $\mathbb{C}$-linear representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Then*

$$V \cong \bigoplus_{i=1}^{r} V_i$$

where each V_i is a irreducible representation.

Proof. By the previous lemma, it is enough to show that V decomposes into irreducible representations of $\mathfrak{su}_n$. As $\mathrm{SU}(n)$ is simply connected, there is a representation $\hat{\rho}$ of $\mathrm{SU}(n)$ on V whose derivative is ρ. $\mathfrak{su}_n$ is compact, so $\hat{\rho}$ decomposes (by Maschke's theorem). Finally, as $\mathfrak{su}_n$ is connected, so $\hat{\rho}$ decomposes on $\mathfrak{su}_n$. $\square$

2.4 Decomposing tensor products

We recall that

$$\begin{aligned}
\{\text{weights of } V \otimes W\} &= \{\text{weights of } V\} + \{\text{weights of } W\}, \\
\{\text{weights of } \mathrm{Sym}^k(V)\} &= \{\text{sum of unordered } k\text{-tuplets of weights of } V\}, \\
\{\text{weights of } \Lambda^k(V)\} &= \{\text{sum of unordered } k\text{-tuplets of distinct weights of } V\}.
\end{aligned}$$

For example, if a representation (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$ has weights $\{-2, 0, 0, 2\}$, then $\Lambda^2(V)$ has weights $\{-2, -2, 0, 0, 2, 2\}$. Recall we are using multisets here. Similarly, $\Lambda^3(V) = \{-2, 0, 0, 2\}$.

We now give a general method for decomposing tensor products into other representations.

Given the multisets of weights of a representation (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$:

1. let k be the biggest weight;
2. any weight vector $v \in V$ of weight k must be a highest weight vector, thus
$$\langle v, Yv, \ldots, Y^k v\rangle \cong \mathrm{Sym}^k(\mathbb{C}^2);$$
3. by complete reducibility
$$V \cong \mathrm{Sym}^k(\mathbb{C}^2) \oplus V'$$
where V' has the weights of V with the weights $\{-k, -k+2, \ldots, k-2, k\}$ removed.

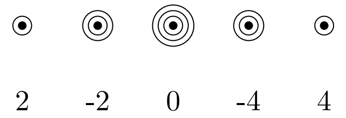
Theorem 2.15. *A representation (ρ, V) of $\mathfrak{sl}_{2,\mathbb{C}}$ is determined up to isomorphism by its weights.*

This theorem allows us to apply the above technique without worry.

Example 2.16. Let $V = \mathrm{Sym}^2(\mathbb{C}^2)$ where $\mathbb{C}^2$ is the standard representation. We will decompose $V \otimes V$ into irreducible representations and irreducible subrepresentations. First, we first the weights of $V \otimes V$.

$$\begin{aligned}\{\text{weights of } V \otimes V\} &= \{\text{weights of } V\} + \{\text{weights of } V\} \\ &= \{-2, 0, 2\} + \{-2, 0, 2\} \\ &= \{-4, -2, -2, 0, 0, 0, 2, 2, 4\}.\end{aligned}$$

We draw the following weight diagram.



So, using our method outlined above, we get the following.

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & & \mathrm{Sym}^4(\mathbb{C}^2) \\ & \bullet & \bullet & \bullet & & & \mathrm{Sym}^2(\mathbb{C}^2) \\ & & \bullet & & & & \mathbb{C} \end{array}$$

Now we decompose into subrepresentations. V has basis $v_2 = e_1^2$, $v_0 = e_1 e{-}1$, and $v_{-2} = e_{-1}^2$. We observe how X and Y acts on these. $X(v_2) = 0$, $X(v_0) = v_2$, and $X(v_{-2}) = 2v_0$. Similarly $Y(v_2) = (2v_0)$, $Y(v_0) = v_{-2}$, and $Y(v_{-2}) = 0$. From here, it is easy to piece the highest weight vectors by looking at how X acts on various combinations (or known highest weight vectors).

3 $\mathfrak{sl}_{3,\mathbb{C}}$

The aims of this section is similar to the previous: classify irreducible finite-dimensional $\mathbb{C}$-linear representations of $\mathfrak{sl}_{3,\mathbb{C}}$ by the highest weights.

Example 3.1 (Some representations of $\mathfrak{sl}_{3,\mathbb{C}}$)**.**

1. The standard representations on $\mathbb{C}^3$, with basis e_1, e_2, e_3.
2. The dual standard representation on $(\mathbb{C}^3)^*$, with basis e_1^*, e_2^*, e_3^* (we note that we did not use this representation in the previous section as dualling on $\mathfrak{sl}_{2,\mathbb{C}}$ reflects the weights about the origin).
3. The adjoint representation $(\mathrm{ad}, \mathfrak{sl}_{3,\mathbb{C}})$.
4. The tensor of symmetric powers $\mathrm{Sym}^a(\mathbb{C}^3) \otimes \mathrm{Sym}^b((\mathbb{C}^3)^*)$ (which is sadly not irreducible).

We proceed similar to before, but we need to redefine our notion of *weights*.

Definition 3.2 (Standard Cartan subalgebra)**.** The *standard Carton subalgebra* is the abelian subalgebra of $\mathfrak{sl}_{3,\mathbb{C}}$

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} : h_1 + h_2 + h_3 = 0 \right\} \subset \mathfrak{sl}_{3,\mathbb{C}}.$$

This is abelian as diagonal matrices commute.

Definition 3.3. If (ρ, V) is a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{3,\mathbb{C}}$, a *weight vector* $v \in V$ is a simultaneous eigenvector of $\{\rho(H) : H \in \mathfrak{h}\}$. The *weight* α of v is a linear map $\alpha : \mathfrak{h} \to \mathbb{C}$ such that $\rho(H)v = \alpha(H)v$. The *weight space* of weight α is

$$V_\alpha = \{v \in V : \rho(H)v = \alpha(H)V \text{ for all } H \in \mathfrak{h}\}.$$

By simultaneous, we mean that it is an eigenvector regardless of the H chosen.

We denote E_{ij} for the matrix with a 1 in entry (i, j) and 0 elsewhere. We note that $E_{ij} \in \mathfrak{sl}_{3,\mathbb{C}}$ if and only if $i \neq j$. We pick a basis of $\mathfrak{h}$ as the elements

$$H_{12} = E_{11} - E_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_{23} = E_{22} - E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and we also define $H_{13} = H_{12} + H_{23}$. It will be enough to study the eigenvectors of $\rho(H_{12})$ and $\rho(H_{23})$.

Example 3.4.

1. Let $(\rho, \mathbb{C}^3)$ be the standard representation with basis e_1, e_2, e_3. Then for $i \in \{1, 2, 3\}$ and $H \in \mathfrak{h}$, we have $\rho(H)e_i = L_i(H)e_i$ where $L_i(H) = h_i$ ($H = h_1E_{11} + h_2E_{22} + h_3E_{33}$). Thus we have the weight vectors being the e_i's with respective weights being the L_i's. Here L_1, L_2, L_3 span $\mathfrak{h}^*$, and there is one relation between them: $L_1 + L_2 + L_3 = 0$. Thus any element of $\mathfrak{h}^*$ can be written as $aL_1 - bL_3$ with $a, b \in \mathbb{C}$.

2. Let $(\rho^*, (\mathbb{C}^3)^*)$ be the dual representation. Then $He_1^* = -h_ie_i$ (should be checked). Thus, the weights of the dual representation are $\{-L_1, -L_2, -L_3\}$.

3. Consider the adjoint representation $(\mathrm{ad}, \mathfrak{g})$ where $\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}}$. We see that
$$\mathrm{ad}_H(H') = [H, H'] = 0$$
for all $H, H' \in \mathfrak{h}$, thus 0 is a weight of the adjoint representation. Thus,
$$\mathfrak{g}_0 := 0\text{-weight space of } \mathfrak{g} = \mathfrak{h}$$

(note we only proved that $\mathfrak{h} \subset \mathfrak{g}_0$, but this is indeed true). See that

$$[H, E_{ij}] = (h_i - h_j)E_{ij}$$

for $H \in \mathfrak{h}$ and $i \neq j$. Thus $E_{ij} \in \mathfrak{sl}_{3,\mathbb{C}}$ for $i \neq j$ is a weight vector with weight $h_i - h_j = L_i - L_j$.

Definition 3.5. A *root* of $\mathfrak{sl}_{3,\mathbb{C}}$ is a non-zero weight of the adjoint representation. A *root vector* is a weight vector of a root, and a *root space* is the weight space of a root.

We write

$$\Phi = \{\pm(L_1 - L_2), \pm(L_2 - L_3), \pm(L_1 - L_3)\}$$

for the set of roots of $\mathfrak{sl}_{3,\mathbb{C}}$. We call

$$\Phi^+ = \{L_1 - L_2, L_2 - L_3, L_1 - L_3\}$$

the *positive roots* and

$$\Phi^+ = \{L_2 - L_1, L_3 - L_2, L_3 - L_1\}$$

the *negative roots*. We write

$$\Delta = \{L_1 - L_2, L_2 - L_3\},$$

these are called the *simple roots*. We may write α_{ij} for the root $L_i - L_j$.

Finally, we have the *root space*, also called the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

3.1 Visualising weights

Theorem 3.6. *Let (ρ, V) be a finite-dimensional $\mathbb{C}$-linear representation of $\mathfrak{sl}_{3,\mathbb{C}}$, then all its weights are elements of*

$$\Lambda_W = \{aL_1 - bL_3 : a, b \in \mathbb{Z}\}$$

called the weight lattice.

Proof. Let $\alpha = aL_1 - bL_3$ for $a, b \in \mathbb{C}$ be a weight of V. We have to prove that $a, b \in \mathbb{Z}$. We sketch the proof here. We consider the embedding

$$\begin{aligned} \mathfrak{sl}_{2,\mathbb{C}} &\hookrightarrow \mathfrak{sl}_{3,\mathbb{C}} \\ H &\mapsto \left(\begin{array}{c|c} H & 0 \\ \hline 0 & 0 \end{array}\right). \end{aligned}$$

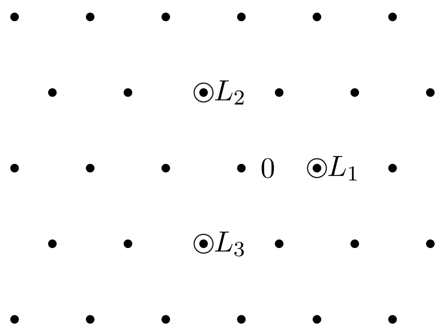
By our $\mathfrak{sl}_{2,\mathbb{C}}$-theorem, all eigenvalues acting on V are integers, thus $a \in \mathbb{Z}$. For $b \in \mathbb{Z}$, we use the similar embedding:

$$\begin{aligned} \mathfrak{sl}_{2,\mathbb{C}} &\hookrightarrow \mathfrak{sl}_{3,\mathbb{C}} \\ H &\mapsto \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & H \end{array}\right). \end{aligned}$$

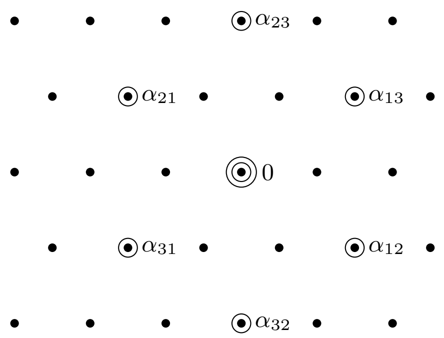
□

To visualise our weights: put L_1, L_2, and L_3 as vertices of an equilateral triangle. Then Λ_W is the lattice generated by these.

Example 3.7. Weights for the standard representation on $\mathbb{C}^3$.



Example 3.8. We now consider the weights of the adjoint representation.



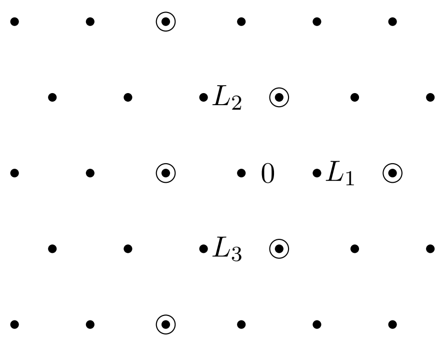
Example 3.9. Consider $\mathrm{Sym}^2(\mathbb{C}^3)$ where $\mathbb{C}^3$ is the standard representation. Our weight vectors are of the form e_ie_j for $1 \le i \le j \le 3$. We have

$$H(e_ie_j) = H(e_i)e_j + e_iH(e_j) = (L_i + L_j)(H)e_ie_j.$$

Thus the weight of e_ie_j is $L_i + L_j$. Considering every i and j, we get

$$\text{weights} = \{2L_1, 2L_2, 2L_3, L_1 + L_2, L_2 + L_3, L_1 + L_3\}.$$

Thus we draw our weights as follows.



We present a fundamental weight calculation, as we did with $\mathfrak{sl}_{2,\mathbb{C}}$.

Theorem 3.10 (Fundamental weight calculation)**.** *Let (ρ, V) be a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{3,\mathbb{C}} = \mathfrak{g}$ and let $v \in V_\beta$ be a weight vector with weight $\beta \in \mathfrak{h}^*$. Let $\alpha \in \mathfrak{h}^*$ be a root and let $X_\alpha \in \mathfrak{g}_\alpha$ be a root vector. Then $\rho(X_\alpha)v = 0$.*

Proof. Let $H \in \mathfrak{h}$. Then

$$\begin{aligned} H(X_\alpha(v)) &= ([H, X_\alpha] + X_\alpha H)v \\ &= \alpha(H)X_\alpha v + X\alpha\beta(H) \\ &= (\alpha + \beta)(H)(X_\alpha v). \end{aligned}$$ □

Definition 3.11. Let (ρ, V) be a $\mathbb{C}$-linear representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Then a weight vector $v \in V$ is a *highest weight vector* $\rho(X)v = 0$ for $X \in \{E_{12}, E_{13}, E_{23}\}$. The *highest weight* of v is the weight of v.

As $E_{13} = [E_{12}, E_{23}]$, we only need to check E_{12} and E_{23}.

Proposition 3.12. *If (ρ, V) is a finite-dimensional representation, then a highest weight exists.*

Proof. Define $l : \mathfrak{h}^* \to \mathbb{C}$ by $l(aL_1 - bL_3) = a + b$. Use this function on contradiction of having a weight vector of maximal l value. □

3.2 Dominant weights

Let (ρ, V) be a representation of $\mathfrak{sl}_{3,\mathbb{C}}$. If $v \in V$ is a highest weight vector of weight $aL_1 - bL_3$, then it is a highest weight vector for V under the restrictions

$$\left(\begin{array}{c|c} \mathfrak{sl}_{2,\mathbb{C}} & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathfrak{sl}_{2,\mathbb{C}} \end{array}\right).$$

Its weight for the top right restriction is a and its weight for the bottom right copy is b.

Definition 3.13 (Dominant weight)**.** A *dominant weight* is an element of $\mathfrak{h}^*$ of the form $aL_1 - bL_3$ with $a, b \in \mathbb{N}_0$.

Theorem 3.14. *For each dominant weight $aL_1 - bL_3$ there is a unique (up to isomorphism) finite-dimensional $\mathbb{C}$-linear irreducible representation of $\mathfrak{sl}_{3,\mathbb{C}}$ with highest weight vector that of the weight.*

We call such a representation $V^{(a,b)}$.

Example 3.15.

- $V^{(0,0)} = \mathbb{C}$ (trivial)
- $V^{(1,0)} = \mathbb{C}^3$ (standard)
- $V^{(0,1)} = (\mathbb{C}^3)^*$
- $V^{(1,1)} = (\mathrm{ad}, \mathfrak{sl}_{3,\mathbb{C}})$
- $V^{(2,0)} = \mathrm{Sym}^2(\mathbb{C}^3)$