Blatt 05

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1c) unterscheidbar: $|n^{(1)}, m^{(2)}\rangle$

Fermion / Boson : $\frac{1}{\sqrt{2}}\left(|n^{(1)}, m^{(2)}\rangle \mp |m^{(1)}, n^{(2)}\rangle \right)$

d) $A_2^{(1)} = A^{(1)} \otimes \mathbb{1}$ $\qquad B_2^{(2)} = \mathbb{1} \otimes B^{(2)}$

$$[A_2^{(1)}, B_2^{(2)}] = [A \otimes \mathbb{1}, \mathbb{1} \otimes B] = \underbrace{A\mathbb{1} \otimes \mathbb{1}B - \mathbb{1}A \otimes B\mathbb{1}}_{} = 0$$

$$A \otimes (B+C) = A \otimes B + A \otimes C$$
$$(B+C) \otimes A = B \otimes A + C \otimes A$$

$A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$

$k, l = 1, \dots, n$

$i, j = 1, \dots, m$

$$(A \otimes \mathbb{1}^{\mathbb{C}^{n\times n}})_{ij} = [A_{ij}\mathbb{1}] = A_{ij}\delta_{kl}$$

$$(\mathbb{1} \otimes B)_{ij} = [\delta_{ij} B] = \delta_{ij} B_{kl}$$

$\uparrow$

$$A_{ij}\mathbb{1}\,\delta_{ji'}B = A_{ii'}\mathbb{1}B$$
$$\delta_{ij}B\,A_{ji'}\mathbb{1} = A_{ii'}B\mathbb{1}$$

1e) $|\psi^\pm\rangle \in \mathcal{H}_N^{(\pm)}$

$$\langle\psi^+|\psi^-\rangle = \langle\psi^+|P^2|\psi^-\rangle = -\langle\psi^+|\psi^-\rangle = 0$$

1f) $P^+ = P$ $\qquad (P^+ = P^{-1})$

$\psi = \psi^- + \psi^+ \qquad \varphi = \varphi^- + \varphi^+$

$$\langle\psi|P|\varphi\rangle$$

$$(\langle\psi^-| + \langle\psi^+|)P(|\varphi^-\rangle + |\varphi^+\rangle) = -\langle\psi^-|\varphi^-\rangle + \langle\psi^+|\varphi^+\rangle$$

$$\langle\psi|P^+|\varphi\rangle = \langle\varphi|P|\psi\rangle^* = \langle\varphi^+|\psi^+\rangle^* - \langle\varphi^-|\psi^-\rangle^* =$$

$$\Longrightarrow \quad P = P^\dagger .$$

$$\langle \psi , P^\dagger \varphi \rangle = \langle P\psi , \varphi \rangle = \langle \varphi , P\psi \rangle^*$$

2. a) $|\psi, \phi\rangle \in \mathcal{H}_2$ b) $\dfrac{|\psi, \phi\rangle + |\phi, \psi\rangle}{\sqrt{2}}$ c) $\dfrac{|\psi, \phi\rangle - |\phi, \psi\rangle}{\sqrt{2}}$

a) $\langle \psi, \phi | x_1^2 - 2x_1x_2 + x_2^2 | \psi, \phi\rangle$

$= \langle x_1^2\rangle_\psi - 2\langle x_1\rangle_\psi \langle x_2\rangle_\phi + \langle x_2^2\rangle_\phi$

$x_1 = x_1^{(1)} \otimes \mathbb{1}$

$x_2 = \mathbb{1} \otimes x_2^{(2)}$ $\Rightarrow [x_1, x_2] = 0$

b) $= \frac{1}{2}\Big[\langle x_1^2\rangle_\psi + \langle x_1^2\rangle_\phi - 2\big(\langle x_1\rangle_\psi \langle x_2\rangle_\phi + \langle x_1\rangle_\phi \langle x_2\rangle_\psi\big) + \langle x_2^2\rangle_\psi + \langle x_2^2\rangle_\phi\Big]$

c) $= \frac{1}{2}\Big(\langle x_1^2\rangle_\psi + \langle x_1^2\rangle_\phi - 2\big(\langle x_1\rangle_\psi \langle x_2\rangle_\phi + \langle x_1\rangle_\phi \langle x_2\rangle_\psi\big) + \langle x_2^2\rangle_\phi + \langle x_2^2\rangle_\psi\Big)$

Bsp. $\langle x^2\rangle_\psi = \langle \psi | x^2 | \psi\rangle$

! Mischterme vergessen, z.B. $\langle \psi | x_1^2 | \phi\rangle$

3. $\hat{\mathcal{H}} = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) - \dfrac{Gm^2}{|\vec{r}_1 - \vec{r}_2|}$

$\vec{r}_1 = \vec{R} + \frac{\vec{r}}{2}$ $\vec{r}_2 = \vec{R} - \frac{\vec{r}}{2}$

$\nabla_1 = \nabla_R + \frac{\nabla_r}{2}$ $\nabla_2 = \nabla_R - \frac{\nabla_r}{2}$

$$\nabla_{1/2} f(\vec{r}_1, \vec{r}_2) = \left(a\nabla_R + b\nabla_r\right) f\left(\vec{R} + \tfrac{\vec{r}}{2}, \vec{R} - \tfrac{\vec{r}}{2}\right)$$

$$= a\Big((\nabla_R \vec{r}_1)(\nabla_1 f) + (\nabla_R \vec{r}_2)(\nabla_2 f)\Big) + b\Big((\nabla_r \vec{r}_1)(\nabla_1 f) + (\nabla_r \vec{r}_2)(\nabla_2 f)\Big)$$

$$\Rightarrow \nabla_{1/2} = a(\nabla_R \vec{r}_1)\nabla_1 + a(\nabla_R \vec{r}_2)\nabla_2 + b(\nabla_r \vec{r}_1)\nabla_1 + b(\nabla_r \vec{r}_2)\nabla_2$$

$$= \Big[a(\nabla_R \vec{r}_1) + b(\nabla_r \vec{r}_1)\Big]\nabla_1 + \Big[a(\nabla_R \vec{r}_2) + b(\nabla_r \vec{r}_2)\Big]\nabla_2$$

$$= \left[a + \frac{b}{2} \right] \nabla_1 + \left[a - \frac{b}{2} \right] \nabla_2$$

$\nabla_1 : \quad a = \frac{1}{2} \quad b = 1 \qquad \nabla_2 : \quad a = \frac{1}{2} \quad b = -1$

$\nabla_1 = \frac{1}{2}\nabla_R + \nabla_r \qquad \nabla_2 = \frac{1}{2}\nabla_R - \nabla_r$

$$\begin{pmatrix} \vec{R} \\ \vec{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ r_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \vec{R} \\ \vec{r} \end{pmatrix}$$

$$\nabla_R f(\vec{r}_1(\vec{R},\vec{r}), \vec{r}_2(\vec{R},\vec{r})) = (\nabla_R \vec{r}_1)(\nabla_1 f) + (\nabla_R \vec{r}_2)(\nabla_2 f)$$

$$\Rightarrow \nabla_R = \left[(\nabla_R \vec{r}_1)\nabla_1 + (\nabla_R \vec{r}_2)\nabla_2 \right]$$

$$= \left[\nabla_1 + \nabla_2 \right]$$

$$\nabla_r = \left[(\nabla_r \vec{r}_1)\nabla_1 + (\nabla_r \vec{r}_2)\nabla_2 \right]$$

$$= \left[\frac{1}{2}\nabla_1 - \frac{1}{2}\nabla_2 \right] = \frac{1}{2}\left[\nabla_1 - \nabla_2 \right]$$

$\Rightarrow \nabla_1 = \frac{\nabla_R}{2} + \nabla_r \qquad \nabla_2 = \frac{\nabla_R}{2} - \nabla_r$

$\Rightarrow \nabla_1^2 + \nabla_2^2 = \frac{\nabla_R^2}{2} + 2\nabla_r^2$

$$\Rightarrow \boxed{\hat{\mathcal{H}} = -\frac{1}{2m}\left[\frac{\nabla_R^2}{2} + 2\nabla_r^2 \right] - \frac{Gm^2}{|\vec{r}|}}$$

$\psi(\vec{R},\vec{r}) = \varphi(\vec{r})\, e^{i\vec{p}_R \cdot \vec{R}}$

$$\Rightarrow \hat{\mathcal{H}} = -\frac{1}{2m}\left[-\frac{\vec{p}_R^{\,2}}{2} + 2\nabla_r^2 \right] - \frac{Gm^2}{|\vec{r}|}$$

$$= -\nabla_r^2 - Gm^2 \quad \vec{p}_R^{\,2}$$

Klassische Mechanik:

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$

$\Leftrightarrow \vec{r}_1 = \vec{R} + \frac{\vec{r}}{2}$

$\vec{r}_2 = \vec{R} - \frac{\vec{r}}{2}$

$\Rightarrow \vec{p}_1 = m(\dot{\vec{R}} + \frac{\dot{\vec{r}}}{2})$

$\Rightarrow \vec{p}_2 = m(\dot{\vec{R}} - \frac{\dot{\vec{r}}}{2})$

$\Rightarrow \vec{p}_1^{\,2} + \vec{p}_2^{\,2} = m^2(2\dot{\vec{R}}^2 + \frac{\dot{\vec{r}}^2}{2})$

$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} = m(\dot{\vec{R}}^2 + \frac{\dot{\vec{r}}^2}{4})$

$$= -\frac{\nabla_r^2}{m} \underbrace{- \frac{G m^2}{|\vec{r}|} + \frac{\vec{p}_R^{\,2}}{4m}}_{V_{eff}(\vec{r})}$$

$$\frac{\ldots^2}{2m} = m(\dot{R}^2 + \frac{\ldots}{4})$$

$$\mathcal{L} = \mathcal{H} \quad \begin{cases} \vec{p}_R = \frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = 2m\dot{R} & \Rightarrow \vec{p}_R^{\,2} = 4m^2 \dot{\vec{R}}^2 \\ \vec{p}_r = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \frac{m\dot{\vec{r}}}{2} & \Rightarrow \vec{p}_r^{\,2} = \frac{m^2 \dot{\vec{r}}^2}{4} \end{cases}$$

$$\Rightarrow \vec{p}_1^{\,2} + \vec{p}_2^{\,2} = \frac{\vec{p}_R^{\,2}}{2} + 2\vec{p}_r^{\,2}$$

$$\psi_{nlm}(\vec{r}) = (-1)^l \psi_{nlm}(-\vec{r})$$

Teilchenvertauschung: $\vec{R} \to \vec{R}$, $\vec{r} \to -\vec{r}$

b) (i) Bosonen

$$\psi_{nlm}(\vec{r}) = \psi_{nlm}(-\vec{r}) \Rightarrow l \text{ gerade}$$

(ii) Fermionen: l ungerade