Quantentheorie II Übung 9

Besprechung: 2021WE25 (KW25)

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1. **Questions**

   (a) Explain the interpretation of the operators $a^\dagger_{\vec{p}}$ and $\hat{\Psi}^\dagger(\vec{x})$ in the context of a non-relativistic Fock space.

   (b) What was the relationship between these two kinds of operators in the case of the lecture (non-relativistic case)?

   (c) What is the Lagrangian in the lecture whose canonical quantization reproduces the above operators and their (anti-)commutation relations?

   (d) Write down the Lagrangian and the Hamiltonian for the hydrogen atom (as given in QM1) and for a 1-dimensional harmonic oscillator.

   (e) Give an example of a key relationship which should be satisfied in relativistic quantum mechanics.

2. **From Lagrangian to Hamiltonian:** in the canonical formalism the canonical conjugate momentum for a given L is defined as $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ and the Hamiltonian (function) is $H(q,p) = \sum_i p_i \dot{q}_i - L(q,\dot{q})$. Write the canonical conjugate momentum and Hamiltonian (function) for each L below.

   (a) The 1-dimensional simple harmonic oscillator: $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\kappa^2 q^2$, where κ is a constant.

   (b) $L = \frac{\alpha}{2}\dot{q}^2 q^2$

   (c) $L = (\alpha\dot{q} + \beta q)^2$

   (d) A particle with charge Q in a 4-potential: $L = \frac{1}{2}m\dot{\vec{q}}^2 + Q\dot{\vec{q}} \cdot \vec{A}(\vec{q}) - Q\phi(\vec{q})$

3. **Linear chain:** we consider a 1-dimensional chain of N connected oscillators with mass $m = 1$. Its Lagrangian is

$$L = \sum_{n=1}^{N} \left(\frac{1}{2}\dot{q}_n^2 - \frac{\kappa}{2}(q_{n+1} - q_n)^2 \right). \tag{1}$$

   The oscillators sit on a lattice with lattice distance a. q_n satisfy the periodic boundary condition $q_{n+N} = q_n$, and N is an odd number.

   (a) Verify the Euler-Lagrange equation

$$\frac{d^2}{dt^2} q_n = \kappa(q_{n+1} + q_{n-1} - 2q_n). \tag{2}$$

   (b) Use the Ansatz $q_n(t) = e^{\pm i(\omega_k t - kna)}$ for Eq. (2). What are the values of k allowed by the boundary condition? Show that there exists a minimum non-vanishing value of $|k|$.

(c) Show that the range of $|k|$ can be restricted as $-\pi < ka \leq \pi$ without losing generality (1. Brillouin-Zone). How many physically different values of k do we obtain? What is the largest $|k|$?

(d) Write the dispersion relation ω_k of the solution obtained by using the Ansatz.

(e) What is the behavior of ω_k when $k \to 0$? Discuss the behavior of ω_k for fixed k when $a \to 0$ and $N \to \infty$! The solution corresponds to sound waves. What is the speed of sound?

(f) Write the canonical conjugate momenta p_n and the Hamiltonian (function) $H(p, q)$

$$p_n \equiv \frac{\partial L}{\partial \dot{q}_n}, \qquad H(p,q) = \sum_n p_n \dot{q}_n - L(q, \dot{q}). \tag{3}$$

(g) Now we turn to canonical quantization. We introduce operators $\hat{q}_n$ and $\hat{p}_n$ which correspond to the variables q_n and p_n respectively, and assume that the operators satisfy the commutation relations

$$[\hat{q}_n, \hat{p}_m] = i\delta_{nm}, \qquad [\hat{q}_n, \hat{q}_m] = 0, \qquad [\hat{p}_n, \hat{p}_m] = 0, \tag{4}$$

We use the Ansatz

$$\hat{q}_n(t) = \sum_k \sqrt{\frac{1}{2\omega_k N m}} \left(\hat{a}_k e^{-i(\omega_k t - kan)} + \hat{a}_k^\dagger e^{i(\omega_k t - kan)} \right). \tag{5}$$

What are the commutation relations that $\hat{a}_k$ and ${a_k}^\dagger$ should satisfy in order that $\hat{q}$ and $\hat{p}$ can fulfill the assumptions in Eq. (4)?

The range of summation in Eq. (5) is $k = 0, \pm\frac{2\pi}{Na}, \cdots, \pm\frac{\pi(N-1)}{Na}$.

(h) Show that the Hamiltonian (operator) can be written as

$$\hat{H} = \sum_k \omega_k {\hat{a}_p}^\dagger \hat{a}_k + \text{ constant}. \tag{6}$$