Quantentheorie II Übung 5

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1. **Questions**
   (a) What are identical particles?
   (b) How do we know that all electrons are identical?
   (c) Write down example two-particle wave functions for two distinguishable particles, two bosons or two fermions respectively.
   (d) Consider an operator $A_2^{(1)} = A^{(1)} \otimes 1$ on a two-particle Hilbert space and an analogous operator $B_2^{(2)} = 1 \otimes B^{(2)}$. Show $[A_2^{(1)}, B_2^{(2)}] = 0$.
   (e) For $|\psi^{\pm}\rangle \in \mathcal{H}_N^{(\pm)}$, show $\langle\psi^+|\psi^-\rangle = 0$.
   (f) Show that the permutation operator P_{ij} is Hermitian and unitary by considering scalar products of basis elements.

2. **Two-particle system:** we consider a system with two particles. The two particles occupy the two one-particle states $|\psi\rangle$ and $|\phi\rangle$ which satisfy orthonormality relations: $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ and $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle = 0$.

   Write down the two-particle states corresponding to the Hilbert spaces $\mathcal{H}_2$, $\mathcal{H}_2^{(+)}$, and ${\mathcal{H}_2}^{(-)}$, and compute $\langle(x_1 - x_2)^2\rangle$ for the following cases.
   (a) Distinguishable particles
   (b) Identical bosons
   (c) Identical fermions

   Express the results in terms of matrix elements such as

   $$\langle x^2\rangle_{AB} \equiv \langle A|x^2|B\rangle, \quad \langle x\rangle_{AB} \equiv \langle A|x|B\rangle\,,$$

   where A and B are either ψ or ϕ.

3. **Gravitationally bound state of two identical particles:** two identical particles with mass m interact with each other through the gravitational force. The Hamiltonian of this system is

   $$\hat{H} = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|}\,.$$

   (a) Rewrite the Hamiltonian using center-of-mass and relative coordinates

   $$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}\,, \quad \vec{r} = \vec{r}_1 - \vec{r}_2\,,$$

   and show that $\hat{H}$ is mathematically equivalent to the hydrogen atom Hamiltonian.

(b) What are the allowed wave functions of the bound state for the following cases?

(i) Spinless Bosons, (ii) Spin-$\frac{1}{2}$ fermions

It is not required to solve the Schrödinger equation $\hat{H}\psi = E\psi$ explicitly. The wave functions of hydrogen atoms satisfy $\psi_{nlm}(\vec{r}) = (-1)^l \psi_{nlm}(-\vec{r})$.

(c) What is the reason that these states are not observed?

4. **Symmetric and antisymmetric Hilbert subspaces:** we know that a Hilbert space of an N-particle system $\mathcal{H}_N$ can have symmetric $\mathcal{H}_N^{(+)}$ and antisymmetric $\mathcal{H}_N^{(-)}$ subspaces. Do the two spaces $\mathcal{H}_N^{(+)}$ and $\mathcal{H}_N^{(-)}$ add up to the full space $\mathcal{H}_N$? Investigate for the following cases.

   (a) $N = 2$

   (b) $N = 3$