1b) $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$

$$x'^\mu x'_\mu = (x^\mu + \omega^\mu{}_\nu x^\nu)(x_\mu + \omega_{\mu\rho} x^\rho)$$
$$= x^\mu x_\mu + \underline{\omega^\mu{}_\nu x^\nu x_\mu} + \underline{x^\mu \omega_{\mu\rho} x^\rho} + \underbrace{\cancel{\omega^\mu{}_\nu \omega_{\mu\rho} x^\nu x^\rho}}_{\mathcal{O}(\omega^2)}$$

$$\omega^\mu{}_\nu x^\nu x_\mu - x^\rho \omega_{\rho\mu} x^\mu = \underbrace{\omega_{\mu\nu} x^\nu x^\mu}_{=0} - \underbrace{\omega_{\nu\mu} x^\nu x^\mu}_{=0}$$

$$a^\mu b_\mu = g^{\mu\nu} a_\nu b_\mu = a_\nu b^\nu$$

1c) $J_i = c\,\sigma_i$

$$[J_x, J_y] = c^2 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \right] = c^2 \left[\begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix} \right]$$
$$= +2ic^2\sigma_z \overset{!}{=} iJ_z = ic\,\sigma_z$$
$$\Rightarrow 2c^2 = c \overset{c\neq 0}{\Longrightarrow} 2c = 1 \Rightarrow c = \tfrac{1}{2}$$

1d) $$(\bar\Psi\chi)' = \bar\Psi'\chi' = \cancel{(\psi^\dagger\gamma^0)'\chi'} = \bar\Psi \underbrace{S^{-1}(\Lambda)S(\Lambda)}_{=\mathbb{1}}\chi \qquad \square$$
$$= \cancel{\psi^{\dagger\prime}\gamma^{0\prime}\chi'}$$

$$\bar\Psi' = (\psi^\dagger\gamma_0)' = \overline{\psi'} = \psi'^\dagger\gamma_0$$

$$\psi' = S(\Lambda)\psi \Rightarrow \overline{\psi'} = (S(\Lambda)\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger(\Lambda)\gamma^0$$
$$= \psi^\dagger\gamma_0\gamma_0 S^\dagger(\Lambda)\gamma^0$$
$$= \bar\Psi S^{-1}(\Lambda)$$

$$S^{-1}(\Lambda) = \mathbb{1} + \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu} = \overset{\gamma^0\gamma^0}{\downarrow}\!\!\mathbb{1} + \frac{i}{2}\omega^{\mu\nu}\gamma^0 S^\dagger_{\mu\nu}\gamma^0 = \gamma_0\left(\mathbb{1} + \frac{i}{2}\omega^{\mu\nu}S^\dagger_{\mu\nu}\right)\gamma^0$$

$$= \gamma_0 \left(1 - \frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}\right)^\dagger \gamma^0$$

$$= \gamma_0 \, S(\Lambda)^\dagger \gamma^0$$

Dirac-Spinor

$$(i\partial_\mu \gamma^\mu - m)\psi = 0 \qquad (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\partial_t \gamma^0 + \partial_x \gamma^1 + \partial_y \gamma^2 + \partial_z \gamma^3 = \partial^\mu \gamma_\mu \qquad \not{p} = \gamma_0 E + \gamma_1 p_x + \dots$$

$$(i\partial_\mu \gamma^\mu - m)\psi = 0$$

Ansatz: $\psi(x) = \omega(p)\, e^{\mp ipx} \qquad px = p_\mu x^\mu = Et - \vec{p}\cdot\vec{x}$

4-spinor

$$(i\gamma^\mu \partial_\mu - m)\,\omega(p)\, e^{\mp i(Et - \vec{p}\cdot\vec{x})}$$

$$(i(\mp\gamma^0 iE \pm i\gamma^1 p_x \pm i\gamma^2 p_y \pm i\gamma^3 p_z) - m)\,\omega(p) = 0$$

$$(\pm(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z) - m)\,\omega(p) = 0$$

$$(\pm p_\mu \gamma^\mu - m)\,\omega(p) = 0 \qquad \omega(p) = \begin{pmatrix} \omega_1(p) \\ \omega_2(p) \\ \omega_3(p) \\ \omega_4(p) \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$(p_\mu \gamma^\mu)^2 = \underbrace{p^2}_{p_\mu p^\mu} \mathbb{1}$$

$$\pm \not{p}\, \omega(p) = m\, \omega(p)$$

$$p^2 \omega(p) = m^2 \omega(p)$$

$$L_{\rho\sigma} = \begin{pmatrix} 0 & -k_x & -k_y & -k_z \\ k_x & 0 & l_z & -l_y \\ k_y & -l_z & 0 & l_x \\ k_z & l_y & -l_x & 0 \end{pmatrix}$$

$L^{12} \quad L^{21} \quad \mu\nu \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dots \end{pmatrix}$

a) $\omega^{12} = -\omega^{21} = \varepsilon \qquad \omega^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\Big[\varepsilon \underbrace{L_{12}}_{l_z} - \varepsilon \underbrace{L_{21}}_{-l_z}\Big]$$

$$= \delta^\mu{}_\nu - i\varepsilon (l_z)^\mu{}_\nu$$

$$= \delta^\mu{}_\nu + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{(1)}$$

$\omega^\mu{}_\nu = g_{\nu\sigma}\omega^{\mu\sigma} = \omega^{\mu\sigma} g_{\sigma\nu}$

Def. $\omega^{\mu\nu}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \overset{\omega^\mu{}_\nu}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

3b) $$S^{-1}(\Lambda)\gamma^\rho S(\Lambda) = \Big[1 + \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\Big]\gamma^\rho\Big[1 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\Big]$$

$$= \gamma^\rho + \frac{i}{2}\omega^{\mu\nu}[S_{\mu\nu}, \gamma^\rho] + \mathcal{O}(\omega^2)$$

$$= \gamma^\rho - \frac{1}{8}\omega^{\mu\nu}\big[[\gamma_\mu, \gamma_\nu], \gamma^\rho\big] \qquad \big(= (\delta^\mu{}_\nu + \omega^\mu{}_\nu)\gamma^\nu$$

$$= (\delta^\rho{}_\nu + \omega^\rho{}_\nu)\gamma^\nu$$

$K = [\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu, \gamma^\rho]$

$$= \gamma_\mu\gamma_\nu\gamma^\rho - \gamma_\nu\gamma_\mu\gamma^\rho - \gamma^\rho\gamma_\mu\gamma_\nu + \gamma^\rho\gamma_\nu\gamma_\mu \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1$$

$$= \gamma_\mu\gamma_\nu\gamma^\rho - (\cancel{2g_{\nu\mu}1} - \gamma_\mu\gamma_\nu)\gamma^\rho - \gamma^\rho\gamma_\mu\gamma_\nu + \gamma^\rho(\cancel{2g_{\nu\mu}1} - \gamma_\mu\gamma_\nu)$$

$$= 2\gamma_\mu\gamma_\nu\gamma^\rho - 2\gamma^\rho\gamma_\mu\gamma_\nu \qquad = -8g_{\lambda\mu}g^{\lambda\rho}\gamma_\nu = \underline{-8\delta^\rho_\mu\gamma_\nu}$$

$$= 2[\gamma_\mu\gamma_\nu, \gamma^\rho]$$

$$2\gamma_\mu[\gamma_\nu, \gamma^\rho] + 2[\gamma_\mu, \gamma^\rho]\gamma_\nu$$

$$= 2\gamma_\mu(\underline{\gamma_\nu\gamma^\rho} - \underline{\gamma^\rho\gamma_\nu}) + \dots$$

$AB = \{A,B\} - BA$

$$= 2\gamma_\mu(\{\gamma_\nu,\gamma^\rho\} - 2\gamma^\rho\gamma_\nu) + \dots$$

$$= 2\gamma_\mu(g^{\rho\lambda}\{\gamma_\nu,\gamma_\lambda\} - 2\gamma^\rho\gamma_\nu) + \dots$$

$$= 2\gamma_\mu(2g^{\rho\lambda}g_{\nu\lambda} - 2\gamma^\rho\gamma_\nu) + 2[\gamma_\mu,\gamma^\rho]\gamma_\nu$$

$$= 4\gamma_\mu\delta^\rho{}_\nu - 4\gamma_\mu\gamma^\rho\gamma_\nu + 2[\gamma_\mu,\gamma^\rho]\gamma_\nu$$

$$= 4\gamma_\mu\delta^\rho{}_\nu - 2\gamma_\mu\gamma^\rho\gamma_\nu - 2\gamma^\rho\gamma_\mu\gamma_\nu$$

$$= 4\gamma_\mu\delta^\rho{}_\nu - 2\{\gamma_\mu,\gamma^\rho\}\gamma_\nu$$

$$= 4\gamma_\mu\delta^\rho{}_\nu - 4g^{\rho\lambda}g_{\mu\lambda}\gamma_\nu$$

$$= 4\gamma_\mu\delta^\rho{}_\nu - 4\delta^\rho{}_\mu\gamma_\nu \quad \longleftarrow \text{Kommutator}$$

$$\rightsquigarrow S^{-1}(\Lambda)\gamma^\rho S(\Lambda) = \gamma^\rho - \frac{1}{2}\omega^{\mu\nu}\underline{(\gamma_\mu\delta^\rho{}_\nu - \gamma_\nu\delta^\rho{}_\mu)}$$

$$= \gamma^\rho - \frac{1}{2}(\omega^{\mu\rho}\gamma_\mu - \omega^{\rho\nu}\gamma_\nu)$$

$$= \gamma^\rho + \frac{1}{2}(\omega^{\rho\mu}\gamma_\mu + \omega^{\rho\nu}\gamma_\nu)$$

$$= \gamma^\rho + \omega^{\rho\mu}\gamma_\mu = \gamma^\rho + \omega^\rho{}_\mu\gamma^\mu$$

$$= (\delta^\rho{}_\mu + \omega^\rho{}_\mu)\gamma^\mu$$

$$= \Lambda^\rho{}_\mu\gamma^\mu \qquad \square$$

c) $$\bar\psi\gamma^\mu\psi \longmapsto \bar\psi S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi$$

$$= \bar\psi\Lambda^\mu{}_\nu\gamma^\nu\psi$$

$$= \Lambda^\mu \quad \bar\psi\gamma^\nu\psi$$

$$= \Lambda^\mu{}_\nu \bar{\Psi}\gamma^\nu\Psi$$

4. (i) $S_z = S_{12} = \frac{i}{4}[\gamma_1, \gamma_2]$

$\gamma_0 = \gamma^0$

$\gamma_i = -\gamma^i$

$\gamma_\mu = g_{\mu\nu}\gamma^\nu$

$$= \frac{i}{4}\left[\begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix}\right]$$

$$\cancel{= \frac{i}{4}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right.} = \frac{i}{4}\begin{pmatrix} \overset{-\sigma_x\sigma_y + \sigma_y\sigma_x}{[\sigma_y, \sigma_x]} & 0 \\ 0 & [\sigma_y, \sigma_x] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

ii) $K_x = S_{10} = \frac{i}{4}[\gamma_1, \gamma_0] = -\frac{i}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}\right]$

$$= -\frac{i}{4}\left(\begin{pmatrix} 2\sigma_x & 0 \\ 0 & -2\sigma_x \end{pmatrix}\right) = -\frac{i}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}$$

iii) $K_y = S_{20} = \frac{i}{4}[\gamma_2, \gamma_0] = -\frac{i}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}$

$$[S_z, K_x] = -\frac{i}{4}\left[\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}\right] = -\frac{i}{4}\begin{pmatrix} 2i\sigma_y & 0 \\ 0 & -2i\sigma_y \end{pmatrix} = i\left(-\frac{i}{2}\begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}\right) = iK_y$$

Dirac: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ $\quad \gamma_i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$

$$(i\partial_\mu\gamma^\mu - m)\Psi = 0$$

$$i\partial_t\gamma^0\Psi = (-i\partial_j\gamma^j + m)\Psi$$

$$i\partial_t\Psi = (-i\partial_j\gamma^0\gamma^j + m)\Psi$$

$$= \underbrace{\left(-i\,\partial_x \gamma^0\gamma^1 - i\,\partial_y \gamma^0\gamma^2 - i\,\partial_z \gamma^0\gamma^3 + \gamma^0 m\right)}_{=\hat{\mathcal{H}}} \psi$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & +\sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\hat{\mathcal{H}} = \begin{pmatrix} m\mathbb{1}_2 & -i(\partial_x\sigma_x + \partial_y\sigma_y + \partial_z\sigma_z) \\ -i(\partial_x\sigma_x + \partial_y\sigma_y + \partial_z\sigma_z) & m\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} m & -i\vec{\sigma}\cdot\nabla \\ -i\vec{\sigma}\cdot\nabla & -m \end{pmatrix}$$

$$(i\vec{\sigma}\cdot\vec{\nabla})^\dagger = i\vec{\sigma}\cdot\vec{\nabla}$$

$$S_{12} = S_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

$$[S_z, \mathcal{H}] = \frac{1}{2}\left[\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -i[\sigma_z, \vec{\sigma}\cdot\vec{\nabla}] \\ -i[\sigma_z, \vec{\sigma}\cdot\vec{\nabla}] & 0 \end{bmatrix}$$

$$[\sigma_z, \vec{\sigma}\cdot\vec{\nabla}]$$
$$= [\sigma_z, \sigma_x]\partial_x + [\sigma_z, \sigma_y]\partial_y$$
$$= 2i\,\sigma_y\partial_x - 2i\,\sigma_x\partial_y$$

$$= \begin{pmatrix} 0 & -\sigma_x\partial_y + \sigma_y\partial_x \\ \sigma_y\partial_x - \sigma_x\partial_y & 0 \end{pmatrix}$$

$$[L_{12}, \mathcal{H}] = \cancel{\tfrac{1}{2}}\, i\left[x\partial_y - y\partial_x, \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix}\right]$$

$$= -i\begin{pmatrix} 0 & -[x\partial_y - y\partial_x, i\vec{\sigma}\cdot\vec{\nabla}] \\ [x\partial_y - y\partial_x, i\vec{\sigma}\cdot\vec{\nabla}] & 0 \end{pmatrix}$$

$$\sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z$$

$$\underline{(x\partial_y - y\partial_x)(i\vec{\sigma}\cdot\vec{\nabla})} - (i\vec{\sigma}\cdot\vec{\nabla})(x\partial_y - y\partial_x) = (i\vec{\sigma}\cdot\vec{\nabla})(x\partial_y - y\partial_x) + (i\vec{\sigma}\cdot\vec{\nabla})(x\partial_y - y\partial_x)$$

$$= -i\sigma_x\partial_y + i\sigma_y\partial_x$$

$$[L_{12}, \mathcal{H}] = -\begin{pmatrix} 0 & -\sigma_x \partial_y + \sigma_y \partial_x \\ \sigma_y \partial_x - \sigma_x \partial_y & 0 \end{pmatrix}$$

$$[L_{12} + S_{12}, \mathcal{H}] = 0 \quad !$$