Quantentheorie II Übung 2

Besprechung: 2021WE17 (KW17)

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1. **Questions**

   (a) Explain the 6 generators of the Lorentz group.

   (b) Consider the infinitesimal Lorentz transformation $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$.
   Show that the scalar product $x^\mu x_\mu$ is Lorentz invariant when $\omega^{\mu\nu}$ is anti-symmetric (up to the first order of ω).

   (c) What do you remember about the 2-dimensional representations of the Lorentz group and why do we prefer the 4-dimensional Dirac spinor representation?

   (d) ψ and χ are Dirac spinors. Is $\bar{\psi}\chi$ Lorentz invariant?

   (e) What are the possible eigenvalues of the matrix $\not{p}$?

2. **Generators of Lorentz group:** a 4-vector x^μ transforms under a Lorentz transformation matrix Λ as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. For infinitesimal parameters $\omega^{\rho\sigma}(= -\omega^{\sigma\rho})$ the Lorentz transformation matrix can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \,. \tag{1}$$

Consider two specific infinitesimal Lorentz transformations:

(a) Rotation around the z-axis: $\omega^{12} = -\omega^{21} = \varepsilon\,$, otherwise $\omega^{\rho\sigma} = 0\,$,
(b) Boost along the x-direction: $\omega^{10} = -\omega^{01} = \beta\,$, otherwise $\omega^{\rho\sigma} = 0\,$.

where ε and β are infinitesimal parameters.

Use the definition of the generator **matrices** $k_x \equiv L_{10} = -L_{01}$ and $l_z \equiv L_{12} = -L_{21}$ from the lecture

$$k_x = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \, l_z = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and show that

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu \,. \tag{3}$$

is equivalent to Eq. (1) for cases of (a) Rotation and (b) Boost by plugging in the ω's and the L's and carrying out the summation over ρ and σ explicitly.

3. **Lorentz transformation of spinors:** the Lorentz transformation of a Dirac spinor associated with the Lorentz transformation matrix in Eq. (1) is written as

$$\psi' = S(\Lambda)\psi \,, \text{with } S(\Lambda) = \left(1 - \frac{i}{2}\omega^{\rho\sigma} S_{\rho\sigma}\right) \tag{4}$$

where

$$S_{\rho\sigma} \equiv \frac{i}{4}[\gamma_\rho, \gamma_\sigma] \,. \tag{5}$$

The γ-matrices are necessary to describe spin $\frac{1}{2}$ particles. The four 4×4 γ-matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4 \,.$$

In *Dirac* representation they are

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} , \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{6}$$

and in *Weyl(chiral)* representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{7}$$

where $i \in \{1, 2, 3\}$. The generalized hermiticity relation is $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

The Dirac adjoint is defined as $\bar{\psi} \equiv \psi^\dagger\gamma^0$.

   (a) Prove explicitly that $\bar{\psi}\psi$ is Lorentz invariant (filling in details which were skipped in the lecture).
   (b) Show $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ for the case of an infinitesimal transformation.
   (c) Use the result of (b) and prove that $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

4. **Commutator of Lorentz group generators:** using Eq. (5) compute the following expressions

$$\text{(i) } S_z \equiv S_{12} \,, \qquad \text{(ii) } K_x \equiv S_{10} \,, \qquad \text{(iii) } K_y \equiv S_{20} \,, \tag{8}$$

in Weyl (chiral) representation by evaluating the commutators explicitly and check whether

$$[S_z \,, K_x] = iK_y$$

is fulfilled.

You can repeat the calculation in the Dirac representation.

5. **Commutators of the Dirac Hamiltonian with angular momenta:** show that by multiplying the Dirac equation with γ^0 you obtain an equation of the form

$$i\partial_t\psi = H_D\psi.$$

Give the explicit 4×4 matrix form of H_D.

Calculate the commutators

$$[S_{12}, H_D] \quad \text{and} \quad [\hat{L}_{12}, H_D] \,,$$

where $\hat{L}_{12} \equiv i(x_1\partial_2 - x_2\partial_1)$ is the orbital angular momentum operator in z-direction.

What is the sum of the commutators and what does that mean?

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(a) 3× Drehung + 3× Boost   Lorentz

6 + 4 (PM) = 10 Generatoren   Poincaré

(b)
$$x'^{\mu} x'_{\mu} = (x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu})(x_{\mu} + \omega_{\mu}{}^{\nu} x_{\nu}) \hat{=} x^{\mu} x_{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} x_{\mu} + x^{\mu} \omega_{\mu}{}^{\nu} x_{\nu}$$

$$= x^{\mu} x_{\mu} + \underbrace{\omega^{\mu\nu} x_{\nu} x_{\mu} + \omega^{\mu\nu} x_{\mu} x_{\nu}}_{=0 \text{ wenn } \omega^{\mu\nu} = -\omega^{\nu\mu}} = x^{\mu} x_{\mu}$$

(c)
$$P: \left.\begin{matrix} \vec{x} \to -\vec{x} \\ \vec{p} \to -\vec{p} \end{matrix}\right\} \vec{J} \to \vec{J} \qquad \underline{\vec{K} \to -\vec{K}}$$

(e) $p^2 = m^2$ on-shell

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$$\hat{J} = \hat{L} + \hat{S}$$

$$H_D = -i\gamma^0\gamma^i\partial_i + m\gamma^0 \qquad (\gamma^0)^2 = \mathbb{1}$$

Dirac

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$H_D = -i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\partial_i + m\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}p^i + m\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$[S_{12}, H_D] = \qquad [\sigma_i,\sigma_j] = i2\varepsilon_{ijk}\sigma_k \qquad S_{12} = S_z = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

$$\frac{1}{2}\left\{\begin{pmatrix} 0 & [\sigma^3,\sigma^1] \\ [\sigma^3,\sigma^1] & 0 \end{pmatrix}p^1 + \begin{pmatrix} 0 & [\sigma^3,\sigma^2] \\ [\sigma^3,\sigma^2] & 0 \end{pmatrix}p^2\right\}$$

$$= -i\begin{pmatrix} 0 & \sigma^1p^2 - \sigma^2p^1 \\ \sigma^1p^2 - \sigma^2p^1 & 0 \end{pmatrix} = -i(\vec{\hat{S}}\times\vec{p})\cdot\vec{e}_z \qquad \vec{\hat{S}} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$[L_{12}, H_D] = \left[i(x_1\partial_2 - x_2\partial_1)\mathbb{1}_4, -i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\partial_i\right]$$

$$= \underbrace{\left[x_1\partial_2, \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\underset{\partial_1}{\partial_i}\right]}_{\sigma^1 p^2} - \underbrace{\left[x_2\partial_1, \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\underset{\partial_2}{\partial_i}\right]}_{\sim\sigma^2 p^1} = \dots = -[S_{12}, H_D]$$

$$[x_i, p_j] = i\delta_{ij}$$

$$\dots [x_1, \partial_1] = \sim i \dots$$

$$\Rightarrow [L_{12} + S_{12}, H_D] = 0$$

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Spinor space
Vektorraum

(a) $\psi \to S\psi \qquad \bar\psi \to \bar\psi S^{-1}$

$\bar\psi\psi \to \bar\psi S^{-1}S\psi = \bar\psi\psi$

$S^\dagger_{\rho\sigma} = \gamma^0 S_{\rho\sigma}\gamma^0$

$\psi' = S\psi = (1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma})\psi$

$\bar\psi' = \bar\psi S^{-1} = \psi^\dagger\underbrace{(1 + \frac{i}{2}\omega^{\rho\sigma}\gamma^0 S_{\rho\sigma}\gamma^0)}\gamma^0$

$\bar\psi'\psi' = \psi^\dagger(1 + \frac{i}{2}\omega^{\rho\sigma}\gamma^0 S_{\rho\sigma}\gamma^0)\gamma^0(1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma})\psi$

$\hat{=} \bar\psi\psi + \frac{i}{2}\psi^\dagger\gamma^0\omega^{\rho\sigma}S_{\rho\sigma}\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\psi$

$-\frac{i}{2}\psi^\dagger\gamma^0\omega^{\rho\sigma}S_{\rho\sigma}\psi = \bar\psi\psi$ Lorentz invariant

b) $S^{-1}(\Lambda)\gamma^\mu S(\Lambda)$

$= (1 + \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma})\gamma^\mu(1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma})$

$\hat{=} \gamma^\mu + \frac{i}{2}(\omega^{\rho\sigma}S_{\rho\sigma}\gamma^\mu - \gamma^\mu\omega^{\rho\sigma}S_{\rho\sigma})$

$= \gamma^\mu + \frac{i}{2}[\omega^{\rho\sigma}S_{\rho\sigma}, \gamma^\mu]$

$$= \gamma^\mu + \frac{i}{2}\frac{i}{4}\left[\omega^{\rho\sigma}[\gamma_\rho,\gamma_\sigma],\gamma^\mu\right]$$

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$

$$= \gamma^\mu - \frac{1}{8}\omega^{\rho\sigma}\underbrace{\left[[\gamma_\rho,\gamma_\sigma],\gamma^\mu\right]}$$

$$[A,BC] = ABC - BCA$$
$$= ABC + BAC - BAC - BCA$$
$$= -B\{A,C\} + \{A,B\}C$$

$$\left((*) = [\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho,\gamma^\mu] = (g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho)4 \right)$$

$$= \gamma^\mu - \frac{1}{2}\omega^{\rho\sigma}(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho)$$

$\rho \leftrightarrow \sigma$

$$= \gamma^\mu - \frac{1}{2}(\omega^{\rho\sigma}g^{\mu\rho}\gamma^\sigma - \overset{\omega^{\sigma\rho}}{\omega^{\rho\sigma}}g^{\mu\sigma}\gamma^\rho)$$

$$= \gamma^\mu + \omega^{\rho\sigma}g^{\mu\rho}\gamma^\sigma = \gamma^\mu + \omega_\mu{}^\sigma\gamma^\sigma$$
$$= (\delta^\mu{}_\sigma + \omega_\mu{}^\sigma)\gamma^\sigma$$
$$= \Lambda^\mu{}_\sigma\gamma^\sigma$$

c) $\bar\psi\gamma^\mu\psi \longrightarrow \Lambda^\mu{}_\sigma\,\bar\psi\gamma^\sigma\psi$

wie ein 4er Vektor

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$\underline{\rho \neq \sigma}$ $\quad S_{\rho\sigma} = \frac{i}{2}\gamma_\rho\gamma_\sigma$

$\underline{\rho = \sigma}$ $\quad S_{\sigma\sigma} = 0$

$$[\gamma_\rho, \gamma_\sigma] = 2\gamma_\rho\gamma_\sigma - \underbrace{\{\gamma_\rho, \gamma_\sigma\}}_{2g_{\rho\sigma}\mathbb{1}}$$

$\gamma^{0^2} = \mathbb{1} \qquad \gamma^{k^2} = -\mathbb{1}$ $\qquad = 0$

Weyl representation

(i) $S_z = S_{12} = \frac{i}{2}\gamma_1\gamma_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$

(ii) $K_x = S_{10} = \frac{i}{2}\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$

(iii) $K_y = S_{20} = \frac{i}{2}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$

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and show that

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu \,. \tag{3}$$

is equivalent to Eq. (1) for cases of (a) Rotation and (b) Boost by plugging in the ω's and the L's and carrying out the summation over ρ and σ explicitly.

Ansatz:

zz. $\omega^\mu{}_\nu = -\frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu$

$\hat{=}$ l_z Ü1

$\rightsquigarrow$ die bekannte 3Dim Rotation wurde in 4Dim eingebettet!

a) $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\left(\omega^{12}L_{12} + \omega^{21}L_{21}\right)^\mu{}_\nu = \delta^\mu{}_\nu - i\left(\omega^{12}L_{12}\right)^\mu{}_\nu$

$= \delta^\mu{}_\nu - i\varepsilon\begin{pmatrix} & l_z & \end{pmatrix} = \delta^\mu{}_\nu + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{= \omega^\mu{}_\nu}$

noch zz. Zusammenhang
$\omega^\mu{}_\nu \longleftrightarrow \omega^{\mu\nu}$ (Aufgabenstellung)