Quantentheorie II Übung 10

Besprechung: 2021WE26 (KW26)

SS 2021 Prof. Dominik Stöckinger (IKTP)

1. **Questions**
   (a) In the lecture simple classical scattering situations were discussed (football against concrete wall, football rolls on/off curb, etc). Map these cases to the cases of quantum mechanical scattering at Yukawa/Coulomb/δ-function potential and on the potential of a Gaussian charge distribution. Answer in particular:
      i. What is the $\cos\theta$-dependence of scattering at the δ-potential — explain in simple terms!
      ii. What is the approximate $\cos\theta$-dependence of scattering at the Yukawa potential for very small k^2 — explain in simple terms!
      iii. Which case(s) are similar to rolling over a curb — consider in particular the possibility of large scattering angles.

   (b) Remind yourself of the key observation in Rutherford scattering — how does Rutherford scattering prove the existence of a hard and small nucleus as opposed to the earlier "plum pudding" model of atoms? Which of our situations is similar to this comparison?

2. **Scattering:** a particle of mass m scatters at a potential

$$V(r) = -V_0 e^{-r/R_0}, \text{ where } V_0 > 0, \text{ and } r \equiv |\vec{x}|.$$

   (a) Compute the scattering amplitude $f(\theta,\cancel{\phi})$ in first Born approximation.
   (b) Compare with the results for the Yukawa potential at small/large momentum transfer $\vec{q} = \vec{k}' - \vec{k}$, small/large angles and small/large r.

3. **Scattering by sphere:** a particle of mass m scatters at a potential

$$V(r) = \begin{cases} -V_0 & \text{for } r < R_0 \\ 0 & \text{for } r \geq R_0 \end{cases} \quad \text{where } , V_0 > 0 \text{ and } r \equiv |\vec{x}|.$$

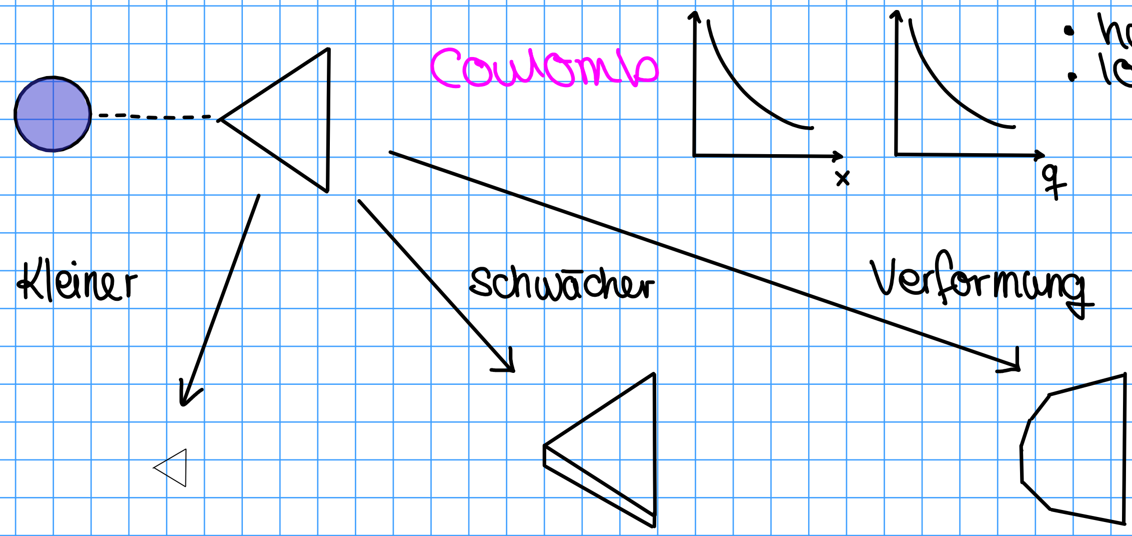
   (a) Compute the scattering amplitude $f(\theta,\cancel{\phi})$ in first Born approximation.
   (b) Compare with the results for the Yukawa potential and δ-function potential at small/large momentum transfer $\vec{q} = \vec{k}' - \vec{k}$, small/large angles and small/large r.
   (c) Determine the differential cross section $d\sigma/d\Omega$ and discuss for small particle energies $kR_0 \ll 1$.

4. **Green function for spherical waves:** look up the expressions for Δ and $\vec{\nabla}$ in spherical coordinates and use the known result $\Delta \frac{-1}{4\pi|\vec{x}|} = \delta^{(3)}(\vec{x})$ to show

$$(\Delta + k^2) \frac{-e^{ik|\vec{x}|}}{4\pi|\vec{x}|} = \delta^{(3)}(\vec{x}). \qquad (1)$$

5. **Fourier transformation:** compute the Fourier transform of the Yukawa potential by explicit integration!

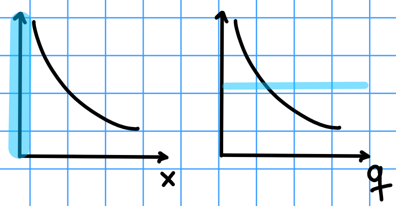
1.



- harter Kern
- long range (∞)

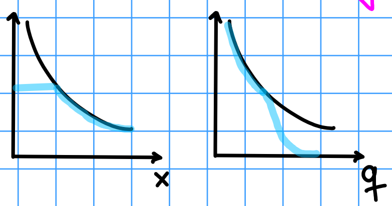
Große Streuteilchen — Stärke Potential — ?

Delta



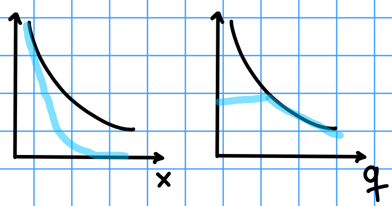
- harter Kern

Gauß-Ladung



- $\neq$ harter Kern
- $\neq$ long range

Yukawa

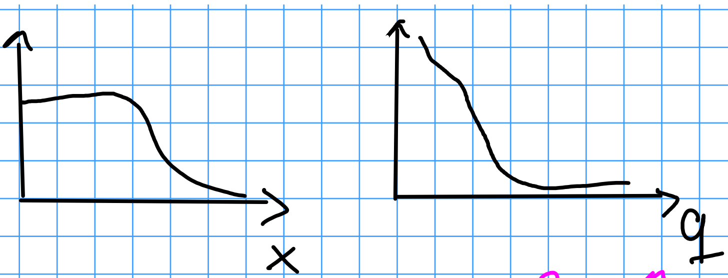


- harter Kern
- kleine q: Delta
- long range

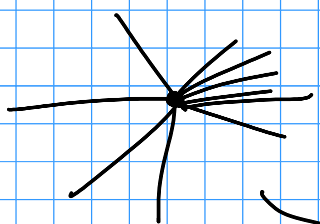
von weit weg (kleine q) wie Delta (Peak)

(b) Remind yourself of the key observation in Rutherford scattering — how does Rutherford scattering prove the existence of a hard and small nucleus as opposed to the earlier "plum pudding" model of atoms? Which of our situations is similar to this comparison?

Plum pudding:



groß q $\hat{=}$ groß θ



$\rightarrow$ "harter Kern" = Protonen

$\alpha \sim 7.7$ MeV

$\hbar c = 0.2$ GeV fm

$1 = 200$ MeV fm

Reminder:

$$\psi \hat{=} e^{i\vec{k}\vec{x}} + f(\theta,\varphi)\,\frac{e^{ikr}}{r} \qquad r = |\vec{x}| \gg \text{Reichweite}$$

$$\hookrightarrow \frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2 \quad !$$

Wie $f(\theta,\varphi)$ berechnen?

Greensche Fkt:
$$\psi \hat{=} e^{i\vec{k}\vec{x}} + \int d^3x'\, G(x-x')\,\psi(x')$$
$$G \sim \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

1. Auswertung G

Reichweite $|\vec{x}'| \ll |\vec{x}|$

$$G \sim \frac{e^{ikr}\, e^{-i\vec{k}'\vec{x}'}}{r} \qquad \underbrace{\vec{k}' \equiv \vec{e}_x\, k}_{\text{Endzustandsimpuls}}$$

2. Bornsche Näherung

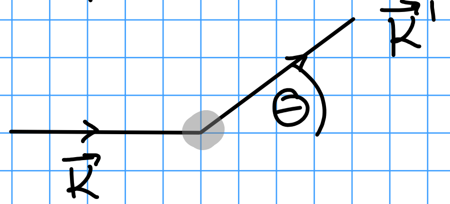
$$f(\theta,\varphi) = f(\vec{e}_{k'}) = -\frac{m}{2\pi}\,\tilde{V}(\vec{q}) \qquad \vec{q} = \vec{k}' - \vec{k}$$

Fouriertransformation:

$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}}\, \tilde{V}(\vec{q})$$

$$\tilde{V}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\vec{x}}\, V(\vec{x})$$

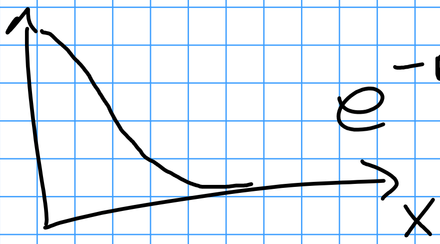
Zentralpotential (kein φ)



$$|k| = |k'|$$

$$\vec{q}^{\,2} = k^2 + |k'|^2 - 2k|k'|\cos\Theta$$
$$= 2k^2(1-\cos\Theta)$$
$$= 4k^2 \sin^2(\Theta/2)$$

2) $$f(\theta) = \frac{4m R_0^3 V_0}{\left[1 + q^2 R_0^2\right]^2}$$

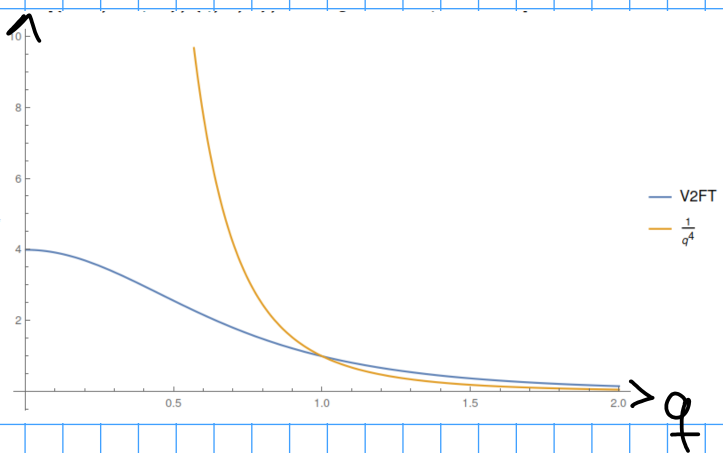


e^{-r/R_0}

große x $\Rightarrow$ Yukawa

$\Leftarrow$ kleine q

$\Rightarrow$ $f(\theta) \approx$ const.



kleine x $V \approx$ const $\lightning$ Yukawa

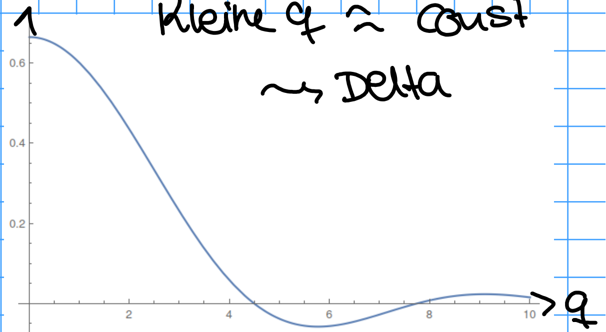
$\rightarrow$ gauß Ladung

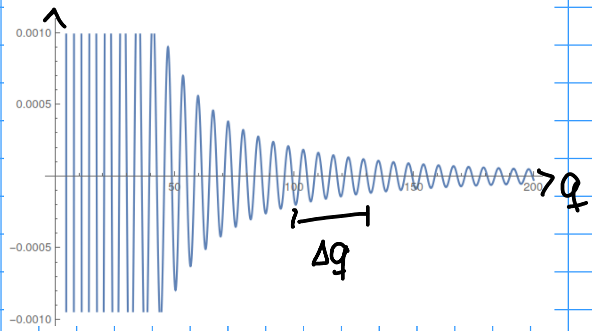
$\Leftrightarrow$ große q $\sim \frac{1}{q^4}$ $\rightarrow$ stark unterdrückt

3) $$f(\theta) = \frac{2mV_0}{q^3} \cdot \left(\sin(qR_0) - qR_0 \cos(qR_0)\right)$$

kleine q $\approx$ const

$\rightsquigarrow$ Delta





$\Rightarrow$ scharf begrenzt

$\Rightarrow$ Oszillationen

4. **Green function for spherical waves:** look up the expressions for Δ and $\vec{\nabla}$ in spherical coordinates and use the known result $\Delta\frac{-1}{4\pi|\vec{x}|} = \delta^{(3)}(\vec{x})$ to show

$$(\Delta + k^2)\underbrace{\frac{-e^{ik|\vec{x}|}}{4\pi|\vec{x}|}}_{G(x)} = \delta^{(3)}(\vec{x}). \qquad (1)$$

$$G = -\frac{1}{4\pi} f_1 \cdot f_2 \qquad f_1 = e^{ikr} \qquad f_2 = \frac{1}{r}$$

$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + 2(\vec{\nabla} f_1)(\vec{\nabla} f_2) + f_1(\Delta f_2)$$

$$\vec{\nabla} f_1 = ik\vec{e}_r f_1 \qquad \vec{\nabla} f_2 = -\frac{1}{r^2}\vec{e}_r = -\frac{1}{r} f_2 \vec{e}_r$$

$$= \left(\vec{\nabla}(ik\vec{e}_r) f_1\right) f_2 + 2\left(ik f_1 \left(-\frac{1}{r}\right) f_2\right) + \underbrace{f_1(\Delta f_2)}_{-4\pi\delta^{(3)}(\vec{x})\cdot \cancel{f_1}}$$

$$\Delta f_2 = \Delta\frac{1}{r} = -4\pi\delta^{(3)}(\vec{x})$$

$$= \left((ik\vec{e}_r)(ik\vec{e}_r) f_1 + \cancel{ik\frac{2}{r} f_1}\right) f_2 \cancel{- 2ik\frac{1}{r}} f_1 f_2 - 4\pi\delta^{(3)}(\vec{x})$$

$$\vec{\nabla}\frac{\vec{r}}{r} = \frac{1}{r}\underbrace{(\vec{\nabla}\vec{r})}_{3} + \underbrace{\vec{r}\,\vec{\nabla}\frac{1}{r}}_{\vec{r}\left(-\frac{1}{r^2}\vec{e}_r\right) = -\frac{1}{r}\vec{e}_r\vec{e}_r} = \frac{2}{r}$$

$$\Delta(f_1 f_2) = -k^2 f_1 f_2 - 4\pi\delta^{(3)}(\vec{x})$$

$$\Rightarrow (\Delta + k^2) f_1 f_2 = -4\pi\delta^{(3)}(\vec{x}) \qquad \square$$

5. **Fourier transformation:** compute the Fourier transform of the Yukawa potential by explicit integration!

$$\tilde{v}(q) = \int d^3x \; e^{-i\vec{q}\vec{x}} \; \frac{e^{-Mr}}{4\pi r} \qquad \bar{\Theta} = \sphericalangle(\vec{q}, \vec{x})$$

$$= \frac{1}{4\pi} \int_0^\infty dr \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\bar{\Theta} \; r^{\cancel{2}} \; \frac{e^{-iqr\cos\bar{\Theta}} \, e^{-Mr}}{\cancel{r}}$$

$$= \frac{1}{2} \int_0^\infty dr \; \cancel{r} \; e^{-Mr} \; \frac{2\sin qr}{q\cancel{r}}$$

$$= \frac{1}{q} \underbrace{\int_0^\infty dr \; e^{-Mr} \sin(qr)}_{\frac{q}{q^2+M^2}} = \frac{1}{\cancel{q}} \frac{\cancel{q}}{q^2+M^2}$$

$$= \frac{1}{q^2+M^2}$$

$$M \to 0 \quad \Rightarrow \quad \frac{1}{q^2}$$

②

erste Born'sche Näherung

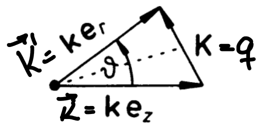
$$f^{(1)}(\vartheta,\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3r'\, V(\boldsymbol{r}')\, e^{-ik(\boldsymbol{e}_r-\boldsymbol{e}_z)\cdot\boldsymbol{r}'}$$

$$\left(z' = \boldsymbol{r}'\cdot\boldsymbol{e}_z\,;\quad \vartheta = \sphericalangle(\boldsymbol{e}_r,\boldsymbol{e}_z)\right)\,. \tag{9.91}$$

In erster Näherung ist die Streuamplitude im wesentlichen gleich der Fourier-Transformierten $V(\boldsymbol{K})$ $\left(\boldsymbol{K} = k(\boldsymbol{e}_r - \boldsymbol{e}_z)\right)$ des Wechselwirkungspotentials. – Beschränken wir uns nun für die weitere Auswertung auf **zentralsymmetrische Potentiale**

$$V(\boldsymbol{r}') = V(r')\,,$$

Abb. 9.12 Winkelbeziehungen für die Berechnung der Streuamplitude in erster Born'scher Näherung



so können wir die Winkelintegrationen in (9.91) explizit durchführen. Aus Abb. 9.12 entnehmen wir:

$$\vec q = K = k(\boldsymbol{e}_r - \boldsymbol{e}_z) := k\vec e_x - k\vec e_z = \vec k' - \vec k$$

$$q = K = 2k\sin\frac{\Theta}{2}\,. \tag{9.92}$$

Legen wir die Polarachse parallel zu $\boldsymbol{r}'$, so folgt für das Integral in (9.91):

$$\int d^3r'\, V(r')\, e^{-ik(\vec e_r - \vec e_z)\cdot\vec r'}$$

$$= 2\pi\int_0^\infty dr'\, r'^2\, V(r')\int_{-1}^{+1} dx\, e^{-iKr'x} = \frac{4\pi}{K}\int_0^\infty dr'\, r'\, V(r')\sin(Kr')\,.$$

$$= \int d^3x\, e^{-i\vec q\,\vec x}\, V(x) \qquad |\vec x| = r$$

$$= \int dr\int d\varphi\int_0^\pi d\bar\Theta\, \underline{\sin\bar\Theta}\; r^2\, \underline{e^{-iqr\cos\bar\Theta}}\; V(x)$$

$$\int_{-1}^{1} d\cos\bar\Theta$$

Nolting 5/12

2. 9.3.2

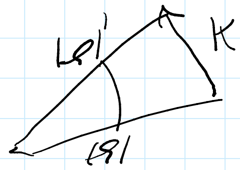
3. 9.3.3

$$-d\bar\Theta\,\sin\bar\Theta = d\cos\bar\Theta$$

$$= -\sin\bar\Theta\, d\bar\Theta$$

Nr 2) $V_0 = -V_0\, e^{-r/R_0}$ $\quad V_0 > 0 \quad r = |\vec{x}| \quad q = \sin$

$$\int d^3r' V(r')\, e^{-i q(\vec{e}_{k'} - \vec{e}_r)\vec{r}\,'} = 2\pi \int_0^\infty dr' r'^2 V(r') \int_{-1}^{+1} dx\, e^{-i q r' x}$$



$q = K$

$$\Rightarrow f^{(1)}(\varphi) = -\frac{2m}{K} \int dr'\, r'\, V(r') \sin(K r')$$

$$= -\frac{2mV_0}{K} \int dr'\, r'\, e^{-r'/R_0} \sin(K r')$$

$u = \frac{r'}{R_0} \qquad dr' = du\, R_0$

$$= \frac{-2m V_0 R_0^2}{K} \int u\, e^{-u} \sin(K R_0 u)\, du \qquad K \cdot R_0 = \beta$$

$$= \frac{+2m R_0^3 V_0}{\beta} \frac{d}{d\beta} \int_0^\infty e^{-u} \cos(\beta u)\, du$$

$$= \frac{2m R_0^3 V_0}{\beta} \frac{d}{d\beta} \mathrm{Re}\left(\int_0^\infty e^{-u + i\beta u}\, du \right)$$

$$= -\frac{2m R_0^3}{\beta} V_0 \frac{d}{d\beta} \mathrm{Re}\left(\frac{1}{1+i\beta}\right) = -\frac{2m R_0^3}{\beta} V_0 \frac{d}{d\beta} \frac{1}{1+\beta^2}$$

$\Rightarrow \frac{2\beta}{(1+\beta^2)^2}$

$$\frac{1}{1+i\beta} \frac{1-i\beta}{1-i\beta} = \frac{1}{1+\beta^2} - \mathrm{Im}(\dots)$$

$$= 4mV_0 R_0^3 \frac{1}{\left[1 + 4 q^2 R_0^2 \sin\left(\frac{\theta}{2}\right)\right]^2}$$

Nr 3) $$f^{(1)}(\theta) = -\frac{2mV_0}{K}\int_0^{R_0} dr'\, r' \sin(Kr')$$

$$= -\frac{2mV_0}{K}\frac{d}{dK}\int_0^{R_0} \cos(Kr')\, dr'$$

$$= -\frac{2mV_0}{K}\frac{d}{dK}\left(\frac{1}{K}\sin(KR_0)\right) = -\frac{2mV_0}{K}\left(-\frac{\sin(KR_0)}{K^2} + \frac{R_0\cos(KR_0)}{K}\right)$$

$$= \frac{2mV_0}{K^3}\left(\sin(KR_0) - KR_0\cos(KR_0)\right)$$

c)

$$\frac{d\sigma^{(1)}}{d\Omega} = |f^{(1)}(\theta)|^2 = \left(\frac{2mV_0R_0^3}{K^3R_0^3}\right)\left(\sin(KR_0) - KR_0\cos(KR_0)\right)^2$$

$qR_0 \ll 1 \Rightarrow KR_0 \ll 1$

L'Hospital

$$\left(\lim_{x\to 0}\frac{\sin(x) - x\cos(x)}{x^3}\right)^2 = \lim_{x\to 0}\frac{\cos(x) - \cos(x) + x\sin(x)}{3x^2}$$

$$= \lim_{x\to 0}\frac{\sin(x)}{3x^2} = \lim_{x\to 0}\frac{\cos(x)}{3} = \frac{1}{3}$$

$$\lim_{R_0 q\to 0}\frac{d\sigma}{d\Omega} = \frac{1}{9}\left(2mV_0R_0^3\right)^2$$