Quantentheorie II Übung 2

Besprechung: 2021WE17 (KW17)

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1. **Questions**
   (a) Explain the 6 generators of the Lorentz group.
   (b) Consider the infinitesimal Lorentz transformation $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$.
   Show that the scalar product $x^\mu x_\mu$ is Lorentz invariant when $\omega^{\mu\nu}$ is anti-symmetric (up to the first order of ω).
   (c) What do you remember about the 2-dimensional representations of the Lorentz group and why do we prefer the 4-dimensional Dirac spinor representation?
   (d) ψ and χ are Dirac spinors. Is $\bar{\psi}\chi$ Lorentz invariant?
   (e) What are the possible eigenvalues of the matrix $\not{p}$?

2. **Generators of Lorentz group:** a 4-vector x^μ transforms under a Lorentz transformation matrix Λ as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. For infinitesimal parameters $\omega^{\rho\sigma}(= -\omega^{\sigma\rho})$ the Lorentz transformation matrix can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \,. \tag{1}$$

Consider two specific infinitesimal Lorentz transformations:

(a) Rotation around the z-axis: $\omega^{12} = -\omega^{21} = \varepsilon\,$, otherwise $\omega^{\rho\sigma} = 0\,$,
(b) Boost along the x-direction: $\omega^{10} = -\omega^{01} = \beta\,$, otherwise $\omega^{\rho\sigma} = 0\,$.

where ε and β are infinitesimal parameters.

Use the definition of the generator **matrices** $k_x \equiv L_{10} = -L_{01}$ and $l_z \equiv L_{12} = -L_{21}$ from the lecture

$$k_x = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \, l_z = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and show that

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu \,. \tag{3}$$

is equivalent to Eq. (1) for cases of (a) Rotation and (b) Boost by plugging in the ω's and the L's and carrying out the summation over ρ and σ explicitly.

3. **Lorentz transformation of spinors:** the Lorentz transformation of a Dirac spinor associated with the Lorentz transformation matrix in Eq. (1) is written as

$$\psi' = S(\Lambda)\psi\,, \text{with } S(\Lambda) = \left(1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma}\right) \tag{4}$$

where

$$S_{\rho\sigma} \equiv \frac{i}{4}[\gamma_\rho, \gamma_\sigma]\,. \tag{5}$$

The γ-matrices are necessary to describe spin $\frac{1}{2}$ particles. The four 4×4 γ-matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4\,.$$

In *Dirac* representation they are

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{6}$$

and in *Weyl(chiral)* representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{7}$$

where $i \in \{1, 2, 3\}$. The generalized hermiticity relation is $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

The Dirac adjoint is defined as $\bar{\psi} \equiv \psi^\dagger\gamma^0$.

   (a) Prove explicitly that $\bar{\psi}\psi$ is Lorentz invariant (filling in details which were skipped in the lecture).

   (b) Show $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ for the case of an infinitesimal transformation.

   (c) Use the result of (b) and prove that $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

4. **Commutator of Lorentz group generators:** using Eq. (5) compute the following expressions

$$\text{(i) } S_z \equiv S_{12}\,, \qquad \text{(ii) } K_x \equiv S_{10}\,, \qquad \text{(iii) } K_y \equiv S_{20}\,, \tag{8}$$

in Weyl (chiral) representation by evaluating the commutators explicitly and check whether

$$[S_z\,, K_x] = iK_y$$

is fulfilled.

You can repeat the calculation in the Dirac representation.

5. **Commutators of the Dirac Hamiltonian with angular momenta:** show that by multiplying the Dirac equation with γ^0 you obtain an equation of the form

$$i\partial_t\psi = H_D\psi.$$

Give the explicit 4×4 matrix form of H_D.

Calculate the commutators

$$[S_{12}, H_D] \quad \text{and} \quad [\hat{L}_{12}, H_D]\,,$$

where $\hat{L}_{12} \equiv i(x_1\partial_2 - x_2\partial_1)$ is the orbital angular momentum operator in z-direction.

What is the sum of the commutators and what does that mean?