Quantentheorie II Übung 10

– Sample solutions –

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2. **Scattering:** for a given potential $V(r)$ where $r \equiv |\vec{x}|$, the first Born approximation of scattering amplitude is

$$f^{(1)}(\theta) = -\frac{2m}{q}\int drrV(r)\sin(qr), \tag{1}$$

where $q = 2k\sin\frac{\theta}{2}$, or equivalently from the lecture we have

$$f^{(1)}(\theta) = -\frac{m}{2\pi}\tilde{V}(\vec{q}), \text{ where } \tilde{V}(\vec{q}) = \int d^3xe^{-i\vec{q}\cdot\vec{x}}V(\vec{x}). \tag{2}$$

(a) The potential is given as $V(r) = -V_0 e^{\frac{-r}{R_0}}$, where $r \equiv |\vec{x}|$ and $V_0 > 0$ and the scattering amplitude is

$$\begin{aligned}
f^{(1)}(\theta) &= -\frac{2m}{q}\int_0^\infty drr(-V_0e^{-\frac{r}{R_0}})\sin qr \\
&= \frac{2m}{q}V_0R_0^2\int_0^\infty dxxe^{-x}\sin(R_0qx) \Longleftarrow x \equiv \frac{r}{R_0} \\
&= \frac{2m}{q}V_0R_0^2\left(-\frac{d}{dp}\int_0^\infty dxe^{-x}\cos(px)\right) \Longleftarrow p \equiv R_0q \\
&= \frac{2m}{q}V_0R_0^2\left(-\frac{d}{dp}\frac{1}{1+p^2}\right) \\
&= 4mV_0R_0^3\frac{1}{(1+4R_0^2q^2\sin^2\frac{\theta}{2})^2}
\end{aligned} \tag{3}$$

(b) For small q ($\sim$ small θ) $f^{(1)}(\theta)$ is constant. For large q, $f^{(1)}(\theta) \propto \frac{1}{\sin^4\frac{\theta}{2}} \implies$ strong superposition. For large r, $f^{(1)}(\theta)$ is similar to that of Yukawa or exponential potential, but for small r, it becomes similar to that of charge distribution potential.

3. **Scattering by shpere:**

(a)

$$\begin{aligned}
f^{(1)}(\theta) &= \frac{-2m}{q}\int drrV(r)\sin(qr) \\
&= \frac{2m}{q}V_0\int_0^{R_0} drr\sin(qr) \\
&= \frac{2m}{q}V_0\Big(\sin(qR_0) - qR_0\cos(qR_0)\Big).
\end{aligned} \tag{4}$$

(b) For small q ($\sim$ large r) $f^{(1)}(\theta) \sim \frac{2}{3} m V_0 R_0^3$, which is constant and independent of θ like δ or Yukawa potential cases, but for large q, $f^{(1)}(\theta) \propto \frac{\cos(qR_0)}{q^2}$, which indicates large interference.

4. **Green function:** in spherical coordinates

$$\nabla^2 = \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial \phi^2}\right), \tag{5}$$

and for $f = f_1 f_2$

$$\nabla^2(f_1 f_2) = (\nabla^2 f_1) f_2 + f_1(\nabla^2 f_2) + 2\nabla f_1 \cdot \nabla f_2. \tag{6}$$

For $f(k,r) = -\frac{e^{ikr}}{r} = e^{ikr} \times \frac{-1}{r}$,

$$\begin{aligned}
\nabla^2(-\frac{e^{ikr}}{r}) &= (\nabla^2 e^{ikr})\left(\frac{-1}{r}\right) + e^{ikr}\nabla\frac{-1}{r} + 2(\nabla e^{ikr})(\nabla\frac{-1}{r}) \\
&= \left(\frac{2ik}{r} - k^2\right)\left(\frac{-e^{ikr}}{r}\right) + e^{ikr}4\pi\delta^{(3)}(\vec{x}) + 2ik\frac{e^{ikr}}{r^2} \\
&= \frac{k^2}{r}e^{ikr} + 4\pi\delta^{(3)}(\vec{x})e^{ikr} \qquad (7) \\
\implies (\nabla^2 + k^2)\frac{-e^{ik|\vec{x}|}}{4\pi|\vec{x}|} &= \delta^{(3)}(\vec{x}). \qquad (8)
\end{aligned}$$

5. **Fourier transformation:** Yukawa potential is defined as

$$V(r) = \alpha\frac{e^{\kappa r}}{r}, \tag{9}$$

and the Fourier transformation is

$$\begin{aligned}
V(\vec{q}) &= \int d^3x e^{-i\vec{x}\cdot\vec{q}} V(\vec{x}) \\
&= \alpha \int dr r^2 d\theta \sin\theta d\phi e^{-irq\cos\theta}\frac{e^{-\kappa r}}{r} \\
&= 2\pi\alpha \int_0^\infty dr r^2 \int_{-1}^1 d\cos\theta e^{-irq\cos\theta}\frac{e^{-\kappa r}}{r} \\
&= 2\pi\alpha \int_0^\infty dr \frac{e^{-\kappa r}}{-iq}(e^{-irq} - e^{irq}) \\
&= 4\pi\frac{\alpha}{q}\int_0^\infty dr e^{-\kappa r}\sin(rq) = 4\pi\frac{\alpha}{q}\,\mathrm{Im}\left(\int_0^\infty dr e^{r(iq-\kappa)}\right) \\
&= 4\pi\alpha\frac{1}{q^2+\kappa^2} \qquad (10)
\end{aligned}$$