Quantentheorie II Übung 11

Besprechung: 2021WE27 (KW27)

SS 2021 Prof. Dominik Stöckinger (IKTP)

1. **Questions**
   (a) What is the "optical theorem" and its interpretation?
   (b) What are the so-called "unitarity bounds" on the partial cross sections σ_l?
   (c) Which partial wave is dominant in case of the δ-function potential? (In other words, which δ_l are largest/non-zero?)
   (d) Which of the spherical harmonics Y_{lm} represents best the temperature distribution of the earth's surface?

2. **Green function and Fourier transformation:** compute the integral
$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x}}}{k^2 - q^2 - i\varepsilon}$$
in the limit $\varepsilon \to 0_+$ (the sign of the imaginary term is opposite to the lecture).

3. **S-wave scattering:** the differential cross section of a pure s-wave scattering is given as
$$\frac{d\sigma}{d\Omega} = a, \text{ where } a > 0.$$
What is the complex scattering amplitude $f(\theta)$?

4. **Scattering on sphere:** we consider s-wave scattering on a homogeneous sphere potential
$$V(r) = \begin{cases} V_0 > 0 & \text{for } r < a, \\ 0 & \text{for } r \geq a. \end{cases}$$
   (a) Take first $V_0 \to \infty$ and solve the radial part of the Schrödinger equation exactly for $l = 0$, i.e. obtain the radial function $u_0(r)$ and the scattering phase δ_0 exactly.
   (b) What is the equation for the scattering phase δ_0 when $E < V_0$ (now $V_0 < \infty$)?
   (c) Estimate the scattering phase δ_0 and the partial cross section σ_0 when the energy of the incoming particle is small, $k \to 0$.

5. **Scattering on exponential potential (again):** a particle with mass m is scattered on the potential
$$V(r) = -V_0 e^{-\frac{r}{R_0}}, \quad V_0 > 0.$$
   (a) Compute the scattering phase of s-wave scattering δ_0. Use the scattering amplitude $f(\theta, \phi)$ obtained in the exercise 10.
   (b) What is the equation to determine the phase of the p-wave scattering δ_1?
   (c) The Born approximation is valid when $|\psi^{(1)}(\vec{r})| \ll |\phi_k(\vec{r})| = 1$, or equivalently
$$\frac{m}{2\pi} \left| \int d^3r' V(r') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} e^{ikz'} \right| \ll 1 \quad \xrightarrow{r\to 0} \quad \left| \int_0^\infty dr V(r) \left(e^{i2kr} - 1\right) \right| \ll \frac{k}{m}, \qquad (1)$$
   What are the conditions to satisfy Eq. (1) for $kR_0 \gg 1$ and $kR_0 \ll 1$ respectively?

1. **Questions**

(a) What is the "optical theorem" and its interpretation?

(b) What are the so-called "unitarity bounds" on the partial cross sections σ_l?

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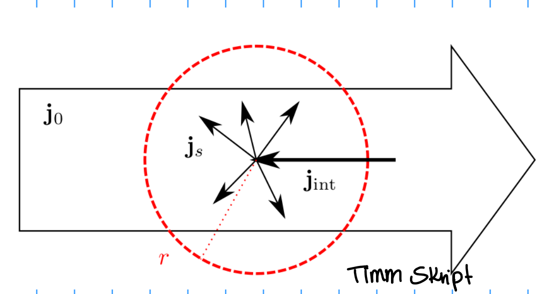
zentral potential $\Rightarrow m=0$

a)



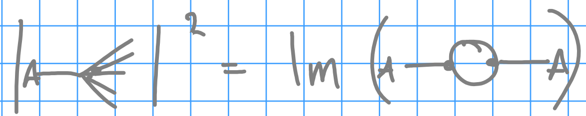
$$\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im}\left[f(\theta=0)\right]$$

$$I = I_0 + I_s + I_{int}$$

ebene Welle Streuung $\sim f$

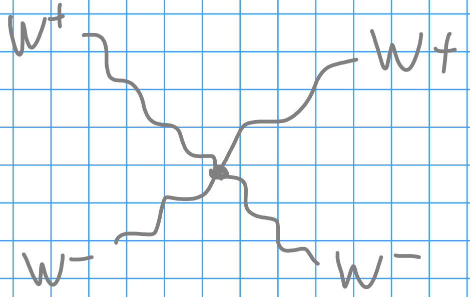
$$\psi = \phi + f\frac{e^{ikr}}{r} \qquad \phi = e^{i\vec{k}\vec{r}} \qquad \theta = 0$$

$$\vec{k}\vec{r} = kr \cdot \cos\theta = kr$$

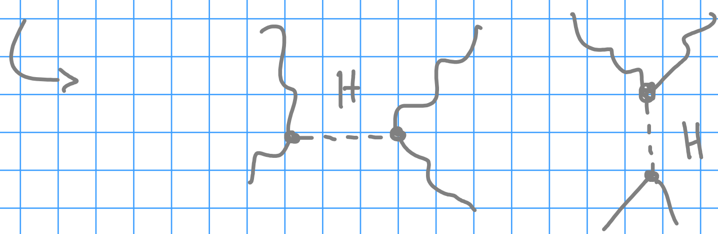


b) $$\sigma = \sum_l \sigma_l = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l$$

$$\sigma_l \leq \frac{4\pi}{k^2}(2l+1)$$



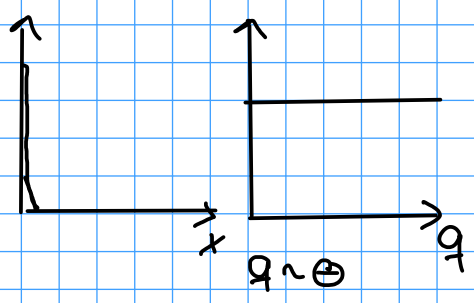
$$\frac{d\sigma}{d\Omega} \sim s = (p_1+p_2)^2$$



c) $$\psi = \sum_l \frac{u_l(r)}{r} P_l(\cos\theta)$$

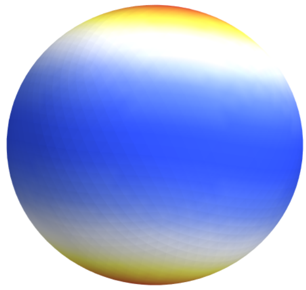
$u_l \sim \sin(x + l\frac{\pi}{2})$ ebene Welle

$u_l \sim \sin(x + l\frac{\pi}{2} + \delta_l)$ gestreute Welle

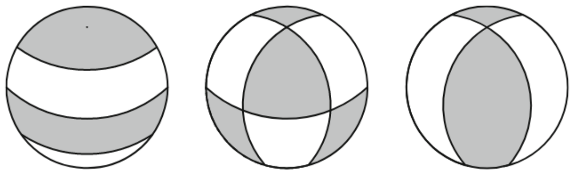


d)

$\text{Re}\left[Y_{l=2,m=0}\right] =$



$P_{l=2}$



(a) n=3 (b) n=3, m=2 (c) n=m=2

Fig. 3 Spherical harmonics: **a** zonal, **b** tesseral, **c** sectoral

2. **Green function and Fourier transformation:** compute the integral

$$\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{x}}}{k^2-q^2-i\varepsilon}$$

in the limit $\varepsilon \to 0_+$ (the sign of the imaginary term is opposite to the lecture).

a) $-\frac{1}{4\pi}\frac{e^{ikr}}{r}$

b) $+\frac{1}{4\pi}\frac{e^{ikr}}{r}$

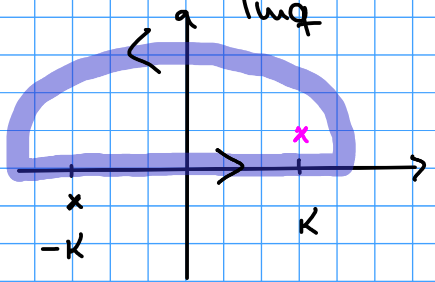
c) $+\frac{1}{4\pi}\frac{e^{-ikr}}{r}$

d) $-\frac{1}{4\pi}\frac{e^{-ikr}}{r}$

Vorlesung:

$$(\#)\ \lim_{\varepsilon\to 0}\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\vec{x}}}{k^2-q^2+i\varepsilon} = -\frac{1}{4\pi}\frac{e^{ikr}}{r}$$

▷ $k^2-q^2+i\varepsilon \Rightarrow q^{*2} = k^2+i\varepsilon \Rightarrow q^*_\pm = \pm(k+i\varepsilon)$



$$q = Re^{i\varphi}$$

$$e^{iqr} = e^{i\,\mathrm{Re}(q)r}\,\underbrace{e^{-\overbrace{\mathrm{Im}(q)}^{\to\text{positiv}}r}}_{\text{exp Unterdrück.}} \xrightarrow[R\to\infty]{} 0$$

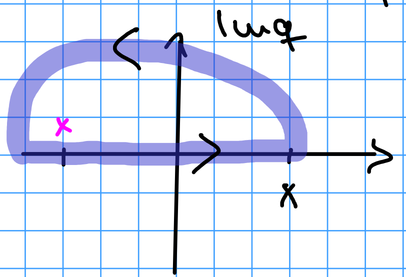
nach Winkel.

$$(\#) = -q\frac{e^{iqr}}{k^2-q^2+i\varepsilon} = \frac{1}{q-q^*}\frac{-q}{q+q^*}e^{iqr}$$

$$\mathrm{Res}\,f = c_{-1} = \lim_{z\to a}(z-a)f(a) \quad a\ \text{Polstelle}$$

$$\mathrm{Res}(f) = -\frac{\cancel{q^*_+}}{2\cancel{q^*_+}}e^{iq^*_+r} = -\frac{1}{2}e^{ikr}$$

▷ $k^2-q^2-i\varepsilon \Rightarrow q^*_\pm = \pm(k-i\varepsilon)$



$$\mathrm{Res} = -\frac{\cancel{q^*_-}}{2\cancel{q^*_-}}e^{iq^*_-r} = -\frac{1}{2}e^{-ikr}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\vec{x}}}{k^2+q^2 \pm i\varepsilon} = -\frac{1}{4\pi} \frac{e^{\pm ikr}}{r}$$

3. **S-wave scattering:** the differential cross section of a pure s-wave scattering is given as

$$\frac{d\sigma}{d\Omega} = a, \text{ where } a > 0.$$

$\rightarrow$ keine Θ Abh.

$\rightarrow$ keine φ Abh.

What is the complex scattering amplitude $f(\theta)$?

$l=0$

$P_l(x) = 1$

- $\frac{d\sigma}{d\Omega} = a = |f(\cancel{\theta}, \cancel{\varphi})|^2 = (\mathrm{Re} f)^2 + (\mathrm{Im} f)^2$

- $\sigma_{tot} = \frac{4\pi}{k} \mathrm{Im}\, f(\Theta=0) = \frac{4\pi}{k} \mathrm{Im}\, f$

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\theta\, d\varphi \sin\Theta \cdot a = 4\pi \cdot a = 4\pi |f|^2$$

II $\quad a = (\mathrm{Re} f)^2 + (\mathrm{Im} f)^2$

I $\quad 4\pi a = \frac{4\pi}{k} \mathrm{Im}\, f \quad \Rightarrow \quad \mathrm{Im}\, f = ak$

II $\rightarrow \quad a = (\mathrm{Re} f)^2 + a^2k^2$

$$\mathrm{Re} f = \pm\sqrt{a - a^2k^2}$$

$$\Rightarrow f = \pm\underbrace{\sqrt{a - a^2k^2}} + i a k$$

Wurzel existiert

$\hookrightarrow a - a^2k^2 \geq 0 \Rightarrow a \geq a^2k^2 \Rightarrow 1 \geq ak^2$

$\frac{d\sigma_0}{d\Omega} = a \leq \frac{1}{k^2} \quad \Rightarrow \quad \sigma_0 = 4\pi a \leq \frac{4\pi}{k^2}$

$\sigma_l \leq \frac{4\pi}{k^2}(2l+1)$ Unitarität ✓

4. **Scattering on sphere:** we consider s-wave scattering on a homogeneous sphere potential

$$V(r) = \begin{cases} V_0 > 0 & \text{for } r < a, \\ 0 & \text{for } r \geq a. \end{cases}$$

(a) Take first $V_0 \to \infty$ and solve the radial part of the Schrödinger equation exactly for $l = 0$, i.e. obtain the radial function $u_0(r)$ and the scattering phase δ_0 exactly.

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Radialgl. $(\partial_r^2 + k^2)\, u_l(r) = V_{eff}(r)\; u_l(r)$

$$V_{eff}(r) = \frac{2m}{\hbar^2} V(r) + \cancel{\frac{l(l+1)}{r^2}} \qquad \hbar = 1$$

outside: $V=0$ $\qquad$ innen $V = V_0$

$(\partial_r^2 + k^2)\, u_0(r) = 0$ (#) $\qquad (\partial_r^2 + k^2)\, u_0(r) = 2mV_0\, u_0(r)$

$\rightsquigarrow e^{\pm ikr}$

a) $V_0 = \infty$

innen: $u_0(r) = 0$

außen: $u_0(r) = \frac{1}{k} e^{i\delta_0} \sin(kr + \delta_0)$

RB: $u_0(r=a) = 0 \Rightarrow kr + \delta_0 = n\pi \xRightarrow{n=0} \delta_0 = -ka$

$$\Rightarrow \quad u_0(r) = \begin{cases} \frac{1}{k} e^{-ika} \sin(k(r-a)) & \text{außen} \\ 0 & \text{innen} \end{cases}$$

b) $V_0 < \infty$ $\qquad$ außen ✓ $\qquad \kappa^2 = -k^2 + 2mV_0$

innen: $(\partial_r^2 + k^2 - 2mV_0)\, u_0 = 0$

$(\partial_r^2 - \kappa^2)\, u_0 = 0 \qquad \rightsquigarrow e^{\pm \kappa r}$

$u_0(r) = A \sinh(\kappa r) \leftarrow u(0) = 0$

RB:
$$\underbrace{\frac{u'}{u}\Big|_{r=a}}_{\text{außen}} = \underbrace{\frac{u'}{u}\Big|_{r=a}}_{\text{innen}} \qquad \boxed{\frac{\kappa \cosh(\kappa a)}{\sinh(\kappa a)} = \frac{k \cos(kr + \delta_0)}{\sin(kr + \delta_0)}}$$

c) Kleine Energien

$K \to 0 \qquad \mathcal{K} \to \sqrt{2mV_0}$

$$\cos(Kr+\delta_0) = \cos(Kr)\cos(\delta_0) - \sin(Kr)\sin(\delta_0)$$

$$\tan(Ka+\delta_0) = \frac{K}{\mathcal{K}} \tanh(\mathcal{K}a)$$

Reihenentwicklung

5. **Scattering on exponential potential (again):** a particle with mass m is scattered on the potential

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What are the conditions to satisfy Eq. (1) for $kR_0 \gg 1$ and $kR_0 \ll 1$ respectively?

$\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$

a) $$f(\theta) = \frac{4mR_0^3 V_0}{(1+4K^2R_0^2\sin^2\theta/2)^2} = \frac{4mR_0^3V_0}{[1+\frac{1}{2}4K^2R_0^2(1-\cos\theta)]^2}$$

$$f(\theta) = \sum_l \frac{2l+1}{K} e^{i\delta_l}\sin(\delta_l) P_l(\cos\theta)$$

$$\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$$

$$\int_{-1}^{1} d\cos\theta\, f(\theta) \underbrace{P_l(\cos\theta)}^{=1} = \sum_l \frac{\cancel{2l+1}}{K} e^{i\delta_l}\sin(\delta_l)\frac{2}{\cancel{2l+1}}$$

$$= \sum_l 2\sin(\delta_l)e^{i\delta_l}\frac{1}{K}$$

$l=0$ $$K\int_{-1}^{1} d\cos\theta \cdot \frac{4mR_0^3V_0}{[1+\frac{1}{2}4K^2R_0^2(1-\cos\theta)]^2} = \frac{K\,8mV_0R_0^3}{1+4K^2R_0^2}$$

$$\Rightarrow 2\sin(\delta_0)e^{i\delta_0} = \frac{K\,8mV_0R_0^3}{1+4K^2R_0^2}$$

$$\int_{-1}^{1} dx \frac{1}{[1+a(1-x)]^2} \Rightarrow \int \frac{1}{(au+1)^2} \rightarrow \int \frac{1}{s^2}$$

$u = 1-x \qquad s = au+1$

$$= \frac{2}{1+2a}$$

$$2\sin(\delta_0)\, e^{i\delta_0} = \frac{K\, 8 m V_0 R_0^3}{1+4K^2R_0^2} = -i\left(e^{i2\delta_0} - 1\right)$$

$$\Rightarrow \underbrace{e^{i2\delta_0}}_{\text{Betrag } 1} = \underbrace{1 + i \underbrace{\frac{K\, 8 m V_0 R_0^3}{1+4K^2R_0^2}}_{\approx 2\delta_0}}_{\text{Betrag} \neq 1} = \underbrace{\cos(2\delta_0)}_{\approx 1} + i \underbrace{\sin(2\delta_0)}_{\approx 2\delta_0}$$

$\Rightarrow$ berechnet mit Bornsche Näherung ($\Rightarrow$ reelles)

$\rightsquigarrow$ Opt. Theorem verletzt

b) p-wave $\quad l=1$

$$2\sin\delta_1\, e^{i\delta_1} = K \int_{-1}^{1} d\cos\, f(\theta) \cdot \overbrace{\cos\theta}^{P_1}$$

(c) The Born approximation is valid when $|\psi^{(1)}(\vec{r})| \ll |\phi_k(\vec{r})| = 1$, or equivalently $\leadsto e^{i\vec{k}\vec{r}}$

$$\frac{m}{2\pi}\left|\int d^3r' V(r') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} e^{ikz'}\right| \ll 1 \quad \xrightarrow{r\to 0} \quad \left|\int_0^\infty dr V(r)(\cancel{e^{ikr}-1})\right| \ll \frac{k}{m}, \qquad (1)$$

$$\left(e^{i2kr}-1\right)$$

What are the conditions to satisfy Eq. (1) for $kR_0 \gg 1$ and $kR_0 \ll 1$ respectively?

$$\vec{k}\vec{r}' \quad \text{für} \quad \vec{k} = k\vec{e}_z$$

$$\psi = \phi + \psi^{(1)} + \dots \qquad \psi^{(1)} = \int d^3x' \, G(\vec{x}-\vec{x}') \, V(\vec{x}') \, \phi(\vec{x}')$$

$$\left| \int_0^\infty dr \; V_0 e^{-r/R_0} \left(e^{i2kr}-1\right) \right| \ll \frac{k}{m}$$

$$\int_{0}^{\infty} dx \, e^{-x^2} = \frac{\sqrt{\pi}}{2} \qquad \int_{-\infty}^{\infty} \quad \sqrt{\pi}$$

$$= V_0 \left| \frac{iR_0}{i+2kR_0} - R_0 \right| = V_0 \left| -\frac{2kR_0^2}{i+2kR_0} \right| = \frac{2V_0 kR_0^2}{\sqrt{1+4k^2R_0^2}} \ll \frac{k}{m}$$

i) $kR_0 \gg 1 \Rightarrow \dfrac{2V_0 \cancel{k} R_0^{\cancel{2}}}{2\cancel{k}\cancel{R_0}} \ll \dfrac{k}{m} \Rightarrow V_0 R_0 \ll \dfrac{k}{m}$

ii) $kR_0 \ll 1 \Rightarrow 2V_0kR_0^2 \ll \dfrac{k}{m} \Rightarrow V_0R_0^2 \ll \dfrac{1}{2m}$

$$2\delta_0 = \frac{k\,8m V_0 R_0^3}{1+4k^2R_0^2} \ll 1$$

Grenzen

$\Rightarrow$ Betrag $|e^{i2\delta_0}| = 1$

ungefähr OK