Quantentheorie II Übung 3

Besprechung: 2021WE18 (KW18)

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1. **Questions**
   (a) How many linearly independent solutions for the Dirac equation for a fixed $\vec{p}$ exist?
   (b) Repeat the calculation to solve the Dirac equation for a particle moving in the z-direction, which was given in the lecture.
   (c) What is the matrix to rotate an ordinary vector around the y-axis?
   (d) If a 4-vector transforms under the Lorentz transformation matrix $L_{31} = l_y$, what is the associated transformation matrix for spinors?

2. **Plane wave solutions of the Dirac equation:** to solve the Dirac equation for a mass m particle we apply the Ansatz $\psi = u(p)e^{-ipx}$ (a plane wave with momentum $\vec{p}$) and obtain the linear equations for $u(p)$ for the positive energy Ansatz

$$(\not{p} - m)u(p) = 0\,.$$

   In the lecture the solutions for a particle moving in the z-direction were shown. Now we consider a particle moving in the x-direction with its momentum $\vec{p} = (p_x, 0, 0)$ and $E^2 = p_x^2 + m^2$.

   (a) Find the linearly independent solutions for $u(p)$ which satisfy the following conditions of orthogonality and normality:

$$\bar{u}_r(p)u_s(p) = 2m\delta_{rs}\,.$$

   (b) Show that

$$\sum_r u_r(p)\bar{u}_r(p) = \not{p} + m\,.$$

   (c) Check whether a simultaneous eigenstate with the following operators is possible:

$$\text{(i)}S_{12}\,, \qquad \text{(ii)}S_{23}\,.$$

   What do the answers mean?

3. **Rotation around y-axis by $\frac{\pi}{2}$:** the Dirac equation solution for a particle moving in the z-direction is given as $u(p_z)$. Use the results for $u(p_z)$ given in the lecture and verify

$$u(p_x) = S(R_y(\pi/2))u(p_z)\,,$$

   where $S_y = S_{31} = \frac{i}{4}[\gamma_3, \gamma_1]$, by comparing with the results for $u(p_x)$ from the previous task.

4. **Dirac equation of a massless particle:** the Dirac equation of a massless particle is

$$\not{\partial}\psi(x) = 0\,. \tag{1}$$

Using the Weyl (chiral) representation of the γ-matrices (see exercise sheet 2), find from Eq. (1) two independent equations for 2-component spinors, $\tilde{\xi}, \tilde{\eta}$ and solve them. Use the Ansatz $\tilde{\xi} = \xi(p)e^{-ipx}$, $\tilde{\eta} = \eta(p)e^{-ipx}$.

5. **Modified Dirac equation:** consider the Dirac equation modified by the so-called Pauli term:

$$(i\not{D} - m - \frac{e}{2m} a S^{\mu\nu} F_{\mu\nu})\psi = 0\,,$$

where $S^{\mu\nu}$ are the generators for the Lorentz transformation of 4-spinors (see exercise sheet 2) and $D^{\mu} \equiv \partial^{\mu} - ieA^{\mu}$. Assume $F_{12} = -F_{21} \equiv -B_z$, otherwise $F_{\mu\nu} = 0$.

   (a) Repeat the steps of the lecture for the non-relativistic limit for this modified equation.

   (b) What is the physical effect of the new term and the parameter a?