QTII – Prof. D. Stöckinger

Übung 13

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5. There are several ways how one can motivate the Klein-Gordon or the Dirac equations. Can you describe some of them?

Klein-Gordon

Vergleich Energie/Impuls Beziehung

$$i\partial_t\phi = -\frac{\Delta}{2m}\phi$$

$$E = \frac{p^2}{2m}$$

$$-(\partial_t)^2\phi = (-\Delta + m^2)\phi$$

$$E^2 = p^2 + m^2$$

Dirac

- Wunsch: 1. Ordnung in Ort und Zeit
- Umsetzung:

$$i\partial_t\psi = \sqrt{\text{Klein-Gordon}}\,\psi$$

- erfüllt durch $\alpha, \beta \cong \gamma^\mu$ Matrizen

5. There are several ways how one can motivate the Klein-Gordon or the Dirac equations. Can you describe some of them?

Alternativ: Lorentzgruppe

- Feld = Darstellung der Lorentzgruppe
- Welche Darstellungen gibt es?
  - skalares Feld $\phi'(x') = \phi(x)$
  - Dirac-Spinorfeld $\psi'(x') = S(\Lambda)\psi(x)$
  - ...
- suche kovariante Feldgleichungen
  - ϕ: Klein-Gordon $(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$
  - ψ: Dirac $(i\partial\!\!\!/ - m)\psi(x) = 0$

1. What are the common versions of the γ^μ matrices? Which properties do you remember?

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$
- $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$
- $(\gamma^\mu)^* = \gamma^\mu$ expect $(\gamma^2)^* = -\gamma^2$

Dirac representation [γ^0 diagonal]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Weyl (chiral) representation [all off-diagonal]

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

2. Can a spinor of the form $\psi(x) = (e^{-iEt}, 0, 0, 0)^T$ be a solution of the free Dirac equation? (conditions?)

- $\psi(x)$ describes particle in rest frame, i.e. $p^\mu = (m, 0, 0, 0)^T$

$\Longrightarrow$ $(-i\gamma^0\partial_0 + m)\psi(x) = 0$

- γ^0 should be diagonal $\rightarrow$ Dirac representation!

$$\begin{pmatrix} -i\partial_0 + m & 0 \\ 0 & i\partial_0 + m \end{pmatrix} \psi(x) = \begin{pmatrix} -E + m \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

- only for particles with positive energy $E = m$

6. What is the Dirac equation in momentum space? Characterize the spinors $u(p,s)$ and $v(p,s)$.

- position space: $(i\not\partial - m)\psi(x) = 0$
- Ansatz: $\psi(x) = w(p)e^{\mp ipx}$
- momentum space: $(\pm\not p - m)w(p) = 0$
  - $(\not p - m)u(p,s) = 0$ (positive energy)
  - $(\not p + m)v(p,s) = 0$ (negative energy)

- $S_z u(p,s) = su(p,s)$ and $S_z v(p,s) = -sv(p,s)$
- $\bar{u}(p,s_1)u(p,s_2) = 2m\delta_{s_1,s_2}$
- $\bar{v}(p,s_1)v(p,s_2) = -2m\delta_{s_1,s_2}$

3. What is the role of the gauge covariant derivative $D^\mu = \partial^\mu - ieA^\mu$ in the context of Dirac or Klein-Gordon equations?

free theory

$$(i\not{\partial} - m)\psi = 0 \quad (\partial_\mu \partial^\mu + m^2)\phi = 0$$

interacting theory

$$(i\not{D} - m)\psi = 0 \quad (D_\mu D^\mu + m^2)\phi = 0$$

- $\partial_\mu \to D_\mu$ is called “minimale Kopplung”
- compact way to describe interaction with electromagnetic field A^μ

4. What is the non-relativistic Hamiltonian for a charged particle in an electromagnetic field?

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 - e\Phi$$

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 + e\Phi - e\dot{\vec{x}}\vec{A}$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} - e\vec{A}$$

$$\implies \dot{\vec{x}} = \frac{\vec{p} + e\vec{A}}{m}$$

7. Sketch how one can derive the value $g = 2$ and spin-orbit coupling from the Dirac equation?

- solving $(i\not{D} - m)\psi = 0$ with $\psi = (\Psi_A, \Psi_B)$, where Ψ_i 2-comp. Pauli spinors
- system of equations

$$(E - m + e\Phi)\Psi_A - \vec{\sigma}(\vec{p} + e\vec{A})\Psi_B = 0$$
$$(-E - m - e\Phi)\Psi_B + \vec{\sigma}(\vec{p} + e\vec{A})\Psi_A = 0$$

- non-rel. limit: $mc^2 >> $ everything else, eliminate Ψ_B: Pauligleichung

$$(E - m)\Psi_A = \left[\frac{\left(\vec{p} + e\vec{A}\right)^2}{2m} + g\frac{e}{2m}\vec{S}\vec{B} + \mathcal{O}\left(\frac{1}{m^2}\right)\right]\Psi_A$$

- $\mathcal{O}(1/m^2)$ includes $\sim \vec{S}\vec{L}$ via Foldy–Wouthuysen

8. What is a possible basis of a 2-particle Hilbert space of two electrons (identical fermions), taking into account both the spin and position degree of freedom?

$$\mathcal{H}_2^{(-)} = \mathcal{H}_{\text{position},2}^{(\pm)} \otimes \mathcal{H}_{\text{spin},2}^{(\mp)}$$

$$\psi(\vec{r}_1, \vec{r}_2) \times \chi(\sigma_1, \sigma_2) \in \mathcal{H}_2^{(-)}$$

antisym. position $\times$ sym. spin

$$\frac{|nm\rangle - |mn\rangle}{\sqrt{2}} \times |\uparrow\uparrow\rangle$$

$$\frac{|nm\rangle - |mn\rangle}{\sqrt{2}} \times |\downarrow\downarrow\rangle$$

$$\frac{|nm\rangle - |mn\rangle}{\sqrt{2}} \times \frac{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}$$

sym. position $\times$ antisym. spin

$$\frac{|nm\rangle + |mn\rangle}{\sqrt{2}} \times \frac{|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle}{\sqrt{2}}$$

9. An electron on the earth has the absolute same rest mass as an electron on the moon. How? How can we describe it in various formalism?

Fockraum, Symmetriesierung

$\downarrow$

Konstruktion Zustände aus $a_p, a_p^\dagger$

$\downarrow$

Feldoperatoren, H

klassische **Feld**theorie $\mathcal{L}$

$\downarrow$

kanonische Quantisierung

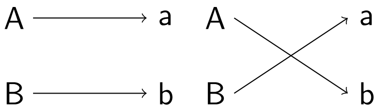
$\downarrow$

Operatoren H, Erzeuger/Vernichter, Zustandsr. = Fockraum

Teilchen $\cong$ Anregungen eines Feldes (translationsinvariant)

10. Consider a $2 \to 2$ process of 2-particle final state. Describe the role of the direct and exchange term contribution to the probability amplitude, and discuss the difference between distinguishable particles and two identical bosons/fermions.

$$(A, B) \to (a, b)$$



$$P = |\mathcal{A}|^2 = |\langle f|i\rangle|^2$$

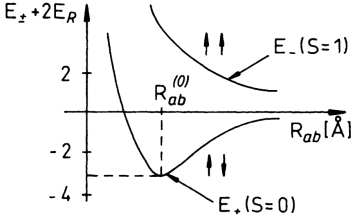
$$\begin{aligned} \text{unterscheidbar} \quad & P = |\mathcal{A}^d|^2 + |\mathcal{A}^e|^2 \\ \text{Bosonen} \quad & P = |\mathcal{A}^d + \mathcal{A}^e|^2 \\ \text{Fermionen} \quad & P = |\mathcal{A}^d - \mathcal{A}^e|^2 \end{aligned}$$

11. Discuss the chemical binding of the H_2 molecule. How does it arise from the fact that the two electrons are identical particles?

- Heitler-London: $|\psi_{H_2}\rangle = |\phi_{H,A}, \phi_{H,B}\rangle^{(\pm)} \otimes |S, M\rangle$ mit $|\phi_{H,i}\rangle$ Grundzustand für freies H-Atom
- Grundzustandsenergie minimal für $S = 0, 1$?

$$E = \frac{\langle\psi_{H_2}|H|\psi_{H_2}\rangle}{\langle\psi_{H_2}|\psi_{H_2}\rangle}$$

- $\exists$ nicht-klassische Austauschterme



$\Longrightarrow$ Ortsanteil symmetrisch, $S = 0$

12. What are some important properties of creation/annihilation operators?

Erzeugung symmetrisierter Zustände aus dem Vakuum mit $b^\dagger = a^\dagger, c^\dagger$

$$|n_1 \dots n_N\rangle^{(\pm)} = \frac{1}{\sqrt{N!}} b^\dagger_{n_1} \dots b^\dagger_{n_N} |0\rangle$$

Bosonen

$$a_n^\dagger : \mathcal{H}_N^{(+)} \to \mathcal{H}_{N+1}^{(+)}$$
$$\left[a_n, a_m^\dagger\right] = \text{``}\delta_{nm}\text{''}$$ [1]

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$
$$a |n\rangle = \sqrt{n}\, |n-1\rangle$$

Fermionen

$$c_n^\dagger : \mathcal{H}_N^{(-)} \to \mathcal{H}_{N+1}^{(-)}$$
$$\left\{c_n, c_m^\dagger\right\} = \text{``}\delta_{nm}\text{''}$$

$$c^\dagger c^\dagger = 0$$

[1] $= \delta_{nm}$, falls Zustände $|n\rangle, |m\rangle$ normiert

13. How can you express the free Hamiltonian (for kinetic energy) in terms of c/a operators?

$$H = T + V$$

$$\hat{T} = \sum_p \frac{p^2}{2m} a_p^\dagger a_p \quad \text{bzw.} \quad \hat{T} = \int \mathrm{d}p \, \frac{p^2}{2m} a_p^\dagger a_p$$

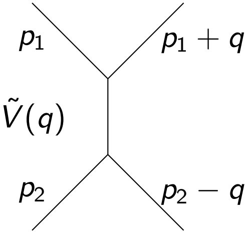
- für jedes p ein HO
- Anregungszahl $\hat{n}_p = a_p^\dagger a_p$
- $\sum_p$ bzw. $\int \mathrm{d}p \implies$ Unabhängigkeit der einzelnen Moden, keine Wechselwirkung

14. How can you express the interaction Hamiltonian of two particles bounded with the Coulomb potential in terms of c/a operators? Which mathematical operation appears?

$$V(x_1, x_2) = \frac{1}{4\pi}\frac{1}{|x_1 - x_2|} \quad \tilde{V}(q) = \frac{1}{q^2}$$

- Coulombpotential translationsinvariant (nur abhängig von relativen Größen)

$$\hat{V} = \frac{1}{2}\int \mathrm{d}^3 p_1 \mathrm{d}^3 p_2 \mathrm{d}^3 q \, \tilde{V}(q) a^\dagger_{p_1+q} a^\dagger_{p_2-q} a_{p_2} a_{p_1}$$



Ü07, #2, Ziel: finde Besetzungszahldarstellung

- Festlegung Reihenfolge

$$|n_1, n_2, \ldots\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots|0\rangle$$

- Reihenfolge definiert Vorzeichen (Fermionen)

$$a_4^\dagger a_7^\dagger a_5 a_1 a_1^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle \implies (-1)^N a_5 a_1 a_1^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_7^\dagger a_8^\dagger |0\rangle$$

- N Permutationen?

$$\{a_{n'}^\dagger, a_n^\dagger\} = \{a_{n'}, a_n\} = 0 \qquad (1)$$

$$\{a_{n'}^\dagger, a_n\} = \delta_{nn'} \qquad (2)$$

  - Tausch mit gleichem *Daggerstatus*, immer (1)
  - Tausch mit unterschiedlichen *Daggerstatus*, falls $n \neq n'$ (2)

$$(-1)^0 a_4^\dagger a_7^\dagger a_5 a_1 a_1^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$(-1)^1 a_4^\dagger a_5 a_7^\dagger a_1 a_1^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$(-1)^2 a_5 a_4^\dagger a_7^\dagger a_1 a_1^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$(-1)^3 a_5 a_4^\dagger a_1 a_7^\dagger a_1^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$(-1)^4 a_5 a_4^\dagger a_1 a_1^\dagger a_7^\dagger a_3^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$(-1)^5 a_5 a_4^\dagger a_1 a_1^\dagger a_3^\dagger a_7^\dagger a_5^\dagger a_8^\dagger |0\rangle$$

$$\cdots$$

$$(-1)^9 a_5 a_1 a_1^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_7^\dagger a_8^\dagger |0\rangle$$

16. Compare the structure of wave functions for (i) 1dim scattering at a potential barrier, (ii) 3dim scattering at short-range potential.

i) 1dim: Ebene Wellen – incoming, reflected and transmitted

$$\psi \cong e^{ikx} + re^{-ikx} + te^{ikx}$$

ii) 3dim: Ebene Welle + Kugelwelle

$$\psi \cong e^{i\vec{k}\vec{r}} + f(\theta, \phi)\frac{e^{ikr}}{r}$$

17. How can you derive the relationship between the scattering amplitude $f(\theta, \phi)$ and the differential cross section?

Definition als Proportionalitätsfaktor/effektive Querschnittsfläche

$$\mathrm{d}I_{\text{aus}} = |\vec{j}_{\text{ein}}| \mathrm{d}\sigma$$

bekannt aus QM

$$\mathrm{d}I_{\text{aus}} = |\vec{j}_{\text{aus}}| r^2 \mathrm{d}\Omega$$

Stromdichten ($\phi \to \vec{j}_{\text{ein}} \quad \psi_{\text{streu}} \to \vec{j}_{\text{aus}}$)

$$\vec{j} = \frac{1}{2mi} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$\mathrm{d}\sigma = |f(\theta, \phi)|^2 \mathrm{d}\Omega$$

15.+18. What are the two approximation methods to describe scattering processes? Describe the basic principles!

Bornsche Näherung

- Störungsreihe $\psi^{(n)} \propto V^n$ $\rightarrow$ Feynmandiagramme
- $\psi^{(1)} \propto \tilde{V}(\vec{q})$
- Gültigkeitsbereich?!
  - optisches Theorem (keine imaginären Beiträge)
  - *schwaches* Potential

Partialwellenmethode

- Entwicklung nach Y_{nlm} (neue Basis, $m = 0$ Zentralpotential)
- einlaufend vs. streu.: Phase δ_l
- oft reicht $l = 0$

Was ist der physikalische Unterschied?

- Entwicklung in Potenzen von V (Näherung) vs. Wahl einer praktischen Basis (keine Näherung)
- Tendenz: Reichweite Potential R
  - $Rk >> 1$ eher Born [z.B. Ü11 #5, Ü12 #3]
  - $Rk << 1$ eher *kleine* l

semiklassisch: Streuparameter b.
keine Streuung wenn

$$b = \frac{l\hbar}{\hbar k} = \frac{l}{k} > R$$
$$\implies l_{max} \approx Rk$$

19. What is the integral equation for the scattering wave function from which you can derive the first Born approximation?

- SGL + Greensche Funktion + RB = Integralgleichung
- $G(\vec{x} - \vec{x}') \sim \frac{e^{ik|\vec{x}_1 - \vec{x}_2|}}{|\vec{x}_1 - \vec{x}_2|}$

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\vec{x}} + \int \mathrm{d}^3x' \, G(\vec{x} - \vec{x}') v(x') \psi_{\vec{k}}(\vec{x}')$$

$$v(x') = 2mV(x')$$

- = Lippmann-Schwinger Gleichung in Ortsdarstellung

20. What is the result of the first Born approximation for the scattering amplitude? Give and discuss some example potentials!

$$f^{(1)}(\theta, \phi) = -\frac{m}{2\pi} \tilde{V}(\vec{q})$$

- Coulomb
- Yukawa
- Delta
- ...

21. Describe the derivation of scattering phase for scattering at a hard sphere (inside $V = \infty$)!

Radialgleichung + RB $\implies$ Bestimmungsgleichung δ_l

$$(\partial_r^2 + k^2)u_l = v_{eff}\, u_l \quad \text{mit} \quad v_{eff} = 2mV + \frac{l(l+1)}{r^2}$$

- meist reicht $l = 0$
- innen: $V = \infty \implies u_l = 0$
- außen: freie Lösungen $\sim e^{\pm ikr}$, Ansatz

$$u_l \sim \sin(kr + \delta_l)$$

- RB: $u_l(r = R) = 0 \implies \delta_l = ?$

Viel Erfolg bei der Klausur! ☺