Blatt 04

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1. a) $\gamma^0 \cdot (\dots) \to$ nach $i\partial_t$ umstellen

b) $(i\not\partial - m)\psi = 0$

$\psi' = e^{-i\alpha(x)}\psi \Rightarrow (i\not\partial - m)e^{i\alpha(x)}\psi' = 0$

$\Rightarrow (i\gamma^\mu\partial_\mu - m)e^{i\alpha(x)}\psi' = 0$

$i\gamma^\mu\partial_\mu(i\alpha(x))\cancel{e^{i\alpha(x)}}\psi' + i\gamma^\mu\partial_\mu\psi'\cancel{e^{i\alpha(x)}} - m\cancel{e^{i\alpha(x)}}\psi' = 0$

$(i\gamma^\mu(\partial_\mu(i\alpha(x)) + \partial_\mu) - m)\psi' = 0$

$\partial'_\mu = \underbrace{\partial_\mu(i\alpha(x))} + \partial_\mu$

c) $D^\mu = \partial^\mu - ieA^\mu$

$(i\not\partial + e\not A - m)\psi = 0$

$\psi' = e^{-i\alpha(x)}\psi \Rightarrow (i\not\partial + e\not A - m)e^{i\alpha(x)}\psi' = 0$

$(i(\underbrace{\not\partial(i\alpha(x))} + \not\partial) + e\not A - m)\cancel{e^{i\alpha}}\psi' = 0$

$\boxed{A^{\mu\prime} = A^\mu - \frac{1}{e}\partial^\mu\alpha(x)}$ $\boxed{\partial^{\mu\prime} = \partial^\mu}$

$D^{\mu\prime} = \partial^{\mu\prime} - ieA^{\mu\prime}$

$= \partial^\mu - ieA^\mu + i\partial^\mu\alpha(x)$

$= D^\mu + i\partial^\mu\alpha(x)$

$D^{\mu\prime}\psi' = (D^\mu + i\partial^\mu\alpha(x))e^{-i\alpha(x)}\psi$

$$= \left(\partial^\mu - ieA^\mu + i\partial^\mu \alpha(x) \right) e^{-i\alpha(x)} \psi$$

$$= \left(\underbrace{-i\partial^\mu \alpha(x) e^{-i\alpha(x)}} + e^{-i\alpha(x)} \partial^\mu - ieA^\mu e^{-i\alpha(x)} + \underbrace{i\partial^\mu \alpha(x) e^{-i\alpha(x)}} \right) \psi$$

$$= e^{-i\alpha(x)} \left(\partial^\mu - ieA^\mu \right) \psi = e^{-i\alpha(x)} D^\mu \psi$$

d) $$(E-m)\psi = \left[\frac{(\vec{p} + e\vec{A})^2}{2m} + \frac{e}{2m} \vec{\sigma} \cdot \vec{B} - e\phi \right] \psi \qquad (e > 0)$$

2. $(i\cancel{\partial} + e\cancel{A} - m)\psi$

a) $\psi' = e^{-ie\alpha(x)} \psi \Rightarrow (i\cancel{\partial} + e\cancel{A} - m) e^{ie\alpha(x)} \psi' = 0$

$$\Rightarrow \left(i\left(\gamma (ie\alpha(x)) + \cancel{\partial} \right) + e\cancel{A} - m \right) \psi' = 0$$

$$\Rightarrow \left(i\cancel{\partial} \underbrace{- e\cancel{\partial}\alpha(x) + e\cancel{A}}_{= e\cancel{A}'} - m \right) \psi' = 0$$

$$\Rightarrow A^{\mu\prime} = A^\mu - \partial^\mu \alpha(x)$$

b) $$\left(\gamma^\mu (i\partial_\mu - qA_\mu) - m \right) \psi = 0 \qquad (\mathrm{I})$$

$$\left(\gamma^\mu (i\partial_\mu + qA_\mu) - m \right) \psi_c = 0 \qquad (\mathrm{II})$$

(I) kompl. konj. $\left[\gamma^{\mu *} (-i\partial_\mu - qA_\mu) - m \right] \psi^* = 0$

$\psi_c = B\psi^* \Rightarrow \left[\gamma^{\mu *} (-i\partial_\mu - qA_\mu) - m \right] B^{-1} \psi_c = 0$

$B^{-1}(\mathrm{II}) \Rightarrow \quad \left[B^{-1} \gamma^\mu (i\partial_\mu + qA_\mu) - B^{-1} m \right] \psi_c = 0$

$\Rightarrow \gamma^{\mu *} B^{-1} = - B^{-1} \gamma^\mu$

$\Rightarrow \gamma^{\mu *} = - B^{-1} \gamma^\mu B$

$\Gamma \psi_c = C \bar{\psi}^T$

$= C \gamma^0 \psi^*$

$\Rightarrow C\gamma^0 = B$

$$\Rightarrow \gamma^{\mu *} = -(C\gamma^0)^{-1}\gamma^\mu C\gamma^0$$

$$(C\gamma^0)^{-1} = \gamma^0 C^{-1}$$

$$= -\gamma_0 C^{-1}\gamma^\mu C\gamma^0$$

$$\Rightarrow \gamma_0\gamma^{\mu *}\gamma_0 = -C^{-1}\gamma^\mu C$$

$$\Gamma \gamma^{\mu * T} = \gamma^0\gamma^\mu\gamma^0$$

$$\Rightarrow \boxed{-\gamma^{\mu T} = C^{-1}\gamma^\mu C}$$

$$\Rightarrow -C\gamma^{\mu T} = \gamma^\mu C \quad \Rightarrow \quad C = -C^T$$

$$(C = i\gamma^2\gamma^0)$$

$$\begin{pmatrix} 0 & \partial_y \\ -\partial_y & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -\partial_y^\dagger \\ \partial_y^\dagger & 0 \end{pmatrix} \quad \cancel{\forall}$$

$$\langle \psi, \varphi \rangle = \underline{\bar{\psi}\gamma^0\varphi}$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

3.

$$\phi = -\frac{\alpha}{r} \qquad \text{K.G.} \quad -\partial_t^2 = m^2 - \nabla^2$$

$$E^2 = m^2 + p^2$$

$$E \to E - q\underbrace{\phi}_{V = -\frac{\alpha}{r}}$$

$$\Rightarrow (E - q\phi)^2 = m^2 + p^2$$

$$\Rightarrow \left(i\partial_t + \frac{\alpha}{r}\right)^2 = m^2 - \nabla^2 \quad \Rightarrow \left[-\partial_t^2 + \frac{2i\alpha\partial_t}{r} + \frac{\alpha^2}{r^2} - m^2 + \nabla^2\right]\psi = 0$$

$$\psi_K(x^\mu) = e^{-i\omega t}\phi_K(\vec{r})$$

$$\Rightarrow \left[\omega^2 + \frac{2\alpha\omega}{r} + \frac{\alpha^2}{r^2} - m^2 + \Delta\right]\phi_k(\vec{r}) = 0$$

$$\Rightarrow \left[\left(\omega + \frac{\alpha}{r}\right)^2 - m^2 + \Delta\right]\phi_k(\vec{r}) = 0 \qquad \begin{matrix}\omega = E_k \\ M = m\end{matrix}$$

c) $H_k \psi = E\psi$

$$\left[E_k^2 + \frac{2E_k\alpha}{r} + \frac{\alpha^2}{r^2} - m^2 + \Delta\right]\phi_k(\vec{r}) = 0$$

$$\left[E_k^2 + \frac{2E\alpha r + \alpha^2}{r^2} - m^2 + \Delta\right]\phi_k(\vec{r}) = 0$$

$E_k = m + E_s \leftarrow$

$$\Rightarrow \left[\cancel{m^2} + 2mE_s + E_s^2 + \frac{2m\alpha}{r} + \frac{2E_s\alpha}{r} + \frac{\alpha^2}{r^2} - \cancel{m^2} + \Delta\right]\phi_k(\vec{r}) = 0$$

$$\left[\Delta + 2m\frac{\alpha}{r} + 2mE_s + \underbrace{\underbrace{\left(E_s + \frac{\alpha}{r}\right)^2}_{E_s - V}}_{\frac{\Delta^2}{4m^2}}\right]\phi_k(\vec{r}) = 0$$

$$\boxed{E_s = \frac{p^2}{2m} + V}$$

$$(E_s - V)^2 = \frac{p^4}{4m^2}$$

?

4.

$$H_D = \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} - \frac{e}{4m^2}\vec{E}\cdot(\vec{\sigma}\times\vec{p})$$

$$= \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} - \frac{e}{4m^2}\mathcal{E}(\sigma_x p_y - \sigma_y p_x) \qquad \vec{x} = (x, y, 0)$$

$H_0 \psi = E\psi$

$\vec{p}^{\,2} = -\nabla^2 = -\partial_x^2 - \partial_y^2$

$\vec{p}^{\,4} = \Delta^2 = \partial_x^4 + \partial_y^4 + 2\partial_x^2\partial_y^2$

$\psi_1(x) := u(x)\, e^{i\vec{k}\cdot\vec{x}}$

$\partial_x \psi_1(x) = \left(\partial_x u(x) + ik_x u(x)\right) e^{i\vec{k}\cdot\vec{x}}$

$\partial_x^2 \psi_1(x) = \left(\partial_x^2 u(x) + ik_x \partial_x u + ik_x(\partial_x u + ik_x u)\right) e^{i\vec{k}\vec{x}}$