Blatt 03

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1. (a) 4.

(b) Siehe RM

(c) $\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$

(d) $S(\Lambda) = \mathbb{1} - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}$ $\qquad \omega^{31} = \varepsilon = -\omega^{13}$

$S_{13} = -S_y = \frac{i}{4}[\gamma^3_3, \gamma_1]$

$\Rightarrow \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

$\gamma_1 = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix}$

$+\frac{i}{4}[\gamma_3, \gamma_1] = -\frac{i}{4}\begin{pmatrix} [\sigma_z, \sigma_x] & 0 \\ 0 & [\sigma_z, \sigma_x] \end{pmatrix} = -\frac{i}{4}\begin{pmatrix} +2i\sigma_y & 0 \\ 0 & +2i\sigma_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +\sigma_y & 0 \\ 0 & +\sigma_y \end{pmatrix}$

$\rightsquigarrow \boxed{S(\Lambda) = \mathbb{1} \cancel{+} - \frac{i}{2}\varepsilon\begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}}$

2.a) $(\not p - m)u(p) = 0 \Rightarrow \left[E\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} - p_x\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} - m\mathbb{1}_4\right]u(p) = 0$

$[(\not p + m)v(p) = 0]$

$\Rightarrow \begin{pmatrix} E-m & 0 & 0 & -p_x \\ 0 & E-m & -p_x & 0 \\ 0 & p_x & -E-m & 0 \\ p_x & 0 & 0 & -E-m \end{pmatrix} u(p) = 0$

$u_1 = N\begin{pmatrix} p_x \\ 0 \\ 0 \\ E-m \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ p_x \\ E-m \\ 0 \end{pmatrix}$

Normierung:

Bedingung: $\bar{u}_r u_s \overset{!}{=} 2m\delta_{rs}$

$\bar{u}_+ u_+ = N^2(p_x^2 - (E-m)^2)$

$= N^2(p^2 - E^2 - m^2 + 2Em)$

$= N^2(-2m^2 + 2Em) \overset{!}{=} 2m$

$\Rightarrow N^2(E-m) = 1$

$\Rightarrow N = 1/\sqrt{E-m}$

(Hinweis: Man kann u_+/u_- anders wählen, wodurch sich N ändert)

$$S_x = \frac{i}{4}[\gamma_2, \gamma_3] = \frac{i}{4}\begin{pmatrix}[\sigma_x,\sigma_y] & 0 \\ 0 & [\sigma_z,\sigma_y]\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}\sigma_x & 0 \\ 0 & \sigma_x\end{pmatrix}$$

$$u_\pm = a_\pm u_1 + b_\pm u_2$$

$$\Rightarrow S_x u_\pm = +\frac{1}{2}\begin{pmatrix}\sigma_x & 0 \\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}a_\pm p_x \\ b_\pm p_x \\ b_\pm (E-m) \\ a_\pm (E-m)\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}b_\pm p_x \\ a_\pm p_x \\ a_\pm (E-m) \\ b_\pm (E-m)\end{pmatrix} \overset{!}{=} \pm\frac{1}{2}\begin{pmatrix}a_\pm p_x \\ b_\pm p_x \\ b_\pm (E-m) \\ a_\pm (E-m)\end{pmatrix}$$

$$\Rightarrow a_+ = +b_+ \qquad a_- = -b_-$$

$$u_- = N\begin{pmatrix}p_x \\ -p_x \\ -(E-m) \\ E-m\end{pmatrix} \qquad u_+ = N\begin{pmatrix}p_x \\ p_x \\ E-m \\ E-m\end{pmatrix}$$

$$\bar{u}_- u_- = N^2\left(2p_x^2 - 2(E-m)^2\right) = N^2\left(\cancel{2E^2} - 2m^2 - \cancel{2E^2} - 2m^2 + 4mE\right) = 2m.$$

$$\Rightarrow N = \frac{1}{\sqrt{2(E-m)}}$$

$\bar{u}_+ u_+$ analog.

b) $\sum_r u_r(p)\bar{u}_r(p) = \left(\sum_r u_r(p) u_r^\dagger(p)\right)\gamma_0$

$$\left(\overset{!}{=} E\gamma^0 - p_x\gamma^1 + m\right)$$
$$= \left(E - p_x\gamma^1\gamma^0 + m\gamma^0\right)\gamma^0$$

$A = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ ← Für $\langle\psi_i|\psi_i\rangle = 1$

$\Rightarrow \sum_i \frac{\lambda_i}{2m} |\psi_i\rangle\langle\psi_i|$ ← Für $\langle\psi_i|\psi_i\rangle = 2m$

Hier: $|\psi_i\rangle = u_r(p)$
$\langle\psi_i| = \bar{u}_r(p)$

$$\not{p} + m = \sum_r u_r(p)\,\bar{u}_r(p)$$

$\not{p}$ EW $\pm m$

$\Rightarrow \not{p}+m$ EW $2m$, 0

EV $u_r(p)$ EV $v_r(p)$

c) ii → Siehe oben.

i) $\begin{pmatrix} E-m & -p_x\sigma_x \end{pmatrix}$ ... $\begin{pmatrix} \sigma_z & 0 \end{pmatrix}$

c) ii → Siehe oben.

i) $$\not p = \begin{pmatrix} E-m & -p_x\sigma_x \\ p_x\sigma_x & -E-m \end{pmatrix} \qquad S_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

$$[\not p, S_z] = \begin{pmatrix} 0 & p_x[\sigma_z,\sigma_x] \\ -p_x[\sigma_z,\sigma_x] & 0 \end{pmatrix} \neq 0$$

Allg. $$[\not p_i, S_j] = \begin{pmatrix} 0 & p^i[\sigma^j,\sigma^i] \\ -p^i[\sigma^j,\sigma^i] & 0 \end{pmatrix} = 2i\varepsilon_{ijk}p^i\begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$$= 2i\varepsilon^{ijk}p^i\gamma_k$$

3.

$$S(R_y(\tfrac{\pi}{2})) = e^{-i\frac{\pi}{2}S_y} \qquad \text{allg. } S(R) = e^{-i\vartheta G_R}$$

mit $G = S\left(i\frac{\partial R}{\partial\vartheta}\Big|_{\vartheta=0}\right)$

$$S_y = \frac{1}{2}\begin{pmatrix} \sigma^y & 0 \\ 0 & \sigma^y \end{pmatrix}$$

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \qquad C^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$S_y = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \Rightarrow e^{-i\frac{\pi}{2}S_y} = \frac{1}{2}\begin{pmatrix} 1&1&0&0 \\ i&-i&0&0 \\ 0&0&1&1 \\ 0&0&i&-i \end{pmatrix}\begin{pmatrix} -i&0&0&0 \\ 0&i&0&0 \\ 0&0&-i&0 \\ 0&0&0&i \end{pmatrix}\begin{pmatrix} 1&-i&0&0 \\ 1&i&0&0 \\ 0&0&1&-i \\ 0&0&1&i \end{pmatrix}$$

$A = C\,D\,C^{-1}$

Matrix EV — diagonal. EW

$$= \frac{1}{2}\begin{pmatrix} 1&1&0&0 \\ i&-i&0&0 \\ 0&0&1&1 \\ 0&0&i&-i \end{pmatrix}\begin{pmatrix} -i&-1&0&0 \\ i&-1&0&0 \\ 0&0&-i&-1 \\ 0&0&i&-1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0&-2&0&0 \\ 2&0&0&0 \\ 0&0&0&-2 \\ 0&0&2&0 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}}$$

$= -2i\,S_y$

$$e^{-i\frac{\pi}{2}S_y}u_+ = N\begin{pmatrix} 0 \\ E+m \\ 0 \\ p_z \end{pmatrix} \qquad e^{-i\frac{\pi}{2}S_y}u_- = N\begin{pmatrix} -(E+m) \\ 0 \\ p_z \\ 0 \end{pmatrix}$$

4.

$$\not p\,\psi = 0$$

$$\not p \psi = \left(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3\right)\psi$$

$$= \begin{pmatrix} 0 & -\vec p\cdot\vec\sigma + E\mathbb{1} \\ +\vec p\cdot\vec\sigma + E\mathbb{1} & 0 \end{pmatrix}\psi \qquad \psi = \begin{pmatrix} \tilde\xi \\ \tilde\eta \end{pmatrix}$$

$$(E - \vec p\cdot\vec\sigma)\,\eta = 0$$
$$(E + \vec p\cdot\vec\sigma)\,\xi = 0$$

EW von $\vec p\cdot\vec\sigma$: $\pm E \begin{cases} +E & \text{Gl. 1} \\ -E & \text{Gl. 2} \end{cases}$

$$\begin{pmatrix} E-p_z & ip_y - p_x \\ -ip_y - p_x & E + p_z \end{pmatrix}\eta = 0 \quad \Rightarrow \quad \eta = \begin{pmatrix} p_x - ip_y \\ E - p_z \end{pmatrix}$$

$$\begin{pmatrix} E+p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix}\xi = 0 \quad \Rightarrow \quad \xi = \begin{pmatrix} ip_y - p_x \\ E + p_z \end{pmatrix}$$

5.

$$\left(i\not\partial - m - \frac{e}{2m}a\,S^{\mu\nu}F_{\mu\nu}\right)\psi = 0$$

$$\Rightarrow \left(i\not\partial - m - \frac{e}{m}a\,B_z S_z\right)\psi = 0 \qquad A^\mu = (0, \overbrace{0, B_z x, 0}^{\vec A})$$

$$\Rightarrow \left(i\not\partial + eB_z x\gamma_2 - m - \frac{e}{m}aB_zS_z\right)\psi = 0 \qquad \Rightarrow \nabla\times\vec A = B_z\vec e_z$$

$$\Rightarrow \begin{pmatrix} i\partial_t - m - \frac{e}{2m}aB_z\sigma_z & i(\sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z) + eB_z\sigma_z \\ ? & -i\partial_t - m - \frac{e}{2m}aB_z\sigma_z \end{pmatrix}$$

$\Rightarrow$ Wir gehen zur Übung ■