Quantentheorie II Übung 4

Besprechung: 2021WE19 (KW19)

SS 2021 Prof. Dominik Stöckinger (IKTP)

1. **Questions**

   (a) How can you obtain the Dirac Hamiltonian from the Dirac equation?

   (b) How can you make the Dirac equation invariant under a local transformation $\psi(x) \to e^{-i\alpha(x)}\psi(x)$?

   (c) What is the covariant derivative D^μ for the Dirac equation with the electromagnetic fields?

   (d) What are the terms appearing in the nonrelativistic Dirac Hamiltonian with nonzero electromagnetic fields up to the order of $\mathcal{O}(\frac{1}{m})$?

2. **Gauge invariance and charge conjugation:** for a given real potential $A_\mu(x)$ the Dirac equation of an electron with its charge $q = -e$ satisfies

$$(i\partial\!\!\!/ + eA\!\!\!/(x) - m)\psi(x) = 0\,.$$

   (a) We now consider a transformation of the electron spinor field ψ where $\psi'(x) = e^{-ie\alpha(x)}\psi(x)$. Find out how the potential $A_\mu(x)$ should transform

$$A(x) \to A'_\mu(x) = A_\mu(x) = A_\mu(x) + \ ?\,,$$

   in order that the Dirac equation is invariant under this transformation, which means

$$(i\partial\!\!\!/ + eA\!\!\!/'(x) - m)\psi'(x) = 0\,.$$

   (b) The corresponding Dirac equation of a positron should be obtained when we replace $q = -e$ with $q = +e$ and $\psi(x)$ with its charge conjugation spinor $\psi_c = C\bar{\psi}^T$. Find out the equation to determine the charge conjugation matrix C.

3. **Relativistic hydrogen atom (spinless):** we can discuss the spinless relativistic hydrogen atom with the Schrödinger equation in the spherical coordinate system,

$$(\Delta + 2M\frac{\alpha}{r} + 2ME_S)\psi_S(\vec{r}) = 0\,, \quad \Delta = \partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat{L}^2}{r^2}\,,$$

   where $\hat{L}^2\psi_S = l(l+1)\psi_S$.

   (a) Write down the Klein-Gordon equation for a particle in the Coulomb potential.

   (b) Use the Ansatz

$$\psi_K(x^\mu) = e^{-i\omega t}\phi_K(\vec{r})\,,$$

   and bring the Klein-Gordon equation into the form

$$\left(\Delta - M^2 + \left(\frac{\alpha}{r} + E_K\right)^2\right)\phi_K(\vec{r}) = 0\,.$$

(c) Which substitutions are required to rewrite the Klein-Gordon equation into the Schrödinger equation?

(d) What are the energy eigenvalues of the Klein-Gordon equation? Compare the results with the Schrödinger equation up to the order of $\mathcal{O}(\alpha^4)$. You can use the known results of the energy eigenvalues of the Schrödinger equation.

4. **Dirac Hamiltonian in nonrelativistic limits:** an electron is moving in a homogeneous electric fields $\vec{E} = \mathcal{E}\hat{z}$. The magnetic field vanishes, and we ignore the movement along the z-axis. The corresponding Dirac equation, rewritten for an energy eigenstate, is $H_D\psi(\vec{x}) = E\psi(\vec{x})$ where $\vec{x} = (x, y, 0)$ as we ignore the movement along the z-axis, and the Dirac Hamiltonian expanded up to the order of $\mathcal{O}(\frac{1}{m^3})$ is

$$H_D = \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3} - \frac{e}{4m^2}\vec{E}\cdot(\vec{\sigma}\times\vec{p})\,.$$

(a) Solve the equation using the Ansatz

$$\psi(\vec{x}) = \begin{pmatrix} u(k_x, k_y) \\ v(k_x, k_y) \end{pmatrix} e^{i\vec{k}\cdot\vec{x}}\,.$$

(b) An electron is moving along the x-axis. Sketch the dispersion $E(k_x)$ which you obtained above along the x-axis. Assume that the p^4 correction term is negligible.

(c) Calculate the expectation value of the spin operators $\vec{s} = \frac{\vec{\sigma}}{2}$. What is the relative orientation of the spin and the momentum?

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(b) How can you make the Dirac equation invariant under a local transformation $\psi(x) \to e^{-i\alpha(x)}\psi(x)$?

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a) $i\partial_0 \psi = H_D \psi$ $\longrightarrow$ via multiplication mit γ^0

b) $\psi' \to e^{-i\alpha(x)}\psi$

c) $$(i\partial\!\!\!/ - m)\psi = 0 \longrightarrow i\gamma^\mu\left(\partial_\mu - i\partial_\mu\alpha(x)\right)\psi e^{-i\alpha(x)} = 0$$

$$\partial_\mu \to D_\mu = \underbrace{\partial_\mu - iA_\mu}_{\text{minimale Kopplung}}$$

d) $$H = -e\Phi + \frac{(\vec{p} + e\vec{A})^2}{2m} + \frac{e}{2m}\vec{\sigma}\vec{B} + \mathcal{O}\left(\frac{1}{m^2}\right)$$

4. **Dirac Hamiltonian in nonrelativistic limits:** an electron is moving in a homogeneous electric fields $\vec{E} = \mathcal{E}\hat{z}$. The magnetic field vanishes, and we ignore the movement along the z-axis. The corresponding Dirac equation, rewritten for an energy eigenstate, is $H_D\psi(\vec{x}) = E\psi(\vec{x})$ where $\vec{x} = (x, y, 0)$ as we ignore the movement along the z-axis, and the Dirac Hamiltonian expanded up to the order of $\mathcal{O}(\frac{1}{m^3})$ is

$\vec{K} = K_x\hat{x} + K_y\hat{y}$

$$H_D = \overbrace{\frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3}}^{\text{Kin Energie}} - \frac{e}{4m^2}\overbrace{\underline{\vec{E}\cdot(\vec{\sigma}\times\vec{p})}}^{\to \text{Spin}}\,.$$

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$E = \mathcal{E}\hat{z}$

$[\vec{\sigma}\vec{p}, \vec{\sigma}\vec{E}] = 2\vec{\sigma}(\vec{E}\times\vec{p})$ $\qquad \vec{p} = -i\partial_x$

a) $\vec{E}\cdot(\vec{\sigma}\times\vec{p})\psi = \vec{E}\cdot(\vec{\sigma}\times\vec{K})\psi = \mathcal{E}(\sigma_x K_y - \sigma_y K_x)\psi$

$\vec{p}^{\,2}\psi = \vec{K}^2\psi$

$$\Rightarrow H_D\psi = \underbrace{\frac{\vec{K}^2}{2m} - \frac{\vec{K}^4}{8m^3}}_{\equiv\, \epsilon_K} - \frac{e}{4m^2}\mathcal{E}\begin{pmatrix} 0 & K_y + iK_x \\ K_y - iK_x & 0 \end{pmatrix}\psi = E\psi$$

$$\pi_K = -\frac{e}{4m^2}\mathcal{E}(K_y + iK_x)$$

$$\Rightarrow |\pi_K| = \left|\frac{e}{4m^2}\mathcal{E}\right|\cdot|\vec{K}|$$

$$\begin{pmatrix} \epsilon_K & \pi_K \\ \pi_K^* & \epsilon_K \end{pmatrix}\begin{pmatrix} u(K_x, K_y) \\ v(K_x, K_y) \end{pmatrix} = E_K\begin{pmatrix} u \\ v \end{pmatrix} \quad \leftarrow (\#)$$

$$E_K = \epsilon_K \pm |\pi_K|$$

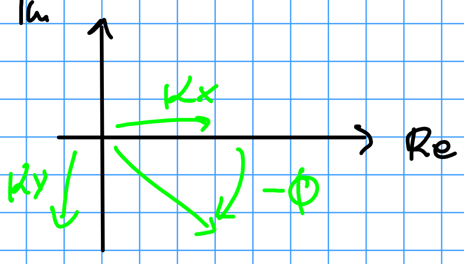
$(\#) \Rightarrow \quad \epsilon_K u + \pi_K v = (\epsilon_K \pm |\pi_K|)u$

$$\Rightarrow \quad u_K = \pm\frac{\pi_K}{|\pi_K|}v_K = \underline{\left(\mp i\,\mathrm{sign}(\mathcal{E})\,e^{-i\phi}\right)}\,v_K$$

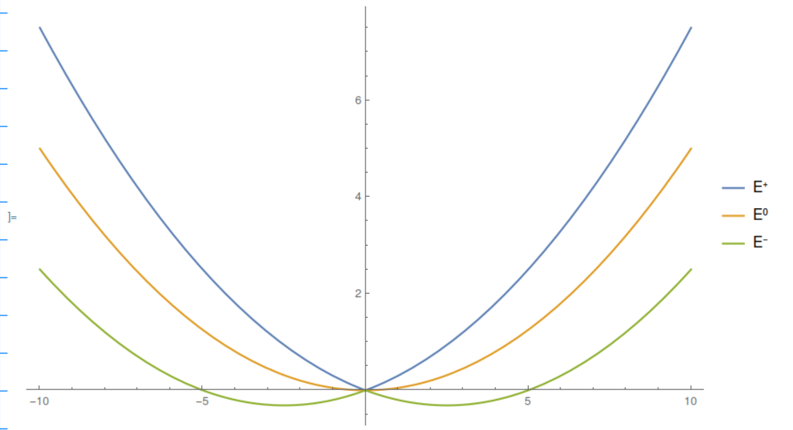
$$\pi_K = |\pi_K|e^{-i\phi}$$

$\pi_K \sim K_y + iK_x$

$\Rightarrow \quad -i(K_y + iK_x) = K_x - iK_y$



b) $$E_k^{\pm} = \underbrace{\frac{k^2}{2m}}_{\sim \epsilon_k} \pm \underbrace{\frac{e|\mathcal{E}|}{4m^2}|\vec{k}|}_{|\pi_k|}$$



c) $$\vec{S} = \frac{\vec{\sigma}}{2} \qquad \langle S \rangle_{\pm} = \int dx\,dy\, \psi^{\dagger}\left(\frac{\vec{\sigma}}{2}\right)\psi$$

$$\psi_k^{\pm} = N \begin{pmatrix} \mp i\,\mathrm{sgn}(\mathcal{E})\, e^{-i\phi} \\ 1 \end{pmatrix} e^{i\vec{k}\vec{r}}$$

$$\langle S^x \rangle_{\pm} = \int d^2r\, (\psi_k^{\pm})^{\dagger} S^x (\psi_k^{\pm})$$

$$= \underbrace{\int d^2r}_{\to A^2} N^2 \left(\pm i\,\mathrm{sgn}(\mathcal{E})\, e^{i\phi}, 1\right) \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mp i\,\mathrm{sgn}(\mathcal{E})\, e^{-i\phi} \\ 1 \end{pmatrix}$$

$$= N^2 A \frac{1}{2}\left(\pm i\,\mathrm{sgn}(\mathcal{E})\left(e^{i\phi} - e^{-i\phi}\right)\right)$$

$$= N^2 A \left(\mp \mathrm{sgn}(\mathcal{E})\right) \underbrace{\sin\phi}_{\frac{k_y}{|\vec{k}|}}$$

$\pi_k \sim k_x - i k_y$

$\pi_k = |\pi_k| e^{-i\phi}$

$= |\pi_k|(\cos\phi - i\sin\phi)$

$$\langle S^y \rangle_{\pm} = N^2 A \left(\pm \mathrm{sgn}(\mathcal{E})\right) \underbrace{\cos\phi}_{\frac{k_x}{|\vec{k}|}}$$

$$\langle S^z \rangle_{\pm} = N^2 A \left(e^{i\phi} e^{-i\phi} - 1\right) = 0$$

$$\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad \vec{E} = \mathcal{E}\hat{e}_z$$

$$\to \langle \vec{S} \rangle_{\pm} = N^2 A \left(\pm \mathrm{sgn}(\mathcal{E})\right) \cdot \left(\hat{e}_z \times \frac{\vec{k}}{|\vec{k}|}\right)$$

$\Rightarrow$ Spin $\perp$ Feld, Impuls

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$C = \gamma^2\gamma^0$

b) $\bar{\psi}^T = (\psi^+\gamma^0)^T = \gamma^{0T}\psi^* = \gamma^0\psi^* \Rightarrow \psi^* = \gamma^0\bar{\psi}^T$

$\left((i\partial\!\!\!/ + eA\!\!\!/ - m)\psi = 0\right)^*$

$= (-i\gamma^{\mu *}\partial_\mu + e\gamma^{\mu *}A_\mu - m)\psi^* = 0$

$(\gamma^\mu)^* = \gamma^\mu$ außer $(\gamma^2)^* = -\gamma^2$

$\gamma^{\mu *} = \gamma^2\gamma^\mu\gamma^2 \qquad (\gamma^0)^2 = \mathbb{1} \quad (\gamma^i)^2 = -\mathbb{1}$

$(-i\gamma^2\gamma^\mu\gamma^2\partial_\mu + e\gamma^2\gamma^\mu\gamma^2A_\mu + m\,\gamma^2\gamma^2)\gamma^0\bar{\psi}^T = 0$

$\gamma^2(-i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu + m)\gamma^2\gamma^0\bar{\psi}^T = 0$

$\gamma^2(i\partial\!\!\!/ - eA\!\!\!/ - m)\underbrace{\gamma^2\gamma^0\bar{\psi}^T}_{} = 0$

$\Rightarrow \psi_c = C\bar{\psi}^T \Rightarrow C = \gamma^2\gamma^0$

a) $(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi = 0$

$\psi' = e^{-i\alpha(x)e}\psi \qquad A'_\mu(x) = A_\mu(x) + \Lambda_\mu(x)$

$(i\partial\!\!\!/ + e(A\!\!\!/ + \Lambda\!\!\!/) - m)\psi' = 0$

$\underbrace{-i\gamma^\mu\partial_\mu\left(e^{-i\alpha(x)e}\psi(x)\right)}_{} + e\gamma^\mu A_\mu e^{-i\alpha(x)e}\psi + e\gamma^\mu\Lambda_\mu(x)e^{-i\alpha(x)e}\psi$

$- m e^{-i\alpha(x)e}\psi = 0$

$$= i\gamma^\mu \left(\left(-ie\, \partial_\mu \alpha(x)\right) e^{-i\alpha(x)e} \psi + e^{-i\alpha(x)e} \partial_\mu \psi \right)$$

$$= e^{-i\alpha(x)e} \Big(\overbrace{i \not{\partial} \psi + e \not{A} \psi - m\psi}^{\text{Dirac Gl. für } \psi \;=0}$$

$$+ e \not{\Lambda} + e \not{\partial} \alpha(x) \Big) \overset{!}{=} 0$$

$$\Rightarrow \quad e \not{\Lambda} + e \not{\partial} \alpha(x) = 0$$

$$\Rightarrow \quad \Lambda_\mu(x) = - \partial_\mu \alpha(x)$$

$$\Rightarrow \quad A'_\mu = A_\mu + \Lambda_\mu$$

$$= A_\mu - \partial_\mu \alpha(x)$$

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a) + b) $\longrightarrow$ Pia

c)

		KG	SG
$\frac{1}{r^2}$	:	$a = \hat{L}^2 - \alpha^2 = \tilde{l}(\tilde{l}+1)$	$a = \hat{L}^2 = l(l+1)$
$\frac{1}{r}$	:	$b = 2E_K\alpha$	$b = 2M\alpha$
r^0	:	$c = E_K^2 - M^2$	$c = 2ME_S$

$E_S = E_n$

d) SG: Wasserstoff $\quad E_n = -\frac{M\alpha^2}{2n^2}$

(SG) $\quad \frac{E_n}{\alpha} = \frac{-M\alpha}{2n^2} = \frac{c}{b} = -\frac{b}{4n^2} \Rightarrow c = \frac{-b^2}{4n^2}$

(KG) $\quad c = \frac{-b^2}{4\tilde{n}^2}$! $\Rightarrow E_K^2 - M^2 = -\frac{\cancel{4}E_K^2\alpha^2}{\cancel{4}\tilde{n}^2}$

$$E_K^2\left(1 + \frac{\alpha^2}{\tilde{n}^2}\right) = M^2 \Rightarrow E_K^2 = M^2\left(1 + \frac{\alpha^2}{\tilde{n}^2}\right)^{-1}$$

KG: $\tilde{\ell}(\tilde{\ell}+1) = \vec{L}^2 - \alpha^2$

SG: $\ell(\ell+1) = \vec{L}^2$

$\rightarrow \tilde{\ell}(\tilde{\ell}+1) = \ell(\ell+1) - \alpha^2$

Ansatz: $\tilde{\ell} = \ell + \delta$

$$(\ell+\delta)(\ell+\delta+1) \overset{!}{=} \ell(\ell+1) - \alpha^2$$

$$\delta^2 + 2(\ell+\tfrac{1}{2})\delta = \alpha^2$$

$$\Rightarrow \underline{\delta = -(\ell+\tfrac{1}{2}) \mp \sqrt{(\ell+\tfrac{1}{2})^2 - \alpha^2}} \quad (\#)$$

Ansatz: $\tilde{n} = \tilde{\ell} + n'$ $\leftarrow$ $n = N + \ell + 1$ (bekannt)

$n = \ell + n'$

$$\tilde{n}^2 = (\tilde{\ell} + n')^2 = (\ell + \delta + n')^2 \qquad n' = -\ell + n$$

$$\overset{(\#)}{=} \left(n - (\ell+\tfrac{1}{2}) + \sqrt{(\ell+\tfrac{1}{2})^2 - \alpha^2}\right)^2$$

$\tilde{n} = \tilde{n}(n, \ell)$

$$\tilde{n}^2 = \left(n - (\ell+\tfrac{1}{2}) + (\ell+\tfrac{1}{2})\sqrt{1 - \frac{\alpha^2}{(\ell+\frac{1}{2})^2}}\right)^2 \qquad \sqrt{1-x} \approx 1 - \tfrac{1}{2}x + \ldots$$

$$= n^2 + n - \frac{\alpha^2}{\ell+\frac{1}{2}} + \mathcal{O}(\alpha^3)$$

$$E_K^2 = M^2\left(1 + \frac{\alpha^2}{\tilde{n}^2}\right)^{-1} = M^2\left(1 - \frac{\alpha^2}{n^2} - \frac{\alpha^4}{n^3(\ell+\frac{1}{2})} + \frac{\alpha^4}{n^4} + \ldots\right)$$

$$E_K = \sqrt{E_K^2} = M\left(\underline{1} - \underline{\frac{\alpha^2}{2n^2}} + \underbrace{\alpha^4\left(\frac{3}{8n^4} - \frac{1}{2n^3(\ell+\frac{1}{2})}\right)}\right) + \mathcal{O}(\alpha^6)$$

Ruheenergie $E = Mc^2$

normaler (nichtrel, SGL) Wasserstoff

rel. Korrekturen

SGL-H^+: E_n hängen nicht von ℓ ab $\left(\sim \frac{1}{r}\right)$

$\Rightarrow$ hier nicht mehr!

3) SGe: $\left(\Delta + 2M\frac{\alpha}{r} + 2M E_s\right)\psi_s(\vec r) = 0$ $\quad \Delta = \partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat L^2}{r^2}$ $\quad$ (mit $\hat L^2 \psi_s = \ell(\ell+1)\psi_s$)

a) KGe for Coulomb potential

$[(\partial^\mu + iqA^\mu)(\partial_\mu + iqA_\mu) + M^2]\psi = 0$

wir betrachten ein Elektron, d.h. $q=-e$ darf man setzen

für Coulomb Potential

b) $[\partial^\mu\partial_\mu + iq\,\partial^\mu A_\mu + iq A^\mu \partial_\mu + M^2 - q^2 A^\mu A_\mu]\psi_k = 0$

$\Leftrightarrow [\partial_t^2 - \Delta + 2iq[\partial_t \phi - \cancel{\vec\nabla \cdot \vec A}] - q^2\phi^2 + \cancel{q^2 A^2} + M^2]\psi_k = 0$ $\qquad q\phi = -\frac{\alpha}{r}$ $\qquad \vec A = 0$

$\psi_k(x^\mu) = e^{-i\omega t}\phi_k(\vec r)$

$\Rightarrow [\partial_t^2 - \Delta - 2i\frac{\alpha}{r}\partial_t - \frac{\alpha^2}{r^2} + M^2]\, e^{-i\omega t}\phi_k = 0$

$\Leftrightarrow [-\omega^2 - \Delta - 2\frac{\alpha}{r}\omega - \frac{\alpha^2}{r^2} + M^2]\,\phi_k = 0$ $\qquad |\cdot(-1)$

$\rightarrow [+\Delta - M^2 + \omega^2 + 2\frac{\alpha}{r}\omega + \frac{\alpha^2}{r^2}]\phi_k = 0$

$\Leftrightarrow [\Delta - M^2 (\omega + \frac{\alpha}{r})^2]\,\phi_k = 0$ $\qquad E_k = \omega \cdot \hbar$ $\qquad$ ($\hbar = 1$ macht das Leben leichter ☺)

$\Rightarrow [\Delta - M^2 (E_k + \frac{\alpha}{r})^2]\phi_k = 0$

c) compare SGe with KGe

	$\partial_r^2 + \frac{2}{r} - \frac{\hat L^2}{r^2} + 2M + 2M\frac{\alpha}{r} + 2ME_s S$ ($\hat L^2 \to \ell(\ell+1)$)	$\partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat L^2}{r^2} - M^2 + \frac{\alpha^2}{r^2} + 2E_k\frac{\alpha}{r} + E_k^2$
r^0	$\partial_r^2 + 2ME_s$	$\partial_r^2 - M^2 + E_k^2$
r^{-1}	$\frac{2}{r}\partial_r + 2M\frac{\alpha}{r}$	$\frac{2}{r}\partial_r + 2E_k\frac{\alpha}{r}$
r^{-2}	$-\frac{\ell(\ell+1)}{r^2}$	$-\frac{\ell(\ell+1)}{r^2} + \frac{\alpha^2}{r^2}$

wenn man $q\phi = -e\phi = -\frac{\alpha}{r} \equiv V_c = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}$ fordert, dann ist $\alpha = \frac{Ze^2}{4\pi}$, $\varepsilon_0 = 1$ (hier $Z=1$)

$E_n = -\frac{M\alpha^2}{2n^2}$

Kopplungskonstante bekannt aus Kern- & Teilchenphysik!

d) $E_{s,n} = \frac{-e^4 M}{2(4\pi)} \cdot \frac{1}{n^2}$ $\Leftrightarrow$ $2ME_{s,n} = -\frac{e^4 M^2}{4\pi} \cdot \frac{1}{(n_r + \ell + 1)^2}$ $\longrightarrow$ $E_k^2 - M^2 = -\frac{E_k^2 \cdot e^4}{4\pi} \cdot \frac{1}{(?)^2}$