Quantentheorie II Übung 1

Besprechung: 2021WE16 (KW16)

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1. **Questions**
   (a) What are the axioms that define Hilbert space?
   (b) Explain the properties of Hermitian matrices.
   (c) What are characteristic properties of unitary transformations?

2. **Observables:** $\hat{A}$ is a Hermitian operator, and its eigenvalues a_n and eigenstates $|n\rangle$ are given:
$$\hat{A}|n\rangle = a_n|n\rangle \, .$$
Show that the eigenvalues a_n are real, and the eigenstates $|n\rangle$ are orthogonal.

3. **Projectors:** The projection operator of a state space is
$$\hat{\Pi} = \sum_i |i\rangle\langle i| \, ,$$
where $|i\rangle$ are a certain set of orthonormal states (not necessarily a complete basis).
   (a) Show that $\hat{\Pi}$ is Hermitian.
   (b) Find the eigenvalues and eigenvectors of $\hat{\Pi}$.

4. **Translation Operator:** Define the translation operator by
$$T(a) := e^{-\frac{i}{\hbar}\hat{p}a}.$$
Show that
$$\hat{x}\left(T(a)|x_0\rangle\right) = (x_0 + a)\left(T(a)|x_0\rangle\right) \, ,$$
and in this way you may identify $T(a)|x_0\rangle = |x_0 + a\rangle$.

5. **Harmonic oscillator:** The harmonic oscillator Hamiltonian is given
$$\mathcal{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \, ,$$
and $[\hat{a}, \hat{a}^\dagger] = 1$.
   (a) Calculate for an eigenstate $|n\rangle$ of HO:
$$\langle \hat{x}^2 \rangle \, , \quad \langle \hat{p}^2 \rangle \, .$$
   (b) What are the results for the groundstate $|0\rangle$?

(c) At the measurement moment the wavefunction of the HO is

$$|\psi\rangle = \frac{1}{\sqrt{74}} \left(\frac{5}{\sqrt{24}} \hat{a}^{\dagger\, 4} + \frac{7}{\sqrt{6}} \hat{a}^{\dagger\, 3} \right) |0\rangle .$$

Which energy values can be measured? And what is the probability to measure those values?

6. **Spin $\frac{1}{2}$ and Pauli matrices:** The angular momentum operator components $\hat{J}_a$, $a \in \{1, 2, 3\}$ satisfy the commutation relations:

$$[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc} \hat{J}_c \,.$$

The ladder operators are defined as $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$.

(a) Show

$$\hat{J}_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle \,. \tag{1}$$

(b) Apply Eq. 1 for the case of $j = \frac{1}{2}$ and show

$$\begin{pmatrix} \langle +|\hat{J}_i|+\rangle & \langle +|\hat{J}_i|-\rangle \\ \langle -|\hat{J}_i|+\rangle & \langle -|\hat{J}_i|-\rangle \end{pmatrix} = \frac{\sigma_i}{2}, \tag{2}$$

where $|\pm\rangle \equiv |\frac{1}{2}, \pm\frac{1}{2}\rangle$, and σ_i , $i \in \{1, 2, 3\}$ are Pauli matrices.

(c) Verify the relation

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot \vec{A} \times \vec{B} \,,$$

where $\vec{A}$ and $\vec{B}$ are vector operators which commute with σ_i, but do not commute with each other.

7. **Rotation matrix:** Take the matrix l_z as defined in the lecture, with matrix elements

$$(l_z)_{ij} = -i\epsilon_{3ij} \,.$$

(a) Compute powers such as $(l_z)^2$, $(l_z)^3$, etc., and then the matrix-valued exponential $e^{-i\theta l_z}$.

(b) Show that this agrees with the rotation matrix $R_z(\theta)$ defined in the lecture.