Quantentheorie II Übung 5

– Sample solutions –

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2. **Two-particle system:** $|\psi\rangle$ and $\phi\rangle$ are orthonormal one-particle states: $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ and $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle = 0$. The operators x_1 and x_2 are defined as $x_1 = x^{(1)} \otimes 1$ and $x_2 = 1 \otimes x^{(2)}$. The unsymmetrized, symmetric and antisymmetric two-particle Hilbert spaces are $\mathcal{H}_2 = \{|\psi^{(1)}\phi^{(2)}\rangle, |\phi^{(1)}\psi^{(2)}\rangle\}$, $\mathcal{H}_2^+ = \{\frac{1}{\sqrt{2}}(|\psi^{(1)}\phi^{(2)}\rangle + |\phi^{(1)}\psi^{(2)}\rangle)\}$ and $\mathcal{H}_2^- = \{\frac{1}{\sqrt{2}}(|\psi^{(1)}\phi^{(2)}\rangle - |\phi^{(1)}\psi^{(2)}\rangle)\}$ respectively, where $|\psi^{(1)}\phi^{(2)}\rangle \equiv |\psi^{(1)}\rangle|\phi^{(2)}\rangle$.

   (a) Distinguishable particles: $|D\rangle = |\psi^{(1)}\phi^{(2)}\rangle$.

$$\begin{aligned}\langle(x_1-x_2)^2\rangle_D &= \langle D|(x_1-x_2)^2|D\rangle = \langle\psi^{(1)}\phi^{(2)}|x_1^2+x_2^2-2x_1x_2|\psi^{(1)}\phi^{(2)}\rangle \\ &= \langle\psi^{(1)}|(x^{(1)})^2|\psi^{(1)}\rangle\langle\phi^{(2)}|\phi^{(2)}\rangle + \langle\psi^{(1)}|\psi^{(1)}\rangle\langle\phi^{(2)}|(x^{(2)})^2|\phi^{(2)}\rangle \\ &\quad - 2\langle\psi^{(1)}|x^{(1)}|\psi^{(1)}\rangle\langle\phi^{(2)}|x^{(2)}|\phi^{(2)}\rangle \\ &= \langle x^2\rangle_\psi + \langle x^2\rangle_\phi - 2\langle x\rangle_\psi\langle x\rangle_\phi \qquad (1) \\ &\Longleftarrow\ (\langle O\rangle_A = \langle A|O|A\rangle),\ \text{for } O = x, x^2,\ \text{and } A = \psi, \phi\end{aligned}$$

   (b) Two identical bosons: symmetric state, $|B\rangle = \frac{1}{\sqrt{2}}(|\psi^{(1)}\phi^{(2)}\rangle + |\phi^{(1)}\psi^{(2)}\rangle)$.

$$\begin{aligned}\langle(x_1-x_2)^2\rangle_B &= \langle B|(x_1-x_2)^2|B\rangle \\ &= \langle\psi^{(1)}\phi^{(2)}|(x_1-x_2)^2|\psi^{(1)}\phi^{(2)}\rangle \\ &\quad + \mathrm{Re}\left(\langle\psi^{(1)}\phi^{(2)}|(x_1-x_2)^2|\psi^{(2)}\phi^{(1)}\rangle\right), \qquad (2)\end{aligned}$$

$$\begin{aligned}\langle\psi^{(1)}\phi^{(2)}|(x_1-x_2)^2|\phi^{(2)}\phi^{(1)}\rangle &= -2\langle\psi^{(1)}|x_1|\phi^{(1)}\rangle\langle\phi^{(2)}|x_2|\psi^{(2)}\rangle \\ &= -2|\langle x\rangle_{\psi\phi}|^2 \Longleftarrow (\langle x\rangle_{\psi\phi} = \langle x\rangle^*_{\phi\psi}) \qquad (3)\end{aligned}$$

$$\therefore \quad \langle(x_1-x_2)^2\rangle_B = \langle(x_1-x_2)^2\rangle_D - 2|\langle x\rangle_{\psi\phi}|^2. \qquad (4)$$

   (c) Tow identical fermions: antisymmetric state, $|F\rangle \equiv \frac{1}{\sqrt{2}}\left(|\psi^{(1)}\phi^{(2)}\rangle - |\phi^{(1)}\psi^{(2)}\rangle\right)$.

$$\begin{aligned}\langle(x_1-x_2)^2\rangle_F &= \langle F|(x_1-x_2)^2|F\rangle \\ &= \langle(x_1-x_2)^2\rangle_D - \mathrm{Re}(\langle\psi^{(1)}\phi^{(2)}|(x_1-x_2)^2|\phi^{(1)}\psi^{(2)}\rangle) \\ &= \langle(x_1-x_2)^2\rangle_D + 2|\langle x\rangle_{\psi\phi}|^2. \qquad (5)\end{aligned}$$

   We can see that the expectation value of identical fermions are larger than that of identical bosons.

$$\langle(x_1-x_2)^2\rangle_B < \langle(x_1-x_2)^2\rangle_D < \langle(x_1-x_2)^2\rangle_F. \qquad (6)$$

3. **Gravitationally bound state:** we consider two identical particles with mass m, which are attracted by the gravitational force. The Hamiltonian is

$$\hat{H} = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{Gm^2}{|\vec{r}_1^2 - \vec{r}_2^2|}. \tag{7}$$

(a) C-M coordinate: $\vec{R} \equiv \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, total mass $M = 2m$
Relative coordinate: $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, reduced mass $M_R = \frac{m}{2}$
$\vec{\nabla}_1 = \frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_r$, $\vec{\nabla}_2 = \frac{1}{2}\vec{\nabla}_R - \vec{\nabla}_r$

$$\hat{H} = -\frac{1}{4m}\nabla_R^2 - \frac{1}{m}\nabla_r^2 - \frac{Gm^2}{r} \qquad = \hat{H}_R + \hat{H}_r,$$

where

$$\hat{H}_R \equiv -\frac{1}{4m}\nabla_R^2 \text{ free particle of mass } 2m, \tag{8}$$

$$\hat{H}_r \equiv -\frac{1}{m}\nabla_r^2 - \frac{Gm^2}{r}. \tag{9}$$

$\hat{H}_r$ is equivalent to the Hamiltonian of the electron in the hydrogen atom with reduced mass $M = \frac{m}{2}$.

$$\hat{H}_{hyd.} = -\frac{1}{2M}\nabla_r^2 - \frac{\alpha}{|\vec{r}|}, \text{ where } M = m_e. \tag{10}$$

The solutions of $\hat{H}_{hyd.}$ are

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\phi), \text{ where } r = |\vec{r}|, \tag{11}$$

and $R_{nl}(r)$ is Laguerre Polynomials and $Y_{lm}(\theta,\phi)$ associated Legendre polynomials

$$R_{nl}(r) = \left(\frac{1}{a_0}\right)^{\frac{3}{2}} \frac{2}{n^2(n+l)!}\sqrt{\frac{(n-l-1)!}{(n+1)!}}e^{-\frac{r}{na_0}}L_{n+l}^{2l+1}\left(\frac{2}{na_0}r\right), \tag{12}$$

$$Y_{lm}(\theta,\phi) \propto P_l^m(\cos\theta), \tag{13}$$

where $L_n^m(r)$ are the Laguerre polynomial and $P_l^m(x)$ associated Legendre polynomial. Also $\psi_{nlm}(-\vec{r}) = (-1)^m\psi_{nlm}(\vec{r})$. The gravitationally bound eigenstates are similarly obtained.

$$\Psi_{\vec{K},nlm}(\vec{K},\vec{r}) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} e^{i\vec{K}\cdot\vec{R}} \left(\frac{a_0}{a_G}\right)^{\frac{3}{2}} \psi_{nlm}(\frac{a_0}{a_G}\vec{r}), \tag{14}$$

The Bohr radius and energy eigenvalues for the hydrogen atom and the gravitational bound states are

$$\left.\begin{aligned} a_0 &= \tfrac{1}{\alpha M} \\ E_n &= -\tfrac{E_R}{n^2} \\ E_R &= \tfrac{1}{2Ma_0^2} \end{aligned}\right\} \xRightarrow{\alpha=Gm^2, M=\frac{m}{2}} \left\{\begin{aligned} a_G &= \tfrac{2}{Gm^3} \\ E_n &= -\tfrac{E_G}{n^2} = -\tfrac{G^2m^5}{4n^2} \\ E_G &= \tfrac{G^2m^5}{4} \end{aligned}\right. . \tag{15}$$

(b) Spinless bosons: symmetric wave functions $\Longrightarrow$ $\psi_{\vec{K},nlm}(\vec{R},\vec{r})$ symmetric under $\vec{r} \to -\vec{r}$
$\therefore$ only zero and even l are allowed. $l = 0, 2, 4, \cdots$.
Spin-$\frac{1}{2}$ fermions: wave functions multiplied by spin-state $\chi(\sigma_1, \sigma_2)$,

$$\chi(\sigma_1, \sigma_2) = \begin{cases} \chi_S(\sigma_1, \sigma_2), & \text{Singlet, antisymmetric} \\ \chi_T(\sigma_1, \sigma_2), & \text{Triplet, symmetric} \end{cases} . \tag{16}$$

The fermion wave functions $\psi_{\vec{K},nlm}(\vec{R},\vec{r})\chi(\sigma_1, \sigma_2)$ should be antisymmetric. For singlet spin states, the spatial functions should be symmetric, therefore zero and even l are allowed, $l = 0, 2, 4, \cdots$. For triplet spin states, the spatial functions should be antisymmetric, therefore l should be an odd integer, $l = 1, 3, 5, \cdots$.

(c) The binding energy is far smaller than the thermal energy of the cosmic microwave background.

to be updated