Quantentheorie II Übung 12

– Sample solutions –

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2. **Scattering on central potential:** see Nolting (9.2.2).

3. **Electron scattering on hydrogen atom (Born approximation):** see Nolting (9.3.4) for calculation in position space. The calculation in momentum space is presented in the following.

   (a) The ground state of the electron in hydrogen atom is $\psi_{100}(\vec{x}) = \frac{1}{\sqrt{\pi}} a_B^{-\frac{3}{2}} e^{-\frac{r}{a_B}}$. The charge distribution in hydrogen atom is

$$\begin{aligned}\rho(\vec{x}) &= \rho_{\text{nucleus}} + \rho_{\text{electron}} \\ &= e\delta^{(3)}(\vec{x}-\vec{x}_0) - e|\psi_{100}(|\vec{x}-\vec{x}_0| = r)|^2 \\ &= e(\delta^{(3)}(\vec{x}) - \frac{1}{\pi a_B^3} e^{\frac{-2r}{a_B}}), \quad \Longleftarrow \quad \vec{x}_0 = 0 \end{aligned} \tag{1}$$

   and the potential from the charge distribution is

$$V(\vec{x}) = -e \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}. \tag{2}$$

   As $\rho(\vec{x}) \propto r$, $V(\vec{x})$ is obviously spherical symmetric.

   (b) The scattering amplitude in first Born approx. is given as

$$f^{(1)}(\theta) = \frac{m}{2\pi}\tilde{V}(\vec{q}), \text{ and } \tilde{V}(\vec{q}) = \int d^3x e^{-i\vec{q}\cdot\vec{x}} V(\vec{x}). \tag{3}$$

   The Fourier transformation of $V(\vec{x})$ is

$$\begin{aligned}\tilde{V}(\vec{q}) &= -e \int d^3x e^{-i\vec{q}\cdot\vec{x}} \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \\ &= -e \int d^3x e^{-i\vec{q}\cdot\vec{x}} \int d^3x' \rho(\vec{x}') \int \frac{d^3q'}{(2\pi)^3} \frac{e^{i\vec{q}'\cdot(\vec{x}-\vec{x}')}}{(\vec{q}')^2} \\ &\Longleftarrow \quad \nabla^2 \frac{-1}{4\pi r} = \delta^{(3)}(\vec{x}) \to \tilde{V}_{\text{Coul.}}(\vec{q}) = \frac{1}{\vec{q}^2} \\ &= -e \int d^3x e^{-i\vec{q}\cdot\vec{x}} \int \frac{d^3q'}{(2\pi)^2} \int d^3x' \rho(\vec{x}') e^{i\vec{q}'\cdot(\vec{x}-\vec{x}')} \\ &= -e \int d^3x' \rho(\vec{x}') \int \frac{d^3q'}{(2\pi)^3} \frac{e^{-i\vec{q}'\cdot\vec{x}'}}{(\vec{q})^2} \delta^{(3)}(\vec{q}-\vec{q}') \\ &\Longleftarrow \int d^3x e^{-i\vec{x}\cdot(\vec{q}-\vec{q}')} = (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{q}') \\ &= \frac{-e}{(\vec{q})^2} \int d^3x' \rho(\vec{x}') e^{-i\vec{q}\cdot\vec{x}'} = \frac{-e}{(\vec{q})^2} \tilde{\rho}(\vec{q}), \end{aligned} \tag{4}$$

and

$$\tilde{\rho}(\vec{q}) = e\Big(1 - \frac{1}{\pi a_B}\int d^3x \frac{1}{\pi a_B^3} e^{\frac{-2r}{a_B}} e^{-i\vec{q}\cdot\vec{x}}\Big)$$
$$= e\Big(1 - \frac{16}{(4+q^2a_B^2)^2}\Big). \tag{5}$$
$$\therefore \tilde{V}(\vec{q}) = \frac{-e}{q^2}\tilde{\rho}(\vec{q}) = -e^2 a_B^2 \frac{8+q^2a_B^2}{(4+q^2a_B^2)^2}. \tag{6}$$

The scattering amplitude is

$$f^{(1)}(\theta) = \frac{-m}{2\pi}\tilde{V}(\vec{q}) = 2a_B\frac{8+q^2a_B^2}{(4+q^2a_B^2)^2}, \tag{7}$$

and $\frac{d\sigma^{(1)}}{d\Omega} = |f^{(1)}(\theta)|^2$.

(c)

$$\sigma^{(1)} = \int d\Omega \frac{d\sigma^{(1)}}{d\Omega} = \int d\Omega |f^{(1)}(\theta)|^2$$
$$= 8\pi a_B^2 \int d\cos\theta \frac{(8+q^2a_B^2)^2}{(4+q^2a_B^2)^4}$$
$$= \frac{4\pi a_B^2}{k^2}\int_0^{4k^2} dq^2 \frac{(8+q^2a_B^2)^2}{(4+q^2a_B^2)^4} \Longleftarrow q^2 = 4k^2\sin^2\frac{\theta}{2}$$
$$= \frac{4\pi}{k^2}\int_0^{4k^2a_B^2} dx \frac{(8+x)^2}{(4+x)^4} \Longleftarrow x = q^2a_B^2$$
$$= \frac{4\pi}{k^2}\left(-\frac{16}{3}\frac{1}{x^3} - 4\frac{1}{x^2} - \frac{1}{x}\right)\Big|_4^{4k^2a_B^2+4}. \tag{8}$$