Quantentheorie II Übung 8

Besprechung: 2021WE24 (KW24)

SS 2021 Prof. Dominik Stöckinger (IKTP)

1. **Questions**

   (a) Which Hilbert space(s) can a bosonic creation operator $a^\dagger_\psi$ act onto: $\mathcal{H}_N$, $\mathcal{H}_N^{(+)}$, $\mathcal{H}_N^{(-)}$? What is (are) the results?

   (b) Write down the (anti)commutation relations for fermionic and bosonic creation/annihilation operators!

   (c) Write down a relationship of the form $|0\rangle = (???)|\psi\rangle$.

   (d) What is the meaning of a "one-particle observable", and how is such an observable represented in terms of a, $a^\dagger$?

   (e) Is the operator $a^\dagger_\psi a_\psi$ an observable/if yes, what does it mean?

   (f) Is the operator $a^\dagger_\psi a^\dagger_\phi a_\psi$ an observable/if yes, what does it mean?

   (g) Suppose a very complicated Hamiltonian

   $$H = \sum_k \frac{p_k^2}{2m} a^\dagger_{p_k} a_{p_k} + \sum_{ijklmn} x_{ijklmn} a^\dagger_i a^\dagger_j a^\dagger_k a_l a_m a_n,$$

   which through some approximation leads to

   $$H = \mathcal{E}_0 + \sum_\alpha \epsilon_\alpha b^\dagger_\alpha b_\alpha\,,$$

   with

   $$\epsilon_\alpha > 0, \qquad [b_\alpha, b^\dagger_\beta] = \delta_{\alpha\beta}, \qquad [b_\alpha, b_\beta] = 0,$$

   and $\mathcal{E}_0$ is a numerical constant.

      i. What does this mean? How does it help? Explain an interpretation using the term "quasi-particle"! What is the ground state and its energy? What is the lowest excited state, and is this unique? How stable is the ground state against very small perturbations?

      ii. In answering, is it useful to distinguish the following cases?

         A. $\epsilon_\alpha > 0$ and nothing else

         B. $\epsilon_\alpha \geq \epsilon_{\min} > 0$

         C. $\epsilon_1 > 0, \epsilon_\alpha > \epsilon_1 \forall \alpha \neq 1$

2. **Weakly interacting bose gases and superfluidity:** the second quantization expression of the Hamiltonian of N interacting particles in a box of volume $V = L^3$ is

$$\hat{H} = \sum_k \frac{p_k^2}{2m} a^\dagger_{p_k} a_{p_k} + \frac{1}{2} \sum_{p} \frac{1}{L^3} \omega(\vec{p}_l - \vec{p}_i) a^\dagger_{p_l} a^\dagger_{p_m} a_{p_i} a_{p_k}\,, \tag{1}$$

where $\omega(\vec{p}) \equiv \int d^3q V(|\vec{q}|) e^{-i\vec{p}\cdot\vec{q}}$, and the summation of the second term in Eq. (1) is subject to the condition $\vec{p}_l + \vec{p}_m = \vec{p}_i + \vec{p}_k$. We consider an almost condensed situation, i.e. most of the particles are found in the ground state, $a_0^\dagger a_0 = n_0 \approx N$.

(a) By considering the assumption, put the Hamiltonian in Eq. (1) into a simpler form

$$\hat{H} = \sum_{p\neq 0}\left[\frac{p^2}{2m} + \frac{n_0}{V}\omega(p)\right] a_p^\dagger a_p + \frac{1}{2}\frac{N^2}{V}\omega(0) + \frac{a_0^2}{2V}\sum_{p\neq 0}\omega(p) a_p^\dagger a_{-p}^\dagger + \frac{a_0^{\dagger 2}}{2V}\sum_{p\neq 0}\omega(p) a_p a_{-p}\,. \tag{2}$$

The particle number in p-state n_p is very small $n_p \ll N$, and $\sum_{p\neq 0} a_p^\dagger a_p = N - n_0 \approx 0$.

(b) Introduce new operators b_p and $b_p^\dagger$ related to a_p and $a_p^\dagger$ (symplectic transformation):

$$a_p = u_p b_p + v_p b_{-p}^\dagger\,, \qquad a_p^\dagger = u_p b_p^\dagger + v_p b_{-p}\,, \tag{3}$$

where u_p and v_p are unknown functions of p, which are determined in such a way that the following conditions are satisfied:

i. They satisfy the usual commutation relations,

$$[b_p, b_{p'}^\dagger] = \delta_{pp'}\,, \qquad [b_p, b_{p'}] = [b_p^\dagger, b_{p'}^\dagger] = 0 \tag{4}$$

ii. The Hamiltonian in Eq. (2) transforms into a diagonal form. In other words all non-diagonal terms vanish.

Determine u_p, v_p accordingly.

(c) Verify that the diagonalized Hamiltonian is

$$\hat{H} = \frac{N^2}{2V}\omega(0) - \frac{1}{2}\sum_{p\neq 0}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) - \epsilon(p)\right] + \sum_{p\neq 0}\epsilon(p) b_p^\dagger b_p\,, \tag{5}$$

where

$$\epsilon(p) = \sqrt{\frac{N\omega(p)}{V}\frac{p^2}{m} + \frac{p^4}{4m^2}}\,, \text{ and } n_0 \approx N. \tag{6}$$

Eq. (5) is the Hamiltonian of the weakly interacting Bose gas in the Bogoliubov approximation, and Eq. (6) is the dispersion relation of the "quasi-particles" which are created (annihilated) by $b_p^\dagger$ (b_p).

(d) Suppose that the potential is $V(x) = \lambda\delta^{(3)}(\vec{x})$. What is the dominant term in $\epsilon(p)$ for $p \approx 0$ and for $p \gg 1$ respectively?

(e) We now consider that the particles are moving with velocity $\vec{v}$ and think of two reference frames. One is moving along with the particles with velocity $\vec{v}$ and the other fixed. E denotes the total kinetic energy in the moving reference frame and E' the total kinetic energy in the fixed reference frame. E and E' are related as

$$E' = E + N\frac{mv^2}{2} + \vec{v}\cdot\vec{P}\,,$$

where $\vec{P}$ is the total momentum of the system. Any decrease in the velocity of the system is equivalent to the creation of a "quasi-particle" having a momentum in a direction opposite to that of $\vec{v}$. What is the kinetic energy change $\Delta E'$? Negative $\Delta E'$ means kinetic energy loss of the system. What is the physical meaning of non-zero value of $\min_p \frac{\epsilon(p)}{p}$? What is the condition that this N particle system can show superfluid properties?

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a) $a^\dagger_\psi : \mathcal{H}_N^{(+)} \to \mathcal{H}_{N+1}^{(+)}$

$\downarrow$ $N=0,1$ $\quad \mathcal{H}_0, \mathcal{H}_1$

$$a^\dagger_n \underbrace{|m\rangle}_{N T.} = \underbrace{\sqrt{2}}_{\sqrt{N+1}} \underbrace{|nm\rangle^+}_{N+1\, T.}$$

b) $\{[a_n, a_m^+]\} = "\delta_{nm}"$

$\{[a_n, a_m]\} = \{[a^+_n, a^+_m]\}$
$= 0$

c) $$|n_1 .. n_N\rangle^+ = \frac{1}{\sqrt{N!}} a^+_{n_1} .. a^+_{n_N} |0\rangle$$

$$|0\rangle = a_\psi \cdot \bar{N} \, |\psi\rangle$$

d) $$T = \sum_{kl} T_{kl} a^+_k a_l$$

e) ja Teilchenzahl

f) nein

Bedingungen Observable

- hermitesch
- symmetrisch $\psi \leftrightarrow \phi$

Reihenfolge egal

(g) Suppose a very complicated Hamiltonian

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which through some approximation leads to

$$H = \underline{\mathcal{E}_0} + \sum_\alpha \epsilon_\alpha b^\dagger_\alpha b_\alpha\,, \qquad (*)$$

with

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i. What does this mean? How does it help? Explain an interpretation using the term "quasi-particle"! What is the ground state and its energy? What is the lowest excited state, and is this unique? How stable is the ground state against very small perturbations?

ii. In answering, is it useful to distinguish the following cases?

A. $\epsilon_\alpha > 0$ and nothing else

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• diagonalisieren $\Rightarrow$ (*)

$b^+, b \sim a^+, a$

• Hartree Fock, mean field

$\langle aaa \rangle \neq 0$

• (*) = HO

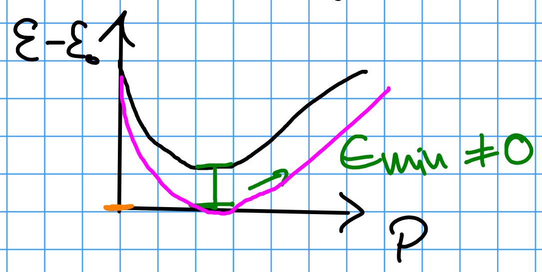
$\rightarrow$ Quasiteilchen b^+, b

bosonisch

Grundzustand: $b\,|\underline{\text{groundst.}}\rangle = 0 \rightsquigarrow \varepsilon_0$

angeregt: $b^+\,|\text{groundst.}\rangle = |\text{Quasiteilchen}\rangle \rightsquigarrow \epsilon_\alpha$

B.



$\Rightarrow$ Grundstabil ggü. Anregungen

A. $\Rightarrow$ Gr.Z. instabil

C. wie A nur ϵ_1 eindeutig.

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where $\omega(\vec{p}) \equiv \int d^3q V(|\vec{q}|)e^{-i\vec{p}\cdot\vec{q}}$, and the summation of the second term in Eq. (1) is subject to the condition $\vec{p}_l + \vec{p}_m = \vec{p}_i + \vec{p}_k$. We consider an almost condensed situation, i.e. most of the particles are found in the ground state, $a_0^\dagger a_0 = n_0 \approx N$.

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The particle number in p-state n_p is very small $n_p \ll N$, and $\sum_{p\neq 0} a_p^\dagger a_p = N - n_0 \approx 0$.

a)

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p$$
$$+ \frac{1}{2V}\,\omega(0)\sum_{p\neq 0} \underbrace{a_0^\dagger a_0^\dagger}_{n_0} a_0 a_0 \rightarrow \frac{1}{2V}\omega(0)\, n_0^2$$
$$+ \frac{1}{2V}\,\omega(0)\sum_{p\neq 0} a_0^\dagger a_0 a_p^\dagger a_p = 0$$
$$+ \frac{1}{2V}\sum \omega(p)\, a_0^\dagger a_0^\dagger a_p a_{-p}$$
$$+ \frac{1}{2V}\sum \omega(p)\, a_0 a_0 a_p^\dagger a_{-p}^\dagger$$
$$+ \frac{1}{2V}\sum \omega(p)\, \overbrace{a_0^\dagger a_0}^{n_0} a_p^\dagger a_p$$
$$+ \frac{1}{2V}\sum \omega(p)\, \underbrace{a_0^\dagger a_0}_{n_0} a_p^\dagger a_p \quad + \mathcal{O}(a_p^3)$$

Impulserhaltung

$\vec{p}_l$	$+\vec{p}_m$	$=\vec{p}_i$	$+\vec{p}_k$
0	0	0	0
0	p	0	p
0	0	p	-p
p	-p	0	0
0	p	p	0
p	0	0	p

weil
1). $a_0^\dagger a_0 \simeq N$
2). schwache WW

$$= \frac{1}{2V}\omega(0)\, n_0^2 + \sum\left[\frac{p^2}{2m} + \frac{n_0}{V}\omega(p)\right] a_p^\dagger a_p$$
$$+ \frac{1}{2V}\underbrace{a_0^\dagger a_0^\dagger}_{N}\sum \omega(p)\, a_p^\dagger a_{-p} + \frac{1}{2V}\underbrace{a_0 a_0}_{N}\sum \omega(p)\, a_p a_{-p}$$

$N \sim 10^{23}$

$a_0^\dagger a_0^\dagger = n_0 = N$

$a_0 a_0 = n_0 = N$

$a_0^\dagger a_0 = N = n_0$

$a_0 \simeq \sqrt{n_0}$
$a_0^\dagger \simeq \sqrt{n_0}$

$$= \frac{N^2}{2V}\omega(0) + \sum_p\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] a_p^\dagger a_p + \frac{N}{2V}\sum \omega(p)\left(a_p a_{-p} + a_p^\dagger a_{-p}^\dagger\right)$$

(b) Introduce new operators b_p and $b_p^\dagger$ related to a_p and $a_p^\dagger$ (symplectic transformation):

$$a_p = u_p b_p + v_p b_{-p}^\dagger\,, \qquad a_p^\dagger = u_p b_p^\dagger + v_p b_{-p}\,, \tag{3}$$

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i. They satisfy the usual commutation relations,

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ii. The Hamiltonian in Eq. (2) transforms into a diagonal form. In other words all non-diagonal terms vanish.

Determine u_p, v_p accordingly.

• Annahme: $u(p) = \underset{= u_p}{u(-p)}$ $\quad V(p) = V(-p)$ $\quad$ weil H sym. $p \leftrightarrow -p$

i) $[a_p, a_p^\dagger] = u_p^2 \underbrace{[b_p, b_p^\dagger]}_{1} + V_p^2 \underbrace{[b_{-p}^\dagger, b_{-p}]}_{-1} + u_p V_p \cancel{[b_p, b_{-p}]} + u_p V_p \cancel{[b_{-p}^\dagger, b_p^\dagger]}$

$= u_p^2 - V_p^2 = 1$

$a_p^\dagger a_p = (u(p)\, b_p^\dagger + V_p\, b_{-p})(u_p b_p + V_p\, b_{-p}^\dagger)$

$= u_p^2\, b_p^\dagger b_p + u_p V_p\, b_p^\dagger b_{-p}^\dagger + V_p^2\, b_{-p} b_{-p}^\dagger + V_p u_p\, b_{-p} b_p$

$a_p a_{-p} = u_p^2\, b_p b_{-p} + u_p V_p\, b_p b_p^\dagger + V_p u_p\, b_{-p}^\dagger b_{-p} + V_p^2\, b_{-p}^\dagger b_p^\dagger$

$a_p^\dagger a_{-p}^\dagger = u_p^2\, b_p^\dagger b_{-p}^\dagger + u_p V_p\, b_p^\dagger b_p + V_p u_p\, b_{-p} b_{-p}^\dagger + V_p^2\, b_{-p} b_p$

$H = \frac{N^2}{2V} \omega(0) + \sum \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] V_p^2 + \omega(p) \frac{N}{V} u_p V_p \right)$

$+ \sum b_p^\dagger b_p \left([\ldots] u_p^2 + \frac{N}{V}\omega(p)\, u_p V_p \right)$

$+ \sum b_{-p}^\dagger b_{-p} \left([\ldots] V_p^2 + \frac{N}{V}\omega(p)\, u_p V_p \right)$

$\left.\begin{aligned} &+ \sum b_p^\dagger b_{-p}^\dagger \left([\ldots] u_p V_p + \frac{N}{V}\omega(p)(u_p^2 + V_p^2) \right) \\ &+ \sum b_p b_{-p} \left([\ldots] u_p V_p + \frac{N}{V}\omega(p)(u_p^2 + V_p^2) \right) \end{aligned}\right\} = 0$

nicht diag.

$$\epsilon(p)^2 = \frac{p^4}{4m^2} + \frac{p^2}{m}\frac{N}{V}\omega(p)$$

$$v_p \quad u_p = \frac{-\epsilon(p) + \frac{p^2}{2m} + \frac{N}{V}\omega(p)}{2\epsilon(p)}$$

$$u_p v_p = -\frac{N\,\omega(p)}{2V\epsilon(p)}$$

$$H = \frac{N^2}{2V}\omega(0) + \sum b_p^\dagger b_p \left(\frac{1}{2}\underbrace{\cancel{\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right]}} + \underline{\frac{\epsilon(p)}{2}} \right)$$

$$+ \sum b_{-p}^\dagger b_{-p} \left(\frac{1}{2}\underbrace{\cancel{[\,..\,]}} + \underline{\frac{\epsilon(p)}{2}} \right)$$

$$- \sum b_{-p}^\dagger b_{-p} \left(\underbrace{\cancel{[\,..\,]}} \right)$$

$$- \frac{1}{2}\sum [\,..\,] + \frac{1}{2}\sum_p \underline{\epsilon(p)}$$

[..] sum unter $p \leftrightarrow -p$

$$= \underbrace{\frac{N^2}{2V}\omega(0) - \frac{1}{2}\sum_p \left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) - \epsilon(p)\right]}_{\substack{E_0 \\ \text{Grundzustand}}} + \underbrace{\sum_p \epsilon(p)\, b_p^\dagger b_p}_{\substack{\text{Quasiteilchen} \\ \text{Anregungen}}}$$

(c) Verify that the diagonalized Hamiltonian is

$$\hat{H} = \frac{N^2}{2V}\omega(0) - \frac{1}{2}\sum_{p\neq 0}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) - \epsilon(p)\right] + \sum_{p\neq 0}\epsilon(p) b_p^\dagger b_p\,, \tag{5}$$

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(d) Suppose that the potential is $V(x) = \lambda\delta^{(3)}(\vec{x})$. What is the dominant term in $\epsilon(p)$ for $p \approx 0$ and for $p \gg 1$ respectively?

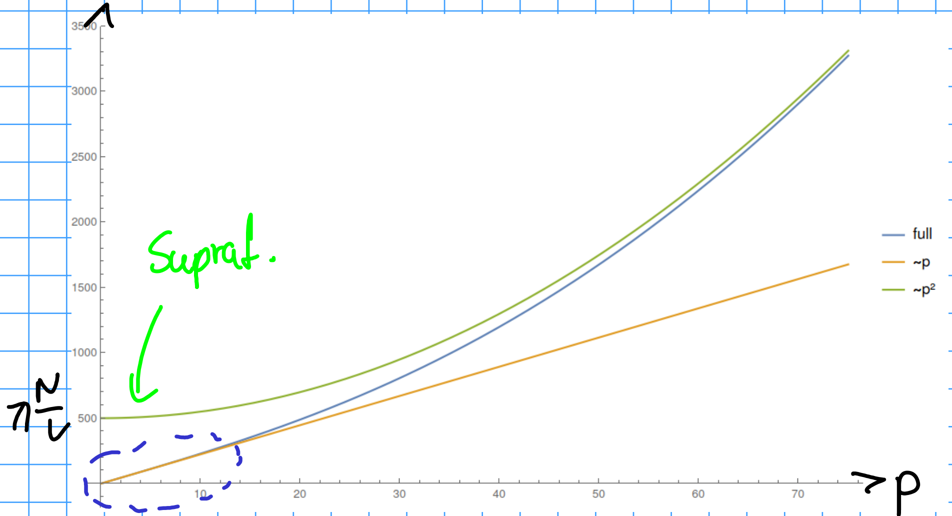
$$\Rightarrow \quad \omega(p) = \lambda$$

$$\epsilon(p) = \sqrt{\frac{N\lambda}{V}\frac{p^2}{m} + \frac{p^4}{4m^2}}$$

$\lambda > 0$

$$\epsilon(p) = \begin{cases} p\sqrt{\frac{N\lambda}{Vm}} & p \approx 0 \\ \frac{p^2}{2m} + \lambda\frac{N}{V} & p \gg 1 \end{cases}$$

$\epsilon(p)$



durch WW $\lambda \neq 0$

① masseloses Teilchen ⇝ Goldstone mode

② freies Teilchen mit Shift

(e) We now consider that the particles are moving with velocity $\vec{v}$ and think of two reference frames. One is moving along with the particles with velocity $\vec{v}$ and the other fixed. E denotes the total kinetic energy in the moving reference frame and E' the total kinetic energy in the fixed reference frame. E and E' are related as

$$E' = E + N\frac{mv^2}{2} + \vec{v}\cdot\vec{P}\,,$$

where $\vec{P}$ is the total momentum of the system. Any decrease in the velocity of the system is equivalent to the creation of a "quasi-particle" having a momentum in a direction opposite to that of $\vec{v}$. What is the kinetic energy change $\Delta E'$? Negative $\Delta E'$ means kinetic energy loss of the system. What is the physical meaning of non-zero value of $\min_p \frac{\epsilon(p)}{p}$? What is the condition that this N particle system can show superfluid properties?

$\vec{P} = \sum^N \vec{p}$

Fixed — moving

$E', V'=0$ — $E, V \leftarrow$ Geschw. Teilchen

$$E' = E + N\frac{mV^2}{2} + \vec{V}\vec{P}$$

$$E = \mathcal{E}_0 + \sum_p \epsilon(p)\underbrace{b_p^+ b_p}_{n_p}$$

(Rest frame)

$$E' = \mathcal{E}_0 + N\frac{mv^2}{2} + \sum_p n_p\left(\epsilon(p) + \vec{v}\vec{p}\right)$$

$\vec{v} \parallel \vec{p}$

$$\Delta E' = \epsilon(p) - |\vec{v}||\vec{p}| < 0$$

$\Rightarrow$ Anregung möglich

nicht möglich ($\neq$ Anregung $\Rightarrow$ GZ $\Rightarrow$ Supraf.)

$$\epsilon - |v||p| > 0 \Rightarrow |v| < \frac{\epsilon}{|p|} \leftarrow v_{crit}$$

bis v_{crit} Supraleitung!