# Warum sich Spinoren so transformieren.

## Spin-0 Teilchen (nicht-relativistisch)

$$i \frac{d}{dt} |\psi\rangle = H\psi$$

$$\Rightarrow \quad |\psi(t)\rangle = U(t)\, |\psi\rangle \quad \longrightarrow \quad \langle\psi| U^{+} = \langle\psi(t)|$$

## Spin 1/2 relativistisch

$$\psi \longrightarrow S\psi$$

$$\psi^{+} \rightarrow \psi^{+}S^{+} \quad \Rightarrow \quad \psi^{+}\gamma^{0} \longrightarrow \psi^{+}S^{+}\gamma^{0}$$

$$= \psi^{+}\gamma^{0}S^{-1}$$

$$S^{+}\gamma^{0} = \gamma^{0}S^{-1} \iff \gamma^{0}S^{+}\gamma^{0} = S^{-1}$$

$$S = 1\!\!1 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}$$

$$S^{-1} = 1\!\!1 + \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu} \quad \Rightarrow S^{-1}S = 1\!\!1 + \frac{i}{2}\cdots - \frac{i}{2}\cdots = 1\!\!1$$

$$\Rightarrow \quad \bar{\psi} \rightarrow \bar{\psi}S^{-1}$$

Übung 2:

2. **Generators of Lorentz group:** a 4-vector $x^\mu$ transforms under a Lorentz transformation matrix $\Lambda$ as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. For infinitesimal parameters $\omega^{\rho\sigma}(= -\omega^{\sigma\rho})$ the Lorentz transformation matrix can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,. \tag{1}$$

Consider two specific infinitesimal Lorentz transformations:

(a) Rotation around the $z$-axis: $\omega^{12} = -\omega^{21} = \varepsilon\,$, otherwise $\omega^{\rho\sigma} = 0\,$,
(b) Boost along the $x$-direction: $\omega^{10} = -\omega^{01} = \beta\,$, otherwise $\omega^{\rho\sigma} = 0\,$.

where $\varepsilon$ and $\beta$ are infinitesimal parameters.

Use the definition of the generator **matrices** $k_x \equiv L_{10} = -L_{01}$ and $l_z \equiv L_{12} = -L_{21}$ from the lecture

$$k_x = i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\; l_z = i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix} \tag{2}$$

$l_z$

and show that

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu\,. \tag{3}$$

is equivalent to Eq. (1) for cases of (a) Rotation and (b) Boost by plugging in the $\omega$'s and the $L$'s and carrying out the summation over $\rho$ and $\sigma$ explicitly.

a) (3)

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\left(\omega^{12}L_{12} + \omega^{21}L_{21}\right)^\mu{}_\nu = \delta^\mu{}_\nu - i\left(\omega^{12}L_{12}\right)^\mu{}_\nu$$

$$= \delta^\mu{}_\nu - i\varepsilon\left( l_z \right) = \delta^\mu{}_\nu + \underbrace{\begin{pmatrix}0&0&0&0\\0&0&-\varepsilon&0\\0&\varepsilon&0&0\\0&0&0&0\end{pmatrix}}_{=\,\omega^\mu{}_\nu}$$

noch zz. Zusammenhang

$$\omega^\mu{}_\nu \longleftrightarrow \omega^{\mu\nu} = \begin{pmatrix}0&0&0&0\\0&0&\varepsilon&0\\0&-\varepsilon&0&0\\0&0&0&0\end{pmatrix}$$

$$\omega^1{}_2 = \omega^{1\sigma}g_{\sigma 2} = -\omega^{12}$$

b) $\omega^{10} = -\omega^{01} = \beta$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\left(\omega^{01}L_{01} + \omega^{10}L_{10}\right)^\mu{}_\nu$$

$$= \delta^\mu{}_\nu + i\beta\left(L_{01}\right)^\mu{}_\nu$$

$$= \delta^\mu{}_\nu + i\beta\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\&0&0&\end{pmatrix} = \delta^\mu{}_\nu + \overbrace{\begin{pmatrix}0&\beta&0&0\\\beta&0&0&0\\0&0&0&0\end{pmatrix}}^{\omega^\mu{}_\nu}$$

$$\omega^{10} = \omega^{01} = \beta$$

$$\omega^{0}{}_{1} = \beta = \omega^{0s} g_{s1} = -\omega^{01}$$

$$\omega^{1}{}_{0} = \beta = \omega^{1s} \underbrace{g_{s0}}_{1} = \omega^{10}$$

$$\Rightarrow \omega^{\mu\nu} = \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# Quantentheorie II Übung 3

Besprechung: 2021WE18 (KW18)

SS 2021 Prof. Dominik Stöckinger (IKTP)

1. **Questions**
   (a) How many linearly independent solutions for the Dirac equation for a fixed $\vec{p}$ exist?
   (b) Repeat the calculation to solve the Dirac equation for a particle moving in the $z$-direction, which was given in the lecture.
   (c) What is the matrix to rotate an ordinary vector around the $y$-axis?
   (d) If a 4-vector transforms under the Lorentz transformation matrix $L_{31} = l_y$, what is the associated transformation matrix for spinors?

2. **Plane wave solutions of the Dirac equation:** to solve the Dirac equation for a mass $m$ particle we apply the Ansatz $\psi = u(p)e^{-ipx}$ (a plane wave with momentum $\vec{p}$) and obtain the linear equations for $u(p)$ for the positive energy Ansatz

$$(\not{p} - m)u(p) = 0\,.$$

In the lecture the solutions for a particle moving in the $z$-direction were shown. Now we consider a particle moving in the $x$-direction with its momentum $\vec{p} = (p_x, 0, 0)$ and $E^2 = p_x^2 + m^2$.

   (a) Find the linearly independent solutions for $u(p)$ which satisfy the following conditions of orthogonality and normality:

$$\bar{u}_r(p)u_s(p) = 2m\delta_{rs}\,.$$

   (b) Show that

$$\sum_r u_r(p)\bar{u}_r(p) = \not{p} + m\,.$$

   (c) Check whether a simultaneous eigenstate with the following operators is possible:

$$(\mathrm{i})S_{12}\,, \qquad (\mathrm{ii})S_{23}\,.$$

   What do the answers mean?

3. **Rotation around $y$-axis by $\frac{\pi}{2}$:** the Dirac equation solution for a particle moving in the $z$-direction is given as $u(p_z)$. Use the results for $u(p_z)$ given in the lecture and verify

$$u(p_x) = S(R_y(\pi/2))u(p_z)\,,$$

where $S_y = S_{31} = \frac{i}{4}[\gamma_3, \gamma_1]$, by comparing with the results for $u(p_x)$ from the previous task.

4. **Dirac equation of a massless particle:** the Dirac equation of a massless particle is

$$\not{\partial}\psi(x) = 0\,. \tag{1}$$

Using the Weyl (chiral) representation of the $\gamma$-matrices (see exercise sheet 2), find from Eq. (1) two independent equations for 2-component spinors, $\tilde{\xi}, \tilde{\eta}$ and solve them. Use the Ansatz $\tilde{\xi} = \xi(p)e^{-ipx}$, $\tilde{\eta} = \eta(p)e^{-ipx}$.

5. **Modified Dirac equation:** consider the Dirac equation modified by the so-called Pauli term:

$$(i\not{D} - m - \frac{e}{2m} a S^{\mu\nu} F_{\mu\nu})\psi = 0\,,$$

where $S^{\mu\nu}$ are the generators for the Lorentz transformation of 4-spinors (see exercise sheet 2) and $D^\mu \equiv \partial^\mu - ieA^\mu$. Assume $F_{12} = -F_{21} \equiv -B_z$, otherwise $F_{\mu\nu} = 0$.

   (a) Repeat the steps of the lecture for the non-relativistic limit for this modified equation.

   (b) What is the physical effect of the new term and the parameter a?

1. **Questions**

   (a) How many linearly independent solutions for the Dirac equation for a fixed $\vec{p}$ exist?

   (b) Repeat the calculation to solve the Dirac equation for a particle moving in the $z$-direction, which was given in the lecture.

   (c) What is the matrix to rotate an ordinary vector around the $y$-axis?

   (d) If a 4-vector transforms under the Lorentz transformation matrix $L_{31} = l_y$, what is the associated transformation matrix for spinors?

c) $R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \exp(-i\theta\, l_y)$

$R_z(\theta) = \exp(-i\theta\, l_z)$

d) $S(R_y(\theta)) = \exp(-i\theta\, S_y)$ $\qquad S_y = \frac{i}{4}[S_3, S_1] = S_{31}$

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(b) Show that

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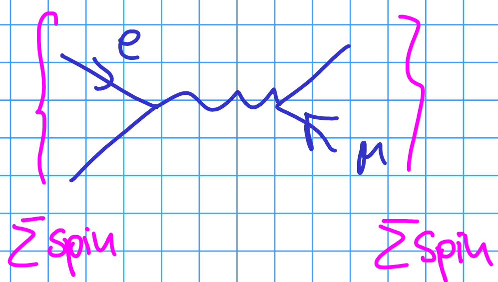
a) $$u_1(p) = N\begin{pmatrix} 1 \\ 1 \\ \frac{p_x}{E+m} \\ \frac{p_x}{E+m} \end{pmatrix} \qquad u_2(p) = N\begin{pmatrix} 1 \\ -1 \\ -\frac{p_x}{E+m} \\ \frac{p_x}{E+m} \end{pmatrix}$$

$$N = \sqrt{\frac{E+m}{2}}$$

b) $$\sum_r u_r(p)\,\bar{u}_r(p) = \not{p} + m$$




$$\left(\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = \not{p} + m\right)\cdot u_s(p)$$

$$= \sum_{r=1}^{2} u_r(p)\underbrace{\bar{u}_r(p)\,u_s(p)}_{2m\delta_{rs}} = \underbrace{\not{p}\,u_s(p)}_{m u_s} + m\,u_s(p)$$

$$2m\,u_s(p) = 2m\,u_s(p)$$

(c) Check whether a simultaneous eigenstate with the following operators is possible:

$$\text{(i)} S_{12}\,, \qquad \text{(ii)} S_{23}\,.$$

$$= S_z \qquad = S_x$$

What do the answers mean?

$$[S_z, \not{p}] \neq 0$$

$$\not{p} = p^0 \gamma_0 - p^x \gamma_1$$

$$S_x = \frac{1}{2}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$$

$$[S_x, \not{p}] = 0$$

$$\gamma_1 \sim \sigma^1$$

$$S_x\, u(p) = \pm\, u(p)$$

erfüllt durch Linearkombination

$$(\not{p} - m)\, u(p) = 0 \qquad E > 0$$

$$(-\not{p} - m)\, v(p) = 0 \qquad E < 0$$

3. **Rotation around $y$-axis by $\frac{\pi}{2}$:** the Dirac equation solution for a particle moving in the $z$-direction is given as $u(p_z)$. Use the results for $u(p_z)$ given in the lecture and verify

$$u(p_x) = S(R_y(\pi/2))u(p_z)\,,$$

where $S_y = S_{31} = \frac{i}{4}[\gamma_3, \gamma_1]$, by comparing with the results for $u(p_x)$ from the previous task.

$$S(R_y(\theta)) = \exp(-i\theta S_y)$$

$$= \sum_{n=0} \frac{1}{n!}(-i\theta S_y)^n$$

$$u(p_z, \tfrac{1}{2}) = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \frac{1}{\sqrt{E+m}}$$

$$u(p_z, -\tfrac{1}{2}) = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \cdot \ldots$$

$$\left( S_y = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right) \qquad (\sigma^2)^2 = \mathbb{1}_2$$

$$= \mathbb{1}_4 - i\frac{\theta}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 \mathbb{1}_4 + \ldots$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & & \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & & 0 \\ & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ & & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad \theta = \frac{\pi}{2}$$

$$S(R_y(\tfrac{\pi}{2}))\,u(p_z, \tfrac{1}{2}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} 1 \\ 1 \\ \frac{p_x}{E+m} \\ \frac{p_x}{E+m} \end{pmatrix}$$

4. **Dirac equation of a massless particle:** the Dirac equation of a massless particle is

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Using the Weyl (chiral) representation of the $\gamma$-matrices (see exercise sheet 2), find from Eq. (1) two independent equations for 2-component spinors, $\tilde{\xi}, \tilde{\eta}$ and solve them. Use the Ansatz $\tilde{\xi} = \xi(p)e^{-ipx}$, $\tilde{\eta} = \eta(p)e^{-ipx}$.

$$p^\mu p_\mu = 0 \qquad p^0 = |\vec{p}| \qquad \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\not{\partial} = \begin{pmatrix} 0 & \partial_t \\ \partial_t & 0 \end{pmatrix} - \begin{pmatrix} 0 & +\partial_x\sigma^1 + \partial_y\sigma^2 + \partial_z\sigma^3 \\ -\partial_x\sigma^1 - \partial_y\sigma^2 - \partial_z\sigma^3 & 0 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} \xi(p)e^{-ipx} \\ \eta(p)e^{-ipx} \end{pmatrix} \qquad \not{\partial}\psi = 0$$

$$p^\mu = i\partial^\mu$$

$$\text{I} \quad (p^0 + p^1\sigma^1 + p^2\sigma^2 + p^3\sigma^3)\,\eta(p) = 0$$

$$\Rightarrow \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \eta(p) = 0$$

$$\eta(p) = \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \eta(p) = \begin{pmatrix} -(p^1 - ip^2) \\ (p^0 + p^3) \end{pmatrix}$$

$$\vec{p} = \vec{p}_z \qquad \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{II} \quad (p^0 - p^1\sigma^1 - p^2\sigma^2 - p^3\sigma^3)\,\xi(p) = 0$$

$$\Rightarrow \xi(p) = \begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix} \quad \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Helizität: $\dfrac{\vec{p}\,\vec{S}}{|\vec{p}|}$ , $|\vec{p}| = p^0$

$$\Rightarrow (p^0 - \vec{p}\,\vec{\sigma})\,\xi(p) = 0 \quad \Rightarrow \quad \frac{\vec{p}\,\vec{\sigma}}{p^0}\,\xi(p) = +\xi(p)$$

$$(p^0 + \vec{p}\,\vec{\sigma})\,\eta(p) = 0 \quad \Rightarrow \quad \frac{\vec{p}\,\vec{\sigma}}{p^0}\,\eta(p) = - \,\eta(p)$$

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(a) Repeat the steps of the lecture for the non-relativistic limit for this modified equation.

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a)

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

$$S^{\nu\mu} = -S^{\mu\nu}$$

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$S^{\mu\nu} F_{\mu\nu} = S^{12} F_{12} + S^{21} F_{21} = 2 S^{12} F_{12} = -\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} B_z$$

$$\Rightarrow \left( i\not{D} - m + \frac{e}{2m} a \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} B_z \right)\psi = 0$$

$$= \left( i\not{D} - m + \frac{e}{2m} 2a \vec{S}\vec{B} \right)\psi = 0$$

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$(E - m + e\Phi)\psi_A = \left( \frac{(\vec{p} + e\vec{A})^2}{2m} + \underbrace{\overset{2}{g_s} \cdot \frac{e}{2m} \vec{S}\vec{B} - \frac{2a\, e}{2m}\vec{S}\vec{B}}_{+\underbrace{2(1-a)}_{g} \frac{e}{2m}\vec{S}\vec{B}} \right)\psi_A$$

historisch

b)

$$\vec{\mu}_S = g \frac{e}{2m} \vec{S}\vec{B}$$

heute:

$$g = 2(1+a)$$

$$\Leftrightarrow a = \frac{g-2}{2}$$