Quantentheorie II Übung 8

– Sample solutions –

SS 2021 Dr. H. Stöckinger-Kim, Prof. Dominik Stöckinger (IKTP)

2. **Weakly interacting bose gases and superfluidity:**

   (a) The volume of the box is $V = L^3$.

$$
\begin{aligned}
\hat{H} &= \sum_k \frac{p_k^2}{2m} {a_{p_k}}^\dagger a_{p_k} + \frac{1}{2V} \sum_{lmik} \omega(|\vec{p}_l - \vec{p}_i|) {a_{p_l}}^\dagger {a_{p_m}}^\dagger a_{p_i} a_{p_k} \\
&= \sum_p \frac{p^2}{2m} {a_p}^\dagger a_p \Longleftarrow (p_k \to p,\ \text{(the index } k \text{ dropped)}) \\
&+ \frac{1}{2V} \omega(0) {a_0}^\dagger {a_0}^\dagger a_0 a_0 \Longleftarrow (p_l = p_m = p_i = p_k = 0) \\
&+ \frac{1}{2V} \omega(0) \sum_{p \neq 0} {a_0}^\dagger a_0 {a_p}^\dagger a_p \Longleftarrow (p_l = p_i = 0, p_m = p_k = p) \\
&+ \frac{1}{2V} \omega(0) \sum_{p \neq 0} {a_0}^\dagger a_0 {a_p}^\dagger a_p \Longleftarrow (p_l = p_i = p, p_m = p_k = 0) \\
&+ \frac{1}{2V} \sum_{p \neq 0} \omega(p) {a_0}^\dagger {a_0}^\dagger a_p a_{-p} \Longleftarrow (p_l = p_m = 0, p_i = -p_k = p) \\
&+ \frac{1}{2V} \sum_{p \neq 0} \omega(p) a_0 a_0 {a_p}^\dagger {a_{-p}}^\dagger \Longleftarrow (p_l = -p_m = p, p_i = p_k = 0) \\
&+ \frac{1}{2V} \sum_{p \neq 0} \omega(p) {a_0}^\dagger a_0 {a_p}^\dagger a_p \Longleftarrow (p_l = p_k = 0, p_m = p_i = p) \\
&+ \frac{1}{2V} \sum_{p \neq 0} \omega(p) {a_0}^\dagger a_0 {a_p}^\dagger a_p \Longleftarrow (p_l = p_k = p, p_m = p_i = 0) \\
&= \frac{1}{2V} \omega(0) n_0^2 + \sum_{p \neq 0} \left[\frac{p^2}{2m} + \frac{n_0}{V} \omega(p) \right] {a_p}^\dagger a_p \\
&+ \frac{1}{2V} {a_0}^\dagger {a_0}^\dagger \sum_{p \neq 0} \omega(p) a_p a_{-p} + \frac{1}{2V} a_0 a_0 \sum_{p \neq 0} \omega(p) {a_p}^\dagger {a_{-p}}^\dagger \\
&+ \frac{n_0}{V} \omega(0) \sum_{p \neq 0} {a_p}^\dagger a_p + (\text{higher order terms of } a \text{ and } a^\dagger) \\
&= \frac{N^2}{2V} \omega(0) + \sum_{p \neq 0} \left[\frac{p^2}{2m} + \frac{N}{V} \omega(p) \right] {a_p}^\dagger a_p + \frac{N}{2V} \sum_{p \neq 0} \omega(p) \left(a_p a_{-p} + {a_p}^\dagger {a_{-p}}^\dagger \right). \qquad (1)
\end{aligned}
$$

$$\Longleftarrow ({a_p}^\dagger a_p = n_p \approx 0, (a_0)^2 \approx n_0, ({a_0}^\dagger)^2 \approx n_0, n_0 \approx N)$$

We assume that the particles are condensed in the zero momentum state, $p = 0$. Note that the Hamiltonian (including the Fourier transformation and the summation ranges) is symmetric under $p \to -p$.

(b) Symplectic transformation: $a_p = u_p b_p + v_p {b_{-p}}^\dagger$, ${a_p}^\dagger = u_p {b_p}^\dagger + v_p b_{-p}$. The usual commutation relations for a and $a^\dagger$ are valid for b and $b^\dagger$, and we assume (motivated by the symmetry under $p \leftrightarrow -p$) $u_p = u_{-p}$ and $v_p = v_{-p}$. The variable change $p \to -p$ leads to

$$a_{-p} = u_p b_{-p} + v_p {b_p}^\dagger, \text{ and } {a_{-p}}^\dagger = u_p {b_{-p}}^\dagger + v_p b_p. \tag{2}$$

From the commutation relations we obtain

$$\begin{aligned}
[a_p, {a_p}^\dagger] &= u_p^2 [b_p, {b_p}^\dagger] + v_p^2 [{b_{-p}}^\dagger, b_{-p}] + u_p v_p ([b_p, b_{-p}] + [{b_{-p}}^\dagger, {b_p}^\dagger]) \\
&= u_p^2 - v_p^2 \Longleftarrow ([b_p, {b_{p'}}^\dagger] = \delta_{pp'}, [b_p, b_{p'}] = [{b_p}^\dagger, {b_{p'}}^\dagger] = 0) \\
\therefore \quad & u_p^2 - v_p^2 = 1.
\end{aligned} \tag{3}$$

The following operators can be expressed in b and $b^\dagger$

$$a_p a_{-p} + {a_p}^\dagger {a_{-p}}^\dagger, \text{ and } {a_p}^\dagger a_p,$$

with which we can rewrite the Hamiltonian in Eq. (1) as

$$\begin{aligned}
\hat{H} = & \frac{N^2}{2V}\omega(0) \\
& + \sum_{p \neq 0} \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] v_p^2 + \frac{N}{V}\omega(p) u_p v_p \right) \Longleftarrow (bb^\dagger \to b^\dagger b + 1) \\
& + \sum_{p \neq 0} {b_p}^\dagger b_p \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] u_p^2 + \frac{N}{V}\omega(p) u_p v_p \right) \\
& + \sum_{p \neq 0} {b_{-p}}^\dagger b_{-p} \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] v_p^2 + \frac{N}{V}\omega(p) u_p v_p \right) \\
& + \sum_{p \neq 0} {b_p}^\dagger {b_{-p}}^\dagger \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] u_p v_p + \frac{N}{2V}\omega(p)(u_p^2 + v_p^2) \right) \\
& + \sum_{p \neq 0} b_p b_{-p} \left(\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] u_p v_p + \frac{N}{2V}\omega(p)(u_p^2 + v_p^2) \right).
\end{aligned} \tag{4}$$

The diagonalized Hamiltonian is obtained when the off-diagonal terms (the last two terms) vanish:

$$\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] u_p v_p + \frac{N}{2V}\omega(p)(u_p^2 + v_p^2) = 0, \tag{5}$$

$$\left(u_p v_p \left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right] \right)^2 = \left(\frac{N}{2V}\omega(p)(2u_p^2 - 1) \right)^2 \Longleftarrow (u_p^2 - v_p^2 = 1) \tag{6}$$

Define

$$\epsilon(p) \equiv \sqrt{\left(\frac{p^2}{2m} + \frac{N}{V}\omega(p) \right)^2 - \left(\frac{N}{V}\omega(p) \right)^2} = \sqrt{\frac{p^4}{4m^2} + \frac{p^2}{m}\frac{N}{V}\omega(p)}, \tag{7}$$

and we obtain from Eqs. (3),(5) and (6)

$$u_p^2 = \frac{\epsilon(p) + \left(\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right)}{2\epsilon(p)}, \tag{8}$$

$$v_p^2 = \frac{-\epsilon(p) + \left(\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right)}{2\epsilon(p)}, \tag{9}$$

$$u_p v_p = -\frac{N\omega(p)}{2V\epsilon(p)} \tag{10}$$

(c) The coefficients in Eq. (4) are

$$u_p^2\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{N}{V}\omega(p)u_p v_p = \frac{1}{2}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{\epsilon(p)}{2}, \tag{11}$$

and

$$v_p^2\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{N}{V}\omega(p)u_p v_p = -\frac{1}{2}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{\epsilon(p)}{2}. \tag{12}$$

After combining all together we obtain

$$\begin{aligned}
\hat{H} &= \frac{N^2}{2V}\omega(0) + \sum_{p\neq 0}\left(-\frac{1}{2}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{\epsilon(p)}{2}\right) \\
&+ \sum_{p\neq 0} {b_p}^\dagger b_p \left(\frac{1}{2}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{\epsilon(p)}{2}\right) \\
&+ \sum_{p\neq 0} {b_{-p}}^\dagger b_{-p} \left(-\frac{1}{2}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p)\right] + \frac{\epsilon(p)}{2}\right) \\
&= \frac{N^2}{2V}\omega(0) - \frac{1}{2}\sum_{p\neq 0}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) - \epsilon(p)\right] + \sum_{p\neq 0}\epsilon(p){b_p}^\dagger b_p.
\end{aligned} \tag{13}$$

The eigenvalues of the diagonal Hamiltonian are

$$E = E_0 + \sum_{p\neq 0}\epsilon(p)\lambda_p, \tag{14}$$

where

$$E_0 = \frac{N^2}{2V}\omega(0) - \frac{1}{2}\sum_{p\neq 0}\left[\frac{p^2}{2m} + \frac{N}{V}\omega(p) - \epsilon(p)\right]. \tag{15}$$

This is the energy measured in the rest frame of the particles (fluid). λ_p is an eigenvalue of the operator ${b_p}^\dagger b_p$, and can be regarded as the number of elementary oscillations ("quasi-particles") in a state of energy $\epsilon(p)$.

(d) For $V(x) = \lambda\delta^{(3)}(\vec{x})$ (contact interaction), $\omega(p) = \lambda$.

$$(a)\ \frac{p}{m} \ll 1 \text{ or } p \approx 0, \epsilon(p) = \sqrt{\frac{p^2}{m}\frac{N}{V}\lambda + \frac{p^4}{4m^2}} \sim p\sqrt{\frac{N\lambda}{Vm}} \tag{16}$$

$$(b)\ \frac{p}{m} \gg 1, \epsilon(p) = \sqrt{\frac{p^2}{m}\frac{N}{V}\lambda + \frac{p^4}{4m^2}} \sim \frac{p^2}{2m} \tag{17}$$

(e) $\vec{v}_i$ and $\vec{v'}_i$ are the velocities in the rest frame of the particles and in the laboratory frame respectively. In the rest frame the kinetic energy is $E = \sum_i \frac{1}{2}mv_i^2$, and by using the Galilean transformation we find the energy in the lab frame

$$\begin{aligned} E' &= \sum_i \frac{1}{2}m(v_i')^2 \\ &= \sum_i \frac{1}{2}mv_i^2 + \sum_{i=1}^{N} \frac{1}{2}mv^2 + \vec{v} \cdot \sum_i m\vec{v}_i \\ &= E + N\frac{1}{2}mv^2 + \vec{v} \cdot \vec{P} \Longleftarrow (\vec{P} \equiv \sum_i m\vec{v}_i). \end{aligned} \tag{18}$$

The energy E is the eigenvalue of Eq. (13), $E = E_0 + \sum_{p\neq 0} \epsilon(p)\lambda_p$, where λ_p is the eigenvalue of ${b_p}^\dagger b_p$, and is the number of the quasi-particles in momentum p-state and E_0 is the sum of the first two terms in Eq. (13). After combining these together,

$$E' = E_0 + N\frac{1}{2}mv^2 + \sum_{p\neq 0} \lambda_p(\epsilon(p) + \vec{v} \cdot \vec{p}) \tag{19}$$

Creation of the quasi-particles moving in the opposite direction to $\vec{v}$ causes the kinetic energy change in the lab frame:

$$\Delta E' = \epsilon(p) - |\vec{v}||\vec{p}|. \tag{20}$$

In the lab frame, during dissipation the velocity $|\vec{v}|$ and the kinetic energy decrease: $\Delta E' < 0$. However, $\Delta E'$ becomes *positive* when the particles flow with a velocity $|\vec{v}| < \frac{\epsilon(p)}{|\vec{p}|}$.

If the minimum of $\frac{\epsilon(p)}{|\vec{p}|}$ is positive for all $p = |\vec{p}|$, and if $v = |\vec{v}|$ is smaller than this quantity, then $\Delta E'$ is *always positive*! However positive $\Delta E'$ means that spontaneous creation of the respective quasi-particles is impossible. This means that no dissipation can occur for a fluid moving at such a velocity, and the fluid shows superfluidity. From Eq. (16), $\frac{\epsilon(p)}{|\vec{p}|} = \sqrt{\frac{N\lambda}{Vm}} > 0$, for $\lambda > 0$. The interaction should be repulsive.