Quantentheorie II Übung 5

Besprechung: 2021WE20 (KW20)

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1. **Questions**
   (a) What are identical particles?
   (b) How do we know that all electrons are identical?
   (c) Write down example two-particle wave functions for two distinguishable particles, two bosons or two fermions respectively.
   (d) Consider an operator $A_2^{(1)} = A^{(1)} \otimes 1$ on a two-particle Hilbert space and an analogous operator $B_2^{(2)} = 1 \otimes B^{(2)}$. Show $[A_2^{(1)}, B_2^{(2)}] = 0$.
   (e) For $|\psi^{\pm}\rangle \in \mathcal{H}_N^{(\pm)}$, show $\langle \psi^+ | \psi^- \rangle = 0$.
   (f) Show that the permutation operator P_{ij} is Hermitian and unitary by considering scalar products of basis elements.

2. **Two-particle system:** we consider a system with two particles. The two particles occupy the two one-particle states $|\psi\rangle$ and $|\phi\rangle$ which satisfy orthonormality relations: $\langle \psi|\psi\rangle = \langle \phi|\phi\rangle = 1$ and $\langle \psi|\phi\rangle = \langle \phi|\psi\rangle = 0$.

   Write down the two-particle states corresponding to the Hilbert spaces $\mathcal{H}_2$, $\mathcal{H}_2^{(+)}$, and ${\mathcal{H}_2}^{(-)}$, and compute $\langle (x_1 - x_2)^2 \rangle$ for the following cases.

   (a) Distinguishable particles
   (b) Identical bosons
   (c) Identical fermions

   Express the results in terms of matrix elements such as

   $$\langle x^2 \rangle_{AB} \equiv \langle A|x^2|B\rangle, \quad \langle x\rangle_{AB} \equiv \langle A|x|B\rangle \,,$$

   where A and B are either ψ or ϕ.

3. **Gravitationally bound state of two identical particles:** two identical particles with mass m interact with each other through the gravitational force. The Hamiltonian of this system is

   $$\hat{H} = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|} \,.$$

   (a) Rewrite the Hamiltonian using center-of-mass and relative coordinates

   $$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \,, \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \,,$$

   and show that $\hat{H}$ is mathematically equivalent to the hydrogen atom Hamiltonian.

(b) What are the allowed wave functions of the bound state for the following cases?

(i) Spinless Bosons, (ii) Spin-$\frac{1}{2}$ fermions

It is not required to solve the Schrödinger equation $\hat{H}\psi = E\psi$ explicitly. The wave functions of hydrogen atoms satisfy $\psi_{nlm}(\vec{r}) = (-1)^l \psi_{nlm}(-\vec{r})$.

(c) What is the reason that these states are not observed?

4. **Symmetric and antisymmetric Hilbert subspaces:** we know that a Hilbert space of an N-particle system $\mathcal{H}_N$ can have symmetric $\mathcal{H}_N^{(+)}$ and antisymmetric $\mathcal{H}_N^{(-)}$ subspaces. Do the two spaces $\mathcal{H}_N^{(+)}$ and $\mathcal{H}_N^{(-)}$ add up to the full space $\mathcal{H}_N$? Investigate for the following cases.

   (a) $N = 2$

   (b) $N = 3$

1. **Questions**

   (a) What are identical particles?

   (b) How do we know that all electrons are identical?

   (c) Write down example two-particle wave functions for two distinguishable particles, two bosons or two fermions respectively.

   (d) Consider an operator $A_2^{(1)} = A^{(1)} \otimes 1$ on a two-particle Hilbert space and an analogous operator $B_2^{(2)} = 1 \otimes B^{(2)}$. Show $[A_2^{(1)}, B_2^{(2)}] = 0$.

   (e) For $|\psi^\pm\rangle \in \mathcal{H}_N^{(\pm)}$, show $\langle\psi^+|\psi^-\rangle = 0$.

   (f) Show that the permutation operator P_{ij} is Hermitian and unitary by considering scalar products of basis elements.

d) $[A_2^{(1)}, B_2^{(2)}] = [A^{(1)} \otimes 1\!\!1, 1\!\!1 \otimes B^{(2)}]$

$= [A^{(1)}, 1\!\!1] + [1\!\!1, B^{(2)}] = 0$

$\mathcal{H}_2 = \mathcal{H}_{(1)}^{(1)} \otimes \mathcal{H}_1^{(2)}$

c) $\psi(x_1, x_2) = N \cdot \phi_A(x_1)\, \phi_B(x_2)$

e) $P_{ij}|\psi^\pm\rangle = \pm|\psi^\pm\rangle$

$\langle\psi^+|\psi^-\rangle = \langle\psi^+|P^+ P|\psi^-\rangle$

$= \langle\psi^+|\psi^-\rangle = -\langle\psi^+|\psi^-\rangle$

$\Rightarrow \langle\psi^+|\psi^-\rangle = 0$

f) $P^2 = 1\!\!1$

$\langle\psi^+|P^+P|\psi^+\rangle = \langle\psi^+|\psi^+\rangle$

$P^+P = 1\!\!1 = P^2 \quad \Rightarrow P = P^{-1}$

3. **Gravitationally bound state of two identical particles:** two identical particles with mass m interact with each other through the gravitational force. The Hamiltonian of this system is

$$\hat{H} = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|}.$$

(a) Rewrite the Hamiltonian using center-of-mass and relative coordinates

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2,$$

and show that $\hat{H}$ is mathematically equivalent to the hydrogen atom Hamiltonian.

a) $\vec{\nabla}_1 = \frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_r \qquad \vec{\nabla}_2 = \frac{1}{2}\vec{\nabla}_R - \vec{\nabla}_r$

$$H = -\frac{1}{2m}\left(\frac{1}{4}\vec{\nabla}_R^2 + \vec{\nabla}_r^2 + \vec{\nabla}_R\vec{\nabla}_r + \frac{1}{4}\vec{\nabla}_R + \vec{\nabla}_r^2 - \vec{\nabla}_R\vec{\nabla}_r\right) - \frac{Gm}{|r|}$$

$$= \underbrace{-\frac{1}{4m}\nabla_R^2}_{} \underbrace{- \frac{1}{m}\nabla_r^2 - \frac{Gm^2}{|r|}}_{} = \underbrace{H_R}_{} + \underbrace{H_r}_{}$$

freies Teilchen $2m$ = Masse Gesamtsystem

Wasserstoff $\mu = \frac{m}{2}$ red. Masse $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2}$

$$H_r = -\frac{1}{2 \cdot \frac{m}{2}}\nabla_r^2 - \frac{Gm^2}{|r|} \qquad Gm^2 = \alpha_G$$

$$H_{\text{Wasserstoff}} = -\frac{1}{2\mu}\nabla_r - \frac{\alpha}{|r|} \qquad \mu \approx m_e$$

(b) What are the allowed wave functions of the bound state for the following cases?

(i) Spinless Bosons, (ii) Spin-$\frac{1}{2}$ fermions

It is not required to solve the Schrödinger equation $\hat{H}\psi = E\psi$ explicitly. The wave functions of hydrogen atoms satisfy $\psi_{nlm}(\vec{r}) = (-1)^l \psi_{nlm}(-\vec{r})$.

$$\psi_{nlm}(\vec{r}) = R_{nl}(r) \cdot Y_{lm}(\theta, \varphi)$$

(c) What is the reason that these states are not observed?

$$\text{b)} \quad \Psi_{nlm}(\vec{K}, \vec{r}) = \underbrace{e^{i\vec{K}\cdot\vec{R}}}_{\text{ebene Welle}} \cdot \psi_{nlm}(\vec{r}) \cdot \left(\frac{1}{2\pi}\right)^{3/2} \cdot \left(\frac{a_0}{a_G}\right)^{3/2}$$

symmetrisch $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$

a_0 Bohrradius $\quad a_0 = \frac{1}{\alpha M}$

hier: $\alpha = G m^2 \quad M = \frac{m}{2} \quad \Rightarrow \quad a_G = \frac{2}{G m^3}$

$\Psi_{nlm}(-\vec{r}) = (-1)^l \, \Psi_{nlm}(\vec{r}) \quad$ (von Y_{lm})

Bosonen: $\quad l = 0, 2, 4 \ldots$

Fermionen:

$$\Psi \cdot \chi(\sigma_1, \sigma_2)$$

χ_S

$$\frac{|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}$$

$\Downarrow$

Ort antisym.
$l = 1, 3, 5 \ldots$

χ_A

$$\frac{|\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle}{\sqrt{2}}$$

$\Downarrow$

Ort summ.
$l = 0, 2, 4 \ldots$

c) $E_H = \frac{1}{2 M a_0^2}$

$$E_G = \frac{G^2 m^5}{4}$$

$$G \sim 7 \cdot 10^{-39} \, (\mathrm{GeV})^{-2}$$

$m = 500$ GeV

neutralino χ_1^0

($m_e = 0.5$ MeV)

$\Rightarrow E_G \sim 10^{-64}$ GeV

CMB $\quad E_\gamma \sim 10^{-13}$ GeV

4. **Symmetric and antisymmetric Hilbert subspaces:** we know that a Hilbert space of an N-particle system $\mathcal{H}_N$ can have symmetric $\mathcal{H}_N^{(+)}$ and antisymmetric $\mathcal{H}_N^{(-)}$ subspaces. Do the two spaces $\mathcal{H}_N^{(+)}$ and $\mathcal{H}_N^{(-)}$ add up to the full space $\mathcal{H}_N$? Investigate for the following cases.

(a) $N = 2$

(b) $N = 3$

a) $N=2$ $\quad |S\rangle = \frac{1}{\sqrt{2}}\left(|\overset{1}{\alpha}\rangle|\overset{2}{\beta}\rangle + |\beta\rangle|\alpha\rangle \right)$

unter P_{12} $\quad \Rightarrow$ dim $=1$

$|A\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle|\beta\rangle - |\beta\rangle|\alpha\rangle \right)$

$\Rightarrow$ dim $=1$

$|\alpha\rangle \otimes |\beta\rangle$ $\quad 2!$ Permutation $\quad \Rightarrow$ dim $=2$

b) $N=3$ $\quad |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$ $\quad 3!$ Permutationen

$$|S\rangle = \frac{1}{\sqrt{6}}\Big(|\alpha\rangle|\beta\rangle|\gamma\rangle + |\beta\rangle|\alpha\rangle|\gamma\rangle + |\alpha\rangle|\gamma\rangle|\beta\rangle + |\gamma\rangle|\beta\rangle|\alpha\rangle + |\gamma\rangle|\alpha\rangle|\beta\rangle + |\beta\rangle|\gamma\rangle|\alpha\rangle \Big)$$

$\rightarrow$ dim $=1$ $\quad$ invariant P_{12} P_{23} P_{31}

$$|A\rangle = \frac{1}{\sqrt{6}}\Big(|\alpha\rangle|\beta\rangle|\gamma\rangle - |\overset{2}{\beta}\rangle|\overset{1}{\alpha}\rangle|\gamma\rangle - |\overset{1}{\alpha}\rangle|\overset{3}{\gamma}\rangle|\overset{2}{\beta}\rangle - |\overset{3}{\gamma}\rangle|\overset{2}{\beta}\rangle|\overset{1}{\alpha}\rangle + |\overset{3}{\gamma}\rangle|\overset{1}{\alpha}\rangle|\overset{2}{\beta}\rangle + |\overset{2}{\beta}\rangle|\overset{3}{\gamma}\rangle|\overset{1}{\alpha}\rangle \Big)$$

$\Rightarrow$ dim $=1$ $\quad$ sign change P_{12} P_{23} P_{31}

$$|M_{1a}\rangle = \frac{1}{\sqrt{12}}\Big(2|\alpha\rangle|\beta\rangle|\gamma\rangle + 2|\beta\rangle|\alpha\rangle|\gamma\rangle - |\alpha\rangle|\gamma\rangle|\beta\rangle - |\gamma\rangle|\beta\rangle|\alpha\rangle - |\gamma\rangle|\alpha\rangle|\beta\rangle - |\beta\rangle|\gamma\rangle|\alpha\rangle \Big)$$

$$P_{23}|M_{1a}\rangle = -\frac{1}{2}|M_{1a}\rangle - \frac{\sqrt{3}}{2}|M_{1b}\rangle$$

$$|M_{1b}\rangle = \frac{1}{2}\Big(-|\alpha\rangle|\gamma\rangle|\beta\rangle + |\gamma\rangle|\beta\rangle|\alpha\rangle + |\gamma\rangle|\alpha\rangle|\beta\rangle - |\beta\rangle|\gamma\rangle|\alpha\rangle

$$a|M_{1a}\rangle = b|S\rangle + c|A\rangle \qquad c=0$$

$$P_{12}\left(a|M_{1a}\rangle\right) = P_{12}\left(b|S\rangle\right) + P_{12}\left(c|A\rangle\right)$$

$$\pm\, a|M_{1a}\rangle = b|S\rangle - c|A\rangle$$

$$a|M_{1a}\rangle = b|S\rangle$$

2. **Two-particle system:** we consider a system with two particles. The two particles occupy the two one-particle states $|\psi\rangle$ and $|\phi\rangle$ which satisfy orthonormality relations: $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ and $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle = 0$.

Write down the two-particle states corresponding to the Hilbert spaces $\mathcal{H}_2$, $\mathcal{H}_2^{(+)}$, and $\mathcal{H}_2{}^{(-)}$, and compute $\langle(x_1 - x_2)^2\rangle$ for the following cases.

(a) Distinguishable particles $\Rightarrow \Gamma$

(b) Identical bosons $\Rightarrow \Gamma_B$

(c) Identical fermions $\Rightarrow \Gamma_F$

Express the results in terms of matrix elements such as

$$\langle x^2\rangle_{AB} \equiv \langle A|x^2|B\rangle, \quad \langle x\rangle_{AB} \equiv \langle A|x|B\rangle\,,$$

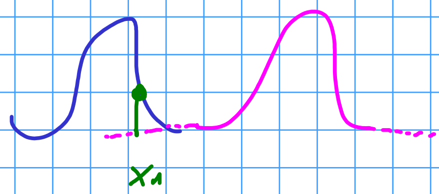
where A and B are either ψ or ϕ.

$\Rightarrow \quad \Gamma_B \leq \Gamma \leq \Gamma_F$

attractive WW

repulsive WW

$\Gamma_{B/F} = \Gamma \mp \underbrace{2|\langle x\rangle_{\psi\phi}|^2}_{=0}$



②. Two-particle system

$\underset{①}{|\psi\rangle}, \underset{②}{|\phi\rangle} \; ; \; \langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1 \; ; \langle\psi,\phi\rangle = \langle\phi,\psi\rangle = 0$

a) unterscheidbare Teilchen

$$\varphi(x_1,x_2) = \psi(x_1)\phi(x_2)$$

$$\langle(x_1-x_2)^2\rangle = \langle x_1^2\rangle_{\psi\psi} + \langle x_2^2\rangle_{\phi\phi} - 2\langle x_1\rangle_{\psi\psi}\langle x_2\rangle_{\phi\phi} \qquad (*)$$

$$= \langle x^2\rangle_{\psi\psi} + \langle x^2\rangle_{\phi\phi} - 2\langle x\rangle_{\psi\psi}\langle x_2\rangle_{\phi\phi}$$

b) identische Bosonen

$$\varphi(x_1,x_2) = \frac{\psi(x_1)\phi(x_2) + \phi(x_1)\psi(x_2)}{\sqrt{2}}$$

$$\langle(x_1-x_2)^2\rangle = \langle x^2\rangle_{\psi\psi} + \langle x^2\rangle_{\phi\phi} - 2\left(\langle x\rangle_{\psi\psi}\langle x\rangle_{\phi\phi} + |\langle x\rangle_{\psi\phi}|^2\right)$$

$$\overset{(*)}{=} \langle(x_1-x_2)^2\rangle_{(a)} - 2|\langle x\rangle_{\psi\phi}|^2$$

c) identische Fermionen

$$\varphi(x_1,x_2) = \frac{\psi(x_1)\phi(x_2) - \phi(x_1)\psi(x_2)}{\sqrt{2}}$$

$$\langle(x_1-x_2)^2\rangle = \langle x^2\rangle_{\psi\psi} + \langle x^2\rangle_{\phi\phi} - 2\left(\langle x\rangle_{\psi\psi}\langle x\rangle_{\phi\phi} - |\langle x\rangle_{\psi\phi}|^2\right)$$

$$\overset{(*)}{=} \langle(x_1-x_2)^2\rangle_{(a)} + 2|\langle x\rangle_{\psi\phi}|^2$$

→ Bosonen sind näher aneinander zu finden als Fermionen!

NR: a) $\langle(x_1-x_2)^2\rangle = \langle\psi^{(x_1)}|\langle\phi^{(x_2)}|(x_1-x_2)^2|\phi^{(x_2)}\rangle|\psi^{(x_1)}\rangle = \langle\psi^{(x_1)}|\langle\phi^{(x_2)}|x_1^2+x_2^2-2x_1x_2|\phi^{(x_2)}\rangle|\psi^{(x_1)}\rangle$

$$= \langle\psi^{(x_1)}|x_1^2|\psi^{(x_1)}\rangle + \langle\phi^{(x_2)}|x_2^2|\phi^{(x_2)}\rangle - 2\langle\psi^{(x_1)}|x_1|\psi^{(x_1)}\rangle\langle\phi^{(x_2)}|x_2|\phi^{(x_2)}\rangle$$

$$= \langle x_1^2\rangle_{\psi\psi} + \langle x_2^2\rangle_{\phi\phi} - 2\langle x_1\rangle_{\psi\psi}\langle x_2\rangle_{\phi\phi}$$

b,c) $\langle(x_1-x_2)^2\rangle = \frac{1}{2}\left(\langle\psi^{(x_1)}|\langle\phi^{(x_2)}| \pm \langle\phi^{(x_1)}|\langle\psi^{(x_2)}|\right)(x_1^2+x_2^2-2x_1x_2)\left(|\psi^{(x_1)}\rangle|\phi^{(x_2)}\rangle \pm |\phi^{(x_1)}\rangle|\psi^{(x_2)}\rangle\right)$

$$= \frac{1}{2}\Big[\langle x_1^2\rangle_{\psi\psi} \pm 0 \pm 0 \mp \langle x_1^2\rangle_{\phi\phi} + \langle x_2^2\rangle_{\phi\phi} + \langle x_2^2\rangle_{\psi\psi} +$$

$$-2\Big(\langle\psi^{(x_1)}|\langle\phi^{(x_2)}|x_1x_2|\psi^{(x_1)}\rangle|\phi^{(x_2)}\rangle \pm \langle\psi^{(x_1)}|\langle\phi^{(x_2)}|x_1x_2|\phi^{(x_1)}\rangle|\psi^{(x_2)}\rangle$$

$$\pm \langle\phi^{(x_1)}|\langle\psi^{(x_2)}|x_1x_2|\psi^{(x_1)}\rangle|\phi^{(x_2)}\rangle \mp \langle\phi^{(x_1)}|\langle\psi^{(x_2)}|x_1x_2|\phi^{(x_1)}\rangle|\psi^{(x_2)}\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big[\langle x_1^2\rangle_{\psi\psi} + \langle x_1^2\rangle_{\phi\phi} + \langle x_2^2\rangle_{\phi\phi} + \langle x_2\rangle^2_{\psi\psi} - 2\Big(\langle x_1\rangle_{\psi\psi}\langle x_2\rangle_{\phi\phi} \pm \langle x_1\rangle_{\psi\phi}\langle x_2\rangle_{\phi\psi}$$

$$\pm \langle x_1\rangle_{\phi\psi}\langle x_2\rangle_{\psi\phi} + \langle x_1\rangle_{\phi\phi}\langle x_2\rangle_{\psi\psi}\Big)\Big]$$

$$= \frac{1}{2}\Big[2\langle x^2\rangle_{\psi\psi} + 2\langle x^2\rangle_{\phi\phi} - 2(2\langle x\rangle_{\psi\psi}\langle x\rangle_{\phi\phi} \pm 2|\langle x\rangle_{\psi\phi}|^2\Big]$$