Quantentheorie II Übung 3

– Sample solutions –

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2. **Plane wave solutions of the Dirac equation:**

(a) The 4-momentum of the particle is $p^\mu = (E, p_x, 0, 0)$ and the mass and the momentum satisfies $E^2 = p_x^2 + m^2$. We use Dirac representation γ-matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{1}$$

After applying the Ansatz, $\psi(x) = u(p)e^{-ipx} = u(p)e^{-ip_\nu x^\nu}$, to the Dirac equation we obtain

$$(i\not{\partial} - m)\psi(x) = 0 \quad \xrightarrow[i\gamma^\mu\partial_\mu e^{-ip_\nu x^\nu} = i\gamma^\mu(-ip_\nu \delta_\mu{}^\nu)e^{-ipx} = \not{p}e^{-ipx}]{\text{Ansatz:}\psi(x)=u(p)e^{-ipx}} \quad (\not{p} - m)u(p) = 0 \tag{2}$$

The explicit expression of $(\not{p} - m)$ for $p^\mu = (E, p_x, 0, 0)$ is

$$\not{p} - m = \gamma^0 E - \gamma^1 p_x - m\mathbb{1}_4 = \begin{pmatrix} E-m & 0 & 0 & -p_x \\ 0 & E-m & -p_x & 0 \\ 0 & p_x & -E-m & 0 \\ p_x & 0 & 0 & -E-m \end{pmatrix}. \tag{3}$$

Assume $u^{\mathrm{T}}(p) = (a_1, a_2, a_3, a_4)$. The fourth (third) rows of the matrix in Eq. (3) is the first (second) row multiplied by $\frac{p_x}{E-m}$. As a consequence we obtain two independent equations for a_1 and a_4, and for a_2 and a_3 respectively:

$$\begin{aligned} (E-m)a_1 - p_x a_4 &= 0\,, \\ (E-m)a_2 - p_x a_3 &= 0\,. \end{aligned} \tag{4}$$

We can either i) set $a_1 = \pm a_2$ and $a_4 = \pm a_3$ or ii) set one of the pairs, (a_1, a_4) and (a_2, a_3), zero.

The two orthogonal spinors are:

(Case i)

$a_1 = a_2,\ a_3 = a_4$	$a_1 = -a_2,\ a_3 = -a_4$
$u_1(p) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} 1 \\ 1 \\ \frac{p_x}{E+m} \\ \frac{p_x}{E+m} \end{pmatrix}$,	$u_2(p) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} 1 \\ -1 \\ -\frac{p_x}{E+m} \\ \frac{p_x}{E+m} \end{pmatrix}$ or $\sqrt{\frac{E+m}{2}} \begin{pmatrix} -1 \\ 1 \\ \frac{p_x}{E+m} \\ -\frac{p_x}{E+m} \end{pmatrix}$,

(Case ii)

$a_2 = a_3 = 0$	$a_1 = a_4 = 0$
$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p_x}{E+m} \end{pmatrix}$	$u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x}{E+m} \\ 0 \end{pmatrix}$

For both cases the spinors are normalized as $\bar{u}_r(p)u_s(p) = 2m\delta_{rs}$ for $r, s = 1, 2$. It is also possible to choose a spinor which is a linear combination of $u_1(p)$ and $u_2(p)$, $u'(p) = au_1(p) + bu_2(p)$, and this is still an eigenstate of $\not{p}$ but not necessarily an eigenstate of S_x. The spinors of both cases are the eigenstates of $\not{p}$. However only the spinors of **(Case i)** are the simultaneous eigenstates of $\not{p}$ and S_x. See if $S_x u(p) = \lambda u(p)$.

(b) You can compute the explicit direct products $\sum_r u_r(p)\bar{u}_r(p) = u_1(p)\otimes\bar{u}_1(p) + u_2(p)\otimes\bar{u}_2(p)$ and obtain

$$\begin{aligned}
\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) &= \sum_{r=1}^{2} u_r(p)\otimes\bar{u}_r(p) \\
&= \begin{pmatrix} E+m & 0 & 0 & -p_x \\ 0 & E+m & -p_x & 0 \\ 0 & p_x & -E+m & 0 \\ p_x & 0 & 0 & -E+m \end{pmatrix} \\
&= p^0\gamma^0 - p_x\gamma^1 + m\mathbb{1}_4 \\
&= \not{p} + m\,.
\end{aligned} \tag{5}$$

Alternatively you can use the normalization condition $\bar{u}_r(p)u_s(p) = 2m\delta_{rs}$. Multiply both sides of $\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = \not{p} + m$ by $u_s(p)$ and obtain

$$\sum_{r=1}^{2} u_r(p)\bar{u}_r(p)u_s(p) = (\not{p}+m)u_s(p)$$

$$\begin{aligned}
\text{Left-hand side: } \sum_{r=1}^{2} u_r(p)\bar{u}_r(p)u_s(p) &= \sum_{r=1}^{2} u_r(p)2m\delta_{rs} \Longleftarrow (\bar{u}_r(p)u_s(p) = 2m\delta_{rs}) \\
&= 2mu_s(p)\,, \\
\text{Right-hand side:} (\not{p}+m)u_s(p) &= mu_s(p) + mu_s(p) (\Longleftarrow \text{ Dirac eq.}) \\
&= 2mu_s(p)
\end{aligned}$$

$$\Longrightarrow \sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = \not{p} + m\,. \tag{6}$$

(c) Simultaneous eigenstates $\Longleftrightarrow [\hat{O}_1, \hat{O}_2] = 0$

(i)

$$S_z = S_{12} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix},$$

$$[S_z, \not{p} - m] = \frac{1}{2}p_x\begin{pmatrix} 0 & [\sigma^1, \sigma^3] \\ -[\sigma^1, \sigma^3] & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \neq 0\,. \tag{7}$$

Therefore simultaneous eigenstates of S_z and $(\not{p} - m)$ do not exist.

(ii)

$$
\begin{aligned}
S_x = S_{23} &= \frac{1}{2}\begin{pmatrix}\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}, \\
[S_x, \not{p} - m] &= -\frac{1}{2}p_x\left\{\begin{pmatrix}\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}\begin{pmatrix}0 & \sigma^1\\ -\sigma^1 & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma^1\\ -\sigma^1 & 0\end{pmatrix}\begin{pmatrix}\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}\right\} = 0
\end{aligned}
\tag{8}
$$

Therefore simultaneous eigenstates of S_x and $(\not{p} - m)$ are possible.

(d) **Rotation around the y-axis by $\frac{\pi}{2}$:** for the spinors of the particle moving along the z-axis we use those from the lecture

$$
u(p_z, \frac{1}{2}) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}E+m\\ 0\\ p\\ 0\end{pmatrix}, u(p_z, -\frac{1}{2}) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}0\\ E+m\\ 0\\ -p\end{pmatrix}. \tag{9}
$$

The matrix to rotate a spinor around the y-axis by θ is $S(R_y(\theta)) = e^{-i\theta S_y}$, where

$$
S_y = S_{31} = \frac{1}{2}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}, \text{ and } \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}^2 = \begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & \mathbb{1}_2\end{pmatrix}. \tag{10}
$$

The Taylor series of $S(R_y(\theta))$ is

$$
\begin{aligned}
S(R_y(\theta)) = e^{-i\theta S_y} &= \sum_{n=0} \frac{1}{n!}(-i\theta S_y)^n \\
&= \mathbb{1}_4 - i\frac{\theta}{2}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2\mathbb{1}_4 + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} + \cdots \\
&= \cos\frac{\theta}{2}\mathbb{1}_4 - i\sin\frac{\theta}{2}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix},
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
S(R_y(\theta_1+\theta_2)) &= e^{-i\theta_1 S_y}e^{-i\theta_2 S_y} = e^{-i(\theta_1+\theta_2)S_y} \\
&= \cos\left(\frac{\theta_1+\theta_2}{2}\right)\mathbb{1}_4 - i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}.
\end{aligned}
\tag{12}
$$

For $\theta = \frac{\pi}{2}$ we obtain

$$
S(R_y(\frac{\pi}{2})) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{pmatrix} \tag{13}
$$

Apply this to $u(p_z, \pm\frac{1}{2})$ in Eq. (9):

$$S(R_y(\frac{\pi}{2})u(p_z, \frac{1}{2}) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} 1 \\ 1 \\ \frac{p}{\sqrt{E+m}} \\ \frac{p}{\sqrt{E+m}} \end{pmatrix} = u_1(p_x),$$

$$S(R_y(\frac{\pi}{2}))u(p_z, -\frac{1}{2}) = -\sqrt{\frac{E+m}{2}} \begin{pmatrix} 1 \\ -1 \\ -\frac{p}{\sqrt{E+m}} \\ \frac{p}{\sqrt{E+m}} \end{pmatrix} = u_2(p_x) \tag{14}$$

for $p_x = p_z = p$, and $u_{1,2}(p_x)$ are the spinors of **Case (i)** in 2.(a).

3. **Dirac equation of a massless particle:** the Dirac equation is $\partial\!\!\!/\psi(x) = 0$ and $p^\mu p_\mu = 0$, which means $p^0 = |\vec{p}|$. In Weyl representation

$$\partial\!\!\!/ = \begin{pmatrix} 0 & \partial_t \mathbb{1}_2 \\ \partial_t \mathbb{1}_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_i \sigma^i \\ -\partial_i \sigma^i & 0 \end{pmatrix} . \tag{15}$$

After applying the Ansatz the Dirac equation becomes

$$\partial\!\!\!/\psi(x) = \begin{pmatrix} 0 & \partial_0 \mathbb{1}_2 + \partial_i \sigma^i \\ \partial_0 \mathbb{1}_2 - \partial_i \sigma^i & 0 \end{pmatrix} \begin{pmatrix} \xi(p) e^{-ipx} \\ \eta(p) e^{-ipx} \end{pmatrix} = 0\,, \tag{16}$$

from which we obtain two independent equations:

$$(\partial_0 \mathbb{1}_2 + \partial_i \sigma^i)\eta(p) e^{-ipx} = 0\,, \tag{17}$$
$$(\partial_0 \mathbb{1}_2 - \partial_i \sigma^i)\xi(p) e^{-ipx} = 0\,. \tag{18}$$

First we solve Eq. (17). After applying the differential operators we obtain

$$(p^0 \mathbb{1}_2 - p^i \sigma^i)\eta(p) = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \eta(p) = 0 \tag{19}$$

The first row of Eq. (19) multiplied by $\frac{-p^1 - ip^2}{p^0 - p^3}$ becomes the second row. Therefore we obtain one independent equation for $\eta(p) = \begin{pmatrix} a \\ b \end{pmatrix}$

$$(p^0 - p^2)a + (-p^1 + ip^2)b = 0, \text{ and } a = p^0 + p^3,\ b = p^1 + ip^2\,, \tag{20}$$

which leads

$$\eta(p) = \begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix} \text{, and when } \vec{p} = p\hat{e}_3,\ \eta(p) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{21}$$

Likewise we can solve Eq. (18)

$$\begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \xi(p) = 0 \tag{22}$$

and obtain

$$\eta(p) = \begin{pmatrix} -p^1 + ip^2 \\ p^0 + p^3 \end{pmatrix} \,, \text{ and when } \vec{p} = p\hat{e}_3,\ \xi(p) \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,. \tag{23}$$

$\eta(p)$ and $\xi(p)$ are orthogonal to each other: $\bar{\eta}\xi = \bar{\xi}\eta = 0$, and they are eigenstates of the operator $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$,

$$\begin{aligned}
&\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\eta(p) = \eta(p)\,,\ \text{eigenvalue} = +1, \text{ left-handed},\\
&\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi(p) = -\xi(p)\,,\ \text{eigenvalue} = -1, \text{ right-handed}\,.
\end{aligned}$$

4. **Modified Dirac equation:** we can first rewrite the Dirac equation with the Pauli term as

$$\begin{aligned}
&(i\not{D} - m - \frac{e}{2m}aS^{\mu\nu}F_{\mu\nu})\psi(x)\\
&= \left(i\not{D} - m + \frac{e}{2m}a\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} B_z\right)\psi(x) \Longleftarrow \left[S^{\mu\nu}F_{\mu\nu} = 2S^{12}F_{12} = -B_z\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right]\\
&= (i\not{D} - m + \frac{e}{2m}2a\hat{s}\cdot\vec{B})\psi(x) \Longleftarrow \left[\hat{s} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\right]\\
&= 0\,.
\end{aligned} \tag{24}$$

After using the 4-spinor decomposed with two 2-spinors (ψ_A and ψ_B), $\psi(x) = \begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix}$, and following the procedure from the lecture, Eq. (24) becomes

$$\left(\begin{pmatrix} iD_0 - m & iD_i\sigma^i \\ -iD_i\sigma^i & -iD_0 - m \end{pmatrix} + \frac{e}{2m}a\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\cdot\vec{B}\right)\begin{pmatrix} \psi_A(x) \\ \psi_B(x) \end{pmatrix} = 0\,, \tag{25}$$

from which we obtain two equations for $\psi_A(x)$ and $\psi_B(x)$

$$(iD_0 - m + \frac{e}{2m}a\,\vec{\sigma}\cdot\vec{B})\psi_A(x) + iD_i\sigma^i\psi_B(x) = 0 \tag{26}$$

$$-iD_i\sigma^i\psi_A(x) + (-iD_0 - m + \frac{e}{2m}a\,\vec{\sigma}\cdot\vec{B})\psi_B(x) = 0\,. \tag{27}$$

After applying the Ansatz

$$\begin{aligned}
iD_0 &= E + e\Phi = m + (E - m + e\Phi) \text{ with } (E - m + e\Phi) \ll 1\,,\\
iD_i\sigma^i &= -\vec{\sigma}\cdot(\vec{p} + e\vec{A})\,, (\Longleftarrow\ iD^\mu = i\partial^\mu + eA^\mu)
\end{aligned} \tag{28}$$

to Eq. (27), we obtain

$$\begin{aligned}
-iD_i\sigma^i\psi_A(x) &= (E + e\Phi + m - \frac{e}{2m}a\,\vec{\sigma}\cdot\vec{B})\psi_B(x)\\
&\approx (2m + \mathcal{O}(m^2))\psi_B(x),
\end{aligned} \tag{29}$$

and the approximate form of $\psi_B(x)$ in terms of $\psi_A(x)$ is

$$\psi_B(x) = \frac{1}{2m}\vec{\sigma}\cdot(\vec{p} + e\vec{A})\psi_A(x) + \mathcal{O}(\frac{1}{m^2})\,. \tag{30}$$

After applying Eq. (30) to Eq. (26) we obtain

$$
\begin{aligned}
&(E-m+e\Phi+\frac{e}{2m}a\,\vec{\sigma}\cdot\vec{B})\psi_A(x)-\frac{1}{2m}(\vec{\sigma}\cdot(\vec{p}+e\vec{A}))(\vec{\sigma}\cdot(\vec{p}+e\vec{A}))\psi_A(x)\\
&=(E-m+e\Phi+\frac{e}{2m}a\,\vec{\sigma}\cdot\vec{B})\psi_A(x)-\frac{1}{2m}((\vec{p}+e\vec{A})^2+e\vec{\sigma}\cdot\vec{B})\psi_A(x)\\
&\qquad\Longleftarrow\ \left[(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot(\vec{A}\times\vec{B})\right]\\
&=0\,,
\end{aligned}
\tag{31}
$$

and

$$
(E-m+e\Phi)\psi_A(x)=\left(\frac{1}{2m}(\vec{p}+e\vec{A})^2+2(1-a)\frac{e}{2m}\vec{s}\cdot\vec{B}\right)\psi_A(x), \tag{32}
$$

where $\vec{s}\equiv\frac{\vec{\sigma}}{2}$.

The g-factor is the coefficient of $\frac{e}{2m}\vec{s}\cdot\vec{B}$, and we obtain

$$
g=2(1-a)\neq 2\,,\ \text{when}\ a\neq 0\,. \tag{33}
$$

The non-zero value of a is obtained from the loop corrections of the Quantum field theory.