Quantentheorie II Übung 2

– Sample solutions –

SS 2021 Dr. H. Stöckinger-Kim, Prof. D. Stöckinger (IKTP)

2. **Generators of Lorentz group:** a 4-vector transforms as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ under Lorentz transformation. The matrix $\Lambda^\mu{}_\nu$ can be expressed in terms of the Lorentz group generators $L_{\rho\sigma}$ as follows:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu\,. \tag{1}$$

We have $l_k \equiv \epsilon_{ijk}L_{ij}$ and $k_i \equiv L_{i0} = -L_{0i}$, and the explicit expressions of l_z and k_x are

$$l_z = l_3 = L_{12} = i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}, \text{ and } k_x = k_1 = L_{10} = i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{2}$$

(a) Rotation around the z-axis: $\omega^{12} = -\omega^{21} = \varepsilon$, otherwise $\omega^{\rho\sigma} = 0$.
After applying $\omega^{12} = \varepsilon$ and L_{12} in Eq. (2) to Eq. (1) we obtain

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - i\omega^{12}(L_{12})^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,,\ \text{where } \omega^\mu{}_\nu = \begin{pmatrix}0&0&0&0\\0&0&-\varepsilon&0\\0&\varepsilon&0&0\\0&0&0&0\end{pmatrix}, \tag{3}$$

and $\omega^1{}_2 = -\varepsilon = \omega^{1\rho}g_{\rho 2} = -\omega^{12}$ as well as $\omega^2{}_1 = \varepsilon = \omega^{2\rho}g_{\rho 1} = -\omega^{21}$.

(b) Boost along the x-axis: $\omega^{10} = -\omega^{01} = \beta$, otherwise $\omega^{\rho\sigma} = 0$.
After applying $\omega^{01} = -\beta$ and $k_x = k_1 = L_{10}$ in Eq. (2) to Eq. (1) we obtain

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - i\omega^{01}(L_{01})^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,,\ \text{where } \omega^\mu{}_\nu = \begin{pmatrix}0&\beta&0&0\\\beta&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \tag{4}$$

and $\omega^0{}_1 = \beta = -\omega^{01}$. Also $\omega^1{}_0 = \beta = \omega^{10}$.

Therefore $\omega^\mu{}_\nu$ is equivalent to $-\frac{i}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu$.

3. **Lorentz transformation of spinors:** $S(\Lambda)$ is the spinor transformation matrix associated with $\Lambda^\mu{}_\nu$, and

$$\psi' = S(\Lambda)\psi\,, \quad S(\Lambda) \equiv 1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma}\,, \quad S_{\rho\sigma} \equiv \frac{i}{4}[\gamma_\rho, \gamma_\sigma]\,.$$

The relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4\,,\ \text{and } \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0,$$

are valid for all γ-matrix representations.

(a) $\psi' = (1 - \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma})\psi$ and equivalently $\psi'^\dagger = \psi^\dagger(1 + \frac{i}{2}\omega^{\rho\sigma}{S_{\rho\sigma}}^\dagger)$, where ${S_{\rho\sigma}}^\dagger = \gamma^0 S_{\rho\sigma}\gamma^0$. The Dirac adjoint is $\bar{\psi}' = \psi^\dagger(1 + \frac{i}{2}\omega^{\rho\sigma}\gamma^0 S_{\rho\sigma}\gamma^0)\gamma^0$. After combining these we obtain $\bar{\psi}'\psi' = \bar{\psi}\psi + \mathcal{O}(\omega^2)$, which means that $\bar{\psi}\psi$ is Lorentz-invariant.

(b) For ${\Lambda^\mu}_\nu = {\delta^\mu}_\nu + {\omega^\mu}_\nu$, $S^{-1}(\Lambda) = 1 + \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma} = 1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}$, which is compatible with $S^{-1}(\Lambda)S(\Lambda) = \mathbb{1}$ up to the first order of ω. Explicitly,

$$\begin{aligned}
S^{-1}(\Lambda)\gamma^\mu S(\Lambda) &= \gamma^\mu + \frac{i}{2}[S^{\rho\sigma}, \gamma^\mu] + \mathcal{O}(\omega^2) \\
&= \gamma^\mu + \frac{1}{8}\omega_{\rho\sigma}[\gamma^\mu, [\gamma^\rho, \gamma^\sigma]] \Longleftarrow \left(S^{\rho\sigma} \equiv \frac{i}{4}[\gamma^\rho, \gamma^\sigma]\right) \\
&= \gamma^\mu + \frac{1}{4}\omega_{\rho\sigma}[\gamma^\mu, \gamma^\rho\gamma^\sigma] \Longleftarrow \left(\{\gamma^\rho, \gamma^\sigma\} = 2g^{\rho\sigma}\mathbb{1}_4\right) \\
&= \gamma^\mu + \frac{1}{4}\omega_{\rho\sigma}(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu) = \gamma^\mu + \frac{1}{2}\omega_{\rho\sigma}(g^{\mu\rho}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho) \\
&= \gamma^\mu + \omega_{\rho\sigma}g^{\mu\rho}\gamma^\sigma = \gamma^\mu + {\omega^\mu}_\sigma\gamma^\sigma \\
&= {\Lambda^\mu}_\sigma\gamma^\sigma \,.
\end{aligned}$$

(c) $\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = {\Lambda^\mu}_\nu\bar{\psi}\gamma^\nu\psi$. Therefore $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector.

4. **Commutator of Lorentz group generators:** note that $\gamma_i = -\gamma^i$ and $\gamma_0 = \gamma^0$.

In Weyl-representation we obtain:

(i) $S_z \equiv S_{12} = \frac{i}{2}\gamma_1\gamma_2 = \frac{i}{2}\begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & -\sigma^1\sigma^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$,

$\Longleftarrow \sigma^i\sigma^j = \delta_{ij}\mathbb{1}_2 + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma^k$, for $i, j = 1, 2, 3$

(ii) $K_x \equiv S_{10} = \frac{i}{2}\gamma_1\gamma_0 = \frac{i}{2}\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$,

(iii) $K_y \equiv S_{20} = \frac{i}{2}\gamma_2\gamma_0 = \frac{i}{2}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$,

and $[S_z, K_x] = \frac{i}{4}\begin{pmatrix} [\sigma^1, \sigma^3] & 0 \\ 0 & -[\sigma^1, \sigma^3] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} = iK_y$.

5. **Commutators of the Dirac Hamiltonian with angular momenta:** after multiplying the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ with γ^0 we obtain

$$\begin{aligned}
&i\partial_0\psi\mathbb{1}_4 + i\gamma^0\gamma^i\partial_i\psi - m\gamma^0\psi = 0\,, \\
&i\partial_0\psi\mathbb{1}_4 = (-i\gamma^0\gamma^i\partial_i + m\gamma^0)\psi = H_D\psi\,,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
H_D &\equiv -i\gamma^0\gamma^i\partial_i + m\gamma^0 = -i\gamma^0\gamma_i\partial^i + m\gamma^0 \\
&= i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\partial^i + m\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \\
&= \begin{pmatrix} m & 0 & -i\partial_3 & -i(\partial_1 - i\partial_2) \\ 0 & m & -i(\partial_1 + i\partial_2) & i\partial_3 \\ -i\partial_3 & -i(\partial_1 - i\partial_2) & -m & 0 \\ -i(\partial_1 + i\partial_2) & i\partial_3 & 0 & -m \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} p^i + m \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \Longleftarrow \ (p^\mu = i\partial^\mu) \\
&= \begin{pmatrix} m & 0 & p^3 & p^1 - ip^2 \\ 0 & m & p^1 + ip^2 & -p^3 \\ p^3 & p^1 - ip^2 & -m & 0 \\ p^1 + ip^2 & -p^3 & 0 & -m \end{pmatrix} .
\end{aligned}
$$

The commutators are

$$
\begin{aligned}
[S_{12}, H_D] &= \frac{1}{2} \left\{ \begin{pmatrix} 0 & [\sigma^3, \sigma^1] \\ [\sigma^3, \sigma^1] & 0 \end{pmatrix} p^1 + \begin{pmatrix} 0 & [\sigma^3, \sigma^2] \\ [\sigma^3, \sigma^2] & 0 \end{pmatrix} p^2 \right\} \\
&= -i \begin{pmatrix} 0 & \sigma^1 p^2 - \sigma^2 p^1 \\ \sigma^1 p^2 - \sigma^2 p^1 & 0 \end{pmatrix} (\Longleftarrow \ [\sigma^i, \sigma^j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma^k, \ \text{for } i, j = 1, 2, 3) \\
&= -i(\hat{s} \times \vec{p}) \cdot \hat{e}_3 \,,
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
[\hat{L}_{12}, H_D] &= - \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} (\partial_i x_1) \partial_2 + \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} (\partial_i x_2) \partial_1 \\
&= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \partial_2 - \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \partial_1 = i \begin{pmatrix} 0 & \sigma^1 p^2 - \sigma^2 p^1 \\ \sigma^1 p^2 - \sigma^2 p1 & 0 \end{pmatrix} \\
&\quad (\Longleftarrow \ \partial_i x_j = -\delta_{ij} \,, (\partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)) \\
&= i(\hat{s} \times \vec{p}) \cdot \hat{e}_3 \,,
\end{aligned}
\tag{7}
$$

where $\hat{s} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, and $\hat{e}_3$ is the unit vector for the z-axis.

We can see that $[S_{12}, H_D] \neq 0$ and $[\hat{L}_{12}, H_D] \neq 0$, but $[S_{12} + \hat{L}_{12}, H_D] = 0$.

$\Longrightarrow$ The total angular momentum is conserved.