

## Black Hole: Problems – 2007

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In physics, whenever we have an equality relation, both sides of the equation should be of the same type i.e. they must have the same dimensions. For example you cannot have a situation where the quantity on the right-hand side of the equation represents a length and the quantity on the left-hand side represents a time interval. Using this fact, sometimes one can nearly deduce the form of a physical relation without solving the problem analytically. For example if we were asked to find the time it takes for an object to fall from a height of  $h$  under the influence of a constant gravitational acceleration  $g$ , we could argue that one only needs to build a quantity representing a time interval, using the quantities  $g$  and  $h$  and the only possible way of doing this is  $T = a(h/g)^{1/2}$ . Notice that this solution includes an as yet undetermined coefficient  $a$  which is dimensionless and thus cannot be determined, using this method. This coefficient can be a number such as 1,  $1/2$ ,  $\sqrt{3}$ ,  $\pi$ , or any other real number. This method of deducing physical relations is called *dimensional analysis*. In dimensional analysis, the dimensionless coefficients are not important and we do not need to write them. Fortunately, in most physical problems these coefficients are of the order of 1, and eliminating them does not change the order of magnitude of the physical quantities. Therefore, by applying the dimensional analysis to the above problem, one obtains  $T = (h/g)^{1/2}$ .

Generally, the dimensions of a physical quantity are written in terms of the dimensions of four fundamental quantities:  $M$  (mass),  $L$  (length),  $T$  (time), and  $K$  (temperature). The dimensions of an arbitrary quantity,  $x$  is denoted by  $[x]$ . As an example, to express the dimensions of velocity  $v$ , kinetic energy  $E_k$ , and heat capacity  $C_V$  we write:  $[v] = LT^{-1}$ ,  $[E_k] = ML^2T^{-2}$ ,  $[C_V] = ML^2T^{-2}K^{-1}$ .

### 1 Fundamental Constants and Dimensional Analysis

1.1	Find the dimensions of the <i>fundamental constants</i> , i.e. the Planck's constant, $h$ , the speed of light, $c$ , the universal constant of gravitation, $G$ , and the Boltzmann constant, $k_B$ , in terms of the dimensions of length, mass, time, and temperature.	0.8
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The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to  $\sigma\theta^4$  where  $\sigma$  is the Stefan-Boltzmann's constant and  $\theta$  is the absolute temperature of the black body.

#### ChatGPT's Answer:

Using dimensional analysis, we can express the dimensions of the fundamental constants in terms of the dimensions of length  $L$ , mass  $M$ , time  $T$ , and temperature  $K$  as follows:

Planck's constant,  $h$ :  $[h] = ML^2T^{-1}$  Speed of light,  $c$ :  $[c] = LT^{-1}$  Universal constant of gravitation,  $G$ :  $[G] = L^3M^{-1}T^{-2}$  Boltzmann constant,  $k_B$ :  $[k_B] = ML^2T^{-2}K^{-1}$  It is important to note that the dimensions of a physical constant are independent of the system of units used to measure it.

#### ChatGPT's Grade: 0.8

All correct. Note that ChatGPT seems to derive all these answers internally without providing the reasoning steps. For human, unless we remember, we would use a relevant equation we know such as  $h\nu = E$  for such analysis.

1.2	Determine the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length, mass, time, and temperature.	0.5
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The Stefan-Boltzmann's constant is not a fundamental constant and one can write it in terms of fundamental constants i.e. one can write  $\sigma = ah^\alpha c^\beta G^\gamma k_B^\delta$ . In this relation,  $a$  is a dimensionless parameter of the order of 1. As mentioned before, the exact value of  $a$  is not significant from our viewpoint, so we will set it equal to 1.

#### ChatGPT's Answer:

Using the relation  $\sigma = ah^\alpha c^\beta G^\gamma k_B^\delta$ , we can determine the dimensions of the Stefan-Boltzmann's constant,  $\sigma$ , in terms of the dimensions of length  $L$ , mass  $M$ , time  $T$ , and temperature  $K$  as follows:

$$\begin{aligned}
[\sigma] &= [a] \cdot [h^\alpha] \cdot [c^\beta] \cdot [G^\gamma] \cdot [k_B^\delta] \\
&= 1 \cdot (ML^2T^{-1})^\alpha \cdot (LT^{-1})^\beta \cdot (L^3M^{-1}T^{-2})^\gamma \cdot (ML^2T^{-2}K^{-1})^\delta \\
&= M^\delta L^{2\alpha+\beta+3\gamma} T^{-\alpha-\beta-2\gamma-2\delta} K^{-\delta}
\end{aligned}$$

Since the expression  $\sigma\theta^4$  in the Stefan-Boltzmann law must have units of power per unit area, we know that  $[\sigma\theta^4] = M^0L^{-2}T^{-3}K^4$ . Therefore, we can equate the dimensions of  $\sigma$  and  $\theta$  to obtain:

$$\begin{aligned}
[\sigma\theta^4] &= [\sigma][\theta]^4 = M^\delta L^{2\alpha+\beta+3\gamma} T^{-\alpha-\beta-2\gamma-2\delta} K^{-\delta} K^4 L^{-2} T^{-3} M^0 \\
&= M^\delta L^{2\alpha+\beta+3\gamma-2} T^{-\alpha-\beta-2\gamma-2\delta-3} K^{4-\delta}
\end{aligned}$$

We want this expression to be equal to  $M^0L^{-2}T^{-3}K^4$ . Therefore, we need to solve the following system of equations:

$$\begin{aligned}
2\alpha + \beta + 3\gamma &= 2 & -\alpha - \beta - 2\gamma - 2\delta \\
&= -3 & 4 - \delta \\
&= 4
\end{aligned}$$

Solving this system of equations, we obtain  $\alpha = 0$ ,  $\beta = -5$ ,  $\gamma = 1$ , and  $\delta = 0$ . Therefore, the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length  $L$ , mass  $M$ , time  $T$ , and temperature  $K$  are:

$$[\sigma] = M^0L^{-2}T^{-3}K^0 = L^{-2}T^{-3}$$

#### ChatGPT's Grade: 0.1

The logic is correct up until "we know that  $[\sigma\theta^4] = M^0L^{-2}T^{-3}K^4$ ." The correct reasoning should be that  $\sigma\theta^4$  has the same unit as power per area (as stated in the problem), then the dimension is  $[E]T^{-1}L^{-2} = MT^{-3}$ . This is because the model does not correctly infer from Stefan-Boltzmann law that  $\sigma\theta^4$  has the same unit as power per area, as stated in "The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to  $\sigma\theta^4$ " (-0.3)

The logic for setting up the equation "Therefore, we can equate the dimensions of  $\sigma$  and  $\theta$  to obtain" and what comes after is also incorrect. (-0.1)

The final answer is incorrect.

1.3	Find $\alpha, \beta, \gamma$ , and $\delta$ using dimensional analysis.	1.0
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**ChatGPT's Answer:** To find the values of  $\alpha, \beta, \gamma$ , and  $\delta$  using dimensional analysis, we can express the quantity to be determined as a product of powers of the fundamental quantities. Let's consider an arbitrary physical quantity  $Q$  that depends on mass  $M$ , length  $L$ , time  $T$ , and temperature  $K$ . We can express  $Q$  in terms of its dimensions as follows:

$$[Q] = M^\alpha L^\beta T^\gamma K^\delta \quad (1)$$

where  $\alpha, \beta, \gamma$ , and  $\delta$  are unknown exponents to be determined. Let's assume that  $Q$  depends on  $n$  fundamental quantities, which means that  $n$  of these exponents are non-zero.

Now, let's consider  $m$  fundamental quantities  $Q$  depends on, where  $m < n$ . We can choose  $m$  quantities to be the independent variables in  $Q$ , and we can express the remaining  $n - m$  variables in terms of these  $m$  variables using the relationships between them. We can then substitute these expressions into the original expression for  $Q$  to obtain an expression for  $Q$  solely in terms of  $m$  independent variables. We can then equate the dimensions of both sides of the resulting equation to determine the unknown exponents.

As an example, let's determine the values of  $\alpha, \beta, \gamma$ , and  $\delta$  for the period  $T$  of a simple pendulum of length  $l$  and mass  $m$ . The period  $T$  of a pendulum is given by:

$$T = 2\pi \sqrt{\frac{l}{g}} \quad (2)$$

where  $g$  is the acceleration due to gravity. We can express the dimensions of  $T$ ,  $l$ ,  $m$ , and  $g$  as:

$$[T] = T, \quad [l] = L, \quad [m] = M, \quad [g] = LT^{-2} \quad (3)$$

We can choose  $m = 2$  and the independent variables to be  $l$  and  $g$ . The period  $T$  can then be expressed as:

$$T = \sqrt{\frac{l}{g}} \cdot 2\pi \quad (4)$$

We can now equate the dimensions of both sides of this equation to determine the unknown exponents:

$$[T] = [l]^a [g]^b T = L^a (LT^{-2})^b = L^{a-b} T^{-2b}$$

Equating the dimensions of both sides, we obtain the following system of equations:

$$a - b = 0 \quad -2b = 1 \quad (5)$$

Solving this system of equations, we obtain  $a = b = 0$  and  $b = -1/2$ . Therefore, the dimensions of the period  $T$  of a simple pendulum in terms of the dimensions of length  $L$ , mass  $M$ , time  $T$ , and temperature  $K$  are:

$$[T] = L^0 M^0 T^1 K^0 = T \quad (6)$$

This result agrees with our previous knowledge that the period of a pendulum is a quantity with dimensions of time.

0

Incorrect. The model did not answer the question that is related to the main problem statement. It is possible that the context is insufficient. **ChatGPT's Grade:**

## 2 Physics of Black Holes

In this part of the problem, we would like to find out some properties of black holes using dimensional analysis. According to a certain theorem in physics known as the *no hair theorem*, all the characteristics of the black hole which we are considering in this problem depend only on the mass of the black hole. One characteristic of a black hole is the area of its event horizon. Roughly speaking, the *event horizon* is the boundary of the black hole. Inside this boundary, the gravity is so strong that even light cannot emerge from the region enclosed by the boundary.

We would like to find a relation between the mass of a black hole,  $m$ , and the area of its event horizon,  $A$ . This area depends on the mass of the black hole, the speed of light, and the universal constant of gravitation. As in 1.3 we shall write  $A = G^\alpha c^\beta m^\gamma$ .

2.1	Use dimensional analysis to find $\alpha, \beta$ , and $\gamma$ .	0.8
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**ChatGPT's Answer:** To find the dimensions of  $A$ , we can recall that the units of area are given by  $L^2$ , where  $L$  represents a length. The units of  $G$  are  $L^3/(MT^2)$ , where  $M$  represents mass and  $T$  represents time, and the units of  $c$  are  $L/T$ . Therefore, we have:

$$\begin{aligned} A &= G^\alpha c^\beta m^\gamma \\ [L^2] &= \left[ \frac{L^3}{MT^2} \right]^\alpha \cdot \left[ \frac{L}{T} \right]^\beta \cdot [M]^\gamma \\ [L^2] &= L^{3\alpha+\beta} \cdot T^{-2\alpha-\beta} \cdot M^\gamma \end{aligned}$$

This gives us three equations relating the exponents of  $L$ ,  $T$ , and  $M$ :

$$\begin{aligned} 3\alpha + \beta &= 2 \\ -2\alpha - \beta &= 0 \\ \gamma &= 0 \end{aligned}$$

Solving for  $\alpha$  and  $\beta$ , we get:

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= 3\alpha = -\frac{3}{2} \end{aligned}$$

Therefore, the relation between the mass of a black hole,  $m$ , and the area of its event horizon,  $A$ , is:

$$A = k \frac{G^{-1/2} c^{3/2}}{\sqrt{m}},$$

where  $k$  is a dimensionless constant.

**ChatGPT's Grade: 0.3**

The step  $[L^2] = L^{3\alpha+\beta} \cdot T^{-2\alpha-\beta} \cdot M^\gamma$  is slightly incorrect. The M term should be  $[M]^{\gamma-\alpha}$  (-0.1)

The model also did not solve the equations correctly. (-0.4)

From the result of 2.1 it becomes clear that the area of the event horizon of a black hole increases with its mass. From a classical point of view, nothing comes out of a black hole and therefore in all physical processes the area of the event horizon can only increase. In analogy with the second law of thermodynamics, Bekenstein proposed to assign entropy,  $S$ , to a black hole, proportional to the area of its event horizon i.e.  $S = \eta A$ . This conjecture has been made more plausible using other arguments.

2.2	Use the thermodynamic definition of entropy $dS = dQ/\theta$ to find the dimensions of entropy. $dQ$ is the exchanged heat and $\theta$ is the absolute temperature of the system.	0.8
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2.3	As in 1.3, express the dimensioned constant $\eta$ as a function of the fundamental constants $h, c, G$ , and $k_B$ .	1.1
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*Do **not** use dimensional analysis for the rest of problem, but you may use the results you have obtained in previous sections.*

### 3 Hawking Radiation

With a semi-quantum mechanical approach, Hawking argued that contrary to the classical point of view, black holes emit radiation similar to the radiation of a black body at a temperature which is called the *Hawking temperature*.

3.1	Use $E = mc^2$ , which gives the energy of the black hole in terms of its mass, and the laws of thermodynamics to express the Hawking temperature $\theta_H$ of a black hole in terms of its mass and the fundamental constants. Assume that the black hole does no work on its surroundings.	0.8
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**ChatGPT's Answer:** To derive the Hawking temperature of a black hole, we can use the laws of thermodynamics, specifically the first and second laws. The first law of thermodynamics relates the change in energy of a system to the heat added to it and the work done on it:

$$dE = TdS - PdV$$

where  $E$  is the internal energy of the system,  $S$  is its entropy,  $T$  is its temperature,  $P$  is its pressure,  $V$  is its volume, and  $d$  denotes a differential change in the corresponding variable.

For a non-rotating, spherically symmetric black hole, its entropy is given by the Bekenstein-Hawking formula:

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}$$

where  $A$  is the area of the black hole's event horizon,  $\hbar = h/(2\pi)$  is the reduced Planck's constant, and  $k_B$  is the Boltzmann constant.

The second law of thermodynamics states that the entropy of a closed system cannot decrease, i.e.,  $dS \geq 0$ . Since a black hole can only gain mass, its area and entropy can only increase, which means that  $dS_{BH} \geq 0$ . Thus, we can rewrite the first law of thermodynamics as:

$$dE = T_H dS_{BH}$$

where  $T_H$  is the temperature of the black hole, which we call the Hawking temperature.

Using the formula for the entropy of a black hole, we can express the temperature of a black hole in terms of its mass  $M$ :

$$T_H = \frac{\hbar c^3}{8\pi k_B G M}$$

Therefore, the Hawking temperature of a black hole can be expressed in terms of its mass and the fundamental constants as:

$$\theta_H = \frac{\hbar c^3}{8\pi k_B G M}$$

This formula shows that the Hawking temperature of a black hole is inversely proportional to its mass. As the black hole loses mass due to Hawking radiation, its temperature increases, and it eventually evaporates completely when its mass reaches a very small value.

**ChatGPT's Grade: 0.4**

Correct. The reasoning steps and derivations are missing in a few places. (1) The effect of the blackhole doing no work on the system is not clearly incorporated in the analysis (2) not clear how the model derives T from the equation of  $S$  and  $E = mc^2$ . More reasoning steps are required for full score.

3.2	The mass of an isolated black hole will thus change because of the Hawking radiation. Use Stefan-Boltzmann's law to find the dependence of this rate of change on the Hawking temperature of the black hole, $\theta_H$ and express it in terms of mass of the black hole and the fundamental constants.	0.7
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**ChatGPT's Answer:** The rate of change of the mass of a black hole due to Hawking radiation can be obtained from the energy conservation law. The energy lost by the black hole due to radiation is given by:

$$dE = -\frac{\hbar c^6}{15360\pi G^2 M^2} dt$$

where  $M$  is the mass of the black hole and  $t$  is time. This expression for  $dE$  can be obtained by integrating the energy flux of the outgoing radiation over the surface area of the black hole's event horizon.

By using the relation  $E = mc^2$ , we can convert the change in energy to the change in mass:

$$dM = -\frac{dE}{c^2} = -\frac{\hbar c^4}{15360\pi G^2 M^2} dt$$

The rate of change of mass  $dM/dt$  can be found by taking the time derivative of the above equation:

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2}$$

We can express the Hawking temperature in terms of the mass of the black hole, using the same formula we derived in problem 3.1:

$$\theta_H = \frac{\hbar c^3}{8\pi k_B G M}$$

Substituting this expression into the equation for the rate of change of mass, we get:

$$\frac{dM}{dt} = -\frac{k_B^4 \theta_H^4}{15360\pi^2 \hbar^3 c^2}$$

Therefore, the rate of change of the mass of a black hole due to Hawking radiation is given by:

$$\boxed{\frac{dM}{dt} = -\frac{k_B^4}{15360\pi^2 \hbar^3 c^2} \left( \frac{\hbar c^3}{8\pi k_B G M} \right)^4}$$

This formula shows that the rate of change of the mass of a black hole due to Hawking radiation is proportional to the fourth power of the Hawking temperature. It also shows that the rate of change of mass is inversely proportional to the square of the mass of the black hole. As the black hole loses mass due to Hawking radiation, its rate of mass loss increases, and it eventually evaporates completely when its mass reaches a very small value.

**ChatGPT's Grade: 0**

incorrect

**ChatGPT's Answer:** Sure, here's a possible analysis of the solution for problem 3.2:

The solution correctly uses the energy conservation law to derive the rate of change of the mass of a black hole due to Hawking radiation. The solution then expresses this rate in terms of the Hawking temperature, which is found in problem 3.1. The final formula for the rate of change of mass due to Hawking radiation is consistent with previous results in the literature.

However, the solution could benefit from clearer and more detailed explanations of the key steps. Here is a possible revised solution with more detailed reasoning steps:

To find the rate of change of the mass of a black hole due to Hawking radiation, we use the energy conservation law, which relates the change in energy of a system to the energy that flows in or out of the system:

$$dE = -dE_{rad}$$

where  $dE$  is the change in the energy of the black hole and  $dE_{rad}$  is the energy carried away by the radiation. We can express  $dE_{rad}$  in terms of the temperature and entropy of the black

hole using the Stefan-Boltzmann law, which relates the energy radiated by a black body to its temperature:

$$dE_{rad} = \sigma AT^4 dt$$

where  $\sigma$  is the Stefan-Boltzmann constant,  $A$  is the area of the black hole's event horizon, and  $T$  is the Hawking temperature of the black hole, which is given by:

$$T = \frac{\hbar c^3}{8\pi k_B GM}$$

Substituting this expression into the equation for  $dE_{rad}$  and integrating over time, we obtain the total energy radiated by the black hole:

$$E_{rad} = \frac{\hbar c^4}{15360\pi G^2 M^2}$$

Using the relation  $E = mc^2$ , we can convert this energy to a change in mass:

$$\frac{dM}{dt} = -\frac{dE_{rad}}{c^2} = -\frac{\hbar c^4}{15360\pi G^2 M^2 c^2}$$

Simplifying this expression and substituting in the expression for the Hawking temperature, we obtain the final formula for the rate of change of mass due to Hawking radiation:

$$\frac{dM}{dt} = -\frac{k_B^4}{15360\pi^2 \hbar^3 c^2} \left( \frac{\hbar c^3}{8\pi k_B GM} \right)^4$$

This formula shows that the rate of change of the mass of a black hole due to Hawking radiation is proportional to the fourth power of the Hawking temperature, and inversely proportional to the square of the mass of the black hole. This means that smaller black holes lose mass more quickly than larger black holes. The formula also shows that the rate of mass loss due to Hawking radiation is extremely slow for astronomical black holes, which have masses on the order of millions or billions of times the mass of the sun.

**ChatGPT's Answer:** I apologize for the confusion. After reviewing my previous response, I realize that it is incorrect. Here's a corrected solution:

To find the rate of change of the mass of a black hole due to Hawking radiation, we can start by considering the energy flux of the outgoing radiation at the black hole's event horizon. This flux can be expressed as:

$$F = \frac{\sigma}{c} T_H^4$$

where  $T_H$  is the Hawking temperature of the black hole, which we found in problem 3.1, and  $\sigma$  is the Stefan-Boltzmann constant.

The total power radiated by the black hole is given by integrating the energy flux over the area of the black hole's event horizon:

$$P = \int F dA = \frac{4\pi G^2 M^2}{c^4} \frac{\sigma}{c} T_H^4 = \frac{\hbar c^6}{15360\pi G^2 M^2}$$

Fixme:  
correct  
reasoning -  
power per  
area =  $\sigma T^4$



where we have used the relation  $T_H = \hbar c^3 / (8\pi G k_B M)$  from problem 3.1.

Using the relation  $E = mc^2$ , we can convert the power to a rate of change of mass:

$$\frac{dM}{dt} = -\frac{P}{c^2} = -\frac{\hbar c^4}{15360\pi G^2 M^2 c^2}$$

Therefore, the rate of change of the mass of a black hole due to Hawking radiation is given by:

$$\boxed{\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2 c^2}}$$

This formula shows that the rate of mass loss due to Hawking radiation is extremely slow for astronomical black holes, which have masses on the order of millions or billions of times the mass of the sun. The rate of mass loss increases as the mass of the black hole decreases, and the black hole eventually evaporates completely when its mass reaches a very small value.

**ChatGPT's Grade:** roughly correct

3.3	Find the time $t^*$ , that it takes an isolated black hole of mass $m$ to evaporate completely i.e. to lose all its mass.	1.1
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**ChatGPT's Answer:** The rate of change of the mass of a black hole due to Hawking radiation, as derived in problem 3.2, is given by:

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2 c^2}$$

We can integrate this equation to find the time  $t^*$  it takes for a black hole to evaporate completely, starting from an initial mass  $M_0 = m$ :

$$\int_m^0 \frac{dM}{dt} dt = \int_0^{t^*} -\frac{\hbar c^4}{15360\pi G^2 M^2 c^2} dt$$

Solving this integral, we get:

$$t^* = \frac{5120\pi G^2 M_0^3}{\hbar c^4}$$

Substituting in the expressions for the fundamental constants, we obtain:

$$t^* = \frac{5120\pi G^3 m^3}{\hbar c^4}$$

Therefore, the time it takes for a black hole of mass  $m$  to evaporate completely due to Hawking radiation is:

$$\boxed{t^* = \frac{5120\pi G^3 m^3}{\hbar c^4}}$$

This formula shows that the evaporation time of a black hole is proportional to the cube of its mass, and inversely proportional to the fourth power of the fundamental constants. For a black hole with the mass of the sun, this time is on the order of  $10^{67}$  years, which is much longer than

the current age of the universe. For smaller black holes, the evaporation time is much shorter, and can be on the order of seconds or even microseconds for black holes with masses close to the Planck mass.

**ChatGPT's Grade:** incorrect constant, but correct relationship with  $m, c, h$ .

From the viewpoint of thermodynamics, black holes exhibit certain exotic behaviors. For example the heat capacity of a black hole is negative.

3.4	Find the heat capacity of a black hole of mass $m$ .	0.6
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**ChatGPT's Answer:** The heat capacity of a system is defined as the rate of change of its internal energy with respect to its temperature:

$$C = \frac{dE}{dT}$$

For a black hole, the internal energy is given by its mass-energy  $E = mc^2$ , and the temperature is given by the Hawking temperature  $T_H$  derived in problem 3.1.

Differentiating the expression for the internal energy with respect to temperature, we get:

$$\frac{dE}{dT} = \frac{d}{dT}(mc^2) = 0$$

since the mass of the black hole is constant.

Using the expression for the Hawking temperature derived in problem 3.1, we can express the heat capacity of a black hole in terms of its mass  $m$  and the fundamental constants:

$$C = \frac{dE}{dT} = \frac{d}{dT}(mc^2) = \frac{dE}{dT_H} \frac{dT_H}{dT} = \frac{-2k_B^2 \pi G m^2 c^4}{\hbar^2 T_H^3}$$

Substituting in the expression for the Hawking temperature, we obtain:

$$C = \left[ -\frac{2k_B^2 \pi G m^2 c^4}{\hbar^2} \left( \frac{8\pi G k_B m}{\hbar c^3} \right)^3 \right]$$

This formula shows that the heat capacity of a black hole is negative. This means that the black hole is a thermodynamically unstable system, since it would radiate away more energy as its temperature increased, leading to a decrease in its mass and a further increase in its temperature. The heat capacity is also proportional to the square of the mass of the black hole, and inversely proportional to the sixth power of the fundamental constants. For astronomical black holes, the heat capacity is extremely small, and it approaches zero as the mass of the black hole increases.

**ChatGPT's Grade:** 0.1

The first equation is set up correctly. however the rest are incorrect.

## 4 Black Holes and the Cosmic Background Radiation

Consider a black hole exposed to the cosmic background radiation. The cosmic background radiation is a black body radiation with a temperature  $\theta_B$  which fills the entire universe. An object with a total area  $A$  will thus receive an energy equal to  $\sigma \theta_B^4 \times A$  per unit time. A

black hole, therefore, loses energy through Hawking radiation and gains energy from the cosmic background radiation.

4.1	Find the rate of change of a black hole's mass, in terms of the mass of the black hole, the temperature of the cosmic background radiation, and the fundamental constants.	0.8
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**ChatGPT's Answer:**

**ChatGPT's Grade:**

4.2	At a certain mass, $m^*$ , this rate of change will vanish. Find $m^*$ and express it in terms of $\theta_B$ and the fundamental constants.	0.4
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**ChatGPT's Answer:**

**ChatGPT's Grade:**

4.3	Use your answer to 4.2 to substitute for $\theta_B$ in your answer to part 4.1 and express the rate of change of the mass of a black hole in terms of $m, m^*$ , and the fundamental constants.	0.2
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**ChatGPT's Answer:**

**ChatGPT's Grade:**

4.4	Find the Hawking temperature of a black hole at thermal equilibrium with cosmic background radiation.	0.4
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**ChatGPT's Answer:**

**ChatGPT's Grade:**

4.5	Is the equilibrium stable or unstable? Why? (Express your answer mathematically)	0.6
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**ChatGPT's Answer:**

**ChatGPT's Grade:**