5.8 Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Second-order linear homogeneous recurrence relations

Definition

A **second-order linear homogeneous recurrence relation with constant coefficients** is a recurrence relation of the form

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for all integers k greater than some fixed integer, where A and B are fixed real numbers with $B \neq 0$.

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Examples

Which of the following examples are second-order linear homogeneous recurrence relations?

1. $a_k = 3a_{k-1} + 4a_{k-2}$.
2. $a_k = 3a_{k-1} + 4a_{k-3}$.
3. $a_k = 4a_{k-2}$. (What about $a_k = 3a_{k-1}$?)
4. $a_k = 3a_{k-1}^2 + 4a_{k-2}$.

Finding explicit formulas for second-order relations

Fact

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Theorem

A sequence $a_n = t^n$ satisfies the second-order linear homogeneous recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$ iff

$$t^2 - At - B = 0.$$

We say that $t^2 - At - B = 0$ is the **characteristic equation of the relation**.

Example

Fact

If the equation $t^2 - At - B = 0$ has two distinct roots r and s, then an explicit formula for a_n is given by

$$a_n = Cr^n + Ds^n,$$

where C and D are the numbers whose values are determined by the values a_0 and a_1.

Example

Find all sequences that satisfy relation $a_k = a_{k-1} + 2a_{k-2}$ for all integers $k \geq 2$ and have the form $1, t, t^2, \ldots,$ where t is nonzero. Then, find a sequence that also satisfies the initial conditions $a_0 = 1$ and $a_1 = 8$.

Fibonacci's sequence revisited

Example

Find all the sequences that satisfy $a_n = a_{n-1} + a_{n-2}$ and have the form $1, t, t^2, \ldots$. Then, find an explicit solution for the Fibonacci sequence that satisfies $a_0 = a_1 = 1$.

Example

Fact

If the equation $t^2 - At - B = 0$ *has only one root* r*, then an explicit formula for* a_n *is given by*

$$a_n = Cr^n + Dnr^n,$$

where C *and* D *are the numbers whose values are determined by the values* a_0 *and* a_1*.*

Example

Find all sequences that satisfy relation $a_k = 4a_{k-1} - 4a_{k-2}$ for all integers $k \geq 2$ and have the form $1, t, t^2, \ldots,$ where t is nonzero. Then, find a sequence that also satisfies the initial conditions $a_0 = 1$ and $a_1 = 8$.