Lecture 2 Mathematics for Computer Science

Benay Tomas

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Continuation of Lecture 1

1 Proof by Contradiction

Essentially: To prove P is true, we assume P is false (i.e., $\neg P$ is True) then use that hypothesis to derive a contradiction; That is, if $\neg P \implies F$; then P must be T.

Example

Prove that $\sqrt{2}$ is irrational.

Proof by Contradiction

Assume for purpose of contradiction that $\sqrt{2}$ is rational.

Then this means, by the rational number definition, that 2 must be able to be represented by the division of 2 values in their lowest terms. So:

$$\implies \sqrt{2} = \frac{A}{B}$$

$$\implies 2 = \frac{A^2}{B^2}$$

$$\implies 2B^2 = A^2$$

$$\implies \text{This means 'A' is even}$$

$$\implies \text{Which means it can divide 4}$$

$$\implies \text{So } \frac{4}{A^2} = \frac{4}{2B^2}$$

$$\implies \text{Therefore } 2B^2 \text{ implies that B is also even}$$

$$\implies \text{So } \sqrt{2} \text{ can't be } \frac{A}{B} \text{ because there's no even value minor than } 2.$$

$$\implies \text{Thus a contradiction.}$$

Conclusion

$\sqrt{2}$ is irrational.

2 Proof by Induction

Let $P(n)$ be a predicate. If $P(0)$ is true and $\forall n \in N$ — $(P(n) \implies P(n+1))$ is true then $\forall n \in N$ — $P(n)$ is true, if $P(0) \implies P(1), ...,$ are true; then $P(0), P(1), P(2), ..., P(n+1)$ are true.

Examples

In this section i'll express 2 examples of use for the Induction method. The first one being the Gauss Sum and the second a special arithmetic expression.

2.1 Gauss Sum

The Gauss Sum states that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof by Induction

Let $P(n)$ be the proposition that encapsulates this expression. To prove it we'll need to do 2 steps the base case and the inductive step.

Base Case

The predicate is tested at the lowest value for n possible

$$\sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0$$

So it is true.
Then we now go for the Inductive Step.

Inductive Step

For $n \geq 0$, show $P(n) \implies P(n+1)$ is True.
To do this we'll assume $P(n)$ is true for purposes of induction.
Therefore, assume

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

With this we need to show that

$$\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2} \implies \frac{n(n+1)}{2} + n + 1$$

If we expand the expression on the right we'll get

$$\frac{n^2 + n + 2n + 2}{2}$$

Which is none other than

$$\frac{n+1(n+2)}{2}$$

Conclusion

This shows that using $P(n)$ we can inductively prove that the expression is valid for $P(n+1)$, and so therefore it is valid for any n.

2.2 Special Arithmetic Expression

Consider the following:

$$\forall n \in N : 3|(n^3 - n)$$

Where the value inside the parenthesis means it's divisible by 3.

Proof by Induction

$Let P(n)$ be $3|(n^2 - n)$

Base Case

The value for n can be 0 for this case.

$$3|(0^3 - 0) = 0$$

$$0/3 = 0 \implies True$$

Inductive Step

For $n > 0$, show that $P(n) \implies P(n+1)$ is True.

Assuming $P(n)$ is True, for $n+1$ we would have

$$(n+1)^3 - (n+1)$$

$$\implies n^3 + 3n^2 + 3n + 1 - n - 1$$

$$\implies n^3 + 3n^2 + 2n$$

$$\implies n^3 - n + 3n^2 + 3n$$

Conclusion

Considering $n^3 - n$ and $3n^2 + 3n$ are both divisible by 3, then there's our answer, the proposition holds.

3 References

1. My personal notes taken
2. Slides Lecture 2 Mathematics for Computer Science - MITOpenCourseware
3. Overleaf LaTeX Help

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