SINGLE VARIABLE CALCULUS SUMMARY

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1 Limits

Limit is a tool for discovering and understanding how functions behave near certain points, usually points where said functions aren't fully defined. The usual examples being, $f(x) = \frac{1}{x}$ and $f(x) = \frac{\sin(x)}{x}$ at $x = 0$. A simple definition of a limit is as follows.

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then,

$$\lim_{x \to a} f(x) = L$$

This means the limit of $f(x)$ is L as x approaches a.

1.1 Continuity

A function $f(x)$ is continuos at x_0 when

$$\lim_{x \to x_0} f(x) = f(x_0)$$

1.1.1 Right- and left-hand limits

Functions may have different limits when x approaches a from values smaller or greater than a. This gives us left-hand limits and right-hand limits. In many cases a function can be continuous at a point x_0 considering the left-hand limit, but not the right-hand limit, or vice-versa. In these cases we have discontinuities.

So to be entirely certain a function is continuos at point a, we need to make sure the limit exist at that point. But not every limit will exist, limits with $x \to a$ will only exist if:

- Left-hand limit exists at a.
- Right-hand limit exists at a.
- Left-hand limit = right-hand limit.

Symbolically, $\lim_{x \to a} f(x)$ exists if

$$\lim_{x \to a_-} f(x) : exists$$

$$\lim_{x \to a_+} f(x) : exists$$

$$\lim_{x \to a_-} = \lim_{x \to a_+}$$

1.2 Discontinuity

When a function is not continuos at a point w, it means there's a discontinuity at $x = w$.

1.2.1 Types of discontinuity

1. Removable discontinuity Left-hand limit and right-hand limit both exist at $x \to a$ but they are not equal to $f(a)$, or $f(a)$ is undefined.

2. Jump discontinuity Left-hand limit and right-hand limit differ.

3. Infinite Discontinuity Left-hand limit equals $\pm\infty$ and right-hand limit $\mp\infty$, respectively.

There are other possible discontinuities but we won't get into detail.

1.3 Trigonometric limits

A few basic trigonometric limits

$$\lim_{x \to 0} \sin x = 0 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\arcsin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 \qquad \lim_{x \to 0} \frac{\arctan x}{x} = 1$$

2 The tangent line

Calculus can be roughly summarized into 2 major problems, (1) how to calculate slope the tangent line for a given function, and, (2) how to calculate the area under a given function.

The tangent line to a graph $f(x)$ at a is given by

$$y \approx f(a) + f'(a)(x - a)$$

Where $f(a)$ is the value we know, and $f(x)$ is the one we want to know.

The slope $f'(a)$ of the tangent line y at the point a is called the Derivative of $f(x)$ at point a.

The Derivative can be defined as follows.

2.1 Derivative definition

Let $f(x)$ be a function with a curvature, take points P and Q as two different points on this curve, with P being fixed and Q being draggable. First, we give coordinates to these points,

$$P = (x_0, y_0), Q = (x_0 + \Delta x, f(x_0 + \Delta x)).$$

Now consider the following limit,

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

This is the formal definition of a derivative. What represents it's 2 points, where the second point is defined in terms of the first point. Then we bring the Q point infinitely close to P by using a limit, where in this limit, the infinitely close representation can be seen as the difference between the points, Δx and $f(x_0 + \Delta x)$ or Δf, come close to zero.

2.2 Physical interpretation of derivatives

You can think of derivatives as representing an instant rate of change at a particular moment in time. I'll explain this better with an example. Consider an object dropped from a 400m tall building, this object height above the ground can be interpreted as the following:

$$y = 400 - 16t^2$$

The average speed can be given as $= \frac{\Delta y}{\Delta t} = \frac{distance}{time}$

When the object hits the ground, $y = 0$, we solve to find $t = 5$, so after 5 seconds the object will have reached the ground. This means the average speed will be:

$$\frac{400}{5} = 80m/s$$

This we already know how to calculate, but what about the instant speed? Well, now we can get to this.

For that, first we calculate the derivative, so considering $f(t) = 400 - 16t^2$ we apply the formal definition:

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

$$f(t + \Delta t) = 400 - 16(t + \Delta t)^2$$

$$f(t) = 400 - 16t^2$$

Expanding $f(t + \Delta t)$ using binomial expansion, we get:

$$\lim_{\Delta t \to 0} \frac{(400 - 16t^2 - 32t\Delta t - 16(\Delta t)^2) - (400 - 16t^2)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{400 - 400 - 16t^2 + 16t^2 - 32t\Delta t - 16(\Delta t)^2}{\Delta t}$$

$$\lim_{\Delta t \to 0} (-32t - 16\Delta t)$$

And so we get to: $f'(t) = -32t$. We can for instance discover what was the final speed just before hitting the ground, $t = 5$, $v = -160m/s$

3 Derivatives

As said in the previous section, derivatives are the slope of tangent lines on function curves.

3.1 Notation

There are 2 major notations for derivatives, the one most used is by LaGrange and the other by Leibniz:

LaGrange notation,

$$f'(x)$$

Leibniz notation,

$$\frac{d}{dx}f(x)$$

We'll see later that Leibniz notation is way more useful. And although they mean the same, sometimes choosing the right notation can make our job easier.

3.2 General rules/Techniques

Derivatives have a few rules that may help us when differentiating (calculating the derivative) functions.

3.2.1 Constant

$$\frac{d}{dx}C = 0$$

$$\frac{d}{dx}Cf(x) = C\frac{d}{dx}f(x)$$

3.2.2 Power

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

3.2.3 Sum/Difference

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

3.2.4 Product

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

3.2.5 Quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

3.2.6 Sine/Cosine

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

3.3 Chain rule

Let $y = f(u)$ and $u(x)$ be two different functions. In order to calculate the derivative of y in terms of x, in this case, $\frac{dy}{dx}$, we'll use the chain rule, which is as follows

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

So basically what we'll do is differentiate the $f(u)$ function in terms of u, then multiplicate with the derivative of $u(x)$ in terms of x. And at the end change u by the function in terms of x.

3.4 Higher order derivatives

These are derivatives of derivatives. When you differentiate a function 3 times, that'll get you a derivative of order 3. The notations for them are as follow:

$f'(x)$	Df	$\frac{df}{dx}$
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$
$f^n(x)$	$D^n f$	$\frac{d^n f}{dx^n}$

3.5 Other trigonometric derivatives

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

4 Implicit differentiation

Implicit differentiation, as the name says, comes from derivatives of implicit functions. There are 2 ways of differentiating implicit functions, (1) we can isolate the dependent variable, usually y, and then differentiate normally, or (2) we differentiate every member of the equation in terms of x and then later isolate the $\frac{dy}{dx}$ part. This second method is called *implicit differentiation.*

4.1 Method

Let $F(x, y) = 0$, where y is implicitly defined as a function of x.

$$\frac{d}{dx}(F(x,y)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial y}{\partial x}$$

4.2 Examples

Considering $x^3y^4 = 1$, find dy/dx.

$$\frac{d}{dx}(x^3y^4) = 3x^2y^4 + x^3 4y^3\frac{dy}{dx}$$

$$3x^2y^4 + x^3 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x^2y^4}{4x^3y^3}$$

$$\frac{dy}{dx} = -\frac{3y}{4x}$$

Considering $x^2 + y^2 = 25$, find dy/dx.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

5 Exponential and logarithmic functions

The most comon exponenetial and logarithm functions in calculus are the natural exponential function, e^x, and the natural logarithmic function, $\ln x$.

Starting with a general exponential function,

$$f(x) = a^x$$

where $a \in R$.

We'll have to use the definition of derivative in order to find some sort of rule to solve for any value. For this consider $h = \Delta x$.

$$\frac{d}{dx}(a^x) = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x(a^h - 1)}{h}$$

$$f'(x) = a^x \lim_{h\to 0}\frac{(a^h - 1)}{h}$$

At this point there's some seriously advanced processes to get the proof for the final definition, but we won't touch that, we'll just jump right to the conclusion, which is

$$\frac{d}{dx}(a^x) = a^x\ln(a)$$

And for the logarithmic functions

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (\ln f(x))' = \frac{f'(x)}{f(x)}$$

6 Maxima and minima

A function $f(x)$ is increasing in intervals where $x_1 < x_2$ results in $f(x_1) < f(x_2)$ and it's decreasing when $x_1 < x_2$ results in $f(x_1) > f(x_2)$.

A better way of saying, using our recently acquired knowledge of derivatives, is that when

$f'(a) > 0$ the function is increasing at $x = a$

and when

$f'(a) < 0$ the function is decreasing at $x = a$.

At the transition points between increasing, and decreasing, the function will have reached its *Maxima* or its *Minima*. If the function went from $f'(x) > 0$ to $f'(x) < 0$ means it went from increasing to decreasing, which means at $f'(x) = 0$ the function reached its *maximum local point*, also known as Maxima. If the opposite happens, the function goes from decreasing to increasing, then at the transition point we would have a Minima.

These points are called *critical points* and they tell us a lot about a function. It's also worth mentioning that not all critical points will be maximum or minimum, they will only be maximum and minimum points if there was the transition between increasing and decreasing or vice-versa.

Taking all of this in consideration, some important things to keep in mind when sketching a graph are

1. Find critical points, $f'(x) = 0$ or undefined.
2. Sign of $f'(x)$ between critical points.
3. Points where $f(x) = 0$.
4. Behavior of $f(x)$ when $x \to \pm\infty$.
5. Behavior of $f(x)$ near undefined points.

There are circumstances where the Maxima and Minima might not be at $f'(x) = 0$. The most common being when the max and min are at cusps, vertical tangents or corners; Which are points where the derivative is undefined/doesn't exist.

6.1 Concavity and points of inflection

What influences the concavity of a curve in a function is it's second derivative.

So for the second derivative

$f''(x) > 0$ means concavity up.

$f''(x) < 0$ means concavity down.

And the transition point between one concavity to another is called *point of inflection.* Which can be found in points where $f''(x) = 0$.

6.2 Example

Let's investigate the function given by:

$$f(x) = 2x^3 - 12x^2 + 18x - 2$$

First derivative : $f'(x) = 6(x-1)(x-3)$

Second derivative : $f''(x) = 12(x-2)$

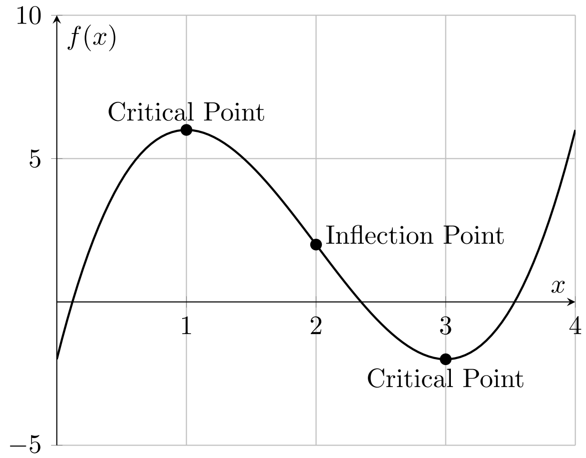
Critical points : $6(x-1)(x-3) = 0$

$$x_c = 1; x_c = 3$$

Inflection point : $12(x-2) = 0$

$$x_i = 2$$

With this information we can already have some pretty good idea of how the function looks. We know there will be 2 concavities, one before $x = 2$ and one after that. Considering the function for $f''(x)$ we know that for smaller than 2 x-values the concavity was down and after that the concavity was up. We also know the Maxima and Minima will be at $x = 1$ and $x = 3$. With this information we can make a pretty good sketch, see it below.



Caution note Sometimes the second derivative = 0 won't give you the real inflection point, if the inflection point is equal to the maxima or minima, then you know that's not the real inflection.

In some other cases $f''(x)$ might not exist or be undefined, if that's the case, take extra attention, even if they do not exist they might still be the inflection.

It's also worth mentioning that the second derivative may also be helpful at identifying if a critical point is a Maxima or a Minima, in cases where might not be clear.

If at $x = x_1$ there's a critical point and $f''(x_1) > 0$, it means that at x_1 the concavity is up, which means x_1 is a minimum point. Works the same if the concavity is down at x_1, but then that will be a maximum point.

6.3 Applied example

A rectangular garden with $450ft^2$ of area is to be fenced.

If one side of the garden is already protected by a wall; What dimensions will require the shortest length of fence?

First, let y be the similar sides of the fence, and let x be the standalone side.

We know the area is given by

$$xy = 450$$

with that we can get

$$y = \frac{450}{x}$$

We also know that the perimeter of the fence will be

$$P = 2y + x$$

$$P = 2\left(\frac{450}{x}\right) + x$$

Considering we want the minimum possible size for the fence, we're gonna need to minimize the perimeter, so for that we take its derivative

$$P'(x) = -\frac{900}{x^2} + 1$$

And for the critical minimum point, we set to zero

$$-\frac{900}{x^2} + 1 = 0$$

And we get

$$x = 30$$

Substituting back into $y = \frac{450}{x}$ we find that

$$y = 15$$

And that's the answer for the problem, we'll get the minimum fence size for $450ft^2$ area when 2 sides are 15 and the other is 30.

6.4 Strategy for max-min problems

George Polya, one of mathematics most prominent teachers had a four step principle for solving problems that required exceptional problem solving skills, such as max-min problems, which are:

1. Understand what the problem want.

2. Sketch the problem whenever possible for better geometry comprehension.

3. Label the variables wisely and keep them in mind.

4. Always try keeping formulas in terms of one variable.

7 Differentials

As we know, the derivative of a function $y = f(x)$ can be written as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

We also introduced the equivalent notation $\frac{dy}{dx}$.

And now our present purpose is to give individual meaning to dy and dx, in such a way that their quotient is indeed $f'(x)$.

This will be necessary to better tackle the *Integration*, and certain techniques for it.

So basically, using the Leibniz notation $f'(x) = \frac{dy}{dx}$ we can get dy as multiplication of $f'(x)$ and dx.

This means :

$$f(x) = x^2$$

$$\frac{dy}{dx} = 2x$$

$$dy = 2xdx$$

And this variable dy is what we call *Differential.*

All of our familiar formulas for differentiation can now be given their equivalent versions.

Let $y = f(u)$, so that $dy = f'(u)du$. Then for any applicable differentiation method we would have:

$$d(u^n) = nu^{n-1}du$$
$$d(\sin(u)) = \cos(u)du$$
$$d(uv) = vdu + udv$$

Every other differentiation rule would also apply.

7.1 Linear approximation

Linear approximation is the process of estimating the value of a function near a point using its tangent line. As we've seen earlier the tangent line is represented by

$$y \approx f(a) + f'(a)(x - a)$$

It provides a simple but good linear estimate of $f(x)$ near $x = a$.

Now using differentials we have the following statement:

"When $dy = f'(x)dx$ is known, then to compute values of said function near values of x we can apply the formula:"

$$f(x + dx) \approx f(x) + dy$$

7.2 Examples

Calculate the approximate value of $\sqrt[3]{28}$.

First, we set out variables,

$$y = x^{\frac{1}{3}}$$
$$dy = \frac{1}{3}x^{-\frac{2}{3}}dx$$

Now, we need a value close to 28 that we can calculate the cubic root, 27 is the perfect value, so 27 will be the value for x. Apply the formula $f(x + dx) \approx f(x) + dy$ considering the variables $x = 27$, $dx = 1$, $f(27) = 3$. We will get:

$$f(27 + 1) \approx f(27) + \frac{1}{3}(27)^{-\frac{2}{3}}.1$$
$$f(28) = 3 + \frac{1}{3}.\frac{1}{9}$$
$$f(28) = 3 + \frac{1}{27}$$
$$f(28) = 3.037$$

And the value by calculator is 3.0366, so we can see how good the linear approximation is.

Calculate the value of $(4.01)^3$ by linear approximation.

Right off the bat we can get these variables

$$f(x) = x^3; x = 4; dx = 0.01; dy = 3x^2dx$$

So now let's apply it

$$f(4.01) = f(4) + 3(4)^2(0.01)$$

$$f(4.01) = 64 + 0.48$$

$$f(4.01) = 64.48$$

And the actual value:

$$(4.01)^3 = 64.4812$$

Which is close enough for almost everything.

In sum, the 2 formulas for *linear approximation* are

$$f(x) \approx f(a) + f'(a)(x - a)$$

$$f(x + dx) \approx f(x) + dy$$

8 Indefinite integrals

Let $y = F(x)$ be a function whose derivative is known, say, for example,

$$\frac{d}{dx}F(x) = 2x,$$

Can we discover what the function $F(x)$ is?

Doesn't take much imagination to write down one function with this property, namely, $F(x) = x^2$. Moreover, adding a constant term doesn't change the derivative, so each of the functions

$$x^2, \quad x^2 - \sqrt{3}, \quad x^2 + 5\pi,$$

And more generally, $x^2 + c$, where c is any constant. So a way of saying can be

If $F(x)$ and $G(x)$ are two functions having the same derivative $f(x)$ on a certain interval, then $G(x)$ differs from $F(x)$ by a constant, that is, there exists a constant c with the property that

$$G(x) = F(x) + c$$

for all x in the interval.

8.1 Antiderivative

If $f(x)$ is given, then a function $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x)$$

is called an *antiderivative* of $f(x)$, and the process of finding $F(x)$ from $f(x)$ is $antidifferentiation$, or $integration$.

The standard notation for an integral of $f(x)$ is

$$\int f(x)dx,$$

which is read 'the integral of $f(x)dx$.' The equation

$$\int f(x)dx = F(x)$$

is therefore completely equivalent to $\frac{d}{dx}F(x) = f(x)$.

To illustrate a point of usage, we remark that the formulas

$$\int x^2dx = \frac{1}{3}x^3 \quad \textit{and} \quad \int x^2dx = \frac{1}{3}x^3 + c$$

are both correct, but the first provides only one integral while the second provides all possible integrals. And for this reason,

$$\int f(x)dx$$

is called an *indefinite integral.*

8.2 Integration rules

Now some basic integration rules, mostly derived from a reverse on the differentiation process.

8.2.1 Constant

$$\int adx = ax + C$$

$$\int af(x)dx = a\int f(x)dx$$

8.2.2 Power

$$\int f(x)dx = \frac{x^{n+1}}{n+1} + C,$$

with $n \neq -1$

8.2.3 Sum/Difference

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

8.2.4 Integration by parts

Let $u = u(x)$, $v = v(x)$, $du = \frac{d}{dx}u$, and $dv = \frac{d}{dx}v$;

$$udv = uv - \int vdu$$

8.2.5 Integration by substitution

Also called 'u-substitution' this method has 3 steps; First we write our integral in this form:

$$\int f(g(x))g'(x)dx,$$

after setting up this way, we can do the 'u-substitution', where $u = g(x)$ and $du = g'(x)dx$:

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

and then,

$$\int f(u)du = F(u) + C$$

after this we substitute u back to $g(x)$.

$$\int f(g(x))dx = F(g(x)) + C$$

9 Common integrals

These are some integrals that frequently are needed, for the sake of practicity i'll display the most used ones.

9.1 Basic functions

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

9.2 Exponential/logarithm functions

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln(a)} + C$$

$$\int ln(u) du = u\ln(u) - u + C$$

9.3 Trigonometric functions

$$\int \cos(u) du = \sin(u) + C$$

$$\int \sin(u) du = -\cos(u) + C$$

$$\int \tan(u) du = \ln|\sec(u)| + C$$

$$\int \sec^2(u) du = \tan(u) + C$$

$$\int \csc^2(u) du = -\cot(u) + C$$

$$\int \sec(u)\tan(u) du = \sec(u) + C$$

$$\int \csc(u)\cot(u) du = -\csc(u) + C$$

$$\int \cot(u)\cot(u) du = -\ln|\csc(u)| + C$$

9.4 Inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}(\frac{u}{a}) + C$$

$$\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C$$

$$\int \sin^{-1}(u) du = u\sin^{-1}(u) + \sqrt{1-u^2} + C$$

$$\int \cos^{-1}(u) du = u\cos^{-1}(u) - \sqrt{1-u^2} + C$$

$$\int \tan^{-1}(u) du = u\tan^{-1}(u) - \frac{1}{2}\ln(1-u^2) + C$$

10 Mean value theorem

Consider $f(x)$ as a defined continuos and differentiable function inside a closed interval $[a, b]$. The mean value theorem says that if these circumstances are matched, then there must be a point c, $a \leq c \leq b$, where $f'(c)$ will be equal to the average slope of the tangent line given by

$$\frac{f(b) - f(a)}{b - a}$$

In sum,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Where $f(x)$ is continuous and differentiable inside the interval $[a, b]$, a and b are the endpoints of said closed interval, and c is a random point between a and b; The derivative of $f(x)$ at $x = c$ will be parallel to the secant between a and $b$of $f(x)$ inside $[a, b]$. See the figure beside for better understanding.