Chap. 6 Elasticity

6.1. Introduction

Because the scope of this book is limited to the reactor pressure vessel and the associated piping and the fuel rods in the core, only a limited portion of the theory of elasticity needs to be addressed.

When a force, or *load*, is applied to a solid, the body changes shape and perhaps size. These changes are called *deformations*. The objective of stress analysis is to quantitatively relate loads and deformations. *Elasticity theory* deals with deformations sufficiently small to be reversible; that is, the body returns to its original size and shape when the load is removed. Within this range, the distances between atoms in the solid, or, alternatively, the atom-atom bonds, are stretched or compressed, but are not broken. In order to remain in the elastic range, the fractional changes in interatomic distances (on a microscopic scale) or in the body's gross dimensions (on a macroscopic scale) must be less than ~ 0.2%

When loads or deformations exceed the elastic range, the changes in shape of the body are not recovered when the load is removed. This type of irreversible deformation is termed *plastic deformation*. On a microscopic level, atomic bonds are broken and reformed between different atoms than in the original configuration. Plastic behavior of solids is treated in Chaps. 7 and 11. The maximum extent of plastic deformation is breakage or *fracture* of the material.

Instead of load (or force) and deformation, elasticity theory utilizes the related quantities *stress* and *strain*. Stresses are forces per unit area acting on internal planes in the body and strains are fractional deformations of the body.

6.1.1 Stresses and strains

Figure 6.1 shows a rod of cross sectional area A acted upon by force F. All planes perpendicular to the rod's axis experience the same force. The stress on plane a-a is:

$$\sigma_n = F/A \qquad (6.1)$$

The stress is called *normal* if it acts in the direction perpendicular to the plane. To determine the sign of the stress requires reference to a set of orthogonal coordinate axes such as that shown in the figure. The normal stress is positive if it also acts in the same direction. The load F in Fig. 6.1 generates a positive normal stress on plane a-a. This stress tends to pull atomic planes apart and is termed *tensile*. If the force F in Fig. 6.1 were reversed, the normal stress on a-a would also flip 180^o and the atoms in the solid would be squeezed together. Such a stress is called *compressive*. By convention, tensile stresses are positive and compressive stresses are negative.

In Fig. 6.2, the force F is applied perpendicular to the rod axis. The stress

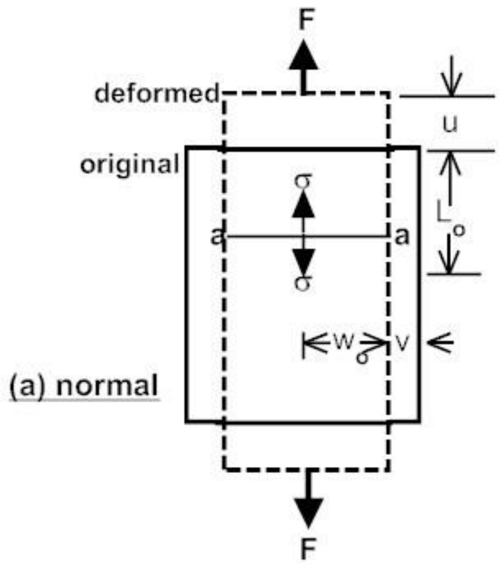

Fig. 6.1 Normal Stress

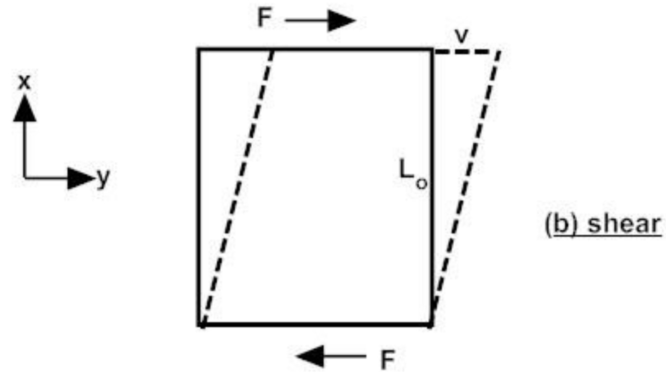

Fig. 6.2 Shear Stress

is generated parallel to the internal surface rather than perpendicular to the surface. Such a stress is termed a *shear* stress:

$$\sigma_s = F/A \tag{6.2}$$

These results are important in many aspects of stress analysis of structures; whether or not the applied load (force) is purely normal or purely shear relative to a coordinate axis, arbitrarily-oriented planes in the body will experience both normal and shear stress components.

Stresses are generated in structures in a number of ways, which include:

<u>Externally applied loads (Membrane stresses)</u>

- Mechanical loads represented by the discrete force F in Fig. 6.1. This type of loading is the basis of the uniaxial tensile test used to measure many mechanical properties of metals.
- Different pressures of a fluid (gas or liquid) on opposite faces of a structure. An example of this type of loading is the high pressure of water or steam in the primary system of a LWR. Components subject to this pressure loading are the primary system piping and the reactor pressure vessel.
- Reaction forces due to connection of a particular structure to its supports and to other components of a complex mechanical system. For example, the lower grid plate of a LWR supports the weight of hundreds of fuel assemblies. In this case, the stresses are induced by gravity.

<u>Thermal Stresses</u>

Stresses are generated when the expansion of a heated body is restrained. This topic is treated in detail in Sect. 6.6.

Residual Stresses

This type of stress arises from two principal sources:

- Fabrication of a component: processes such as cold-working (reduction in cross-sectional area by passing between dies) introduce internal stresses in the finished piece. These stresses remain during operation unless the piece is annealed at high temperature prior to use.
- Welding of two components: welding involves melting of a metal, and introduces large thermal stresses in the adjacent metal that does not melt (called the "heat-affected zone"). These stresses persist in the cooled piece and are supplemented by additional stresses arising from the solidification and cooling of the weld.

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6.1.2 Notation for stresses

In general, any point within a solid subjected to one or more of the loads just enumerated may possess as many as six components of stress. Three are normal and three are shear. The simple designations σ_n for normal stresses and σ_s for shear stresses need refinement in order to cover complex stress patterns. The convention adopted here is as follows: stress components are denoted with respect to the orthogonal coordinate system by which they are described (x, y, z for Cartesian; r, ϕ, z for cylindrical; r, θ, ϕ for spherical). *The stress components are labeled* $\sigma_{i\,j}$, *where i is the plane on which the stress component acts and j is its direction.*

The stress labeled σ in Fig 6.1 is properly denoted by σ_{xx}: it acts on the y-z plane, which is called the x plane after the direction of its normal; the stress component acts in the same direction, so the second subscript is also x. All normal stresses bear the generic designation σ_{ii}, which is often shortened to σ_i, with the understanding that the stress component acts on the i plane in the i direction. In situations where the type of stress component is obvious, subscripts are often entirely dispensed with. Because space is three-dimensional, there are at most three nonzero normal components of the stress state at a point.

Proper designation of shear stress components cannot be reduced to a single subscript because i and j in σ_{ij} are always different. The shear stress in Fig. 6.1b should be written as σ_{xz} because the stress acts on the x plane and in the z direction. A *moment* is the product of a stress and its distance from an axis. Equilibrium of the moments in a solid require that $\sigma_{ij} = \sigma_{ji}$ so that there are at most three nonzero shear components of the general state of stress.

6.1.3 Displacements and Strains

In what follows, definitions and derivations are given for two-dimensional Cartesian coordinates (i.e., x and y). This is done in order to minimize the complexity of the theory. Extensions to three dimensions or to other coordinate systems are stated without proof.

Displacements are changes in the position of a point in a body between the unstressed and stressed states. *Strain* is a fractional displacement. In common with stresses, displacements and strains come in two varieties, normal and shear.

Figure 6.1 shows the normal displacements u and v in the x and y directions, respectively, of a body subjected to a normal force acting uniformly on both horizontal surfaces. The original dimensions of the piece are L_o and w_o. The solid rectangle represents the stress-free solid and the dashed rectangle is its shape following application of the axial force. The displacement u (positive) corresponds to the outward movement of the horizontal surfaces and the displacement v (negative) represents the shrinkage of the body's sides.

The strains in the x and y directions are defined as fractional displacements:

$$\varepsilon_x = u/L_o \qquad (6.3)$$

$$\varepsilon_y = {}_v v/w_o \qquad (6.4)$$

The point (or in this case, the plane) selected to follow the displacement need not be an outer surface; displacements and strains of interior points are defined in the same manner.

Figure 6.2 shows a shear displacement v resulting from a force F acting on the side of the body at a distance L_o from the fixed bottom. The shear strain is defined as the ratio of these two lengths:

$$\varepsilon_{xy} = v/L_o \qquad (6.5)$$

The two-digit subscript notation has been used – shear movement occurs on the plane normal to the x axis and in the y direction. A more general definition of shear strain is the angular distortion of both edges of an initially rectangular section (see below).

Generalization of strain definition

Equations (6.4) and (6.5) need to be generalized for stress analyses. Figure 6.3 shows how this is done.

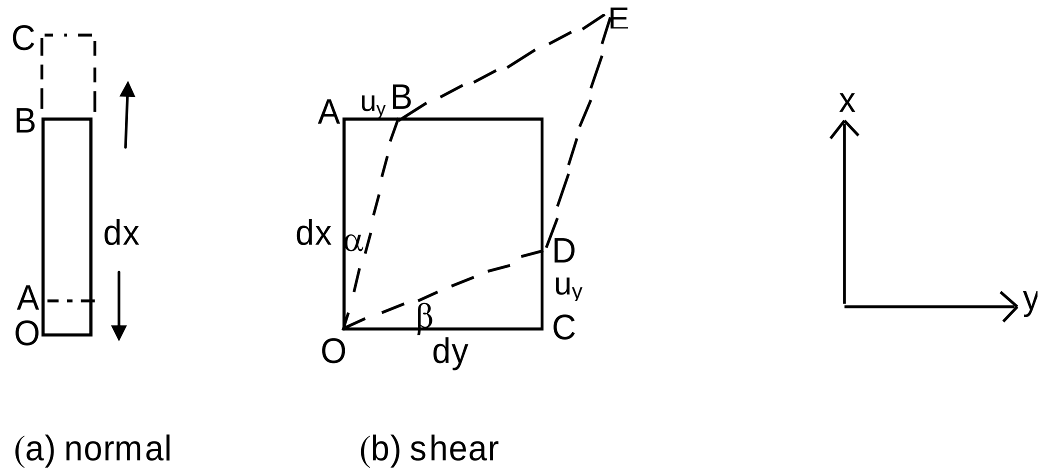


Fig. 6.3 Definitions of strains

The solid rectangles are the original shapes and the dashed figures represent the deformed shapes.

Normal strain

Figure 6.3a shows a deformation in the x direction. The original length of the body OB is taken to be a differential element dx. Upon application of a normal stress, the displacement of the bottom surface is OA = u_x. The displacement of the upper surface is BC = $u_x+(\partial u_x/\partial x)dx$, assuming that the body is continuous. The strain in the x direction, ε_x, is the change in length, (OB+BC-OA) – OB = $(\partial u_x/\partial x)dx$, divided by the initial length OB = dx. Expressing the lengths by their equivalents in terms of displacement (u_x) and normal strain (ε_x) yields:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \tag{6.6a}$$

The analogous equation for the y-direction gives:

$$\varepsilon_y = \frac{\partial u_y}{\partial y} \tag{6.6b}$$

The analogous formulas for the normal strains in cylindrical coordinates with angular symmetry (i.e., $\partial/\partial\theta = 0$) are:

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \qquad \varepsilon_\theta = \frac{u_r}{r} \qquad \varepsilon_z = \frac{\partial u_z}{\partial z} \tag{6.7}$$

For spherical geometry with spherical symmetry (i.e., $\partial/\partial\varphi = 0$, $\partial/\partial\theta = 0$)

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \qquad\qquad \varepsilon_\theta = \varepsilon_\varphi = \frac{u_r}{r} \tag{6.8}$$

In these equations, u_r is the radial displacement, φ is the polar angle, and θ is the azimuthal angle.

<u>Shear strain</u>
Generalization of Eq (6.5) is accomplished with the aid of Fig. 6.3b. The shear strain on the x – y (or y – x) plane is defined as the sum of the angles (in radians) of the deformed figure (dashed) relative to the stress-free shape (solid square); that is, $\varepsilon_{xy} = \varepsilon_{yx} \equiv \alpha + \beta$. Similar definitions apply to shear strains on the x-z and y-z planes. What remains is to connect the angles α and β to the displacements u and v.

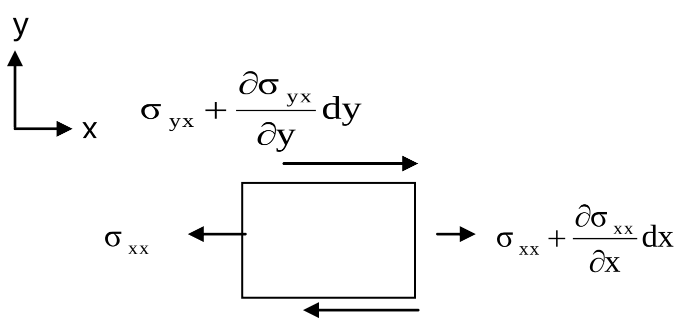
For simplicity, the lower left-hand corners of the original and deformed figures in Fig. 6.3b are superimposed at point O. Also, the corners B and D of the deformed figure are taken to lie on the sides of the original shape. With these simplifications,
$AB = (\partial u_y/\partial x)dx$ and $\tan\alpha \cong \alpha = AB/dx = \partial u_y/\partial x$.
$CD = (\partial u_x/\partial y)dy$ and $\tan\beta \cong \beta = CD/dy = \partial u_x/\partial y$

$$\varepsilon_{xy} + \varepsilon_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{6.9}$$

Whereas normal strain is the fractional deformation parallel to a chosen direction, shear strain consists of fractional deformations <u>perpendicular</u> to particular directions; that is, the tangents of the angles α and β in Fig. 6.3b are components of ε_{xy}. The shear-strain formulas for cylinders and spheres are not given here because all applications in this book involve only normal stresses in these geometries.

6.2 Equilibrium Conditions

The so-called equilibrium conditions of elasticity theory are consequences of Newton's third law: if a body is to remain stationary, the sum of the forces acting on it must be zero. This condition applies to all volume elements in a stressed body, and Fig. 6.4 provides relations between stress components.



σ_{yx}

Fig. 6.4 x-direction stresses on a volume element dx in length and dy in height

Figure 6.4 shows the basis for deriving the x-direction force balance for a two-dimensional Cartesian body. The balance of x-direction forces is:

$$\text{net x force } = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx - \sigma_{xx}\right)dy + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy - \sigma_{yx}\right)dx = 0$$

or:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad (6.10)$$

Note that the normal stresses σ_{ii} (where i=x,y,z,r,θ,z) can also be written as σ_i. Both expressions are used interchangeably in this chapter. Note also that Eq (6.10) is a <u>force</u> balance expressed in terms of stresses.

Comparable equilibrium conditions apply to the y- and z-directions, and the extension to three dimensions is straightforward (Ref. 1, Appendix)

In axisymmetric cylindrical coordinates, the radial equilibrium condition is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0 \qquad (6.11a)$$

and for the z direction:

$$\frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad (6.11b)$$

In spherical coordinates with spherical symmetry, the radial equilibrium condition is:

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad (6.12a)$$

and, by symmetry:

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} \qquad (6.12b)$$

6.3 Stress-Strain Relations

The final set of equations that forms the basis of elasticity theory relates stresses and strains. Contrary to the strain-displacement relations (Eqs (6.6) – (6.8)) and the equilibrium conditions

(Eqs (6.10) – (6.12)), the connection between stresses and strains involves material properties called *elastic moduli*.

6.3.1. Elastic Constants

Figure 6.1 represents the simplest loading configuration because it produces only one component of stress, the normal component σ_x. The axial strain defined by the first of Eqs (6.4) is related to σ_x by:

$$\varepsilon_x = \sigma_x/E \qquad (6.13)$$

where E is a material property called *Young's modulus* or the *modulus of elasticity*. For steels, E ~ $2x10^5$ MPa; E for aluminum is about one third that for steel. The nuclear fuel UO_2 has approximately the same Young's modulus as steel, but this correspondence has little to do with its mechanical performance in a reactor environment, as will be seen in subsequent chapters.

There is a stress limit (and consequently a strain limit) for the applicability of Eq (6.13) (see chapter 11). For steel, the proportionality of stress and strain implied in this formula fails at a stress of about 500 MPa, which is called the *yield stress*. At this point the strain is 0.2%. These conditions define the *elastic limit* of the material.

As suggested in Fig. 6.1, a positive displacement (or strain) in the x direction produces a negative displacement (or strain) in the y direction. In an isotropic solid, the other transverse direction (z) experiences the same strain as does the y direction, or $\varepsilon_z = \varepsilon_y$. The ratio of the magnitudes of the lateral strains to the axial strain in the uniaxial tensile situation of Fig. 6.1 defines *Poisson's ratio,* ν:

$$\nu = -\varepsilon_y/\varepsilon_x = -\varepsilon_z/\varepsilon_x \qquad (6.14)$$

Like E, ν is a material property, which, for most materials is ~ 1/3."

It is not surprising that the deformed body in Fig. 6.1 shrinks in transverse dimensions as it elongates axially, as otherwise, a substantial volume change would occur. However, the transverse shrinkage does not quite compensate for the axial elongation, and the solid does change volume as it deforms elastically.

6.3.2 Young's Modulus

On a microscopic scale, macroscopic elastic strains are caused by the stretching or contraction of interatomic bonds. Figure 6.5 depicts the microscopic response to a tensile stress applied to a crystal structure.

The origin of Young's modulus lies in the increase of the distance between the adjacent atoms a and b from ab in the unstressed condition to the a'b' with the applied stress. This stretching of the a-b bond

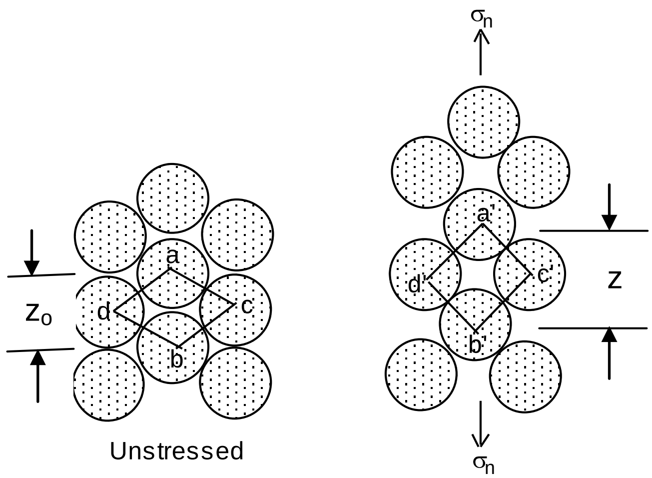


Fig. 6.5 Movement of atoms as a result of application of a tensile stress

is accompanied by an increase in the potential energy between the two atoms as shown in Fig. 6.6[1]. In the unstressed state, the atom separation ab lies close to the minimum on the

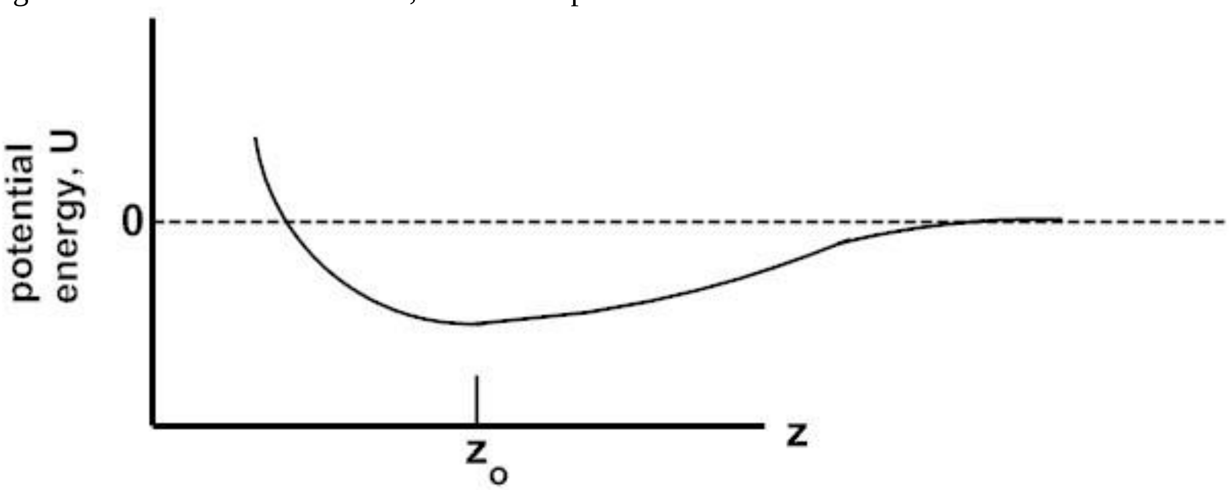


Fig. 6.6 Potential energy between atoms a and b in Fig. 6.5

[1] The curve in Fig. 6.6 represents the interaction energy between a pair of atoms; however interactions in metals, for example, are rarely exclusively pairwise.

potential curve (z_o). Increasing the separation distance from ab to a'b' is accompanied by an increase in the potential energy between the two atoms (U). Around its minimum, the potential energy curve around z_o can be approximated by a parabola:

$$U = U_o + \frac{1}{2}\left(\frac{d^2U}{dz^2}\right)_0 (z - z_o)^2 \qquad (6.15)$$

The force between the two atoms is the the gradient of the potential energy:

$$F = \frac{dU}{dz} = \left(\frac{d^2U}{dz^2}\right)_{z_o} (z - z_o) \qquad (6.16)$$

Macroscopically, the force is equal to the stress σ_n times the projected area of an atom, $\pi(z_o/2)^2$, where ½z_o is the atomic radius.

The normal stress is: $$\sigma_n = \frac{F}{\pi(z_o/2)^2} = \frac{(d^2U/dz^2)_{z_o}(z - z)_o}{\pi(z_o/2)^2} \qquad (6.17)$$

The normal strain is:

$$\varepsilon_n = \frac{z - z_o}{z_o} \qquad (6.18)$$

Combining the above two equations yields the expression for Young's modulus in atomic terms:

$$E = \frac{\sigma_n}{\varepsilon_n} = \frac{4}{\pi z_o}\left(\frac{d^2U}{dz^2}\right)_{z_o} \qquad (6.19)$$

Knowing the potential energy curve of Fig. 6.6 is required to calculate the second derivative in Eq (6.19). Such interatomic potentials can be accurately estimated by a type of microscopic computation called *molecular dynamics*.

6.3.3 Poisson's ratio

Poisson's ratio can be derived by the same detailed analysis used for the Young's modulus in the preceding section. However, the following simple method gives a close estimate.

First, the two diamond-shaped figures in Fig. 6.5 are superimposed, as shown in Fig. 6.7. The normal strain in the z direction, ε_n, and the normal strains in the transverse (y or z) directions, ε_t, are:

$$\varepsilon_n = \frac{aa'}{oa} \qquad \varepsilon_t = -\frac{cc'}{oc} \qquad (6.20a)$$

Where "o" is the midpoint between atoms a and b in Fig. 6.5. The deformation consists of moving point a to a', a positive displacement and point c moved to c', a negative displacement. The minus sign in the above equation accounts for the reduction of the segment oc to oc'.

Now oc is the height of the equilateral triangle abc. Since all sides of the quadrilateral adbc are equal to twice the atomic radius (see Fig. 6.5):

$$\frac{oc}{ac} = \sin 60 = \frac{\sqrt{3}}{2} \qquad \frac{oa}{ac} = \cos 60 = \frac{1}{2} \qquad (6.20b)$$

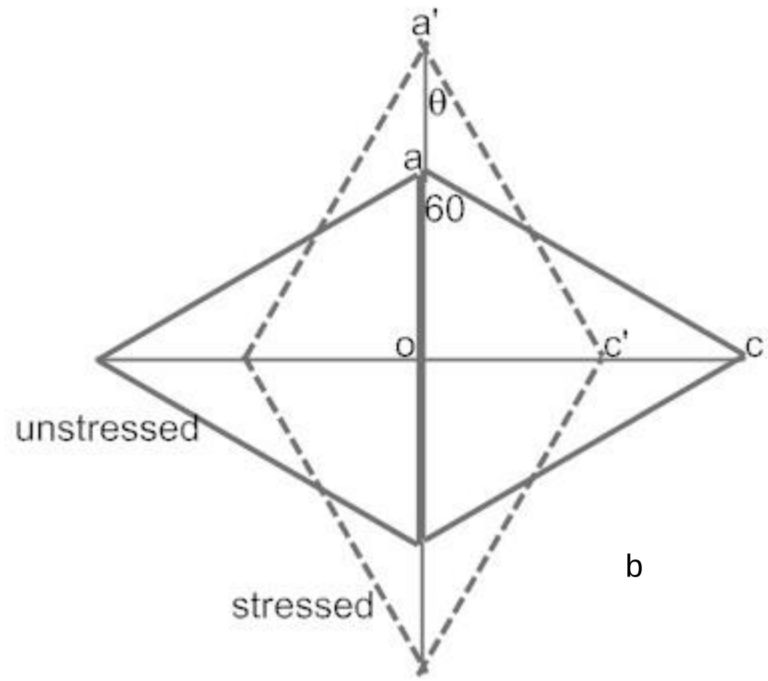


Fig 6.7 Diagram for calculating Poisson's ratio in the fcc structure
The solid quadrilateral is the left-hand drawing of Fig. 6.5 and the dashed one is the right-hand figure

Second, considering the angle θ in Fig. 6.7 and noting that from Fig. 6.5, a'c'= ac = atomic diameter:

$$\sin\theta = \frac{oc'}{a'c'} = \frac{oc - cc'}{ac} = \sin 60 - \frac{oc}{ac}\frac{cc'}{oc} = \sin 60(1 + \varepsilon_t)$$

$$\cos\theta = \frac{oa'}{a'c'} = \frac{oa + aa'}{ac} = \cos 60 + \frac{oa}{ac}\frac{aa'}{oa} = \cos 60\left(1 + \varepsilon_n\right)$$

Squaring and adding the above two equations yields:

$$1 = \sin^2 60(1 + \varepsilon_t)^2 + \cos^2 60(1 + \varepsilon_n)^2$$

Since the strains are always << 1 (for elastic deformations), the squared parenthetical terms can be expanded in a one-term Taylor series, which yields:

$$1 = {}^{3}\!/_{4}\,(1 - 2\varepsilon_t) + {}^{1}\!/_{4}\,(1 + 2\varepsilon_n) \qquad \text{or} \qquad \frac{\varepsilon_t}{\varepsilon_n} = \nu = \frac{1}{3}$$

$$1 = {}^{3}\!/_{4}\,(1 + 2\varepsilon_t) + {}^{1}\!/_{4}\,(1 + 2\varepsilon_n) \qquad or \qquad -\frac{\varepsilon_t}{\varepsilon_n} \equiv \nu = {}^{1}\!/_{3}$$

(6.21)

Although the above derivation was restricted to deformation of the close-packed plane of the fcc structure, similar results are obtained for other planes and other lattice types. Nearly all crystalline solids exhibit Poisson's ratios close to the above value of 1/3 of Eq (6.21).

6.3.4 Complete Stress-Strain Relations

Equation (6.13) is applicable when only one normal stress component acts on a body. When all six stress components are nonzero, the corresponding formulation is known as the *generalized Hooke's Law.* These equations are the same in all coordinate systems; instead of the subscripts x, y, and z, the coordinate axes are labeled 1, 2, and 3 so as to include cylindrical and spherical geometries as well as Cartesian coordinates. For the normal stresses and strains:

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\,(\sigma_2 + \sigma_3)\right] \qquad (6.22a)$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\,(\sigma_1 + \sigma_3)\right] \qquad (6.22b)$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu\,(\sigma_1 + \sigma_2)\right] \qquad (6.22c)$$

and for shear stresses and strains:

$$\varepsilon_{12} = \sigma_{12}/G \qquad \varepsilon_{13} = \sigma_{13}/G \qquad \varepsilon_{23} = \sigma_{23}/G \qquad (6.23)$$

6.3.5 Shear Modulus and Bulk Modulus

Shear modulus

In Eq (6.23), G is the *shear modulus,* which can be expressed in terms of E and ν as shown in the following analysis.

Figure 6.8(a) is a cross section of a square prism of side length L = EE' that is acted upon by a shear stress σ_S.

In Fig 6.8(b), the applied shear stresses are resolved into normal stresses of magnitude $\sigma_S/\sqrt{2}$ acting on the four EE' sides of the prism of side-length L and unit-length down the sides (into the paper).

In Fig. 6.8(c) the normal stresses in Fig. 6.8(b) are combined into components parallel to and perpendicular to the E-E diagonal of the prism. One, labeled σ_n'is tensile and the other, σ_n, is compressive. The area of the plane on which they act is the length of the EE diagonal ($\sqrt{2}$ L) times the unit length into the plane of the paper, or ($\sqrt{2}$ L) (1).

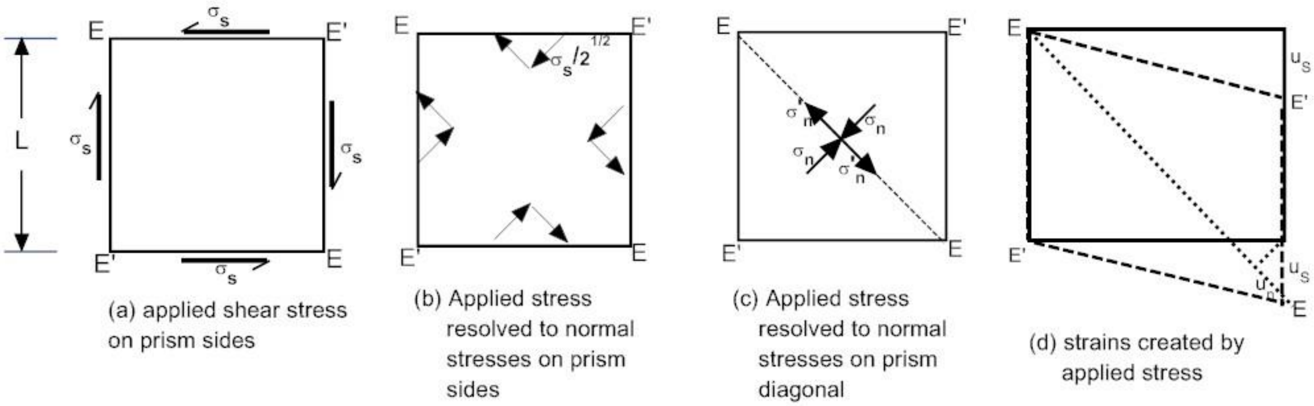


Fig. 6.8 Stresses and strains of a square prism subjected to pure applied shear stress σ_S

The magnitude of each stress component in (b) is $\sigma_S/\sqrt{2}$ and the force per unit length on the prism sides due to each is $(\sigma_S/\sqrt{2})L(1)$. Since there are two such components in each direction, the total normal force on the diagonal plane E-E is $2(\sigma_S/\sqrt{2})L(1)$. The length of the diagonal is $\sqrt{2}$ L , so conversion of the normal force to normal stress results in the stress components on the diagonal plane:

$$\sigma_n = -2\frac{\sigma_S}{\sqrt{2}}L(1) \times \frac{1}{\sqrt{2}\,L\,(1)} = -\sigma_s \tag{6.24a}$$

The minus sign indicates that the stress is compressive.

The same analysis applies to the components in Fig. 6.8(b) perpendicular to the compressive components. This yields:

$$\sigma'_n = \sigma_s \tag{6.24b}$$

This stress on the diagonal is tensile.

Figure 6.8(d) depicts the strains resulting from the applied shear stress in Fig. 6.8(a) σ_n and σ'_n produce displacements u_n of the diagonal and u_s of the downward location of the side. The normal strain of the diagonal is $u_n/(\sqrt{2}\,L)$, which can also be expressed by Eq (6.22a) with $\sigma_1 = \sigma'_n$, $\sigma_2 = \sigma_n$ and $\sigma_3 = 0$. Equating these two forms of the normal strain and using Eq (6.24) yields:

$$\frac{u_n}{\sqrt{2}L} = \frac{1}{E}(\sigma'_n - \nu o_n) = \frac{\sigma_S}{E}(1+\nu) \tag{6.25}$$

In the unstrained solid, the angle between the diagonal EE and a side E'E is 45°. When strained, shape changes are sufficiently small that u_n and u_s form the 45° right triangle shown in the lower right hand corner of Fig. 6.8d. Because u_S is the hypotenuse of this triangle, the two displacements are related by $u_S = \sqrt{2}\,u_n$.

From its definition (as ε_y in Eq (6.4)), the shear strain of the initially square figure is $\varepsilon_S = u_S/L$, or, with Eq (6.25),

$$\varepsilon_S = \frac{u_S}{L} = \frac{\sqrt{2}u_n}{L} = \frac{\sqrt{2}}{L}\left(\sqrt{2L}\right)\frac{\sigma_S}{E}(1+\nu) = \frac{2}{E}(1+\nu)\sigma_S$$

Comparing this result with the *shear modulus* G as defined by Eq (6.23) yields the desired connection between G, E and ν:

$$G = \frac{E}{2(1+\nu)} \tag{6.26}$$

Since Poisson's ratio is ~ 1/3 for most materials, this relation shows that the shear modulus is approximately 3/8 of Young's modulus.

Bulk Modulus

Another elastic constant, called the *bulk modulus*, K, describes the volume change due to normal stresses . If the normal stresses are due to the pressure in a fluid in which the body is immersed, $\sigma_1 = \sigma_2 = \sigma_3 = -p$. The stress in a solid corresponding to the pressure in a fluid is the *mean hydrostatic stress*, defined by:

$$\sigma_h = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{6.27}$$

The change in volume from initial value V to final value V_f is related to the normal strains by:

$$V_f/V = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) \cong 1 + (\varepsilon_1+\varepsilon_2+\varepsilon_3)$$

or, the fractional volume change is approximately equal to the sum of the normal strains:

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{6.28}$$

Adding Eqs (6.22a) – (6.22c) gives:

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{E}\left[(\sigma_1 + \sigma_2 + \sigma_3) - 2\nu(\sigma_1 + \sigma_2 + \sigma_3)\right]$$

Substituting the above equation and Eq (6.27) into Eq (6.28) yields:

$$\frac{\Delta V}{V} = \frac{3(1-2\nu)}{E}\sigma_h = \frac{\sigma_h}{K} \tag{6.29}$$

where the bulk modulus is[2]

$$K = \frac{E}{3(1-2\nu)} \tag{6.30}$$

Note the similarity between Eq (6.29) and the uniaxial form of Hooke's law, Eq (6.13). The linear strain in the latter is replaced by the volumetric strain in the former. In place of the axial stress in Hooke's law, the bulk compressibility formula employs the mean of the three orthogonal normal stresses. The bulk modulus K is the three-dimensional analog of Young's modulus.

6.4 Elastic energy density

Work is required to produce the elongation of the rod depicted in Fig. 6.1. No work is expended in creating the transverse displacement because no forces act on the rod's vertical sides. The external work performed by the force F is converted to the increase in the internal energy of the solid by stretching the interatomic bonds, as shown in Fig. 6.6. When divided by the volume of the body, the internal energy stored in the solid is called the *elastic energy density,* or sometimes simply the *strain energy*. This excess energy (over a perfect, unstressed solid) is the source of thermodynamic instability of defects such as, dislocations, grain boundaries, and voids.

Considering only the upper half of the deformed rod in Fig. 6.1, the work needed to produce the displacement u is $\int_0^u F du'$. This work is stored internally in the solid, and when divided by $L_o A$ to convert to an internal energy density yields:

$$E_{el} = \int_0^u \frac{F}{A}\frac{du'}{L_o} = \int_0^{\varepsilon_x} \sigma_x d\varepsilon'_x = \frac{1}{E}\int_0^{\sigma_x} \sigma'_x \, d\sigma'_x = \frac{\sigma_x^2}{2E}$$

where Hooke's law has been used to convert the variable of integration from strain to stress.

Similarly, the strain energy in pure shear can be deduced from the forces and displacements shown in Fig. 6.8. This body is a square prism of unit depth and side length L. Work is done only by the action of the force $\sigma_S L$ acting on the right hand face; the other three faces do no work because the dimensions remain fixed. The analog of the above equation for this case is:

$$E_{el} = \int_0^u \frac{(\sigma_S L)}{L}\frac{dv'}{L} = \int_0^{\varepsilon_s} \sigma_S d\varepsilon'_S = \frac{1}{G}\int_0^{\sigma_s} \sigma'_S \, d\sigma'_S = \frac{\sigma_S^2}{2G}$$

where the shear analog of Hooke's law (Eq (6.23)) has been used to convert strain to stress in the integral.

---

[2] Like the thermal expansion coefficient α, the bulk modulus is a thermodynamic property related to the coefficient of compressibility $\beta = -(1/V)(\partial V/\partial p)_T$. Comparison of this definition (in its integrated form) with Eq (6.29) wherein σ_h is replaced by $-p$ shows that $K = 1/\beta$

For the general case in which all six components of the stress are nonzero, the above equations are extended to:

$$E_{el} = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\nu}{E}\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) + \frac{1}{2G}\left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right) \quad (6.31)$$

This equation applies to all coordinate systems, whose orthogonal axes are designated by 1, 2 and 3.

6.5 Membrane Stresses in Cylinders and Spheres

The preceding sections dealt with the fundamentals of elasticity theory, including stresses, strains and displacements arising from externally applied loads (membrane stresses). In this section, the general theory is applied to membrane stresses in the simple geometries most frequently encountered in nuclear systems. For example, the fuel-rod cladding, the reactor pressure vessel and piping of the primary coolant circuit can be accurately modeled as annular cylinders with large radius-to-wall thickness ratios and infinite in length. Similarly, the hemispherical upper and lower heads connected to the pressure vessel wall are treated in spherical geometry.

These components operate essentially isothermally, so the stresses are due only to the difference in pressure across the wall. Thermal stresses due to temperature nonuniformities in a component are analyzed in Sect. 6.6.

6.5.1 Thin-Wall Cylinders and Spheres

If the radius of a hollow cylinder or sphere is much larger than the thickness of the wall, it can be treated in the so-called *thin-wall* approximation. This method replaces the mathematical machinery of elasticity theory with simple force balances. Because the wall is thin, the spatial stress distributions in it are replaced by average values.

Figure 6.9 shows a cross section of a thin-wall cylinder of radius R and wall thickness δ that is internally stressed by a pressure p. The pressure acts radially on the inner surface, and so must be resolved in the vertical direction in order to balance the azimuthal stress acting on the midplane section of the tube wall. The middle diagram of Fig. 6.9 shows the resolved upward force dF_{press} due to the pressure on an arc length $Rd\theta$ of the inner wall. The calculated upward pressure force per unit length perpendicular to the diagram, is:

$$F_{press} = pR\int_0^{\pi} \sin\theta \, d\theta = 2pR$$

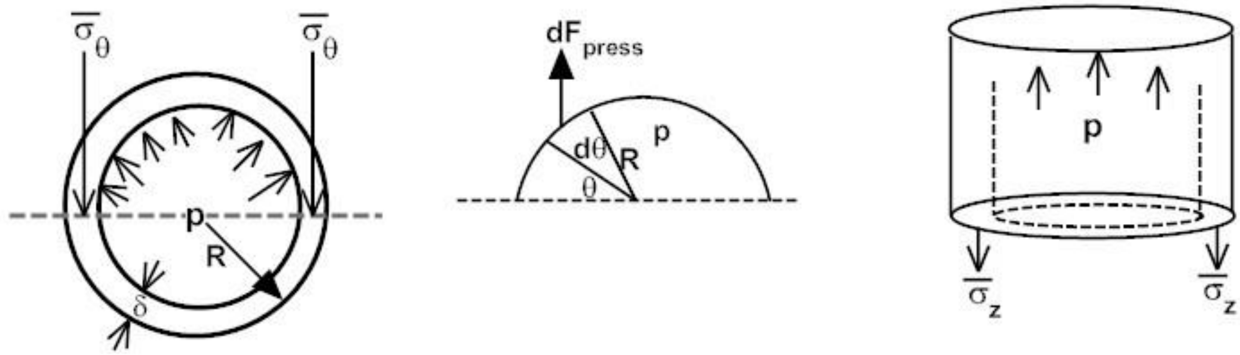


Fig. 6.9 Average stresses in a thin-wall cylinder

F_{press} is opposed by the azimuthal stress acting on the area δ (per unit length) on both sides of the cross section. This force is $F_{stress} = 2\delta\ \overline{\sigma}_\theta$. At equilibrium, $F_{press} = F_{stress}$, or:

$$\overline{\sigma}_\theta = \frac{pR}{\delta} \tag{6.32}$$

The average azimuthal stress is tensile.

In order to maintain a pressure in the tube, its ends must be closed. The right hand sketch in Fig. 6.9 shows the closed top of the tube and a cross section far enough removed from the top to avoid end effects that would invalidate the simple force-balance analysis. The force exerted on the closed top, $\pi R^2 p$, is balanced by the tensile force in the tube wall, $2\pi R\delta\ \overline{\sigma}_z$. Equating these two yields:

$$\overline{\sigma}_z = \frac{pR}{2\delta} \tag{6.33}$$

which is tensile and one-half as large as the azimuthal stress.

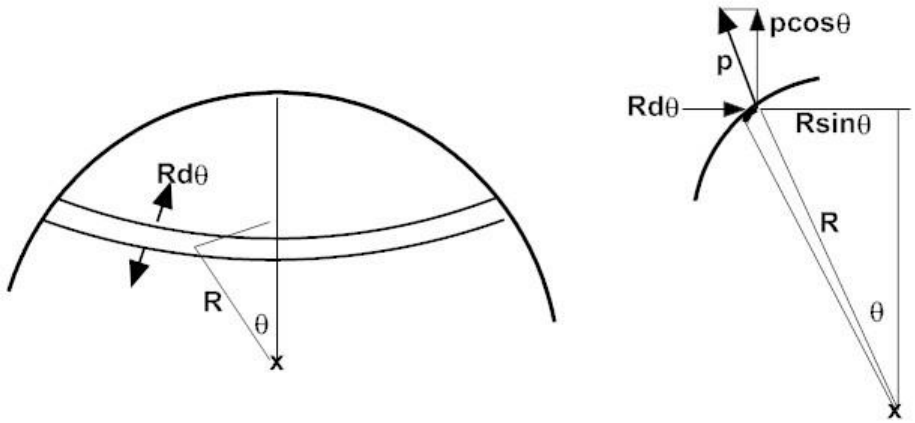
The radial stress on the inner wall is –p. Assuming zero external pressure, the average radial stress is:

$$\overline{\sigma}_r = -\frac{1}{2}p \tag{6.34}$$

which is small compared to $\overline{\sigma}_\theta$ and $\overline{\sigma}_z$ because the factor R/δ is large in thin-wall tubing.

<u>Example #1</u> The upper head of the reactor pressure vessel in a PWR is a hemisphere 4 m in diameter and 24 cm thick (see Fig.1.8). It is attached to the cylindrical wall of the pressure vessel by a flange with 8 cm diameter bolts. The internal pressure in the vessel is 15 MPa. How many bolts are required so that the stress in each does not exceed 2/3 of the yield stress (350 MPa)?

The bolts must resist the vertical component of the force due to the internal steam pressure acting on the inside of the hemispherical upper head. The differential area on the hemisphere on which the pressure acts is an annular strip of width Rdθ and periphery 2πRsinθ, or an area equal to $2\pi R^2 \sin\theta d\theta$. Here R is the radius of the pressure vessel and θ is the angle with respect to the vessel axis. Resolving the pressure stress in a vertical direction by multiplying p by cosθ and integrating the polar angle θ from 0 to π/2 yields:



$$F_{press} = 2\pi R^2 p \int_0^{\pi/2} \cos\theta \sin\theta d\theta = 2\pi R^2 p \int_0^1 \cos\theta d(\cos\theta) = \pi R^2 p$$

The stress in N_{bolts} of radius R_{bolt} resisting the pressure force is:

$$\sigma_{bolt} = \frac{\pi R^2 p}{N_{bolt} \pi R_{bolt}^2}$$

if σ_{bolt} is not to exceed 2/3 of the yield stress σ_Y, the required number of bolts is:

$$N_{bolt} = \frac{3}{2} \frac{p}{\sigma_Y} \left(\frac{R}{R_{bolt}} \right)^2 = \frac{3}{2} \frac{15}{350} \left(\frac{2}{0.04} \right) = 160$$

or, the spacing between bolts is $2\pi R/N_{bolt}$ = 7.8 cm.

Although the thin-wall approximation is useful for calculating average stresses, there is no way to compute the strains or displacements by the simple force-balance method used above. Full elasticity theory is needed to calculate the strains and displacements. This is treated in the following section.

6.5.2 Thick-wall Cylinders

In order to calculate stresses and strains in internally-pressurized cylindrical components which do not satisfy the thin-wall criterion employed above, the full set of elasticity equations must be utilized. As long as the analysis is not applied near the ends of the cylinder, only normal stresses and strains are developed. In addition, what was a planar cross section in the unstressed state remains planar when loaded. This condition is equivalent to requiring that the axial strain is not a function of radius. It is called the *plane strain* restriction.

The starting point is Eqs (6.8), (6.11a) and (6.22). The approach is to combine these equations so as to sequentially eliminate one variable at a time until a single equation with only one variable is obtained. The remaining variable is the radial stress so that the boundary conditions on the inner and outer walls of the cylinder can be applied.

<u>Example # 2a Radial stress in a long cylinder</u>

First, u_r is eliminated between the first two of Eqs (6.8):

$$\frac{d\varepsilon_\theta}{dr} = \frac{d}{dr}\left(\frac{u_r}{r}\right) = \frac{1}{r}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) = \frac{\epsilon_r - \epsilon_\theta}{r} \qquad (6.35a)$$

Next, the strains in Eq (6.35a) are replaced by the stresses using Eqs (6.22a) - (6.22c) with 1 = r, 2 = θ and 3 = z:

$$E\,\frac{d\epsilon_\theta}{dr} = \frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_z}{dr}\right) \qquad (6.22bb)$$

$$E\,\frac{d\epsilon_z}{dr} = \frac{d\sigma_z}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right) = 0 \text{ (plane strain)} \qquad (6.22cc)$$

eliminating $d\sigma_z/dr$ from Eq (6.22bb) with Eq (6.22cc):

$$E\,\frac{d\epsilon_\theta}{dr} = (1-\nu^2)\,\frac{d\sigma_\theta}{dr} - \nu(1+\nu)\,\frac{d\sigma_r}{dr}$$

Subtract Eq (6.22b) from Eq (6.22a):

$$E(\epsilon_r - \epsilon_\theta) = (1+\nu)(\sigma_r - \sigma_\theta)$$

Insert the above two equations into Eq (6.35a):

$$(1-\nu)\,\frac{d\sigma_\theta}{dr} - \nu\,\frac{d\sigma_r}{dr} - \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (6.35b)$$

From Eq (6.11a) (with $\sigma_{rz} = 0$):

$$\sigma_\theta = \sigma_r + r\,\frac{d\sigma_r}{dr} \qquad \frac{d\sigma_\theta}{dr} = r\,\frac{d^2\sigma_r}{dr^2} + 2\,\frac{d\sigma_r}{dr}$$

Substitute the above equations into Eq (6.35b):

$$r\,\frac{d^2\sigma_r}{dr^2} + 3\,\frac{d\sigma_r}{dr} = 0$$

or:

$$\frac{d}{dr}\left(r^3\,\frac{d\sigma_r}{dr}\right) = 0 \qquad (6.36)$$

which is readily integrated twice. Using the boundary conditions $\sigma_r(R_o) = 0$ and $\sigma_r(R) = -p$, where R and R_o are the inner and outer radii of the cylinder, respectively, the solution is:

$$\sigma_r = -p\frac{(R_o/r)^2 - 1}{(R_o/R)^2 - 1} \qquad (6.37)$$

The hoop stress is obtained by substituting the above equation into Eq (6.11a) (without the shear stress term):

$$\sigma_\theta = p\frac{(R_o/r)^2 + 1}{(R_o/R)^2 - 1} \qquad (6.38)$$

There remains to specify the axial end conditions in order to calculate σ_z and from this and Eqs (6.37) and (6.38), the strains and displacements. Three possibilities are shown schematically in Fig.6.10.

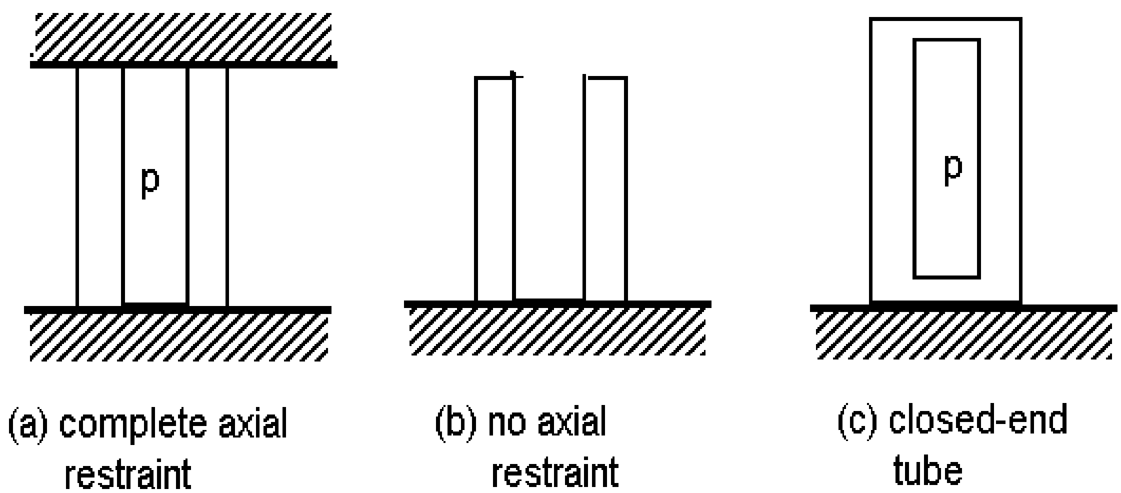


Fig. 6.10 End Conditions on a thick-wall cylinder

In case (a), the axial strain is zero. Setting $\varepsilon_3 = \varepsilon_z = 0$ in Eq (6.22c), inserting the radial and azimuthal stresses from Eqs (6.37) and (6.38) gives:

$$\sigma_{z(a)} = p\frac{2\nu}{(R_o/R)^2 - 1} \qquad (6.39a)$$

which is tensile, independent of r, and smaller than σ_θ.

In case (b), there is no axial stress because there is no stress exerted on the upper end of the annulus, or:

$$\sigma_{z(b)} = 0 \qquad (6.39b)$$

In case (c), the axial stress is obtained by equating the force on the inner surface of the upper end, $\pi R^2 p$, with the counterbalancing force in the annular cross section, $\pi(R_o^2 - R^2)\sigma_z$. This yields:

$$\sigma_{z(c)} = p\frac{1}{(R_o/R)^2 - 1} \qquad (6.39c)$$

Once all the stresses have been computed, the strains follow from Eqs (6.22a) – (6.22c).

<u>Example #2b</u> Limiting case of the thin-wall cylinder
With $\delta = R_o - R$, the denominator of Eqs ((6.37), (6.38) and (6.39c) reduces to:
$(R_o/R)^2 - 1 = (1 + \delta/R)^2 - 1 \sim 1 + 2(\delta/R) - 1 = 2(\delta/R)$
Eq (6.39c) reduces to:

$\sigma_z = pR/2\delta$, in agreement with Eq (6.33)

Because R/r is very close to unity for a thin wall, the numerator of Eq (6.38) is ~ 2 and $\sigma_\theta = pR/\delta$, which checks Eq (6.32)
Radial stress:

Approximate $r = R + ½\delta$, so $\frac{R_o}{r} = \frac{R_o/R}{1+\frac{1}{2}\delta/R} \cong \frac{R_o}{R}\left(1-\frac{1}{2}\frac{\delta}{R}\right)$ and

$$\left(\frac{R_o}{r}\right)^2 = \left(\frac{R_o}{R}\right)^2\left(1-\frac{1}{2}\frac{\delta}{R}\right)^2 \cong \left(\frac{R_o}{R}\right)^2\left(1-\frac{\delta}{R}\right)$$

$$\left(\frac{R_o}{r}\right)^2 - 1 = \left(\frac{R_o}{R}\right)^2 - 1 - \left(\frac{R_o}{R}\right)^2\frac{\delta}{R}$$

$$\frac{(R_o/r)^2 - 1}{(R_o/R)^2 - 1} = 1 - \frac{(R_o/R)^2(\delta/R)}{(R_o/R)^2 - 1} = 1 - \frac{(R_o/R)^2(\delta/R)}{2(\delta/R)} = 1 - \frac{1}{2}\left(\frac{R_o}{R}\right)^2$$

Since $R_o/R = 1 + \delta/R$, the above expression can be approximated by ½ Eq (6.37) reduces to σ_r = - ½ p, in agreement with Eq (6.34)

6.5.3 Spherical Shapes

Three spherically-shaped objects subject to mechanical loads are especially important in nuclear technology:

- The spherical shell with different internal and external gas pressures, such as the top head of the reactor pressure vessel analyzed in Example #1;
- Spherical voids or gas bubbles created by irradiation in solids (Problem 6.4)
- Solid spheres in the surrounding host solid. Included are atomic-size spheres such as interstitial atoms and macroscopic objects such as precipitates.

In all of these examples, determination of the stresses (generally all normal stresses) in the spherical object, or in the medium in which the sphere is embedded, is needed.

The normal stress components active in spherical geometry are the radial component σ_r, the polar component σ_θ and the azimuthal component σ_φ. None of the stress components are functions of φ or θ but depend on r only. This does not mean that σ_θ and σ_φ are zero, only that they are equal.

The present analysis is limited to the internally-pressurized shell. The mathematical manipulations of the elasticity equations are analogous to those employed for cylindrical geometry (Sect 6.5.2). The result is the ordinary differential equation (Prob.6.4a):

$$\frac{d}{dr}\left(r^4 \frac{d\sigma_r}{dr}\right) = 0 \qquad (6.40)$$

Integrating twice yields the general solution $\sigma_r = C_1 + C_2/r^3$, with the integration constants depending on the particular problem.

<u>Internally-pressurized spherical shell of inner radius R and outer radius R_o</u>

The boundary conditions are $\sigma_r(R) = -p$ and $\sigma_r(R_o) = 0$. The radial stress component is:

$$\sigma_r = -p\frac{(R_o/r)^3 - 1}{(R_o/R)^3 - 1} \qquad (6.41)$$

Substituting this result into the equilibrium condition of Eq (6.12a) yields the tangential stresses:

$$\sigma_\varphi = \sigma_\theta = p\frac{\tfrac{1}{2}(R_o/r)^3 + 1}{(R_o/R)^3 - 1} \qquad (6.42)$$

Note the strong similarity between these equations and the corresponding equations for the thick-wall cylindrical annulus (Eqs (6.36) – (6.38)).

In the thin-wall limit ($R_o - R = \delta << R$), Eq (6.42) reduces to:

$$\bar{\sigma}_\theta = \frac{pR}{2\delta} \qquad (6.43)$$

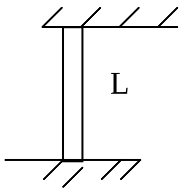
which is half of the hoop stress for the thin-wall cylinder (Eq (6.32)).

6.6 Thermal Stresses

Almost every component of a reactor core and the primary pressure system is subject to a nonuniform temperature distribution. Thermal gradients in the fuel are generated by internal heating due to the energy released by slowing-down of fission fragments. Gamma-rays heat metal structures. In the absence of internal heat sources, spatial temperature variations are produced by unequal surface temperatures. It does not matter whether the thermal gradients are steady-state or transient; the thermal stresses are dependent only on the instantaneous temperature distribution.

An unrestrained (i.e., stress-free) body heated from temperature T_o to a higher temperature $T_o + \Delta T$, expands in each of the three orthogonal coordinate directions by an amount $\alpha\Delta TL$, where L is the original length and α is the *linear coefficient of thermal expansion.*[3] When expansion is impeded, either internally or externally, *thermal stresses* are generated, as shown in the following example .

Example #3a: Bar between rigid ends A bar of length L with a linear thermal expansion coefficient α is placed between two rigid ends. At temperature T_o the bar is stress-free. The bar is then heated to temperature $T_o + \Delta T$ but the end pieces remain the same distance apart. What stresses develop?
First imagine that the top end piece is removed and the bar allowed to expand freely by an amount $\alpha\Delta TL$, or a thermal strain $\varepsilon_{th} = \alpha\Delta T$. Now a stress σ is applied to the top of the bar that is just sufficient to return



the bar to its original length and the top end piece replaced. The elastic strain in this step is $\varepsilon_{el} = \sigma/E$. The total strain is zero: $\varepsilon_{tot} = \varepsilon_{th} + \varepsilon_{el} = 0$, or the thermal stress in the bar is $\sigma = -\alpha E\Delta T$. The negative sign indicates that the stress is compressive.

- Another example of restrained thermal expansion is the generation of compressive stresses by uniform heating of a solid containing inclusions (precipitates) with a larger value of α than that of the host solid.

- Reactor power changes during startup or shutdown must be slow to avoid thermal stresses exceeding failure stresses.

- rapid injection of cold emergency cooling water is a potentially dangerous source of thermal stresses in the reactor pressure vessel

Thermal stresses are readily incorporated into the framework of elasticity theory by adding the thermal strain to the elastic strains given by Eqs (6.22)[4]:

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] + \alpha(T - T_o) \qquad (6.44a)$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right] + \alpha(T - T_o) \qquad (6.44b)$$

---

[3] α is a thermodynamic property; for isotropic solids, it is 1/3 of the volumetric coefficient of thermal expansion,
$\alpha_{vol} = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$, where v is the specific volume and p is pressure. α_{vol} is obtained from the equation of state of the solid. Uniform expansion occurs only in isotropic solids such as UO_2, but not non-cubic metals such as zirconium

[4] In the general case of an anisotropic solid, α, E and ν are different for the three coordinate directions.

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right] + \alpha(T - T_o) \qquad (6.44c)$$

T_o is the temperature at which thermal strains vanish.

6.6.1 Axi-Symmetric Cylindrical Geometry

The following analysis applies to infinitely long cylindrical annuli as well as to solid cylinders at locations far removed from the ends. In both cases, the temperature distribution is assumed to be a function of radial position only. The surface temperatures are not functions of azimuthal angle θ or axial location z.

The temperature distribution T(r) can be due to different temperatures at the inner and outer surfaces, internal heat generation in the wall or a transient temperature change in an initially isothermal cylinder. The radial and azimuthal thermal stress components are independent of the axial end conditions, although these affect the axial component of the stress (Example #3).

The thermal stresses produced by temperature gradients can be understood using a simple physical picture. Expansion of the hotter part of the body is resisted by the cooler part. Similarly expansion of the cold portion is enhanced by the greater volume increase of the contiguous hot part. As a result of these interactions, the hot zone is compressed and the cold zone is placed in tension. Between these two stressed zones lies a surface of zero stress.

Thermal stresses in long thick-wall and thin-wall tubes are analyzed below.

Radial stress

Following the same sequence of manipulations starting from Eq (6.35) but using Eqs (6.44a) and (6.44b) instead of Eqs (6.22a) and (6.22b) changes Eq (6.36) to:

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_r}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr} \qquad (6.45)$$

This equation is integrated for a long, thick-wall cylinder with inner radius R and outer radius R_o. Pressure loading is considered independently, so the boundary conditions for the thermal-stress state are :

$$\sigma_r(R) = \sigma_r(R_o) = 0$$

The result is:

$$\sigma_r = \frac{\alpha E}{1-\nu}\left[\frac{(1-R^2)/r^2}{R_o^2 - R^2}\int_R^{R_o} rT(r)dr - \frac{1}{r^2}\int_R^r r'T(r')dr'\right] \qquad (6.46)$$

As shown in Problem 6.2, for a thin-wall cylinder, the temperature profile is approximately linear:

$$T(r) = T(R) + (\Delta T/\delta)(r - R) = \Delta T(R/\delta)y \qquad (6.47)$$

where $\Delta T = T(R_o) - T(R)$ and

$$y = r/R - 1 << 1 \qquad (6.47a)$$

T(R) in Eq (6.47) has been dropped because constant temperature does not generate thermal stresses.

The thin-wall approximation reduces Eq (6.46) to:

$$\sigma_r = \tfrac{1}{2}\Delta T \frac{\alpha E}{1-\nu} y \left[1 - \frac{R}{\delta} y\right] \qquad (6.46a)$$

Azimuthal (hoop) stress

Using Eq (6.46) in Eq (6.11a), without the shear stress term, gives the hoop stress:

$$\sigma_\theta = \frac{\alpha E}{1-\nu}\left[\frac{(1-R^2)/r^2}{R_o^2 - R^2}\int_R^{R_o} rT(r)dr - \frac{1}{r^2}\int_R^r r'T(r')dr'\right] \qquad (6.48)$$

Substituting Eqs (6.47) and (6.47a) into Eq (6.48) and using the thin-wall condition where needed results in the hoop stress given by Eq (6.48a). The thermal hoop stress is negative (compressive) on the inner half of the cladding wall (because $\Delta T < 0$) and tensile on the outer half.

$$\sigma_\theta = \tfrac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R}{\delta} y\right) \qquad (6.48a)$$

Axial stress

The axial component of the thermal stress depends on the axial end conditions. The solution method for no axial restraint is outlined below.

Step 1: Eq (6.44c) with axial restraint ($\varepsilon_z = 0$) (prime denotes axial restraint):

$$\sigma'_z = \nu(\sigma_r + \sigma_\theta) - \alpha E[T(r) - T(R)]jh \qquad (6.49)$$

Step 2: Cross-section average:

$$\bar{\sigma}'_z = \frac{1}{\pi(R_O^2 - R^2)}\int_R^{R_O} 2\pi r \sigma'_z(r)dr$$

Step 3: remove axial restraint:

$$\sigma_z = \sigma'_z - \bar{\sigma}'_z$$

Example #4 Axial thermal stresses in thin-wall cladding

Step 1: The radial and azimuthal stresses are given by Eqs (6.46a) and (6.48a). Substituting these into the first term of Eq (6.49) and Eqs (6.47) and (6.47a) into the second term results in:

Step 1:

$$\sigma'_z = \tfrac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\nu\left[y\left(1 - \frac{R}{\delta}y\right) + 1 - 2\frac{R}{\delta}y - 2\frac{1-\nu}{\nu}\frac{R}{\delta}y\right] \cong A\left(\nu - 2\frac{R}{\delta}y\right)$$

where

A = $\frac{1}{2}\Delta T\,\frac{\alpha E}{1-\nu}$

Step 2: Change integration variable from r to y using Eq (6.47a).

$$\bar{\sigma}'_z = A\left(\frac{R}{\delta}\right)\int_0^{\frac{\delta}{R}}(1+y)\sigma'_z(y)dy = A\left(\frac{R}{\delta}\right)\int_0^{\frac{\delta}{R}}(1+y)\left(\nu-2\frac{R}{\delta}y\right)dy$$

$$\bar{\sigma}'_z \cong A\left[\nu\left(1+\frac{1\delta}{2R}\right)-1\right] \cong -A(1-\nu)$$

Step 3:

$$\sigma_z = \sigma'_z - \bar{\sigma}'_z = A\left(1-2\frac{R}{\delta}y\right) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R}{\delta}y\right) \qquad (6.49a)$$

σ_z is equal to the azimuthal stress (Eq (6.48a)) at all radial positions in the cladding wall. Because of the y preceding the bracketed term in Eq (6.46a), σ_r is very small compared to σ_θ and σ□ for thin-wall tubes.

6.6.2 Fuel-pellet cracking due to thermal stresses

Another important application of cylindrical thermal stresses is to the ceramic fuel pellets. Uniform volumetric heating in the solid cylinder cooled to temperature T_S at its periphery (R_o) generates a parabolic temperature distribution (see Chap. 9). With T_o on the axis, the distribution is given by Eq (9.10):

$$\frac{T-T_S}{T_o-T_S} = 1-\frac{r^2}{R_o^2} \qquad (6.50)$$

and

$$\frac{dT}{dr} = -\frac{2r}{R_o^2}(T_o-T_S)$$

Substituting the temperature gradient into Eq (6.45) results in:

$$\frac{d}{d\eta}\left(\eta^3\frac{d\sigma_r}{d\eta}\right) = 8\sigma^*\eta^3$$

where: $$\sigma^* = \frac{\alpha E(T_0-T_s)}{4(1-\nu)} \quad \text{and} \quad \eta = \frac{r}{R_o} \qquad (6.50a)$$

With the boundary conditions:

$$\frac{d\sigma_r}{d\eta} = 0 \;\; at \;\; \eta = 0 \qquad and \qquad \sigma_r = 0 \;\; at \;\; \eta = 1$$

the solution is:

$$\sigma_r^{th} = -\sigma^*(1-\eta^2) \qquad \text{and} \qquad \sigma_\theta^{th} = -\sigma^*(1-3\eta^2) \qquad (6.51)$$

The hoop stress was obtained from Eq (6.11a) without the last term.

Calculation of the axial stress distribution is a messy process which we do not repeat here. The analysis can be found on p. 393 of Ref. 3, leading to the result:

$$\sigma_z^{th} = -\sigma^*(2-4\eta^2) \qquad (6.52)$$

restriction means that The thermal stress solutions satisfy the *plane strain* condition ($d\varepsilon_z/dr = 0$) only near the midplane of the solid cylinder; the ends satisfy the *plane stress* condition (i.e., the stress is independent of radial position).

Example # 5 Thermal stresses in a UO_2 fuel pellet
A fuel pellet sustains a centerline-to-surface temperature difference of 530 K. The stresses due to the temperature gradient in the pellet are to be calculated. The important properties of UO_2 are:
Young's Modulus: E = 170 GPa
Poisson's ratio: $\nu = 0.3$
Thermal expansion coefficient: $\alpha = 1.5\times10^{-5}\ ^{o}C^{-1-}$

From Eq (6.50a):

$$\sigma^* = \frac{(1.5\times10^{-5})(170)(530)}{4(1-0.3)} = 0.5\ \text{GPA} = 500\ \text{MPa}$$

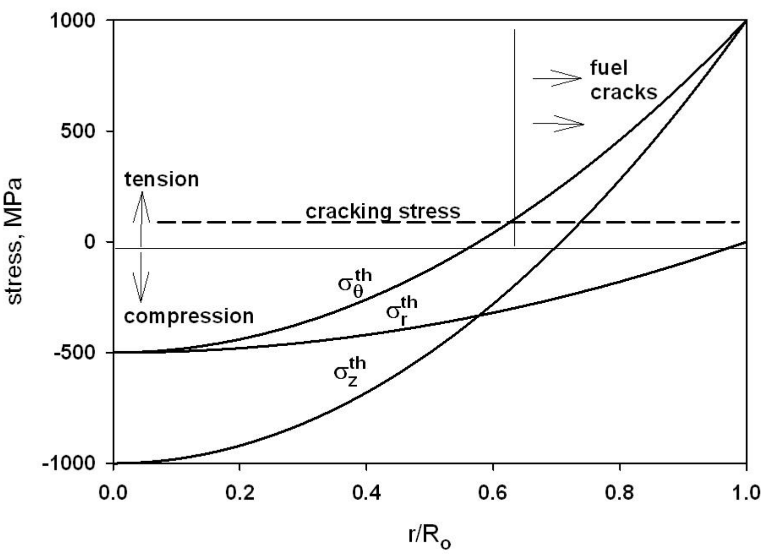


Fig. 6.11 Thermal stresses in a fuel pellet under irradiation

Figure 6.11 shows the thermal stress components developed in the UO_2 pellet. The stress at which UO_2 cracks (130 MPa) is shown as the horizontal dashed line in the plot. This stress is exceeded by σ_θ^{th} at a fractional radius of ~ 0.58, at which point radial cracks appear. Similarly all fuel beyond this location experiences radial and horizontal cracking. Similarly, σ_z^{th} becomes greater than the fracture stress at

$r/R_o > 0.75$, beyond which horizontal and vertical cracks extend to the pellet surface. The micrographs in Fig. 6.12 show the cracking pattern that results from the stress distribution in Fig. 6.11.

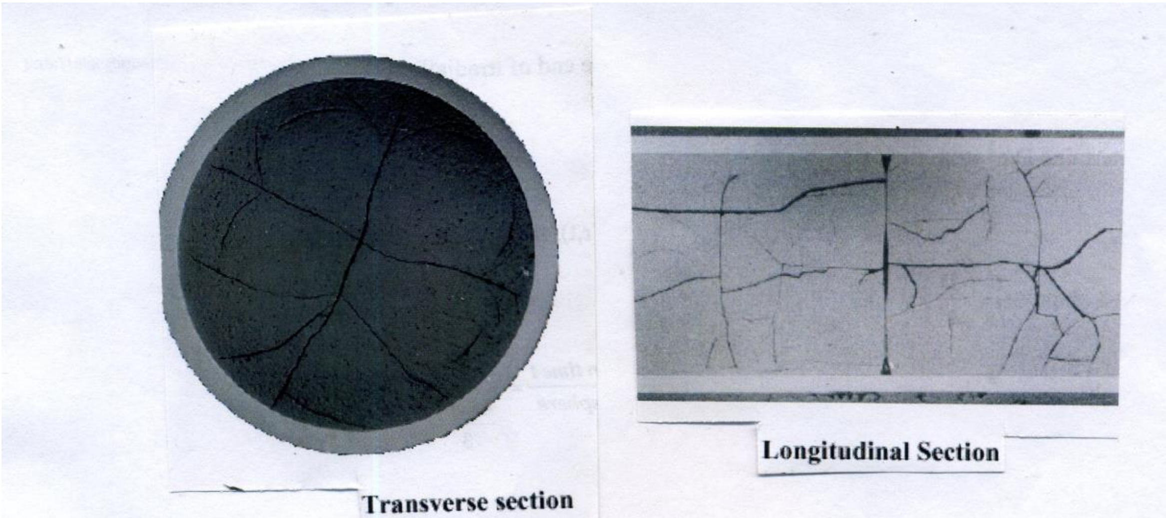


Fig. 6.12 Cracking in a UO_2 fuel pellet

Figure 6.13 illustrates another consequence of the temperature gradients in irradiated fuel. The "hourglass" shape of the pellet is due to the change from the plane-strain condition near the midplane to the plane-stress condition at the ends. The pellet ends contact the cladding resulting in an external shape that resembles a stalk of bamboo. Because hourglassing/bambooing risks breaching the cladding, specially-designed pellets have been developed to eliminate this potential problem (Chap 16).

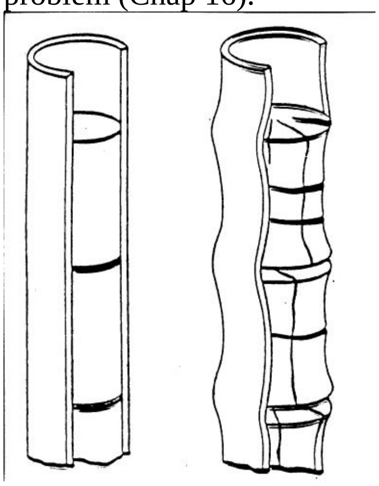


Fig. 6.12 Cracking and "hourglassing" in fuel pellets with uniform heat generation

6.6.3 Pellet Expansion

The thickness of the gas-filled gap between the fuel pellet and the cladding tube strongly affects heat transfer from, and consequently, the temperature of the fuel. The gap thickness decreases from its initial "cold" value as a result of the temperature distribution given by Eq (6.49) during operation. The gap thickness reduction is largely due to thermal expansion of the pellets, although the wedges of cracked pellets also tend to move out.

According to Eq (6.8), the strain component ε_θ evaluated at r = R is the fractional increase in the cylinder radius. ε_θ is expressed by Eq (6.44b) evaluated at r = R, where $\sigma_r = 0$ and $T = T_S$:

$$\varepsilon_\theta(R) = \frac{1}{E}\left[\sigma_\theta(R) - \nu\sigma_z(R)\right] + \alpha(T_S - T_{ref}) \qquad (6.53)$$

where T_{ref} is the pellet temperature in the cold state. The stresses are obtained from Eqs (6.50), (6.51) and (6.52):

$$\sigma_\theta(R) = \sigma_z(R) = \frac{\alpha E(T_o - T_S)}{2(1-\nu)} \qquad (6.54)$$

Combining the above two equations yields:

$$\varepsilon_\theta(R) = \Delta R / R = \alpha(\overline{T} - T_{ref}) \qquad (6.55)$$

where $\overline{T}$ = ½ ($T_o + T_S$) is the average fuel-pellet temperature. This simple result is valid only for the parabolic temperature distribution of Eq (6.49).

6.6.4 Thermal Stress Parameter

σ^* of Eq (6.51) is the basis of a measure of the ability of a material to withstand thermal stresses without cracking. It can be generalized by expressing the temperature difference in terms of Fourier's law as ΔT = qLk, where q is the heat flux, L is the distance over which ΔT occurs, and k is the thermal conductivity. The first two of these parameters are operational, in the sense that they can be changed at will. The thermal conductivity k, however, is a material property. Substituting the Fourier-law expression for ΔT into the definition of σ^* (Example #5), dividing by the fracture stress σ_F and omitting the product qL leaves a grouping of material properties that serves as a general measure of the thermal-stress resistance of the material:

$$\frac{\sigma_F(1-\nu)k}{\alpha E} \qquad (6.56)$$

This quantity is called the *thermal stress parameter*. Although developed from the solution for a solid cylinder, the concept is applicable to any geometry, with or without heat generation. The larger the thermal stress parameter, the more resistant is the solid to thermal stress failure. A high thermal conductivity is beneficial because, for a fixed heat flux, it reduces ΔT. A small coefficient of thermal expansion is desirable because it reduces the thermal strain (last term in Eq (6.11)). For zirconium, the thermal stress parameter is ~ 2×10^4 W/m. Ceramic oxides generally exhibit low thermal stress parameters, which is why they are susceptible to cracking in a thermal gradient. For UO_2, the thermal stress parameter is ~ 200 W/m.

References

1. D. R. Olander, "Fundamentals of Nuclear Reactor Fuel Elements", TID-26711-P1, Nat'l Technical Information Service (1976)

2. C. F. Bonilla, "Nuclear Engineering", Chap. 11, McGraw Hill (1957)

3. J. H. Rust, "Nuclear Power Plant Engineering", Chap. 8, Haralson Pub. Co. (1979)

Problems

6.1
Show that the solution in Sect. 6.6.2 satisfies the plain-strain condition.

6.2
Zircaloy cladding of a fuel rod is R = 0.5 cm radius and $\delta = 1$ mm thick. The coolant pressure is 15 MPa and the internal rod pressure is 20 MPa. The fuel rod is operating at a linear heat rate LHR = 200 W/cm. The cladding ID temperature is 370^o C,
Using the thin-wall approximation, calculate
(a) the elastic energy stored in a unit length of cladding. Include both membrane and thermal stresses.
(b) the thermal energy (relative to 25^{o}C) stored in the cladding.

Use the following properties of Zircaloy:

Linear thermal expansion coefficient @ 380^{o}C: $\alpha = 7.4\text{x}10^{-6}$ K^{-1}
Thermal conductivity: k = 0.14 W/cm-K
Heat capacity: C_P = 0.34 J/g-K
Density: ρ = 6.5 g/cm^3
Yield strength @ 380^{o}C: 1000 MPa (longitudinal); 1300 (transverse)
Young's modulus: E = 8x10^5 MPa
Poisson's ratio: $\nu = 0.41$

6.3
What error is incurred in calculating the average membrane hoop stress by the thin-wall approximation (Eq(6.32)) for Zircaloy cladding 1 cm in diameter and 1 mm wall thickness?

6.4
Spherical defects in solids, such as gas bubbles, voids and precipitates, create stress fields in the surrounding solid. Only the normal stress components, σ_r, σ_θ and σ_φ, are nonzero, and the last two are equal.

Following the method in Problem #2a of Sect. 6.5.2, derive Eq (6.40).

What are σ_r and σ_θ in the solid around a bubble of radius R and surface tension γ with internal pressure p?
What is the inward displacement of the radius of an equilibrium bubble if all gas is removed?
Solid properties: $E = 2x10^5$ MPa $\nu = 0.3$ $\gamma = 0.6$ N/m

6.5

A spherical precipitate particle of radius R pushes the surrounding stress-free solid outward by a distance $u_r(R) = \psi R$, where ψ is a small number.

Using the results of problem 6.4, what is the stress in the solid at the interface as a function of ψ?
What is u_r as a function of r?

6.6

Starting with Eq (6.9), prove that the thermal expansion of a solid cylinder with the parabolic radial temperature distribution of Eq (6.28) is given by Eq (6.32) of the notes. Here T_o and T_s are the centerline and surface temperatures, respectively, and T_{ref} is a reference temperature. The linear thermal expansion coefficient is α.

6.7

Determine the radial depth of penetration of vertical thermal-stress cracks in a fuel pellet with a parabolic temperature distribution
$T_o = 2000^oC$, $T_s = 500^oC$. Assume the following elastic properties for UO_2:
$\sigma_f = 140$ MPa
$\alpha = 1.0x10^{-5}\ K^{-1}$
$E = 1.6x10^5$ Mpa
$G = 5.9x10^4$ Mpa

6.8

Derive the expression for the stored elastic energy per unit length of a tube of radius R and wall thickness δ that is internally pressurized at a pressure p. The material has a Young's modulus E and a Poisson's ratio ν. Use the thin-wall approximation.

6.9

What error is incurred in calculating the average hoop stress by using the thin-wall approximation for cladding 1 cm inside diameter and 1 mm wall thickness? *Hint: this is most efficiently done by expanding the terms in Eq (6.23) in one-term Taylor series before taking the average.*
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