Nuclear Fuel Performance

NE-533

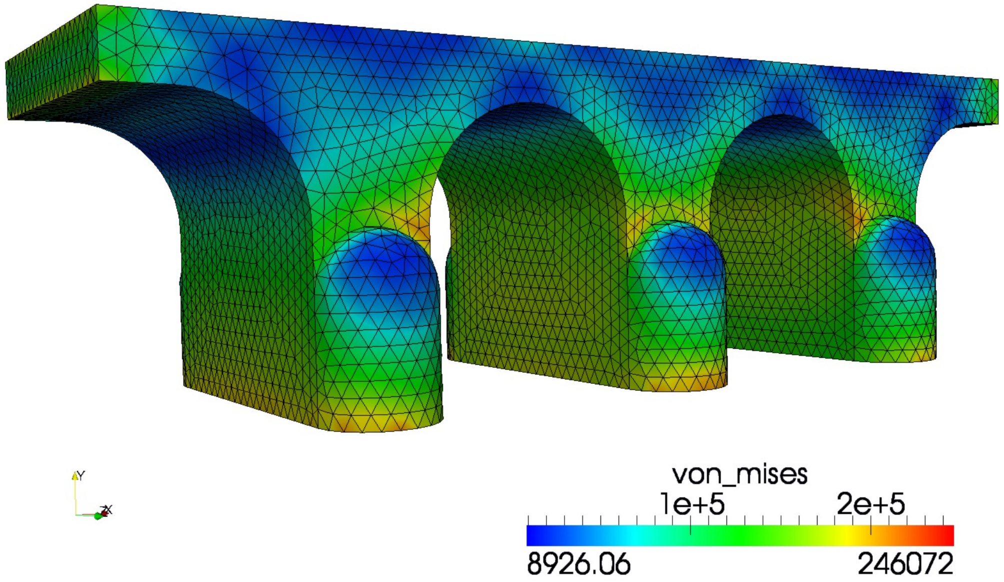
Spring 2024

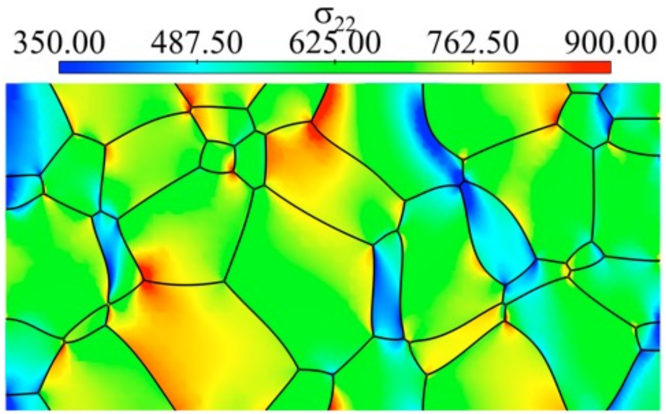
Last Time

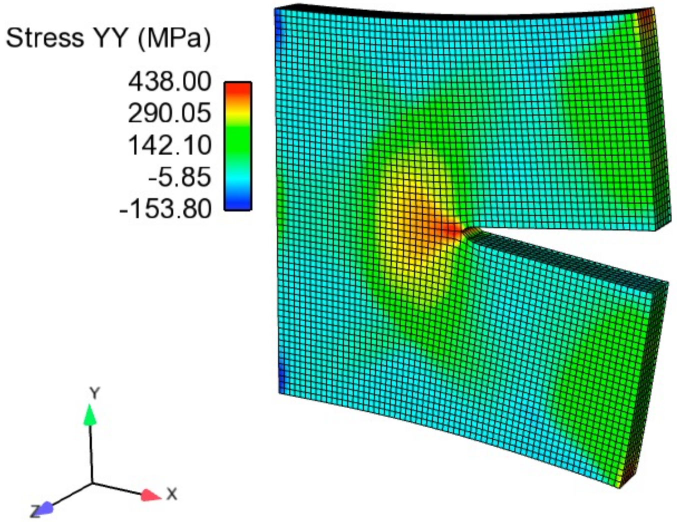
- Solid mechanics predicts the deformation of a body from its applied load
- Elastic deformation and plastic deformation
- Concepts of stress and strain
- Elastic moduli and Elastic constants
- Work hardening/strain hardening
- Toughness, ductility, DBTT, etc.

MECHANICS

Determine the stress and strain throughout a body



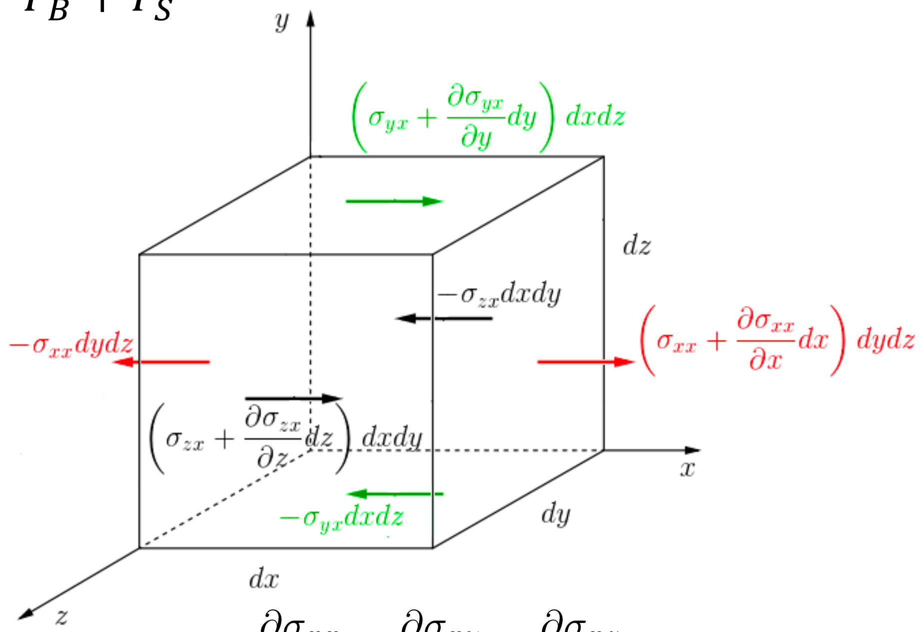




The stress divergence equation derivation

- Generalized momentum conservation: $\frac{d\vec{p}}{dt} = \vec{F} = \overrightarrow{F_B} + \overrightarrow{F_S}$
- Considering a cubic element, surface forces act on the walls of the cube
- Force on a wall is the product of the stress and the surface area
  – For wall at dx, approximate stress via Taylor expansion

$$\sigma_{xx}(x + dx) = \sigma_{xx}(x) + dx\frac{\partial \sigma_{xx}}{\partial x}$$



$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

$$F_p^x = (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx) dydz - \sigma_{xx} dydz + (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy) dxdz - \sigma_{yx} dxdz + (\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz) dxdy - \sigma_{zx} dxdy$$

$$F_p^x = \frac{\partial \sigma_{xx}}{\partial x} dxdydz + \frac{\partial \sigma_{yx}}{\partial y} dydxdz + \frac{\partial \sigma_{zx}}{\partial z} dzdxdy$$

$$0 = \nabla \cdot \boldsymbol{\sigma}$$

Simplified Cauchy for our typical system

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\boldsymbol{\sigma} + \rho\,\mathbf{g}$$

- Assumption 1: We have a static body

$$0 = \nabla\cdot\boldsymbol{\sigma} + \rho\,\mathbf{g}$$

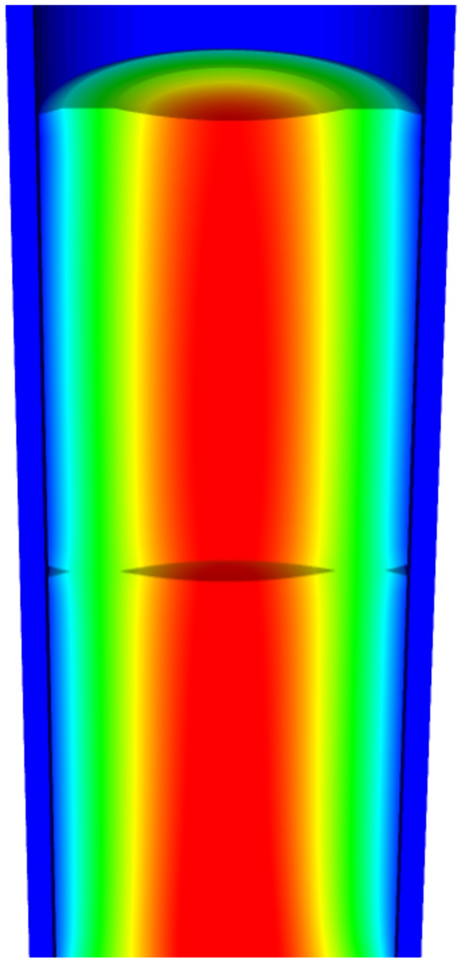
- Assumption 2: Gravity is negligible

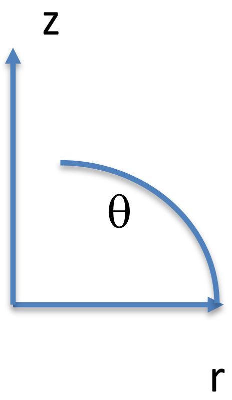
$$0 = \nabla\cdot\boldsymbol{\sigma}$$

- Assumption 3: The problem is axisymmetric

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial\sigma_{rz}}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial(r\sigma_{rz})}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} = 0$$





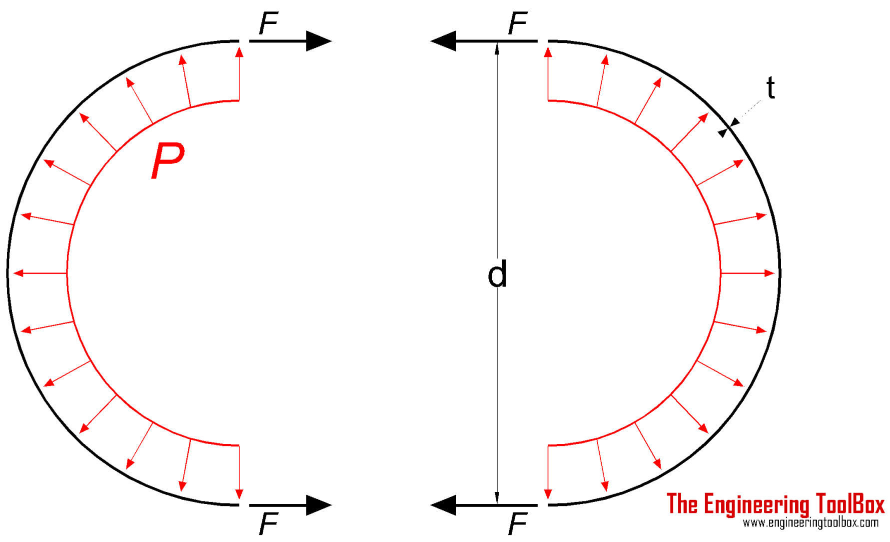
Solve for the stress throughout a pressurized cylinder (cladding tube) with thin walls

- Because our cylinder has such thin walls (δ = thickness), we can simplify everything to simple force balance
- The hoop stress is the force exerted circumferentially in both directions on every particle in the cylinder wall
- First, we need the Force per unit length due to the pressure

$$F_{press} = pR\int_0^{\pi} \sin\theta \, d\theta$$

- Utilize force to hoop stress relation: $F_{stress} = 2\delta \, \overline{\sigma_\theta}$
- Then we equate the forces and solve for the hoop stress

$$\overline{\sigma_\theta} = \frac{pR}{\delta}$$



Other two stresses for a thin-walled closed cylinder

- To find the stress in the z-direction we do another force balance

$$F_{top} = p\,\pi R^2 \qquad F_{wall} = 2\pi R\delta\bar{\sigma}_z$$

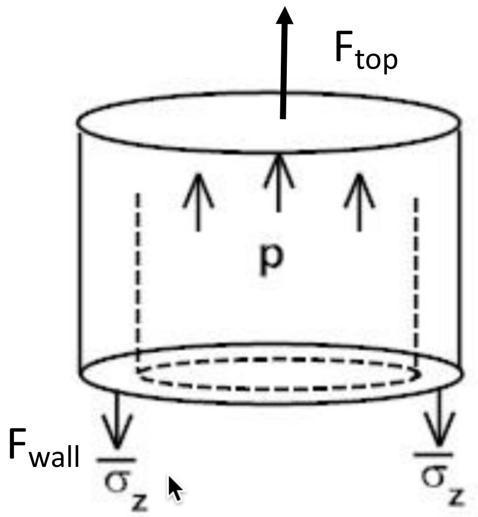
- Again, we equate the forces and solve for the stress

$$\bar{\sigma}_z = \frac{pR}{2\delta}$$

- Last, the stress on the inside of the wall is –p and on the outside is zero

$$\bar{\sigma}_r = -\frac{1}{2}p$$

$$\boxed{\bar{\sigma}_\theta = \frac{pR}{\delta} \qquad \bar{\sigma}_z = \frac{pR}{2\delta} \qquad \bar{\sigma}_r = -\frac{1}{2}p}$$



Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R = 0.55 cm, δ = 0.05 cm, σ_y = 381 MPa

$$\overline{\sigma_\theta} = \frac{pR}{\delta} \qquad \overline{\sigma_z} = \frac{pR}{2\delta} \qquad \overline{\sigma_r} = -\frac{1}{2}p$$

- The largest stress will be the hoop stress
- The hoop stress is P*(0.55/.05)
  - For 5 MPa, $\sigma\theta$ = 55 Mpa
  - For 9 MPa, $\sigma\theta$ = 99 Mpa
- With these pressures, we don't come even close to the yield stress of the cladding

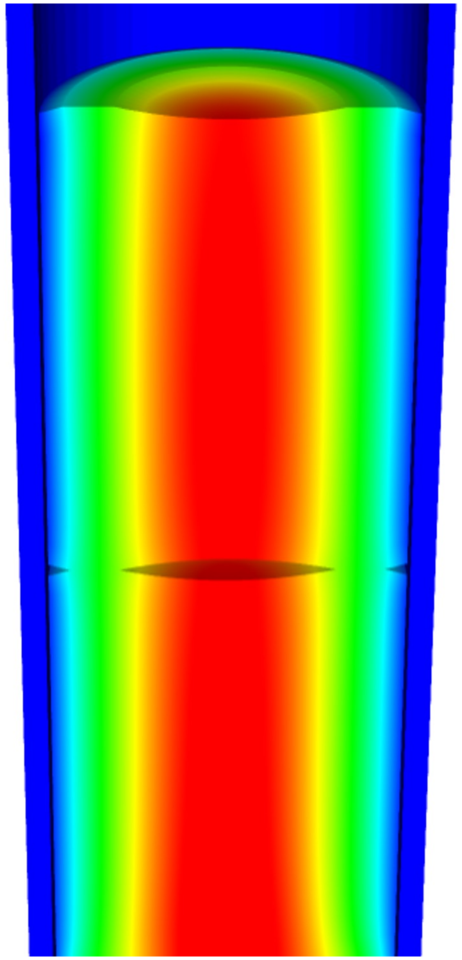
Develop constitutive relations for thick-walled

- We assume small strains, so the strain is defined with respect to displacement: u

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \;\; \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

- We assume isotropic material response:

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$
$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right)$$
$$\epsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$



Small strain theory

- In continuum mechanics, the infinitesimal strain theory is a mathematical approach to the description of the deformation of a solid body in which the displacements of the material particles are assumed to be much smaller than any relevant dimension of the body so that its geometry and the constitutive properties of the material at each point of space can be assumed to be unchanged by the deformation
- With this assumption, the equations of continuum mechanics are considerably simplified
- The small strain theory is commonly adopted in civil and mechanical engineering for the stress analysis of structures built from relatively stiff elastic materials

Stress within a pressurized cylinder that has thick walls (radius/thickness<20)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0 \qquad \frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- The cylinder as an inner radius R_i and an outer Radius R_o
- We assume there is no shear stress, so $\sigma_{rz} = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- We will begin with the r equation

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr}$$

- Now we need our constitutive law

$$E\,\epsilon_{\theta\theta,r} = (\sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \sigma_{zz,r}))$$
$$E\,\epsilon_{zz,r} = (\sigma_{zz,r} - \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r})) = 0$$

$$\epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz}))$$
$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}))$$
$$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr}))$$

- We combine these equations to get

$$E\,\epsilon_{\theta\theta,r} = \sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r})) \qquad E\,\epsilon_{\theta\theta,r} = (1-\nu^2)\,\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r}$$
$$E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

Develop equations for the stress within a pressurized cylinder with thick walls

- We need one more relationship from the definition of the strain

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \;\; \epsilon_{\theta\theta} = \frac{u_r}{r}; \qquad \epsilon_{\theta\theta,r} = \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r = \frac{1}{r}(\epsilon_{rr} - \epsilon_{\theta\theta})$$

- From the previous slide, we have

$$E\,\epsilon_{\theta\theta,r} = \left(1-\nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} \qquad E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- If we combine these, we get

$$(1-\nu^2)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} = \frac{1}{r}(1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- Again, from the previous slide

$$\sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr} \qquad \sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$$

- We end up with

$$r\,\sigma_{rr,rr} + 3\sigma_{rr,r} = 0$$

$$\frac{\partial}{\partial r}\left(r^3\frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

We now have an ODE so we can solve to get the stress

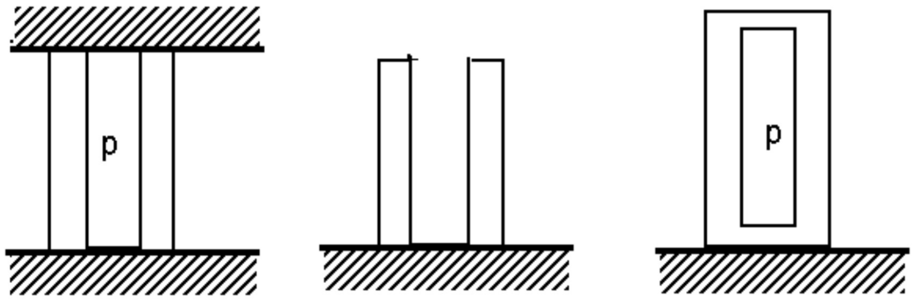
$$\frac{\partial}{\partial r}\left(r^3 \frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

- Our boundary conditions are $\sigma_{rr}(R_i)$ = -p and $\sigma_{rr}(R_o)$ = 0
- After integrating twice, we get $$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1}$$
- From σ_{rr}, we can get $\sigma_{\theta\theta}$ from $$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1}$$
- From the end condition, we determine σ_{zz}

(a) $$\sigma_{zz} = p\frac{2\nu}{(R_o/R)^2 - 1}$$

(b) $$\sigma_{zz} = 0$$

(c) $$\sigma_{zz} = p\frac{1}{(R_o/R)^2 - 1}$$



(a) complete axial restraint   (b) no axial restraint   (c) closed-end tube

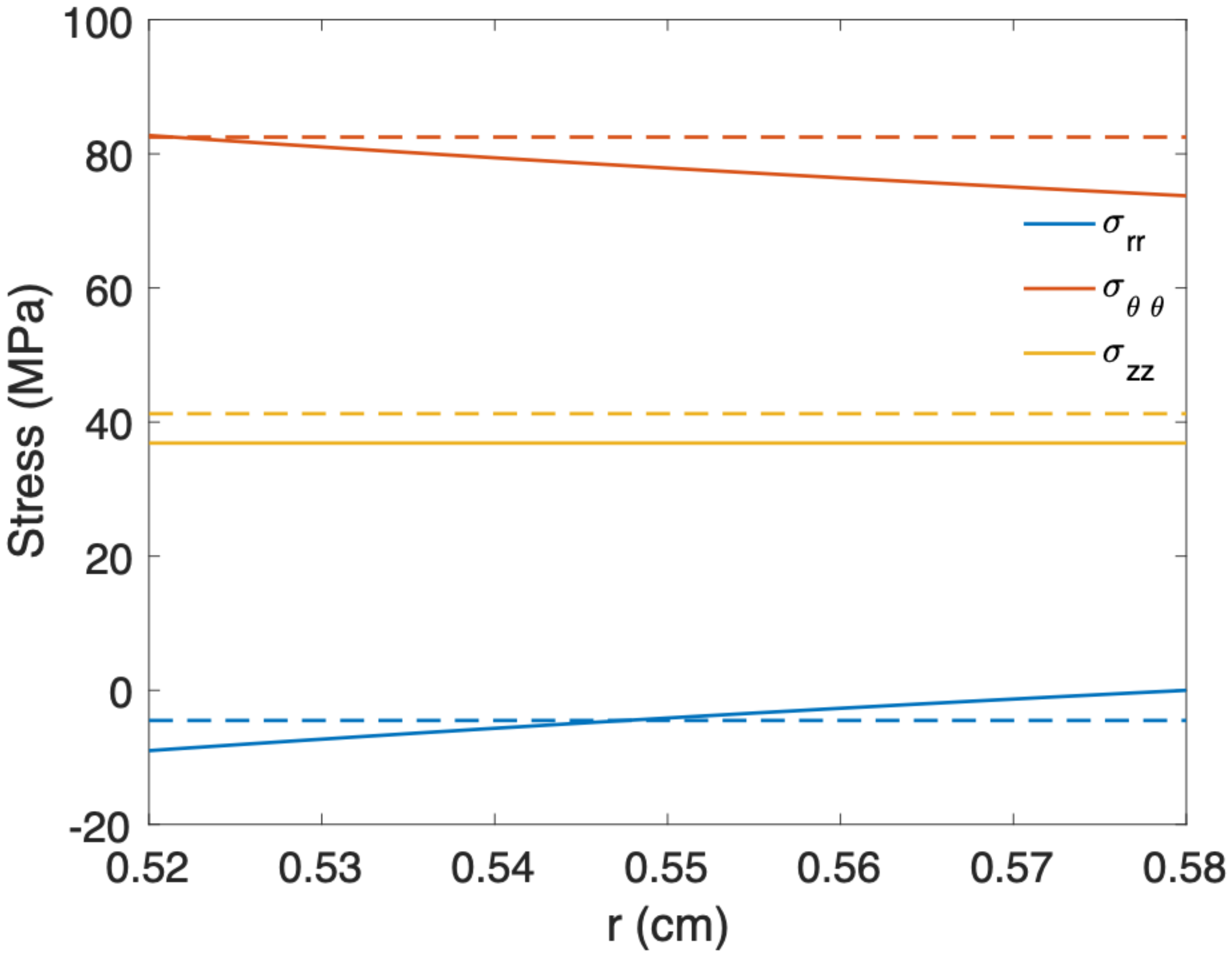
Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R_i = 0.52, R_o = 0.58 cm, σ_y = 381 MPa

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1} \quad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1} \quad \sigma_{zz} = p\frac{1}{(R_0/R_i)^2 - 1}$$

- Which stress is the largest?
  - $\sigma_{\theta\theta}$
- At what position r will the hoop stress be the largest?
  - r = R_i = 0.52 cm
- What is the stress at p = 5 MPa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 46.0 MPa
- What is the stress at p = 9 Mpa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 82.7 Mpa
- Again, we don't even get close to yielding

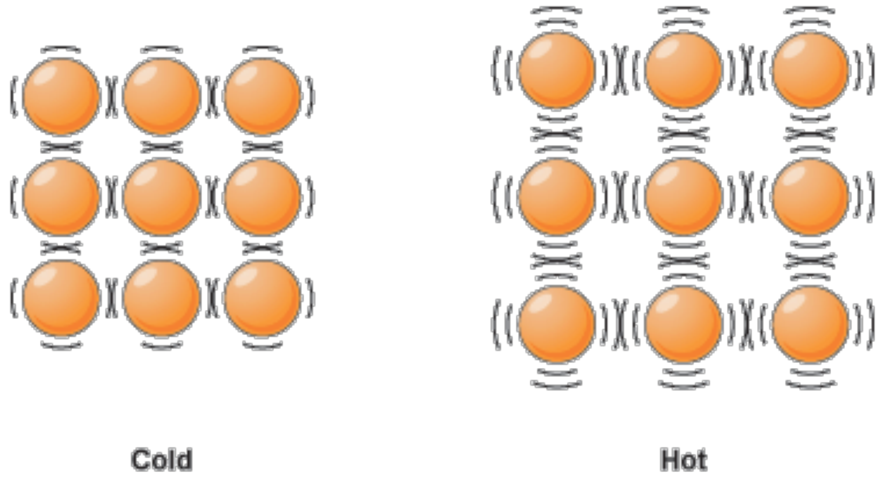
We've now solved this same problem assuming thin and thick-walled cylinders



THERMO-MECHANICS

Thermal Expansion

- As the temperature increases, atoms have larger vibrations, causing the material to expand
- This expansion doesn't raise the energy of the material when unrestrained
- This expansion can be treated as a strain, but as one that doesn't cause stress



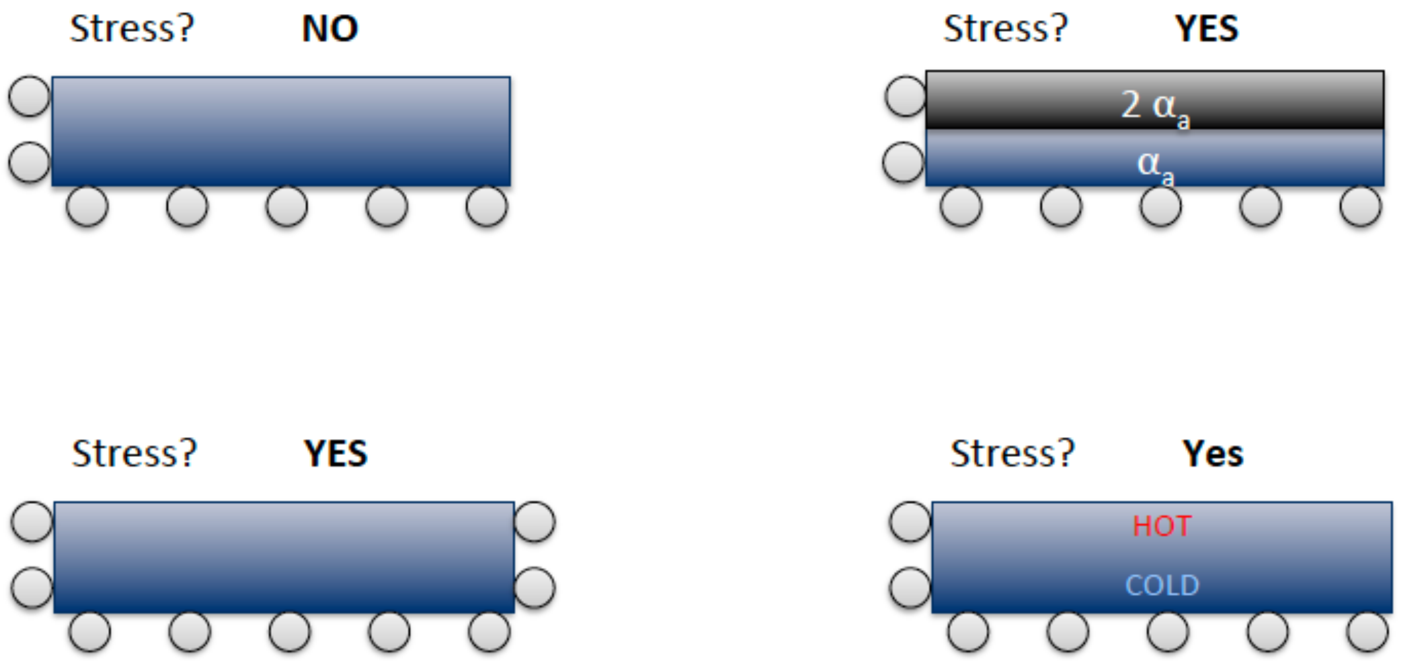
Thermal Expansion

- In isotropic materials, thermal expansion happens equally in all directions and is linear with temperature
- $\epsilon_0 = (T - T_0)\alpha I$
- In this equation
  - T is the current temperature
  - T_0 is the temperature the original size was measured
  - α is the linear thermal expansion coefficient
  - I is the identity tensor

Material	α (× 10^{-6} 1/K)
Aluminum	24
Copper	17
Steel	13
UO_2	11
Zircaloy (Axial)	5.5
Zircaloy (radial)	7.1

Thermal Expansion

- Though thermal expansion doesn't directly cause stress, it can still lead to thermal stress

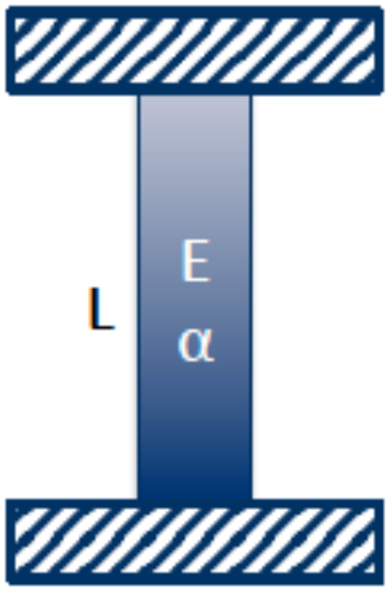


What is the stress in a thin constrained rod of length L when it is heated to ΔT?

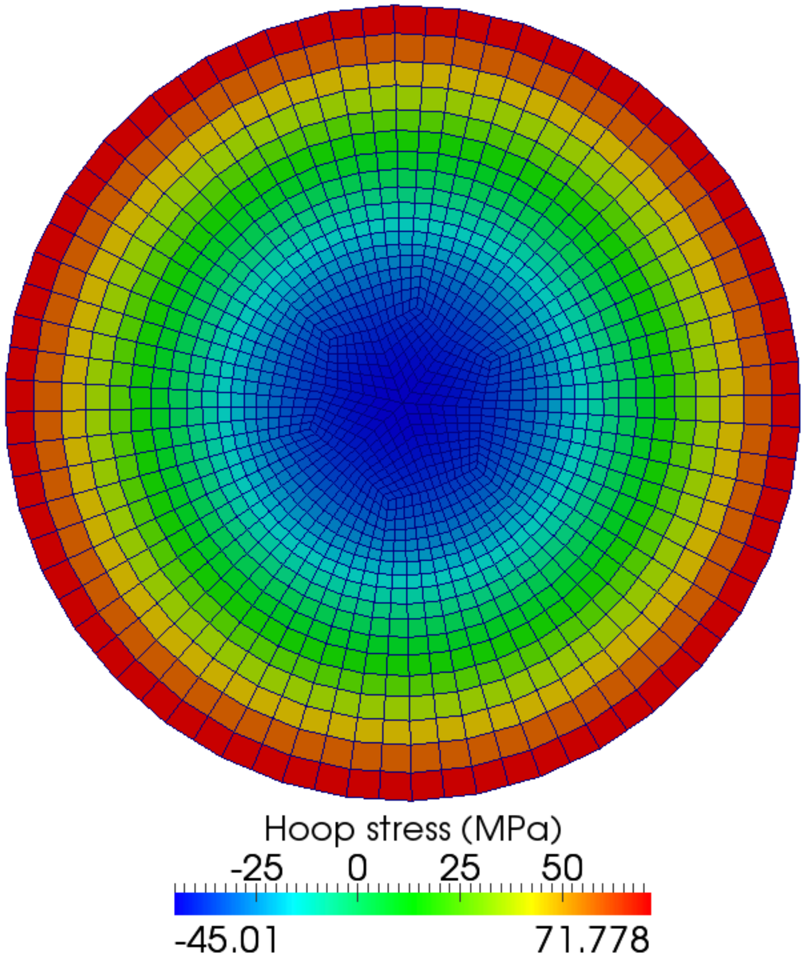
- The rod has a Young's modulus of E and an expansion coefficient of α

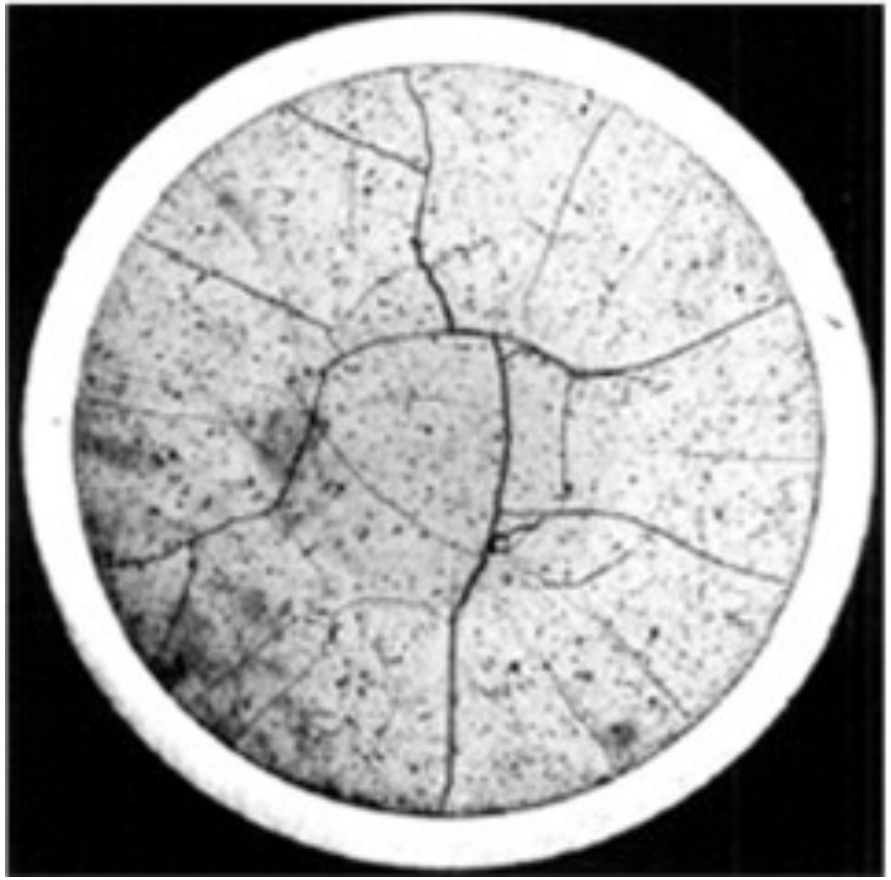
$\epsilon_T = (T - T_0)\alpha I \qquad \sigma = C\,(\epsilon - \epsilon_T)$

$$\epsilon_0 = (T - T_0)\alpha$$
$$\sigma = E\,(0 - \Delta T\alpha)$$
$$\sigma = -E\Delta T\alpha$$



The large temperature gradient within a fuel pellet results in large thermal stresses





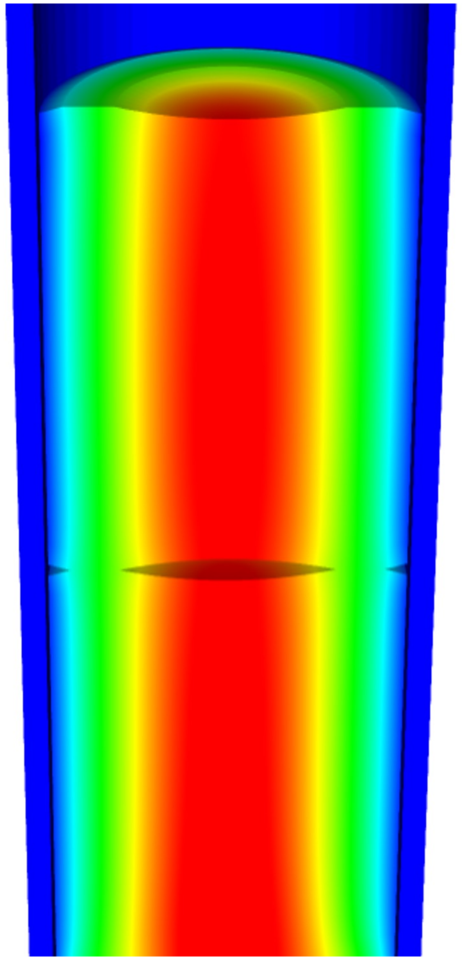
Consider the constitutive relations plus T

- We assume small strains, so the strain is defined as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r},\ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

- We assume isotropic material response, so

$$\begin{aligned}
\epsilon_{rr} &= \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})) + \alpha\Delta T \\
\epsilon_{\theta\theta} &= \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})) + \alpha\Delta T \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})) + \alpha\Delta T \\
\epsilon_{rz} &= \frac{1}{2G}\sigma_{rz}
\end{aligned}$$

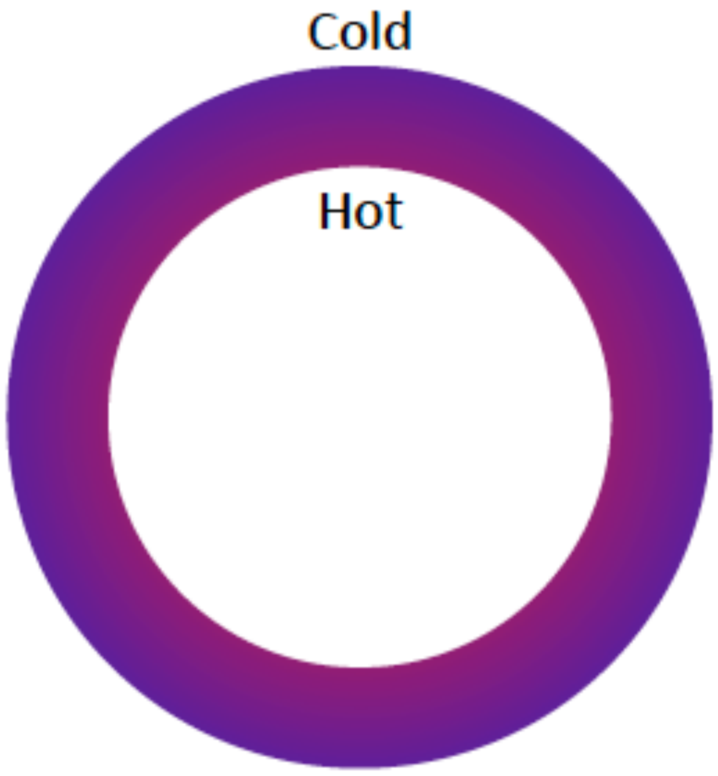


Consider a cylinder with thermal expansion but not pressure

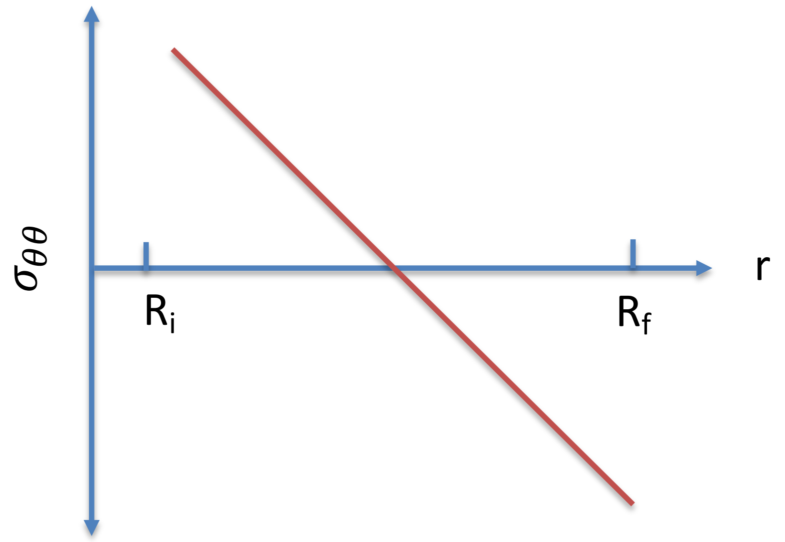
- $\sigma_{rr}(R_i) = \sigma_{rr}(R_0) = 0$
- Similar to the equations we worked through before
- $\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$
- Solving this ODE:

$$\sigma_{rr}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(\frac{r}{R_i}-1\right)\left(1-\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{zz}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$

What is the hoop stress in the cladding?



$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$



Where is hoop stress equal to zero?

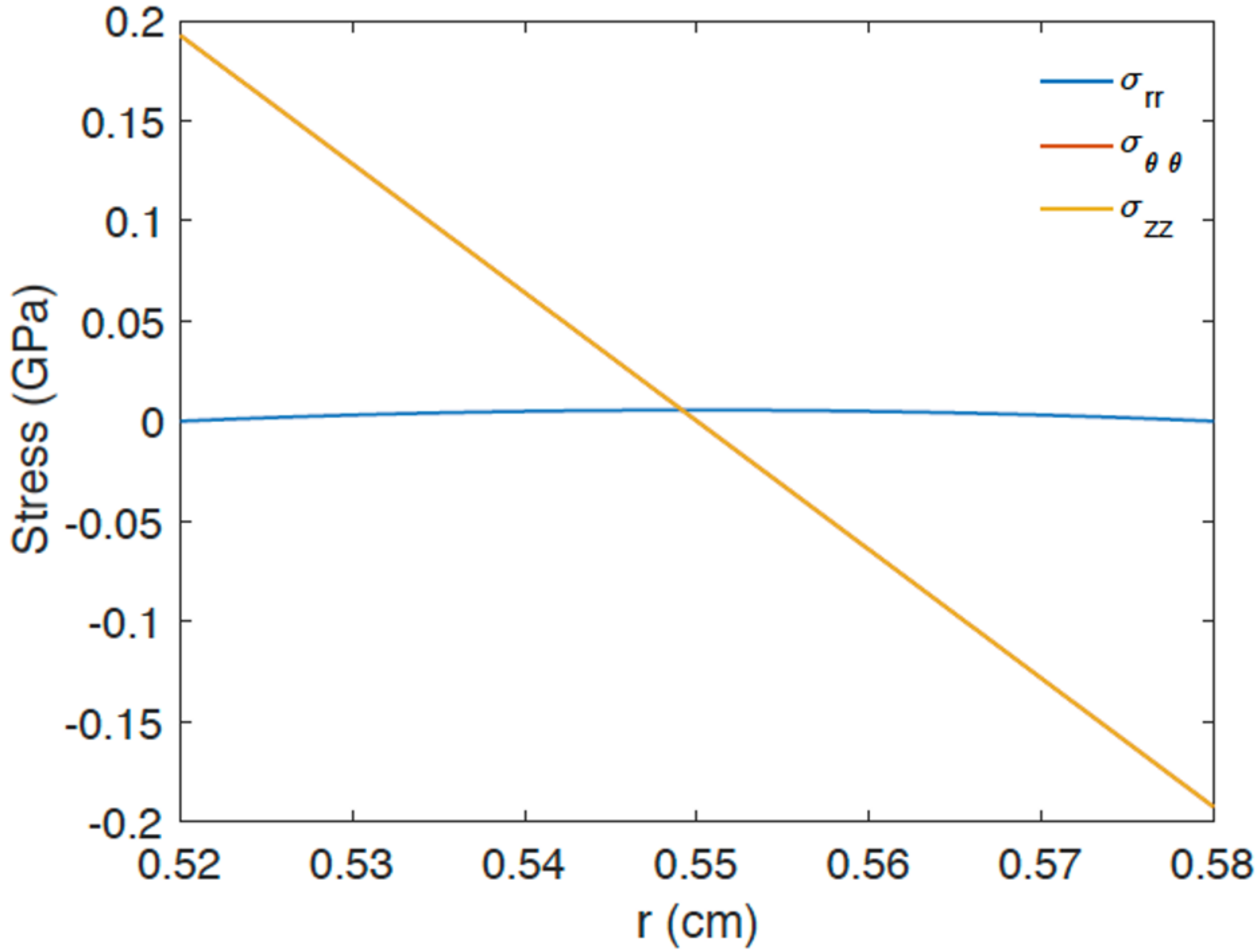
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$

$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0$$

$$\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0 \longrightarrow 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right) = 1 \longrightarrow \left(\frac{r}{R_i} - 1\right) = \frac{\delta}{2R_i}$$

$$r = \frac{\delta}{2} + R_i$$

The linear temperature gradient across the cladding causes axial thermal stresses



Same approach to the thermal stress in a fuel pellet

- The thermal stress is due to the temperature gradient

$$T(r) = \frac{LHR}{4\pi k}\left(1 - \frac{r^2}{R_f^2}\right) + T_s$$

$$T_0 - T_s = \frac{LHR}{4\pi k}$$

$$T - T_s = (T_0 - T_s)\left(1 - \frac{r^2}{R_f^2}\right)$$

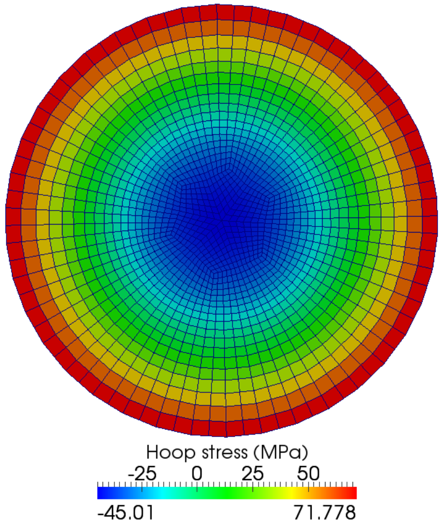
$$\frac{\partial T}{\partial r} = 2(T_0 - T_s)\left(\frac{r}{R_f^2}\right)$$

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$$

$$\eta = \frac{r}{R_f}$$

$$\sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

$$\frac{d}{d\eta}\left(\eta^3\frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^*\eta^3$$



Solve this stress ODE

- The boundary conditions are:

$$\frac{d\sigma_{rr}}{d\eta} = 0 \text{ at } \eta = 0$$

$$\sigma_{rr} = 0 \text{ at } \eta = 1$$

$$\frac{d}{d\eta}\left(\eta^3 \frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^* \eta^3 \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

- Once we solve it, we obtain

$$\boxed{\sigma_{rr}(\eta) = -\sigma^*(1-\eta^2)}$$

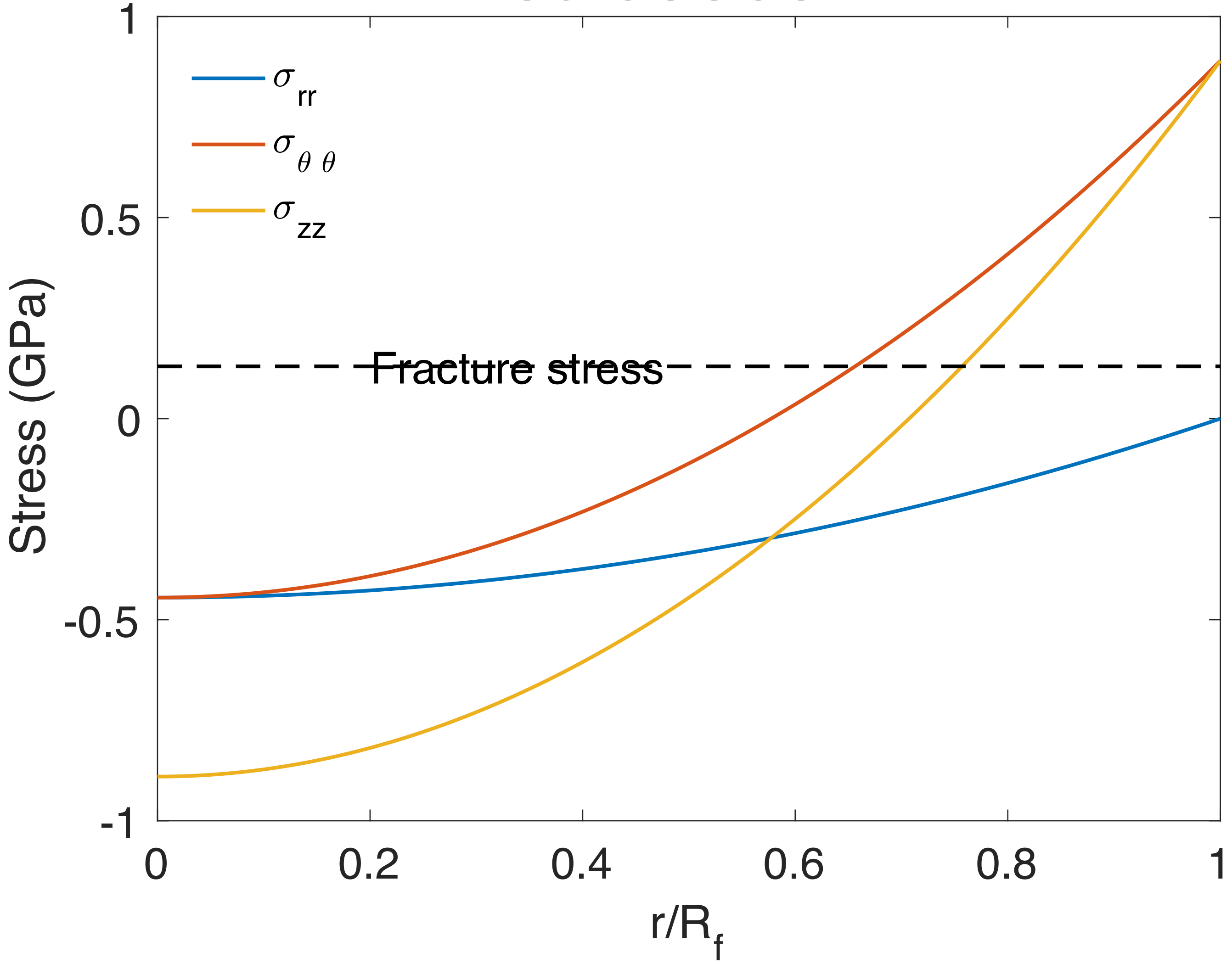
- Then we can solve the hoop stress

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \boxed{\sigma_{\theta\theta}(\eta) = -\sigma^*(1-3\eta^2)}$$

- The axial stress is more complicated to obtain, but you end up with

$$\boxed{\sigma_{zz}(\eta) = -2\sigma^*(1-2\eta^2)}$$

The fuel temperature gradient causes large thermal stresses



Example

How far do fuel cracks extend?

$$\sigma_{\theta\theta}(\eta) = -\sigma^*(1 - 3\eta^2) \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)} \qquad \eta = \frac{r}{R_f}$$

- E = 200 GPa, v = 0.345, α = 11.0e-6 1/K, σ_{fr} = 130 MPa, ΔT = 550 K
- Solve for η
  - $-\sigma_{fr} / \sigma^* = 1 - 3\eta^2$
  - $3\eta^2 = 1 + \sigma_{fr} / \sigma^*$
  - $\eta = ((1 + \sigma_{fr} / \sigma^*)/3)^{½}$
- σ* = 11.0e-6*200*550/(4*(1 – 0.345)) = 461.8 MPa
- η = sqrt((1 + 130/461.8)/3) = 0.65

Summary

- We derived two analytical solutions for the stress in a pressurized cylinder
  - Thin-walled cylinder
  - Thick-walled cylinder
- Increases in temperature cause most materials to expand
- We call this thermal expansion, which is a strain that doesn't inherently cause stress
- Thermal expansion can cause stress when
  - Deformation is constrained
  - There are gradients in the expansion coefficient
  - There is a temperature gradient
- We developed analytical equations for thermal stresses
  - in the cladding
  - in the fuel