PennState
College of Engineering

NucE 497: Reactor Fuel Performance

Lecture 13: Analytical solutions of the solid mechanics equations

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Mechanical and Nuclear Engineering

Content taken from Chapter 6 of Professor Motta's book

Today we will finish up the basic theory of solid mechanics and develop analytical solutions

- Module 1: Fuel basics
- Module 2: Heat transport
- Module 3: Mechanical behavior
  - **Introduction to solid mechanics**
  - **Analytical solutions of the mechanics equations**
  - Thermomechanics, thermal expansion
  - Solving equations in 1D numerically
  - Solving in multiple dimensions with FEM
  - Summary of fuel performance codes
- Module 4: Materials issues in the fuel
- Module 5: Materials issues in the cladding
- Module 6: Accidents, used fuel, and fuel cycle

Here is some review from last time

- What is the purpose of solid mechanics?
  a) To torture students
  b) To predict material transport
  c) To predict the deformation due to an applied load
  d) To predict the stretching of the atomic lattice

a) What is a stress? Choose all that apply
  a) A force per unit area
  b) A property of a deformed material
  c) A response of a material
  d) The derivative of the energy density

Quiz Question: Which elastic constant defines the ratio of decrease in a lateral measurement to the increase in length?

a) Young's modulus
b) Shear modulus
c) Poisson's ratio
d) Zener ratio

Attempts: 39 out of 39

+0.26

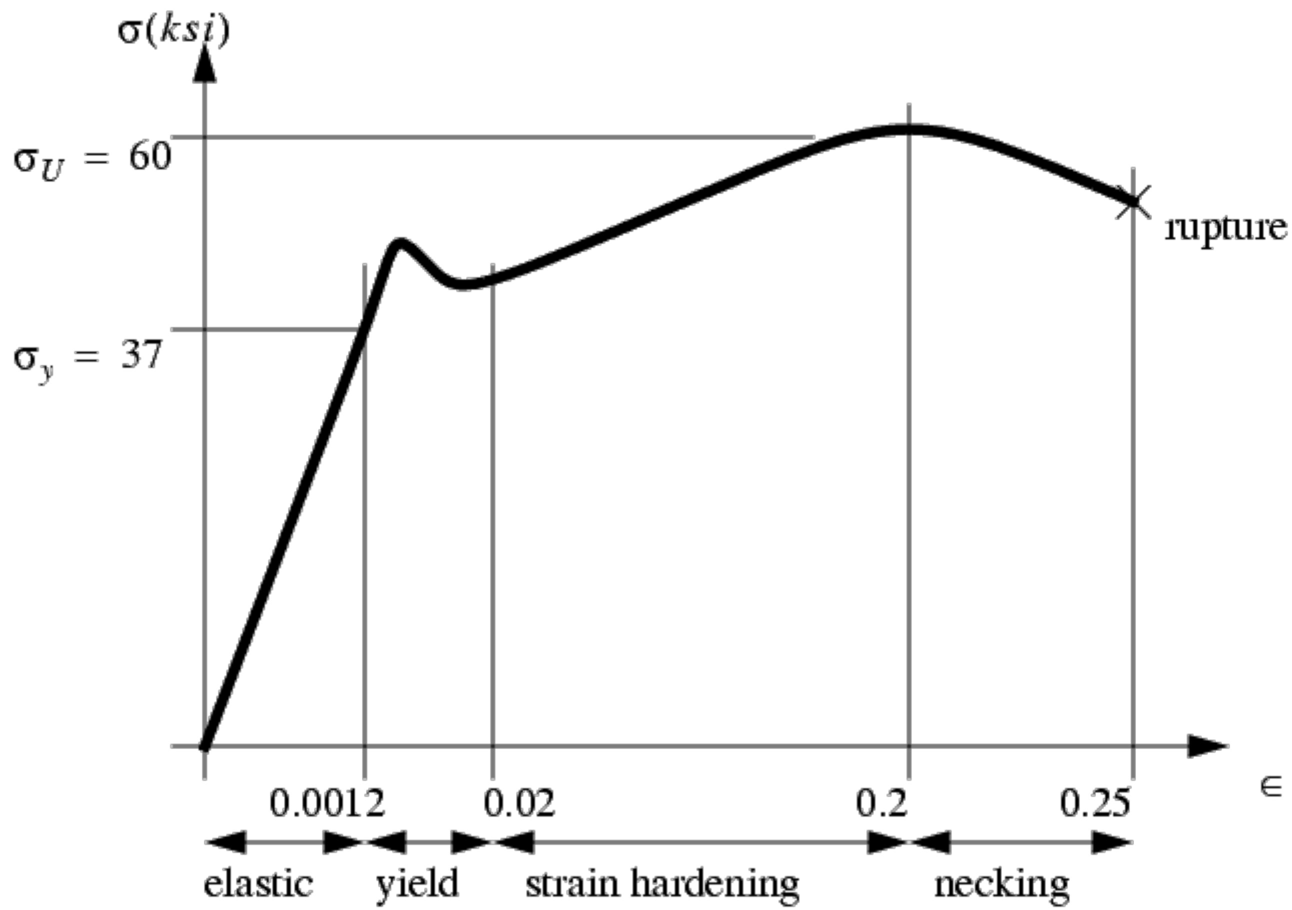
Discrimination Index

Which elastic constant defines the ratio of decrease in a lateral measurement to the increase in length?

Young's modulus	1 respondents	3 %
Shear modulus		0 %
Poisson's ratio	38 respondents	97 % ✓
Zener ratio		0 %

Once the stress reaches the yield stress, it plastically deforms

- σ_y is the yield stress
- σ_U is the ultimate tensile stress
- The final stress before rupture is called the fracture stress



Quiz question: Which statement about elastic and plastic deformation is FALSE?

a) Elastic deformation is recoverable but plastic deformation is permanent
b) Plastic deformation only occurs in metals and other ductile materials
c) The yield stress defines the transition from elastic to plastic deformation
d) Plastic deformation occurs due to the migration of line defects in the crystal lattice called dislocations

Attempts: 39 out of 39

+0.54

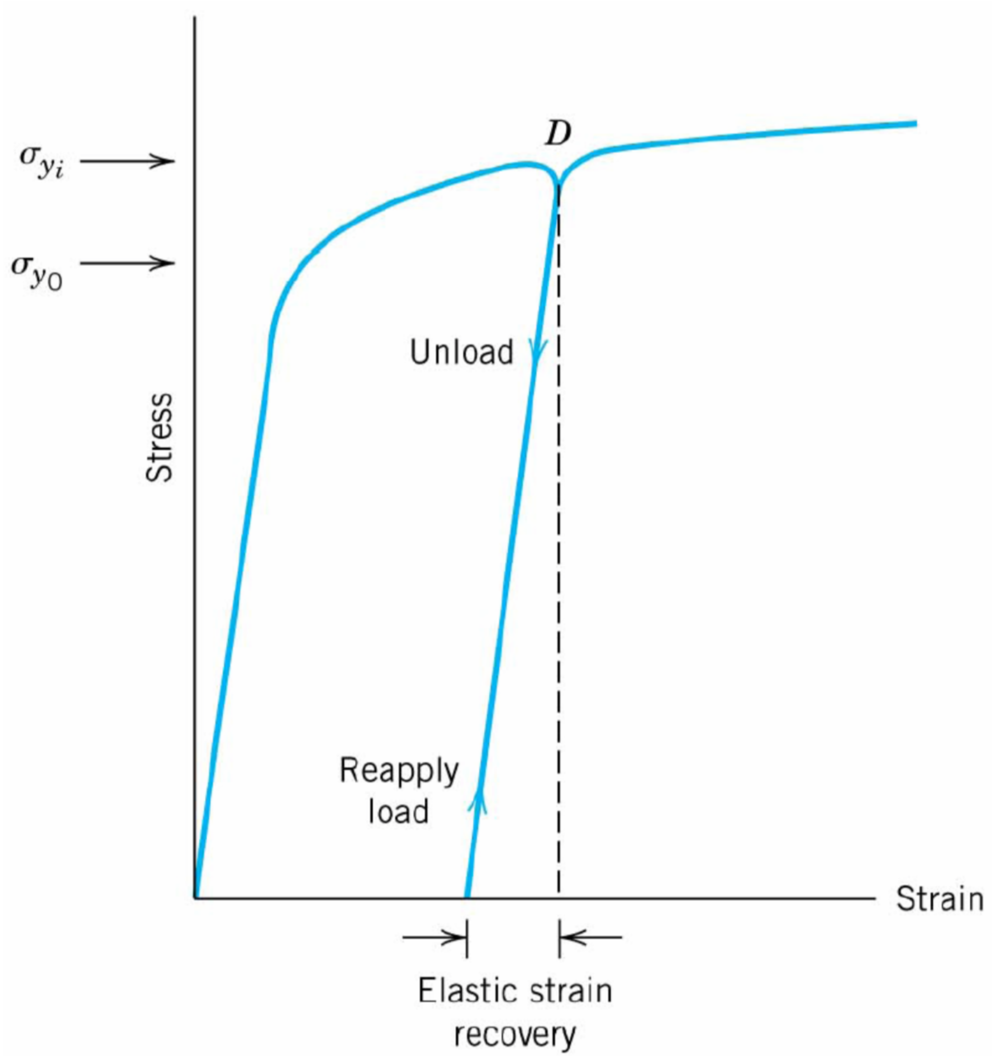
Discrimination Index

Which statement about elastic and plastic deformation is FALSE?

Elastic deformation is recoverable but plastic deformation is permanent		0 %
Plastic deformation only occurs in metals and other ductile materials	33 respondents	**85 %** ✓
The yield stress defines the transition from elastic to plastic deformation	2 respondents	5 %
Plastic deformation occurs due to the migration of line defects in the crystal lattice called dislocations	4 respondents	10 %

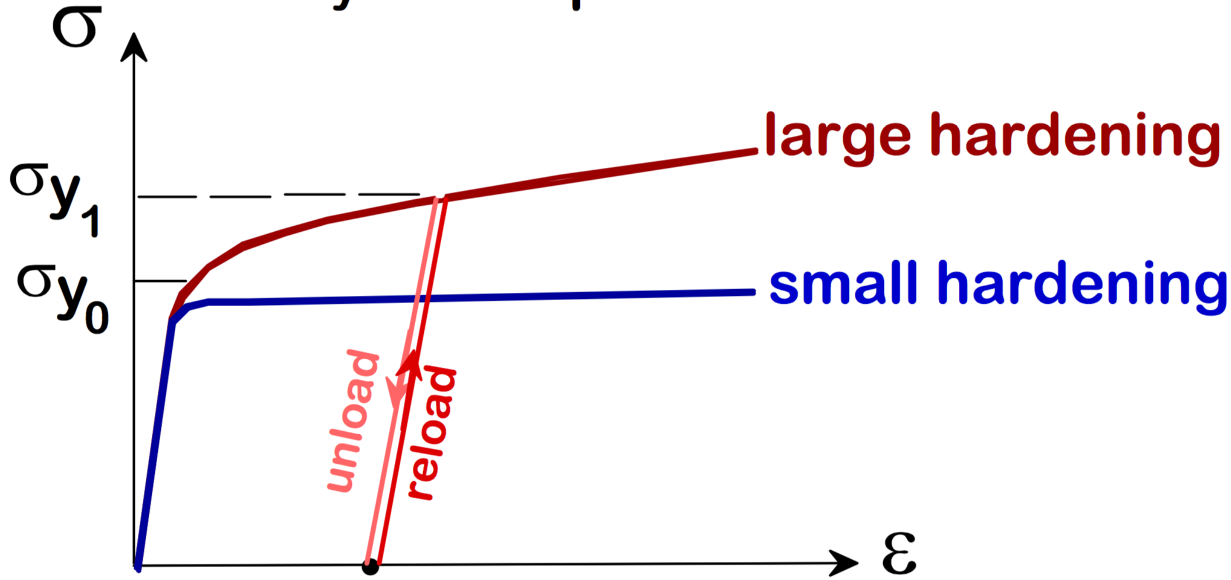
After plastic deformation, if unload the sample, it still has the permanent strain

- After unloading, the strain hardening has changed the material
- Thus, if you reload, the yield strain is often higher than it was previously



The hardening behavior changes for different materials

- An increase in σ_y due to plastic deformation.



- Curve fit to the stress-strain response:

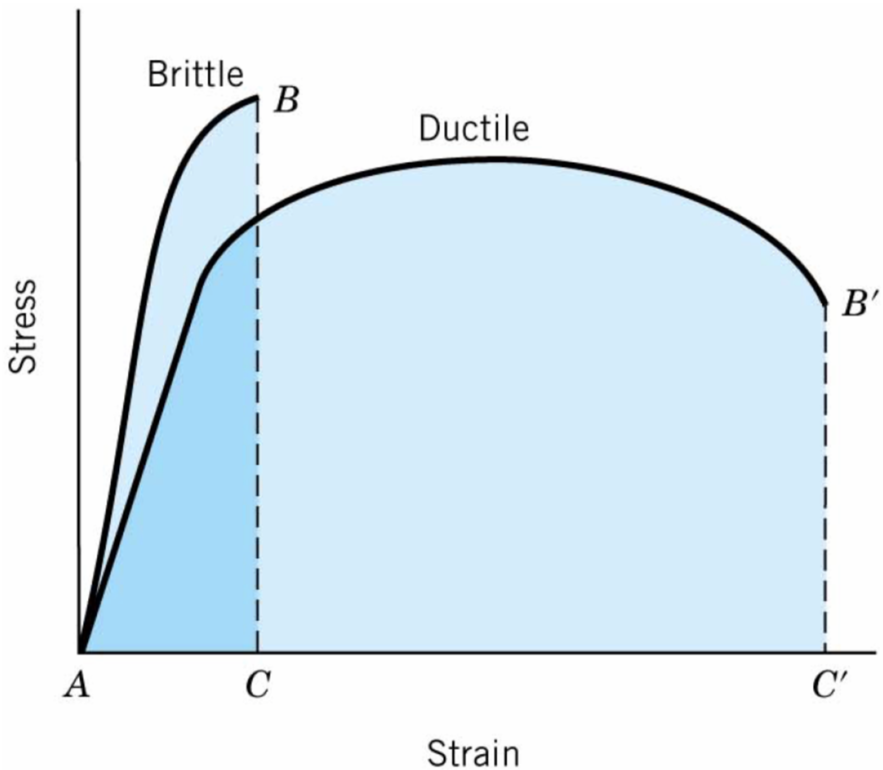
$$\sigma_T = C\left(\varepsilon_T\right)^n$$

hardening exponent:
n=0.15 (some steels)
to n=0.5 (some copper)

"true" stress (F/A)

"true" strain: $\ln(L/L_o)$

Ductile materials plastically deform significantly, brittle materials do not



Ductility

$$\%EL = \frac{(l_f - l_0)}{l_0} \times 100$$

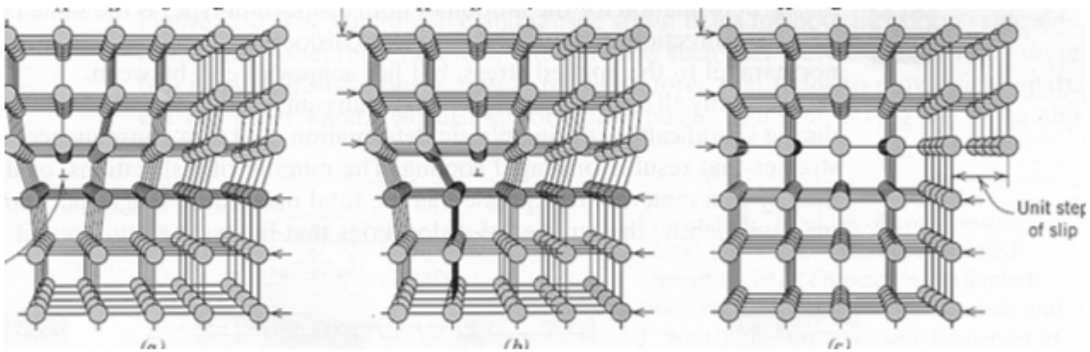
$$\%RA = \frac{(A_0 - A_f)}{A_0} \times 100$$

l_f = length at fracture
A_f = section area at fracture

Typical Ductility for soft metals %EL
25 – 75%

Plastic deformation occurs due to dislocation motion

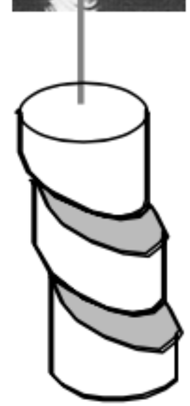
- A dislocation is a line defect.
- When it moves, only a small number of bonds are broken at a time



Adapted from Fig. 7.1, *Callister 6e.* (Fig. 7.1 is adapted from A.G. Guy, *Essentials of Materials Science*, McGraw-Hill Book Company, New York, 1976. p. 153.)

Plastically stretched zinc single crystal.

Adapted from Fig. 7.9, *Callister 6e.* (Fig. 7.9 is from C.F. Elam, *The Distortion of Metal Crystals*, Oxford University Press, London, 1935.)

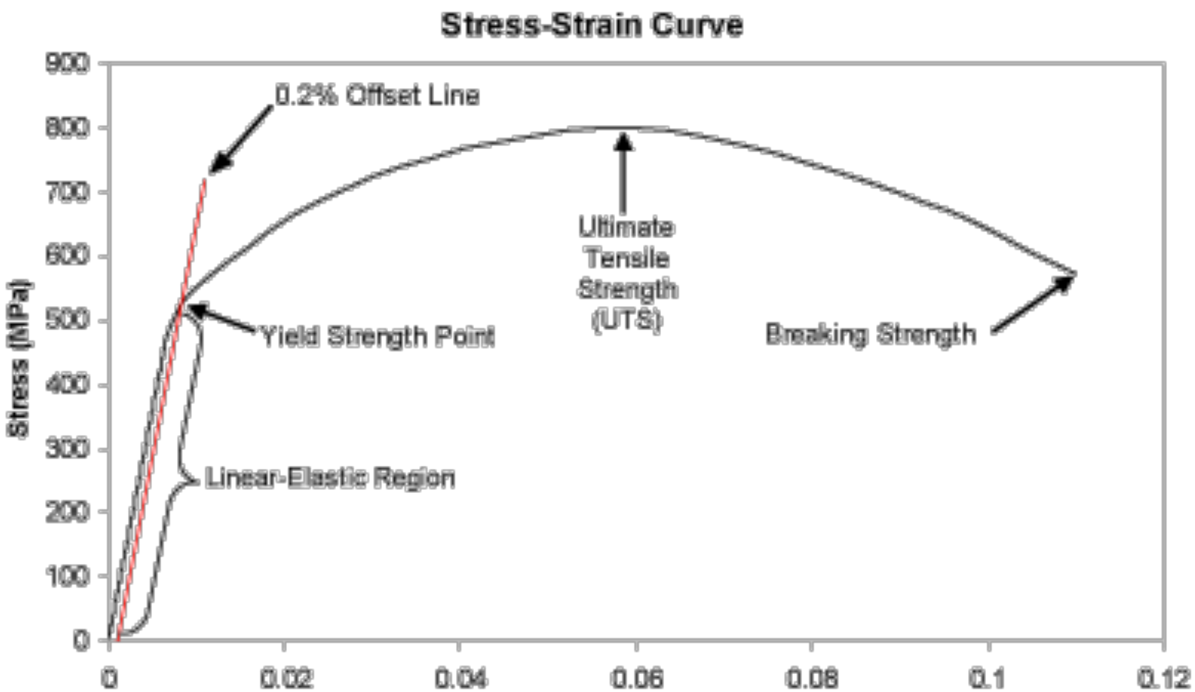


Adapted from Fig. 7.8, *Callister 6e.*

Dislocations multiply during deformation due to dislocation sources

Dislocation motion causes plastic deformation, dislocation pileup causes hardening

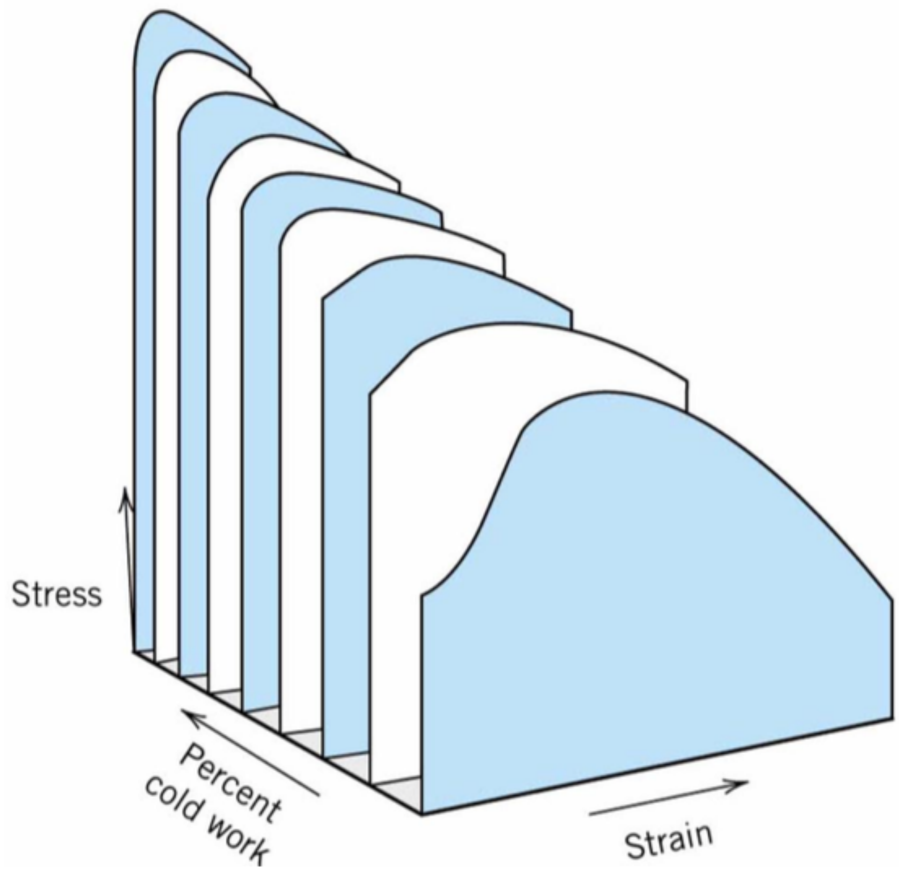
As the dislocation density increases, dislocations cannot move freely, raising the yield stress



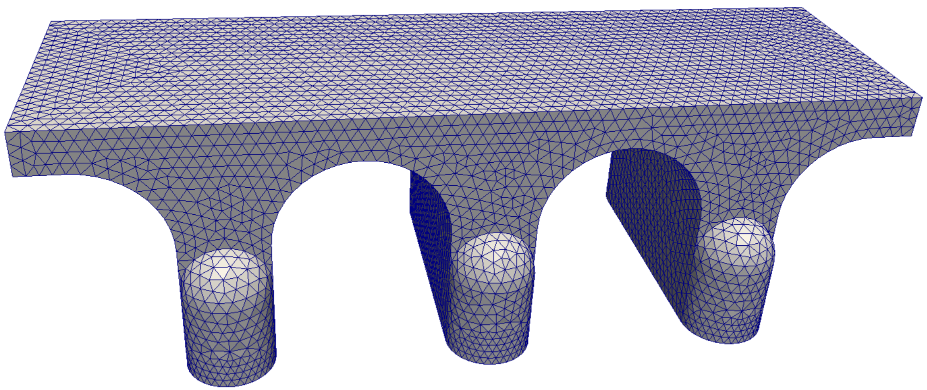
As the strength increases due to dislocation pile-up, the ductility decreases

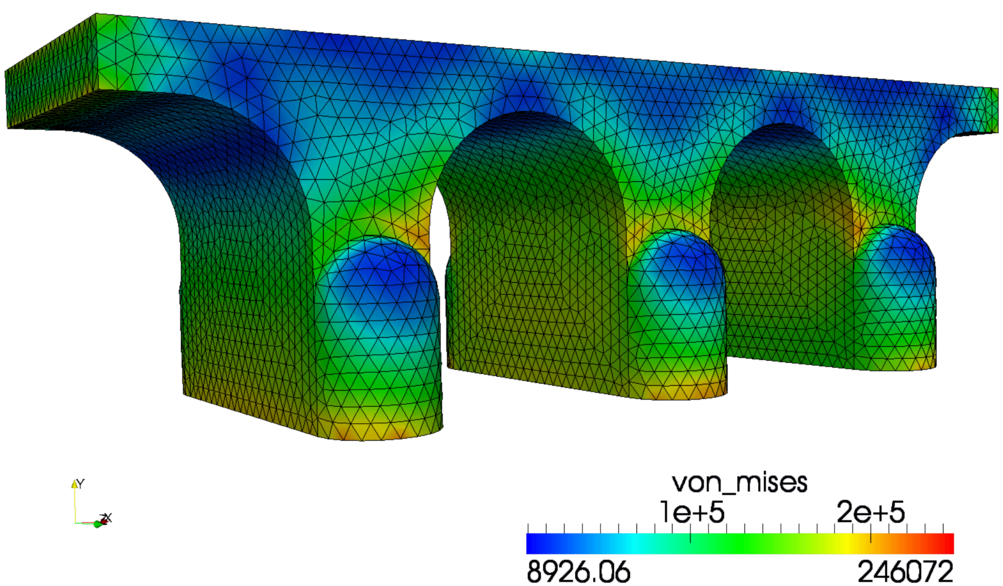
Stress-strain curves for a material with progressively increasing cold work

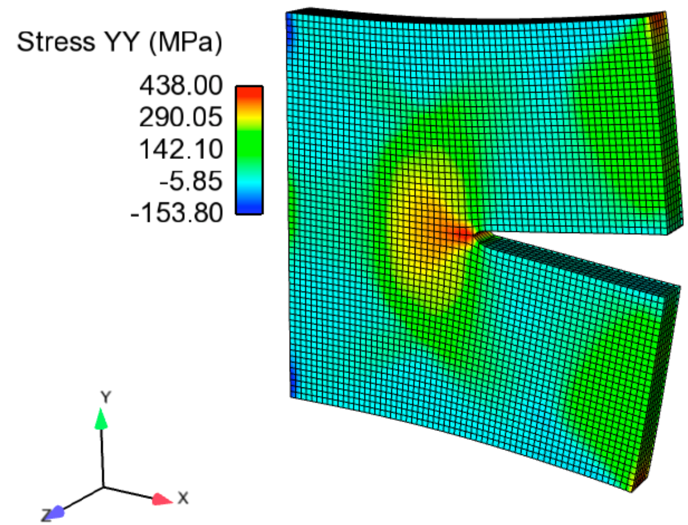
What happens to the energy to failure (toughness) with increasing cold work?

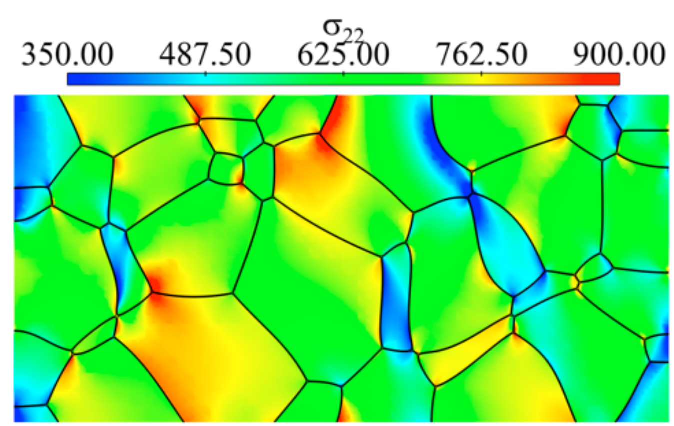


We need to determine the stress and strain throughout a body



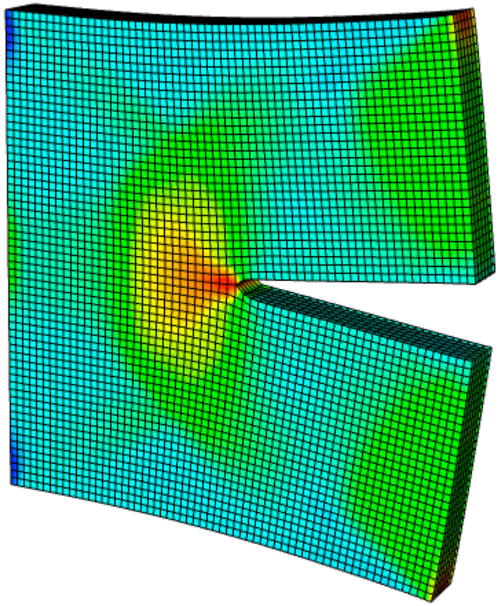






We solve for the displacements using the Cauchy momentum equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\boldsymbol{\sigma} + \rho\,\mathbf{g}$$



- In this equation, **u** is the displacement vector, ρ is the density, **σ** is the stress, and **g** is the acceleration of gravity.
- Often, we are determining the stress and strain in a static body. So the equation becomes:

$$0 = \nabla\cdot\boldsymbol{\sigma} + \rho\,\mathbf{g}$$

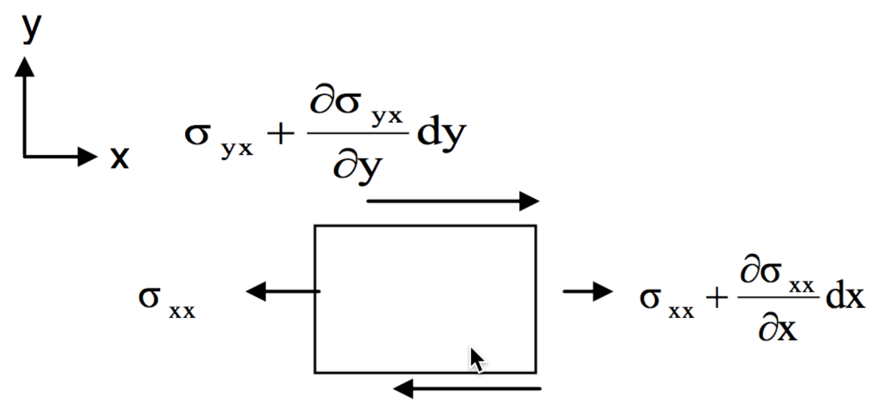
- Finally, we can simplify things more by making our first assumption*: the impact of gravity is negligible*

$$0 = \nabla\cdot\boldsymbol{\sigma}$$

The stress divergence equation is really just a force balance

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx - \sigma_{xx}\right) dy + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy - \sigma_{yx}\right) dx = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$



$$0 = \nabla \cdot \boldsymbol{\sigma}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Now we will consider an axisymmetric body

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \rho\,\mathbf{g}$$

- Assumption 1: We have a static body

$$0 = \nabla \cdot \boldsymbol{\sigma} + \rho\,\mathbf{g}$$

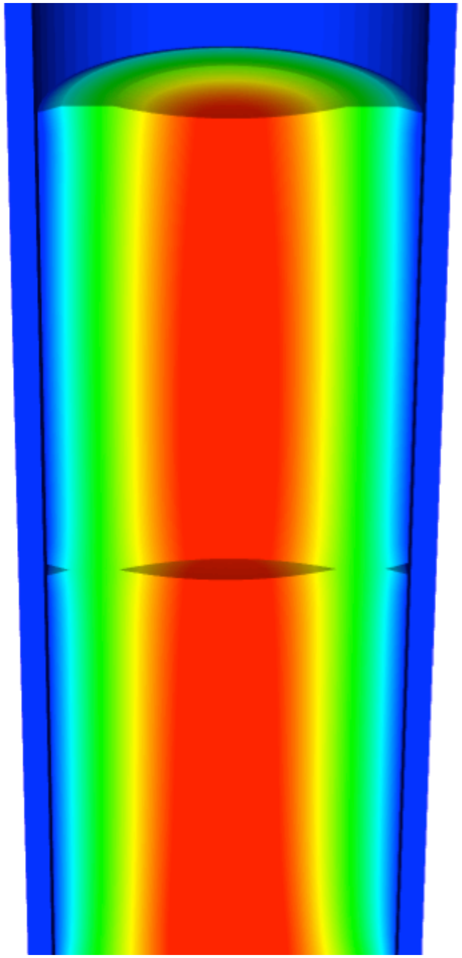
- Assumption 2: Gravity is negligible

$$0 = \nabla \cdot \boldsymbol{\sigma}$$

- Assumption 3: The problem is axisymmetric

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0$$

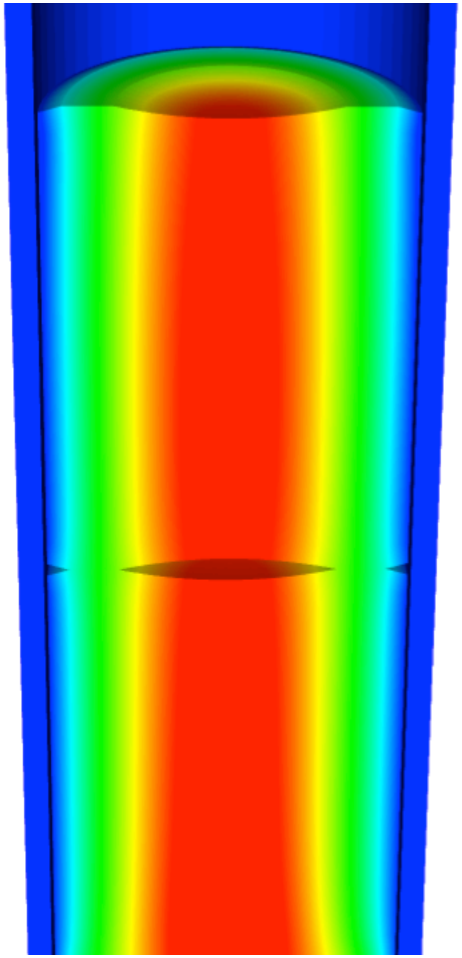
$$\frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$



Now we need to consider the material response of the axisymmetric body

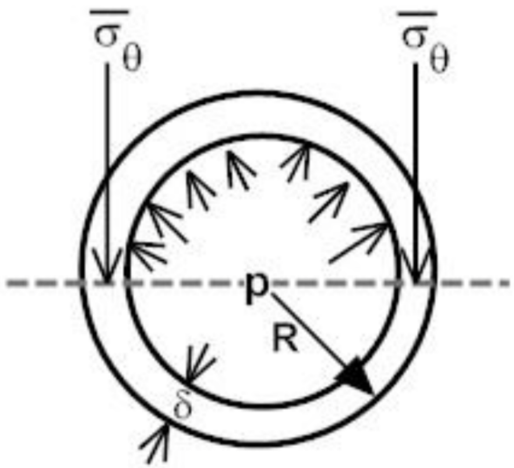
- We assume small strains, so the strain is defined as
  $\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$
- We assume isotropic material response, so

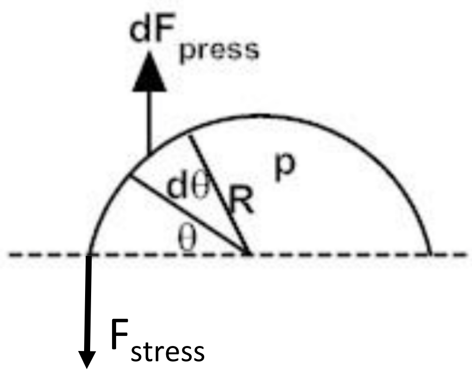
$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$
$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right)$$
$$\epsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$



First, we will solve for the stress throughout a pressurized cylinder (cladding tube) with thin walls

- Because our cylinder has such thin walls (δ = thickness), we can simplify everything to simple force balance





- First, we need the Force per unit length due to the pressure

$$F_{press} = pR\int_0^{\pi} \sin\theta \, d\theta = 2pR \qquad F_{stress} = 2\delta \, \overline{\sigma_\theta}$$

- Then we equate the forces and solve for the hoop stress $\overline{\sigma_\theta} = \dfrac{pR}{\delta}$

Now we find the other two stresses for a thin walled cylinder

- To find the stress in the z-direction we do another force balance

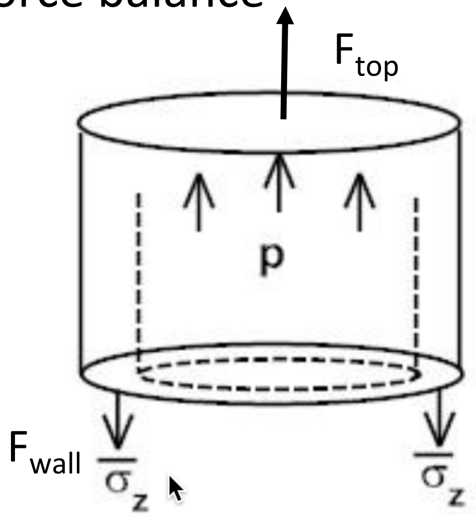
$$F_{top} = p\,\pi R^2 \qquad F_{wall} = 2\pi R\delta\bar{\sigma}_z$$

- Again, we equate the forces and solve for the stress

$$\bar{\sigma}_z = \frac{pR}{2\delta}$$

- Last, the stress on the inside of the wall is –p and on the outside is zero, so the average

$$\bar{\sigma}_r = -\frac{1}{2}p$$



$$\boxed{\bar{\sigma}_\theta = \frac{pR}{\delta} \qquad \bar{\sigma}_z = \frac{pR}{2\delta} \qquad \bar{\sigma}_r = -\frac{1}{2}p}$$

Example problem: We need to determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R = 0.55, δ = 0.06 cm, σ_y = 381 MPa

$$\overline{\sigma_\theta} = \frac{pR}{\delta} \qquad \overline{\sigma_z} = \frac{pR}{2\delta} \qquad \overline{\sigma_r} = -\frac{1}{2}p$$

- The largest stress will be the hoop stress
- The hoop stress is P*(0.55/.06)
  - For 5 MPa, σθ = 45.8 Mpa
  - For 9 MPa, σθ = 82.5 Mpa
- With these pressures, we don't come even close to the yield stress of the cladding

Now we will develop equations for the stress within a cylinder under pressure that has thicker walls

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0 \qquad \frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- The cylinder as an inner radius R_i and an outer Radius R_o
- We assume there is no shear stress, so $\sigma_{rz} = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- We will begin with the r equation

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$$
$$\sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr}$$

- Now we need our constitutive law

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$
$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right)$$

$$E\,\epsilon_{\theta\theta,r} = \left(\sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \sigma_{zz,r})\right)$$
$$E\,\epsilon_{zz,r} = \left(\sigma_{zz,r} - \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r})\right) = 0$$

- We combine these equations to get

$$E\,\epsilon_{\theta\theta,r} = \sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r}))$$
$$E\,\epsilon_{\theta\theta,r} = \left(1 - \nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r}$$

$$E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

Now we will develop equations for the stress within a cylinder under pressure that has thicker walls

- We need one more relationship from the definition of the strain

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}; \qquad \epsilon_{\theta\theta,r} = \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r = \frac{1}{r}(\epsilon_{rr} - \epsilon_{\theta\theta})$$

- From the previous slide, we have

$$E\,\epsilon_{\theta\theta,r} = \left(1-\nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} \qquad E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- If we combine these, we get

$$(1-\nu^2)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} = \frac{1}{r}(1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- Again, from the previous slide

$$\sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr} \qquad \sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$$

- We end up with $r\,\sigma_{rr,rr} + 3\sigma_{rr,r} = 0$

$$\frac{\partial}{\partial r}\left(r^3\frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

We now have an ODE so we can solve to get the stress

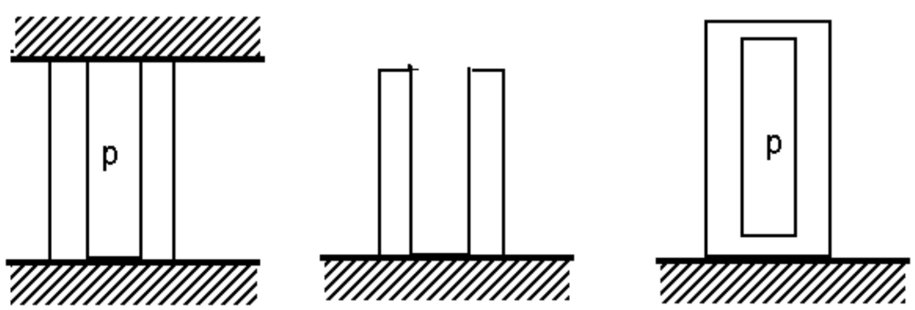
$$\frac{\partial}{\partial r}\left(r^3 \frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

- Our boundary conditions are $\sigma_{rr}(R_i) = -p$ and $\sigma_{rr}(R_o) = 0$
- After integrating twice, we get $$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1}$$
- From σ_{rr}, we can get $\sigma_{\theta\theta}$ from $\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$ $$\sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1}$$
- From the end condition, we determine σ_{zz}

(a) $$\sigma_{zz} = p\frac{2\nu}{(R_o/R)^2 - 1}$$

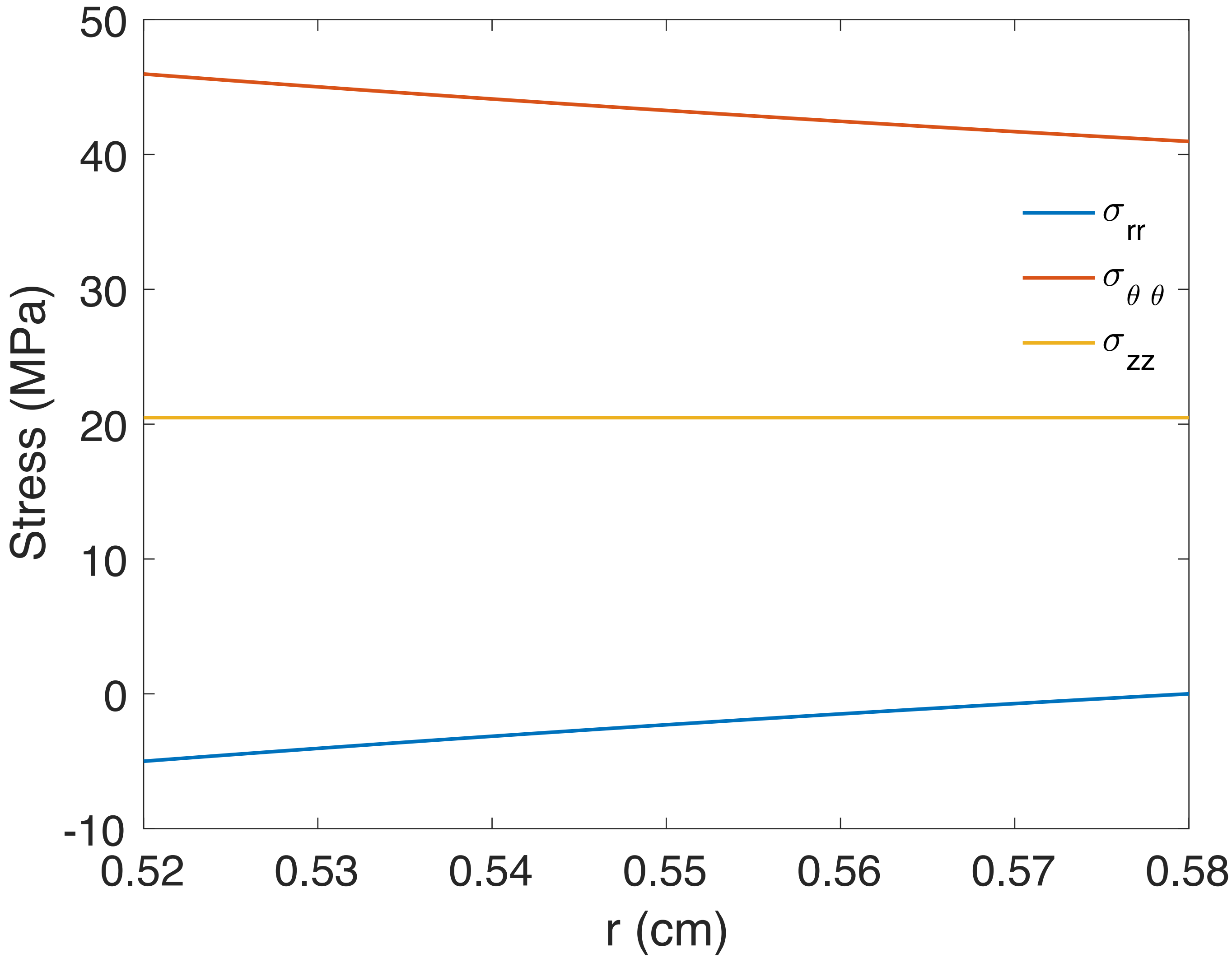
(b) $$\sigma_{zz} = 0$$

(c) $$\sigma_{zz} = p\frac{1}{(R_o/R)^2 - 1}$$



(a) complete axial restraint
(b) no axial restraint
(c) closed-end tube

The stresses vary through the thickness of the cladding tube



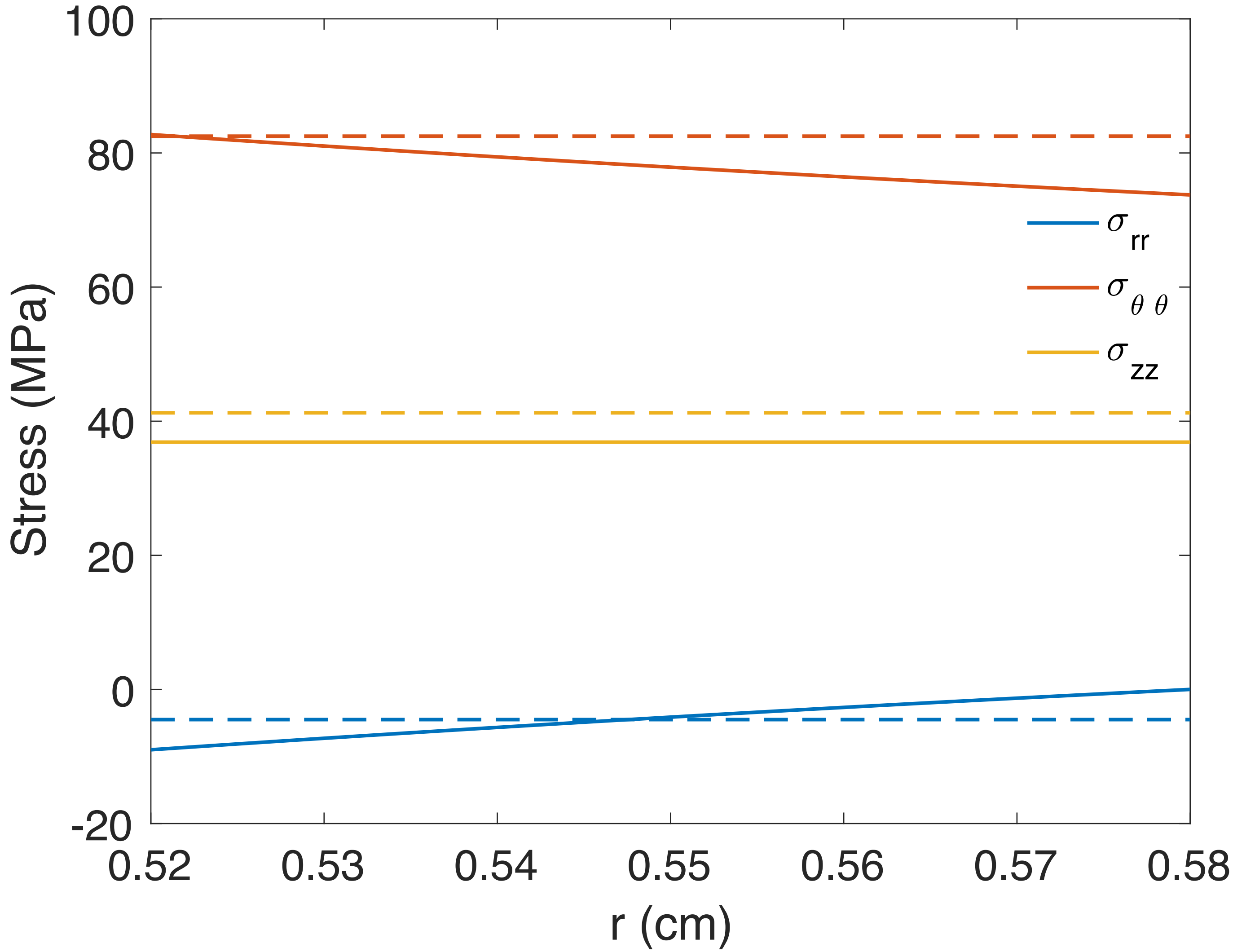
Example problem: We need to determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R_i = 0.52, R_o = 0.58 cm, σ_y = 381 MPa

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1} \quad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1} \quad \sigma_{zz} = p\frac{1}{(R_o/R)^2 - 1}$$

- Which stress is the largest?
  - $\sigma_{\theta\theta}$
- At what position r will the hoop stress be the largest?
  - r = R_i = 0.52 cm
- What is the stress at p = 5 MPa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 46.0 MPa
- What is the stress at p = 9 Mpa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 82.7 Mpa
- Again, we don't even get close to yielding

We’ve now solved this same problem assuming thin and thick walled cylinders



Summary

- Solid mechanics predicts the deformation of a body from its applied load
  - The strain defines the deformation
  - The stress defines the material's internal response to the strain
- Materials can have recoverable and permanent deformation
  - Elastic deformation is recoverable and occurs due to the stretching of the atomic lattice
  - Plastic deformation is permanent and results form the breaking of bonds during dislocation (line defect) migration
- We derived two analytical solutions for the stress in a pressurized cylinder
  - Thin walled cylinder
  - Any size wall