Thermo-Mechanics

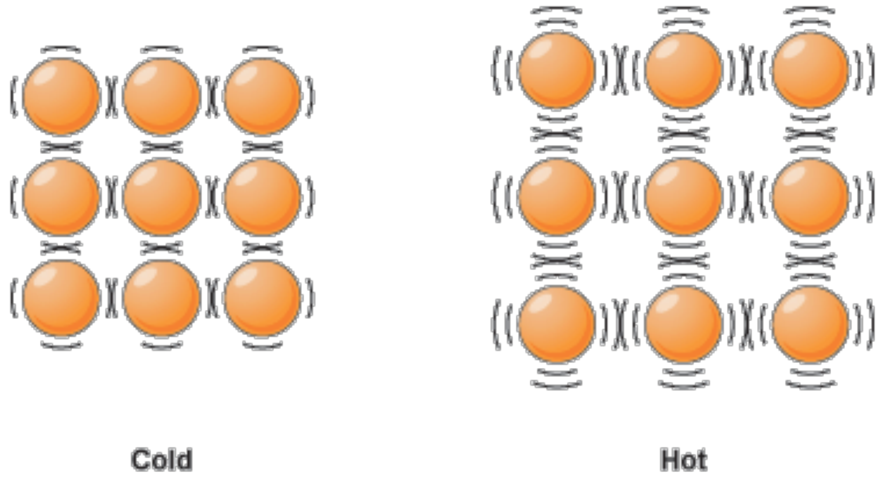
NE 591

Last Time

- Solid mechanics predicts the deformation of a body from its applied load
  - The strain defines the deformation
  - The stress defines the material's internal response to the strain
- Materials can have recoverable and permanent deformation
  - Elastic deformation is recoverable and occurs due to the stretching of the atomic lattice
  - Plastic deformation is permanent and results form the breaking of bonds during dislocation (line defect) migration
- We derived two analytical solutions for the stress in a pressurized cylinder
  - Thin walled cylinder
  - Any size wall

Thermal Expansion

- As the temperature increases, atoms have larger vibrations, causing the material to expand
- This expansion doesn't raise the energy of the material when unrestrained
- This expansion can be treated as a strain, but as one that doesn't cause stress



$$\sigma = C\,(\epsilon - \epsilon_0)$$

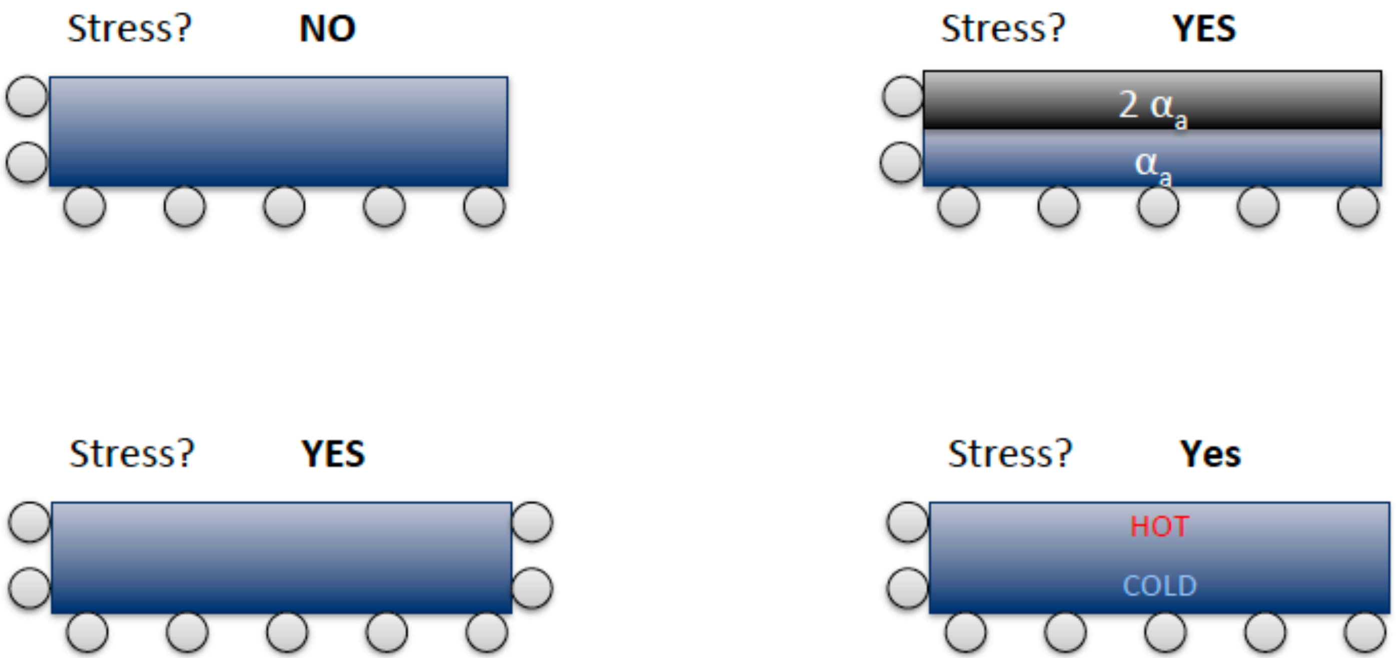
Thermal Expansion

- In isotropic materials, thermal expansion happens equally in all directions and is linear with temperature
- $\epsilon_0 = (T - T_0)\alpha I$
- In this equation
  - T is the current temperature
  - T_0 is the temperature the original size was measured
  - α is the linear thermal expansion coefficient
  - I is the identity tensor

Material	α (× 10^{-6} 1/K)
Aluminum	24
Copper	17
Steel	13
UO_2	11
Zircaloy (Axial)	5.5
Zircaloy (radial)	7.1

Thermal Expansion

- Though thermal expansion doesn’t directly cause stress, it can still lead to thermal stress

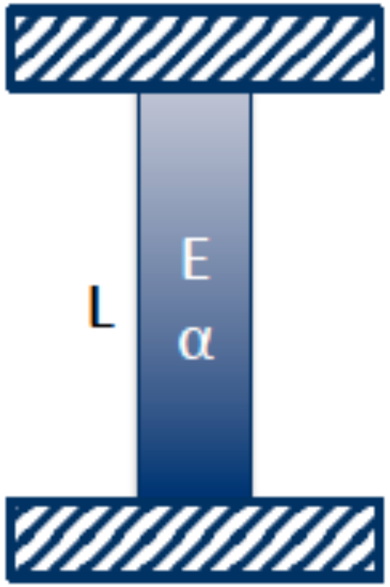


What is the stress in a thin constrained rod of length L when it is heated to ΔT?

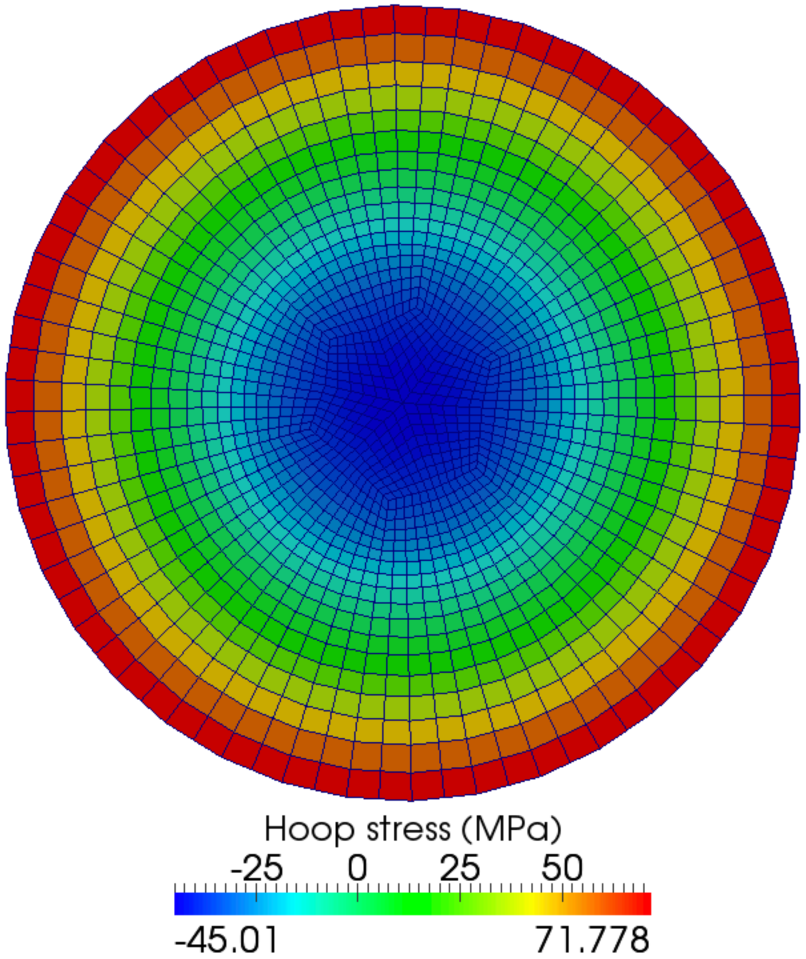
- The rod has a Young's modulus of E and an expansion coefficient of α

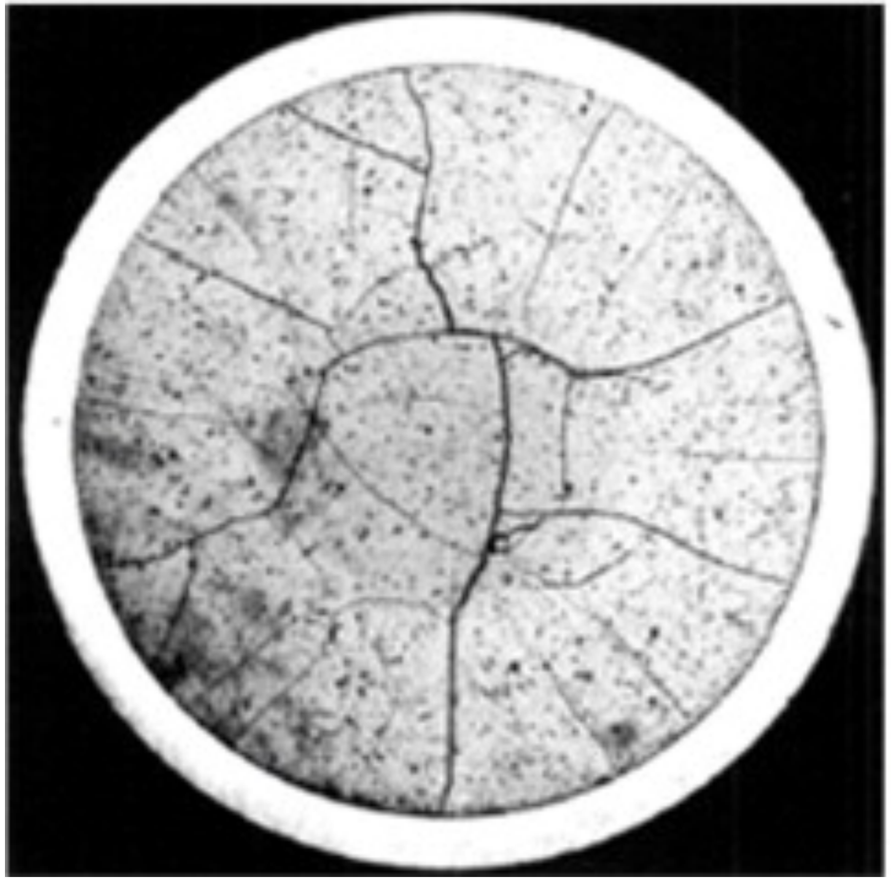
$$\epsilon_0 = (T - T_0)\alpha I \qquad \sigma = C\,(\epsilon - \epsilon_0)$$

$$\epsilon_0 = (T - T_0)\alpha$$
$$\sigma = E\,(0 - \Delta T\alpha)$$
$$\sigma = -E\Delta T\alpha$$



The large temperature gradient within a fuel pellet results in large thermal stresses





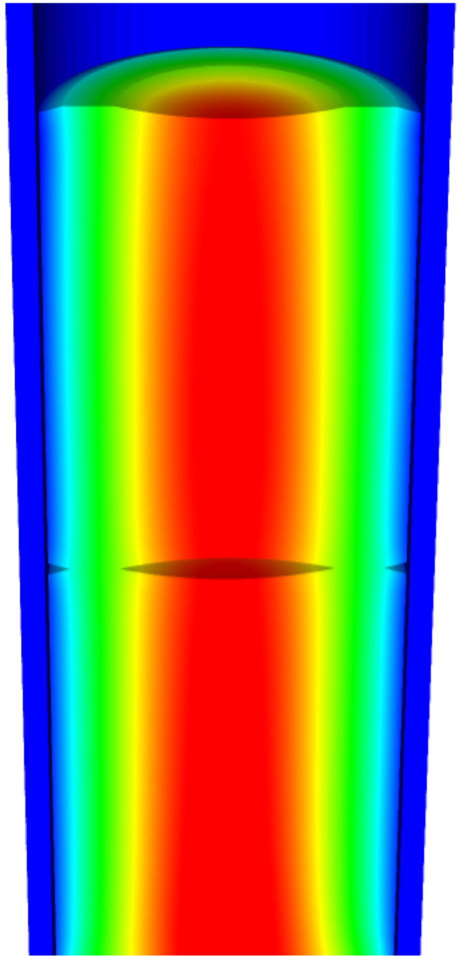
Consider the material response of the axisymmetric body

- We assume small strains, so the strain is defined as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r},\ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

- We assume isotropic material response, so

$$\begin{aligned}
\epsilon_{rr} &= \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})) + \alpha\Delta T \\
\epsilon_{\theta\theta} &= \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})) + \alpha\Delta T \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})) + \alpha\Delta T \\
\epsilon_{rz} &= \frac{1}{2G}\sigma_{rz}
\end{aligned}$$

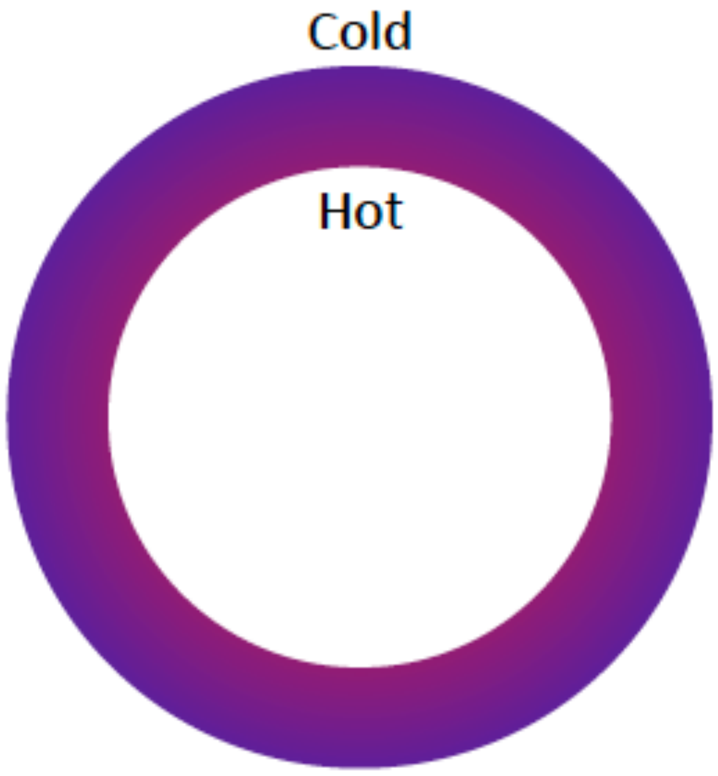


Consider a cylinder with thermal expansion but not pressure

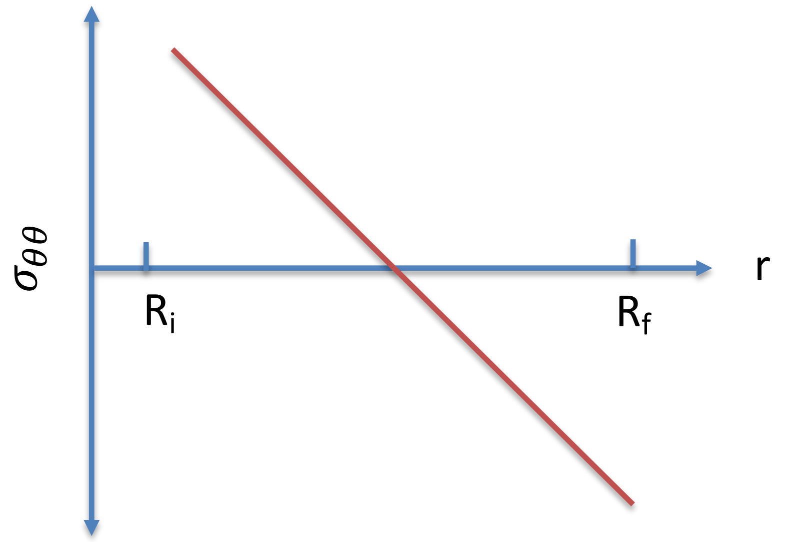
- $\sigma_{rr}(R_i) = \sigma_{rr}(R_0) = 0$
- Similar to the equations we worked through last time
- $\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$
- Solving this ODE:

$$\sigma_{rr}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(\frac{r}{R_i}-1\right)\left(1-\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{zz}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$

What is the hoop stress in the cladding?



$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$



Where is hoop stress equal to zero?

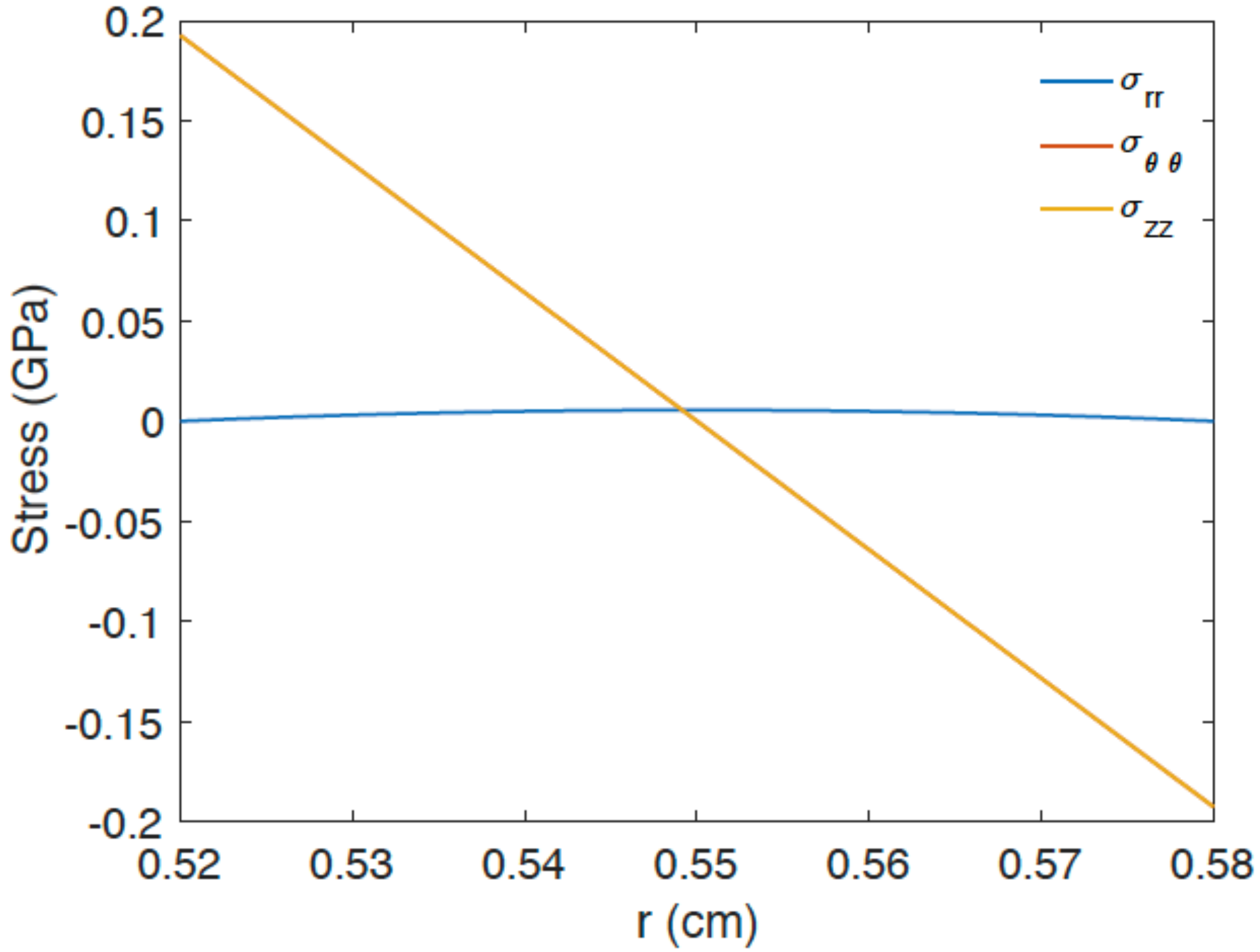
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$

$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0$$

$$\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0 \longrightarrow 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right) = 1 \longrightarrow \left(\frac{r}{R_i} - 1\right) = \frac{\delta}{2R_i}$$

$$r = \frac{\delta}{2} + R_i$$

The linear temperature gradient across the cladding causes axial thermal stresses



Same approach to the thermal stress in a fuel pellet

- The thermal stress is due to the temperature gradient

$$T(r) = \frac{LHR}{4\pi k}\left(1 - \frac{r^2}{R_f^2}\right) + T_s$$

$$T_0 - T_s = \frac{LHR}{4\pi k}$$

$$T - T_s = (T_0 - T_s)\left(1 - \frac{r^2}{R_f^2}\right)$$

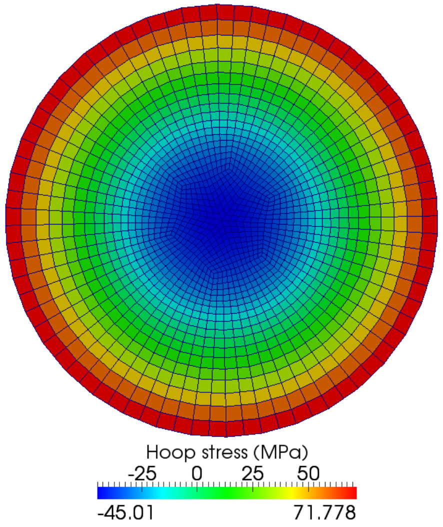
$$\frac{\partial T}{\partial r} = 2(T_0 - T_s)\left(\frac{r}{R_f^2}\right)$$

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$$

$$\eta = \frac{r}{R_f}$$

$$\sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

$$\frac{d}{d\eta}\left(\eta^3\frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^*\eta^3$$



Solve this stress ODE

- The boundary conditions are:

$$\frac{d\sigma_{rr}}{d\eta} = 0 \text{ at } \eta = 0$$

$$\sigma_{rr} = 0 \text{ at } \eta = 1$$

$$\frac{d}{d\eta}\left(\eta^3 \frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^*\eta^3 \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

- Once we solve it, we obtain

$$\boxed{\sigma_{rr}(\eta) = -\sigma^*(1-\eta^2)}$$

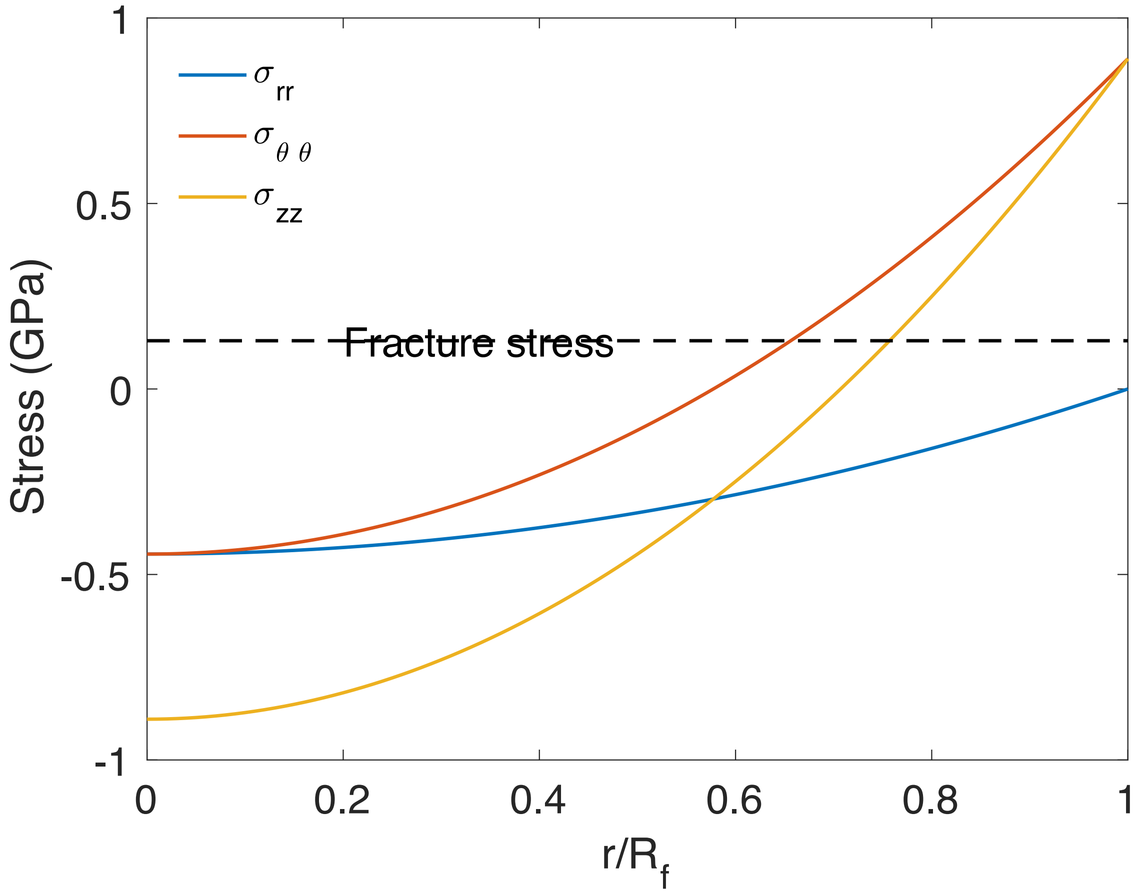
- Then we can solve the hoop stress

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \boxed{\sigma_{\theta\theta}(\eta) = -\sigma^*(1-3\eta^2)}$$

- The axial stress is much more complicated to obtain, but you end up with

$$\boxed{\sigma_{zz}(\eta) = -2\sigma^*(1-2\eta^2)}$$

The fuel temperature gradient causes large thermal stresses



How far do fuel cracks extend?

$$\sigma_{\theta\theta}(\eta) = -\sigma^*(1 - 3\eta^2) \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1 - \nu)} \qquad \eta = \frac{r}{R_f}$$

- E = 200 GPa, v = 0.345, α = 11.0e-6 1/K, σ_{fr} = 130 MPa, ΔT = 550 K
- Solve for η
  - $-\sigma_{fr} / \sigma^* = 1 - 3\eta^2$
  - $3\eta^2 = 1 + \sigma_{fr} / \sigma^*$
  - $\eta = \left((1 + \sigma_{fr} / \sigma^*)/3 \right)^{1/2}$
- σ* = 11.0e-6*200*550/(4*(1 – 0.345)) = 461.8 MPa
- η = sqrt((1 + 130/461.8)/3) = 0.65

Analytical Thermomechanics Summary

- Increases in temperature cause most materials to expand
- We call this thermal expansion, which is a strain that doesn’t inherently cause stress
- Thermal expansion can cause stress when
  - Deformation is constrained
  - There are gradients in the expansion coefficient
  - There is a temperature gradient
- We have analytical equations for thermal stresses in the cladding and in the fuel

The gap changes as a function of time

- Both the pellet and the cladding swell

$$\Delta\delta_{gap} = \delta_{gap} - \delta_{gap}^{0}$$
$$\Delta\delta_{gap} = \Delta\bar{R}_C - \Delta R_f$$
$$\frac{\Delta R_f}{\bar{R}_C} = \alpha_f\left(\bar{T}_f - T_{fab}\right)$$
$$\frac{\Delta R_C}{\bar{R}_C} = \alpha_C\left(\bar{T}_C - T_{fab}\right)$$
$$\Delta\delta_{gap} = \bar{R}_c\alpha_C\left(\bar{T}_C - T_{fab}\right) - \bar{R}_f\alpha_f\left(\bar{T}_f - T_{fab}\right)$$

- But, as the gap decreases, the temperature changes, which again makes the gap change
- The solution using the analytical equations is iterative, due to the dependence of the gap size and temperature.

Calculate the steady state temperature profile in the rod, including thermal expansion

- LHR = 200 W/cm, δ^0_{gap} = 30 µm, R_f = 0.5, T_{cool} = 580 K, $T_{EXP,0}$ = 373 K, k_{gap} = 0.0026 W/cm-K, t_C= 0.06 cm, α_f = 11.0e-6 1/K, α_C = 7.1e-6 1/K

$$\Delta\delta_{gap} = \bar{R}_c\alpha_C\left(\bar{T}_C - T_{fab}\right) - \bar{R}_f\alpha_f\left(\bar{T}_f - T_{fab}\right) = \Delta R_c - \Delta R_f$$

$$\Delta T_{gap} = \frac{LHR}{2\pi R_f k_{gap}/\delta_{gap}}$$

- ΔT_{cool} = 25.5 K, ΔT_{clad} = 22.5 K, ΔT_{fuel} = 530.5 K
- So, T_{IC} = 580 + 25.5 + 22.5 = 628.0 K, T_s = 701.5 K, T_0 = 1232.0 K
- First, we will deal with expansion in the cladding
  - Av(R_c) = 0.5 + 30e-4 + 0.06/2 = 0.533 cm
  - Av(T_C) = 580 + 25.5 + 22.5/2 = 616.75 K
  - ΔR_c = 0.533*7.1e-6*(616.75 – 373) = 9.22e-4 cm

Calculate the steady state temperature profile in the rod, including thermal expansion

- Second, we deal with the fuel
  - Av(Tf) = (1232 + 701.5)/2 = 966.7 K
  - ΔR_f = 0.5*11e-6*(966.7 – 373) = 0.0033 cm

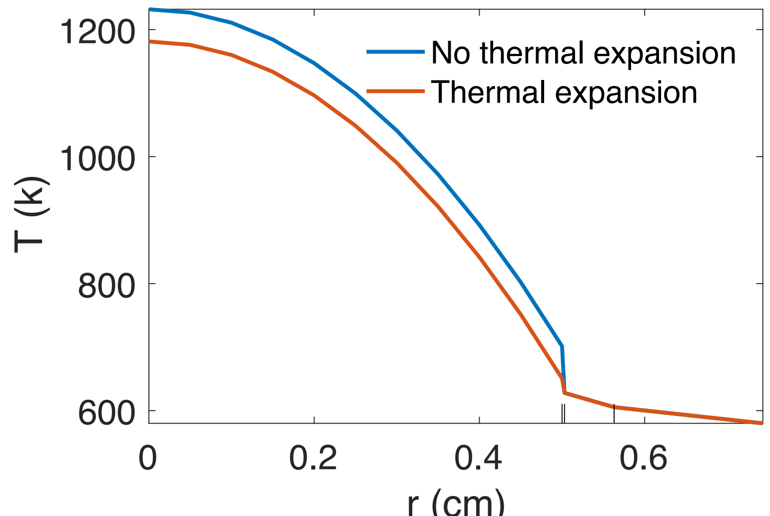
$$\Delta\delta_{gap} = \bar{R}_c \alpha_C \left(\bar{T}_C - T_{fab}\right) - \bar{R}_f \alpha_f \left(\bar{T}_f - T_{fab}\right)$$

- The total change in the gap is 9.22e-4 – 0.0033 = -0.0023
- However, that means the gap is smaller and so our temperatures were wrong!

This calculation is repeated until the gap width stops changing significantly

- The change in the gap does NOT effect the coolant or cladding temperatures, just the gap and fuel temperatures.
- We only need to repeat the calculation of the fuel and cladding temperatures and the change in the gap

Iteration	δ_{gap} (cm)	T_s (K)	T_0 (K)
0	0.003	701	1232
1	0.00066	644	1174
2	0.00097	652	1182
3	0.00094	651	1181
4	0.00094	651	1181



Next time

- Solve the temperature and the displacement vector for the full thermomechanical problem