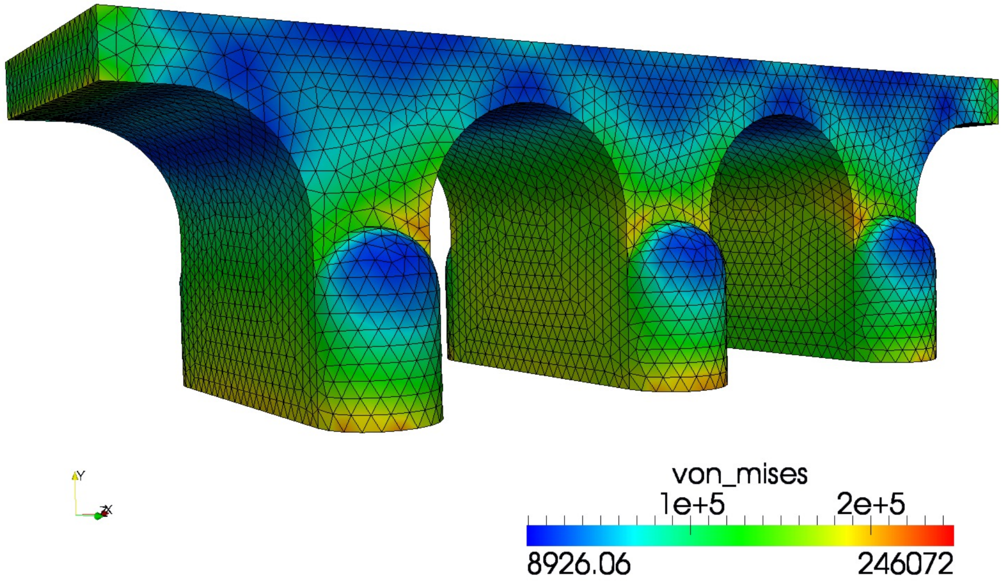
# Nuclear Fuel Performance

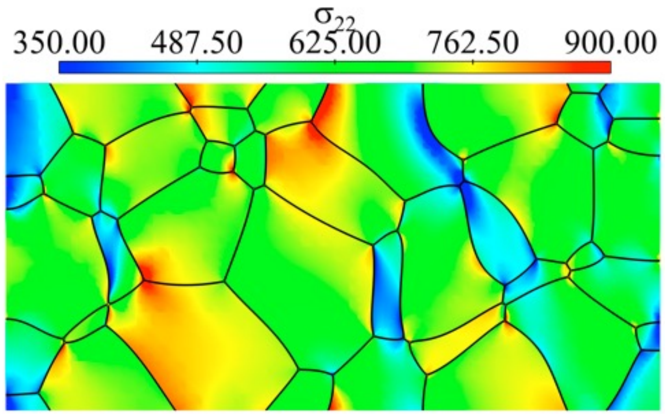
NE-591-010

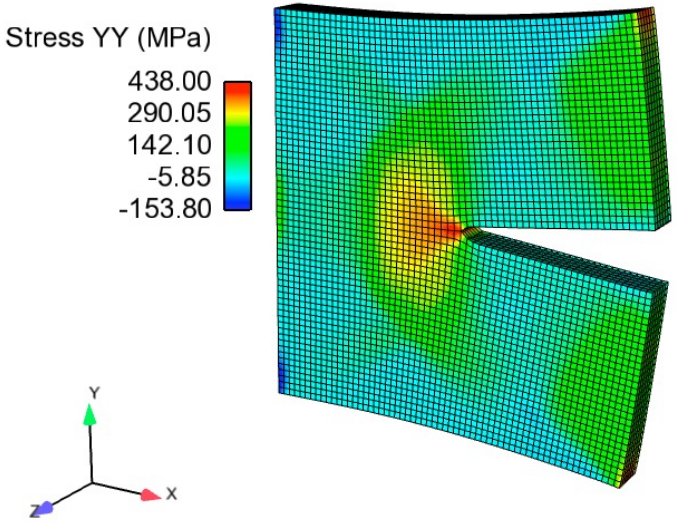
Spring 2021

# MECHANICS

# Determine the stress and strain throughout a body



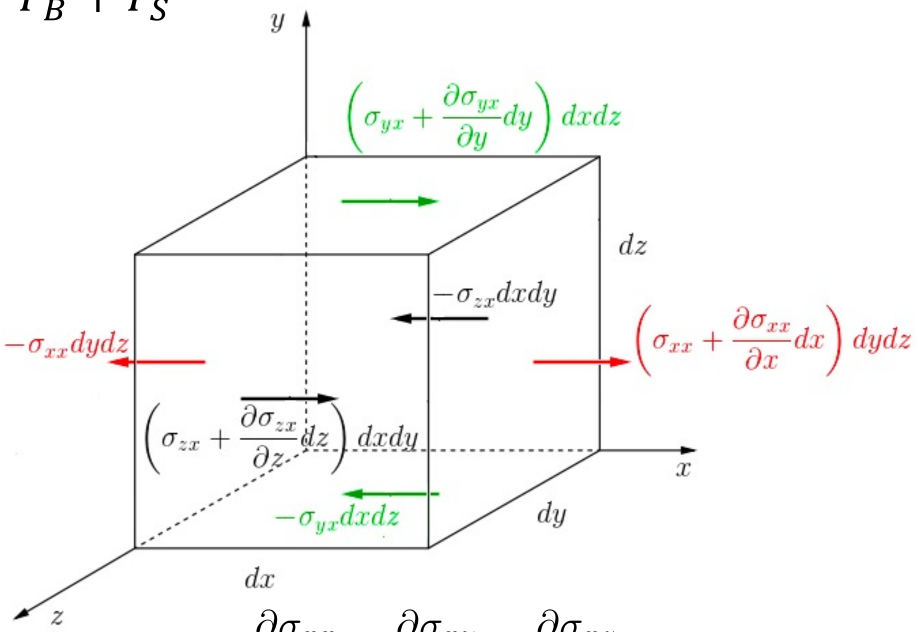



# The stress divergence equation derivation

- Generalized momentum conservation: $$\frac{d\vec{p}}{dt} = \vec{F} = \overrightarrow{F_B} + \overrightarrow{F_S}$$
- Considering a cubic element, surface forces act on the walls of the cube
- Force on a wall is the product of the stress and the surface area
  - For wall at dx, approximate stress via Taylor expansion

$$\sigma_{xx}(x+dx) = \sigma_{xx}(x) + dx\frac{\partial \sigma_{xx}}{\partial x}$$




$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

$$F_p^x = (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx)dydz - \sigma_{xx}dydz + (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy)dxdz - \sigma_{yx}dxdz + (\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}dz)dxdy - \sigma_{zx}dxdy$$

$$F_p^x = \frac{\partial \sigma_{xx}}{\partial x}dxdydz + \frac{\partial \sigma_{yx}}{\partial y}dydxdz + \frac{\partial \sigma_{zx}}{\partial z}dzdxdy$$

$$0 = \nabla \cdot \boldsymbol{\sigma}$$

# Simplified Cauchy for our typical system

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{g}$$

- Assumption 1: We have a static body

$$0 = \nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{g}$$

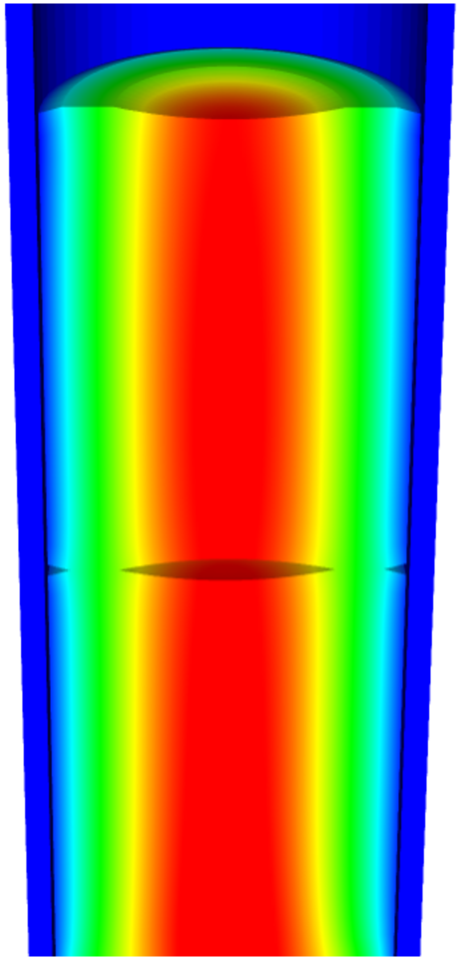
- Assumption 2: Gravity is negligible

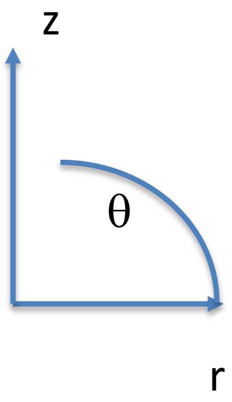
$$0 = \nabla \cdot \boldsymbol{\sigma}$$

- Assumption 3: The problem is axisymmetric

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$





# Consider the material response of the axisymmetric body

- We assume small strains (elasticity), so the strain is defined as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r},\ \ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

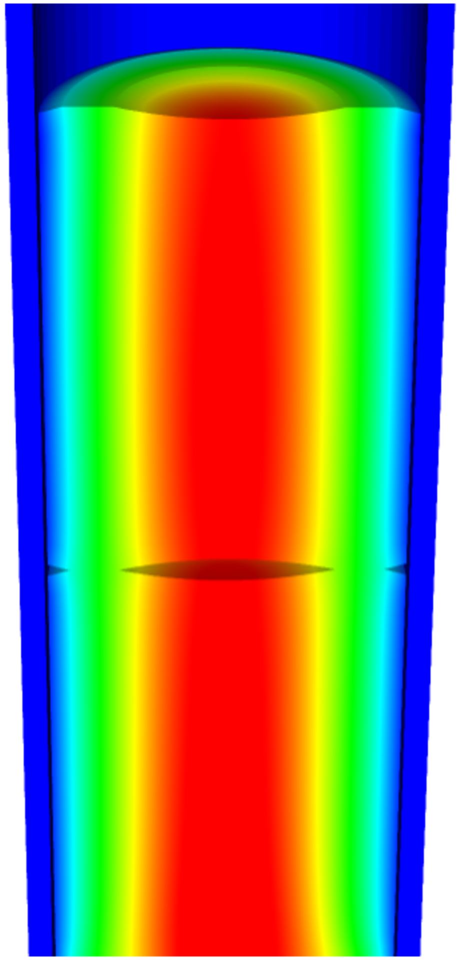
- We assume isotropic material response, so

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$

$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right)$$

$$\epsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$



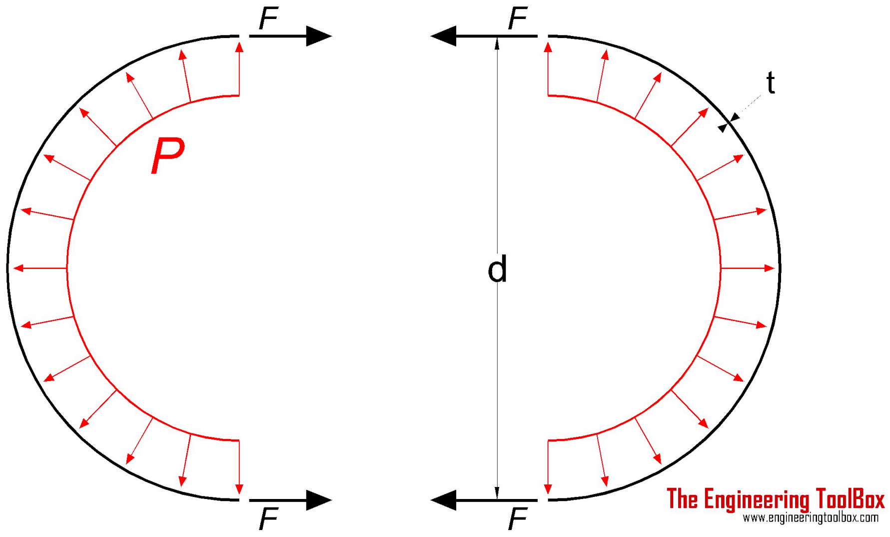
# Solve for the stress throughout a pressurized cylinder (cladding tube) with thin walls

- Because our cylinder has such thin walls (δ = thickness), we can simplify everything to simple force balance
- The hoop stress is the force exerted circumferentially in both directions on every particle in the cylinder wall
- First, we need the Force per unit length due to the pressure

$$F_{press} = pR\int_0^{\pi} \sin\theta \, d\theta$$

- Utilize force to hoop stress relation: $F_{stress} = 2\delta \, \overline{\sigma_\theta}$
- Then we equate the forces and solve for the hoop stress

$$\overline{\sigma_\theta} = \frac{pR}{\delta}$$



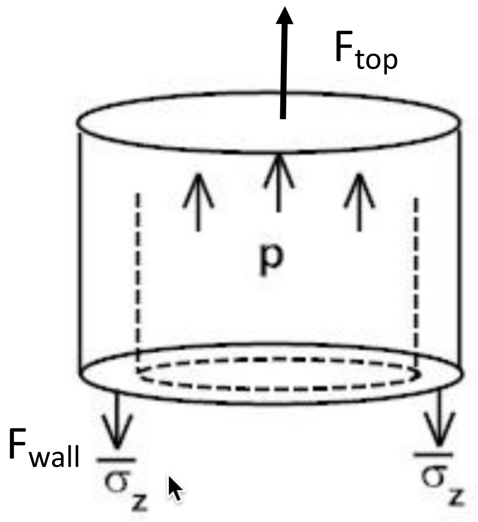
# Find the other two stresses for a thin walled cylinder

- To find the stress in the z-direction we do another force balance

$$F_{top} = p\,\pi R^2 \qquad F_{wall} = 2\pi R\delta\bar{\sigma}_z$$

- Again, we equate the forces and solve for the stress

$$\bar{\sigma}_z = \frac{pR}{2\delta}$$




- Last, the stress on the inside of the wall is –p and on the outside is zero, so the average

$$\bar{\sigma}_r = -\frac{1}{2}p$$

$$\boxed{\bar{\sigma}_\theta = \frac{pR}{\delta} \qquad \bar{\sigma}_z = \frac{pR}{2\delta} \qquad \bar{\sigma}_r = -\frac{1}{2}p}$$

# Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R = 0.55 cm, $\delta$ = 0.05 cm, $\sigma_y$ = 381 MPa

$$\overline{\sigma_\theta} = \frac{pR}{\delta} \qquad \overline{\sigma_z} = \frac{pR}{2\delta} \qquad \overline{\sigma_r} = -\frac{1}{2}p$$

- The largest stress will be the hoop stress
- The hoop stress is P*(0.55/.05)
  - For 5 MPa, σθ = 55 Mpa
  - For 9 MPa, σθ = 99 Mpa
- With these pressures, we don’t come even close to the yield stress of the cladding

# Stress within a pressurized cylinder that has thick walls (radius/thickness<20)

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}=0 \qquad \frac{1}{r}\frac{\partial(r\sigma_{rz})}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}=0$$

- The cylinder as an inner radius $R_i$ and an outer Radius $R_o$
- We assume there is no shear stress, so $\sigma_{rz} = 0$

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \qquad \frac{\partial\sigma_{zz}}{\partial z}=0$$

- We will begin with the r equation

$$\sigma_{\theta\theta}=\sigma_{rr}+r\sigma_{rr,r} \qquad \sigma_{\theta\theta,r}=2\,\sigma_{rr,r}+r\sigma_{rr,rr}$$

- Now we need our constitutive law

$$E\,\epsilon_{\theta\theta,r}=\ (\sigma_{\theta\theta,r}-\nu(\sigma_{rr,r}+\sigma_{zz,r}))$$

$$E\,\epsilon_{zz,r}=\ (\sigma_{zz,r}-\nu(\sigma_{\theta\theta,r}+\sigma_{rr,r}))=0$$

$$\epsilon_{rr}=\frac{1}{E}(\sigma_{rr}-\nu(\sigma_{\theta\theta}+\sigma_{zz}))$$

$$\epsilon_{\theta\theta}=\frac{1}{E}(\sigma_{\theta\theta}-\nu(\sigma_{rr}+\sigma_{zz}))$$

$$\epsilon_{zz}=\frac{1}{E}(\sigma_{zz}-\nu(\sigma_{\theta\theta}+\sigma_{rr}))$$

- We combine these equations to get

$$E\,\epsilon_{\theta\theta,r}=\sigma_{\theta\theta,r}-\nu(\sigma_{rr,r}+\nu(\sigma_{\theta\theta,r}+\sigma_{rr,r})) \qquad E\,\epsilon_{\theta\theta,r}=\ (1-\nu^2)\,\sigma_{\theta\theta,r}+\nu(1+\nu)\sigma_{rr,r}$$

$$E(\epsilon_{rr}-\epsilon_{\theta\theta})=(1+\nu)(\sigma_{rr}-\sigma_{\theta\theta})$$

# Develop equations for the stress within a pressurized cylinder with thick walls

- We need one more relationship from the definition of the strain

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \;\; \epsilon_{\theta\theta} = \frac{u_r}{r}; \qquad \epsilon_{\theta\theta,r} = \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r = \frac{1}{r}(\epsilon_{rr} - \epsilon_{\theta\theta})$$

- From the previous slide, we have

$$E\,\epsilon_{\theta\theta,r} = \left(1-\nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} \qquad E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- If we combine these, we get

$$(1-\nu^2)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} = \frac{1}{r}(1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- Again, from the previous slide

$$\sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr} \qquad \sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$$

- We end up with

$$r\,\sigma_{rr,rr} + 3\sigma_{rr,r} = 0$$

$$\frac{\partial}{\partial r}\left(r^3\frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

# We now have an ODE so we can solve to get the stress

$$\frac{\partial}{\partial r}\left(r^3 \frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

- Our boundary conditions are $\sigma_{rr}(R_i) = -p$ and $\sigma_{rr}(R_o) = 0$
- After integrating twice, we get

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1}$$

- From $\sigma_{rr}$, we can get $\sigma_{\theta\theta}$ from

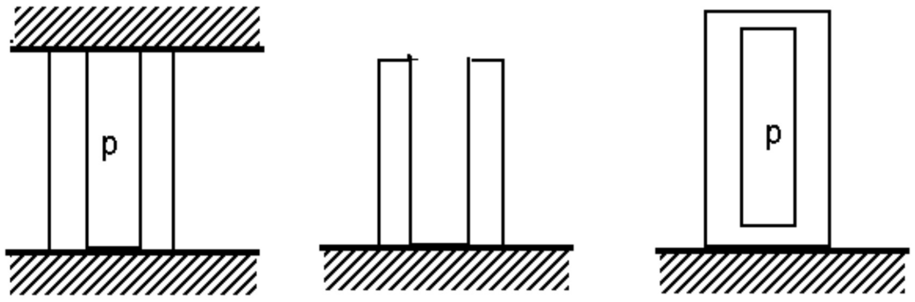
$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1}$$

- From the end condition, we determine $\sigma_{zz}$

(a) $\sigma_{zz} = p\dfrac{2\nu}{(R_o/R)^2 - 1}$

(b) $\sigma_{zz} = 0$

(c) $\sigma_{zz} = p\dfrac{1}{(R_o/R)^2 - 1}$




(a) complete axial restraint (b) no axial restraint (c) closed-end tube

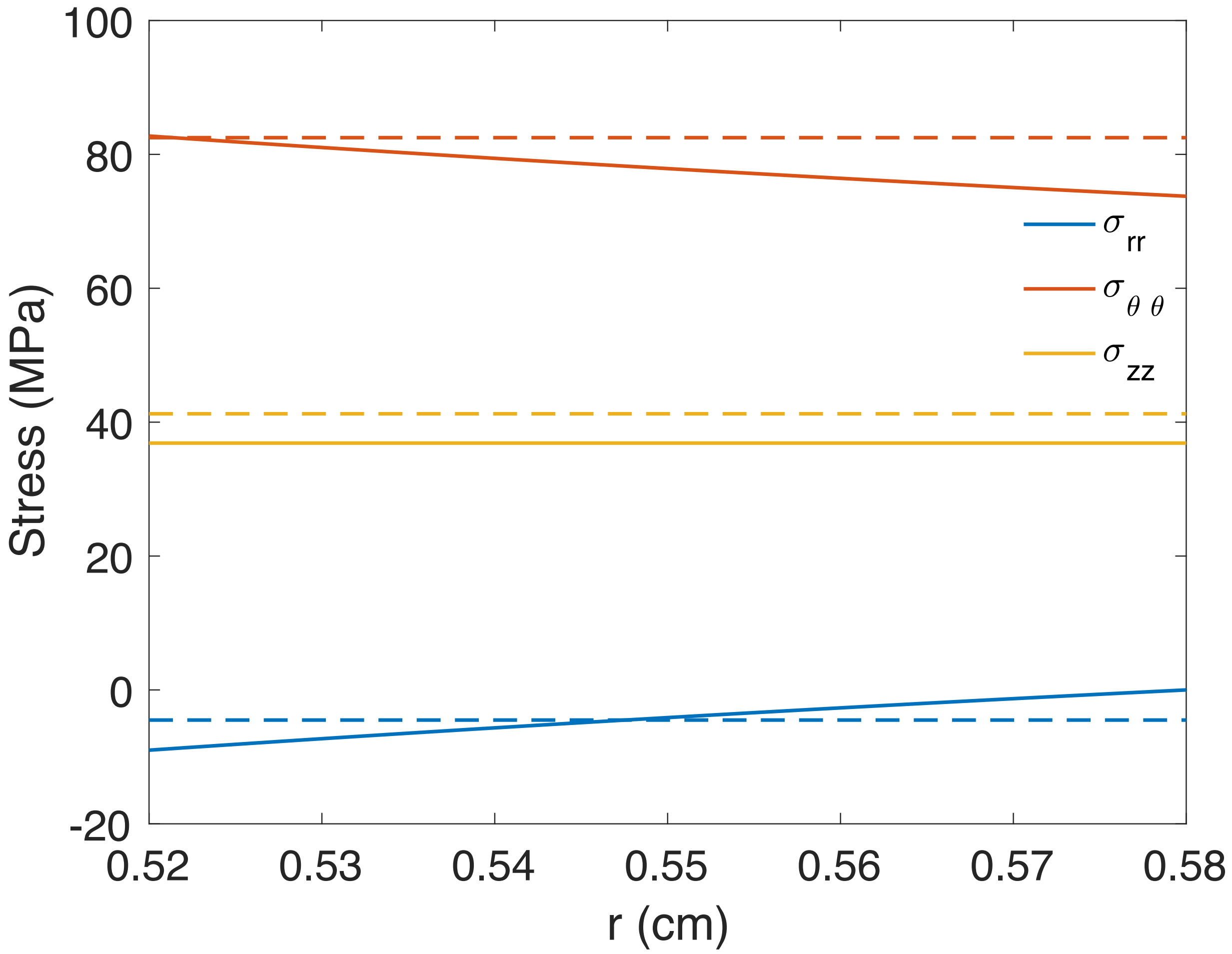
# Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - $R_i$ = 0.52, $R_o$ = 0.58 cm, $\sigma_y$ = 381 MPa

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1} \quad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1} \quad \sigma_{zz} = p\frac{1}{(R_0/R_i)^2 - 1}$$

- Which stress is the largest?
  - $\sigma_{\theta\theta}$
- At what position r will the hoop stress be the largest?
  - r = $R_i$ = 0.52 cm
- What is the stress at p = 5 MPa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 46.0 MPa
- What is the stress at p = 9 Mpa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 82.7 Mpa
- Again, we don’t even get close to yielding

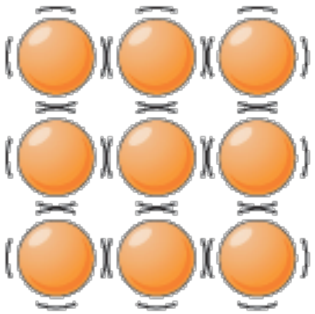
# We've now solved this same problem assuming thin and thick walled cylinders



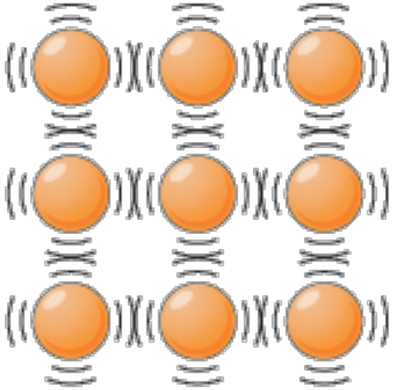
# THERMO-MECHANICS

# Thermal Expansion

- As the temperature increases, atoms have larger vibrations, causing the material to expand
- This expansion doesn't raise the energy of the material when unrestrained
- This expansion can be treated as a strain, but as one that doesn't cause stress



Cold



Hot

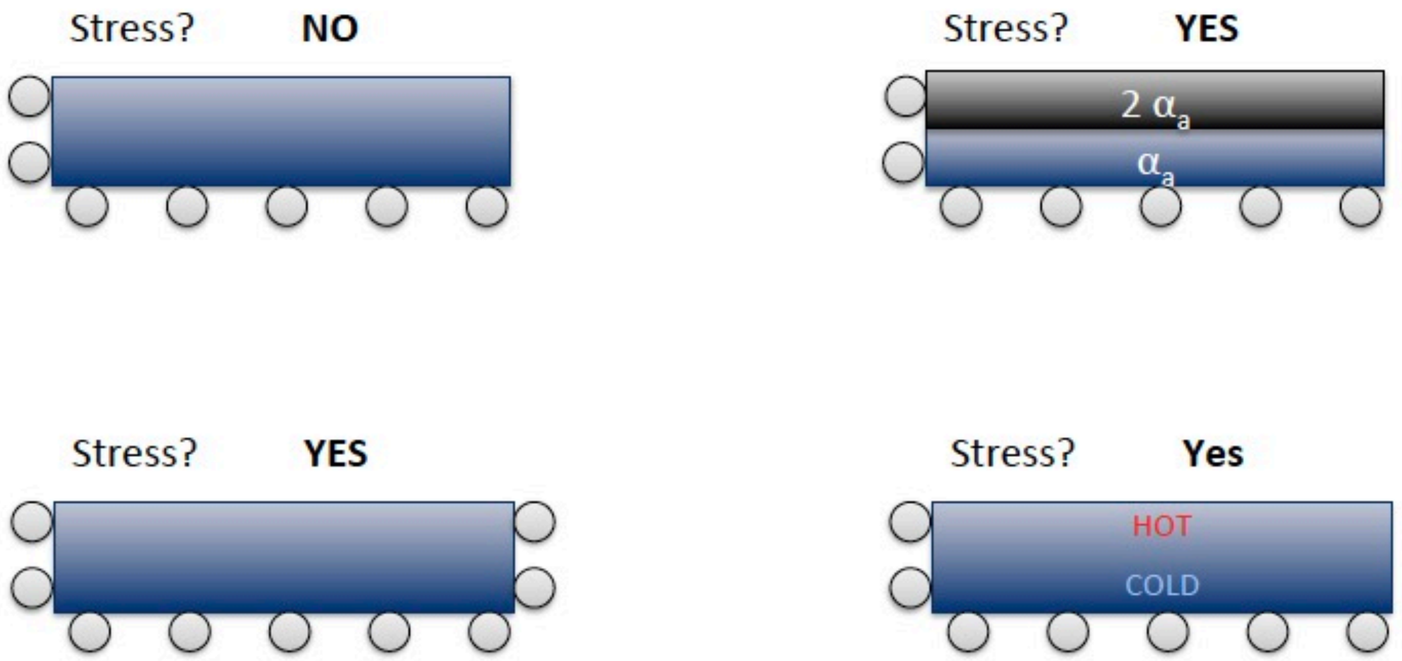
# Thermal Expansion

- In isotropic materials, thermal expansion happens equally in all directions and is linear with temperature
- $\epsilon_0 = (T - T_0)\alpha I$
- In this equation
  - T is the current temperature
  - $T_0$ is the temperature the original size was measured
  - α is the linear thermal expansion coefficient
  - I is the identity tensor

| Material | α ( × $10^{-6}$ 1/K) |
|---|---|
| Aluminum | 24 |
| Copper | 17 |
| Steel | 13 |
| $UO_2$ | 11 |
| Zircaloy (Axial) | 5.5 |
| Zircaloy (radial) | 7.1 |

# Thermal Expansion

- Though thermal expansion doesn't directly cause stress, it can still lead to thermal stress

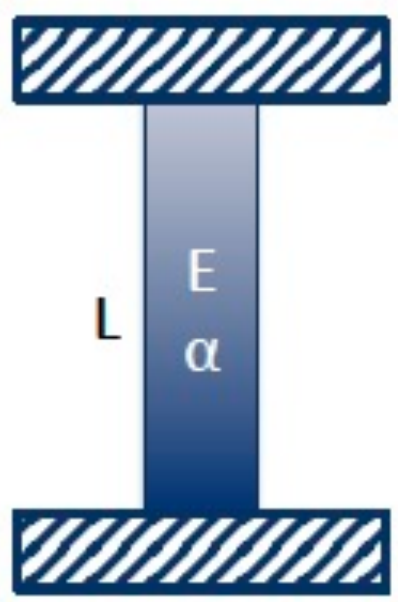


# What is the stress in a thin constrained rod of length L when it is heated to ΔT?

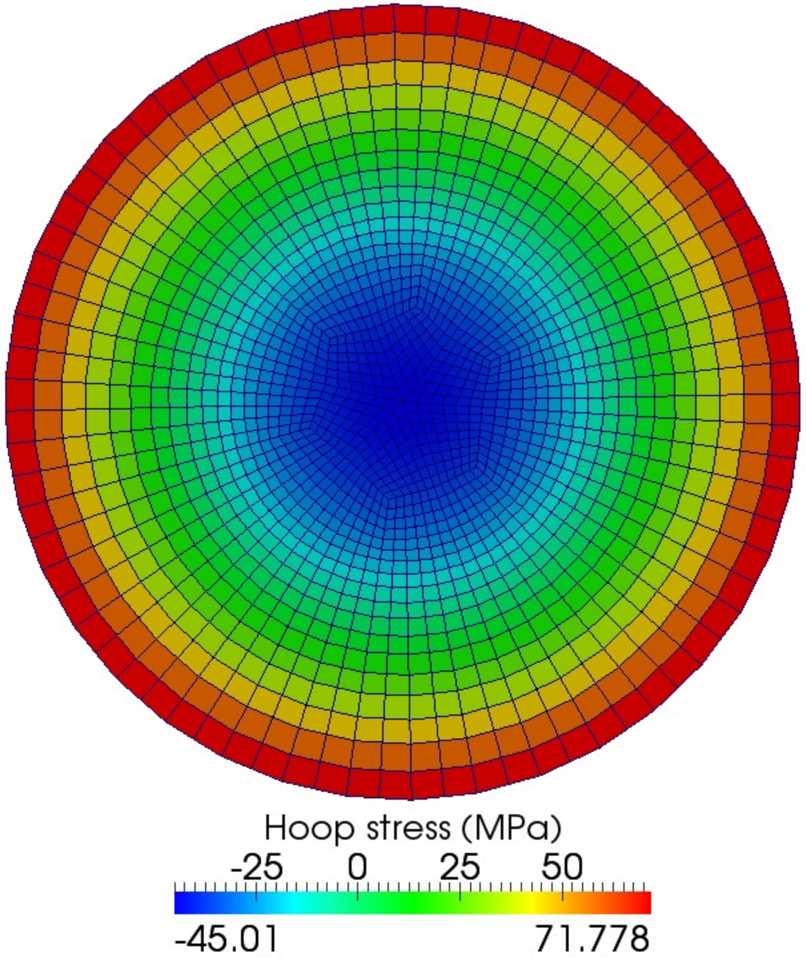
- The rod has a Young's modulus of E and an expansion coefficient of α

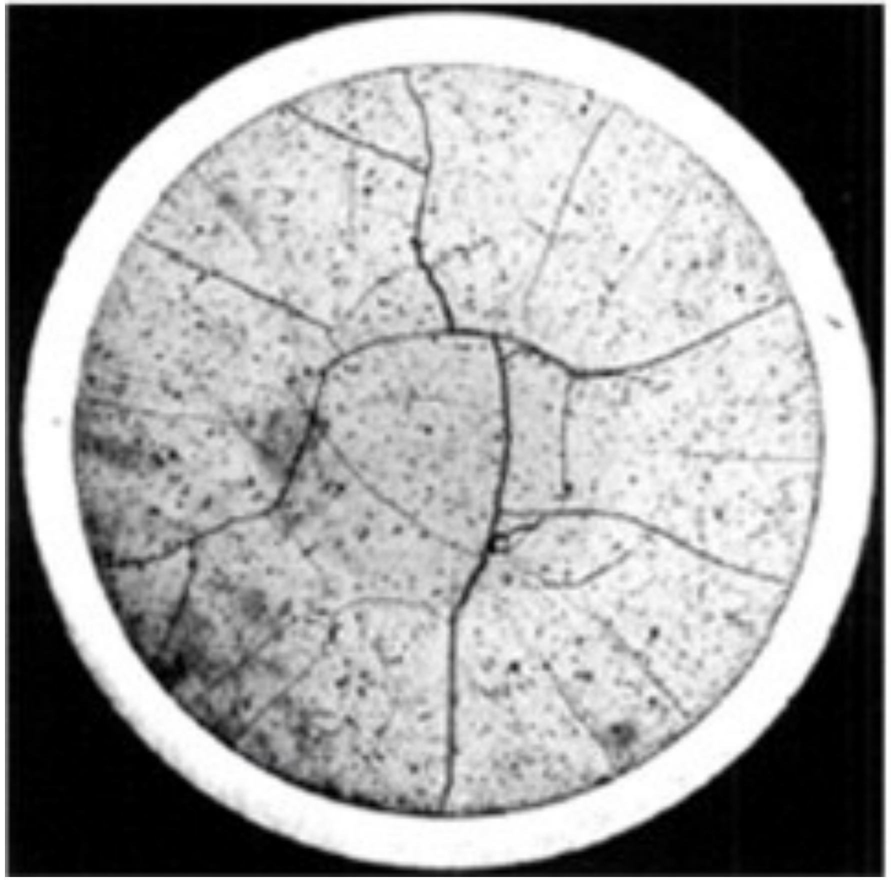
$$\epsilon_0 = (T - T_0)\alpha I \qquad \sigma = C\,(\epsilon - \epsilon_0)$$

$$\epsilon_0 = (T - T_0)\alpha$$
$$\sigma = E\,(0 - \Delta T\alpha)$$
$$\sigma = -E\Delta T\alpha$$



# The large temperature gradient within a fuel pellet results in large thermal stresses



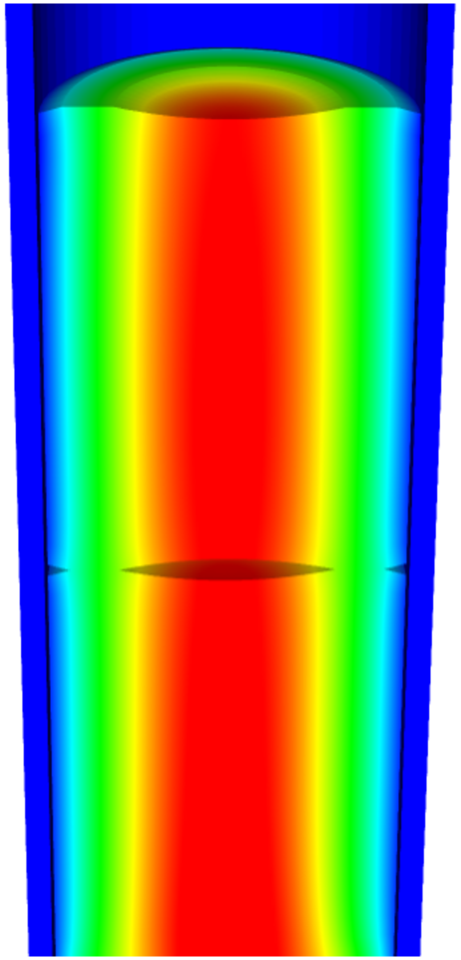
# Consider the material response of the axisymmetric body

- We assume small strains, so the strain is defined as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r},\ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

- We assume isotropic material response, so

$$\epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})) + \alpha\Delta T$$
$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})) + \alpha\Delta T$$
$$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})) + \alpha\Delta T$$
$$\epsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$

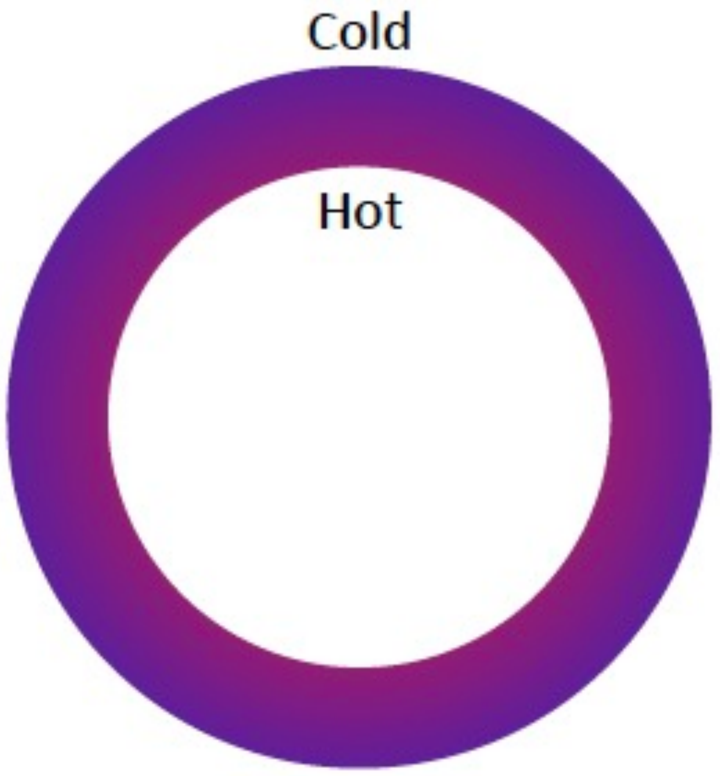


# Consider a cylinder with thermal expansion but not pressure

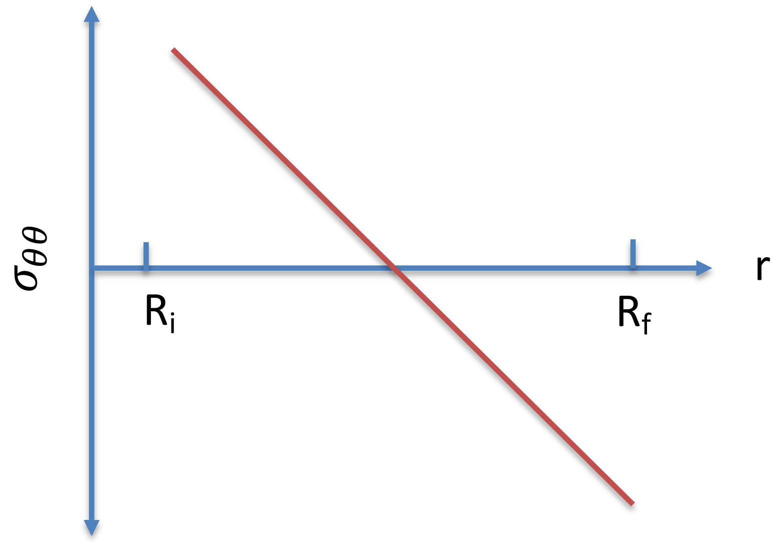
- $\sigma_{rr}(R_i) = \sigma_{rr}(R_0) = 0$
- Similar to the equations we worked through before
- $\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$
- Solving this ODE:

$$\sigma_{rr}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(\frac{r}{R_i}-1\right)\left(1-\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$
$$\sigma_{zz}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-2\frac{R_i}{\delta}\left(\frac{r}{R_i}-1\right)\right)$$

# What is the hoop stress in the cladding?




$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$



# Where is hoop stress equal to zero?

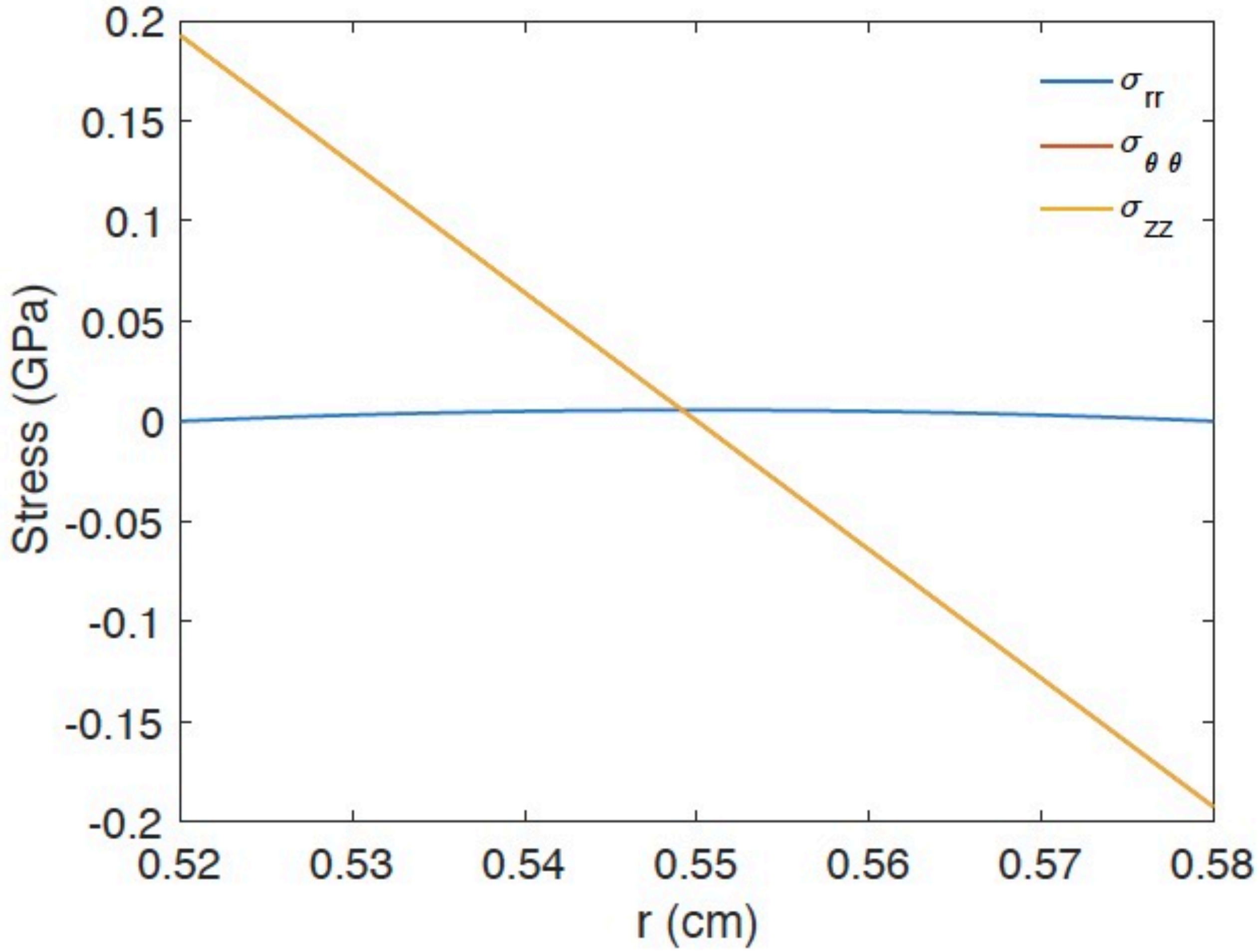
$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right)$$

$$\sigma_{\theta\theta}(r) = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0$$

$$\left(1 - 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right)\right) = 0 \longrightarrow 2\frac{R_i}{\delta}\left(\frac{r}{R_i} - 1\right) = 1 \longrightarrow \left(\frac{r}{R_i} - 1\right) = \frac{\delta}{2R_i}$$

$$r = \frac{\delta}{2} + R_i$$

# The linear temperature gradient across the cladding causes axial thermal stresses



# Same approach to the thermal stress in a fuel pellet

- The thermal stress is due to the temperature gradient

$$T(r) = \frac{LHR}{4\pi k}\left(1 - \frac{r^2}{R_f^2}\right) + T_s$$

$$T_0 - T_s = \frac{LHR}{4\pi k}$$

$$T - T_s = (T_0 - T_s)\left(1 - \frac{r^2}{R_f^2}\right)$$

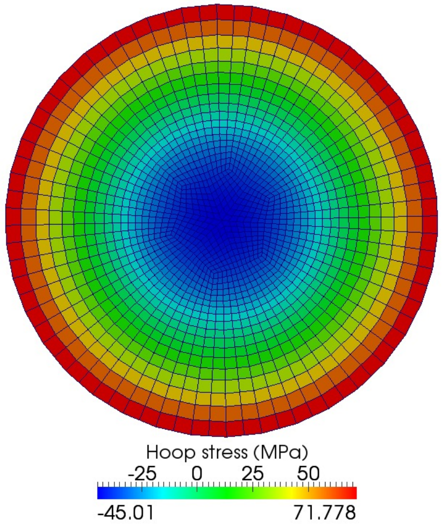
$$\frac{\partial T}{\partial r} = 2(T_0 - T_s)\left(\frac{r}{R_f^2}\right)$$

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr}$$

$$\eta = \frac{r}{R_f}$$

$$\sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

$$\frac{d}{d\eta}\left(\eta^3\frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^*\eta^3$$



# Solve this stress ODE

$$\frac{d}{d\eta}\left(\eta^3 \frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^* \eta^3 \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)}$$

- The boundary conditions are:

$$\frac{d\sigma_{rr}}{d\eta} = 0 \text{ at } \eta = 0$$

$$\sigma_{rr} = 0 \text{ at } \eta = 1$$

- Once we solve it, we obtain

$$\boxed{\sigma_{rr}(\eta) = -\sigma^*(1-\eta^2)}$$

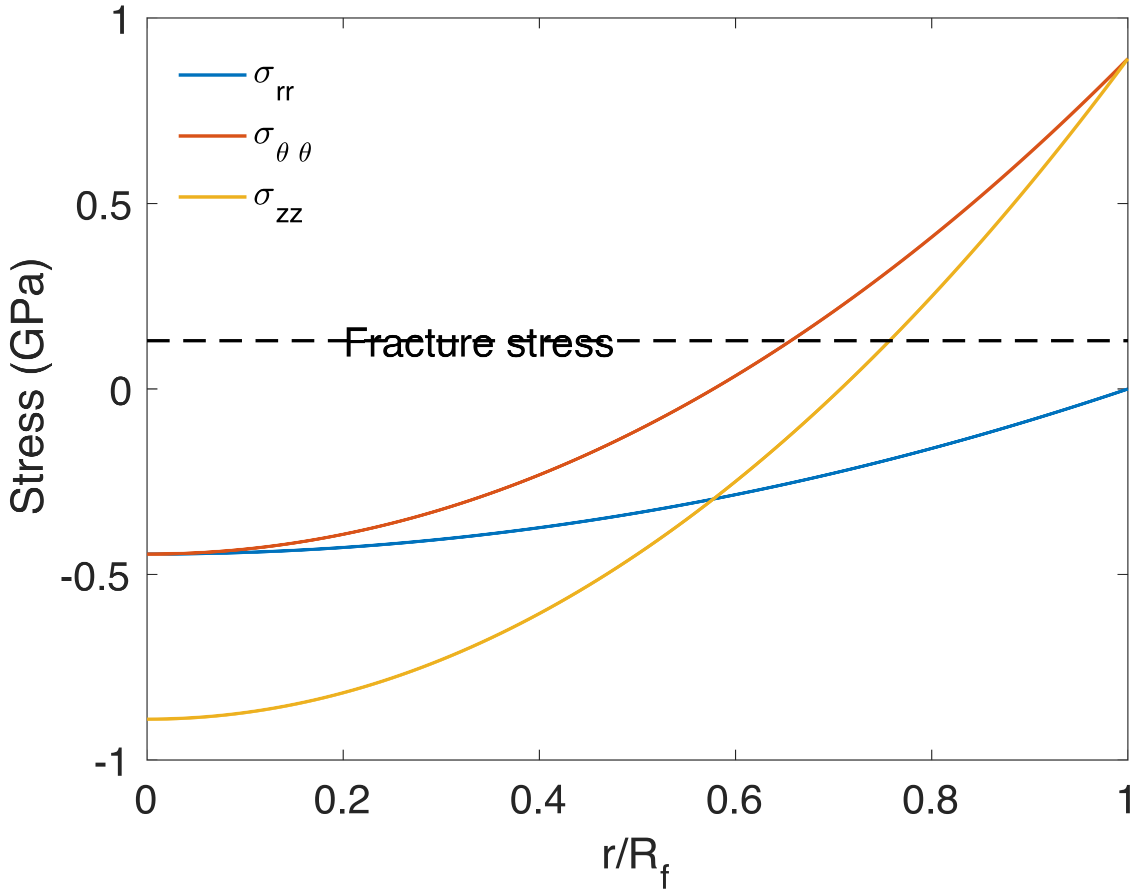
- Then we can solve the hoop stress

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \boxed{\sigma_{\theta\theta}(\eta) = -\sigma^*(1-3\eta^2)}$$

- The axial stress is more complicated to obtain, but you end up with

$$\boxed{\sigma_{zz}(\eta) = -2\sigma^*(1-2\eta^2)}$$

# The fuel temperature gradient causes large thermal stresses



# How far do fuel cracks extend?

$$\sigma_{\theta\theta}(\eta) = -\sigma^*(1-3\eta^2) \qquad \sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)} \qquad \eta = \frac{r}{R_f}$$

- E = 200 GPa, v = 0.345, α = 11.0e-6 1/K, $\sigma_{fr}$ = 130 MPa, ΔT = 550 K
- Solve for η
  - – $\sigma_{fr}$ / σ* = 1 – 3 $\eta^2$
  - 3 $\eta^2$ = 1 + $\sigma_{fr}$ / σ*
  - η = ( (1 + $\sigma_{fr}$ / σ*)/3 )$^{½}$
- σ* = 11.0e-6*200*550/(4*(1 – 0.345) ) = 461.8 MPa
- η = sqrt( (1 + 130/461.8)/3 ) = 0.65

# Summary

- We derived two analytical solutions for the stress in a pressurized cylinder
  - Thin walled cylinder
  - Thick walled cylinder
- Increases in temperature cause most materials to expand
- We call this thermal expansion, which is a strain that doesn't inherently cause stress
- Thermal expansion can cause stress when
  - Deformation is constrained
  - There are gradients in the expansion coefficient
  - There is a temperature gradient
- We have analytical equations for thermal stresses in the cladding and in the fuel