# Mechanics

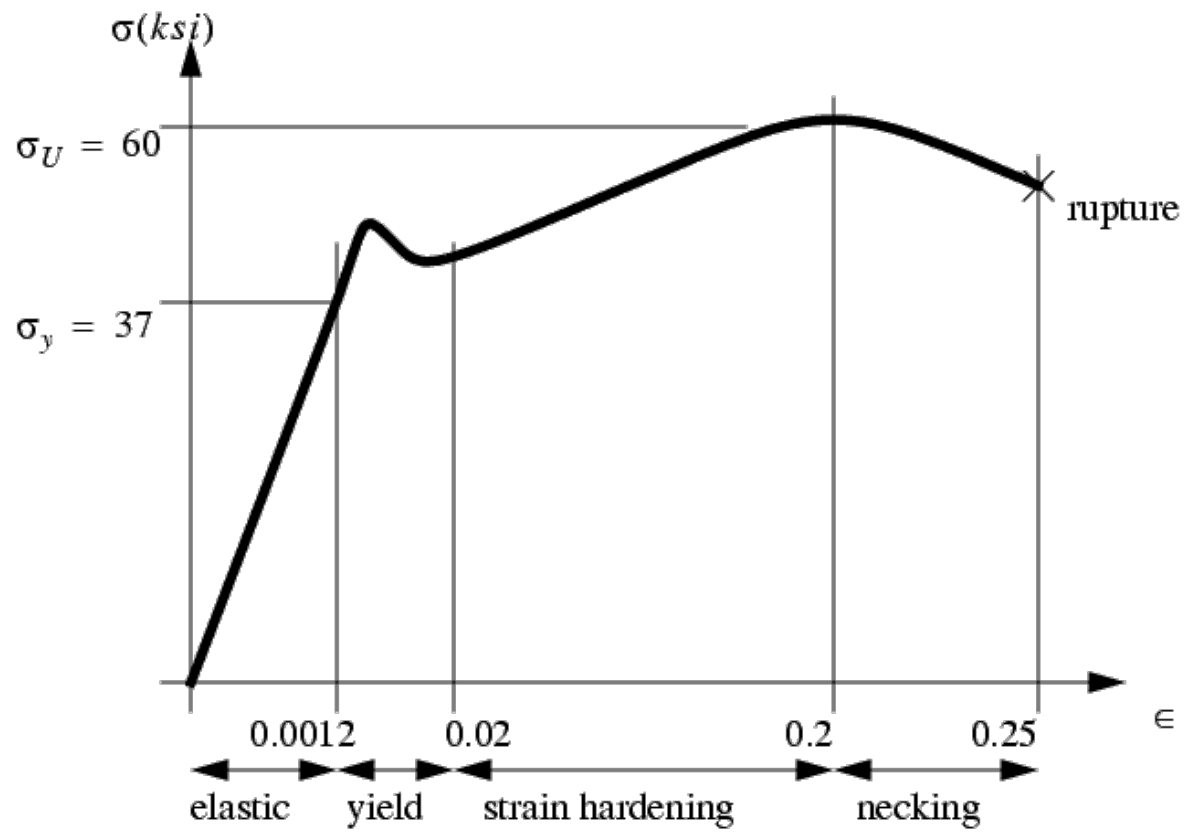
NE 591

# Last Time

- Solid mechanics predicts the deformation of a body from its applied load
- The strain defines the deformation
- The stress defines the material’s response to the strain
- Materials can have recoverable and permanent deformation
  - Elastic deformation is recoverable and occurs due to the stretching of the atomic lattice
  - Plastic deformation is permanent and results form the breaking of bonds

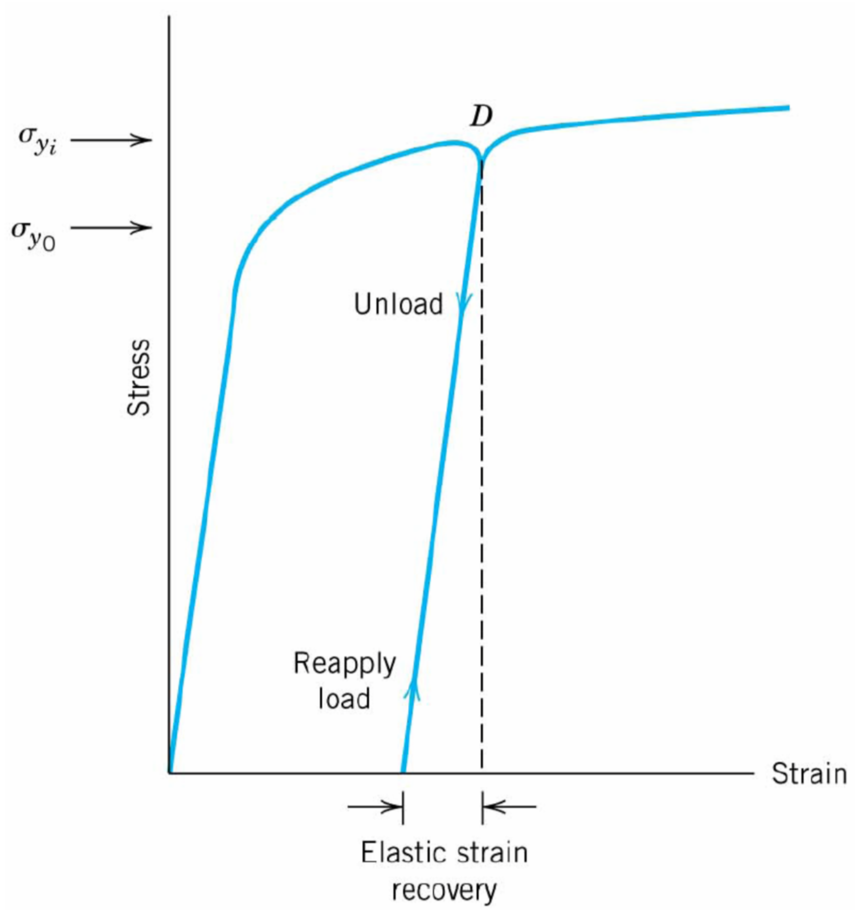
# Stress/Strain Regions

- Once the stress reaches the yield stress, it plastically deforms
- $\sigma_y$ is the yield stress
- $\sigma_U$ is the ultimate tensile stress
- The final stress before rupture is called the fracture stress



# Permanent Strain

- After plastic deformation, if you unload the sample, it still has the permanent strain
- After unloading, the strain hardening has changed the material
- Thus, if you reload, the yield strain is often higher than it was previously

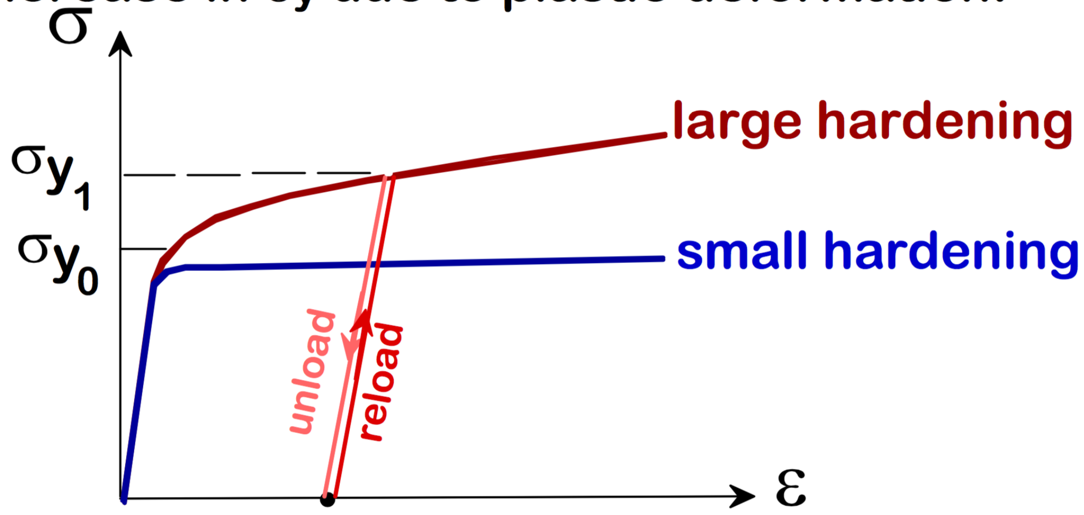


# The hardening behavior changes for different materials

$\sigma = K \varepsilon^{n}$

The value of the strain hardening exponent lies between 0 and 1. A value of 0 means that a material is a perfectly plastic solid, while a value of 1 represents a 100% elastic solid.

- **An increase in $\sigma_y$ due to plastic deformation.**




- **Curve fit to the stress-strain response:**

$$\sigma_T = C(\varepsilon_T)^n$$

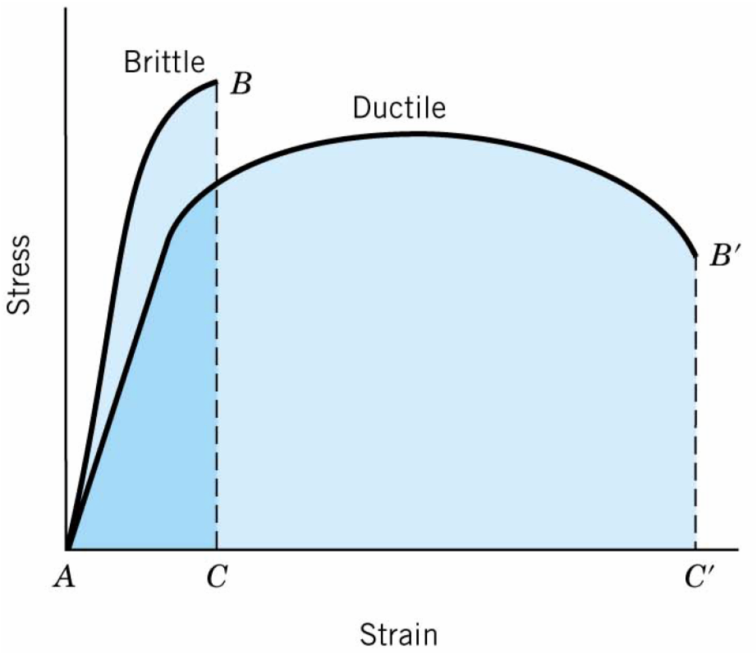
**hardening exponent:**
**n=0.15 (some steels)**
**to n=0.5 (some copper)**

**"true" stress (F/A)**

**"true" strain: $\ln(L/L_o)$**

# Ductility

- Ductile materials plastically deform significantly, brittle materials do not
- The ductile–brittle transition temperature (DBTT) of a metal is the temperature at which stress/strain response changes




**Ductility**

$$\%EL = \frac{(l_f - l_0)}{l_0} \times 100$$

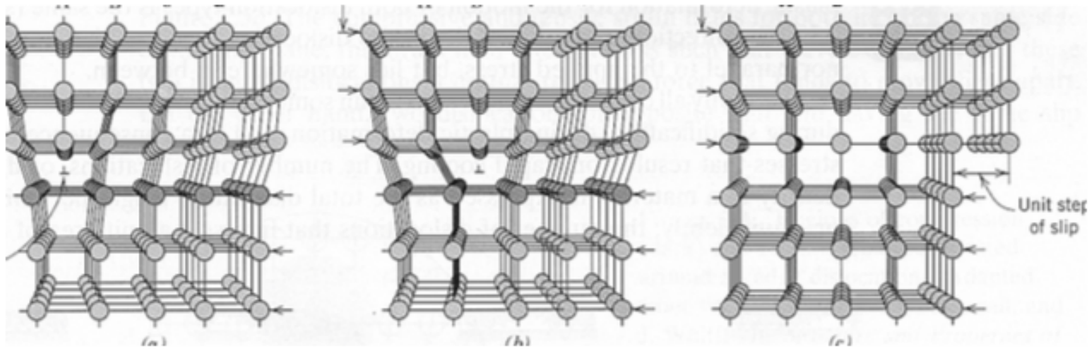
$$\%RA = \frac{(A_0 - A_f)}{A_0} \times 100$$

$l_f$ = length at fracture
$A_f$ = section area at fracture

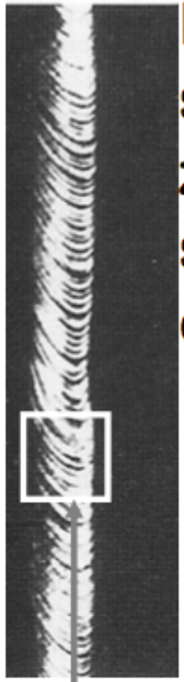
**Typical Ductility for soft metals %EL**
25 – 75%

# Dislocation motion

- Plastic deformation occurs due to dislocation motion
- A dislocation is a line defect
  - Edge and screw type
- When it moves, only a small number of bonds are broken at a time

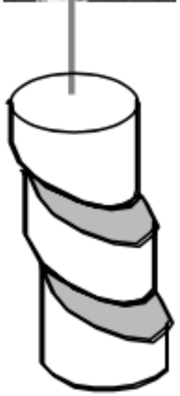



Adapted from Fig. 7.1, *Callister 6e.* (Fig. 7.1 is adapted from A.G. Guy, *Essentials of Materials Science*, McGraw-Hill Book Company, New York, 1976. p. 153.)



**Plastically stretched zinc single crystal.**

Adapted from Fig. 7.9, *Callister 6e.* (Fig. 7.9 is from C.F. Elam, *The Distortion of Metal Crystals*, Oxford University Press, London, 1935.)

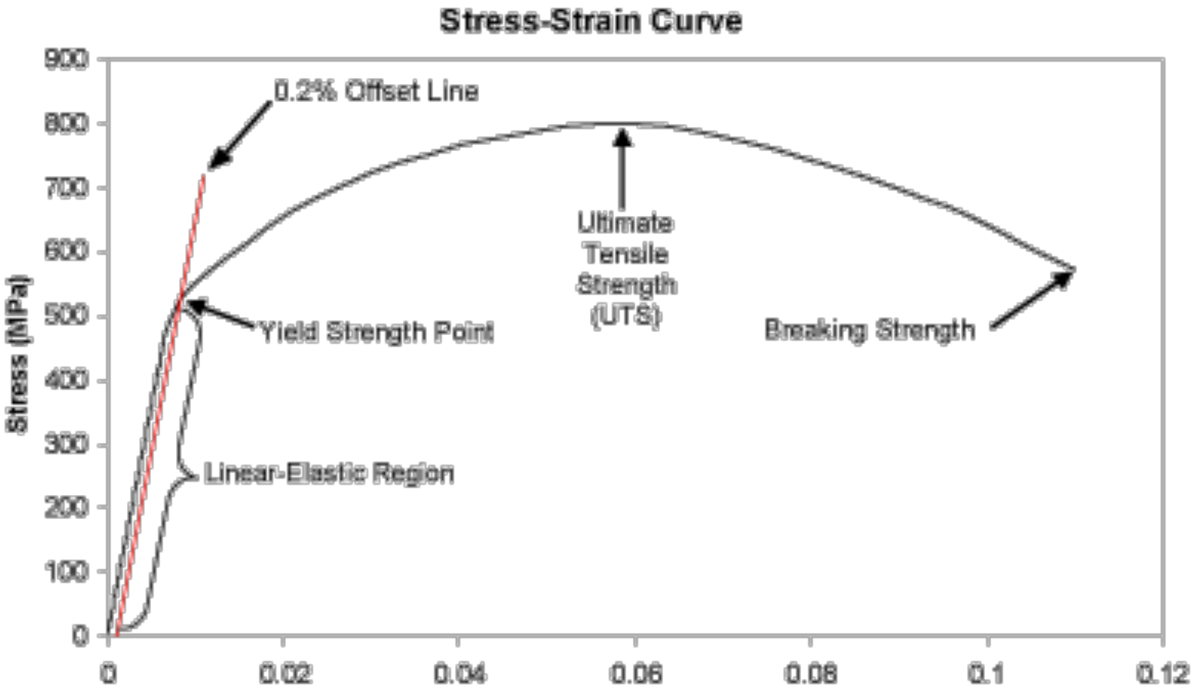


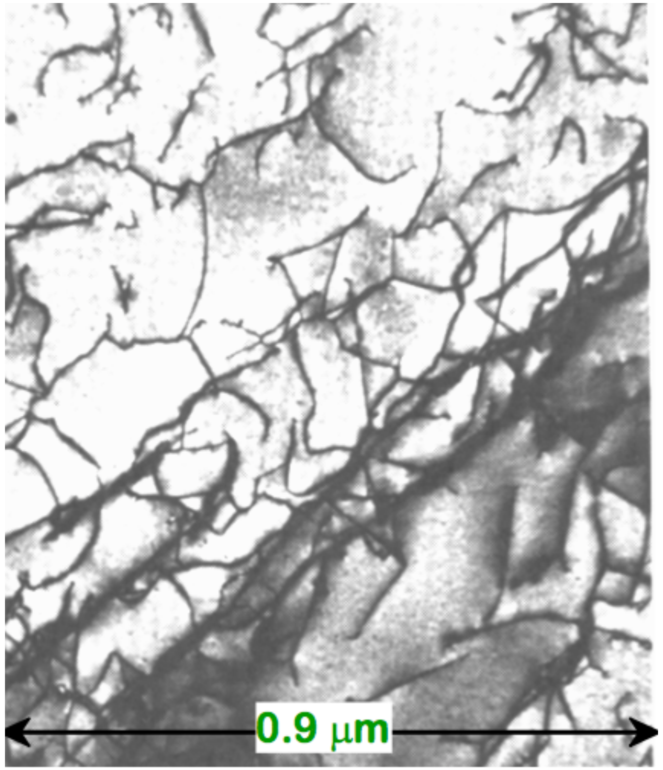
Adapted from Fig. 7.8, *Callister 6e.*

# Dislocation are produced during deformation

# Dislocation motion causes plastic deformation, dislocation pileup causes hardening

# As the dislocation density increases, dislocations cannot move freely, raising the yield stress



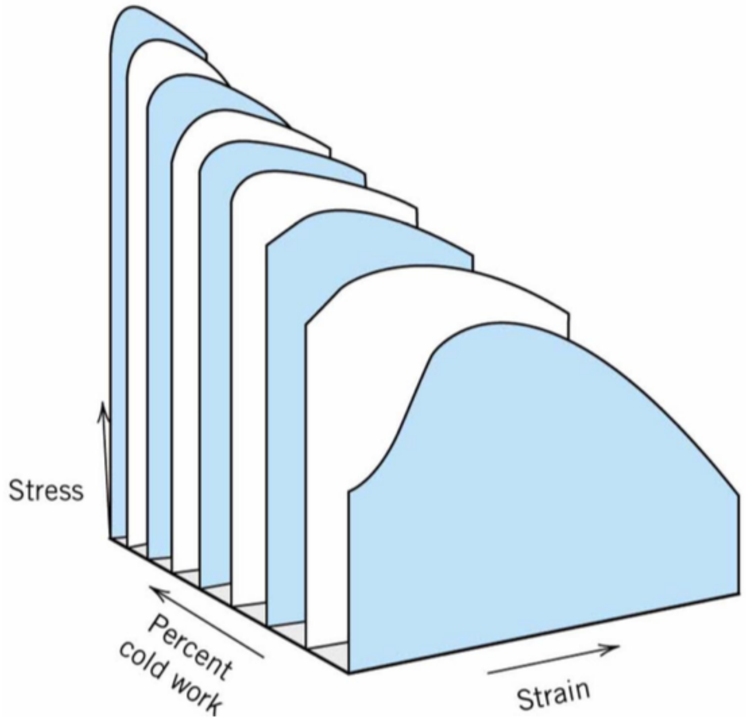



# As the strength increases due to dislocation pile-up, the ductility decreases

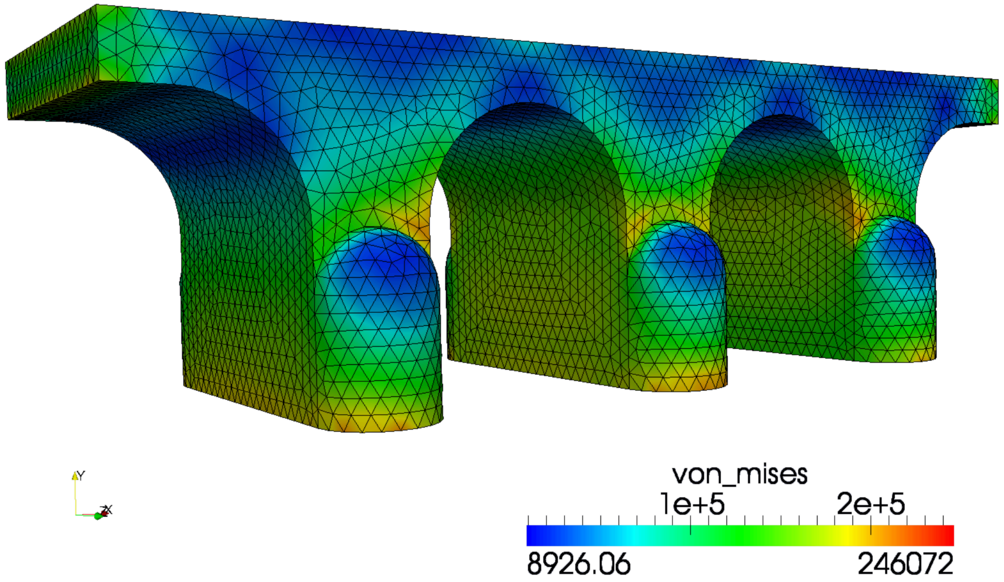
- Toughness is the ability of a material to absorb energy and plastically deform without fracturing
- Toughness is related to the area under the stress–strain curve
- In order to be tough, a material must be both strong and ductile

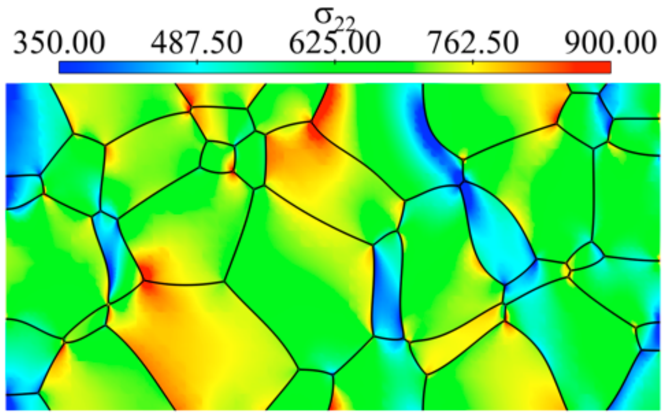
**Stress-strain curves for a material with progressively increasing cold work**

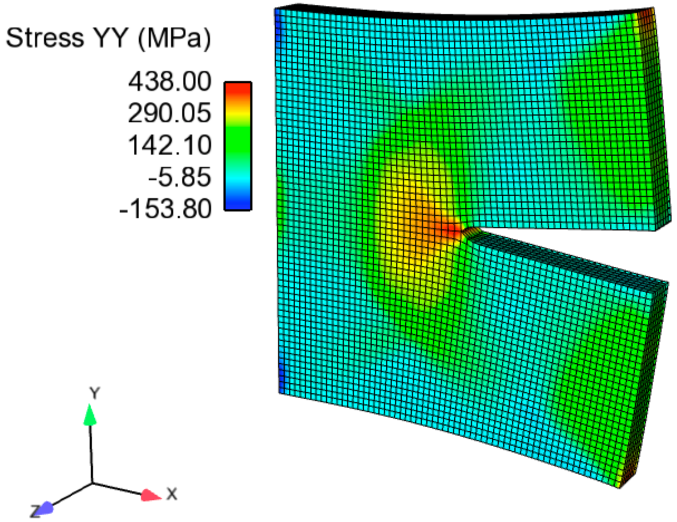
What happens to the energy to failure (toughness) with increasing cold work?



# We need to determine the stress and strain throughout a body



# Cauchy momentum equation

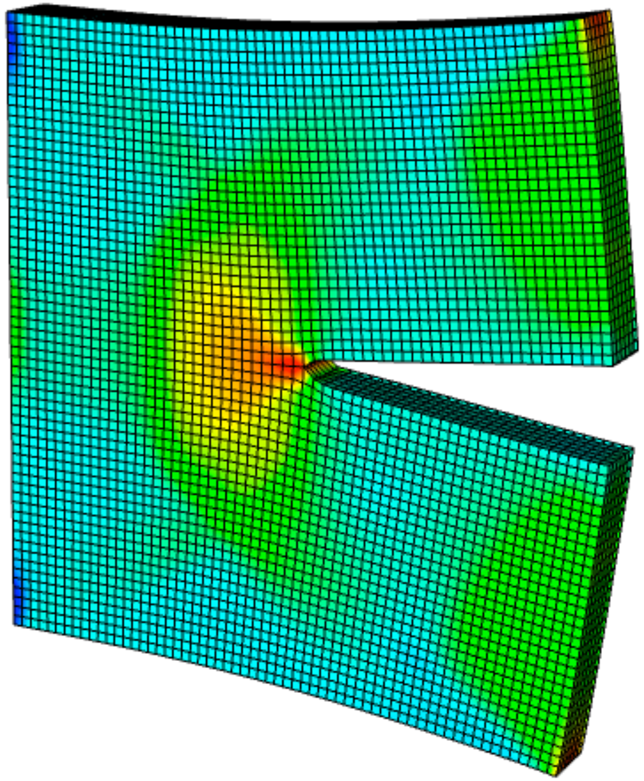
$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{g}$$

- In this equation, **u** is the displacement vector, ρ is the density, **σ** is the stress, and **g** is the acceleration of gravity.
- Often, we are determining the stress and strain in a static body. So the equation becomes:

$$0 = \nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{g}$$

- Finally, we can simplify things more by making our first assumption*: the impact of gravity is negligible*

$$0 = \nabla \cdot \boldsymbol{\sigma}$$



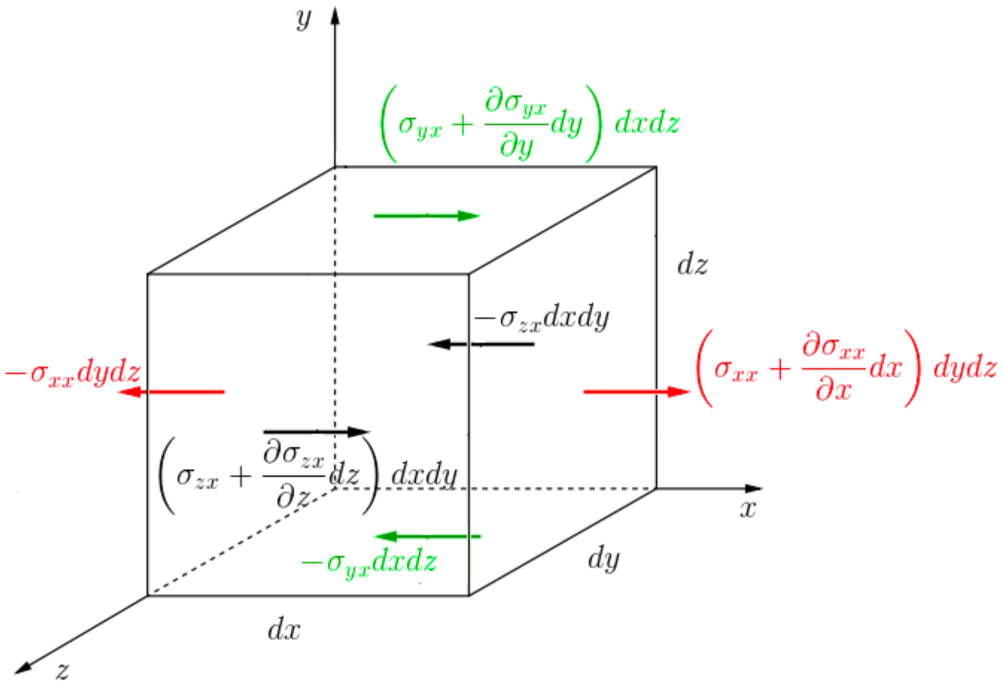
# The stress divergence equation is really just a force balance

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx - \sigma_{xx}\right) dy + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy - \sigma_{yx}\right) dx = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

$$0 = \nabla \cdot \boldsymbol{\sigma}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$



# Consider an axisymmetric body

- Assumption 1: We have a static body

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{g}$$

- Assumption 2: Gravity is negligible

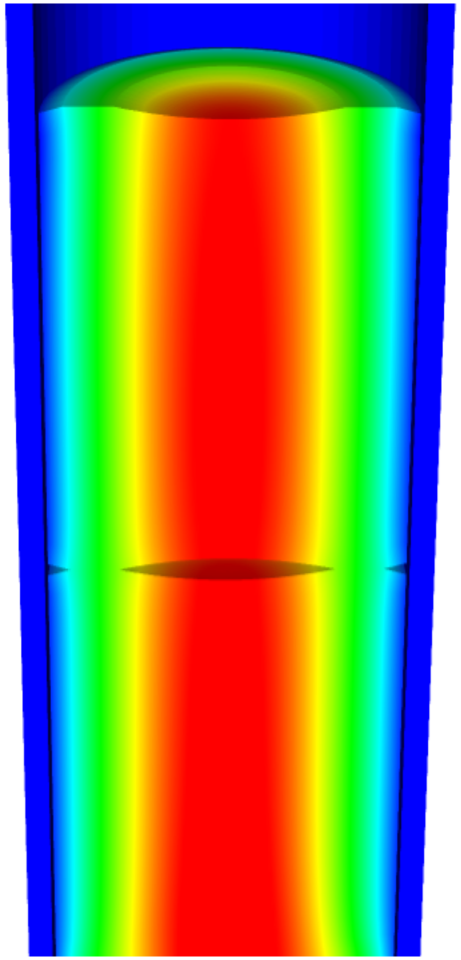
$$0 = \nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{g}$$

- Assumption 3: The problem is axisymmetric

$$0 = \nabla \cdot \boldsymbol{\sigma}$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$



# Consider the material response of the axisymmetric body

- We assume small strains (elasticity), so the strain is defined as

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \ \epsilon_{\theta\theta} = \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

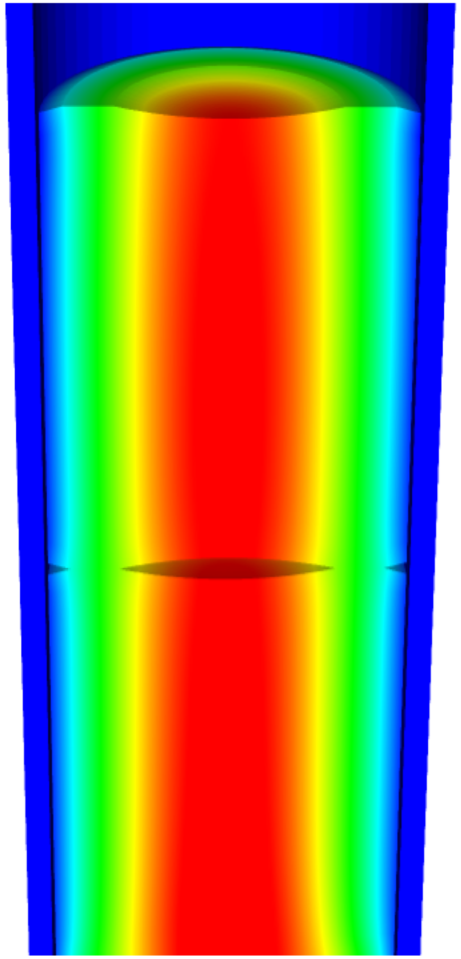
- We assume isotropic material response, so

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right)$$

$$\epsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right)$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right)$$

$$\epsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$

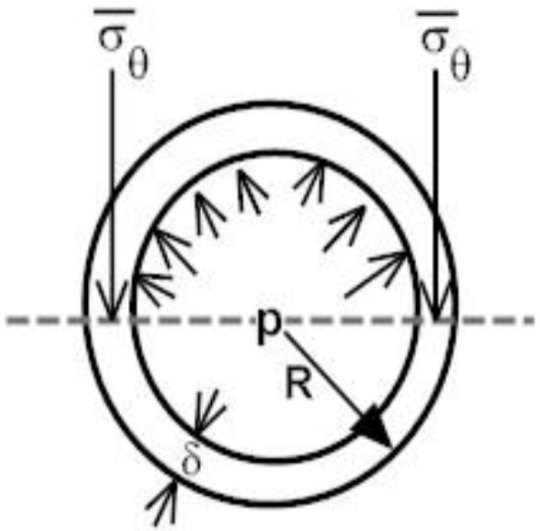


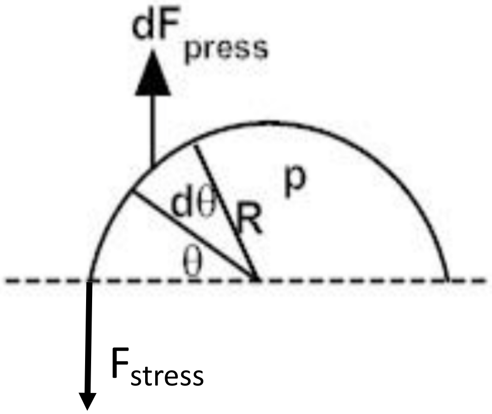
# Solve for the stress throughout a pressurized cylinder (cladding tube) with thin walls

- Because our cylinder has such thin walls ($\delta$ = thickness), we can simplify everything to simple force balance
- The hoop stress is the force exerted circumferentially
- First, we need the Force per unit length due to the pressure

$$F_{press} = pR \int_0^{\pi} \sin\theta \; d\theta \qquad\qquad F_{stress} = 2\delta \; \overline{\sigma_\theta}$$

- Then we equate the forces and solve for the hoop stress




$$\overline{\sigma_\theta} = \frac{pR}{\delta}$$

# Find the other two stresses for a thin walled cylinder

- To find the stress in the z-direction we do another force balance

$$F_{top} = p\,\pi R^2 \qquad F_{wall} = 2\pi R\delta\bar{\sigma}_z$$

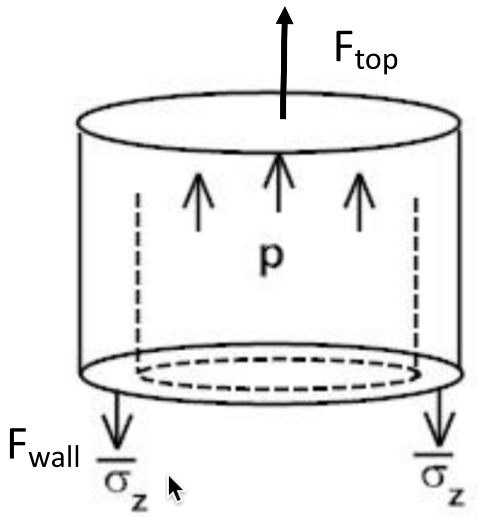
- Again, we equate the forces and solve for the stress

$$\overline{\sigma}_z = \frac{pR}{2\delta}$$

- Last, the stress on the inside of the wall is –p and on the outside is zero, so the average

$$\overline{\sigma}_r = -\frac{1}{2}p$$

$$\boxed{\overline{\sigma}_\theta = \frac{pR}{\delta} \qquad \overline{\sigma}_z = \frac{pR}{2\delta} \qquad \overline{\sigma}_r = -\frac{1}{2}p}$$



# Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - R = 0.55 cm, δ = 0.05 cm, $\sigma_y$ = 381 MPa

$$\overline{\sigma_\theta} = \frac{pR}{\delta} \qquad \overline{\sigma_z} = \frac{pR}{2\delta} \qquad \overline{\sigma_r} = -\frac{1}{2}p$$

- The largest stress will be the hoop stress
- The hoop stress is P*(0.55/.05)
  - For 5 MPa, σθ = 55 Mpa
  - For 9 MPa, σθ = 99 Mpa
- With these pressures, we don't come even close to the yield stress of the cladding

# Stress within a cylinder under pressure that has thicker walls (radius/thickness>10)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0 \qquad \frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- The cylinder as an inner radius $R_i$ and an outer Radius $R_o$
- We assume there is no shear stress, so $\sigma_{rz} = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad \frac{\partial \sigma_{zz}}{\partial z} = 0$$

- We will begin with the r equation

$$\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r} \qquad \sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr}$$

- Now we need our constitutive law

$$E\,\epsilon_{\theta\theta,r} = (\sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \sigma_{zz,r}))$$

$$E\,\epsilon_{zz,r} = (\sigma_{zz,r} - \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r})) = 0$$

$$\boxed{\begin{aligned} \epsilon_{rr} &= \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})) \\ \epsilon_{\theta\theta} &= \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})) \\ \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})) \end{aligned}}$$

- We combine these equations to get

$$\boxed{\begin{gathered} E\,\epsilon_{\theta\theta,r} = \sigma_{\theta\theta,r} - \nu(\sigma_{rr,r} + \nu(\sigma_{\theta\theta,r} + \sigma_{rr,r})) \qquad E\,\epsilon_{\theta\theta,r} = \left(1-\nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} \\ E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta}) \end{gathered}}$$

# Develop equations for the stress within a cylinder under pressure that has thicker walls

- We need one more relationship from the definition of the strain

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \ \epsilon_{\theta\theta} = \frac{u_r}{r}; \qquad \epsilon_{\theta\theta,r} = \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r = \frac{1}{r}(\epsilon_{rr} - \epsilon_{\theta\theta})$$

- From the previous slide, we have

$$E\,\epsilon_{\theta\theta,r} = \left(1-\nu^2\right)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} \qquad E(\epsilon_{rr} - \epsilon_{\theta\theta}) = (1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- If we combine these, we get

$$(1-\nu^2)\sigma_{\theta\theta,r} + \nu(1+\nu)\sigma_{rr,r} = \frac{1}{r}(1+\nu)(\sigma_{rr} - \sigma_{\theta\theta})$$

- Again, from the previous slide

$$\sigma_{\theta\theta,r} = 2\,\sigma_{rr,r} + r\sigma_{rr,rr} \qquad \sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$$

- We end up with

$$r\,\sigma_{rr,rr} + 3\sigma_{rr,r} = 0$$

$$\frac{\partial}{\partial r}\left(r^3\frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

# We now have an ODE so we can solve to get the stress

$$\frac{\partial}{\partial r}\left(r^3 \frac{\partial \sigma_{rr}}{\partial r}\right) = 0$$

- Our boundary conditions are $\sigma_{rr}(R_i) = -p$ and $\sigma_{rr}(R_o) = 0$
- After integrating twice, we get

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1}$$

- From $\sigma_{rr}$, we can get $\sigma_{\theta\theta}$ from $\sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{rr,r}$

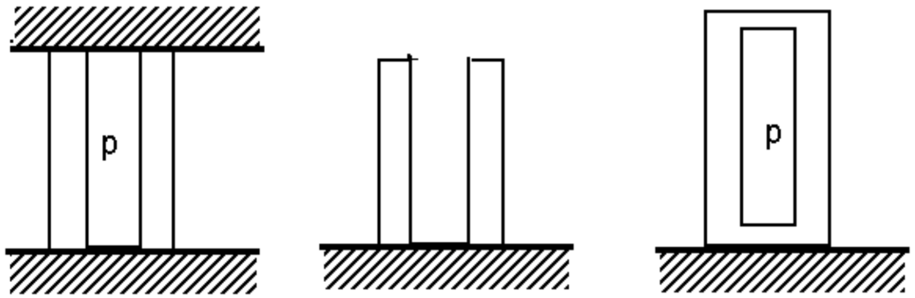
$$\sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1}$$

- From the end condition, we determine $\sigma_{zz}$

(a) $\sigma_{zz} = p\dfrac{2\nu}{(R_o/R)^2 - 1}$

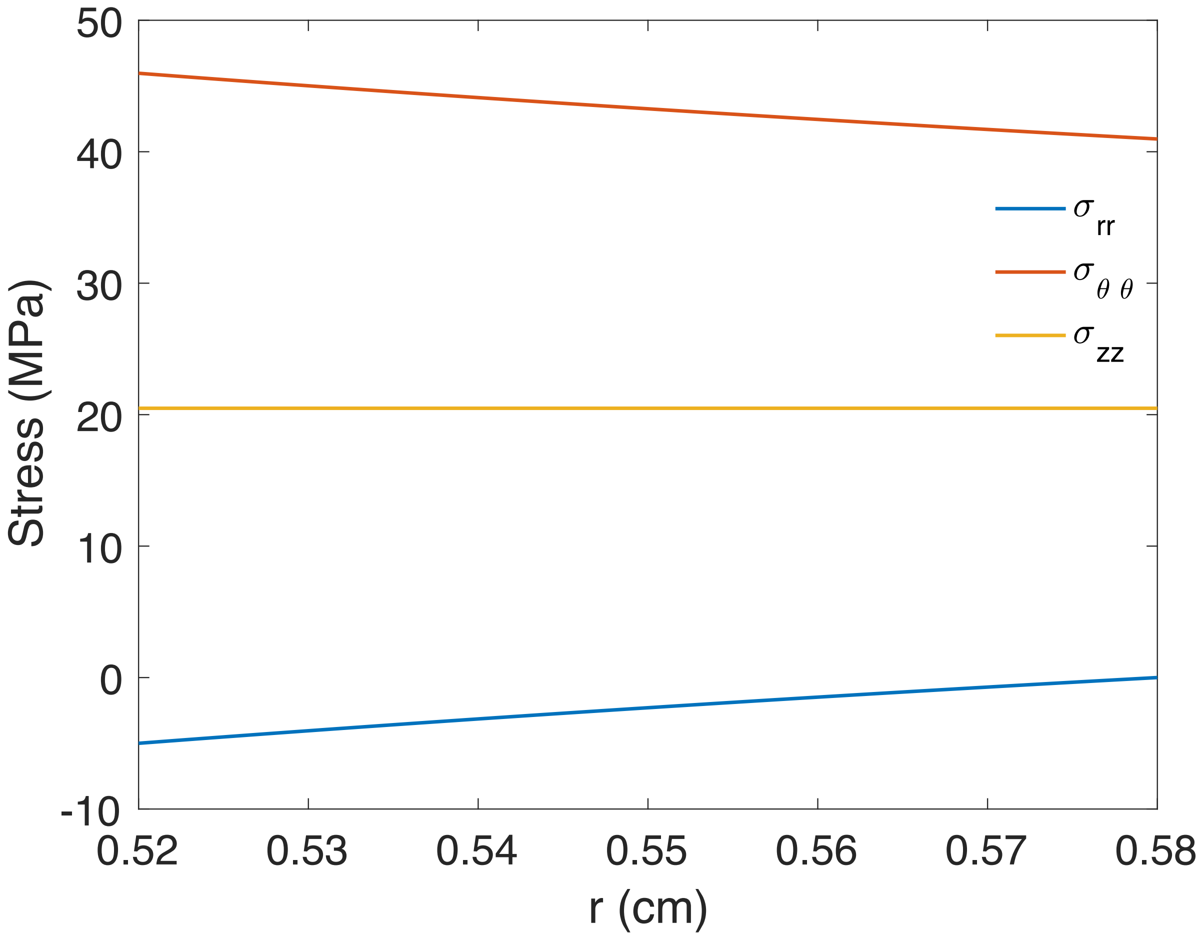
(b) $\sigma_{zz} = 0$

(c) $\sigma_{zz} = p\dfrac{1}{(R_o/R)^2 - 1}$




(a) complete axial restraint (b) no axial restraint (c) closed-end tube

# Stresses vary over the thickness of the cladding tube



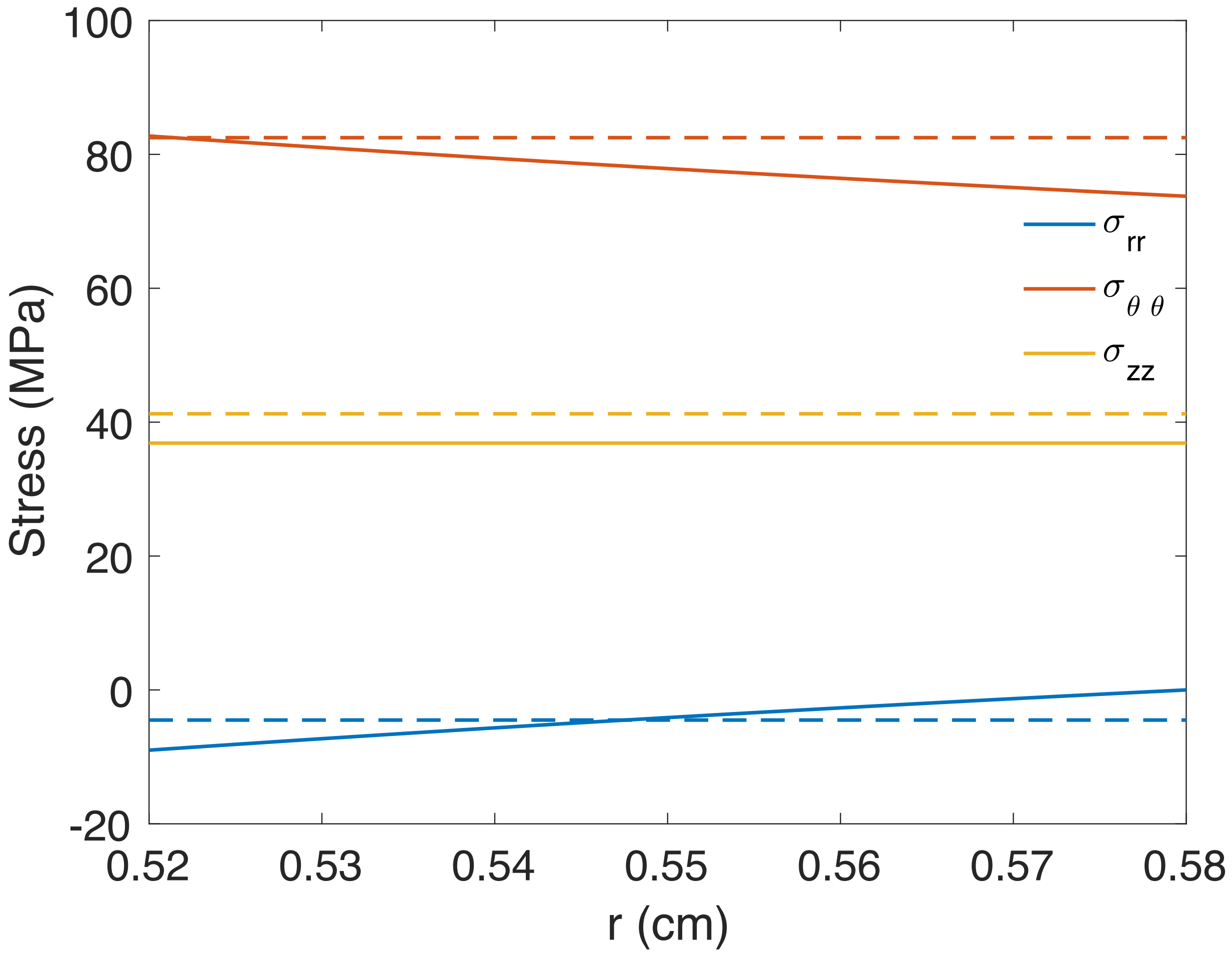
# Determine if a fuel rod will exceed its yield stress

- A fuel rod has an initial pressure of 5 MPa that increases to 9 MPa during operation. Will it exceed its yield stress?
  - $R_i$ = 0.52, $R_o$ = 0.58 cm, $\sigma_y$ = 381 MPa

$$\sigma_{rr}(r) = -p\frac{(R_o/r)^2 - 1}{(R_o/R_i)^2 - 1} \quad \sigma_{\theta\theta}(r) = p\frac{(R_o/r)^2 + 1}{(R_o/R_i)^2 - 1} \quad \sigma_{zz} = p\frac{1}{(R_0/R_i)^2 - 1}$$

- Which stress is the largest?
  - $\sigma_{\theta\theta}$
- At what position r will the hoop stress be the largest?
  - $r = R_i = 0.52$ cm
- What is the stress at p = 5 MPa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 46.0 MPa
- What is the stress at p = 9 Mpa?
  - $\sigma_{\theta\theta}$ = 5*((0.58/0.52)^2 + 1)/((0.58/0.52)^2 – 1) = 82.7 Mpa
- Again, we don’t even get close to yielding

# We've now solved this same problem assuming thin and thick walled cylinders



# Summary

- Solid mechanics predicts the deformation of a body from its applied load
  - The strain defines the deformation
  - The stress defines the material's internal response to the strain
- Materials can have recoverable and permanent deformation
  - Elastic deformation is recoverable and occurs due to the stretching of the atomic lattice
  - Plastic deformation is permanent and results form the breaking of bonds during dislocation (line defect) migration
- We derived two analytical solutions for the stress in a pressurized cylinder
  - Thin walled cylinder
  - Thick walled cylinder