On Squares Inscribed on Continuous Embeds in the Real Plane

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1 Introduction

The Inscribed Square Problem was formed in 1911 by Oliver Toeplitz. Here, Toeplitz conjectures that any continuous embedding of a circle onto the real plane λ always admits at least one inscribed square. While seemingly true in nature, the lack of a proof for this statement has caused the conjecture to gain popularity. In this paper, we shall consider a plane simple closed curve

$$\lambda : S^1 \hookrightarrow \mathbb{R}^2,$$

a non-self-intersecting Jordan curve that divides $\mathbb{R}^2$ into an inner and outer region, as shown in Figure 1.

Such a curve will be proved to admit an inscribed square given a certain topological mapping of a curve on the surface of a circular cylinder onto a Mobius strip; a self-intersection that always occurs during the mapping from a non-contractible nature justifies Toeplitz' statement.

We will start by discussing a continuous existence of a certain class of right triangles along λ. Next, we will show that some of these triangles share edges in

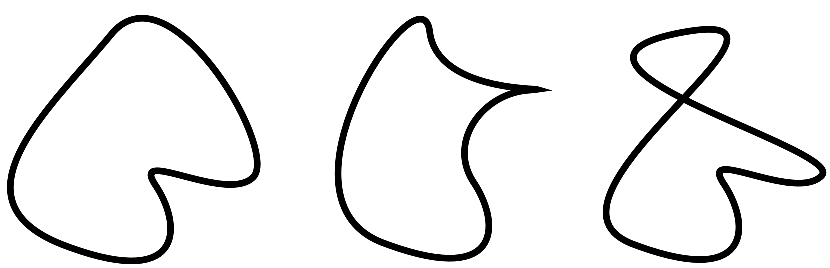

Figure 1: Three closed curves. The leftmost and middle curves can be classified as Jordan curves. The rightmost one cannot, as it crosses over iteslf.

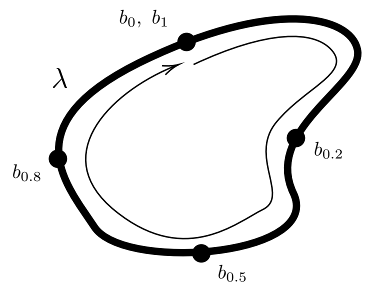

Figure 2: Each point on λ can be identified by a real number 0 to 1.

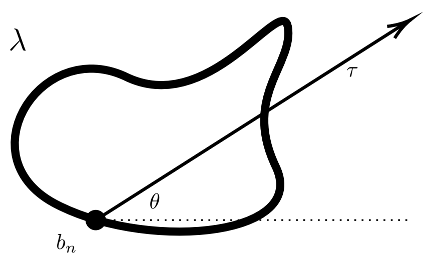

Figure 3: A ray τ extended from b_n with an angle θ to the horizontal.

a way that forms an inscribed square. Finally, we will use a topological mapping to show that this occurrence must always be present.

2 Parametrizing Distances within λ

Consider the set B, the set of all points b on λ. A map of all real numbers on $n \in [0, 1]$ to each item in B allows each point b_n to be classified by a number, as shown in Figure 2. These points will be classified in such a way that $b_{n+\Delta n}$ changes continuously. Hence,

$$b_0 \cong b_1$$

since λ is a closed curve; the start and end points are the same.

Definition 2.1. *Allow a ray τ to be extended from any point b_n. Let τ have a initial point at b_n with an angle θ with respect to the horizontal, as shown in Figure 3. Considering the set of intersection points*

$$I_{n,\theta} : \tau \cap \lambda,$$

let $L_{n,\theta}$ be defined as the set of distances between

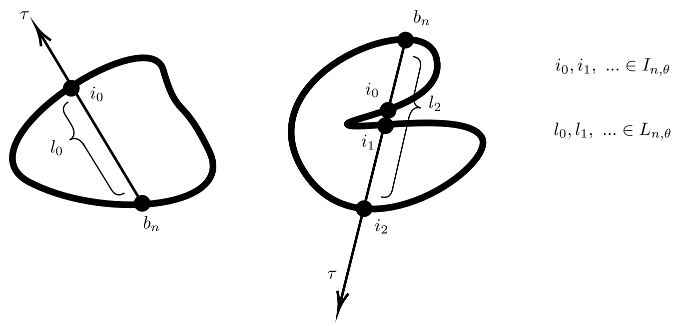

Figure 4: The lengths of the segments from b_n and $\forall i \in I_{n,\theta}$ are visualised.

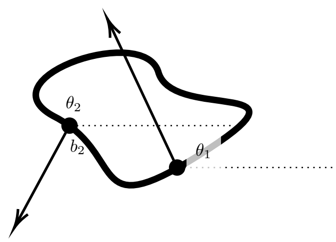

Figure 5: $g(n_1, \theta_1)$ meets condition C; $g(n_2, \theta_2)$ does not.

$$b_n \textit{ and } i, \forall i \in I \ \forall l \in L_{n,\theta}.$$

As shown in Figure 4. The function

$$g : g(n, \theta), n \in [0, 1], \theta \in [0, 2\pi] = L_{n,\theta}.$$

When τ is extended in a fashion where it never meets with λ, g returns an empty set. We will consider the converse, or the condition C when g returns a non-null set of real numbers, as shown in Figure 5.

Lemma 2.1. *For any chosen value n_0, $g(n_0, \theta)$ has a one unbroken θ domain whose outputs meet condition C and one unbroken θ domain whose outputs fail to meet C.*

Proof. The reason for this lemma is intuitive. Choosing a base point determined by n_0 and revolving the terminal point of ray around b_{n0} will have a period where it intersects γ, and usually a period where it does not:

- There is no instance where g will always return $\emptyset$, as the curve cannot be a single point.
- There is no instance where g will contain more than one range of outputs that meet C, as the curve is unbroken and occupies a limited space that can be intersected by τ.

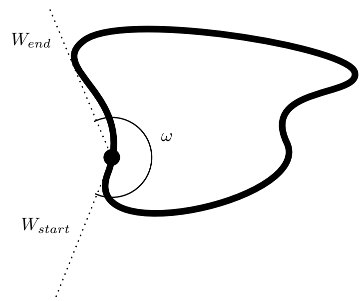

Figure 6: The width of W defined as ω measures from W_{start} to W_{end}.

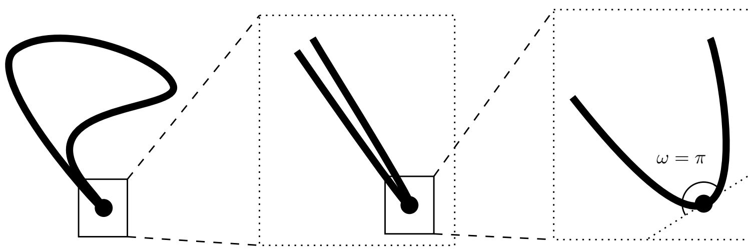

Figure 7: Base points at a sharp turn have an ω_{min} value of π.

$\square$

Hence, we will define the single continuous set of outputs from g with one n chosen $\forall \theta \in [0, 2\pi]$.

Definition 2.2. *Let W be the unbroken domain where $g(n_0, \theta)$ meets condition C when varying θ. Allow ω, then, to represent the "width" of W, regarding the fact that ω is an angle width, as shown in Figure 6.*

When a ray is extended from a base point b_n, the range ω where τ meets C is quite large, with respect to all of the possible angle values of θ. In fact, we can make a conclusion about the minimum angle width ω_{min} for any b_n on any γ.

Lemma 2.2. *The angle width ω as a result of any base point b_n on any λ must always have a value of π or greater.*

Proof. Any point given by b_n on a smooth part of the curve has tangent lines along the curve that extend in both directions from b_n. Hence the angle between them is π radians, and there must always be curve on at least one side of the tangent line due to the nature of λ.

> If b_n is located at a sharp turn, one can assume that the curve is smooth once infinitely zoomed in upon, meaning the above situation occurs. See Figure 7.

$\square$

3 Establishing a 3-Dimensional Relationship

Since g is defined as a two-variable function, it can be graphed in an three-dimensional space.

Consider the collection of surfaces Φ in $x-y-z$ space, where

$$(n, \theta, g_0 \forall g_0 \in g(n, \theta) \hookrightarrow (x, y, z).$$

Φ is a *collection* of surfaces since g results in a set of real numbers. See Figure 8.

> If γ is convex, then Φ has only one z value for every pair of x and y values (n, θ). Conversely, if γ is non-convex, Φ may have multiple z values for pairs of inputs.

Regardless of whether Φ contains multiple z values, we can find a special class of triangles by transforming Φ by $\frac{\pi}{2}$ radians.

This transformation, in turn, represents extending another ray τ' perpendicular to τ. All g' outputs from the newly transformed θ' value form the surface Φ'.

$$\Phi' : (n, \theta + \tfrac{\pi}{2}, g_0 \forall g_0 \in g(n, \theta + \tfrac{\pi}{2}) \hookrightarrow (x, y, z)$$

$$\implies (n, \theta', l) \hookrightarrow (x, y, z)$$

And because $W \geq \pi$ for all x-slices of Φ, all x-slices of Φ' must also have this same width. We will now consider the intersection of Φ and Φ'.

> Any intersection point of of these two surfaces represents an isosceles right triangle whose congruent legs extend from b_n with angles θ and $\theta' := \theta + \frac{\pi}{2}$ from the horizontal.

Definition 3.1. *If the two surfaces intersect at a point (n, θ, l), then an isosceles right triangle with base legs of length l can be defined as $T_{n,\theta}$. See Figure 9.*

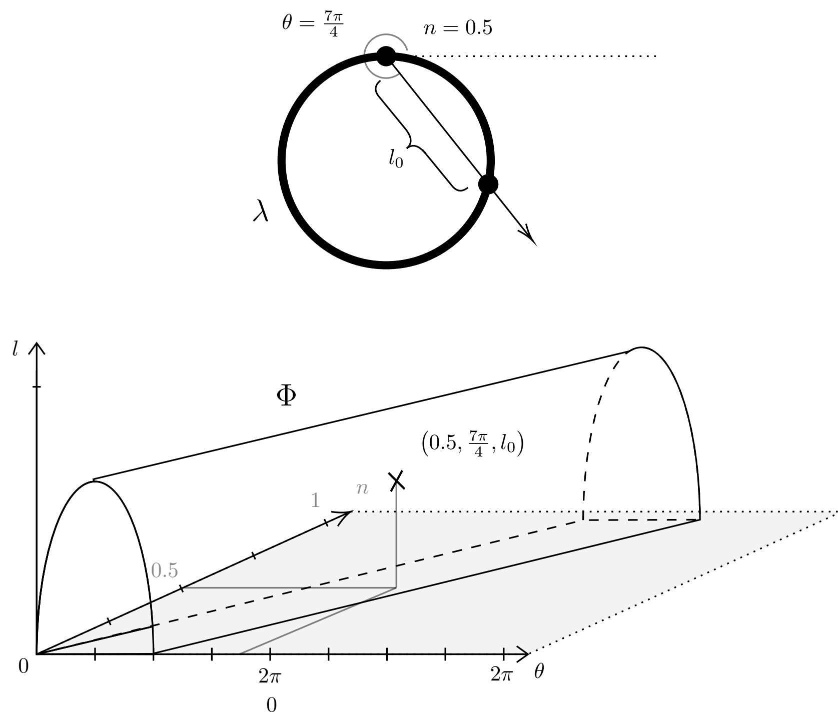

Figure 8: Surface Φ formed by $z = g(n, \theta)$. This surface is an example when λ is a circle. We have chosen two sample inputs and shown the output on Φ for reference.

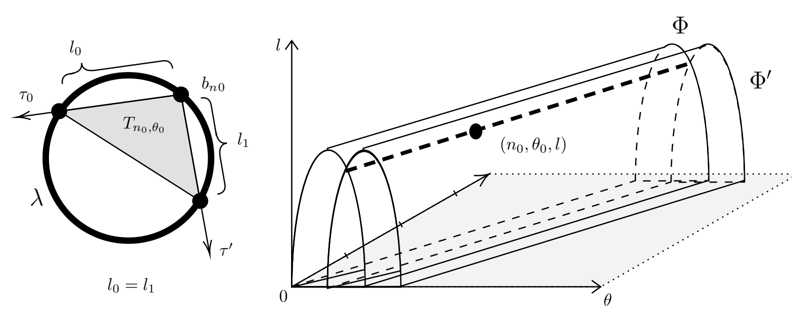

Figure 9: Any point where Φ and Φ' intersect is representative of one isosceles right triangle $T_{n,\theta}$.

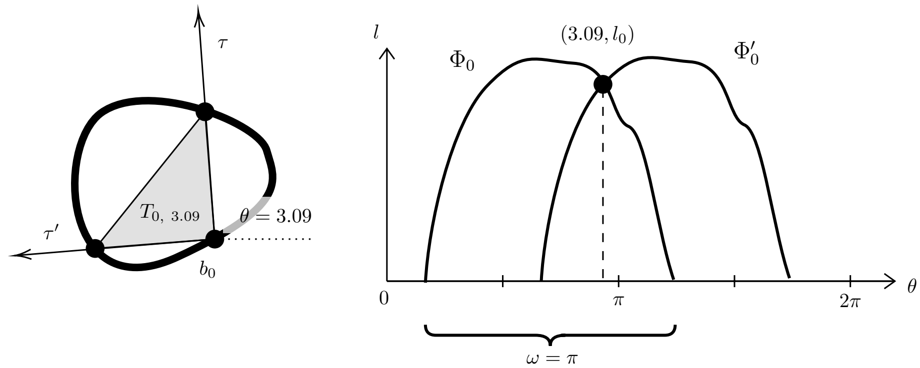

Figure 10: Consider a cross-section of Φ when $n = 0$. Φ_0 and Φ'_0 must intersect since $\omega > 0 \forall b_n \in B$. The point $(3.09,\ l_0)$ is such an intersection; it represents $T_{0,3.09}$.

Theorem 3.1. *For any base point b_n chosen on λ, it must act as the point between two base legs of an inscribed isosceles right triangle $T_{n,\theta}$ for one or more θ values.*

Proof. Since a chosen base point b_n restricts Φ and Φ' to a x-cross-section Φ_x where $x = n$, and since the width of Φ_x and Φ'_x is always greater than or equal to π, Φ_x and Φ'_x must always intersect at a minimum of one point because Φ' is a translation of $\frac{\pi}{2}$ radians in the y-direction. Every intersection point of these two surfaces represents the existence of an isosceles right triangle. See Figure 10.

□

And since the intersections of Φ and Φ' form a line or set of lines, we can make conclusions about the continuity of these triangles.

Theorem 3.2. *As n increments continuously, b_n acts as the base point between two congruent base legs of a continually existing inscribed isosceles right triangle.*

Proof. Each intersection of Φ and Φ' represents one inscribed isosceles right triangle. The intersection of these two surfaces always results in a continuous curve or set of curve.

□

At this point, we will aim to use these continually existing triangles to locate an inscribed square.

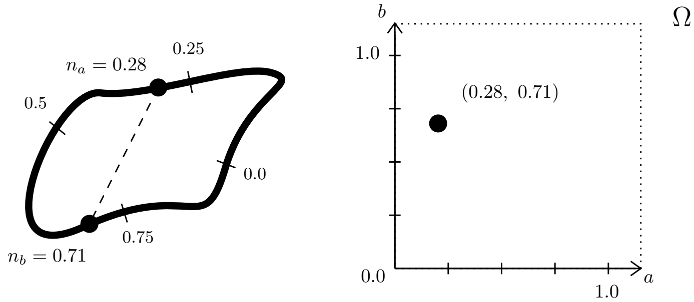

Figure 11: Any ordered pair (a, b) on Ω represents an pair of points on λ.

4 Forming an Inscribed Square from Triangles

Consider an inscribed isosceles right triangle $T_{n,\theta}$. We will observe the three vertices of

$$T_{n,\theta}\colon\ b_n,\ i_m, j_p,$$

where i_m and j_p are the vertices on the ends of hypotenuse of

$$\triangle b_n i_m j_p.$$

Each of these points can be represented by an n value, as each point on the curve corresponds to a real number on $[0, 1]$.

Consider the 2-dimensional plane Ω with axis $x \in [0, 1]$ and $y \in [0, 1]$ where values for x and y wrap at 0 and 1. Each of these axes, then, can represent the number line where points exist on λ. Any point in this 2-dimensional plane, therefore, represents a pair of points on λ, where order matters. See Figure 11.

We will now consider the entire domain of Ω, restricting it down to order-dependent pairs of points that acts as vertices on the ends of legs of an ambiguous triangle $T_{n,\theta}$.

Hence, if the domain of Ω is restricted to curves where all points (n, m) and (n, p) represent inscribed isosceles right triangles $\triangle b_n i_m j_p$, then two primary curves α and β (some that can branch off and merge if λ is not convex) are the only objects graphed on Ω, as shown in Figure 12.

Lemma 4.1. *Since each point on α and β represent side legs of some $\triangle b_n i_m j_p$, α and β have certain properties:*

- *They cannot cross the line given by $a = b$, as points that lie on this line represent a pair of points that occupy the same n value.*
- *The curves wrap around a. Any n-value at $a = 1$ is the same at $a = 0$.*

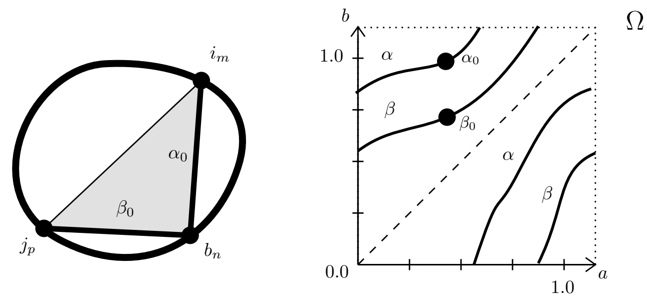

Figure 12: Triangle $\triangle b_n i_m j_p$ has sides α_0 and β_0 that are represented by two points on ω that share the same a-coordinate.

- *The curves wrap around b. Any m- or p-value at* $b = 1$ *is that same at* $b = 0$.

Proof. The second and third properties of α and β are implied, given the nature of Ω. Since the line $a = b$ represents all pairs of points where $n = m$ or $n = p$, α and β cannot cross this line, as a point on this line implies a triangle with a side length of 0.

□

Recall that each point on α represents the most counterclockwise leg of any inscribed isosceles right triangle T_1. If we find another inscribed isosceles right triangle T_2 that shares a leg with T_1, we can show that a square exists if we meet a few conditions.

Theorem 4.1. *At any point where the graph of* α^{-1} *intersects* β*, a square exists on* λ*.*

Proof. Allow two inscribed isosceles right triangles to be defined as

$$T_1 : \triangle b_{n1} i_{m1} j_{p1}, \text{ and}$$

$$T_2 : \triangle b_{n2} i_{m2} j_{p2}$$

where any point b is between the two legs of T.

inscribed on λ. If T_1 and T_2 share a leg in opposing directions from the base points such that

$$\overline{b_{n1} j_{p1}} \cong \overline{b_{n2} i_{m2}},$$

then the points form an inscribed square, shown in Figure 13, defined as

$$S : \square b_{n1} i_{m1} j_{p2} b_{n2}$$

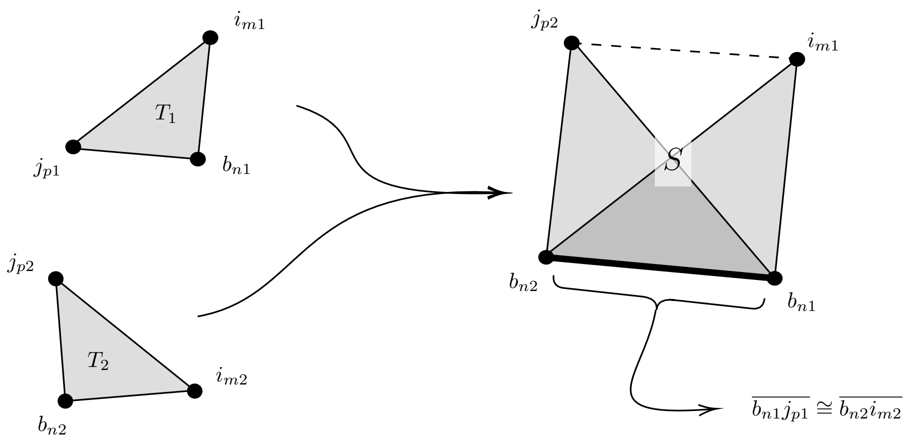

Figure 13: Square $S : \square b_{n1}i_{m1}j_{p2}b_{n2}$ is formed from two isosceles right triangles. Note that $\overline{b_{n1}i_{m1}}$ and $\overline{b_{n2}j_{p2}}$ are on the same side of $\overline{b_{n1}b_{n2}}$.

Since each point on α represents a segment $\overline{b_{n1}j_{p2}}$ and each point on β represents a segment $\overline{b_{n2}i_{m2}}$, then $\alpha^{-1} \cap \beta$ represents an instance where T_1 and T_2 share a leg such that $\overline{b_{n2}j_{p2}}$ and $\overline{b_{n1}i_{m1}}$ are parallel and on the same side of $\overline{b_{n1}b_{n2}}$.

□

5 Intersections in Topological Mappings

At this point, the intersection of α^{-1} and β must be proven for any λ. We will map the curves α and β to various topological structures, showing that a particular map always forces an intersection of the two curves.

Since Ω can be split along the line $y = x$, it can be represented topologically as a circular cylinder. See Figure 14.

And since we are looking for an inverse-independent intersection of curves α^{-1} and β, we can map one-half of Ω, divided by the symmetry $y = x$, to a Mobius strip. See Figure 15.

Due to the nature of its representation in topological space, α and β can be said to be non-contractible when represented on a topological circular cylinder. α and β are both continuous collections of curves, and they must wrap their a- and b-values.

Theorem 5.1. *Toeplitz' conjecture holds for any given continuous embedding $\lambda : S^1 \hookrightarrow \mathbb{R}^2$.*

Proof. The mapping of two non-contractible curves α and β must intersect one another due to the nature of a map from a order-dependent circular cylinder to

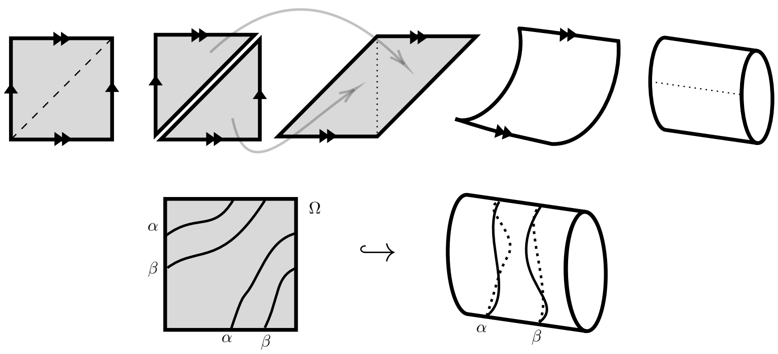

Figure 14: Ω is mapped from $\mathbb{R}^2$ to a circular cylinder.

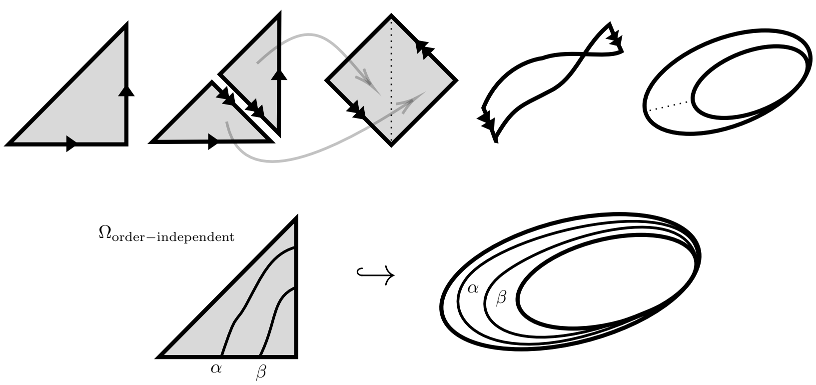

Figure 15: $\Omega_{\text{order}-\text{independent}}$ is mapped from $\mathbb{R}^2$ to a Mobius strip.

an order-independent Mobius strip. Hence, this intersection must occur for any λ given, as a intersection of α^{-1} and β through the topological mapping must always occur. $\square$