Syntax for $\mathcal{L}^{\text{LP}}$

LOGIC I
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September 10, 2024

Object Language and Metalanguage

Object Language: $\mathcal{L}^{\text{PL}}$ is the OBJECT LANGUAGE under study.

Metalanguage: Mathematical English is the METALANGAUGE with which we will conduct our study.

Quotation: To talk about $\mathcal{L}^{\text{PL}}$ we will take a quoted expression to be the CANONICAL NAME for the expression quoted.

Use/Mention: We MENTION expressions by putting them in quotes, whereas otherwise they are USED.

- 'Sue' is a nickname for Susanna.
- The complex sentence '$A \to B$' includes the sentence letters 'A' and 'B'.
- 'A' belongs to $\mathcal{L}^{\text{PL}}$, but " 'A' " and A do not.

The Expressions of $\mathcal{L}^{\text{PL}}$

Sentential Operators: '$\neg$','$\wedge$','$\vee$','$\to$', and '$\leftrightarrow$'.

- '$\sim$', '&', '.', '$|$', '$\supset$', and '$\equiv$' are also sometimes used.

Punctuation: '(' and ')'.

Sentence Letter: 'A_0', 'A_1', ..., 'B_0', 'B_1', ..., 'Z_0', 'Z_1', ...

Question: How can we specify all sentence letter explicitly?

- A SENTENCE LETTER is the result of subscripting a capital English letter with a numeral.

Corner Quotes: Let $\ulcorner \varphi_x \urcorner$ refer to the result of concatenating φ with $_x$.

- $\ulcorner \varphi_x \urcorner$ is a SENTENCE LETTER for any capital letter φ and numeral for a natural number x.

Primitive Symbols: The sentential operators, punctuation, and sentence letters are the PRIMITIVE SYMBOLS of $\mathcal{L}^{\text{PL}}$.

Expressions: The EXPRESSIONS of $\mathcal{L}^{\text{PL}}$ are defined recursively:

- The primitive symbol of $\mathcal{L}^{\text{PL}}$ are expression of $\mathcal{L}^{\text{PL}}$.
- If φ and ψ are expressions of $\mathcal{L}^{\text{PL}}$, then so is $\ulcorner \varphi\psi \urcorner$.
- Nothing else is an expression of $\mathcal{L}^{\text{PL}}$.

The Sentences of $\mathcal{L}^{\text{PL}}$

Uninterpretable: The expressions '$\neg\neg\neg\neg$', 'B_3A_0', '$))\leftrightarrow$', and '$A_4\vee$' cannot be assigned truth-values in a meaningful way.

- Compare 'MIT is in session' and '$A_4 \wedge P_1$'.

Well-Formed Sentences: Letting $\varphi, \psi, \chi, \ldots$ be variables with expressions for values, we may define the WFSS of $\mathcal{L}^{\text{PL}}$ as follows:

- Every sentence letter of $\mathcal{L}^{\text{PL}}$ is a wfs of $\mathcal{L}^{\text{PL}}$.
- If the expressions φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$, then:
  1. $\ulcorner\neg\varphi\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  2. $\ulcorner(\varphi \wedge \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  3. $\ulcorner(\varphi \vee \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  4. $\ulcorner(\varphi \rightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$; and
  5. $\ulcorner(\varphi \leftrightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$.
- Nothing else is a wff of $\mathcal{L}^{\text{PL}}$.

Sentential Variables: We will often restrict 'φ', 'ψ', 'χ',... to the wfs of $\mathcal{L}^{\text{PL}}$.

Main Operator: The MAIN OPERATOR is the last operator used in the construction of a sentence.

Arguments: The inputs to a main operator are its ARGUMENTS.

Scope: The main operator has SCOPE over its arguments.

Metalinguistic Conventions

Subscripts: We will suppress the subscript '$_0$' to ease exposition.

Task: Build increasingly complex sentences from just A.

Naming: We will refer to the NEGAND in a NEGATION, the CONJUNCTS in a CONJUNCTION, the DISJUNCTS in a DISJUNCTION, the ANTECEDENT and CONSEQUENT in a MATERIAL CONDITIONAL, and the ARGUMENTS in a MATERIAL BICONDITIONAL.

Quotation: We will sometimes drop quotes and corner quotes when the intended meaning is clear from the context.

- We will only do so when this improves readability.

Punctuation: We will drop outermost parentheses for ease.

- Compare $A \wedge B$, $A \vee B \vee C$, and $A \vee B \wedge C$.

Therefore: We will use '$\because$' for inline arguments.

Metalinguistic: These abbreviations all happen in the metalanguage.

Truth Functionality

Interpretations: Improving on last time, an INTERPRETATION $\mathcal{I}$ is an assignment of truth-values to sentence letters of $\mathcal{L}^{\text{PL}}$.

Valuation: We may then define a VALUATION function $\mathcal{V}_{\mathcal{I}}$ which assigns truth-values to every sentence of $\mathcal{L}^{\text{PL}}$ by way of the following semantic clauses:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Observe: These clauses resemble the composition rules for $\mathcal{L}^{\text{PL}}$.

Homophonic Semantics: The clauses for $\neg$, $\wedge$, and $\vee$ use analogous operators in the metalanguage, but not so for $\rightarrow$ and $\leftrightarrow$.

Truth Tables: Use the semantics to fill out the TRUTH TABLES below:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Truth Functions: The sentential operators express truth-functions, and so are often called TRUTH-FUNCTIONAL OPERATORS.

Question: How many unary/binary truth-functions are there?

Adequacy: Given these limitations, what should we hope to be able to adequately regiment in $\mathcal{L}^{\text{PL}}$?

Logical Truths: φ is a LOGICAL TRUTH of $\mathcal{L}^{\text{PL}}$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.