Existential Elimination and Soundness

LOGIC I
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Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

QD Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

($=$I) $\vdash \alpha = \alpha$ for any constant α.

($=$E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Existential Elimination

Task 1: Regiment and derive the following in QD.

1. The elephant would not obey.
   Patrick is an elephant.

   Patrick would not obey.
2. $\forall x(Jx \supset Kx)$
   $\exists x\forall yLxy$
   $\forall xJx$

   $\exists x(Kx \wedge Lxx)$.
3. $\exists x(Px \supset \forall xQx)$

   $\forall xPx \supset \forall xQx$.
4. $\exists xPx \vee \exists xQx$

   $\exists x(Px \vee Qx)$.
5. Every nonempty asymmetric relation is non-symmetric.

Natural to Normative

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

1. Shows that we can trust QD to establish validity.
2. Easier to derive a conclusion that to provide a semantic argument.
3. The natural rules of deduction preserve validity.

Natural: QD describes (approximately) how we in fact reason.

Normative: Soundness explains why we ought to use QD to reason.

Soundness of QD

Assume: $\Gamma \vdash_{\text{QD}} \varphi$, so there is a QD proof X of φ from Γ.

Lines: Let φ_i be the ith line of X.

Dependencies: Let Γ_i be the undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

Base: $\Gamma_1 \vDash \varphi_i$.

Induction: If $\Gamma_k \vDash \varphi_k$ for all $k \leq n$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\Gamma_m = \Gamma$ and $\varphi_m = \varphi$, so $\Gamma \vDash \varphi$.

Base Case

Proof: Every line in a QD proof is either a premise or follows by the rules.

Assume: φ_1 is either a premise or follows by AS or =I.

Premise: If φ_1 is a premise or assumption, then $\Gamma_1 = \{\varphi_1\}$, and so $\Gamma_1 \vDash \varphi_1$.

Identity: If φ_1 follows by =I, then φ_1 is $\alpha = \alpha$ for some constant α.

- Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model, $\mathcal{I}(\alpha) = \mathcal{I}(\alpha)$.
- Letting a be a variable assignment, $\mathcal{V}^a_{\mathcal{I}}(\alpha) = \mathcal{V}^a_{\mathcal{I}}(\alpha)$.
- So $\mathcal{V}^a_{\mathcal{I}}(\alpha = \alpha) = 1$, and so $\vDash \alpha = \alpha$.
- Thus $\Gamma_1 \vDash \varphi_1$ since $\Gamma_1 = \varnothing$.

Induction Case

Assume: $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

Undischarged: If φ_{n+1} is a premise or assumption, then the argument above applies.

Rules: If φ_{n+1} follows from Γ_{n+1} by the QD rules, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Cases: There are 12 rules in SD and an additional 6 in QD.

Further Problems: Relations

Task 1: Regiment and derive the following in QD.

1. Every transitive and symmetric relation is quasi-reflexive.
2. Only the empty relation is symmetric and asymmetric.
3. Every intransitive relation is irreflexive.
4. Every intransitive relation is asymmetric.