Satisfaction Semantics for QL: Key concepts and results

A QL-model $\mathfrak{M} := (D, I)$ consists of a non-empty set D of objects—called the domain of $\mathfrak{M}$—and a map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of QL to objects and ordered pairs from D such that (1) For each constant c, $I(c)$ is an element of D, called the *referent* or denotation of c, and (2) For each k-place predicate P, $I(P)$ is a set of ordered k-tuples of objects in D, called the *extension* of P. (I maps atomic sentences to true or false)

A **variable assignment** for I is a function d_I that maps each variable to an object in the domain D. e.g. $d_I(y)$ might equal the object $5 \in \mathbb{N}$

Satisfaction for Atomic wffs: $\mathcal{Q}$ of the form $\mathcal{P}t_1 \dots t_k$ where each t_i is a term. d_I satisfies $\mathcal{Q}$ provided that the k-tuple $\langle t_1^D, \dots, t_k^D \rangle$ lies in the extension of $\mathcal{Q}$, i.e. in $I(\mathcal{Q})$

Satisfaction for conjunctions: d_I satisfies $\mathcal{Q} \,\&\, \mathcal{R}$ iff it satisfies both $\mathcal{Q}$ and $\mathcal{R}$

Shorthand: if Pc is true in $\mathfrak{M}$, then $I(c) = r$ **satisfies** Px in $\mathfrak{M}$. We can write $\mathfrak{M}_{\mathbf{d}_I} \vDash Px$

A **variant** of a variable assignment d_I is a modified function $d_I[r/x]$ that assigns object $r \in D$ to x and otherwise assigns all other non-x variables the same objects as d_I.
e.g. $d_I[r/x](x) = r$ and $d_I[r/x](y) = d_I(y)$

Satisfaction conditions for **Existentially Quantified** wff $(\exists x)\,\mathcal{Q}$: a d_I satisfies $(\exists x)\,\mathcal{Q}$ provided there is SOME object $r \in D$ such that $d_I[r/x]$ satisfies $\mathcal{Q}$ (i.e. some x-variant of d_I)
Intuition: provided there's at least one thing you can plug in for x s.t. $\mathcal{Q}$ comes out true

Satisfaction conditions for **Universally Quantified** wff $(\forall x)\,\mathcal{Q}$: a d_I satisfies $(\forall x)\,\mathcal{Q}$ provided $d_I[r/x]$ satisfies $\mathcal{Q}$ for EACH object $r \in D$ (i.e. ALL the x-variants of d_I).
Intuition: provided no matter what you plug in for x, $\mathcal{Q}$ comes out true

All-or-nothing Lemma (11.1.3): given a model $\mathfrak{M} = (D, I)$ and a QL *sentence* $\mathcal{P}$ (i.e. a wff with no free variables), either all variable assignments d_I satisfy $\mathcal{P}$ or none do.

Truth-in-QL: A sentence $\mathcal{P}$ of QL is *true* on model $\mathfrak{M}$ iff some variable assignment d_I satisfies $\mathcal{P}$ in $\mathfrak{M}$. In this case, we write $\mathfrak{M} \vDash \mathcal{P}$. Otherwise, a sentence $\mathcal{P}$ of QL is *false* on model $\mathfrak{M}$, i.e. if no variable assignment d_I satisfies $\mathcal{P}$ in $\mathfrak{M}$.

Substitution Lemma (11.1.1): let $\mathcal{Q}$ be a wff of QL. The variable assignment d_I satisfies $\mathcal{Q}[\mathfrak{c}/\chi]$ if and only if $d_I[I(\mathfrak{c})/\chi]$ satisfies $\mathcal{Q}$ (intuition: χ refers to the same thing as $\mathfrak{c}$)
e.g. let $\mathcal{Q} = Fxx$, then $\mathcal{Q}[c/x] = Fcc$. Let $I(c)$ be some object $r \in D$ s.t. $\langle (r, r) \rangle \in Ext(F)$

Locality Lemma (11.1.7): consider a QL-sentence $\mathcal{P}$ and two QL-models $\mathfrak{M}^1 := (D, I_1)$ and $\mathfrak{M}^2 := (D, I_2)$ with the same domain D, whose interpretation functions I_1 and I_2 give the same interpretations for any constants or predicates appearing in QL-sentence $\mathcal{P}$.
Then $\mathfrak{M}^1 \vDash \mathcal{P}$ if and only if $\mathfrak{M}^2 \vDash \mathcal{P}$. (Note that any differences between $\mathfrak{M}^1$ and $\mathfrak{M}^2$ arise from how they interpret QL-symbols NOT appearing in $\mathcal{P}$).

Derivation Schemas for Proving Soundness of QND

All the rules of SND, plus the following quantifier rules. The rules of SND govern inferences where the main logical operator is one of the connectives from *SL*. Reiteration also allowed.

Substitution instance: "$\mathcal{Q}[c/\chi]$" is the sentence you get from $(\forall\chi)\,\mathcal{Q}$ or $(\exists\chi)\,\mathcal{Q}$ by dropping the quantifier and putting c in place of every χ in $\mathcal{Q}$. The other variables are untouched! Read "$[c/x]$" as saying "substitute c for every x"

Partial Substitution instance:"$\mathcal{Q}\lceil\chi/c\rceil$" is the sentence you get by replacing some but not necessarily all instances of the constant c in $\mathcal{Q}$ with the variable χ.
We may write "$\mathcal{Q}[c]$" to indicate that the constant c appears in $\mathcal{Q}$

Universal Elimination ($\forall$E)

$$
\begin{array}{l|ll}
h & (\forall\chi)\,\mathcal{Q} & \\
\vdots & \quad\vdots & \\
k+1 & \mathcal{Q}[c/\chi] & :h\ \forall\text{E}
\end{array}
$$

Existential Introduction ($\exists$I)

$$
\begin{array}{l|ll}
h & \mathcal{Q} & \\
\vdots & \quad\vdots & \\
k+1 & (\exists\chi)\,\mathcal{Q}\lceil\chi/c\rceil & :h\ \exists\text{I}
\end{array}
$$

Provided that χ does not occur in $\mathcal{Q}[c]$ (auto-enforced by (a) needing $\mathcal{Q}$ to be a sentence and (b) our recursion clause for QL-wffs)

Universal Introduction ($\forall$I)

$$
\begin{array}{l|ll}
h & \mathcal{Q} & \\
\vdots & \quad\vdots & \\
k+1 & (\forall\chi)\,\mathcal{Q}[\chi/c] & :h\ \forall\text{I}
\end{array}
$$

Provided that both
(i) c does not occur in any undischarged assumptions that $\mathcal{Q}$ is in the scope of.
(ii) χ does not occur already in $\mathcal{Q}$
(auto-enforced by needing to replace EVERY instance of c with χ in $\mathcal{Q}[\chi/c]$)

Existential Elimination ($\exists$E)

$$
\begin{array}{l|lll}
h & (\exists\chi)\,\mathcal{Q} & & \\
 & \quad\vdots & & \\
j & & \multicolumn{1}{|l}{\underline{\mathcal{Q}[c/\chi]}} & :\text{AS for }\exists\text{E} \\
 & & \multicolumn{1}{|l}{\quad\vdots} & \\
m & & \multicolumn{1}{|l}{\mathcal{P}_{k+1}} & \\
k+1 & \mathcal{P}_{k+1} & & :h,\ j\text{–}m\ \exists\text{E}
\end{array}
$$

Provided that
(i) c does not occur in any other undischarged assumptions that $\mathcal{Q}[c/\chi]$ is in the scope of
(ii) c does not occur already in $(\exists\chi)\,\mathcal{Q}$
(iii) c does not occur in $\mathcal{P}_{k+1}$