What is Logic?

LOGIC I
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Definitions

Proposition:	A PROPOSITION is a way for things to be which is either true or false.
Declarative Sentence:	A DECLARATIVE SENTENCE is a grammatical string of symbols which, on an interpretation, expresses a proposition that is either true or false.
Argument:	An ARGUMENT is a finite sequence of declarative sentences where the final sentence is the CONCLUSION and the preceding sentences are the PREMISES.

Examples

Snow

(1) It's snowing.
$\therefore$ John drove to work.

This argument may be compelling, but not certain.

Red

(1) The ball is crimson.
$\therefore$ The ball is red.

This argument provides certainty, but not on all interpretations.

Museum

(1) Kate is either at home or at the Museum.
(2) Kate is not at home.
$\therefore$ Kate is at the Museum.

This argument's certainty is independent of the interpretation.

Informal Validity

Question 1: What goes wrong if we assume the premises but deny the conclusion in *Red* and *Museum*?

Answer: Nature of 'crimson' and 'red' *vs.* meaning of 'or' and 'not'.

Informal Semantics: Give informal semantics for disjunction and negation.

Complex Sentences: Observe that complex sentences in *Museum* are composed from simpler sentences *via* 'not' and 'or'.

Atomic Sentences: A declarative sentence is ATOMIC just in case it is not composed of simpler declarative sentences.

Task 1: Identify atomic sentences in *Museum*.

Informal Interpretation: Let an INFORMAL INTERPRETATION assign every atomic sentence of English to exactly one TRUTH-VALUE 1 or 0.

Informal Validity: An argument in English is INFORMALLY VALID just in case its conclusion is true in every informal interpretation in which its premises are true.

Task 2: Use semantics to show that *Museum* is informally valid.

Formal Languages

Problem 1: There is no set of all atomic sentences of English, and so no clear notion of an informal interpretation of English.

Suggestion: Could choose some large set of atomic English sentences, but this would be arbitrary and hard to specify.

Solution 1: We will *regiment* English arguments in a formal language that is both general and easy to specify precisely.

Sentential Logic: The SENTENCES of SL are composed of sentence letters $A, B, C, \ldots$ and sentential operators $\neg, \vee, \wedge, \supset$, and $\equiv$.

Task 3: Regiment *Museum* in SL: $A \vee B, \neg A \vDash B$.

Task 4: Provide semantic clauses for SL.

Interpretation: An INTERPRETATION $\mathcal{I}$ of the sentences of SL assigns exactly one truth-value (1 or 0) to each sentence letter.

Disjunction: $\mathcal{I}(\varphi \vee \psi) = 1$ just in case $\mathcal{I}(\varphi) = 1$ or $\mathcal{I}(\psi) = 1$ (or both).

Negation: $\mathcal{I}(\neg\varphi) = 1$ just in case $\mathcal{I}(\varphi) = 0$.

Logical Validity: An argument in SL is LOGICALLY VALID just in case its conclusion is true in every interpretation in which its premises are true.

Task 5: Show that *Museum* is logically valid.

Logical Form

Picasso

(1) The painting is either a Picasso or a counterfeit and illegally traded.

(2) The painting is not a Picasso.

$\therefore$ The painting is a counterfeit and illegally traded.

This argument is also logically valid.

Question 2: How does this argument relate to *Museum*?

Task 6: Regiment *Picasso* in SL: $A \vee (B \wedge C), \neg A \vDash B \wedge C$.

Logical Form: Both arguments are instances of $\varphi \vee \psi, \neg\varphi \vDash \psi$ which is a logically valid argument form.

Question 3: How many logically valid argument forms are there, and how could we hope to describe this space?

Suggestion: Logical validity in SL describes the space of logically valid arguments, where the logically valid argument forms are patterns in this space.

Problem 2: SL cannot regiment all logically valid arguments.

Socrates: All men are mortal, Socrates is a man $\vDash$ Socrates is mortal.

Solution 2: Rather, logical validity in SL provides a partial answer, where we may extend the language to provide a broader description of logical validity, e.g. QL.

Proof Theory

Model Theory: We have characterized logical reasoning in terms of truth-preservation across a space of interpretations of the formal language by providing elements of a model theoretic semantics for SL.

Task 7: Can we make *Snow* and *Red* logically valid?

Syntactic Account: Another approach focuses entirely on syntax, using rules to specify which inferences are deductively valid given the meanings of the logical constants.

Metalogic: Amazingly, these two strategies coincide for both SL and QL, and we will prove these important results later in this course.

Neutrality: These methods accommodate reasoning about anything whatsoever, though not all logical constants are equally well understood.

The Connectives

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Definitions

Previously: We considered the sentences that could be constructed from the sentence letters with the connectives. We will now seek to specify this construction precisely.

Object Language: We will be concerned to define the sentences of SL, where the language of SL $= \langle \mathbb{L}, \neg, \wedge, \vee, \supset, \equiv, (,) \rangle$ will be referred to as the OBJECT LANGUAGE.

Strings: An EXPRESSION of SL is any finite string of symbols from the language of SL.

Quotation: To talk about strings we will need to name them, where a quoted string is the CANONICAL NAME for the string quoted.

Use/Mention: We mention expressions by putting them in quotes, where otherwise they are used.

Example 1: 'Sue' is a three letter name but Sue is not.

Example 2: The complex sentence '$A \supset B$' includes the sentence letters 'A' and 'B'.

Metalanguage: We talk about the expressions of SL with the resources of our METALANGAUGE mathematical English.

Metalinguistic Variables: Letting $\varphi, \psi, \chi, \ldots$ be variables whose values are expressions, we may quantify over the expressions of SL in order to define the wffs of SL.

The Sentences of SL

1. Every atomic sentence in $\mathbb{L}$ is a wff of SL.
2. If φ and ψ are wffs of SL, then:
   (a) $\neg\varphi$ is a wff of SL;
   (b) $(\varphi \wedge \psi)$ is a wff of SL;
   (c) $(\varphi \vee \psi)$ is a wff of SL;
   (d) $(\varphi \supset \psi)$ is a wff of SL; and
   (e) $(\varphi \equiv \psi)$ is a wff of SL.
3. Nothing else is a wff of SL.

Observations and Conventions

Corner Quotes:	Strictly speaking, this definition is non-sense and we need to use corner quotes to fix it.
Well-formed Formulas:	The wffs are the grammatical expressions of SL of type t, and so candidates for interpretation.
Sentences:	Since all wffs in SL are *good* candidates for interpretation (it makes sense to assign them truth-values), we may identify the wffs with the sentences of SL. By contrast, not all the wffs of QL are sentences of QL.
Sentential Variables:	Restrict $\varphi, \psi, \chi, \ldots$ to sentences of SL.
Task 1:	Build increasingly complex sentences from just A.
Conventions:	We will often drop quotes and parentheses for ease: $A \vee B \vee C$ vs $A \vee B \wedge C$.
Therefore:	$\therefore$ is not part of SL.

Truth Functionality

Sentential Operators:	The connectives are SENTENTIAL OPERATORS which map sentences to sentences.
Interpretations:	Last time we said that an INTERPRETATION $\mathcal{I}$ assigns truth-values to sentence letters.
Valuation:	We may then define a VALUATION function $\mathcal{V}_{\mathcal{I}}$ which assigns truth-values to every sentence of SL by way of the following semantic clauses:
(A)	$\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ iff φ is a sentence letter of SL.
$(\neg)$	$\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
$(\wedge)$	$\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
$(\vee)$	$\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
$(\supset)$	$\mathcal{V}_{\mathcal{I}}(\varphi \supset \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
$(\equiv)$	$\mathcal{V}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.
Homophonic Semantics:	The semantics for $\neg$, $\wedge$, and $\vee$ use analogous operators in the metalanguage, but not so for $\supset$ and $\equiv$.
Quantified Logic:	Later in the course, we will provide semantic clauses for the quantifiers 'for all' and 'there is' which will not have homophonic semantic clauses.

Negation

Uninitiated

(1) If Sam attended the gathering, then he has been initiated.
(2) Sam is uninitiated.
∴ Sam did not attend the gathering.

Observe: Being uninitiated is the same as not being initiated.

Uninvited

(1) Arden is not invited.
∴ Arden is uninvited.

Observe: Arden can fail to be invited without being uninvited.
Question: What about the converse?

The Material Conditional

Square

(1) The rug is square.
∴ If the rug is triangular, then it is square.

Regiment: Use S and T to regiment this argument.
Observe: It is natural to provide a modal reading for the conclusion.
Conclude: The material conditional does not track possibilities.

1 Valid Arguements

1.1 Definitions

V1: An argument is **valid if and only if** the premises are true and you can't imagine a scenario where the conclusion is false.

- Problem: requires the premises to be true. But we don't necessarily know if the premises are true. And we still want to be able to draw logical inferences from the premises.

V2: An argument is **valid if and only if** the premises logically entail the conclusion.

- Problem: this is better, but 'logically entails' is just a synonym for validity. So this is not that informative. Since we'd need to know what logical entailment is.

(Good) V3: An argument is **valid if and only if** in *every interpretation/conceptual-scenario/"possibility"* in which the premises are true, then the conclusion must also be true.

- Correct: it is important that the "possibility" is something conceptual/interpretational not something about physics or the way the world must be as a matter of its physical nature.

Argument 1: P1 Gyre is a mome rath.
$\therefore$ Gyre or Jake is a mome rath.

(Unsound because premises are false)

P1 Ben is sitting.
P2 If Ben is sitting, then Jake is standing.
$\therefore$ Ben or Jake is sitting.

P1 Ben is sitting.
P2 If Ben is sitting, then Jake is standing.
$\therefore$ Jake is standing.

P1 Jake is not standing.
P2 If Ben is sitting, then Jake is standing.
$\therefore$ Ben is not sitting.

Try 1: to invalidate by making conclusion false and premises true.

Since conclusion is false: then Ben is sitting.

But if P2 is true: then P1 is false

So: no way to make premises both true if conclusion is false.

Try 2: assume premises are true; check to see if conclusion is true.

So: **assume P1** and P2 are true.

Assume that the conclusion is false (to see if we get into trouble).

If the conclusion is false, then Ben is sitting.

But by P2, it follows that P1 must then be false, contrary to our assumption.

So we were wrong to think it is possible for the conclusion to be false given that the premises are true.

(Unsound because premises are false)

P1 Ben is sitting.

P2 If Ben is sitting, then Jake is standing

P3 Pigs can fly.

∴ Ben or Jake is sitting.

2 Invalid Arguements

Argument 1: P1 The speed of light is c.

Truth Tables

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Truth Functions

Previously: For an interpretation $\mathcal{I}$, a VALUATION function $\mathcal{V}_{\mathcal{I}}$ is the smallest function to assign truth-values to every sentence of SL that satisfies the semantic clauses:

(A) $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ iff φ is a sentence letter of SL.

($\neg$) $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).

($\wedge$) $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.

($\vee$) $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).

($\supset$) $\mathcal{V}_{\mathcal{I}}(\varphi \supset \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).

($\equiv$) $\mathcal{V}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Truth Tables: Use the semantics to fill out the CHARACTERISTIC TRUTH TABLES given below:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \supset \psi$	$\varphi \equiv \psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Sentential Operators: The connectives are SENTENTIAL OPERATORS which map sentences to sentences.

Truth Functional: The connectives express truth-functions:
$\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1 - \mathcal{V}_{\mathcal{I}}(\varphi)$;
$\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = \mathcal{V}_{\mathcal{I}}(\varphi) \times \mathcal{V}_{\mathcal{I}}(\psi)$.

HOMEWORK: Given an interpretation $\mathcal{I}$, specify the truth-values of $\varphi \vee \psi$, $\varphi \supset \psi$, and $\varphi \equiv \psi$ as a function of the truth-values of φ and ψ in a similar fashion as above.

Task 1: How many unary/binary truth-functions are there?

Adequacy: Given the expressive limitations of SL, what should we hope to be able to adequately regiment?

Examples

COMPLEX ARGUMENTS

Rain

(1) If it is raining on a week day, Sam took his car.
(2) Kate borrowed Sam's car only if Sam did not take it.
(3) Kate borrowed Sam's car just in case she visited her parents.
(3) It is raining and Kate visited her parents.
∴ It is not a week day.

Task 2: Regiment this argument and construct its truth table.
Observe: This argument can be adequately regimented and evaluate in SL.

CONJUNCTION

Gym

(1) Kate took a shower and went to the gym.
∴ Kate went to the gym and took a shower.

Task 3: Regiment this argument and construct its truth table.
Observe: Conjunction in English can track temporal order.
Question: How can we capture the invalidity of this argument in SL?

DISJUNCTION

Vault

(1) If Kin uses the remote, the trunk will open.
(2) If Adi tries the handle, the trunk will open.
(3) If Kin uses the remote and Adi tries the handle, the trunk won't open.
∴ If Kin uses the remote or Adi tries the handle, the trunk will open.

Task 4: Regiment this argument and construct its truth table.
Observe: We cannot regiment the conclusion with inclusive-'or'.
Question: Can we salvage the validity of this argument?

THE MATERIAL CONDITIONAL

Roses

(1) Sugar is sweet.
∴ The roses are only red if sugar is sweet.

Task 5: Regiment this argument and construct its truth table.

Observe: The locution 'only if' appears to assert something stronger than ⊃.

Vacation

(1) Casey is not on vacation.
∴ If Casey is on vacation, then he is in Paris.

Crimson

(1) Mary doesn't like the ball unless it is crimson.
(2) Mary likes the ball.
∴ If the ball is blue, then Mary likes it.

THE BICONDITIONAL

Rectangle

(1) The room is a square.
(2) The room is a rectangle.
∴ The room is a square if and only if it is a rectangle.

Work

(1) Kin isn't a professor.
(2) Sue isn't a chef.
∴ Kin is a professor just in case Sue is a chef.

Applications

Objection: The semantics for SL is not good for anything.

Response: SL is perfect for necessary claims (like in mathematics), as well as sentences where we only care about their truth-value as opposed to their modal profile or subject-matter.

Logical Entailment

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Logical Entailment

Satisfaction: An interpretation $\mathcal{I}$ of SL *satisfies* a set of SL sentences Γ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Gamma$. Derivatively, an interpretation $\mathcal{I}$ of SL *satisfies* a sentence φ of SL *iff* $\mathcal{I}$ satisfies $\{\varphi\}$.

Logical Entailment: $\Gamma \vDash \varphi$ *iff* every SL interpretation $\mathcal{I}$ that satisfies Γ also satisfies φ.

Validity: An argument in SL is *valid* just in case its conclusion is true in any interpretation in which its premises are true.

Question: How are we to describe the space of all valid arguments?

Answer: In terms of entailment.

Task 1: Show that validity and entailment are distinct:

- $\Gamma \vDash \varphi$ does not determine a unique argument.
- Entailment does not order the premises.
- Entailment admits of infinitely many premises.
- Entailment admits of no premises.

Tautology: An SL sentence φ is a *tautology* just in case $\vDash \varphi$.

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

Unsatisfiable

Absurdity: A contradiction entails everything: $A \wedge \neg A \vDash B$.

Bottom: Let '$\bot$' abbreviate any contradiction.

Unsatisfiable: A sentence is *unsatisfiable* just in case $\Gamma \vDash \bot$.

Task 2: Show that a set of SL sentences is unsatisfiable just in case no SL interpretation satisfies it.

Consistency: Recall: a set of SL sentences is *consistent* just in case there is a line on the complete truth table for those sentences which makes them all true, and *inconsistent* otherwise.

Task 3: Show that consistency and satisfiability are co-extensional.

Examples

Which sets of sentences are consistent? (e.g., is $\{(1),(2)\}$ consistent?)

Taller

(1) Liza is taller than Sue.

(2) Sue is taller than Paul.

(3) Paul is taller than Liza.

Lost

(4) Kim is either in Somerville or Cambridge.

(5) If Kim is in Somerville, then she is not far from home.

(6) If Kim is not far from home, then she is in Cambridge.

(7) Kim is not in Cambridge.

Methods

Truth Tables: Mechanical but tedious.

- Bad if there are lots of sentence letters.
- Good for counterexamples.
  $A \equiv (B \supset C),\ A \wedge \neg B,\ D \vee \neg A \therefore C.$

Semantic Arguments: Good if there are lots of sentence letters.
$(A \vee B) \supset (C \wedge D),\ \neg C \wedge \neg E \therefore \neg A.$

Task 4: Provide a semantic argument.

Inference Rules: Suppose we were to schematize inferences.

- $\varphi \wedge \psi \vdash \varphi.$
- $\neg\varphi \vdash \neg(\varphi \wedge \psi).$
- $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi.$
- $\neg(\varphi \vee \psi) \vdash \neg\varphi.$

Observe: Rules are valid.

Task 5: Use rules to derive above.

Proof Theory: How many rules are there, and how should we describe the space of all of them?

Sentential Tree Logic

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Constructing the Root

Previously: $\Gamma \vDash \varphi$ if and only if $\Gamma, \neg\varphi \vDash \bot$.

Proof: To show $\Gamma, \neg\varphi \vDash \bot$, we will show $\Gamma, \neg\varphi \vdash \bot$.

Resolution Rules

Root: A *root* is any finite sequence of SL sentences.

Tree: An SL *tree* consists of a root followed by any number of applications of the resolution rules given below:

Conjunction

$$\begin{array}{c} \varphi \wedge \psi \\ \varphi \\ \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \wedge \psi) \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \neg\psi \end{array}$$

The Material Conditional

$$\begin{array}{c} \varphi \supset \psi \\ \diagup \quad \diagdown \\ \neg\varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \supset \psi) \\ \varphi \\ \neg\psi \end{array}$$

Disjunction

$$\begin{array}{c} \varphi \vee \psi \\ \diagup \quad \diagdown \\ \varphi \qquad \psi \end{array} \qquad \begin{array}{c} \neg(\varphi \vee \psi) \\ \neg\varphi \\ \neg\psi \end{array}$$

The Biconditional

$$\begin{array}{c} \varphi \equiv \psi \\ \diagup \quad \diagdown \\ \begin{array}{cc} \varphi & \neg\varphi \\ \psi & \neg\psi \end{array} \end{array} \qquad \begin{array}{c} \varphi \equiv \psi \\ \diagup \quad \diagdown \\ \begin{array}{cc} \varphi & \neg\varphi \\ \neg\psi & \psi \end{array} \end{array}$$

Double Negation

$$\begin{array}{c} \neg\neg\varphi \\ \varphi \end{array}$$

Child: A *child* of φ in an SL tree is any sentence ψ immediately following φ.

Leaf: A *leaf* in an SL tree is any sentence which does not have a child.

Branch: A *branch* in an SL tree is any sequence of sentences beginning with the root of the tree and ending with a leaf of the tree where every sentence in the sequence besides the first is a child of its predecessor.

Note: Officially, an SL tree is an *ordered dyadic tree* of SL sentences where every sentence in the tree either belongs to the root, or results from resolving one of its ancestors.

Closure and Completion

Branch Closure: A branch in an SL tree is *closed* just in case it includes φ and $\neg\varphi$ for some SL sentence φ, and *open* otherwise.

Tree Closure: A tree is *closed* just in case every branch is closed, and *open* otherwise.

Resolvable: A sentence is *resolvable* in a branch just in case it has a resolution rule and the branch is open.

Resolved: A sentence is *resolved* in a branch just in case the resolution rule for that sentence has been applied in that branch.

Branch Completion: A branch is *complete* just in case every resolvable sentence in that branch has been resolved in that branch.

Tree Completion: A tree is *complete* if and only if every branch in the tree is complete.

Derivability

STL: $\Gamma \vdash \bot$ just in case there is a closed tree with root Γ.

Derivability: $\Gamma \vdash \varphi$ just in case $\Gamma, \neg\varphi \vdash \bot$.

Question 1: Why should we care about $\vdash$?

Answer: So far, we shouldn't, but soon we will show that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$.

Suppose: Let's suppose $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$ for now.

Question 2: How can we determine whether Γ is satisfiable?

Answer: Show that $\Gamma \nvdash \bot$.

Examples

1. Evaluate the following argument for validity:

$$\begin{array}{ll} & \neg R \supset \neg Q \\ & P \wedge Q \\ \therefore & P \wedge R \end{array}$$

2. Show that $A \vee B, B \supset C, A \equiv C \vDash C$.
3. Show that $(P \supset Q) \equiv (\neg Q \supset \neg P)$ is a tautology.
4. Show that $A \equiv \neg A$ is a contradiction.
5. Show that $\{P, P \supset Q, Q \supset \neg P\}$ is unsatisfiable.
6. Show that $\{P \supset Q, \neg P \vee \neg Q, Q \supset P\}$ is satisfiable.
7. Evaluate $P, P \supset Q, \neg Q \vDash A$.

Mathematical Induction

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Review from Last Time

1. Show that $A \vee B, B \supset C, A \equiv C \vDash C$.
2. Show that $\{P, P \supset Q, Q \supset \neg P\}$ is unsatisfiable.
3. Show that $\{P \supset Q, \neg P \vee \neg Q, Q \supset P\}$ is satisfiable.
4. Evaluate $P, P \supset Q, \neg Q \vDash A$.
5. Evaluate $(A \wedge B) \supset C, C \equiv (D \wedge E), \neg D \wedge B \vdash \neg A$.

Soundness and Completeness

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Induction: These proofs will require mathematical induction.

Definitions: We will also need a few more recursive definitions.

Recursive Definitions

Complexity: We define $\texttt{Comp}(\varphi)$ to be the number of connectives in φ.

- If φ is a sentence letter of SL, then $\texttt{Comp}(\varphi) = 0$.
- For any SL sentences φ and ψ:

  ($\neg$) $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$;
  ($\wedge$) $\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;

  $\vdots$

Note: Could avoid redundancy by taking $\star$ to be any binary connective.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ.

- If $\texttt{Comp}(\varphi) = 0$, then $[\varphi] = \{\varphi\}$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$;
  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$;

Simplicity: We define $\texttt{Simple}(\varphi)$ to hold just in case the SL sentence φ has at most one occurrence of each sentence letter in SL.

- If $\texttt{Comp}(\varphi) = 0$, then $\texttt{Simple}(\varphi)$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$;

  ($\star$) $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$;

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If $\texttt{Comp}(\varphi) = 0$, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$;

  ($\star$) $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$;

Strong Induction

Step 1: Identify the set of elements and the property in question.

Step 2: Partition the set into a sequence of stages to run induction on.

Step 3: Establish that every element in the base stage has the property.

Step 4: Assume every element in stage n and below have the property.

Step 5: Show that every element in stage $n + 1$ have the property.

Examples

Task 1: Show that every SL sentence has an even number of parentheses.

Task 2: Show that for every SL sentence φ, if $\texttt{Simple}(\varphi)$, then there are SL interpretations $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{J}}(\varphi) = 0$.

Task 3: For any SL sentences φ, ψ, χ and SL sentence letter α, if $\vDash \varphi \equiv \psi$, then $\vDash \chi_{[\varphi/\alpha]} \equiv \chi_{[\psi/\alpha]}$.

Task 4: Let $\mathcal{I}^+(\alpha) = 1$ for every sentence letter α in SL. Show that $\mathcal{V}_{\mathcal{I}^+}(\varphi) = 1$ for every SL sentence φ that does not include negation.

Mathematical Induction

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Continuing from Last Time...

Task 1: Let $\mathcal{I}^+(\alpha) = 1$ for every sentence letter α in SL. Show that $\mathcal{V}_{\mathcal{I}^+}(\varphi) = 1$ for every SL sentence φ that does not include negation.

Task 2: Every tree has a finite number of branches.

Task 3: Show that for every SL sentence φ, if $\texttt{Simple}(\varphi)$, then there are SL interpretations $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{J}}(\varphi) = 0$.

Task 4: For any SL sentences φ, ψ, χ and SL sentence letter α, if $\vDash \varphi \equiv \psi$, then $\vDash \chi_{[\varphi/\alpha]} \equiv \chi_{[\psi/\alpha]}$.

Task 5: Every tree can be completed in a finite number of steps.

Recursive Definitions

Length: For any SL tree X, we define $\texttt{Length}(X)$ to be the number of resolution rules that have been applied.

- $\texttt{Length}(X) = 0$ for any root X.
- For any SL tree X, if $\texttt{Length}(X) = n$ and X' is the result of resolving a sentence in exactly one branch in X, then $\texttt{Length}(X') = n + 1$.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ.

- If $\texttt{Comp}(\varphi) = 0$, then $[\varphi] = \{\varphi\}$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$; and

  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$.

Simplicity: We define $\texttt{Simple}(\varphi)$ to hold just in case the SL sentence φ has at most one occurrence of each sentence letter in SL.

- If $\texttt{Comp}(\varphi) = 0$, then $\texttt{Simple}(\varphi)$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$; and

  ($\star$) $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$.

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If $\texttt{Comp}(\varphi) = 0$, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$; and
  ($\star$) $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$.

Resolution: We define $\texttt{Res}(\varphi)$ to be the maximum number of times that we may resolve φ and any of its descendants.

- $\texttt{Res}(\varphi) = 0$ if φ is a literal.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $\texttt{Res}(\neg\neg\varphi) = \texttt{Res}(\varphi) + 1$; and
  ($\star$) $\texttt{Res}(\varphi \star \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
  ($\star$) $\texttt{Res}(\neg(\varphi \star \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.

Set Binary: We extend the definition of Res to sets of sentences as follows:

$$\texttt{Res}(\Gamma) = \sum_{\varphi \in \Gamma} \texttt{Res}(\varphi).$$

Unresolved: We define $[X]$ to be the set of unresolved sentences in the SL tree X.

- $\varphi \in [X]$ if X is a root and φ occurs in X.
- For any SL tree X' which results from resolving $\varphi \in [X]$ on every open branch in X in which φ occurs:

  ($\neg$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi\}$ if $\varphi = \neg\neg\psi$;
  ($+$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \chi\}$ if $\varphi \in \{\psi \wedge \chi, \psi \vee \chi\}$;
  ($-$) $[X'] = ([X]/\{\varphi\}) \cup \{\neg\psi, \neg\chi\}$ if $\varphi \in \{\neg(\psi \wedge \chi), \neg(\psi \vee \chi)\}$;
  ($\supset$) $[X'] = ([X]/\{\varphi\}) \cup \{\neg\psi, \chi\}$ if $\varphi = \psi \supset \chi$;
  ($\not\supset$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \neg\chi\}$ if $\varphi = \neg(\psi \supset \chi)$;
  ($\equiv$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \chi, \neg\psi, \neg\chi\}$ if $\varphi \in \{\psi \equiv \chi, \neg(\psi \equiv \chi)\}$.

Resolvable: Letting $\mathbb{L}$ be the set of SL literals, we define $X_U = [X]/\mathbb{L}$ to be the set of SL sentences in X that can be resolved.

Task 5

Proof: Given any SL tree X, let X_U be the set of resolvable sentences in X. The proof goes by induction on $\texttt{Res}(X_U)$ for any SL tree X.

Base Case: Let X be an SL tree where $\texttt{Res}(X_U) = 0$. By definition, every branch is complete, and so the tree is complete. Accordingly, X can be completed in a finite number of steps, namely 0.

Hypothesis: Assume that for every SL tree X, if $\texttt{Res}(X_U) \leq n$, then X can be completed in a finite number of steps.

Induction: Let X be an SL tree where $\texttt{Res}(X_U) = n + 1$. Thus there is some $\varphi \in X_U$. Letting X' be the SL tree that results from resolving φ on every open branch in X, we may observe that $\varphi \notin X'_U$. Consider the following cases where $\star \in \{\wedge, \vee, \supset, \equiv\}$ is a binary connective:

Case 1: If $\varphi = \neg\neg\psi$ and $\texttt{Res}(\psi) \neq 0$, then $X'_U = (X_U/\{\varphi\}) \cup \{\psi\}$. If instead $\texttt{Res}(\psi) = 0$, then $X'_U = X_U/\{\varphi\}$. Since $\texttt{Res}(\psi) = \texttt{Res}(\varphi) - 1$, it follows either way that $\texttt{Res}(X'_U) \leq \texttt{Res}(X_U) - 1 = n$.

Case 2: If $\varphi = \psi \star \chi$ and $\texttt{Res}(\psi) \neq 0$ and $\texttt{Res}(\chi) \neq 0$, then we know that $X'_U = (X_U/\{\varphi\}) \cup \{\psi, \chi\}$. Since $\texttt{Res}(\psi) + \texttt{Res}(\chi) = \texttt{Res}(\varphi) - 1$, it follows that $\texttt{Res}(X'_U) = \texttt{Res}(X_U) - 1 = n$. If instead $\texttt{Res}(\psi) = 0$ or $\texttt{Res}(\chi) = 0$, then $\texttt{Res}(X'_U)$ will be even smaller, and so $\texttt{Res}(X'_U) \leq n$.

Case 3: If $\varphi = \neg(\psi \star \chi)$ and $\texttt{Res}(\neg\psi) \neq 0$ and $\texttt{Res}(\neg\chi) \neq 0$, then $X'_U = (X_U/\{\varphi\}) \cup \{\neg\psi, \neg\chi\}$. Since $\texttt{Res}(\neg\psi) + \texttt{Res}(\neg\chi) = \texttt{Res}(\varphi) - 1$, it follows that $\texttt{Res}(X'_U) = \texttt{Res}(X_U) - 1 = n$. If instead $\texttt{Res}(\neg\psi) = 0$ or $\texttt{Res}(\neg\chi) = 0$, then $\texttt{Res}(X'_U)$ will be even smaller, and so $\texttt{Res}(X'_U) \leq n$.

Since in all cases $\texttt{Res}(X'_U) \leq n$, it follows by hypothesis that X' can be completed in a finite number of steps. We know by **Task 2** that X' is the result of resolving φ in at most a finite number of branches in X. Since the sum of finite numbers is finite, we may conclude that X may be completed in a finite number of steps. Thus it follows by induction that every tree X can be completed in a finite number of steps.

The Soundness of SL Tree Proofs

LOGIC I
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October 5, 2023

Informal Proof

Motive: We want to know which arguments are valid.

Equivalence: $\Sigma \vDash \varphi$ *iff* $\Sigma, \neg\varphi \vDash \bot$.

Soundness: Letting $\Gamma = \Sigma \cup \{\neg\varphi\}$, we want to show that $\Gamma \vDash \bot$ if $\Gamma \vdash \bot$.

Informally: We want to show that every closed tree has an unsatisfiable root.

Question 1: Why can't we use our tree method (or similar) to prove soundness?

Definitions

Root: An SL tree whose root contains the sentences in Γ is a tree *with* root Γ.

Branch Satisfaction: An SL interpretation $\mathcal{I}$ *satisfies* a branch $\mathcal{B}$ in an SL tree X just in case $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every φ which occurs in $\mathcal{B}$.

Setting up the Proof

Contrapositive: Every SL tree with a satisfiable root is closed.

Lemma 3: Every SL tree with a satisfiable root has a satisfiable branch.

Question 2: How can we derive soundness from this stronger claim?

Question 3: How can we prove *Lemma 4*?

Supporting Lemmas

Lemma 1: Every satisfiable branch $\mathcal{B}$ in an SL tree X is open.

Lemma 2: If X is an SL tree with a satisfiable branch $\mathcal{B}$, then any tree X' which is the result of resolving a sentence in $\mathcal{B}$ has a satisfiable branch $\mathcal{B}'$.

- Assume X has a satisfiable branch $\mathcal{B}$.
- So there is some $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all φ in $\mathcal{B}$.
- By *Lemma 1*, $\mathcal{B}$ is open.
- If $\mathcal{B}$ is complete, then the consequent holds vacuously.
- If $\mathcal{B}$ is not complete, then $\mathcal{B}$ has a resolvable sentence φ.
- There are nine cases to check given our nine resolution rules.

Lemma 3

Proof: Every SL tree with a satisfiable root has a satisfiable branch.

Antecedent: Assume $\Gamma \nvDash \bot$.

Base: Let X be an tree with root Γ where $\texttt{Length}(X) = 0$.

Hypothesis: Every tree X with root Γ of $\texttt{Length}(X) = n$ has a satisfiable branch $\mathcal{B}$.

Induction: Assume X' is a tree with root Γ of $\texttt{Length}(X') = n + 1$.

1. Let X be any tree with root Γ where X' is the result of resolving a sentence φ in a branch $\mathcal{B}$ of X.
2. So X is a tree with root Γ of $\texttt{Length}(X) = n$.
3. By hypothesis, X has a satisfiable branch $\mathcal{B}^\star$.
4. So either $\mathcal{B}^\star = \mathcal{B}$ or not.

Case 1: Assume $\mathcal{B}^\star = \mathcal{B}$.

(a) We know that X' is the result of resolving φ in $\mathcal{B}$.
(b) By the case assumption $\mathcal{B} = \mathcal{B}^\star$.
(c) Since $\mathcal{B}^\star$ is satisfiable, X' has a satisfiable branch $\mathcal{B}'$ by *Lemma 2*.

Case 2: Assume $\mathcal{B}^\star \neq \mathcal{B}$.

(a) We know $\mathcal{B}^\star$ is a satisfiable branch of X.
(b) X' is the result of resolving φ in $\mathcal{B} \neq \mathcal{B}^\star$ of X.
(c) So $\mathcal{B}^\star$ is also a branch of X'.
(d) Since $\mathcal{B}^\star$ is satisfiable, X' has a satisfiable branch.

5. Thus X' has a satisfiable branch whether $\mathcal{B}^\star = \mathcal{B}$ or not.
6. Every tree X' with root Γ of $\texttt{Length}(X') = n + 1$ has a satisfiable branch $\mathcal{B}$.

Conclusion: By weak induction, QED.

Proving Soundness

Proof: If there is a closed SL tree with root Γ, then Γ is unsatisfiable.

1. Assume Γ is satisfiable.
2. Let X be an SL tree with root Γ.
3. So X has a satisfiable branch $\mathcal{B}$ by *Lemma 3*.
4. So $\mathcal{B}$ is open by *Lemma 1*.
5. So X is not closed.
6. More generally, there is no closed SL tree with root Γ.
7. By contraposition, QED.

The Completeness of SL Tree Proofs

LOGIC I
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The Proof

Completeness: Every unsatisfiable root has a closed tree: $\Gamma \vDash \bot \Rightarrow \Gamma \vdash \bot$.

Contrapositive: If there is no closed tree with root Γ, then Γ is satisfiable.

Lemma 6: For any tree X with root Γ, there is a complete tree X' with root Γ.

- Assume there is no closed tree with root Γ.
- Roots are trees, and so Γ has a complete tree X.
- So X is a complete open tree with a complete open branch $\mathcal{B}$.

Note: This result is purely syntactic.

Lemma 7: Every complete open branch in an SL tree is satisfiable.

- So $\mathcal{B}$ is satisfiable, and so Γ is satisfiable.
- By contraposition, if $\Gamma \vDash \bot$, then $\Gamma \vdash \bot$.

Resolution

Let the *resolution* $\texttt{Res}(\varphi)$ provide an upper bound on the number of times that φ and its descendants could be resolved in an SL tree.

1. $\texttt{Res}(\varphi) = 0$ if φ is a literal.
2. For any SL sentences φ and ψ:
   - $\texttt{Res}(\neg\neg\varphi) = \texttt{Res}(\varphi) + 1$.
   - $\texttt{Res}(\varphi \wedge \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \wedge \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \vee \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \vee \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \supset \psi) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \supset \psi)) = \texttt{Res}(\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \equiv \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \equiv \psi)) = \texttt{Res}(\varphi) + \texttt{Res}(\neg\psi) + \texttt{Res}(\neg\varphi) + \texttt{Res}(\psi) + 1$.

Resolution Set: Let $[X]$ be the set of SL sentences that are resolvable in a branch of X.

Tree Resolution: Let $\texttt{Res}(X) = \sum_{\varphi \in [X]} \texttt{Res}(\varphi)$ be an upper bound on resolutions in X.

Supporting Lemmas

Lemma 4: Every SL tree X has a finite number of branches.

Lemma 5: For any SL tree X with root Γ and $\varphi \in [X]$, there is an SL tree Y with root Γ where $\texttt{Res}(Y) < \texttt{Res}(X)$.

- Let X be an SL tree with root Γ where $\varphi \in [X]$.
- By *Lemma 4*, φ is resolvable in finitely many branches of X.
- So there is a tree Y with root Γ that resolves φ throughout X.
- So $\varphi \notin [Y]$ but the children of φ could be in $[Y]$.

Case 1: Assume $\varphi = \neg\neg\psi$ where $\psi \in [Y]$ and $\psi \notin [X]$.

- So $\texttt{Res}(\psi) < \texttt{Res}(\varphi)$, and so $\texttt{Res}(Y) < \texttt{Res}(X)$.

Case n: The other cases are similar.

Lemma 6

Proof: For any Γ-tree X, there is a complete Γ-tree X'.

Base: Assume X is a Γ-tree where $\texttt{Res}(X) = 0$.

- So every $[X]$ is empty, so X is complete.

Hypothesis: Every Γ-tree X where $\texttt{Res}(X) \leq n$ has a complete Γ-tree X'.

Induction: Let X be a Γ-tree where $\texttt{Res}(X) = n + 1$.

- Since $\texttt{Res}(X) > 0$, there is some $\varphi \in [X]$.
- By *Lemma 5*, there is some Γ-tree Y where $\texttt{Res}(Y) < \texttt{Res}(X)$.
- By hypothesis, there is a complete Γ-tree Y'.

Conclusion: By strong induction, QED.

Finite Lemma

Proof: Every branch $\mathcal{B}$ in an SL tree contains finitely many sentences.

Base: Assume $\mathcal{B}$ belongs to an SL tree X where $\texttt{Length}(X) = 0$, so finite.

Hypothesis: Assume that every branch $\mathcal{B}$ of an SL tree X of $\texttt{Length}(X) = n$ has a finite number of sentences.

Induction: Assume that $\mathcal{B}'$ belongs to an SL tree X' of $\texttt{Length}(X) = n + 1$.

- Let X be a tree where X' is the result of resolving a sentence in X.
- So $\texttt{Length}(X) = n$.

- By hypothesis, every branch $\mathcal{B}$ of X has a finite number of branches.
- $\mathcal{B}'$ includes at most two more sentences than any branch $\mathcal{B}$ in X.
- Thus $\mathcal{B}'$ has a finite number of sentences.

Lemma 7

Proof: Every complete open branch in an SL tree is satisfiable.

Assume: Let $\mathcal{B}$ be a complete open branch in an SL tree.

- Let $\mathcal{I}(\varphi) = 1$ *iff* φ is a sentence letter in $\mathcal{B}$.
- By the *Finite Lemma*, we may assign sentences in $\mathcal{B}$ a position number where the leaf is 0.

Base: Assume φ has position 0.

- Since $\mathcal{B}$ is complete and open, φ is a literal.

Case 1: If φ is a sentence letter, $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi) = 1$.

Case 2: Assume $\varphi = \neg\psi$ where ψ is a sentence letter.

- Since $\mathcal{B}$ is open, ψ does not occur in $\mathcal{B}$.
- So $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{I}(\psi) = 0$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg\psi) = 1$.

Hypothesis: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every φ with position $k \leq n$ in $\mathcal{B}$.

Induction: Assume φ has position $n + 1$ in $\mathcal{B}$.

Case 1: φ is a literal, so $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ as above.

Case 2: $\varphi = \neg\neg\psi$.

- Since $\mathcal{B}$ is complete, ψ occurs in $\mathcal{B}$ in position $k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\psi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg\neg\psi) = 1$.

Case 3: $\varphi = \psi \wedge \chi$.

Case 4: $\varphi = \neg(\psi \wedge \chi)$.

- Since $\mathcal{B}$ is complete, $\neg\psi, \neg\chi$ occur in $\mathcal{B}$ in positions $j, k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\neg\psi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\neg\chi) = 1$.
- So $\mathcal{V}_{\mathcal{I}}(\psi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\chi) = 0$, and so $\mathcal{V}_{\mathcal{I}}(\psi \wedge \chi) = 0$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg(\psi \wedge \chi)) = 1$.

Case n: $\varphi = \neg(\psi \equiv \chi)$.

- Since $\mathcal{B}$ is complete, ψ and $\neg\chi$ occur in $\mathcal{B}$ in positions $j, k \leq n$, or else $\neg\psi$ and χ occur in $\mathcal{B}$ in positions $j, k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{I}}(\neg\chi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\neg\psi) = \mathcal{V}_{\mathcal{I}}(\chi) = 1$.
- In either case, $\mathcal{V}_{\mathcal{I}}(\psi) \neq \mathcal{V}_{\mathcal{I}}(\chi)$, and so $\mathcal{V}_{\mathcal{I}}(\psi \equiv \chi) = 0$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg(\psi \equiv \chi)) = 1$.

Natural Deduction in SL

LOGIC I
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Motivation

Proof Trees: Proof trees provide an efficient way to evaluate validity.

- If an argument is valid, the tree will close.
- If an argument is invalid, the tree will give us an interpretation.

Unnatural: But proof trees do not provide a natural line of reasoning.

- Proof trees go by *reductio* which are not explanatory.
- Rules for proof trees are not entirely unnatural.
- But trees do not resemble natural reasoning.

Natural Deduction: How would we describe the patterns of natural deduction?

- Identify a range of intuitively compelling basic inferences in SL.
- Such inferences hold in virtue of the meanings of the connectives.
- Define a proof to be any composition of basic inferences.

Rules: Our system will include introduction and elimination rules.

- These rules will describe how to reason with the connectives.

Conditional

Elimination: $A,\ A \supset B,\ B \supset C \vdash C$.

- Premises justified by ':PR'.
- Easy to derive C.
- What if A was excluded from the premises?

Introduction: $A \supset B,\ B \supset C \vdash A \supset C$.

- Need something to work with.
- Want to conclude with a conditional claim.
- Assumption of A justified by ':AS'.

Subproofs: Lines in a closed subproof are dead and all else are live.

- $\supset$E can only cite to live lines.
- $\supset$I can only cite an appropriate subproof.

Reiteration

Example: $A \vdash D \supset [C \supset (B \supset A)]$.

Conjunction

Elimination: $A \supset (B \wedge C),\ B \supset D \vdash A \supset D$.

Introduction: $A \wedge B,\ B \supset C \vdash A \wedge C$.

Disjunction

Introduction: $A \vdash B \vee ((A \vee C) \vee D)$.

Elimination: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$.

Biconditional

Elimination: $A \equiv (B \supset [(A \wedge C) \equiv D]) \vdash (A \wedge B) \supset (D \supset C)$.

Introduction: $A \supset (B \wedge C),\ C \supset (B \wedge A) \vdash A \equiv C$.

Negation

Elimination: $\neg\neg A \vdash A$.

Introduction: $A \supset (B \wedge C),\ C \supset (B \wedge A) \vdash A \equiv C$.

Proof

Proof: A natural deduction PROOF (or DERIVATION) of a conclusion φ from a set of premises Γ in SD is any sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) a premise in Γ;
(2) a discharged assumption; or
(3) follows from previous lines by the rules for SD.

Provable: An SL sentence φ is PROVABLE (or DERIVABLE) from Γ in SD *iff* there is a natural deduction proof (derivation) of φ from Γ in SD, i.e., $\Gamma \vdash \varphi$.

Equivalent: Two sentences φ and ψ are PROVABLY EQUIVALENT (or INTERDERIVABLE) if and only if both $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Inconsistent: A set of sentences Γ is PROVABLY INCONSISTENT if and only if $\Gamma \vdash \perp$ where $\perp$ is our arbitrarily chosen contradiction, e.g., $A \wedge \neg A$.

Natural Deduction in SL: Part II

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October 19, 2023

Negation

Elimination Rule: $\neg\neg A \vdash A$. (*Double Negation Elimination*)

1. $A \vee \neg A$. (*Law of Excluded Middle*)
2. $A, \neg A \vdash B$. (*Ex Falso Quodlibet*)

Introduction Rule: $\neg(A \wedge \neg A)$. (*Law of Non-Contradiction*)

3. $A \vdash \neg\neg A$. (*Double Negation Introduction*)

Proof

Proof: A natural deduction PROOF (or DERIVATION) of a conclusion φ from a set of premises Γ in SD is any finite sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) A premise in Γ;

(2) A discharged assumption; or

(3) Follows from previous lines by the rules for SD.

Provable: An SL sentence φ is PROVABLE (or DERIVABLE) from Γ in SD *iff* there is a natural deduction proof (derivation) of φ from Γ in SD, i.e., $\Gamma \vdash \varphi$.

Theorems: An SL sentence φ is a THEOREM of SD *iff* $\vdash \varphi$.

Equivalent: Sentences φ and ψ are PROVABLY EQUIVALENT (or INTERDERIVABLE) if and only if both $\varphi \vdash \psi$ and $\psi \vdash \varphi$, i.e., $\varphi \dashv\vdash \psi$.

Inconsistent: A set of sentences Γ is PROVABLY INCONSISTENT *iff* $\Gamma \vdash \bot$ where $\bot$ is the arbitrarily contradiction we chose, i.e., $A \wedge \neg A$.

Soundness and Completeness

Assume: $\Gamma \vdash \varphi$ *iff* $\Gamma \vDash \varphi$.

Tautologies: Coextensive with the theorems.

Validity: The valid SL arguments are derivable in SD, and *vice versa*.

Task 1: Can we ever use SD to determine that an argument is invalid?

Uncertainty: If we haven't found a proof, that doesn't mean one doesn't exist.

Logical Analysis

Task 2: How can we tell if an argument is valid?

- Use a semantic argument: true premises and false conclusion.
- Construct a tree proof.

Pro: Both methods provide a countermodel if there is one.

Con: Neither method derives the conclusion from the premises if valid.

Task 3: How can we tell if a theorem is valid?

Tautology?	If YES, prove $\vdash \varphi$.	If NO, provide a countermodel.
Contradiction?	If YES, prove $\vdash \neg\varphi$.	If NO, provide a model.
Contingent?	If YES, provide a models.	If NO, prove $\vdash \varphi$ or $\vdash \neg\varphi$.
Equivalent?	If YES, prove $\varphi \dashv\vdash \psi$.	If NO, provide a countermodel.

Schemata

Observe: Compare rules of inference in SD to SL proofs in SD.

- Whereas the rules are general, SL proofs are particular.
- But nothing in our SL proofs depend on the particulars.

Task 3: How might we generalise our proofs beyond any instance?

Rule Schemata: Replace sentence letters in SL proofs with metavariables.

- Premises are replaced with the lines cited by that rule.
- New rules require new names if we are to use them.

Task 4: Can we also generalise proofs of theorems?

Axiom Schemata: Amount to lines that can be added without citing lines.

Goal: We want to derive intuitive rule schemata.

Derivable Schemata

Double Negation: $\neg\neg\varphi \dashv\vdash \varphi$.

Ex Falso Quodlibet: $\varphi,\ \neg\varphi \vdash \psi$.

Law of Excluded Middle: $\vdash \varphi \vee \neg\varphi$.

Law of Non-Contradiction: $\vdash \neg(\varphi \wedge \neg\varphi)$.

Hypothetical Syllogism: $\varphi \supset \psi,\ \psi \supset \chi \vdash \varphi \supset \chi$.

Modus Tollens: $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi$.

Contraposition: $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$.

Dilemma: $\varphi \vee \psi,\ \varphi \supset \chi,\ \psi \supset \chi \vdash \chi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

$\vee$*-Commutativity:* $\varphi \vee \psi \vdash \psi \vee \varphi$.

$\wedge$*-Commutativity:* $\varphi \wedge \psi \vdash \psi \wedge \varphi$.

Biconditional MP: $\varphi \equiv \psi,\ \neg\varphi \vdash \neg\psi$.

$\equiv$*-Commutativity:* $\varphi \equiv \psi \vdash \psi \equiv \varphi$.

$\wedge$*-De Morgan's:* $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.

$\vee$*-De Morgan's:* $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.

$\vee\wedge$*-Distribution:* $\varphi \vee (\psi \wedge \chi) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$.

$\wedge\vee$*-Distribution:* $\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

$\vee\wedge$*-Absorption:* $\varphi \vee (\varphi \wedge \psi) \dashv\vdash \varphi$.

$\wedge\vee$*-Absorption:* $\varphi \wedge (\varphi \vee \psi) \dashv\vdash \varphi$.

$\wedge$*-Associativity:* $\varphi \wedge (\psi \wedge \chi) \dashv\vdash (\varphi \wedge \psi) \wedge \chi$.

$\vee$*-Associativity:* $\varphi \vee (\psi \vee \chi) \dashv\vdash (\varphi \vee \psi) \vee \chi$.

Axiom System for SL

Axiom System: Consider the axiom and rule schemata, writing '/' for deduction.

- $\varphi \supset (\psi \supset \varphi)$.
- $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$.
- $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$.
- $\varphi \supset \psi,\ \varphi \ /\ \psi$.

PL-Proof: $\Gamma \vdash_{PL} \varphi$ *iff* there is a finite sequence of SL sentences where every sentence in the sequence is either: (1) a member of Γ; (2) an axiom schemata; or (3) follows from previous sentences in the sequence by the single rule schemata given above.

Equivalence: Amazingly, it is possible to show that $\Gamma \vdash_{PL} \varphi$ *iff* $\Gamma \vdash_{SD} \varphi$.

Definitions: Given that the axioms and rule schemata only include $\neg$ and $\supset$, we may take these to be the *only* primitive logical connectives, defining all other connectives in their terms.

- This makes for a very compact description of the same logic.
- This logic is much less natural to use, requiring that a lot of derived rules be added to system.
- We don't have this problem, though our system is more complex.

Midterm Review

LOGIC I
Benjamin Brast-McKie
October 24, 2023

Derivable Schemata

Contraposition: $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$.

Hypothetical Syllogism: $\varphi \supset \psi,\ \psi \supset \chi \vdash \varphi \supset \chi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

$\vee$*-Conditional:* $\varphi \supset \psi \dashv\vdash \neg\varphi \vee \psi$.

$\neg$*-Conditional:* $\neg(\varphi \supset \psi) \dashv\vdash \varphi \wedge \neg\psi$.

Conditional Weakening: $\psi \vdash \varphi \supset \psi$.

Double Negation: $\neg\neg\varphi \dashv\vdash \varphi$.

$\wedge$*-De Morgan's:* $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.

$\vee$*-De Morgan's:* $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.

Modus Tollens: $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi$.

Regimentation

Complete the following tasks for arguments (A) and (B):

Task 1: Write a symbolization key and regiment the argument.

Task 2: Determine if the argument is valid.

Task 3: Provide an SD proof if valid, and a countermodel otherwise.

(A) If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

(B) If Cam remembered to do his chores, then things are clean but not neat. Cam forgot only if things are neat but not clean. Therefore, things are clean just in case they are not neat.

Lemmas

`Lemma 1:` Every satisfiable branch $\mathcal{B}$ in an SL tree X is open.

`Lemma 2:` If X is an SL tree with a satisfiable branch $\mathcal{B}$, then any tree X' which is the result of resolving a sentence in $\mathcal{B}$ has a satisfiable branch $\mathcal{B}'$.

`Lemma 3:` Every SL tree with a satisfiable root has a satisfiable branch.

`Lemma 4:` Every SL tree X has a finite number of branches.

`Lemma 5:` For any SL tree X with root Γ and $\varphi \in [X]$, there is an SL tree Y with root Γ where $\texttt{Res}(Y) < \texttt{Res}(X)$.

`Lemma 6:` For any tree X with root Γ, there is a complete tree X' with root Γ.

`Lemma 7:` Every complete open branch in an SL tree is satisfiable.

Soundness

1. Assume Γ is satisfiable.
2. Let X be an SL tree with root Γ.
3. So X has a satisfiable branch $\mathcal{B}$ by *Lemma 3*.
4. So $\mathcal{B}$ is open by *Lemma 1*.
5. So X is not closed.
6. More generally, there is no closed SL tree with root Γ.
7. By contraposition, QED.

Completeness

1. Assume there is no closed tree with root Γ.
2. Roots are trees, and so Γ has a complete tree X by `Lemma 6`.
3. So X is a complete open tree with a complete open branch $\mathcal{B}$.
4. By *Lemma 7*, $\mathcal{B}$ is satisfiable, and so Γ is satisfiable.
5. By contraposition, if $\Gamma \vDash \bot$, then $\Gamma \vdash \bot$.

Quantifier Logic

LOGIC I
Benjamin Brast-McKie
October 31, 2023

Expressive Limitations

Socrates: Consider the following argument:

(a) Every human is mortal.
(b) Socrates is human.
(c) $\therefore$ Socrates is mortal.

Mammals: Consider the following argument:

(a) All humans are mammals.
(b) All mammals are multi-celled organisms.
(c) $\therefore$ All humans are multi-celled organisms.

SL Regimentation: Neither argument is valid in SL.

Predicates, Variables, and Quantifiers

Mammals (a): Everything is such that if it is human then it is a mammal.

Mammals (b): Everything is such that if it is a mammal then it is a multi-celled organism.

Mammals (c): Everything is such that if it is human then it is a multi-celled organism.

Predicates: 'is human',
'is a mammal', and
'is a multi-celled organism'.

Properties: Predicates express properties.

Variables: 'it'.

Reference: What does 'it' refer to?

Atomic Formulas: 'it is human',
'it is a mammal', and
'it is a multi-celled organism'.

Complex Formulas: 'if it is human then it is a mammal',
'if it is a mammal then it is a multi-celled organism', and
'if it is human then it is a multi-celled organism'.

Quantifiers: 'Everything is such that'.

Constants

Socrates (a): Everything is such that if it is human then it is mortal.

Socrates (b): Socrates is human.

Socrates (c): Socrates is mortal.

Predicates: 'is human' and 'is mortal'.

Variables: 'it'.

Constants: 'Socrates'.

Reference: Constants refer to objects.

Atomic Formulas: 'it is human', 'it is mortal', 'Socrates is human', and 'Socrates is mortal'.

Complex Formulas: 'if it is human then it is mortal'.

Quantifiers: 'Everything is such that'.

Binary Predicates

Height: Kin is taller than Prema.
∴ Prema is shorter than Kin.

Task 1: Regiment the argument above.

Predicates: 'is taller than', 'is shorter than', and 'is the same height as'.

Relations: Binary predicates express 2-place properties, i.e., *relations*.

- $Tkp \therefore Spk$.
- $Tkp \therefore \neg Tpk$.
- $Tkp \therefore \neg Tpk \wedge \neg Epk$.

Question 1: Is this argument valid, and if not how can we make it valid?

- $Tkp,\ Tkp \supset Spk \therefore Spk$.
- $Tkp,\ \forall x \forall y (Txy \supset Syx) \therefore Spk$.

Age: Jon is older than Sara.
Sara is older than Ethan.
∴ Jon is older than Ethan.

Task 2: Regiment the argument above.

Predicates: 'is older than'.

- $Ojs,\ Ose \therefore Oje$.

Question 2: Is this argument valid, and if not how can we make it valid?

- $Ojs,\ Ose,\ (Ojs \wedge Ose) \supset Oje \therefore Oje$.
- $Ojs,\ Ose,\ \forall x \forall y \forall z ((Oxy \wedge Oyz) \supset Oxz) \therefore Oje$.

Polyadic Predicates

Triadic: 'x is between y and z',
'x is more similar to y than to z',
'x is closer to y than to z', ...

Polyadic: We may refer to predicates as n-place or n-adic.

Properties: n-place predicates express n-place properties.

Primitive Symbols of QL

Predicates: n-place predicates $A^n, \ldots, Z^n$ for $n \geq 0$ possibly with subscripts.

Constants: $a, b, c, \ldots$ possibly with subscripts.

Variables: $x, y, z, \ldots$ possibly with subscripts.

Connectives: $\neg, \wedge, \vee, \supset, \equiv$.

Quantifiers: $\forall, \exists$.

Parentheses: $(,)$.

Well-Formed Formulas of QL

Singular Terms: Constants and variables are called *singular terms*.

Well-Formed Formulas: We may define the well-formed formulas (wffs) of QL as follows:

1. $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ is a wff if $\mathcal{F}^n$ is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. If φ and ψ are wffs and α is a variable, then:
   (a) $\exists\alpha\varphi$ is a wff;
   (b) $\forall\alpha\varphi$ is a wff;
   (c) $\neg\varphi$ is a wff;
   (d) $(\varphi \wedge \psi)$ is a wff;
   (e) $(\varphi \vee \psi)$ is a wff;
   (f) $(\varphi \supset \psi)$ is a wff; and
   (g) $(\varphi \equiv \psi)$ is a wff.
3. Nothing else is a wff.

Atomic Formulas: The wffs defined by (1) are *atomic*.

Arguments: The singular terms in an atomic wff are the *arguments* of the predicate.

Composition Rules: The clauses in (2) are called *composition rules*.

Scope: φ is the *scope* of the quantifier in $\exists\alpha\varphi$ and $\forall\alpha\varphi$.

- Compare the scope of negation.

Question 3: Does the definition above make sense as stated?

Task 3: How can we fix the definition above to respect use/mention?

The Sentences of QL

Free Variables: We define the *free variables* recursively:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leq i \leq n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. If φ and ψ are wffs and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;
   (d) α is free in $(\varphi \wedge \psi)$ if α is free in φ or α is free in ψ;
   (e) α is free in $(\varphi \vee \psi)$ if α is free in φ or α is free in ψ;
   (f) α is free in $(\varphi \supset \psi)$ if α is free in φ or α is free in ψ;
   (g) α is free in $(\varphi \equiv \psi)$ if α is free in φ or α is free in ψ;
3. Nothing else is a free variable.

Bound Variables: Every free occurrence of α in φ is *bound* in $\exists\alpha\varphi$ and $\forall\alpha\varphi$.

Binding: The variable α is the *binding variable* in $\exists\alpha\varphi$ and $\forall\alpha\varphi$.

Open Sentences: An *open sentence* of QL is any wff with free variables.

Sentences: A *sentence* of QL is any wff without free variables.

Interpretation: Only the sentences of QL will have truth-values on an interpretation independent of an assignment function.

Regimentation and Relations

LOGIC I
Benjamin Brast-McKie
November 2, 2023

Restricting Quantifiers

Universals Quantifiers: Regiment the following sentences:

- All dogs go to heaven.
- Jim took every chance he got.
- All the monkeys that Amar loves love him back.
- Everyone who trained hard or got lucky made it to the top or else didn't compete.

Hidden Quantifiers: Regiment the following sentences:

- At least the guests that remained were pleased with the party.
- I haven't met a cat that likes Merra.
- Kiko's only friends are animals.

Existential Quantifiers: Regiment the following sentences:

- Something great is around the corner.
- One of Ken's statues is very old.
- Kate found a job that she loved.

Mixed Quantifiers

1. Nothing is without imperfections.
2. Every dog has its day.
3. Everyone loves someone.
4. Nobody knows everybody.
5. Everybody everybody loves loves somebody.
6. No set is a member of itself.
7. There is a set with no members.

Arguments

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
- May loves everyone who loves themselves.

$\therefore$ If Cam loves himself, he doesn't love May.

Bigger: Regiment the following argument:

- Whenever something is bigger than another, the latter is not bigger than the former.

$\therefore$ Nothing is bigger than itself.

Relations

Domain: Let the *domain* D be any set.

Relation: A *relation* R on D is any subset of D^2.

Reflexive: A relation R is *reflexive* on D *iff* $\langle x, x\rangle \in R$ for all $x \in D$.

Non-Reflexive: A relation R is *non-reflexive* on D *iff* R is not reflexive on D.

Question 1: What is it to be *irreflexive*?

Irreflexive: A relation R is *irreflexive* on D *iff* $\langle x, x\rangle \notin R$ for all $x \in D$.

Symmetric: A relation R is *symmetric iff* $\langle y, x\rangle \in R$ whenever $x, y \in R$.

Question 2: Why don't we need to specify a domain?

Question 3: Why is a relation reflexive or irreflexive with respect to a domain?

Asymmetric: A relation R is *asymmetric iff* $\langle y, x\rangle \notin R$ whenever $\langle x, y\rangle \in R$.

Question 4: What is it to be non-symmetric? How about non-asymmetric?

Task 1: Show that every asymmetric relation is irreflexive.

Transitive: A relation R is *transitive iff* $\langle x, z\rangle \in R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Intransitive: A relation R is *intransitive iff* $\langle x, z\rangle \notin R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Question 5: Is every symmetric transitive relation reflexive? (No: $R = \varnothing$)

Task 2: Show that every transitive irreflexive relation asymmetric?

Euclidean: A relation R is *euclidean iff* $\langle y, z\rangle \in R$ whenever $\langle x, y\rangle, \langle x, z\rangle \in R$.

Task 3: Show that every transitive symmetric relation is euclidean.

The Semantics for QL

LOGIC I
Benjamin Brast-McKie
November 7, 2023

Examples

Monadic: Casey is dancing.

Dyadic: Al loves Max.

Triadic: Kim is between Boston and New York.

Constants and Referents

Constants: Constants are interpreted as referring to individuals.

Existence: Thus we need to know what things there are.

Domain: A *domain* is any nonempty set $\mathbb{D}$.

Referents: Interpretations assign constants to elements of $\mathbb{D}$.

Question 1: How are we going to interpret predicates?

Predicates and Extensions

Example: 'Al loves Max' is true *iff* Al bears the loves-relation to Max.

Dyadic Predicates: Dyadic predicates are interpreted by sets of *ordered pairs* in $\mathbb{D}^2$.

Question 2: How are we to interpret n-place predicates?

Cartesian Power: $\mathbb{D}^n = \{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle : \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{D}\}$.

Extensions: n-place predicates are interpreted by subsets of $\mathbb{D}^n$.

Singletons: 1-place predicates are interpreted by subsets of $\mathbb{D}^1 = \{\langle \mathbf{x} \rangle : \mathbf{x} \in \mathbb{D}\}$.

Question 3: How are we to interpret 0-place predicates? What is $\mathbb{D}^0$?

n-Tuples: Let $\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle = \{\langle 1, \mathbf{x}_1 \rangle, \ldots, \langle n, \mathbf{x}_n \rangle\}$.

0-*Tuple:* $\langle\rangle = \varnothing$.

Truth-Values: 0-place predicates are interpreted by subsets of $\mathbb{D}^0 = \{\varnothing\}$.

Ordinals: Let $1 = \{\varnothing\}$ and $0 = \varnothing$ be the first two von Neumann ordinals.

QL Models

Interpretations: $\mathcal{I}$ is an QL interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Question 4: What happens if $\mathbb{D} = \varnothing$?

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL *iff* $\mathcal{I}$ is a QL interpretation over $\mathbb{D} \neq \varnothing$.

Task 1: Regiment and interpret the sentences above.

- Dc, Lam, $Bkbn$.
- $\mathbb{D} = \{c, a, m, k, b, n\}$.
- $\mathcal{I}(D) = \{\langle c \rangle\}$.
- $\mathcal{I}(L) = \{\langle a, m \rangle\}$.
- $\mathcal{I}(B) = \{\langle k, b, n \rangle\}$.
- $\mathcal{I}(c) = c,\ \mathcal{I}(a) = a,\ \ldots$

Lagadonian: We often take constants to name themselves.

Question 5: Do models give us truth-values?

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL.

Singular Terms: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Example: Let $\mathbb{D} = \{1, 2, 3, 4, 5\}$ where $\hat{a}(x) = 1$, $\hat{a}(y) = 2$, and $\hat{a}(z) = 3$.

Task 2: If $\hat{c}$ is a y-variant of $\hat{a}$, what is $\hat{c}(1)$, $\hat{c}(2)$, and $\hat{c}(3)$?

Example

Universal: Al loves everything, i.e., $\forall x Lax$.

Existential: Someone is dancing, i.e., $\exists x(Px \wedge Dx)$.

Mixed: Everyone loves someone, i.e., $\forall x(Px \supset \exists y Lxy)$.

Complex: Everything everything loves loves something, i.e., $\forall x(\forall y Lyx \supset \exists z Lxz)$.

Semantics for QL

(A) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1,\ldots,\alpha_n) = 1$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1),\ldots,\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.

$(\forall)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

$(\exists)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

$(\neg)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$.

$(\vee)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

$(\wedge)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$.

$(\supset)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \supset \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

$(\equiv)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi)$.

Truth and Entailment

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL.

Satisfaction: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I}\rangle$ satisfies Γ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every $\varphi \in \Gamma$.

Singletons: As before $\mathcal{M}$ satisfies φ *iff* $\mathcal{M}$ satisfies $\{\varphi\}$.

Entailment: $\Gamma \vDash \varphi$ just in case every model $\mathcal{M}$ that satisfies Γ also satisfies φ.

Tautology: φ is a tautology *iff* $\vDash \varphi$.

Contradiction: φ is a contradiction *iff* $\vDash \neg\varphi$.

Contingent: φ is contingent *iff* $\vDash$ and $\nvDash \neg\varphi$.

Consistent: Γ is consistent *iff* Γ is satisfiable.

Minimal Models

Task 3: Provide minimal models in which the examples above are true/false.

Regimentation

- Every rose has its thorn.
- At least the guests that remained were pleased with the party.
- I haven't met a cat that likes Merra.
- Kate found a job that she loved.
- Everybody everybody loves loves somebody.
- No set is a member of itself.
- There is a set with no members.

Arguments

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
- May loves everyone who loves themselves.

$\therefore$ If Cam loves himself, he doesn't love May.

Bigger: Regiment the following argument:

- Whenever something is bigger than another, the latter is not bigger than the former.

$\therefore$ Nothing is bigger than itself.

Relations

Domain: Let the *domain* D be any set.

Relation: A *relation* R on D is any subset of D^2.

Reflexive: A relation R is *reflexive* on D *iff* $\langle x, x\rangle \in R$ for all $x \in D$.

Non-Reflexive: A relation R is *non-reflexive* on D *iff* R is not reflexive on D.

Question 1: What is it to be *irreflexive*?

Irreflexive: A relation R is *irreflexive* on D *iff* $\langle x, x\rangle \notin R$ for all $x \in D$.

Symmetric: A relation R is *symmetric iff* $\langle y, x\rangle \in R$ whenever $x, y \in R$.

Question 2: Why don't we need to specify a domain?

Question 3: Why is a relation reflexive or irreflexive with respect to a domain?

Asymmetric: A relation R is *asymmetric iff* $\langle y, x\rangle \notin R$ whenever $\langle x, y\rangle \in R$.

Question 4: What is it to be non-symmetric? How about non-asymmetric?

Task 1: Show that every asymmetric relation is irreflexive.

Transitive: A relation R is *transitive iff* $\langle x, z\rangle \in R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Intransitive: A relation R is *intransitive iff* $\langle x, z\rangle \notin R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Question 5: Is every symmetric transitive relation reflexive? (No: $R = \varnothing$)

Task 2: Show that every transitive irreflexive relation asymmetric?

Euclidean: A relation R is *euclidean iff* $\langle y, z\rangle \in R$ whenever $\langle x, y\rangle, \langle x, z\rangle \in R$.

Task 3: Show that every transitive symmetric relation is euclidean.

Minimal Models and Variable Assignments

LOGIC I
Benjamin Brast-McKie
November 9, 2023

QL Models

Interpretations: $\mathcal{I}$ is an QL interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL *iff* $\mathcal{I}$ is a QL interpretation over $\mathbb{D} \neq \varnothing$.

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL.

Singular Terms: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Semantics for QL

(A) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n \alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.

($\forall$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall \alpha \varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

($\exists$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists \alpha \varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

($\neg$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg \varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$.

($\vee$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).

($\wedge$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$.

($\supset$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \supset \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).

($\equiv$) $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \equiv \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi)$.

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL.

Assignment Lemmas

Lemma 1: If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

- Goes by routine induction on complexity.

Lemma 2: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

LTR: Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some v.a. $\hat{c}$ over $\mathbb{D}$.

- Let $\hat{a}$ be any v.a. over $\mathbb{D}$.
- Since φ has no free variables, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$ by *Lemma 1*.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for all v.a. $\hat{c}$ over $\mathbb{D}$.

RTL: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for all v.a. $\hat{a}$ over $\mathbb{D}$.

- Since $\mathbb{D}$ is nonempty, there is some v.a. $\hat{a}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$.

Lemma 3: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Minimal Models

Task 1: Provide minimal models in which the following are true/false.

- Al loves everything, i.e., $\forall x Lax$.

True: Let $\hat{a}$ be a v.a. over $\mathbb{D} = \{a\}$.

- Let $\hat{c}$ be any x-variant of $\hat{a}$.
- So $\hat{c}(x) = a$ and $\mathcal{I}(a) = a$.
- Since $\mathcal{I}(L) = \{\langle a, a\rangle\}$, we know $\langle \mathcal{V}^{\hat{c}}_{\mathcal{I}}(a), \mathcal{V}^{\hat{c}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(L)$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(Lax) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x Lax) = 1$.

False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(L) = \varnothing$.

- Assume $\mathcal{V}_{\mathcal{I}}(\forall x Lax) = 1$ for contradiction.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x Lax) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Lax) = 1$ since $\hat{a}$ is an x-variant of itself.
- So $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(a), \mathcal{V}^{\hat{a}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(L)$, and so $\mathcal{I}(L) \neq \varnothing$.

- Someone is dancing, i.e., $\exists x(Px \wedge Dx)$.

True: Let $\hat{a}$ be a v.a. over $\mathbb{D} = \{a\}$ where $a(x) = a$.

- Since $\mathcal{I}(P) = \mathcal{I}(D) = \{\langle a\rangle\}$, we know $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(P) = \mathcal{I}(D)$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Px) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(Dx) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Px \wedge Dx) = 1$.
- Since $\hat{a}$ is a x-variant of itself, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$.
- Thus $\mathcal{V}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$.

False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(P) = \varnothing$.

- Assume $\mathcal{V}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$ for contradiction.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x(Px \wedge Dx)) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Px \wedge Dx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Px) = 1$, and so $\langle \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x) \rangle \in \mathcal{I}(P)$.
- Thus $\mathcal{I}(P) \neq \varnothing$.

- No set is a member of itself. `[contingent]`
  $\neg\exists x(Sx \wedge x \in x)$
- There is a set with no members. `[contingent]`
  $\exists x(Sx \wedge \forall y(y \notin x))$
- Everyone loves someone. `[contingent]`
  $\forall x(Px \supset \exists yLxy)$.
- The guests that remained were pleased with the party. `[contingent]`
  $\forall x(Rxp \supset Px)$.
- I haven't met a cat that likes Merra. `[contingent]`
  $\neg\exists x(Mbx \wedge Cx \wedge Lmx)$
- Kate found a job that she loved. `[contingent]`
  $\exists x(Fkx \wedge Jx \wedge Lkx)$
- Everything everything loves loves something. `[contingent]`
  $\forall x(\forall yLyx \supset \exists zLxz)$.

Quantifier Exchange

($\neg\forall$) $\neg\forall x\varphi \vDash \exists x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ satisfy $\neg\forall x\varphi$.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\forall x\varphi) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x\varphi) \neq 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) \neq 1$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\neg\varphi) = 1$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\forall x\neg\varphi) = 1$.

($\neg\exists$) $\neg\exists x\varphi \vDash \forall x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ satisfy $\neg\exists x\varphi$.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\exists x\varphi) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x\varphi) \neq 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) \neq 1$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\neg\varphi) = 1$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\forall x\neg\varphi) = 1$.

Arguments

Bigger: Regiment the following argument:

- Whenever something is bigger than another, the latter is not bigger than the former.
  $\forall x \forall y(Bxy \supset \neg Byx)$.

$\therefore$ Nothing is bigger than itself.
  $\neg \exists x Bxx$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model which satisfies the premise.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x \forall y(Bxy \supset \neg Byx)) = 1$ for some v.a. $\hat{a}$.
- Assume $\mathcal{V}_{\mathcal{I}}(\neg \exists x Bxx) \neq 1$ for contradiction.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg \exists x Bxx) \neq 1$ in particular.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x Bxx) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Bxx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\langle \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x), \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x) \rangle \in \mathcal{I}(B)$, and so $\langle \hat{c}(x), \hat{c}(x) \rangle \in \mathcal{I}(B)$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\forall y(Bxy \supset \neg Byx)) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy \supset \neg Byx) = 1$ for y-variant $\hat{e}$ where $\hat{e}(y) = \hat{c}(x)$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\neg Byx) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Byx) \neq 1$.
- So $\langle \hat{e}(x), \hat{e}(y) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{e}(y), \hat{e}(x) \rangle \notin \mathcal{I}(B)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ since $\hat{e}(x) = \hat{c}(x)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$, contradicting the above.

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
  $\forall x(Lxc \supset \neg Lcx)$.
- May loves everyone who loves themselves.
  $\forall y(Lyy \supset Lmy)$.

$\therefore$ If Cam loves himself, he doesn't love May.
  $Lcc \supset \neg Lcm$.

Taller: Regiment the following argument:

- If a first is taller than a second who is taller than a third, then the first is taller than the third.
  $\forall x \forall y \forall z((Txy \wedge Tyz) \supset Txz)$.
- Nothing is taller than itself.
  $\neg \exists x Txx$.

$\therefore$ If a first is taller than a second, the second isn't taller than the first.
  $\forall x \forall y(Txy \supset \neg Tyx)$.

Quantified Logic with Identity

LOGIC I
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Logical Terms

Extensions: QL extends SL, but we needn't stop there.

Question 1: How far could we go? What terms could we include?

Logicality: The primitive symbols of SL and QL can be divided in three:

`Logical Terms:` $\neg, \wedge, \vee, \supset, \equiv, \forall\alpha, \exists\alpha, x_n, y_n, z_n \ldots$ for $n \geq 0$.

`Non-Logical Terms:` $a_n, b_n, c_n, \ldots$ and $A^n, B^n, \ldots$ for $n \geq 0$.

`Punctuation:` $(,)$

Extensions: The "meanings" of the non-logical terms are fixed by an interpretation.

Semantics: The "meanings" of the logical terms are fixed by the semantics.

Question 2: How many logical terms are there?

Identity: At least one more, namely identity which we symbolize by '$=$'.

Syntax for QL$^=$

Identity: We include '$=$' in the primitive symbols of the language.

Well-Formed Formulas: We may define the well-formed formulas (wffs) of QL$^=$ as follows:

1. $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ is a wff if $\mathcal{F}^n$ is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. $\alpha = \beta$ is a wff if α and β are singular terms.
3. If φ and ψ are wffs and α is a variable, then:
   (a) $\exists\alpha\varphi$ is a wff;
   (b) $\forall\alpha\varphi$ is a wff;
   (c) $\neg\varphi$ is a wff;
   (d) $(\varphi \wedge \psi)$ is a wff;
   (e) $(\varphi \vee \psi)$ is a wff;
   (f) $(\varphi \supset \psi)$ is a wff; and
   (g) $(\varphi \equiv \psi)$ is a wff.
4. Nothing else is a wff.

Atomic Formulas: The wffs defined by (1) and (2) are *atomic*.

Complexity: $\texttt{Comp}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = \texttt{Comp}(\alpha = \beta) = 0$.
$\texttt{Comp}(\exists\alpha\varphi) = \texttt{Comp}(\forall\alpha\varphi) = \texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$.
$\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi \vee \psi) = \ldots = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$.

Free Variables

Free Variables: We define the *free variables* recursively:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leq i \leq n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. α is free in $\beta = \gamma$ if $\alpha = \beta$ or $\alpha = \gamma$ where α is a variable.
3. If φ and ψ are wffs and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;
   $\vdots$
4. Nothing else is a free variable.

Sentences of QL$^=$

Sentences: A *sentence* of QL$^=$ is any wff without free variables.

Interpretation: Only the sentences of QL$^=$ will have truth-values on an interpretation independent of an assignment function.

QL$^=$ Models

Question 3: What in the semantics will have to change?

Interpretations: $\mathcal{I}$ is an QL$^=$ interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL$^=$.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL$^=$ *iff* $\mathcal{I}$ is a QL$^=$ interpretation on $\mathbb{D} \neq \varnothing$.

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL$^=$.

Referents: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Semantics for QL$^{=}$

(A) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1,\ldots,\alpha_n) = 1$ *iff* $\langle\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1),\ldots,\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.

($=$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha = \beta) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

($\forall$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

($\exists$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

($\neg$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$.

$\vdots$

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL$^{=}$.

Example

Task 1: Prove that the following argument is valid.

(1) Hesperus is Phosphorus.
(2) Phosphorus is Venus.
$\therefore$ Hesperus is Venus.

Task 2: Prove that $\forall x\forall y\forall z((x = y \wedge y = z) \supset x = z)$ is a tautology.

Logical Predicates

Taller-Than: Suppose we were to take 'taller than' (T) to be logical.

Question 4: Could we provide its semantics?

(T) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(T\alpha\beta) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha)$ is taller than $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

Theory: The semantics would have to rely on a theory of being taller than.

- Providing such a theory lies outside the subject-matter of logic.
- By contrast, identity is something we already grasp.
- Compare our pre-theoretic grasp of negation, conjunction, and the quantifiers.

Question 5: Could we take set-membership $\in$ to be a logical term?

Question 6: What is it to be a logical term?

Existence: Observe that $\exists x(x = x)$ is a tautology.

Question 7: Could we take a term in sentence position to be logical?

($\bot$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\bot) = 1$ *iff* $1 \neq 1$.

($\top$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\top) = 1$ *iff* $1 = 1$.

Assignment Lemmas

Lemma 1: If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

Base: Assume $\texttt{Comp}(\varphi) = 0$, so $\varphi = (\alpha = \beta)$ or $\varphi = \mathcal{F}^n\alpha_1, \ldots, \alpha_n$.

$(\alpha = \beta)$: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha = \beta) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta) \ldots$

$(\mathcal{F}^n\alpha_1, \ldots, \alpha_n)$: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(F^n) \ldots$

Lemma 2: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

Lemma 3: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Leibniz's Law

Believes: Regiment the following argument:

(1) Lois Lane believes that Superman can fly.
(2) Superman is Clark Kent.
$\therefore$ Lois Lane believes that Clark Kent can fly.

Sees: Regiment the following argument:

(1) Lois Lane sees Superman.
(2) Superman is Clark Kent.
$\therefore$ Lois Lane sees Clark Kent.

Question 8: Are these arguments intuitively valid?

Opacity: Whereas 'sees' admits substitution, 'believes' does not.

Transparency: We may say that 'sees' is transparent and that 'believes' is opaque.

Mathematics: Importantly, mathematics is transparent insofar as it does not include any opaque contexts.

Uniqueness and Quantity

LOGIC I
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Uniqueness

Uniqueness: Ingmar trusts Albert, but no one else.

Only: Regiment the following argument:

(1) Lois Lane only loves Clark Kent.
(2) Only Clark Kent is Superman.
$\therefore$ Lois Lane loves Superman.

Definite Descriptions

Question 1: Regiment the following sentences.

- Socrates is guilty.
- Socrates is not guilty.
- Socrates is guilty or not.

Question 2: Regiment the following sentences.

- The king of France is bald.
- The king of France is not bald.
- The king of France is bald or not.

Question 3: What is the difference between these two cases?

Existence: If the king of France is Bald, then the king of France exists.

Definite Article: 'The king of France' can't be a name.

Regimentation: Russell offered the following analysis:

- $\exists x(Kxf \wedge \forall y(Kyf \supset x = y) \wedge Bx)$.
- $\exists x(\forall y(Kyf \equiv x = y) \wedge Bx)$.

Negation: Negation applies to the predicate, not the sentence.

Task 1: Regiment the following:

1. Superman is keeping something from his lover.
2. The man with the axe is not Jack.
3. The Ace of diamonds is not the man with the axe.
4. One-eyed jacks and the man with the axe are wild.
5. No spy knows the combination to the safe.
6. The one Ingmar trusts is lying.
7. The person who knows the combination to the safe is not a spy.

At Least:

Task 2: Regiment the following claims.

1. There is at least one wild card.
2. There are at least two clubs.
3. There are at least three hearts on the table.

Question 4: How can we define these quantifiers in general?

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Constants: If β is a constant, then β is free for any α and φ.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Examples: Consider the following cases:

(a) z is free for x in $\forall y(Fxy \supset Fyx)$

(b) y is not free for x in $\forall y(Fxy \supset Fyx)$

Inequality Quantifiers Defined

Definition: We may define the following abbreviations recursively:

Base: $\exists_{\geq 1}\alpha\varphi := \exists\alpha\varphi$.

Recursive: $\exists_{\geq n+1}\alpha\varphi := \exists\alpha(\varphi \wedge \exists_{\geq n}\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$ where β is free for α.

Infinite: $\Gamma_\infty := \{\exists_{\geq n}x(x = x) : n \in \mathbb{N}\}$.

Question 5: What is the smallest model to satisfy Γ_∞?

At Most: Regiment the following claims.

1. There is at most one wild card.
2. There are at most two one-eyed jacks.
3. There are at most three black jacks.

Definition: $\exists_{\leq n}\alpha\varphi := \neg\exists_{\geq n+1}\alpha\varphi$.

Cardinality Quantifiers

Task 3: Regiment the following.

1. There is one wild card.
2. There are two winning hands.
3. There are three hearts on the table.

Question 6: How can we define the cardinality quantifiers in general?

Base: $\exists_0\alpha\varphi := \forall\alpha\neg\varphi$.

Recursive: $\exists_{n+1}\alpha\varphi := \exists\alpha(\varphi \wedge \exists_n\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$.

Question 7: How do the cardinality quantifiers relate to the inequality quantifiers?

Between: $\exists_{(n,m)}\alpha\varphi := \exists_{\geq n}\alpha\varphi \wedge \exists_{\leq m}\alpha\varphi$ where $n \leq m$.

Exact: $\exists_n\alpha\varphi := \exists_{(n,n)}\alpha\varphi$.

Examples

1. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa)\}$ is satisfiable.
2. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa), \forall x\exists yRxy\}$ is satisfiable.
3. Show that $\forall x\forall y\ x{=}y \vdash \neg\exists x\ x \neq a$.

Relations

Task 4: Is the following argument valid?

- $\forall x\forall y(Rxy \supset Ryx)$.
- $\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$.

$\therefore$ $\forall xRxx$.

Task 5: Is the following argument valid?

- $\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$.
- $\forall x\neg Rxx$.

$\therefore$ $\forall x\forall y(Rxy \supset \neg Ryx)$.

Natural Deduction in $\mathrm{QL}^=$

LOGIC I
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Motivation

Entailment: We have defined entailment in $\mathrm{QL}^=$.

Completeness: We want a complete natural deduction system for $\mathrm{QL}^=$.

Question 1: What rules do we need to derive the following?

- All humans are mortal.	- $\forall x(Hx \supset Mx)$
- Socrates is human.	- Hs
- Socrates is mortal.	- Ms
$\therefore$ Someone is mortal.	$\therefore$ $\exists x Mx$

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Constants: If β is a constant, then β is free for any α and φ.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Instance: $\varphi[\beta/\alpha]$ is a substitution instance of $\forall\alpha\varphi$ and $\exists\alpha\varphi$ if β is a constant.

Universal Elimination and Existential Introduction

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

Task 1: Derive the argument above.

Universal: Everyone is either great or unfortunate $\forall x(Gx \vee Ux)$.

Existential: Tom is either great or unfortunate ($Gt \vee Ut$).

$\therefore$ $\exists x(Gx \vee Ux)$.	$\therefore$ $\exists y \exists x(Gy \vee Uy)$.
$\therefore$ $\exists x(Gx \vee Ut)$.	$\therefore$ $\exists y \exists x(Gx \vee Uy)$.
$\therefore$ $\exists x(Gt \vee Ut)$.	# $\exists x \exists x(Gx \vee Ux)$.

Universal Introduction

Generalising: It would seem that we cannot universally generalise from instances.

Invalid: The following argument is invalid and should not be derivable.

- Socrates is mortal. (Ms)

\# Everything is mortal. $(\forall x Mx)$

Valid: Compare the following valid argument which should be derivable:

- $\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$.
- $\forall x \neg Rxx$.

$\therefore$ $\forall x \forall y(Rxy \supset \neg Ryx)$.

Task 2: Use the rules we have to derive as much as we can.

1. $\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$
2. $\underline{\forall x \neg Rxx}$
3. $\forall y \forall z((Ray \wedge Ryz) \supset Raz)$:$\forall$E
4. $\forall z((Rab \wedge Rbz) \supset Raz)$:$\forall$E
5. $(Rab \wedge Rba) \supset Raa$:$\forall$E
6. $\neg Raa$:$\forall$E
7. $\mid Rab$:AS for $\supset$I
8. $\mid \mid Rba$:AS for $\neg$I
9. $\mid \mid Rab \wedge Rba$:$\wedge$I
10. $\mid \mid Raa$:$\supset$E
11. $\mid \neg Rba$:$\neg$I
12. $Rab \supset \neg Rba$:$\supset$I
13. $\forall y(Ray \supset \neg Rya)$:$\forall$I
14. $\forall x \forall y(Rxy \supset \neg Ryx)$:$\forall$I

Question 2: How are we going to introduce universal quantifiers without making the invalid argument above derivable?

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall \alpha \varphi$ where β is a constant, α is a variable, and β does not occur in $\forall \alpha \varphi$ or in any undischarged assumption.

Arbitrary: The constraints on ($\forall$E) require β to be arbitrary.

Review: Bad inference above is blocked.

In Premise: Anu loves every dog.
$\forall x(Dx \supset Lax) \vdash Da \supset Laa \nvdash \forall x(Dx \supset Lxx)$.

In Conclusion: All dogs love themselves.
$\forall x(Dx \supset Lxx) \vdash Da \supset Laa \nvdash \forall x(Dx \supset Lax)$.

Existential Elimination

Task 3: Compare the following invalid inference.

- Someone is mortal.

\# Zeus is mortal.

Question 3: How are we going to eliminate existential quantifiers without making the argument above derivable?

Example: Consider the following argument:

- Everyone who applied found a position $\forall x(Ax \supset \exists y Fxy)$.
- Someone applied $\exists x Ax$.

$\therefore$ Someone found a position $\exists x \exists y Fxy$.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Derivation: We can derive the example without deriving the invalid inference.

Quantifier Exchange Rules

($\neg\exists$) $\neg\exists\alpha\varphi \vdash \forall\alpha\neg\varphi$.

($\neg\forall$) $\neg\forall\alpha\varphi \vdash \exists\alpha\neg\varphi$.

($\forall\neg$) $\forall\alpha\neg\varphi \vdash \neg\exists\alpha\varphi$.

($\exists\neg$) $\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$.

Task 4: $\forall\alpha\neg\varphi \vdash \neg\exists\alpha\varphi$.

1. $\underline{\forall\alpha\neg\varphi}$
2. $\mid \underline{\exists\alpha\varphi}$
3. $\mid \mid \underline{\varphi[\beta/\alpha]}$
4. $\mid \mid \mid \underline{\exists\alpha\varphi}$
5. $\mid \mid \mid \varphi[\beta/\alpha]$
6. $\mid \mid \mid \neg\varphi[\beta/\alpha]$
7. $\mid \mid \neg\exists\alpha\varphi$
8. $\mid \neg\exists\alpha\varphi$
9. $\neg\exists\alpha\varphi$

Task 5: $\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$.

10. $\underline{\exists\alpha\neg\varphi}$
11. $\mid \underline{\forall\alpha\varphi}$
12. $\mid \mid \underline{\neg\varphi[\beta/\alpha]}$
13. $\mid \mid \mid \underline{\forall\alpha\varphi}$
14. $\mid \mid \mid \neg\varphi[\beta/\alpha]$
15. $\mid \mid \mid \varphi[\beta/\alpha]$
16. $\mid \mid \neg\forall\alpha\varphi$
17. $\mid \neg\forall\alpha\varphi$
18. $\neg\forall\alpha\varphi$

Task 6: Prove the rules below:

(MCP) If $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

($\forall$DN) $\forall\alpha\neg\neg\varphi \vdash \forall\alpha\varphi$.

($\exists$DN) $\exists\alpha\neg\neg\varphi \vdash \exists\alpha\varphi$.

Task 7: Use the rules above to derive ($\neg\exists$) and ($\neg\forall$).

Natural Deduction in QL$^=$

LOGIC I
Benjamin Brast-McKie
November 30, 2023

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Quantifier Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Identity Rules

(=I) $\vdash \alpha = \alpha$ for any constant α.

Axiom: This rule is better referred to as an axiom schema.

Note: Easy to use, but not always obvious when to use.

Task 1: Derive the following in QD:

- $\forall x(x = x \supset \exists y Fyx) \vdash \exists y(Fyy)$.
- Everything is something.
- Something exists.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Note: Also easy to use, but not always obvious how to use.

Task 2: Derive the following in QD:

- $m = n \vee n = o,\ An\ \vdash Am \vee Ao$
- Every symmetric antisymmetric relation is lonely.
- Every irreflexive antisymmetric relation is asymmetric.

Relations

Task 4: Regiment and derive the following in QD.

1. Every transitive symmetric relation is left and right euclidean.
2. Every nonempty transitive and symmetric relation is reflexive.
3. Only the empty relation is symmetric and asymmetric.
4. Every intransitive relation is irreflexive.
5. Every intransitive relation is asymmetric.

Further Examples

Task 3: Regiment and derive the following in QD.

1. $\forall x(x = m), Rma \vdash \exists x Rxx$
2. $\forall x(x{=}n \equiv Mx), \forall x(Ox \lor \neg Mx) \vdash On$
3. $\exists x(Kx \land \forall y(Kyx{=}y) \land Bx), Kd \vdash Bd$
4. $\vdash Pa \supset \forall x(Px \lor x{\neq}a)$

Existential Elimination and Soundness

LOGIC I
Benjamin Brast-McKie
November 29, 2023

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

QD Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

(=I) $\vdash \alpha = \alpha$ for any constant α.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Existential Elimination

Task 1: Regiment and derive the following in QD.

1. The elephant would not obey.
   Patrick is an elephant.
   ―――――
   Patrick would not obey.
2. $\forall x(Jx \supset Kx)$
   $\exists x\forall yLxy$
   $\forall xJx$
   ―――――
   $\exists x(Kx \wedge Lxx)$.
3. $\exists x(Px \supset \forall xQx)$
   ―――――
   $\forall xPx \supset \forall xQx$.
4. $\exists xPx \vee \exists xQx$
   ―――――
   $\exists x(Px \vee Qx)$.
5. Every nonempty asymmetric relation is non-symmetric.

Natural to Normative

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

1. Shows that we can trust QD to establish validity.
2. Easier to derive a conclusion that to provide a semantic argument.
3. The natural rules of deduction preserve validity.

Natural: QD describes (approximately) how we in fact reason.

Normative: Soundness explains why we ought to use QD to reason.

Soundness of QD

Assume: $\Gamma \vdash_{\text{QD}} \varphi$, so there is a QD proof X of φ from Γ.

Lines: Let φ_i be the ith line of X.

Dependencies: Let Γ_i be the undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

Base: $\Gamma_1 \vDash \varphi_i$.

Induction: If $\Gamma_k \vDash \varphi_k$ for all $k \leq n$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\Gamma_m = \Gamma$ and $\varphi_m = \varphi$, so $\Gamma \vDash \varphi$.

Base Case

Proof: Every line in a QD proof is either a premise or follows by the rules.

Assume: φ_1 is either a premise or follows by AS or =I.

Premise: If φ_1 is a premise or assumption, then $\Gamma_1 = \{\varphi_1\}$, and so $\Gamma_1 \vDash \varphi_1$.

Identity: If φ_1 follows by =I, then φ_1 is $\alpha = \alpha$ for some constant α.

- Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model, $\mathcal{I}(\alpha) = \mathcal{I}(\alpha)$.
- Letting a be a variable assignment, $\mathcal{V}^a_{\mathcal{I}}(\alpha) = \mathcal{V}^a_{\mathcal{I}}(\alpha)$.
- So $\mathcal{V}^a_{\mathcal{I}}(\alpha = \alpha) = 1$, and so $\vDash \alpha = \alpha$.
- Thus $\Gamma_1 \vDash \varphi_1$ since $\Gamma_1 = \varnothing$.

Induction Case

Assume: $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

Undischarged: If φ_{n+1} is a premise or assumption, then the argument above applies.

Rules: If φ_{n+1} follows from Γ_{n+1} by the QD rules, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Cases: There are 12 rules in SD and an additional 6 in QD.

Further Problems: Relations

Task 1: Regiment and derive the following in QD.

1. Every transitive and symmetric relation is quasi-reflexive.
2. Only the empty relation is symmetric and asymmetric.
3. Every intransitive relation is irreflexive.
4. Every intransitive relation is asymmetric.

Soundness: Part II

LOGIC I
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December 5, 2023

Soundness of QD

Assume: $\Gamma \vdash_{\mathrm{QD}} \varphi$, so there is a QD proof X of φ from Γ.

Lines: Let φ_i be the i^{th} line of X.

Dependencies: Let Γ_i be the undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

BASE: $\Gamma_1 \vDash \varphi_i$.

HYPOTHESIS: Assume $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

INDUCTION: If φ_{n+1} follows by the proof rules for QD from sentences in Γ_{n+1}, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\Gamma_m = \Gamma$ and $\varphi_m = \varphi$, so $\Gamma \vDash \varphi$.

SD Lemmas

L12.1 If $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vDash \varphi$.

L12.2 For any QD proof X, if φ_k is live at line n where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

L12.3 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

L12.4 If $\Gamma \cup \{\varphi\}$ is unsatisfiable, then $\Gamma \vDash \neg\varphi$.

L12.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$ if $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ.

L12.6 $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

L12.7 If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \supset \psi$.

SD Rules

(R) $\varphi_k = \varphi_{n+1}$ for live $k \leq n$. Thus $\Gamma_k \vDash \varphi_k$ by hypothesis and $\Gamma_k \subseteq \Gamma_{n+1}$ by **L12.2**. Thus $\Gamma_{n+1} \vDash \varphi_k$ by **L12.1**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\neg$I)
- There is a proof of ψ at line h and $\neg\psi$ at line j from φ on line i.
- By hypothesis $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$, where $\Gamma_h, \Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By **L12.1**, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$.
- So $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable by **L12.3**, so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by **L12.4**.

($\wedge$E) • $\varphi_{n+1} \wedge \psi$ is live on line $i \leq n$.
- By hypothesis, $\Gamma_i \vDash \varphi_{n+1} \wedge \psi$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1} \wedge \psi$ by **L12.1**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by semantics.

($\supset$I) • There is a proof of ψ at line j from φ on line i.
- By hypothesis $\Gamma_j \vDash \psi$, where $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi\}$.
- So $\Gamma_{n+1} \cup \{\varphi\} \vDash \psi$, and so $\Gamma_{n+1} \vDash \varphi \supset \psi$ by **L12.7**.

QD Lemmas

L12.8 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

Base: Assume φ is $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ or $\alpha_1 = \alpha_2$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.
- Let $\gamma_i = \beta$ if $\alpha_i = \alpha$ and otherwise $\gamma_i = \alpha_i$.
- $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_1) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_2)$.

Induction: If $\texttt{Comp}(\varphi) \leq n$, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ whenever $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

Case 2: Assume $\varphi = \psi \wedge \chi$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ for all $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\chi) = 1$ *iff* …

Case 6: Assume $\varphi = \forall\gamma\psi$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.
- If $\gamma = \alpha$, then $\varphi = \varphi[\beta/\alpha]$.
- If $\gamma \neq \alpha$, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\gamma\psi) = 1$ *iff* $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff*…
- Let $\hat{e}$ be an arbitrary γ-variant of $\hat{a}$.
- Since $\gamma \neq \alpha$, $\hat{e}(\alpha) = \hat{a}(\alpha)$ if α is a variable, so $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha)$.
- Thus $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ follows from the assumption.
- Since β is free for α in $\forall\gamma\psi$, we know that $\gamma \neq \beta$.
- If β is a variable, then $\hat{e}(\beta) = \hat{a}(\beta)$ since $\hat{e}$ is a γ-variant of $\hat{a}$.
- Thus $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\beta) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{e}}_{\mathcal{I}}(\beta)$.
- By hypothesis, $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi[\beta/\alpha])$, where $\hat{e}$ was arbitrary.
- …*iff* $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi[\beta/\alpha]) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha]) = 1$.

L12.9 If $\mathcal{M} = \langle \mathbb{D}, \mathcal{I}\rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}'\rangle$ where $\mathcal{I}$ and $\mathcal{I}'$ agree about every constant α and n-place predicate $\mathcal{F}^n$ that occurs in φ, it follows that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$.

Base: $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_n)\rangle \in \mathcal{I}'(\mathcal{F}^n)$.
- $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ is immediate from the assumption.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \mathcal{I}(\alpha_i) = \mathcal{I}'(\alpha_i) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a constant.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \hat{a}(\alpha_i) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a variable.

L12.10 For any constant β that does not occur in $\forall\alpha\varphi$ or in any sentence $\psi \in \Gamma$, if $\Gamma \vDash \varphi[\beta/\alpha]$, then $\Gamma \vDash \forall\alpha\varphi$.

1. Assume $\Gamma \vDash \varphi[\beta/\alpha]$ for constant β not in $\forall\alpha\varphi$ or Γ.
2. Assume $\Gamma \nvDash \forall\alpha\varphi$, and so $\mathcal{M}$ satisfies Γ but $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) \neq 1$.
3. So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) \neq 1$ for some α-variant $\hat{c}$ of $\hat{a}$.
4. Let $\mathcal{M}'$ by like $\mathcal{M}$ but for $\mathcal{I}'(\beta) = \hat{c}(\alpha)$.
5. By **L12.9**, $\mathcal{M}'$ satisfies Γ since β does not occur in Γ.
6. So $\mathcal{M}'$ satisfies $\varphi[\beta/\alpha]$ since $\Gamma \vDash \varphi[\beta/\alpha]$.
7. By **L12.6**, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$ for all $\hat{c}$, and so for $\hat{c}$ in particular.
8. Since β is not in $\forall\alpha\varphi$, we know β is not in φ.
9. So $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) \neq 1$ by **L.12.9** given (3) above.
10. By (4) above, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\alpha) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\beta)$ where β is free for α.
11. By **L12.8**, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha])$.
12. Thus $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) \neq 1$, contradicting the above.

L12.11 $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$ where α is a variable and $\varphi[\beta/\alpha]$ is a sentence.

- Let $\mathcal{M}$ satisfy $\forall\alpha\varphi$, so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ for some $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ where $\hat{c}(\alpha) = \mathcal{I}(\beta)$ for an α-variant $\hat{c}$ of $\hat{a}$.
- By **L12.8**, $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi[\beta/\alpha])$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$.

L12.12 If $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then $\Gamma \cup \Sigma \vDash \psi$.

L12.13 $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ where α is a variable and $\varphi[\beta/\alpha]$ is a sentence.

L12.14 For any constant β that does not occur in $\exists\alpha\varphi$, ψ, or in any sentence $\chi \in \Gamma$, if $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$, then $\Gamma \vDash \psi$.

L12.15 If α and β are constants, then $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$.

QD Rules

($\forall$I)
- $\varphi_i = \varphi[\beta/\alpha]$ for $i \leq n$ live at $n+1$ where β is not in φ_{n+1} or Γ_{n+1}.
- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L12.1**, so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$.
- So $\Gamma_{n+1} \vDash \forall\alpha\varphi$ by **L12.10** since β not in $\forall\alpha\varphi$ or Γ_{n+1}.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\forall$E)
- $\varphi_i = \forall\alpha\varphi$ for $i \leq n$ live at $n+1$ where $\varphi_{n+1} = \varphi[\beta/\alpha]$.
- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L12.1**, so $\Gamma_{n+1} \vDash \forall\alpha\varphi$.
- By **L12.11** $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$, and so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$ by **L12.12**.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Completeness of QD

LOGIC I
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Basic Lemmas

L13.1 If α is a constant and X is a proof in which the constant β does not occur, then $X[\beta/\alpha]$ is also a proof.

L13.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L13.5 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L13.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L13.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L13.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Satisfiability

T13.1 Every consistent set of $\text{QL}^=$ sentences Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

1. Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.
2. So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T13.1**.
3. So $\Gamma \vdash \neg\neg\varphi$ by **L13.3**, and so $\Gamma \vdash \varphi$ by DN and **L13.5**.

Saturation

Free: Let $\varphi(\alpha)$ be a wff of $\text{QL}^=$ with at most one free variable α.

Saturated: A set of sentences Σ is saturated in $\text{QL}^=_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\text{QL}^=_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \supset \varphi[\beta/\alpha]) \in \Sigma$.

Constants: Let $\mathbb{C}$ be the constants of $\text{QL}^=_{\mathbb{N}}$ where $\mathbb{N} \subseteq \mathbb{C}$ are new constants.

L13.2 Assuming Γ is consistent in $\text{QL}^=$, we know Γ is consistent in $\text{QL}^=_{\mathbb{N}}$.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\text{QL}^=_{\mathbb{N}}$ with one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \supset \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.

$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \supset \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in θ_j for any $j \leq k$.

Saturation: Let $\Sigma_1 = \Gamma$, $\Sigma_{n+1} = \Sigma_n \cup \{\theta_n\}$, and $\Sigma_\Gamma = \bigcup_{i \in \mathbb{N}} \Sigma_n$.

L13.4 Σ_Γ is consistent and saturated in $\text{QL}^=_{\mathbb{N}}$.

1. If Σ_{m+1} is inconsistent, then $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$.
2. So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I, and so $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$.
3. If Σ_Γ is inconsistent, then $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$.

Maximization

Maximal: A set of sentences Δ is maximal in $\mathrm{QL}^{=}_{\mathbb{N}}$ just in case as either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every sentence ψ in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all sentences in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Maximization: Let $\Delta_0 = \Sigma$, $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$, and $\Delta_\Sigma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

L13.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal consistent in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Case 1: $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

1. If $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent, then Δ_n is inconsistent by **L13.6**.
2. So Δ_{n+1} is consistent in both cases.
3. If Δ_Σ is inconsistent, then $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$.
4. Maximality is immediate.

L13.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

1. Immediate from the definitions.

L13.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

1. Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L13.9**.
2. So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
3. Thus $\varphi \in \Delta$ by maximality.

Henkin Model

Element: $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

Domain: $\mathbb{D}_\Delta = \{[\alpha]_\Delta : \alpha \in \mathbb{C}\}$.

L13.13 If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

1. Assuming $\alpha = \beta \in \Delta$ where $\Gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
2. So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L13.11**.
3. Thus $\beta = \gamma \in \Delta$ by **L13.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

Constants: $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

Predicates: $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}^n_\Delta : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

L13.14 If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

1. Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
2. $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L13.10**.
3. Parity of reasoning completes the proof.

Henkin Lemmas

L13.15 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

1. Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$ for some $\hat{a}$, $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some α-variant $\hat{c}$.
2. So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
3. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.9**.
4. So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L12.6**.
5. Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.
6. Let $\hat{c}$ be the α-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$.
7. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.9**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

L13.16 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

1. Similar to **L13.15**.

L13.17 $\mathcal{M}_\Delta$ satisfies φ just in case $\varphi \in \Delta$.

Base: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

1. If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L13.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
2. Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L13.13**.

Induction: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leq n$.

1. Let φ be a sentence of $\mathrm{QL}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

Case 2: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

Case 6: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L13.15**.

1. *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.
2. *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L13.10** given saturation.

Conclusion: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

Restriction

Restriction: $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant α in $\mathrm{QL}^{=}$.

L13.18 For all $\mathrm{QL}^{=}$ sentences φ, $\mathcal{M}'_\Delta$ satisfies φ just in case $\mathcal{M}_\Delta$ satisfies φ.

T13.1 Every consistent set of $\mathrm{QL}^{=}$ sentences Γ is satisfiable.

Compactness

C13.2 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

C13.3 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

Completeness of QD: Part II

LOGIC I
Benjamin Brast-McKie
December 12, 2023

Basic Lemmas

L13.1 If α is a constant and X is a proof in which the constant β does not occur, then $X[\beta/\alpha]$ is also a proof.

L13.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L13.5 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L13.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L13.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L13.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Satisfiability

T13.1 Every consistent set of $\text{QL}^=$ sentences Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

1. Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.
2. So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T13.1**.
3. So $\Gamma \vdash \neg\neg\varphi$ by **L13.3**, and so $\Gamma \vdash \varphi$ by DN and **L13.5**.

Saturation

Free: Let $\varphi(\alpha)$ be a wff of $\text{QL}^=$ with at most one free variable α.

Saturated: A set of sentences Σ is saturated in $\text{QL}^=_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\text{QL}^=_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \supset \varphi[\beta/\alpha]) \in \Sigma$.

Constants: Let $\mathbb{C}$ be the constants of $\text{QL}^=_{\mathbb{N}}$ where $\mathbb{N} \subseteq \mathbb{C}$ are new constants.

L13.2 Assuming Γ is consistent in $\text{QL}^=$, we know Γ is consistent in $\text{QL}^=_{\mathbb{N}}$.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\text{QL}^=_{\mathbb{N}}$ with one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \supset \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.

$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \supset \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in θ_j for any $j \leqslant k$.

Saturation: Let $\Sigma_1 = \Gamma$, $\Sigma_{n+1} = \Sigma_n \cup \{\theta_n\}$, and $\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

L13.4 Σ_Γ is consistent and saturated in $\text{QL}^=_{\mathbb{N}}$.

1. If Σ_{m+1} is inconsistent, then $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$.
2. So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I, and so $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$.
3. If Σ_Γ is inconsistent, then $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$.

Maximization

Maximal: A set of sentences Δ is maximal in $\mathrm{QL}^{=}_{\mathbb{N}}$ just in case as either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every sentence ψ in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all sentences in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Maximization: Let $\Delta_0 = \Sigma$, $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$, and $\Delta_\Sigma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

L13.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal consistent in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Case 1: $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

1. If $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent, then Δ_n is inconsistent by **L13.6**.
2. So Δ_{n+1} is consistent in both cases.
3. If Δ_Σ is inconsistent, then $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$.
4. Maximality is immediate.

L13.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

1. Immediate from the definitions.

L13.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

1. Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L13.9**.
2. So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
3. Thus $\varphi \in \Delta$ by maximality.

Henkin Model

Element: $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

Domain: $\mathbb{D}_\Delta = \{[\alpha]_\Delta : \alpha \in \mathbb{C}\}$.

L13.13 If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

1. Assuming $\alpha = \beta \in \Delta$ where $\gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
2. So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L13.11**.
3. Thus $\beta = \gamma \in \Delta$ by **L13.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

Constants: $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

Predicates: $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}^n_\Delta : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

L13.14 If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

1. Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
2. $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L13.10**.
3. Parity of reasoning completes the proof.

Henkin Lemmas

L13.15 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

1. Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$ for some $\hat{a}$, $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some α-variant $\hat{c}$.
2. So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
3. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.8**.
4. So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L12.6**.
5. Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.
6. Let $\hat{c}$ be the α-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$.
7. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.8**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

L13.16 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

1. Similar to **L13.15**.

L13.17 $\mathcal{M}_\Delta$ satisfies φ just in case $\varphi \in \Delta$.

Base: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

1. If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L13.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
2. Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L13.13**.

Induction: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leqslant n$.

1. Let φ be a sentence of $\mathrm{QL}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

Case 2: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

Case 6: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L13.15**.

1. *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.
2. *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L13.10** given saturation.

Conclusion: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

Restriction

Restriction: $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant α in $\mathrm{QL}^{=}$.

L13.18 For all $\mathrm{QL}^{=}$ sentences φ, $\mathcal{M}'_\Delta$ satisfies φ just in case $\mathcal{M}_\Delta$ satisfies φ.

T13.1 Every consistent set of $\mathrm{QL}^{=}$ sentences Γ is satisfiable.

Compactness

C13.2 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

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Final Exam Review

Regimentation: (a) No two individuals are at least as tall as each other. Sanna is at least as tall as the finalist, and the finalist is at least as tall as Sanna. Thus, Sanna is the finalist.

Models: (a) $Qab, Qba \not\models a = b$.

(b) $\forall x \forall y (Px \supset (Py \supset x \neq y)) \not\models \exists x \exists y\, x \neq y$.

Equivalence: $\exists x(\forall y(Py \supset x = y) \wedge Px) \dashv\models \exists x \forall y(Py \equiv x = y)$.

Relations: (a) R is symmetric and antisymmetric. Therefore R is reflexive.

(b) R is asymmetric. Therefore R is antisymmetric.