Logic I (24.241)

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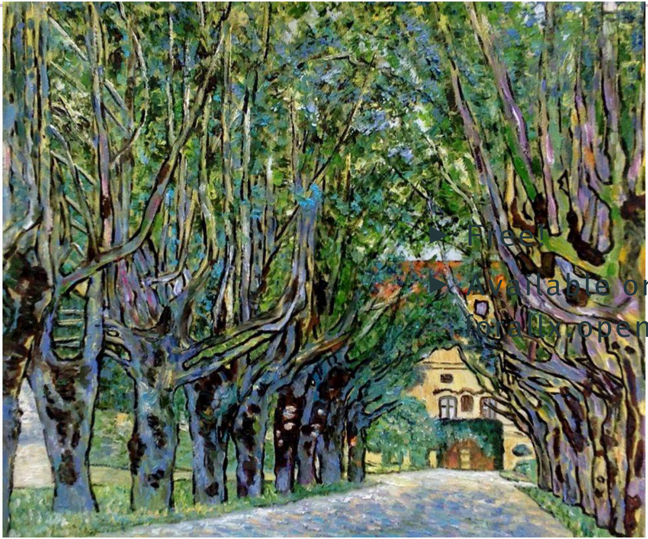
Massachusetts Institute of Technology

Course components

- Textbook
- Lecture videos on D2L
- Synchronous Zoom sessions
  - WF 12:00-12:50
- Tutorials
  - WF 9:00-10:50
- D2L website & discussion forums
- PASS Sessions

Textbook



- Free!
- Available on D2L, and at forallx.openlogicproject.org

Problem sets

- Practice the material.
- Mostly online using *Carnap* system.
- Collaboration ok.
- Should be done by Friday, will take till Monday midnight.
- Don't give away solutions.

Quizzes

- Online on D2L.
- Multiple choice, randomized questions.
- Open book, three attempts, untimed.
- Due Mondays at midnight.
- Test-like: must do them on your own.

Timed problems

- Mostly online using *Carnap* system.
- Open book, one attempt, timed.
- Due Mondays at midnight.
- Test–like: must do them on your own.

A typical week

Monday & Tuesday Watch videos, do reading reading.

Wednesday Tutorials in the morning, lecture at noon. Start working on the problem set.

Thursday Work on problem sets.

Friday Tutorials in the morning, lecture at noon. Finish problem set.

Saturday–Monday Complete the quiz and timed problem.

Grading philosophy

- We're using **proficiency-based evaluation**.
- There are twelve weeks of material.
- Each week corresponds to one learning goal.
- You show proficiency in a learning goal by completing three assessments (problem set, quiz, timed problem).
- Your grade depends on **how many** learning goals you complete.
- Ideally, you complete each goal during the week we cover it.
- But you get two chances to **re-do assessments**.

Completing assessments

- Each week covers one learning goal.
- Our aim is for you to achieve proficiency in these learning goals.
- Proficiency corresponds roughly to a B grade ("good performance").
- Scoring 80% or more on a problem set or quiz is "complete."
- Scoring 95% or more is "complete+" (A–level, excellent performance).
- Timed problems are just complete/not complete: to meet the bar for a B, you have to submit a correct solution.

Earning your final grade

- For a B (good performance), you must complete (score at the level of a B) 10/12 of each type of assessement (10 problem sets, 10 quizzes, 10 timed problems).
- For an A (excellent performance), you must complete **all** assessments and score complete+ on at least 10/12 quizzes and problem sets.
- For C, complete at least 8/12; and for a D, 6/12 of each assessment.
- For $+$ and $-$ grade criteria, see the outline.

Do-overs

- Multiple chances to show what you’ve learned.
- Immediate feedback on everything.
- Unlimited attempts on problem set questions.
- Three attempts on quizzes, no time limit.
- Only timed problems have time limits.
- Re-do timed problems or buy more attempts at quizzes using a token.

Tokens

Six tokens to spend on:

- One more shot at a timed problem.
- Two day extensions in one week.
- Completing problem sets after deadline.
- Three more attempts on a quiz.
- Up to two do-overs per assessment.

House rules

- Be civil and behave like adults: no sexist, racist, etc. jokes.
- Collaborate and study together, but turn in only your own independent work.
- Don’t give away answers.
- Don’t cheat on quizzes and Timed Problems.

Read the course outline!

- Official outline covers all policy questions.
- Outline is binding agreement and you are responsible for knowing policies.
- Available on D2L and Philosophy Department website.

I. What is logic?

I. What is logic?

a. Arguments and validity

An easy puzzle

Where does Sanjeev live?

Sanjeev lives in Calgary or in Edmonton.
Sanjeev doesn’t live in Edmonton.

A: Obviously, in Calgary.

Arguments and sentences

Argument 1

Sanjeev lives in Calgary or in Edmonton.
Sanjeev doesn't live in Edmonton.
Therefore, Sanjeev lives in Calgary.

- Such an argument consists of **sentences**.
- Individual sentences are the kinds that can be **true** or **false**.
- "Therefore" ($\therefore$) indicates that the last sentence (supposedly) **follows from** the first two.
- The last sentence is called the **conclusion**.
- The others are called the **premises**.

Valid and invalid arguments

Argument 2

Mandy enjoys skiing or hiking (or both).

Mandy doesn’t enjoy hiking.

∴ Mandy enjoys skiing.

Argument 3

Mandy enjoys skiing or hiking (or both).

Mandy enjoys skiing.

∴ Mandy doesn’t enjoy hiking.

What’s the difference?

Validity

Definition

An argument is **valid** if there is no case where all its premises are true and the conclusion is false.

Definition

An argument is **invalid** if there is at least one case where all its premises are true and the conclusion is false (i.e., if it is not valid).

Definition

A **case** is some hypothetical scenario that makes each sentence in an argument either true or false.

Argument 2 is valid

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Argument 2 is **valid**: whenever the premises are true, the conclusion is also true.

Argument 3 is not valid

Argument 3

Mandy enjoys skiing or hiking.
Mandy enjoys skiing.
$\therefore$ Mandy doesn't enjoy hiking.

Argument 3 is **invalid**: there is a possible case where the premises are true and the conclusion isn't (Mandy enjoys both skiing and hiking).

A harder puzzle

Where does Sarah live?

Sarah lives in Calgary or Edmonton.
Amir lives in Calgary unless he enjoys hiking.
If Amir lives in Calgary, Sarah doesn't.
Neither Sarah nor Amir enjoy hiking.

I. What is logic?

b. Cases and determining validity

Validity

Definition

An argument is **valid** if there is no case where all its premises are true and the conclusion is false.

Definition

An argument is **invalid** if there is at least one case where all its premises are true and the conclusion is false (i.e., if it is not valid).

Cases

Definition

A **case** is some hypothetical scenario that makes each sentence in an argument either true or false.

- E.g., imagine you have a friend, her name is Mandy, she loves hiking but hates skiing.
- That's a case where "Mandy enjoys hiking or skiing" is true.
- Some cases can be imagined even though they never happen IRL, e.g, "It is raining and the skies are clear."
- Some things you can't imagine, e.g., "There is a blizzard but there is no wind."

Determining validity

- Imagine a case where the conclusion is false.
- Are the premises true? You’re done: invalid.
- Otherwise, change or expand the case to make them true (without making the conclusion also true).
- Can’t? (Probably) valid.

OR

- Imagine a case where all premises are true.
- Is the conclusion false? You’re done: invalid.
- Otherwise, change or expand the case to make it false (without making the premises false).
- Can’t? (Probably) valid.

Valid?

Some rodents have bushy tails.
All squirrels are rodents.
∴ Some squirrels have bushy tails.

- Imagine squirrels evolving so that they no longer have bushy tails. Then conclusion is false.
- But premises still true:
  - Imagine chinchillas still have bushy tails.
  - Imagine also that squirrels have not evolved too much—they are still rodents.

Valid?

All rodents have bushy tails.
All squirrels are rodents.
∴ All squirrels have bushy tails.

- If it were invalid, you'd have a case that makes the conclusion false: some squirrels without bushy tails.
- They would have to be rodents still (otherwise premise 2 false).
- And that would require that they have bushy tails (otherwise premise 1 false).

I. What is logic?

c. Other logical notions

Logical Consistency

Definition

Sentences are (logically) **consistent** if there is a case where they are all true.

- also called ‘jointly possible’ or ‘satisfiable’

Definition

Sentences are (logically) **inconsistent** if there is no case where they are all true.

- also called ‘jointly impossible’ or ‘unsatisfiable’

Consistent?

Some carnivores have bushy tails.
All carnivores are mammals.
No mammals have bushy tails.

- No case makes them all true at the same time, so **jointly impossible**.

Valid?

Some carnivores have bushy tails.
All carnivores are mammals.
No mammals have bushy tails.
∴ All birds are carnivores.

- The premises cannot all be true in the same case, so jointly impossible.
- So: no case makes all the premises true.
- So also: no case makes the premises true and the conclusion false.
- **Arguments with jointly impossible premises are automatically valid**, regardless of what the conclusion is.

Tautology (logically necessary truth)

Definition

A sentence is a **necessary truth** if there is no case where it is false.
also called a 'necessary truth' or 'truth–functionally true'

- If it's snowing, it's snowing.
- Every fawn is a deer.
- The number 5 is prime.
- Physical objects are extended.

Tautology

What can you say about an argument where the conclusion is a necessary truth?

- If the conclusion is a necessary truth, there is no case where it is false.
- So there is no case where it is false, and the premises of the argument are true.
- **Arguments with necessary truths as conclusions are automatically valid**, regardless of what the premises are.

Necessary equivalence

Definition

Two sentences are **equivalent** if there is no case where one is true and the other is false.

- What can you say about an argument where one of the premises is equivalent to the conclusion? Is it automatically valid?
- Can you have two equivalent sentences that are jointly impossible?

I. What is logic?

d. Symbolization and TFL

Validity in virtue of form

Argument 1

Sanjeev lives in Calgary or Edmonton.

Sanjeev doesn't live in Edmonton.

$\therefore$ Sanjeev lives in Calgary.

Argument 2

Mandy enjoys skiing or hiking.

Mandy doesn't enjoy hiking.

$\therefore$ Mandy enjoys skiing.

Form of arguments 1 & 2

X or Y.

Not Y.

$\therefore X$.

Some valid argument forms

Disjunctive syllogism

X or Y.

Not Y.

$\therefore\ X$.

Modus ponens

If X then Y.

X.

$\therefore\ Y$.

Hypothetical syllogism

If X then Y.

If Y then Z.

$\therefore$ If X then Z.

Symbolizing arguments

Symbolization key

S: Mandy enjoys skiing

H: Mandy enjoys hiking

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.	$(S \vee H)$
Not: Mandy enjoy hiking.	$\neg H$
∴ Mandy enjoys skiing.	$\therefore S$

The language of TFL

- **Sentence letters**, such as 'H' and 'S', to symbolize basic sentences ('Mandy likes hiking')
- **Connectives**, to indicate how basic sentences are connected

  $\vee$ either ... or ...
  $\wedge$ both ... and ...
  $\rightarrow$ if ... then ...
  $\neg$ not ...

  This can get complicated, e.g.:
  "Mandy enjoys skiing or hiking, and if she lives in Edmonton, she doesn't enjoy both."

$$((S \vee H) \wedge (E \rightarrow \neg(S \wedge H)))$$

I. What is logic?

e. What are we going to learn, and why?

What is logic?

- **Logic is the science of what follows from what.**
- Sometimes a conclusion follows from the premises, sometimes it does not:
  - Mandy lives in Calgary.
    Everyone who lives in Calgary likes hiking.
    ∴ Mandy likes hiking.
  - Mandy lives in Calgary.
    Everyone who likes hiking lives in Calgary.
    ∴ Mandy likes hiking.
- Logic investigates what makes the first argument **valid** and the second **invalid**.

What is formal logic?

- Studies logical properties of **formal languages** (TFL and FOL, not English).
  - Logical consequence (what follows from what?)
  - Logical consistency (when do sentences contradict one another?)
- Expressive power (what can be expressed in a given formal language, and how?)
- Formal models (mathematical structures described by formal language)
- Inference and proof systems (how can it be proved that something follows from something else?)
- (Meta-logical properties of logical systems)

Plan for the course

- Truth-functional logic (TFL)
  - Symbolization in the formal language of TFL ($H, \vee, \wedge, \rightarrow, \neg$)
  - Testing for validity: truth-tables
  - Proofs in natural deduction
- First-order logic (FOL)
  - More fine-grained symbolization ($E(m, h)$, $\forall$ 'every', $\exists$ 'some', $=$)
  - Semantics: interpretations
  - Proofs in natural deduction
- Some advanced topics: expressive completeness, normal forms

What is logic good for? (Philosophy)

- Logic originates in philosophy (Aristotle), traditionally considered a sub-discipline of philosophy.
- Valid arguments are critical in philosophical research.
- Formal tools of logic are useful to make intuitive philosophical notions precise, e.g.,
  - Possibility and necessity
  - Time
  - Composition and parthood (mereology)
  - Moral obligation and permissibility
  - Belief and knowledge
- Logic applies to semantics of natural language (philosophy of language, linguistics).

What is logic good for? (Mathematics)

- Formal logic was developed in the quest for a foundations of mathematics (19th C.).
- Logical systems provide precise foundational framework for mathematics:
  - Axiomatic systems (e.g, geometry)
  - Algebraic structures (e.g., groups)
  - Set theory (e.g, Zermelo–Fraenkel with Choice)
- Precision
  - Formal language makes claims more precise.
  - Formal structures can point to alternatives, unveil gaps in proofs.
  - Formal proof systems make proofs rigorous.
  - Formal proofs make mechanical **proof checking** and **proof search** possible.

What is logic good for? (Computer Science)

- Computer science deals with lots of formal languages.
- Logic is a good example of how to set up and use formal languages.
- Logic : Computer Science = Calculus : Natural Science
- Applications of logical systems in CS are numerous:
  - Combinational logic circuits
  - Database query languages
  - Logic programming
  - Knowledge representation
  - Automated reasoning
  - Formal specification and verification (of programs, of hardware designs)
  - Theoretical computer science (theory of computational complexity, semantics of programming languages)

II. Symbolization in TFL

II. Symbolization in TFL

a. Symbolization keys and paraphrase

Symbolizing arguments

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.

Not: Mandy enjoy hiking.

$\therefore$ Mandy enjoys skiing.

Form of argument 2

S or H.

Not H.

$\therefore S$.

Symbolization of argument 2 in TFL

$(S \vee H)$

$\neg H$

$\therefore S$

Symbolization keys

Definition

A symbolization key is a list that pairs **sentence letters** with the basic English sentences they represent.

For instance:

Symbolization key

S: Mandy enjoys skiing
H: Mandy enjoys hiking

Symbolization keys

- Sentence letters are uppercase letters, possibly with subscripts (e.g., H_1, H_2).
- Usually the symbolization key is given to you.
- It should not be possible to break down the "basic sentences" represented by sentence letters.

  For instance:

  A: Mandy enjoys skiing or hiking

  is bad.

Paraphrase

- Successful symbolization sometimes requires **paraphrase** to ensure basic sentences appear explicitly.
- Two things to watch for: pronouns and coordination.
- Pronouns stand in for singular terms (e.g., names): replace pronouns by those.
- "or", "both ... and", "neither ... nor" can connect sentences but also noun phrases and verb phrases: paraphrase those so they connect sentences.

Pronouns

Example

If Mandy enjoys hiking, **she** also enjoys skiing.

Replace “she” by “Mandy”:
If [Mandy enjoys hiking] then [Mandy enjoys skiing].

Coordination of noun phrases

Example

Mandy and Sanjeev enjoy hiking.

Both [Mandy enjoys hiking] and [Sanjeev enjoys hiking].

Example

Sanjeev lives in **Edmonton or Calgary**.

Either [Sanjeev lives in Edmonton] or [Sanjeev lives in Calgary].

Exercise caution!

Good

Mandy and Sanjeev ate pizza.

Both [Mandy ate pizza] and [Sanjeev ate pizza].

Bad

Mandy and Sanjeev ate the whole pizza.

Both [Mandy ate the whole pizza] and [Sanjeev ate the whole pizza].

Coordination of verb phrases

Example

Mandy enjoys **skiing or hiking**.

Either [Mandy enjoys skiing] or [Mandy enjoys hiking].

Example

If Sanjeev enjoys **skiing and hiking**, he lives in Calgary.

If [Sanjeev enjoys skiing] and [Sanjeev enjoys hiking], then [Sanjeev lives in Calgary].

II. Symbolization in TFL

b. Basic symbolization

Negation

- **Paraphrase** grammatical negation (“is not”, “does not”) using the corresponding basic sentence prefixed by “**it is not the case that**.”
- **Symbolize** “it is not the case that A” as $\neg A$.

Example

Mandy **doesn’t** enjoy skiing.
It is not the case that [**Mandy enjoys skiing**].
It is not the case that S.
$\neg S$

Conjunction

- **Paraphrase** sentences connected by “and”, “but”, “even though”, “yet”, and “although” using “**both A and B**”
- **Symbolize** “both A and B” as $(A \wedge B)$.

Example

Even though Mandy lives in Edmonton**,** she enjoys hiking.
Both [**Mandy lives in Edmonton**] and [**Mandy enjoys hiking**].
Both E and H.
$(E \wedge H)$

Disjunction

- **Paraphrase** sentences connected by "or" using "**either A or B**"
- **Symbolize** "either A or B" as $(A \lor B)$.

Example

Sanjeev lives in Calgary **or** Edmonton.
Either [**Sanjeev lives in Calgary**] or [**Sanjeev lives in Edmonton**].
Either C or E.
$(C \lor E)$

Ignore the suggestion that "either . . . or . . . " is exclusive. We'll always treat it as inclusive unless explicitly stated.

Conditional

- **Paraphrase** using "**if** A **then** B" any sentence of the form:
  - "if A, B"
  - "B if A" (note order is reversed!)
  - "B provided A"
- **Symbolize** "if A then B" as $(A \to B)$.

Example

- Mandy enjoys hiking **if** Sanjeev lives in Calgary.
- If [**Sanjeev lives in Calgary**] then [**Mandy enjoys hiking**].
- **If** C **then** H.
- $(C \to H)$

The parts of a conditional

- $(A \to B)$ symbolizes:
  - "if A, B"
  - "B if A" (note order is reversed!)
  - "B provided A"
- A is the **antecedent**: it symbolizes the condition that has to be met for the "then" part to apply.
- B is the **consequent**: it symbolizes what must be true if the antecedent condition is true.

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

If [Sanjeev lives in Calgary **or** Edmonton] then [Mandy **doesn't** enjoy hiking].

If [either [**Sanjeev lives in Calgary**] or [**Sanjeev lives in Edmonton**]] then [it is not the case that [**Mandy enjoys hiking**]].

If [either C or E] then [it is not the case that H].

$((C \vee E) \rightarrow \neg H)$

II. Symbolization in TFL

c. Conditionals

A logic puzzle

- Every card has a letter on one side and a number on the other side.
- You're a card inspector tasked with making sure that cards satisfy this quality standard:

  If a card has an even number on one side, then it has a vowel on the other.

A logic puzzle

Which card(s) do you have to turn over to make sure that:

> *If a card has an even number on one side, then it has a vowel on the other.*

E	K	3	4
(1)	(2)	(3)	(4)

Another logic puzzle

- At an all-ages event where everyone has a drink
- You know how old some of the people are, and you can tell what some of them are drinking
- You're tasked with making sure that the following rule is followed:

  If a person is drinking alcohol, then they are at least 18 years old.

Another logic puzzle

Which of these people do you have to check (age or drink) to ensure that:

> *If a person is drinking alcohol, then they must be at least 18 years old.*

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If *A*, then *B*" can only be **false** if:
  - *A* is **true**: we check age if X is drinking beer (*A* true), not if drinking pop; **and**
  - *B* is **false**: we check drink if X underage (*B* false), not if over 18 (*B* true)
- "If *A*, then *B*" is true if:
  - *A* is **false** (we don't check people drinking pop); **or**
  - *B* is **true** (we don't card if X is over 18);
  - (or both)

II. Symbolization in TFL

d. “Only if” and “unless”

'If' and 'only if'

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \rightarrow B$$

- False if Sue is underage, but drinks beer.
- Sue drinks beer (A) if she is over 18 (B).

$$B \rightarrow A$$

- False if she's 25 but drinks pop.
- Not false if she's 16 and drinking beer.

Conditional

- **Paraphrase** "A only if B" as "**if** A **then** B".
- **Symbolize** "A only if B" as $(A \to B)$.
- Note:
  - "A if B" is $(B \to A)$
  - "A only if B" is $(A \to B)$
- **Symbolize** "A if and only if B" as $(A \leftrightarrow B)$.

Unless

Which of these people do you have to check (age or drink) to ensure that:

People are drinking pop unless they are over 18.

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

- "*A* unless *B*" can only be **false** if:
  - *A* is **false**
    (we check age if person is drinking beer), **and**
  - *B* is **false**
    (we check drink if person not at least 18)
- "*A* unless *B*" is true (test OK) if *A* or *B* or both are true.
- "*A* unless *B*" can be paraphrased and symbolized by:
  - "*A* if not *B*" ($\neg B \to A$)
  - "either *A* or *B*" ($A \vee B$)

Unless

Treat "**unless**" the same way you would treat "or"

Example

Mandy enjoys hiking unless Sanjeev lives in Calgary.

$(H \vee C)$

II. Symbolization in TFL

e. More connectives

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

Both [if S then H] and [if H then S].

$((S \to H) \wedge (H \to S))$

$(H \leftrightarrow S)$

Exclusive or

Paraphrase sentences containing "either A or B but not both" using
"**both [either A or B] and**
[it is not the case that [both A and B]]"

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

$((H \lor S) \land \neg(H \land S))$

Neither . . . nor . . .

Paraphrase sentences containing "neither A nor B" using
"**both [it is not the case that A] and**
[it is not the case that B]"

Example

Mandy enjoys neither hiking nor skiing.

Both [it is not the case that H] and
[it is not the case that S].

$(\neg H \wedge \neg S)$

Mix & match

Example

Sarah lives in Calgary or Edmonton.

Either [Sarah lives in Calgary] or [Sarah lives in Edmonton].

Amir lives in Calgary unless he enjoys hiking.

Either [Amir lives in Calgary] or [Amir enjoys hiking].

If Amir lives in Calgary, Sarah doesn't.

If [Amir lives in Calgary] then [it is not the case that [Sarah lives in Calgary]].

Neither Sarah nor Amir enjoy hiking.

Both [it is not the case that [Sarah enjoys hiking]] and [it is not the case that [Amir enjoys hiking]].

$\therefore$ Sarah lives in Edmonton.

Mix & match

Example

Sarah lives in Calgary or Edmonton.

$(C \lor E)$

Amir lives in Calgary unless he enjoys hiking.

$(A \lor M)$

If Amir lives in Calgary, Sarah doesn't.

$(A \to \neg C)$

Neither Sarah nor Amir enjoy hiking.

$(\neg S \land \neg M)$

$\therefore$ Sarah lives in Edmonton.

$\therefore\ E$.

II. Symbolization in TFL

f. Ambiguity

Types of ambiguity

- Lexical ambiguity: one word—many meanings
  e.g., “bank”, “crane”
- Syntactic ambiguity: one sentence—many readings
  e.g.,
  - “Flying planes can be dangerous” (Chomsky)
  - “One morning I shot an elephant in my pajamas.
    How he got in my pajamas, I don’t know.” (Groucho Marx)

The man who was hanged by a comma

- Sir Roger Casement (1864–1916)
- British consul to Congo and Peru
- Tried to recruit Irish revolutionaries in Germany during WWI
- Tried for treason

Treason Act of 1351

ITEM, Whereas divers Opinions have been before this Time in what Case Treason shall be said, and in what not; the King, at the Request of the Lords and of the Commons, hath made a Declaration in the Manner as hereafter followeth, that is to say; When a Man doth compass or imagine the Death of our Lord the King, or of our Lady his Queen or of their eldest Son and Heir; or if a Man do violate the King's Companion, or the King's eldest Daughter unmarried, or the Wife of the King's eldest Son and Heir; or **if a Man** do levy War against our Lord the King in his Realm, or **be adherent to the King's Enemies in his Realm, giving to them Aid and Comfort in the Realm, or elsewhere**, and thereof be probably attainted of open Deed by the People of their Condition: . . . And it is to be understood, that in the Cases above rehearsed, that ought to be judged Treason which extends to our Lord the King, and his Royal Majesty: . . .

R v. Casement in FOL

- Symbolization key:

  A: Casement was adherent to the King's enemies in the realm.

  G: Casement gave aid and comfort to the King's enemies in the realm.

  B: Casement was adherent to the King's enemies abroad.

  H: Casement gave aid and comfort to the King's enemies abroad.

- Not treason:

  $A \lor (G \lor H)$

- Treason:

  $(A \lor B) \lor (G \lor H)$

Ambiguity of $\land$ and $\lor$

- English sentences don't have parentheses.
- This can lead to ambiguity, e.g.,

  Ahmed admires Brit and Cara or Dina.
- It might mean one of:

  Ahmed admires either [both Brit and Cara] or Dina.
  Ahmed admires both Brit and [either Cara or Dina].
- In TFL, symbolizations are unambiguous:

  $((B \land C) \lor D)$
  $(B \land (C \lor D))$

III. TFL and truth tables

III. TFL and truth tables

a. Characteristic truth tables

Sentence letters and connectives

- Symbolization involves **sentence letters** like H and **connectives** ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$)
- Recall that a **case** makes basic sentences **true** or **false** (and never both).
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

When is $(H \wedge S)$ true?

$(H \wedge S)$ is true if and only if H is true and S is also true.
Suppose a case makes H true and S false.
In that case, $(H \wedge S)$ would be **false**.

Negation $\neg$

Definition

$\neg\mathscr{A}$ is true iff $\mathscr{A}$ is false.

Characteristic truth table:

$\mathscr{A}$	$\neg\mathscr{A}$
T	F
F	T

Conjunction $\wedge$

Definition

$(\mathscr{A} \wedge \mathscr{B})$ is true iff $\mathscr{A}$ is true and $\mathscr{B}$ is true, and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \wedge \mathscr{B})$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction $\vee$

Definition

$(\mathscr{A} \vee \mathscr{B})$ is true iff $\mathscr{A}$ is true or $\mathscr{B}$ is true (or both), and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \vee \mathscr{B})$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A logic puzzle

Which card(s) do you have to turn over to make sure that:

If a card has an even number on one side, then it has a vowel on the other.

E	K	3	4
(1)	(2)	(3)	(4)

The material conditional $\to$

Definition

$(\mathscr{A} \to \mathscr{B})$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true (or both), and false otherwise.

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \to \mathscr{B})$
T	T	T
T	F	F
F	T	T
F	F	T

The material biconditional $\leftrightarrow$

Definition

$(\mathscr{A} \leftrightarrow \mathscr{B})$ is true iff $\mathscr{A}$ and $\mathscr{B}$ have the same truth value, and false otherwise.

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \leftrightarrow \mathscr{B})$
T	T	T
T	F	F
F	T	F
F	F	T

III. TFL and truth tables

b. Sentences of TFL

Sentences of TFL

Definition

1. Every sentence letter is a sentence.
2. If $\mathscr{A}$ is a sentence, then $\neg\mathscr{A}$ is a sentence.
3. If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then
   - $(\mathscr{A} \wedge \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \vee \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \rightarrow \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a sentence.
4. Nothing else is a sentence.

The indicated connective is called the **main connective**.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \wedge S)$ is a sentence.
- $\neg(H \wedge S)$ is a sentence.
- $((H \vee S) \wedge \neg(H \wedge S))$ is a sentence.

(Main connective is **highlighted**.)

Examples of non-sentences

- $HikesMandy$

  single sentence letters
- $(H \neg S)$

  $\neg$ can't go between sentences
- $(H \wedge S \wedge C)$

  $\wedge$ combines only two sentences
- $(\neg H)$

  no parentheses around $\neg H$
- $(H \to (S \wedge C)$

  missing closing parenthesis
- $H \vee S$

  missing parentheses
- $[H \to (S \wedge C)]$

  only one kind of parentheses

III. TFL and truth tables

c. Valuations

Valuations

Definition

A **valuation** is an assignment of **T** or **F** to each sentence letter in a sentence or sentences.

Definition

The **truth value of a sentence** $\mathscr{S}$ on a valuation is:

1. if $\mathscr{S}$ is a sentence letter: the truth value assigned to it
2. if $\mathscr{S}$ is $\neg\mathscr{A}$: opposite of the truth value of $\mathscr{A}$
3. if $\mathscr{S}$ is $(\mathscr{A} * \mathscr{B})$: result of characteristic truth table of $*$ for truth values of $\mathscr{A}$ and $\mathscr{B}$.

Computing truth values

Valuation: H is **T**, S is **F**.

On this valuation:

- H is **T**.
- S is **F**.
- $(H \vee S)$ is **T** (because **T** $\vee$ **F** gives **T**).
- $(H \wedge S)$ is **F** (because **T** $\wedge$ **F** gives **F**).
- $\neg(H \wedge S)$ is **T** (because $\neg$**F** is **T**).
- $((H \vee S) \wedge \neg(H \wedge S))$ is **T** (because **T** $\wedge$ **T** gives **T**).

Computing truth values

H	S	$((H$	$\lor$	$S)$	$\land$	$\neg$	$(H$	$\land$	$S))$
T	**F**	**T**	**T**	**F**	**T**	**T**	**T**	**F**	**F**
					$\uparrow$				

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.
- Use computed values for larger parts.
- Done when you have the value under the main connective.

III. TFL and truth tables

d. Validity and truth tables

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.
- A **case** must make every **basic sentence** true or false (and not both).
- In TFL, **valuations** make every **sentence letter** true or false (and not both).
- Also: every valuation makes every **sentence** true or false (and not both), and we can compute the truth value.

Validity in TFL

Definition

An argument is **valid in TFL** if there is **no** valuation in which all premises are **T** and the conclusion is **F**.

An argument is **invalid in TFL** if there is **at least one** valuation in which all premises are **T** and the conclusion is **F**.

Disjunctive syllogism

$$H \vee S$$
$$\neg S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\neg$	S	H	
T	**T**	**T**	**T**	**T**	**F**	**T**	**T**	✓
T	**F**	**T**	**T**	**F**	**T**	**F**	**T**	✓
F	**T**	**F**	**T**	**T**	**F**	**T**	**F**	✓
F	**F**	**F**	**F**	**F**	**T**	**F**	**F**	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

An invalid argument

$$\begin{array}{l} H \vee S \\ H \\ \therefore \neg S \end{array}$$

H	S	$(H$	$\vee$	$S)$	H	$\neg$	S	
T	**T**	**T**	**T**	**T**	**T**	**F**	**T**	✗
T	**F**	**T**	**T**	**F**	**T**	**T**	**F**	✓
F	**T**	**F**	**T**	**T**	**F**	**F**	**T**	✓
F	**F**	**F**	**F**	**F**	**F**	**T**	**F**	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

III. TFL and truth tables

e. Large truth tables

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations
  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.
  - For two letters A, B: B can be **T** or **F** for every possible valuation (2) of A: $2 \times 2 = 2^2 = 4$ valuations
  - For three letters A, B, C: C can be **T** or **F** for every possible valuation (4) of A and B: $2 \times 4 = 2^3 = 8$ valuations
  - Etc.
- In the ith reference column, alternate **T** and **F** every 2^{n-i} lines

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	…
1	**T**	**T**	**T**	…
2	**T**	**T**	**F**	…
3	**T**	**F**	**T**	…
4	**T**	**F**	**F**	…
5	**F**	**T**	**T**	…
6	**F**	**T**	**F**	…
7	**F**	**F**	**T**	…
8	**F**	**F**	**F**	…

↑ ↑ ↑

alternate every …

4 2 1

rows

Example (simplified)

Sarah lives in Calgary or Edmonton.
Amir lives in Calgary unless he enjoys hiking.
If Amir lives in Calgary, Sarah doesn't.
Amir doesn't enjoy hiking.
$\therefore$ Sarah lives in Edmonton.

$$C \vee E$$

$$A \vee M$$

$$A \to \neg C$$

$$\neg M$$

$$\therefore E$$

A	C	E	M	$C \vee E$	$A \vee M$	$A \to \neg C$	$\neg M$	E
T	T	T	T	T T T	T T T	T F F T	F T	T
T	T	T	F	T T T	T T F	T F F T	T F	T
T	T	F	T	T T F	T T T	T F F T	F T	F
T	T	F	F	T T F	T T F	T F F T	T F	F
T	F	T	T	F T T	T T T	T T T F	F T	T
T	F	T	F	F T T	T T F	T T T F	T F	T
T	F	F	T	F F F	T T T	T T T F	F T	F
T	F	F	F	F F F	T T F	T T T F	T F	F
F	T	T	T	T T T	F T T	F T F T	F T	T
F	T	T	F	T T T	F F F	F T F T	T F	T
F	T	F	T	T T F	F T T	F T F T	F T	F
F	T	F	F	T T F	F F F	F T F T	T F	F
F	F	T	T	F T T	F T T	F T T F	F T	T
F	F	T	F	F T T	F F F	F T T F	T F	T
F	F	F	T	F F F	F T T	F T T F	F T	F
F	F	F	F	F F F	F F F	F T T F	T F	F

Every valuation makes at least one premise false, or makes the conclusion true: III.e.4
the argument is valid.

III. TFL and truth tables

f. Entailment, equivalence, tautologies

Validity of arguments

Definition

An argument is **valid in TFL** iff every valuation either makes one or more of the premises false or it makes the conclusion true.

An argument is **invalid in TFL** iff at least one valuation makes all the premises true and it makes the conclusion false.

Entailment

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ **entail** a sentence $\mathscr{B}$ iff every valuation either makes at least one of $\mathscr{A}_1, \ldots, \mathscr{A}_n$ false or makes $\mathscr{B}$ true.

In that case we write $\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B}$.

We have:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B} \text{ iff the argument } \mathscr{A}_1, \ldots, \mathscr{A}_n \therefore \mathscr{B} \text{ is valid.}$$

Entailment

Does $\neg(\neg A \vee \neg B), A \to \neg C \vDash A \to (B \to C)$?

Entailment

A	B	C	$\neg$	$(\neg$	A	$\vee$	$\neg$	$B)$	A	$\rightarrow$	$\neg$	C	A	$\rightarrow$	$(B$	$\rightarrow$	$C)$	
T	T	T	T	F	T	F	F	T	T	F	F	T	T	T	T	T	T	
T	T	F	T T	F	T	F	F	T	T	T T	T	F	T	F F	T	F	F	$\leftarrow$
T	F	T	F	F	T	T	T	F	T	F	F	T	T	T	F	T	T	
T	F	F	F	F	T	T	T	F	T	T	T	F	T	T	F	T	F	
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	T	T	T	
F	T	F	F	T	F	T	F	T	F	T	T	F	F	T	T	F	F	
F	F	T	F	T	F	T	T	F	F	T	F	T	F	T	F	T	T	
F	F	F	F	T	F	T	T	F	F	T	T	F	F	T	F	T	F	

Equivalent sentences

Definition

Two sentences $\mathscr{A}$ and $\mathscr{B}$ are **equivalent in TFL** iff every valuation either makes both $\mathscr{A}$ and $\mathscr{B}$ true or it makes both $\mathscr{A}$ and $\mathscr{B}$ false.

In other words: $\mathscr{A}$ and $\mathscr{B}$ agree in truth value, for every valuation.

Equivalent sentences

A	B	$\neg$	A	$\vee$	B	A	$\rightarrow$	B
T	**T**	**F**	**T**	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**T**	**F**	**F**	**T**	**F**

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.
- If $\mathscr{A}$ is true, $\mathscr{B}$ is also true (since $\mathscr{A}$ and $\mathscr{B}$ agree in truth value on every valuation).
- So if $\mathscr{A}$ is true, the valuation is also not a counterexample.
- So, no valuation can be a counterexample to $\mathscr{A} \vDash \mathscr{B}$.

Tautologies

Definition

A sentence $\mathscr{A}$ is a **tautology** iff it is true on every valuation.

A	A	$\to$	A
T	T	T	T
F	F	T	F

III. TFL and truth tables

g. Joint satisfiability

Joint satisfiability

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are **jointly satisfiable** in TFL if there is at least one valuation that makes all of them true.

If they are not satisfiable, we say they are **jointly unsatisfiable**.

$A \lor B$, $\neg A$, B are jointly satisfiable.

$A \lor B$, $\neg A$, $\neg B$ are unsatisfiable.

Unsatisfiability and validity

- Any argument with jointly unsatisfiable premises is valid.
  - If premises are jointly unsatisfiable, no valuation makes them all true.
  - No valuation makes them all true and the conclusion false.
  - No valuation can be a counterexample.
- An argument is valid if, and only if, the premises together with the negation of the conclusion are jointly unsatisfiable.

LSAT puzzle

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

$$A \to C$$

Ching only goes if at least one of Betty and Dana goes too.

$$C \to (B \vee D)$$

Amir and Dana can't be in the boat together.

$$\neg(A \wedge D)$$

$$A \to \neg D$$

$$\neg A \vee \neg D$$

Dependency resolution by SAT checking

A, B, C, D: package A, B, C, D is installed.

Package A depends on package C.

$$A \to C$$

Package C requires either package B or D.

$$C \to (B \vee D)$$

Package A is incompatible with package D.

$$\neg(A \wedge D)$$

$$A \to \neg D$$

$$\neg A \vee \neg D$$

Solution as satisfiability question

Can you send **Amir** in the boat?

Can package **A** be installed?

Same as: Are these sentences jointly satisfiable?

$$\mathbf{A}$$

$$A \to C$$

$$C \to (B \vee D)$$

$$\neg(A \wedge D)$$

More complex satisfiability questions

Can you send **Amir without Betty** in the boat?

Can package **A** be installed **without installing B**?

Same as: Are these sentences jointly satisfiable?

$$A \wedge \neg B$$

$$A \rightarrow C$$

$$C \rightarrow (B \vee D)$$

$$\neg(A \wedge D)$$

(Exercise: construct a complete truth table. Which valuations, if any, satisfy all four sentences?)

Complexity of logical testing

- Testing for validity, satisfiability, tautology, etc., requires making a complete truth table
  - Testing for validity requires checking **every** valuation.
  - Testing for satisfiability requires finding **at least one** valuation.
- If there are n sentence letters, there are 2^n valuations to check.
- Computer scientists have yet to find a method that can (always) do this faster than truth tables ("P vs NP problem").

IV. Proofs in TFL

IV. Proofs in TFL

a. Why proofs?

Showing that an argument is valid

- Construct a truth table; verify there is no valuation where premises are true and conclusion is false.
- Truth tables can get very large very quickly.
- E.g., the example argument

$$C \lor E, A \lor M, A \to \neg C, \neg S \land \neg M \therefore E$$

  requires 32 lines and 608 individual truth values.
- Is there a better way?

Proofs

- Idea: work our way from premises to conclusion using steps we know are entailed by the premises.
- For instance:
  - From "Neither Sarah nor Amir enjoys hiking" we can conclude "Amir doesn't enjoy hiking."
  - From "Either Amir lives in Calgary or he enjoys hiking" and "Amir doesn't enjoy hiking" we can conclude "Amir lives in Calgary" (Disjunctive syllogism DS).
  - etc.
- If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

1. Sarah lives in Calgary or Edmonton.
2. Amir lives in Calgary unless he enjoys hiking.
3. If Amir lives in Calgary, Sarah doesn't.
4. Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Edmonton.

5. Amir doesn't enjoy hiking (from 4).
6. Amir lives in Calgary (from 2 and 5).
7. Sarah doesn't live in Calgary (from 3 and 6).
8. Sarah lives in Edmonton (from 1 and 7).

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \to \neg C$
4. $\neg S \wedge \neg M$

$\therefore E$

5. $\neg M$ (from 4, since $\mathscr{P} \wedge \mathbb{Q} \vDash \mathbb{Q}$)
6. A (from 2 and 5, since $\mathscr{P} \vee \mathbb{Q}, \neg\mathbb{Q} \vDash \mathscr{P}$)
7. $\neg C$ (from 3 and 6, since $\mathscr{P} \to \mathbb{Q}, \mathscr{P} \vDash \mathbb{Q}$)
8. E (from 1 and 7, since $\mathscr{P} \vee \mathbb{Q}, \neg\mathscr{P} \vDash \mathbb{Q}$)

Formal proofs

- Numbered lines containing sentences of TFL.
- A line may be a **premise**.
- If it's not a premise, it must be **justified**.
- Justification involves:
  - a **rule**, and
  - prior lines (referred to by line number).
- But: what's a rule?

IV. Proofs in TFL

b. Rules for $\wedge$

Rules of natural deduction

- Rules should be ...
  - **Simple**: cite just a few lines as justification.
  - **Obvious**: justified line should clearly be entailed by justifications.
  - **Schematic**: can be described just by **forms** of sentences involved.
  - **Few in number**: want to make do with just a handful.
- We'll have two rules per connective: an **introduction** and an **elimination** rule.
- They'll be used to either:
  - justify (say) $\mathscr{P} \wedge \mathbb{Q}$ (introduce $\wedge$), or
  - justify something **using** $\mathscr{P} \wedge \mathbb{Q}$ (eliminate $\wedge$).

Eliminating $\wedge$

- What can we **justify using** $\mathscr{P} \wedge \mathbb{Q}$?
- A conjunction entails each conjunct:

$$\mathscr{P} \wedge \mathbb{Q} \vDash \mathscr{P}$$

$$\mathscr{P} \wedge \mathbb{Q} \vDash \mathbb{Q}$$

- Already used this above to get $\neg M$ from $\neg S \wedge \neg M$, i.e., from "Neither Sarah nor Amir enjoys hiking" we concluded "Amir doesn't enjoy hiking".
- (Role of $\mathscr{P}$ played by $\neg S$ and that of $\mathbb{Q}$ played by $\neg M$.)

Introducing $\wedge$

- What do we **need to justify** $\mathscr{P} \wedge \mathbb{Q}$?
- We need both $\mathscr{P}$ and $\mathbb{Q}$:

$$\mathscr{P}, \mathbb{Q} \vDash \mathscr{P} \wedge \mathbb{Q}$$

- For instance, if we have "Sarah doesn't enjoy hiking" and also "Amir doesn't enjoy hiking", we can conclude "Neither Sarah nor Amir enjoys hiking".
- (Role of $\mathscr{P}$ played by $\neg S$ and $\mathbb{Q}$ played by $\neg M$: $\neg S, \neg M \vDash \neg S \wedge \neg M$.)

Rules for $\wedge$

$$\begin{array}{r|ll} m & \mathscr{P} & \\ n & \mathbb{Q} & \\ & \mathscr{P} \wedge \mathbb{Q} & \wedge\mathrm{I}\ m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \wedge \mathbb{Q} & \\ & \mathscr{P} & \wedge\mathrm{E}\ m \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \wedge \mathbb{Q} & \\ & \mathbb{Q} & \wedge\mathrm{E}\ m \end{array}$$

We'll illustrate using the exercises in Carnap.

1	$A \wedge B$	
2	A	$\wedge$E 1
3	B	$\wedge$E 1
4	$B \wedge A$	$\wedge$I 2, 3

1	$A \wedge (B \wedge C)$	
2	A	$\wedge$E 1
3	$B \wedge C$	$\wedge$E 1
4	C	$\wedge$E 3
5	$A \wedge C$	$\wedge$I 2, 4

IV. Proofs in TFL

c. Rules for $\to$

Eliminating $\to$

- What can we **justify using** $\mathscr{P} \to \mathbb{Q}$?
- We used the conditional "If Amir lives in Calgary, Sarah isn't" to justify "Sarah doesn't live in Calgary".
- What is the general rule? What can we justify using $\mathscr{P} \to \mathbb{Q}$? What do we need in addition to $\mathscr{P} \to \mathbb{Q}$?
- The principle is **modus ponens**:

$$\mathscr{P} \to \mathbb{Q}, \mathscr{P} \vDash \mathbb{Q}$$

- (In inference from $A \to \neg C$ and A to $\neg C$, role of $\mathscr{P}$ is played by A and role of $\mathbb{Q}$ by $\neg C$.)

Elimination rule for $\to$

$$
\begin{array}{l|ll}
m & \mathscr{P} \to \mathbb{Q} & \\
n & \mathscr{P} & \\
 & \mathbb{Q} & \to\text{E } m, n
\end{array}
$$

We'll illustrate using the exercise in Carnap: we show that $A \wedge B, A \to C, B \to D \models C \wedge D$.

1	$A \wedge B$	
2	$A \rightarrow C$	
3	$B \rightarrow D$	
4	A	$\wedge$E 1
5	C	$\rightarrow$E 2, 4
6	B	$\wedge$E 1
7	D	$\rightarrow$E 3, 6
8	$C \wedge D$	$\wedge$I 5, 7

Introducing $\to$

- How do we justify a conditional? What should we require for a proof of $\mathscr{P} \to \mathbb{Q}$ (say, from some premise $\mathscr{R}$)?
- We need a proof that shows that $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$.
- Idea: show instead that $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$.
- The conditional $\to$ no longer appears, so this seems easier.
- It's a good move, because if $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$.

Justifying $\to$I

Fact

If $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$.

- If $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then every valuation makes one of $\mathscr{R}, \mathscr{P}$ false or it makes $\mathbb{Q}$ true.
- Let's show that no valuation is a counterexample to $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$:
  1. A valuation that makes $\mathscr{R}$ and $\mathscr{P}$ true, and $\mathbb{Q}$ false, is impossible if $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$.
  2. So any valuation must make $\mathscr{R}$ false, $\mathscr{P}$ false, or $\mathbb{Q}$ true.
  3. If it makes $\mathscr{R}$ false, it's not a counterexample to $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$.
  4. If it makes $\mathscr{P}$ false, it makes $\mathscr{P} \to \mathbb{Q}$ true, so it's not a counterexample.
  5. If it makes $\mathbb{Q}$ true, it also makes $\mathscr{P} \to \mathbb{Q}$ true, so it's not a counterexample.
- So, there are no counterexamples to $\mathscr{R} \vDash \mathscr{P} \to \mathbb{Q}$.

Subproofs

- We want to justify $\mathscr{P} \to \mathbb{Q}$ by giving a proof of $\mathbb{Q}$ from $\mathscr{P}$ as a premise.
- How to do this in a proof? We can't just add something as a premise and then remove it later!
- Solution: add $\mathscr{P}$ as a premise in the middle, and keep track of what depends on that premise (say, by a indenting and vertical line).
- Once we're done (have proved $\mathbb{Q}$), close this "subproof".
- Justification of $\mathscr{P} \to \mathbb{Q}$ is the **entire** subproof.
- **Important**: nothing **inside** a subproof is available outside as a justification (it depends on the assumption)

Introduction rule for $\to$

$$
\begin{array}{r|l}
m & \quad \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \mathbb{Q} \end{array} \\
n & \\
 & \mathscr{P} \to \mathbb{Q} \qquad \to\mathrm{I}\ m\text{–}n
\end{array}
$$

We'll illustrate using the exercises here

- Show: $A \to B, B \to C \models A \to C$.
- Show: $A \to (B \to C) \models (A \wedge B) \to (A \wedge C)$

1	$A \to B$	
2	$B \to C$	
3	$\quad A$	
4	$\quad B$	$\to$E 1, 3
5	$\quad C$	$\to$E 2, 4
6	$A \to C$	$\to$I 3-5

1	$A \to (B \to C)$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad B \to C$	$\to$E 1, 3
5	$\quad B$	$\wedge$E 2
6	$\quad C$	$\to$E 4, 5
7	$\quad A \wedge C$	$\wedge$I 3, 6
8	$(A \wedge B) \to (A \wedge C)$	$\to$I 2-7

IV. Proofs in TFL

d. Use of subproofs

Reiteration

$\mathscr{P} \models \mathscr{P}$, so “Reiteration” R is a good rule:

$$\begin{array}{r|ll} m & \mathscr{P} & \\ \hline k & \mathscr{P} & \mathrm{R}\ m \end{array}$$

Uses of reiteration:

- Proof of $A \models A$.
- Proof that $A \to (B \to A)$ is a tautology.

$$
\begin{array}{r|l|ll}
1 & & \quad A & \\
\cline{3-3}
2 & & \quad A & \quad \text{R } 1 \\
3 & A \to A & & \quad \to\text{I } 1\text{–}2
\end{array}
$$

1	$\quad \vert\ A$	
2	$\quad \vert\ \quad \vert\ B$	
3	$\quad \vert\ \quad \vert\ A$	R 1
4	$\quad \vert\ B \to A$	$\to$I 2-3
5	$A \to (B \to A)$	$\to$I 1-4

Rules for justifications and subproofs

- When a rule calls for a subproof, we cite it as n–m, with first and last line numbers of the subproof.
- Assumption line **and** last line have to match rule.
- After a subproof is done, you can only cite the whole thing, and not any individual line in it.
- Subproofs can be nested.
- When that happens, you also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is done.

Reiteration

Which are correct applications of R?

1		A		
2			A	
3			A	✓ R 1
4		A		✓ R 1
5		A		✗ R 2
6	A			✗ R 2
7	A			✗ R 1

IV. Proofs in TFL

e. Rules for $\vee$

Introduction rule for $\vee$

We have $\mathscr{P} \vDash \mathscr{P} \vee \mathscr{Q}$. So:

$$\begin{array}{r|ll} m & \mathscr{P} & \\ & \mathscr{P} \vee \mathscr{Q} & \vee\text{I } m \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{Q} & \\ & \mathscr{P} \vee \mathscr{Q} & \vee\text{I } m \end{array}$$

1	$\quad A$	
2	$\quad B \vee A$	$\vee$I 1
3	$A \rightarrow (B \vee A)$	$\rightarrow$I 1-2

Eliminating $\vee$

- What can we justify with disjunction $\mathscr{P} \vee \mathbb{Q}$?
- Not $\mathscr{P}$ and also not $\mathbb{Q}$: neither is entailed by $\mathscr{P} \vee \mathbb{Q}$.
- But: if both $\mathscr{P}$ and $\mathbb{Q}$ separately entail some third sentence $\mathscr{R}$, then we know that $\mathscr{R}$ follows!
- To show this, we need **two** proofs that show $\mathscr{R}$, but in each proof we are allowed to use only one of $\mathscr{P}$, $\mathbb{Q}$.

Elimination rule for ∨

$$
\begin{array}{l|l}
m & \mathscr{P} \vee \mathscr{Q} \\
i & \quad \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
j & \\
k & \quad \begin{array}{|l} \mathscr{Q} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
l & \\
 & \mathscr{R} \qquad\qquad \vee\mathrm{E}\ m,\ i\text{–}j,\ k\text{–}l
\end{array}
$$

1	$A \vee B$	
2	$\quad A$	
3	$\quad B \vee A$	$\vee$I 2
4	$\quad B$	
5	$\quad B \vee A$	$\vee$I 4
6	$B \vee A$	$\vee$E 1, 2–3, 4–5

1	$A \vee B$	
2	$A \to B$	
3	A	
4	B	$\to$E 2, 3
5	B	
6	B	R 5
7	B	$\vee$E 1, 3-4, 5-6

IV. Proofs in TFL

f. Contradictions

Contradictions

- In proofs, we don't just use the premises of the argument, but also sentences we've proved, and sentences we've assumed (for $\to$I, $\vee$E).
- Sometimes it happens that assumptions we must make for correct applications of these rules are incompatible with the premises.
- For instance, to prove the disjunctive syllogism $A \vee B, \neg B \models A$ using $\vee$E, the assumption B for the second case conflicts with the premise $\neg B$.

Disjunctive syllogism

1	$A \vee B$	
2	$\neg B$	
3	$\quad A$	
4	$\quad A$	R 3
5	$\quad B$	
	$\quad \vdots$	
k	$\quad A$	
$k+1$	A	$\vee$E 1, 3-4, 5-k

Contradictions: eliminating $\neg$

We highlight the situation where inside a subproof we have run into a contradictory situation by the symbol

$$\bot$$

m	$\neg\mathscr{P}$	
n	$\mathscr{P}$	
	$\bot$	$\neg$E m, n

Since this also eliminates a $\neg$, we'll call it $\neg$E.

Explosion

- Any argument with jointly unsatisfiable premises is valid.
- So whenever we can justify $\bot$ in a proof, we should be able to justify **anything**.
- "From a contradiction, anything follows."

$$\begin{array}{r|l}
m & \bot \\
 & \\
k & \mathscr{P} \qquad \mathrm{X}\ m
\end{array}$$

Disjunctive syllogism

1	$A \vee B$	
2	$\neg B$	
3	$\quad A$	
4	$\quad A$	R 3
5	$\quad B$	
6	$\quad \bot$	$\neg$E 2, 5
7	$\quad A$	X 6
8	A	$\vee$E 1, 3-4, 5-7

IV. Proofs in TFL

g. Introducing ¬

Introducing $\neg$

- An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.
- For instance:
  - $\mathbb{Q} \vDash \mathscr{P}$ iff $\mathbb{Q}$ and $\neg\mathscr{P}$ are jointly unsatisfiable.
  - $\mathbb{Q} \vDash \neg\mathscr{P}$ iff $\mathbb{Q}$ and $\mathscr{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\neg$I rule: To justify $\neg\mathscr{P}$, show that $\mathscr{P}$ (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction ($\bot$) follows!

Introduction rule for ¬

$$
\begin{array}{r|l}
m & \quad \begin{array}{|l} \mathscr{P} \\ \hline \end{array} \\
 & \quad \begin{array}{|l} \vdots \end{array} \\
n & \quad \begin{array}{|l} \bot \end{array} \\
 & \neg\mathscr{P} \qquad \neg\mathrm{I}\ m\text{–}n
\end{array}
$$

1	$A \to B$	
2	$\quad \neg B$	
3	$\quad\quad A$	
4	$\quad\quad B$	$\to$E 1, 3
5	$\quad\quad \bot$	$\neg$E 2, 4
6	$\quad \neg A$	$\neg$I 3-5
7	$\neg B \to \neg A$	$\to$I 2-6
8	$(A \to B) \to (\neg B \to \neg A)$	$\to$I 1-7

Indirect proof rule

$$
\begin{array}{r|l}
m & \quad\begin{array}{|l} \neg\mathscr{P} \\ \hline \vdots \\ \bot \end{array} \\
n & \\
 & \mathscr{P} \qquad\qquad \text{IP } m\text{–}n
\end{array}
$$

1	$\neg A \to \neg B$	
2	B	
3	$\neg A$	
4	$\neg B$	$\to$E 1, 3
5	$\bot$	$\neg$E 2, 4
6	A	IP 3-5
7	$B \to A$	$\to$I 2-6
8	$(\neg A \to \neg B) \to (B \to A)$	$\to$I 1-7

IV. Proofs in TFL

h. Strategies and examples

The rules, one more time: $\wedge$

$$\begin{array}{r|ll} m & \mathscr{P} & \\ n & \mathscr{Q} & \\ & \mathscr{P} \wedge \mathscr{Q} & \wedge\text{I } m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \wedge \mathscr{Q} & \\ & \mathscr{P} & \wedge\text{E } m \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \wedge \mathscr{Q} & \\ & \mathscr{Q} & \wedge\text{E } m \end{array}$$

The rules, one more time: $\to$

m	$\quad\mid \mathscr{P}$	
	$\quad\mid \vdots$	
n	$\quad\mid \mathbb{Q}$	
	$\mathscr{P} \to \mathbb{Q}$	$\to$I m–n

m	$\mathscr{P} \to \mathbb{Q}$	
n	$\mathscr{P}$	
	$\mathbb{Q}$	$\to$E m, n

The rules, one more time: $\vee$

$$
\begin{array}{r|l}
m & \mathscr{P} \vee \mathscr{Q} \\
i & \quad \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
j & \\
k & \quad \begin{array}{|l} \mathscr{Q} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
l & \\
 & \mathscr{R} \qquad \vee\text{E } m,\ i\text{–}j,\ k\text{–}l
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathscr{P} & \\
 & \mathscr{P} \vee \mathscr{Q} & \vee\text{I } m
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathscr{Q} & \\
 & \mathscr{P} \vee \mathscr{Q} & \vee\text{I } m
\end{array}
$$

The rules, one more time: ¬

$$
\begin{array}{r|l}
m & \neg\mathscr{P} \\
n & \mathscr{P} \\
 & \bot \qquad \neg\text{E } m,\ n
\end{array}
$$

$$
\begin{array}{r|l}
m & \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \bot \end{array} \\
n & \\
 & \neg\mathscr{P} \qquad \neg\text{I } m\text{–}n
\end{array}
$$

The rules, one more time: R, X, and IP

$$\begin{array}{r|ll} m & \mathscr{P} & \\ k & \mathscr{P} & \mathrm{R}\ m \end{array}$$

$$\begin{array}{r|ll} m & \bot & \\ k & \mathscr{P} & \mathrm{X}\ m \end{array}$$

$$\begin{array}{r|l} m & \begin{array}{|l} \neg\mathscr{P} \\ \hline \vdots \\ \bot \end{array} \\ n & \\ & \mathscr{P} \qquad \mathrm{IP}\ m\text{–}n \end{array}$$

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding I rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)
  - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\wedge$I, $\to$I, $\leftrightarrow$I, $\neg$I, or forward using $\vee$E
  - Work forward using $\wedge$E
  - Work forward from $\neg$E if your goal is $\bot$
  - Work forward using $\to$E, $\leftrightarrow$E
  - Work backward from $\vee$I
  - Try IP
- Repeat for each new goal from top

1	$\neg(A \vee B)$	
2	$\quad A$	
3	$\quad A \vee B$	$\vee$I 2
4	$\quad \bot$	$\neg$E 1, 3
5	$\neg A$	$\neg$I 2-4
6	$\quad B$	
7	$\quad A \vee B$	$\vee$I 6
8	$\quad \bot$	$\neg$E 1, 7
9	$\neg B$	$\neg$I 6-8
10	$\neg A \wedge \neg B$	$\wedge$I 5, 9

1	$\neg A \wedge \neg B$	
2	$A \vee B$	
3	A	
4	$\neg A$	$\wedge$E 1
5	$\bot$	$\neg$E 4, 3
6	B	
7	$\neg B$	$\wedge$E 1
8	$\bot$	$\neg$E 7, 6
9	$\bot$	$\vee$E 2, 3-5, 6-8
10	$\neg(A \vee B)$	$\neg$I 2-9
11	$(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$	$\rightarrow$I 1-10

1	$\neg(A \vee \neg A)$	
2	A	
3	$A \vee \neg A$	$\vee$I 2
4	$\bot$	$\neg$E 1, 3
5	$\neg A$	$\neg$I 2-4
6	$A \vee \neg A$	$\vee$I 5
7	$\bot$	$\neg$E 1, 6
8	$A \vee \neg A$	IP 1-7

V. Introduction to first-order logic

V. Introduction to first-order logic

a. The goals of FOL

Limits of symbolization in TFL

- Consider the argument:

  Greta is a hero.
  $\therefore$ There is a hero.

- It's clearly valid: in any case in which Greta is a hero, someone (or something, at least) is a hero, so there must be a hero.
- But its symbolization in TFL is invalid in TFL:

  G
  $\therefore H$

The problem

- Symbolization in TFL allows us to break down sentences containing “and,” “or,” “if-then” and determine validity in virtue of these **connectives**.
- Anything that can’t be further broken down must be symbolized by a sentence letters.
- That includes basic sentences like “Greta is a hero,” but also:
  - Everyone is a hero.
  - No one is a hero.
  - All heroes wear capes.

The goals of FOL

- Finer-grained symbolization
- Combines with TFL
- Allows for precise semantics (like truth tables for TFL)
- Works with proof rules (add new rules)
- Be simple & expressive (few new symbols!)
- New language: **first-order logic FOL**

The goals of FOL

- Consider the valid argument:

  Greta is a hero.
  Greta does not wear a cape.
  $\therefore$ Not all heroes wear capes.

- We'll need way to connect the occurrences of the name "Greta" in the premises.
- We'll need to connect "hero" in the premise and conclusion.
- We want to retain use of $\neg$ for "not"
- We want the symbolization to have a proof.

V. Introduction to first-order logic

b. Beginning symbolization in FOL

First steps: names

- Purpose of a **proper name**: to pick out a single, specific thing.
- (Contrast with common nouns like "hero" or "rock" which pick out collections of things)
- For simplicity, we'll only consider names that pick out an actual object (in any case we're considering)
- Later on, we'll be able to deal with other expressions that play a similar role to names, e.g., "the prime minister of Canada"
- In FOL, names will be lowercase letters a–r

First steps: predicates

- Remove a name from a sentence. What's left over is a **predicate**, e.g.,

$$\begin{array}{rl} & \text{Greta is a hero} \\ \Rightarrow & \underline{\qquad}_x \text{ is a hero.} \\ & \text{Greta admires Autumn} \\ \Rightarrow & \underline{\qquad}_x \text{ admires } \underline{\qquad}_y. \end{array}$$

- In FOL, predicates are symbolized using uppercase letters A–Z plus a number of argument slots (marked with variables), e.g., $H(x)$ or $A(x, y)$.
- Argument slots correspond to blanks.

Symbolization keys

- **Names**: lowercase letters for proper names of English
- **Predicates**: uppercase letters with variables marking blanks.

a: Autumn

g: Greta

$H(x)$: $\_\_\_\_\__x$ is a hero

$V(x)$: $\_\_\_\_\__x$ is a villain

$I(x)$: $\_\_\_\_\__x$ inspires

$C(x)$: $\_\_\_\_\__x$ wears a cape

$W(x, y)$: $\_\_\_\_\__x$ watched $\_\_\_\_\__y$

$A(x, y)$: $\_\_\_\_\__x$ admires $\_\_\_\_\__y$

$Y(x, y)$: $\_\_\_\_\__x$ is younger than $\_\_\_\_\__y$

- **Domain**: what things we're talking about

  e.g., people alive in 2022

Symbolization

- Basic sentences: predicates with names replacing variables.
  - Greta is a hero. $H(g)$
  - Greta admires Autum. $A(g, a)$
- Combinations using connectives:
  - Greta and Autumn are heroes. $H(g) \wedge H(a)$

Symbolization

- Replacing pronouns by antecedents:
  - If Autumn is a hero, Greta admires **her**. $H(a) \to A(g, a)$
  - Greta doesn't admire **herself**. $\neg A(g, g)$
  - Greta and Autumn watched **each other**. $W(g, a) \wedge W(a, g)$
- Modification:
  - Autumn is an **inspiring hero**.
    Autumn inspires and is a hero. $I(a) \wedge H(a)$
  - Greta is a **hero who doesn't wear a cape**.
    Greta is a hero and it's not the case that Greta wears a cape.
    $H(g) \wedge \neg C(g)$

Careful with modification

- Greta is an international hero.
  - Can’t be paraphrased as “Greta is international and a hero.”
  - So “_____ is an international hero” needs its own predicate.
- The Piltdown Man is a fake fossil.
  - Can’t be paraphrased as “The Piltdown Man is fake and a fossil.”
  - “Fake” and other privative adjectives (“pretend,” “fictitious”) imply opposite!

Examples

- Autumn and Greta are inspiring heroes.
  $(I(a) \wedge H(a)) \wedge (I(g) \wedge H(g))$
- Greta admires Autumn but not herself.
  $A(g, a) \wedge \neg A(g, g)$
- Greta inspires only if Autumn does.
  $I(g) \to I(a)$
- Greta and Autumn watched each other.
  $W(g, a) \wedge W(a, g)$
- Greta is older than Autumn.
  $Y(a, g)$
- One of Greta and Autumn watched the other.
  At least one: $W(g, a) \vee W(a, g)$
  Exactly one: $(W(g, a) \vee W(a, g)) \wedge \neg(W(g, a) \wedge W(a, g))$

V. Introduction to first-order logic

c. The existential quantifier

Existential quantifier

- In English: “something,” “someone,” “there is . . . ”
- For instance:
  - **Someone** wears a cape.
  - **There is** a hero.
  - **Something** inspires.
- Note: can (often) go where names and pronouns also go.
- But works differently from names (“something” doesn’t pick out a unique, specific object).

How to symbolize “something”

- Idea: introduce a special term sg?
- Problem: now we can’t distinguish between
  - Someone is a hero and wears a cape.
  - Someone is a hero and someone wears a cape.

  as both would be symbolized by $H(sg) \wedge C(sg)$.
- Better idea: symbolize (complex) **properties** and introduce mechanism for expressing that properties are **instantiated**.

Expressing properties

- One-place predicates **express** properties, e.g.,
  - $H(x)$ expresses property "being a hero."
  - $I(x)$ expresses "is inspiring."
- Combinations of predicates (with connectives, names) can express **derived** properties, e.g.,
  - $A(x, g)$ expresses "admires Greta"
  - $W(a, x)$ expresses "is watched by Autumn"
  - $H(x) \wedge C(x)$ expresses "is a hero who wears a cape"
- Note: all contain a **single** variable x

The existential quantifier $\exists$

- Symbol for "there is": $\exists$
- Combine $\exists$ with expression for a property (e.g., $(H(x) \wedge C(x))$) to say "something (someone) has that property"
- Put the variable that serves as a marker for the gap also after $\exists$. E.g.,

$$\exists x\,(H(x) \wedge C(x))$$

  says "Someone is a hero and wears a cape"

Quantifiers and variables

- Compare:

$$\exists x\,(H(x) \wedge C(x)) \text{ to}$$
$$\exists x\, H(x) \wedge \exists x\, C(x)$$

- In first case, the same person must be a hero and wear a cape.
- In second case, one person can be the hero, and another wears a cape.
- Multiple $\exists x$ are independent, even if they use the same x. No difference in meaning between

$$\exists x\, H(x) \wedge \exists x\, C(x) \text{ and}$$
$$\exists x\, H(x) \wedge \exists y\, C(y).$$

The domain and quantifiers

- Symbolization key gives a domain of objects being talked about.
- Quantifier **ranges over** this domain.
- That means: $\exists x \ldots x \ldots$ is true iff some object **in the domain** has the property expressed by $\ldots x \ldots$.
- Domain makes a difference: Consider $\exists x\, W(x, g)$.
  - True if someone watched Greta (say, Autumn did).
  - Now take the domain to include only Greta.
  - Relative to that domain, $\exists x\, W(x, g)$ is true iff Greta watched herself.

Quantifier restriction in English

- "something" and "someone" work grammatically like singular terms (go where names can also go).
- "some" (on its own) does not: it is a **determiner** and needs a **complement**, e.g.,
  - a common noun ("some hero"), or
  - a noun phrase ("some admirer of Greta").
- "some" + complement works grammatically like "someone", e.g., "**Some hero** wears a cape"
- General form: "Some F is G."

Quantifier restriction in FOL

- "Some F is G" **restricts** the "something" quantifier to Fs.
- We could (and linguists often do) mark restrictions in the quantifier, e.g., $(\exists x : F(x))G(x)$
- We won't because we can do without.
- "Some F is G" is true iff there is something which is both F and also G, so:
- "Some F is G" can be symbolized as

$$\exists x(F(x) \land G(x))$$

- We'll also symbolize the plural form this way ("Some Fs are Gs").
- And more generally (most) sentences of the form: "G(some F)" or "G(something that Fs)".

Examples

- Some hero wears a cape.
  Some heroes wear capes.
  $\exists x(H(x) \wedge C(x))$
- Someone who wears a cape watched Greta.
  $\exists x(C(x) \wedge W(x, g))$
- Greta admires some hero who wears a cape.
  $\exists x((H(x) \wedge C(x)) \wedge A(g, x))$
- Autumn watched someone who watched Greta.
  $\exists x(W(x, g) \wedge W(a, x))$

V. Introduction to first-order logic

d. The universal quantifier

Universal quantifier

- "**Something** is F" is true iff **at least one** element of domain is F.
- "**Everything** is F" is true iff **every element** of the domain is F.
- In FOL: $\forall x\, F(x)$.
- E.g.:
  - "Everyone wears a cape" $\forall x\, C(x)$
  - "Everyone watched Greta or Autumn" $\forall x(W(x, g) \vee W(x, a))$

Universal determiners: all, every, any

- Determiners with universal meaning: **all, every, any**.
- Take complements (just like “some” does), e.g.,
  - Every hero inspires.
  - All heroes inspire.
  - Any hero inspires.
- These are true in the same cases (mean the same).
- “Every F is G” is true iff everything which is F is G.
- Watch out for “any”: not always universal.

Restricted $\forall$ in FOL

- Suppose we can symbolize F and G.
- How do we symbolize "Every F is G"?
- Options:
  - $\forall x(F(x) \wedge G(x))$
    If true, everything must be F.
    So can be false when "Every F is G" is true.
  - $\forall x(F(x) \vee G(x))$
    True if everything is F (without being G).
    So can be true when "Every F is G" is false.
  - $\forall x(F(x) \to G(x))$
    If x is F, x must also be G.
    (If x is not F, doesn't matter if it's G or not.)

Symbolizing "all Fs are Gs"

Symbolize the following as

$$\forall x(F(x) \to G(x))$$

- All Fs are Gs.
- Every F is G.
- Any F is G.

Examples

- **Every hero** wears a cape.
  All heroes wear capes.
  $\forall x(H(x) \to C(x))$
- **Every hero who wears a cape** watched Greta.
  $\forall x((H(x) \wedge C(x)) \to W(x, g))$
- Greta and Autumn admire **anyone who wears a cape**.
  $\forall x(C(x) \to (A(g, x) \wedge A(a, x)))$
- Autumn watched **everyone who watched Greta**.
  $\forall x(W(x, g) \to W(a, x))$
- All heroes and villains watched Greta.
  $\forall x((H(x) \vee V(x)) \to W(x, g))$

V. Introduction to first-order logic

e. No, only, a, and some & any again

No F is G

- "**No Fs are Gs**" can be paraphrased as
  - "**Every** F is **not**–G," or as
  - "**Not: some** F is G."
- So symbolize it using:
  - $\forall x(F(x) \to \neg G(x))$ or
  - $\neg \exists x(F(x) \wedge G(x))$

Examples

- **No hero** wears a cape.
  No heroes wear capes.
  $\forall x(H(x) \to \neg C(x))$
- **No hero who wears a cape** watched Greta.
  $\forall x((H(x) \land C(x)) \to \neg W(x, g))$
- Greta admires **no one who wears a cape**.
  $\neg \exists x(C(x) \land A(g, x))$
- Autumn watched **no one who watched Greta**.
  $\neg \exists x(W(x, g) \land W(a, x))$

Only Fs are G

- When is "Only Fs are Gs" false?
- When there is a **non**-F that's G.
- So symbolize it as

$$\neg\exists x(\neg F(x) \land G(x))$$

- Or, paraphrase it as: "Any x is G **only if** it is F"
- So another symbolization is:

$$\forall x(G(x) \to F(x))$$

Examples

- Only heroes wear capes.

  $\forall x(C(x) \rightarrow H(x))$

- Only heroes who wear capes watched Greta.

  $\forall x(W(x, g) \rightarrow (H(x) \wedge C(x)))$

- Greta admires only people who wear capes.

  $\neg\exists x(\neg C(x) \wedge A(g, x))$

- Autumn watched only heroes and villains.

  $\neg\exists x(\neg(H(x) \vee V(x)) \wedge W(a, x))$

  $\forall x(W(a, x) \rightarrow (H(x) \vee V(x)))$

The indefinite article

- We use “is a” to indicate predication, e.g., “Greta is a hero.”
- Often “a” is used to claim existence, e.g.,

  Greta admires a hero.

  $\exists x(H(x) \land A(g, x))$
- But a **generic** indefinite is closer to a universal quantifier:

  A hero is someone who inspires.

  $\forall x(H(x) \to I(x))$
- Be careful if the indefinite article is in the antecedent of a conditional:

  If **a hero** wears a cape, **they** inspire.

  That means: all heroes who wear capes inspire.

  $\forall x((H(x) \land C(x)) \to I(x))$

Universal “some”, existential “any”

- “Someone,” “something” can require a **universal** quantifier: if it’s in the antecedent of a conditional, with a pronoun in the consequent referring back to it, e.g.,

  If **someone** is a hero, Autum admires **them**.

  Roughly: Autumn admires all heroes.

  $\forall x(H(x) \rightarrow A(a, x))$

- “Any” in antecedents but **without** pronouns referring back to them are **existential**:

  If **anyone** is a hero, Greta is.

  Roughly: if there are heroes (at all), Greta is a hero.

  $\exists x\, H(x) \rightarrow H(g)$

V. Introduction to first-order logic

f. Mixed domains

Mixed domains

- Sometimes you want to talk about more than one kind of thing.
- The domain can include any mix of things (e.g., people, animals, items of clothing)
- Proper symbolization then needs predicates for these kinds, e.g.:

Domain: people alive in 2022 and items of clothing

$P(x)$: $\_\_\_\_\__x$ is a person

$L(x)$: $\_\_\_\_\__x$ is an item of clothing.

$E(x)$: $\_\_\_\_\__x$ is a cape

$R(x, y)$: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Quantification in mixed domains

- Not everyone is wearing a cape.
  - In domain of people only:
    $\neg\forall x\, C(x)$
  - In mixed domain:
    $\neg\forall x\, (P(x) \to C(x))$
- Some people inspire.
  - In domain of people only:
    $\exists x\, I(x)$
  - In mixed domain:
    $\exists x (P(x) \wedge I(x))$
- Greta wears something.

  $\exists x (L(x) \wedge R(g, x))$

VI. Semantics of FOL

VI. Semantics of FOL

a. Arguments and validity in FOL

Validity of arguments

Everyone is either good or evil.
Not everyone is a villain.
Only villains are evil.
∴ Some heroes are good.

Validity in FOL

- Want to capture validity **in virtue of the meanings of the connectives and the quantifiers** (but ignoring meanings of predicate symbols)
- So we want to ignore any restrictions the predicate symbols place on their **extensions**
- Hence: allow **any** extension in a potential counterexample
- An argument is **first-order valid** if there is no **interpretation** in which the premises are true and the conclusion false

Forms of arguments

Everyone is either good or evil.
Not everyone is a villain.
Only villains are evil.
∴ Some heroes are good.

$$\forall x(G(x) \vee E(x))$$
$$\neg\forall x\, V(x)$$
$$\forall x(E(x) \to V(x))$$
$$\therefore\ \exists x(H(x) \wedge G(x))$$

(In)validity of arguments

$$\begin{array}{rl} & \forall x(G(x) \vee E(x)) \\ & \neg\forall x\, V(x) \\ & \forall x(E(x) \to V(x)) \\ \therefore & \exists x(H(x) \wedge G(x)) \end{array}$$

Domain: the inner planets

$G(x)$: x is smaller than Earth

$E(x)$: x is inhabited

$V(x)$: x has a moon

$H(x)$: x has rings

VI. Semantics of FOL

b. Interpretations

Interpretations

- Domain: collection of objects (not empty)
- **Referents** for each name (which object it names)
- Properties of each object
  - **Extension** of each 1-place predicate symbol: which objects it applies to
- Relations of each pair of objects
  - **Extension** of each 2-place predicate symbol: which pairs of objects standing in the relation

Extensions

Domain: the inner planets
$G(x)$: x is smaller than Earth
$E(x)$: x is inhabited
$V(x)$: x has a moon
$H(x)$: x has rings

Domain: Mercury, Venus, Earth, Mars
$G(x)$: Mercury, Venus, Mars
$E(x)$: Earth
$V(x)$: Earth, Mars
$H(x)$: —

(In)validity of arguments

$$\begin{array}{rl} & \forall x(G(x) \vee E(x)) \\ & \neg\forall x\, V(x) \\ & \forall x(E(x) \rightarrow V(x)) \\ \therefore & \exists x(H(x) \wedge G(x)) \end{array}$$

Domain: Mercury, Venus, Earth, Mars

$G(x)$: Mercury, Venus, Mars

$E(x)$: Earth

$V(x)$: Earth, Mars

$H(x)$: —

(In)validity of arguments

$$
\begin{array}{rl}
 & \forall x(G(x) \vee E(x)) \\
 & \neg\forall x\, V(x) \\
 & \forall x(E(x) \rightarrow V(x)) \\
\therefore & \exists x(H(x) \wedge G(x))
\end{array}
$$

Domain: 1, 2, 3, 4
$G(x)$: 1, 2, 4
$E(x)$: 3
$V(x)$: 3, 4
$H(x)$: —

(In)validity of arguments

$$\begin{array}{rl} & \forall x(G(x) \vee E(x)) \\ & \neg\forall x\, V(x) \\ & \forall x(E(x) \to V(x)) \\ \therefore & \exists x(H(x) \wedge G(x)) \end{array}$$

Domain: 1

$G(x)$: 1

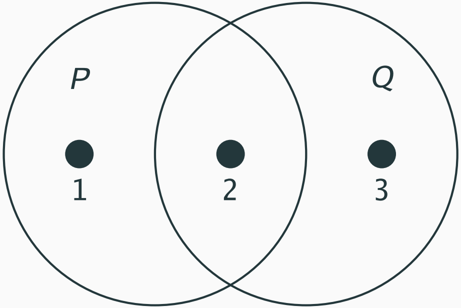
$E(x)$: —

$V(x)$: —

$H(x)$: —

Extensions of predicates

Domain: 1, 2, 3
$P(x)$: 1, 2
$Q(x)$: 2, 3
$R(x)$: —



$R = \emptyset$

(In)validity of arguments

$$\begin{array}{rl} & \forall x(G(x) \vee E(x)) \\ & \neg\forall x\, V(x) \\ & \forall x(E(x) \rightarrow V(x)) \\ \therefore & \exists x(H(x) \wedge G(x)) \end{array}$$

Domain: 1, 2

$G(x)$: 1

$E(x)$: 2

$V(x)$: 2

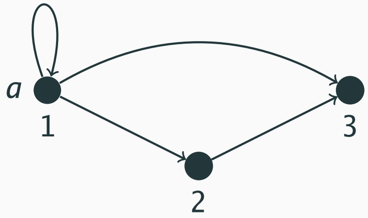
$H(x)$: 2

Extensions of predicates

Domain: 1, 2, 3

a: 1

$A(x, y)$: $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$



VI. Semantics of FOL

c. Truth of sentences of FOL

Truth of sentences of FOL

- Given an interpretation I ...
- An **atomic sentence** is true iff the referents of the constants are in the extension of the predicate:
  - $P(a)$ is true iff referent r of a is in extension of P
  - $R(a, b)$ is true iff $\langle r, p\rangle$ is in extension of R (where r is referent of a, p is referent of b)
- $\neg\mathscr{A}$ is true iff $\mathscr{A}$ is false
- $\mathscr{A} \lor \mathscr{B}$ is true iff at least one of $\mathscr{A}$, $\mathscr{B}$ is true
- $\mathscr{A} \land \mathscr{B}$ is true iff both $\mathscr{A}$, $\mathscr{B}$ are true
- $\mathscr{A} \to \mathscr{B}$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true

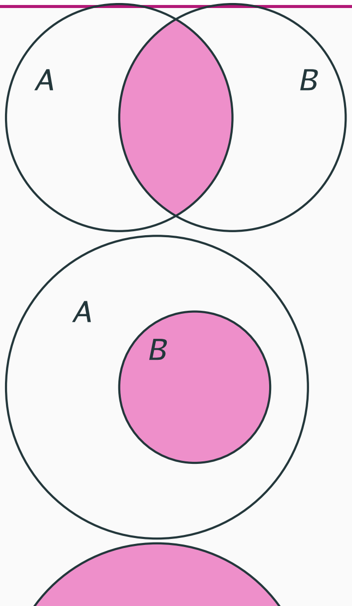
Truth of quantified sentences

- $\exists x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is **satisfied** by **at least one** object in the domain
  - o satisfies $\mathscr{A}(x)$ iff $\mathscr{A}(c)$ is true in interpretation just like I, but with o as referent of c
- $\forall x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is **satisfied** by **every** object in the domain

Truth of quantified sentences

- $\exists x\,(\mathscr{A}(x) \wedge \mathscr{B}(x))$ is true iff some object satisfies $\mathscr{A}(x) \wedge \mathscr{B}(x)$
  - o satisfies $\mathscr{A}(x) \wedge \mathscr{B}(x)$ iff it satisfies both $\mathscr{A}(x)$ and $\mathscr{B}(x)$
- $\forall x\,(\mathscr{A}(x) \to \mathscr{B}(x))$ is true iff every object satisfies $\mathscr{A}(x) \to \mathscr{B}(x)$
  - o satisfies $\mathscr{A}(x) \to \mathscr{B}(x)$ iff either
    - o does not satisfy $\mathscr{A}(x)$ or
    - o does satisfy $\mathscr{B}(x)$

Making “Some *A*s are *B*s” true

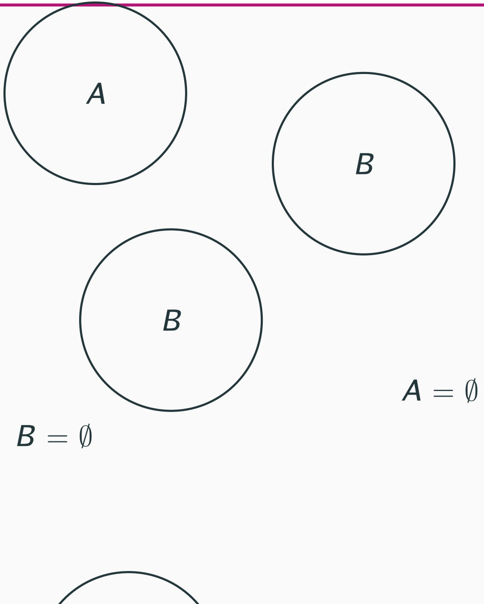


- $\exists x\,(A(x) \wedge B(x))$
- Extension of *A* and *B* must have something in common.

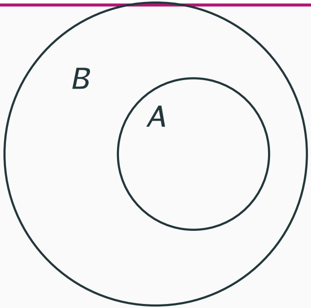
  (Filled area must contain at least one object)

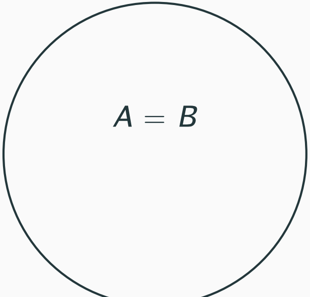
Making “Some As are Bs” false

- $\neg\exists x\,(A(x) \wedge B(x))$
- Extension of A and B must have nothing in common.
- A and B don’t overlap, or one or both is empty.
- Same situations make “No As are Bs” **true**.



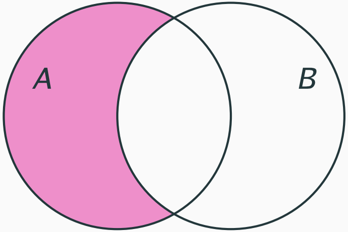
Making "All As are Bs" true



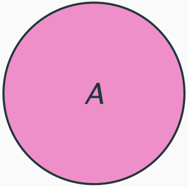


- $\forall x\,(A(x) \to B(x))$
- Extension of A must be contained in extension of B.
- Extensions of A and B can be the same.
- Extension of A can be empty.

Making “All *A*s are *B*s” false



- $\forall x\,(A(x) \rightarrow B(x))$
- Extension of A must contain something not in B.
- Extensions of A cannot be empty, but B may be empty.
- Same situations make …
  - “Only *B*s are *A*s” **false**



VI. Semantics of FOL

d. Testing for validity

Arguments involving quantifiers

1. If an action x is morally wrong then A is blameworthy for freely doing x.
2. If x is rationally optimal (there is no action which A has reason to think there is more reason for A to do), then A is not blameworthy for freely doing x.
3. Therefore, if x is morally wrong, then x is not rationally optimal. (Principle of moral categoricity.)

(John Skorupski, *Ethical Explorations*, 2000 (link)

Symbolizing Skorupski

1. If an action x is morally wrong then A is blameworthy for freely doing x.
2. If x is rationally optimal, then A is not blameworthy for freely doing x.
3. Therefore, if x is morally wrong, then x is not rationally optimal.

Domain: actions

$W(x)$: x is morally wrong

$B(x)$: A is blameworthy for freely doing x

$O(x)$: x is rationally optimal

$$\forall x(W(x) \to B(x))$$
$$\forall x(O(x) \to \neg B(x))$$
$$\therefore \forall x(W(x) \to \neg O(x))$$

Symbolizing Skorupski

Domain: actions

$W(x)$: x is morally wrong

$B(x)$: A is blameworthy for freely doing x

$O(x)$: x is rationally optimal

$$\forall x(W(x) \rightarrow B(x))$$
$$\forall x(O(x) \rightarrow \neg B(x))$$
$$\therefore \forall x(W(x) \rightarrow \neg O(x))$$

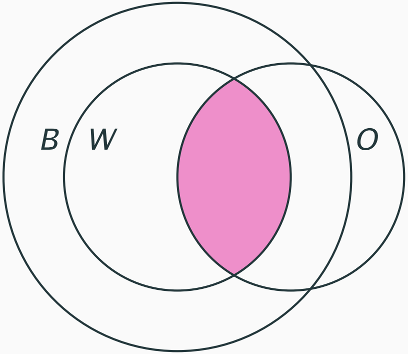
All Ws are Bs

No Os are Bs (iff No Bs are Os)

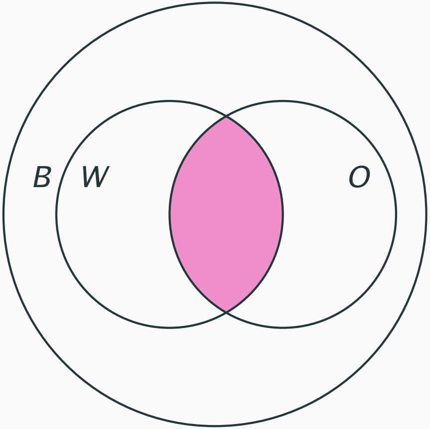
∴ No Ws are Os

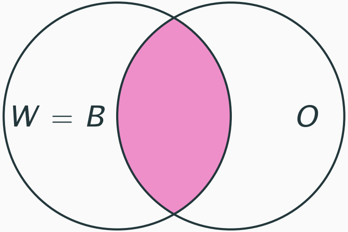
Determining validity

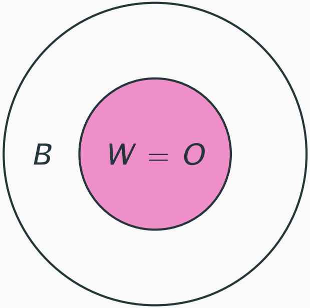
- Make conclusion $\forall x(W(x) \to \neg O(x))$ false.
- Make $\exists x(W(x) \wedge O(x))$ true.
- Make $\forall x(W(x) \to B(x))$ true.
- $\exists x(O(x) \wedge B(x))$ is now forced to be true.
- So, $\forall x(O(x) \to \neg B(x))$ is false.
- But those are not the only possibilities!



Other configurations







VI. Semantics of FOL

e. Semantic notions in FOL

Semantics notions in FOL

- $\mathscr{P}_1, \ldots, \mathscr{P}_n \models \mathbb{Q}$ if no interpretation makes all of $\mathscr{P}_1, \ldots, \mathscr{P}_n$ true and $\mathbb{Q}$ false.
- $\mathscr{P}$ is a **validity** ($\models \mathscr{P}$) if it is true in every interpretation.
- $\mathscr{P}$ and $\mathbb{Q}$ are **equivalent in FOL** if no interpretation makes one true but the other false.
- $\mathscr{P}_1, \ldots, \mathscr{P}_n$ are **jointly satisfiable in FOL** if some interpretation makes all of them true at the same time.

Using interpretations

- By providing one suitable interpretation we **can** show that...
  - an argument is **not valid** in FOL
  - a sentence is **not a validity** in FOL
  - two sentences are **not equivalent** in FOL
  - some sentences **are satisfiable** in FOL
- But we **cannot** show using any number of interpretations that...
  - an argument **is valid** in FOL
  - a sentence **is a validity** in FOL
  - two sentences **are equivalent** in FOL
  - some sentences **are not satisfiable** in FOL

Examples

- $\forall x(A(x) \lor B(x))$ and $\forall x\, A(x) \lor \forall x\, B(x)$ are not equivalent.
- $\forall x(A(x) \to B(x)), \forall x(A(x) \to \neg B(x))$ are jointly satisfiable.
- $\forall x(\neg A(x) \to B(x)), \exists x(B(x) \land C(x, b)) \not\models \exists x(\neg A(x) \land C(x, b))$.
- $\not\models \exists x\, A(a, x) \to \exists x\, A(x, x)$.

Test solutions on carnap.io

VI. Semantics of FOL

f. Arguing about interpretations

Arguing about Interpretations

- No interpretation(s) can show that an argument is valid.
- That's because there is no way to inspect all possible interpretations.
- But we can show that arguments are valid, by:
  - a formal proof (next time)
  - an informal argument
- The informal argument makes use of the **truth conditions** for sentences of FOL.
- Analogous to arguing about valuations in TFL.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.
- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.
- Suppose it's the second, i.e., $\forall x\, \mathscr{B}(x)$ is true: Similarly.
- These are the only possibilities: the interpretation must make the conclusion also true.

VII. Proofs in FOL

VII. Proofs in FOL

a. Rules for $\forall$

Rules for formal proofs

- Need rules for $\forall$ and $\exists$ for formal proofs
- Formal proofs now more important, because no alternative (truth-table method)
- Intro and Elim rules should be
  - simple
  - elegant (not involve other connectives or quantifiers)
  - yield only valid arguments

Candidates for rules

- Only simple sentence close to $\forall x\, \mathscr{A}(x)$ is $\mathscr{A}(c)$
- Gives simple, elegant $\forall$E rule:

k	$\forall x\, \mathscr{A}(x)$	
	$\mathscr{A}(c)$	$\forall$E k

- This is a good rule: $\forall x\, \mathscr{A}(x) \models \mathscr{A}(c)$.

Candidates for rules

- Problem: corresponding "intro rule" isn't valid:

$$\begin{array}{r|l} k & \mathscr{A}(c) \\ & \\ & \forall x\, \mathscr{A}(x) \quad \textbf{doesn't follow from } k \end{array}$$

- Diagnosis: the c in $\mathscr{A}(c)$ is a name for a **specific object**.
- We need a name for an **arbitrary, unspecified object**.
- If $\mathscr{A}(c)$ is true for whatever c **could** name, then $\mathscr{A}(x)$ is satisfied by **every** object.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.
  Only people who wear capes admire Greta.
  ∴ All heroes wear capes.

  Proof: Let Carl be any hero. Since all heroes admire Greta, Carl admires Greta. Since only people who wear capes admire Greta, Carl is wears a cape. But "Carl" stands for **any** hero. So all heroes wear capes.

Universal generalization

$$\begin{array}{l|ll} k & \mathscr{A}(c) & \\ & \forall x\, \mathscr{A}(x) & \forall\mathrm{I}\ k \end{array}$$

- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathscr{A}(x)$ is obtained from $\mathscr{A}(c)$ by replacing **all** occurrences of c by x.
- In other words, c must also not occur in $\forall x\, \mathscr{A}(x)$.

General conditional proof

Proving “All As are Bs”

k	$\quad A(c)$	
l	$\quad B(c)$	
$l+1$	$A(c) \rightarrow B(c)$	$\rightarrow$I k-l
$l+2$	$\forall x(A(x) \rightarrow B(x))$	$\forall$I $l+1$

Example

All heroes admire Greta.
Only people who wear capes admire Greta.
$\therefore$ All heroes wear capes.

$$\forall x(H(x) \to A(x, g))$$
$$\forall x(A(x, g) \to C(x))$$
$$\therefore \forall x(H(x) \to C(x))$$

Let’s do it on carnap.io

Example

1	$\forall x(H(x) \to A(x,g))$	
2	$\forall x(A(x,g) \to C(x))$	
3	$\quad H(c)$	
4	$\quad H(c) \to A(c,g)$	$\forall$E 1
5	$\quad A(c,g)$	$\to$E 4, 3
6	$\quad A(c,g) \to C(c)$	$\forall$E 2
7	$\quad C(c)$	$\to$E 6, 5
8	$H(c) \to C(c)$	$\to$I 3–7
9	$\forall x(H(x) \to C(x))$	$\forall$I 8

Example

1	$\forall x\, A(x) \lor \forall x\, B(x)$	
2	$\quad \forall x\, A(x)$	
3	$\quad A(c)$	$\forall$E 2
4	$\quad A(c) \lor B(c)$	$\lor$I 3
5	$\quad \forall x\, B(x)$	
6	$\quad B(c)$	$\forall$E 5
7	$\quad A(c) \lor B(c)$	$\lor$I 6
8	$A(c) \lor B(c)$	$\lor$E 1, 2–4, 5–7
9	$\forall x(A(x) \lor B(x))$	$\forall$I 8

VII. Proofs in FOL

b. Rules for ∃

Rules for $\exists$

- If we know of a specific object that it satisfies $\mathscr{A}(x)$, we know that at least one object satisfies $\mathscr{A}(x)$.
- So this rule is valid:

$$
\begin{array}{r|ll}
k & \mathscr{A}(c) & \\
 & \exists x\,\mathscr{A}(x) & \exists\mathrm{I}\ k
\end{array}
$$

Arbitrary objects again

- Problem: corresponding "elim rule" isn't valid:

$$\begin{array}{l|ll} k & \exists x\, \mathscr{A}(x) & \\ & & \\ & \mathscr{A}(c) & \quad \textbf{doesn't follow from } k \end{array}$$

- If we know that $\exists x\, \mathscr{A}(x)$ is true, we know that **some** objects satisfy $\mathscr{A}(x)$, but not which ones.
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathscr{A}(x)$.

Reasoning from existential information

- To use $\exists x\, \mathscr{A}(x)$, pretend the x has a name c, and reason from $\mathscr{A}(c)$.
- This is what we'd do if we reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
  $\therefore$ Some heroes admire Greta.

  Proof: We know there are heroes who wear capes. Let Cate be an arbitrary one of them. So Cate wears a cape. Since anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

Existential elimination

- If
  - we know that some object satisfies $\mathscr{A}(x)$,
  - we assume for the time being that c is one of them (i.e., assume $\mathscr{A}(c)$), and
  - we can prove that $\mathscr{B}$ follows from this assumption,

  then $\mathscr{B}$ follows already from $\exists x\, \mathscr{A}(x)$.
- Rule for existential elimination:

$$
\begin{array}{r|l}
k & \exists x\, \mathscr{A}(x) \\
m & \quad \begin{array}{|l} \mathscr{A}(c) \\ \hline \end{array} \\
n & \quad \begin{array}{|l} \mathscr{B} \end{array} \\
 & \mathscr{B} \qquad\qquad\qquad \exists\mathrm{E}\ k,\ m\text{–}n
\end{array}
$$

- c is special: c must not appear outside subproof

Example

There are heroes who wear capes.
Anyone who wears a cape admires Greta.
∴ Some heroes admire Greta.

$$\exists x(H(x) \wedge C(x))$$
$$\forall x(C(x) \rightarrow A(x, g))$$
$$\therefore \exists x(H(x) \wedge A(x, g))$$

Example

1	$\exists x(H(x) \wedge C(x))$	
2	$\forall x(C(x) \to A(x, g))$	
3	$H(c) \wedge C(c)$	
4	$C(c)$	$\wedge$E 3
5	$C(c) \to A(c, g)$	$\forall$E 2
6	$A(c, g)$	$\to$E 4, 5
7	$H(c)$	$\wedge$E 3
8	$H(c) \wedge A(c, g)$	$\wedge$I 4, 7
9	$\exists x(H(x) \wedge A(x, g))$	$\exists$I 8

VIII. Multiple quantifiers

VIII. Multiple quantifiers

a. Two quantifiers

Formulas expressing relations

- A formula $\mathscr{A}(x)$ with one free variable expresses a **property**.
- A formula $\mathscr{B}(x, y)$ with two free variables expresses a **relation**
- $\forall x \forall y\, \mathscr{B}(x, y)$ is a sentence;
- It's true iff **every pair** of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$.
- $\exists x \exists y\, \mathscr{B}(x, y)$ is a sentence.
- It's true iff **at least one pair** of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$.

Multiple uses of a single quantifier: $\forall$

- $A(x, y)$ ... x admires y.
- $\forall x \forall y\, A(x, y)$ ... for every pair $\langle \alpha, \beta \rangle$, α admires β.
- In other words: everyone admires everyone.
- Note: "every pair" includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
- $\forall x \forall y\, A(x, y)$ is true only if all pairs $\langle \alpha, \alpha \rangle$ satisfy $A(x, y)$.
- That means, everyone admires themselves, in addition to everyone else.
- So: $\forall x \forall y\, A(x, y)$ does **not** symbolize "everyone admires everyone **else**."

Multiple uses of single quantifier: $\exists$

- $\exists x \exists y\, A(x, y)$ ... for at least one pair $\langle \alpha, \beta \rangle$, α admires β.
- In other words: at least one person admires at least one person.
- Note: includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
- $\exists x \exists y\, A(x, y)$ is already true if a single pair $\langle \alpha, \alpha \rangle$ satisfies $A(x, y)$.
- That means, we could just have one person admiring themselves.
- So: $\exists x \exists y\, A(x, y)$ does **not** symbolize "someone admires someone **else**."

Alternating quantifiers

1. $\forall x \exists y\, A(x, y)$
   Everyone admires someone
   (possibly themselves)
2. $\forall y \exists x\, A(x, y)$
   Everyone is admired by someone
   (not necessarily the same person)
3. $\exists x \forall y\, A(x, y)$
   Someone admires everyone
   (including themselves)
4. $\exists y \forall x\, A(x, y)$
   Someone is admired by everyone
   (including themselves)

Convergence vs. uniform convergence

- A function f **point-wise continuous** if

$$\forall\epsilon\forall x\forall y\exists\delta(|x-y|<\delta\to|f(x)-f(y)|<\epsilon)$$

- A function f **uniformly continuous** if

$$\forall\epsilon\exists\delta\forall x\forall y(|x-y|<\delta\to|f(x)-f(y)|<\epsilon)$$

VIII. Multiple quantifiers

b. Using quantifiers to express properties

Our symbolization key

Domain: people alive in 2022 and items of clothing

a: Autumn

g: Greta

$P(x)$: $\_\_\_\_\__x$ is a person

$L(x)$: $\_\_\_\_\__x$ is an item of clothing.

$E(x)$: $\_\_\_\_\__x$ is a cape

$R(x, y)$: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

$H(x)$: $\_\_\_\_\__x$ is a hero

$I(x)$: $\_\_\_\_\__x$ inspires

$Y(x, y)$: $\_\_\_\_\__x$ is younger than $\_\_\_\_\__y$

$A(x, y)$: $\_\_\_\_\__x$ admires $\_\_\_\_\__y$

$O(x, y)$: $\_\_\_\_\__x$ owns $\_\_\_\_\__y$

Expressing properties, revisited

- One-place predicates express properties, e.g.,

  $H(x)$ expresses property "being a hero"
- Combinations of predicates (with connectives, names) can express derived properties, e.g.,

  $A(x, g)$ expresses "x admires Greta"
  $H(x) \wedge C(x)$ expresses "x is a hero who wears a cape"
- Using quantifiers, we can express even more complex properties, e.g.,

  $\exists y(P(y) \wedge A(x, y))$ expresses "x admires someone"

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
  - Greta admires a hero.
    $\exists y(H(y) \wedge A(g, y))$
  - x admires a hero.
    $\exists y(H(y) \wedge A(x, y))$
- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $\exists y(E(y) \wedge R(x, y))$
  - Greta wears a cape.
    $\exists y(E(y) \wedge R(g, y))$

$E(x)$: $\_\_\_\_\__x$ is a cape
$R(x, y)$: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.

  $\exists y(E(y) \wedge R(x,y))$
- x is admired by everyone.

  $\forall y(P(y) \rightarrow A(y,x))$
- x admires a hero.

  $\exists y(H(y) \wedge A(x,y))$
- x admires only heroes.

  $\forall y(A(x,y) \rightarrow H(y))$
- x is naked.

  $\neg\exists y(L(y) \wedge R(x,y))$

  $\forall y(L(y) \rightarrow \neg R(x,y))$

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x,y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

VIII. Multiple quantifiers

c. Multiple determiners

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?
- Deal with each determiner separately!
- Think of determiner phrase as replaced with name or variable—result has one less determiner.
- When you're down to one determiner, apply known methods for single quantifiers.
- This results in formulas that express properties or relations, but themselves contain quantifiers.

Two separate determiner phrases

- All heroes wear a cape
- All heroes satisfy "x wears a cape"

$$\forall x(H(x) \to \text{“}x \text{ wears a cape”})$$

- x wears a cape

$$\exists y(E(y) \wedge R(x, y))$$

- Together:

$$\forall x(H(x) \to \exists y(E(y) \wedge R(x, y)))$$

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.
- All things that satisfy "x is a hero who wears a cape" admire Greta.

$$\forall x(\text{“x is a hero who wears a cape”} \to A(x, g))$$

- x is a hero who wears a cape

$$H(x) \wedge \exists y(E(y) \wedge R(x, y))$$

- Together:

$$\forall x((H(x) \wedge \exists y(E(y) \wedge R(x, y))) \to A(x, g))$$

Mary Astell, 1666–1731

- British political philosopher
- *Some Reflections upon Marriage* (1700)
- In preface to 3rd ed. 1706 reacts to William Nicholls' claim (in *The Duty of Inferiors towards their Superiors, in Five Practical Discourses* (London 1701), Discourse IV: The Duty of Wives to their Husbands), that women are naturally inferior to men.

Astell TL;DR

- What can Nicholls possibly mean by "women are naturally inferior to men"?
- It can't be that some women is inferior to some man, since that's "no great discovery."
- After all, surely some men are inferior to some women.
- The obviously intended meaning must be: **all** women are inferior to **all** men.
- But that can't be right, for then "the greatest Queen ought not to command but to obey her Footman."
- It can't even be just: **all** women are inferior to **some** men.
- Since "had they been pleased to remember their Oaths of Allegiance and Supremacy, they might have known that *One* Women is superior to *All* the Men in these Nations."

Symbolizing Astell

- Some woman is superior to every man
- Some woman satisfies "x is superior to every man"

$$\exists x(W(x) \wedge \text{“}x \text{ is superior to every man”})$$

- x is superior to every man

$$\forall y(M(y) \rightarrow S(x, y))$$

- Together:

$$\exists x(W(x) \wedge \forall y(M(y) \rightarrow S(x, y))$$

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \wedge \exists y(M(y) \wedge S(x, y)))$
- Every woman is superior to every man.

  $\forall x(W(x) \rightarrow \forall y(M(y) \rightarrow S(x, y)))$
- Every woman is superior to some man.

  $\forall x(W(x) \rightarrow \exists y(M(y) \wedge S(x, y)))$
- Some woman is superior to every man.

  $\exists x(W(x) \wedge \forall y(M(y) \rightarrow S(x, y)))$

"Any"

- Any (every) cape is worn by a hero.

$$\forall x(E(x) \to \exists y(H(y) \wedge R(y, x)))$$

- No hero wears any cape.

$$\forall x(H(x) \to \neg\exists y(E(y) \wedge R(x, y))$$

- No hero wears every cape.

$$\forall x(H(x) \to \neg\forall y(E(y) \to R(x, y))$$

VIII. Multiple quantifiers

d. Quantifier scope ambiguity

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.
- Autumn admires Isra or Luisa, and so does Greta.

$$(A(a,i) \vee A(a,l)) \wedge$$
$$(A(g,i) \vee A(g,l))$$

- Autumn and Greta both admire Isra, or they both admire Luisa.

$$(A(a,i) \wedge A(g,i)) \vee$$
$$(A(a,l) \wedge A(g,l))$$

Negation and the quantifiers

- "All heroes don't inspire"
  - Denial of "all heroes inspire"
    ("Do all heroes inspire? No, all heroes don't inspire")

$$\neg\forall x(H(x) \to I(x))$$
$$\exists x(H(x) \land \neg I(x))$$

  - All heroes are: not inspiring, i.e.,
    No heroes inspire

$$\forall x(H(x) \to \neg I(x))$$
$$\neg\exists x(H(x) \land I(x))$$

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e., "For every hero, there is a cape they wear"

$$\forall x(H(x) \to \exists y(E(y) \land R(x, y)))$$

  - "All heroes" in scope of "a cape", i.e., "There is a cape which every hero wears"

$$\exists y(E(y) \land \forall x(H(x) \to R(x, y)))$$

- Compare the joke: "Every day, a tourist is mugged on the streets of New York. We will interview him tonight."

VIII. Multiple quantifiers

e. Donkey sentences

Happy farmers

"Every farmer who owns a donkey is happy"

- Step-by-step symbolization: "All *A*s are *B*s"
- x is a farmer who owns a donkey ...

$$F(x) \land \exists y(D(y) \land O(x, y))$$

- Every farmer who owns a donkey is happy

$$\forall x((F(x) \land \exists y(D(y) \land O(x, y))) \to H(x))$$

Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step-by-step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$F(x) \wedge \exists y(D(y) \wedge O(x, y))$$

- Every farmer who owns a donkey beats it

$$\forall x((F(x) \wedge \exists y(D(y) \wedge O(x, y))) \rightarrow B(x, y))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\neg\exists x(F(x) \land \exists y(D(y) \land O(x,y) \land \neg B(x,y)))$$

- For every farmer and every donkey they own: the farmer beats the donkey.

$$\forall x \forall y((F(x) \land (D(y) \land O(x,y))) \to B(x,y)))$$

- Every farmer beats every donkey they own.

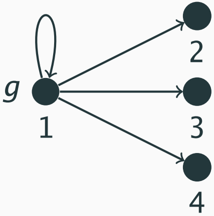
$$\forall x(F(x) \to \forall y((D(y) \land O(x,y)) \to B(x,y)))$$

IX. Identity

IX. Identity

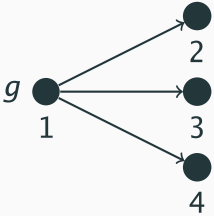
a. The identity predicate

Greta admires everyone (else)



Greta admires everyone.

$\forall x\, A(g, x)$



Greta admires everyone **else**.

$\forall x(\text{“}x \text{ is not Greta”} \rightarrow A(g, x))$

$\forall x(\neg x = g \rightarrow A(g, x))$

The identity predicate

- A new, special two-place predicate: $=$
  - Written between arguments, **without parentheses**.
  - Needs no mention in symbolization key.
  - Always interpreted the same: extension of $=$ is all pairs $\langle \alpha, \alpha \rangle$.
- $a = b$ true iff a and b are names for one and the the same object.
- $x = y$ satisfied by all and only the pairs $\langle \alpha, \alpha \rangle$.
- $\neg x = y$ is satisfied by a pair $\langle \alpha, \beta \rangle$ iff α and β are different objects.
- $x = \neg y$ is not grammatical.
  ($\neg$ can only go in front of a formula, and y is not one.)
- $\neg(x = y)$ is also not grammatical.
  ($(x = y)$ is also not a formula.)

Something else/everything else

- Remember: different variables does not mean different objects.
- $\exists x \exists y\, A(x, y)$ doesn't mean that someone admires someone else.
- It just means that someone admires someone (possibly themselves).
- To symbolize "someone else" add $\neg x = y$:

$$\exists x \exists y (\neg x = y \wedge A(x, y))$$

- $\forall x \forall y\, A(x, y)$ says that everyone admires everyone (including themselves).
- To symbolize "everyone else" add $\neg x = y$:

$$\forall x \forall y (\neg x = y \rightarrow A(x, y))$$

Something else/everything else

- The closest quantifier (typically) determines if you should use $\wedge$ or $\rightarrow$:

$$\forall x \exists y(\neg x = y \wedge A(x, y)) \qquad \exists x \forall y(\neg x = y \rightarrow A(x, y))$$

- If you have mixed domains, it works the same way:
- Everyone admires someone **else**:

$$\forall x(P(x) \rightarrow \exists y((P(y) \wedge \neg x = y) \wedge A(x, y)))$$

- Someone admires everyone **else**:

$$\exists x(P(x) \wedge \forall y((P(y) \wedge \neg x = y) \rightarrow A(x, y))$$

Other than, except

- "**Someone other than Greta** is a hero":

  $\exists x(\neg x = g \wedge H(x))$
- "**Everyone other than Greta** is a hero",
- "**Everyone except Greta** is a hero":

  $\forall x(\neg x = g \rightarrow H(x))$

Singular "only"

- "**No-one other than Greta** is a hero":

  $\neg\exists x(H(x) \wedge \neg x = g)$

  $\forall x(H(x) \rightarrow x = g)$

- "**Only Greta** is a hero":
- No-one other than Greta is a hero, and Greta is a hero:

  $\forall x(H(x) \rightarrow x = g) \wedge H(g)$

  $\forall x(H(x) \leftrightarrow x = g))$

Uniqueness

- There is at least one hero.

$$\exists x\, H(x)$$

- There is exactly one hero.
  - There's at least one hero, and
  - There are no others:

$$\exists x\,(H(x) \wedge \neg\exists y\,(\neg y = x \wedge H(y)))$$
$$\exists x\,(H(x) \wedge \forall y(H(y) \to x = y))$$

  - Or more succinctly: $\exists x \forall y (H(y) \leftrightarrow x = y)$

IX. Identity

b. Numerical quantification

Numerical Quantification

- Cardinal numbers can be determiners:
  - **Three heroes** wear capes.
- Not always clear if "three heroes" means **exactly** or **at least** three.
- We'll assume the latter.
  - Do you have two dollars? Yes, I have two dollars. (Uncontroversially true even if you have more than 2$)
- FOL can express all of:
  - **At least** n people are ...
  - **Exactly** n people are ...
  - **At most** n people are ...

At least n

- At least 1 hero is inspiring:

$$\exists x(H(x) \wedge I(x))$$

- At least 2 heroes are inspiring:

$$\exists x \exists y(\neg x = y \wedge ((H(x) \wedge I(x)) \wedge (H(y) \wedge I(y))))$$

- At least 3 heroes are inspiring:

$$\begin{aligned}&\exists x \exists y \exists z((\neg x = y \wedge (\neg y = z \wedge \neg x = z)) \wedge \\ &\quad ((H(x) \wedge I(x)) \wedge ((H(y) \wedge I(y)) \wedge (H(z) \wedge I(z)))))\end{aligned}$$

At least n

- There are at least n As ("$\exists^{\geq n} x\, A(x)$")::

$$
\begin{aligned}
\exists x_1 \ldots \exists x_n((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\
(\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\
\ddots \\
\neg x_{n-1} = x_n) \ldots) \wedge \\
(A(x_1) \wedge (A(x_2) \wedge \cdots \wedge A(x_n)) \ldots))
\end{aligned}
$$

At least n

- Note: must state that **every pair** of variables is different, e.g.,

$$\exists x_1 \exists x_2 \exists x_3((\neg x_1 = x_2 \wedge \neg x_2 = x_3) \wedge$$
$$(H(x_1) \wedge (H(x_2) \wedge H(x_3))))$$

  only says "There are at least two heroes"!
  - Take extension of $H(x)$ to be: 1, 2
  - Then 1 can play role of x_1 and x_3, 2 role of x_2.
  - Both "$\neg 1 = 2$" and "$\neg 2 = 3$" are true.
- At least n Bs are Cs: take $B(x) \wedge C(x)$ for $A(x)$:

$$\exists^{\geq n} x(B(x) \wedge C(x))$$

Exactly one

- There is exactly one hero:

$$\exists x(H(x) \wedge \neg\exists y(H(y) \wedge \neg x = y))$$

- This is equivalent to:

$$\exists x(H(x) \wedge \forall y(H(y) \rightarrow x = y))$$

- In general: "x has property A **uniquely**":

$$A(x) \wedge \forall y(A(y) \rightarrow x = y)$$

or just: $$\forall y(A(y) \leftrightarrow x = y)$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\begin{aligned}\exists x_1 \ldots \exists x_n ((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge{} \\ (\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge{} \\ \ddots \\ \neg x_{n-1} = x_n) \ldots) \wedge{} \\ (A(x_1) \wedge (A(x_2) \wedge \cdots \wedge A(x_n)) \ldots)) \wedge{} \\ \forall y (A(y) \to (y = x_1 \vee \cdots \vee y = x_n)))\end{aligned}$$

- Exactly n Bs are Cs:

$$\exists^{=n} x (B(x) \wedge C(x))$$

Exactly n

- There are exactly n As ("$\exists^{=n} x\, A(x)$"):

$$\begin{aligned}
\exists x_1 \ldots \exists x_n ((\neg x_1 = x_2 \wedge (\neg x_1 = x_3 \wedge \cdots \wedge (\neg x_1 = x_n \wedge \\
(\neg x_2 = x_3 \wedge \cdots \wedge (\neg x_2 = x_n \wedge \\
\ddots \\
\neg x_{n-1} = x_n) \ldots) \wedge \\
\\
\forall y (A(y) \leftrightarrow (y = x_1 \vee \cdots \vee y = x_n)))
\end{aligned}$$

- Exactly n Bs are Cs:

$$\exists^{=n} x (B(x) \wedge C(x))$$

At most n

- There are **at most n** As $\Leftrightarrow$ There are **not at least $n+1$** As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow \neg\exists^{\geq(n+1)} x\, A(x)$$

- For instance: There are at most two heroes:

$$\neg\exists x \exists y \exists z((H(x) \wedge (H(y) \wedge H(z))) \wedge (\neg x = y \wedge (\neg x = z \wedge \neg y = z)))$$

$$\forall x \forall y \forall z((H(x) \wedge (H(y) \wedge H(z))) \rightarrow (x = y \vee (x = z \vee y = z)))$$

- $\neg\exists^{\geq(n+1)} x\, A(x)$ is equivalent to:

$$\begin{aligned} \forall x_1 \ldots \forall x_{n+1}((A(x_1) \wedge \cdots \wedge A(x_{n+1})) \rightarrow \\ (x_1 = x_2 \vee (x_1 = x_3 \vee \cdots \vee (x_1 = x_{n+1} \vee \\ (x_2 = x_3 \vee \cdots \vee (x_2 = x_{n+1} \vee \\ \ddots \\ x_n = x_{n+1}) \ldots))) \end{aligned}$$

IX. Identity

c. “The”, “both”, “neither”

Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say

  “The A is B”

  is to say:
- There is something, which:
  - is A,
  - is the only A,
  - is B.
- In FOL:

$$\exists x(A(x) \wedge \forall y(A(y) \rightarrow x = y) \wedge B(x))$$

- or more succinctly:

$$\exists x(\forall y(A(y) \leftrightarrow x = y) \wedge B(x))$$

“The” vs. “exactly one”

- Compare:
  1. The hero inspires:

$$\exists x(H(x) \wedge \forall y(H(y) \to x = y) \wedge I(x))$$

  2. There is exactly one inspiring hero:

$$\exists x(H(x) \wedge \forall y((H(y) \wedge I(y)) \to x = y) \wedge I(x))$$

- (2) can be true without (1), but not vice versa.
- (Namely when there is exactly one inspiring hero, but also a non-inspiring hero.)
- So (1) entails (2), but not vice versa.

Strawson's analysis

- According to Russell, "The hero wears a cape" is false if there is no hero, or if there is more than one.
- P. F. Strawson disagrees: we only succeed in making a statement if there is a unique hero.
- "There is a unique hero" is not part of what is **said**, but is only **presupposed**.

Singular possessive

- Singular possessives make noun phrases, e.g., “Joe’s cape”
- They work like definite descriptions: Joe’s cape is the cape Joe owns. E.g.:
  - “Autumn wears Joe’s cape” symbolizes the same as: “Autumn wears the cape Joe owns”:

$$
\begin{aligned}
&\exists x[((E(x) \wedge O(j,x)) \wedge \\
&\quad \forall y((E(y) \wedge O(j,y)) \rightarrow x = y)) \wedge \\
&\quad W(a,x)]
\end{aligned}
$$

Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:
  - "Autumn wears Joe's **capes**" symbolizes the same as: "Autumn wears every cape that Joe owns":

$$\forall x[(E(x) \land O(j, x)) \rightarrow W(a, x)]$$

Both

- “Both heroes inspire”: There are exactly 2 heroes, and both inspire:

$$\begin{aligned}\exists x \exists y[((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z(H(z) \rightarrow (z = x \vee z = y))) \wedge \\ (I(x) \wedge I(y))]\end{aligned}$$

- Note: “Both heroes inspire” implies “There are exactly two inspiring heroes”, but not vice versa!

Neither

- "Neither hero inspires": There are exactly 2 heroes, and neither of them inspires:

$$\begin{aligned}\exists x \exists y[((\neg x = y \wedge (H(x) \wedge H(y))) \wedge \\ \forall z(H(z) \rightarrow (z = x \vee z = y))) \wedge \\ (\neg I(x) \wedge \neg I(y))]\end{aligned}$$

X. Proofs for full FOL

X. Proofs for full FOL

a. Proofs with multiple quantifiers

Eliminating $\forall$

$$
\begin{array}{r|ll}
m & \forall x \mathscr{A}(\ldots x \ldots c \ldots) & \\
& & \\
& \mathscr{A}(\ldots c \ldots c \ldots) & \quad \forall\text{E } m
\end{array}
$$

- No restriction on c:
  - May be in an assumption.
  - May also be in $\forall x \mathscr{A}(\ldots x \ldots x \ldots)$ already!

Working forward from $\forall$

- If you **have** $\forall x \mathscr{A}(x)$, replace every x by the same c.
- The result is $\mathscr{A}(c)$, justified by $\forall E$.
- You can pick any c.
- Good candidates: c which occur in assumptions or in the sentences you're trying to prove.
- You may need to try multiple candidates.

Introducing $\forall$

$$\begin{array}{l|ll} m & \mathscr{A}(\ldots c \ldots c \ldots) & \\ & \forall x \mathscr{A}(\ldots x \ldots x \ldots) & \forall\mathrm{I}\ m \end{array}$$

- Restrictions on c:
  - must not occur in any undischarged assumption.
  - must not occur in $\forall x \mathscr{A}(\ldots x \ldots x)$.

Working backward from $\forall$

- If you **want** $\forall x \mathscr{A}(x)$, replace every x by the same c.
- The result is $\mathscr{A}(c)$.
- Try to prove this **above** $\forall x \mathscr{A}(x)$.
- Justify $\forall x \mathscr{A}(x)$ by $\forall I$.
- You must pick a **new** c not already in the proof constructed so far.
- As long as c is fresh, this will work if you can prove $\forall x \mathscr{A}(x)$ at all.

Introducing $\exists$

$$
\begin{array}{l|ll}
m & \mathscr{A}(\ldots c \ldots c \ldots) & \\
\hline
 & \exists x \mathscr{A}(\ldots x \ldots c \ldots) & \exists\mathrm{I}\ m
\end{array}
$$

- No restriction on c:
  - May be in an assumption.
  - May also be in $\exists x \mathscr{A}(\ldots x \ldots c \ldots)$.
  - So you can also justify $\exists x \mathscr{A}(\ldots x \ldots x \ldots)$ or $\exists x \mathscr{A}(\ldots c \ldots x \ldots)$.

Working backward from $\exists$

- If you **want** $\exists x \mathscr{A}(x)$, replace every x by the same c.
- The result is $\mathscr{A}(c)$.
- Try to prove this **above** $\exists x \mathscr{A}(x)$
- Justify $\exists x \mathscr{A}(x)$ by $\exists I$.
- You can pick any c.
- Good candidates: c which occur in assumptions or in the sentences you're trying to prove.
- That includes $\exists x \mathscr{A}(x)$!
- You may need to try multiple candidates.
- This may not work (especially at the beginning, or if you need IP)!
- Try other strategies first, especially strategies that put more c into play ($\exists E$).

Eliminating $\exists$

m	$\exists x \mathscr{A}(\ldots x \ldots x \ldots)$	
i	$\quad \mathscr{A}(\ldots c \ldots c \ldots)$	
j	$\quad \mathscr{B}$	
	$\mathscr{B}$	$\exists$E m, i–j

- Restrictions on c:
  - must not occur in any assumption still open when you apply $\exists E$.
  - must not occur in $\exists x \mathscr{A}(\ldots x \ldots x)$.
  - must not occur in $\mathscr{B}$.

Working forward from $\exists$

- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.
  - Prove $\mathscr{B}$ on its last line
  - Justify $\mathscr{B}$ after the subproof using $\exists E$.
- You must pick a **new** c not already in the proof constructed so far.
- As long as c is fresh, this will work if you can prove $\mathscr{B}$ from $\exists x \mathscr{A}(x)$ at all.

Admirers and admired

Someone is admired by everyone

$\therefore$ Everyone admires someone

$$\begin{array}{|l} \exists y \forall x\, A(x, y) \\ \hline \forall x \exists y\, A(x, y) \end{array}$$

Let's do it on carnap.io

Admirers and admired

1	$\exists y \forall x\, A(x, y)$	
2	$\quad \forall x\, A(x, c)$	
3	$\quad A(d, c)$	$\forall$E 2
4	$\quad \exists y\, A(d, y)$	$\exists$I 3
5	$\quad \forall x \exists y\, A(x, y)$	$\forall$I 4
6	$\forall x \exists y\, A(x, y)$	$\exists$E 1, 2-5

All hail Queen Anne

Some woman is superior to every man.
$\therefore$ Every man is inferior to some woman.

$$\begin{array}{|l}
\exists y(W(y) \land \forall x(M(x) \to S(y,x))) \\
\hline
\forall x(M(x) \to \exists y(W(y) \land S(y,x)))
\end{array}$$

1	$\exists y(W(y) \wedge \forall x(M(x) \to S(y,x)))$	
2	$\quad W(c) \wedge \forall x(M(x) \to S(c,x))$	
3	$\quad\quad M(d)$	
4	$\quad\quad \forall x(M(x) \to S(c,x))$	$\wedge$E 2
5	$\quad\quad M(d) \to S(c,d))$	
6	$\quad\quad S(c,d)$	
7	$\quad\quad W(c)$	$\wedge$E 2
8	$\quad\quad W(c) \wedge S(c,d)$	$\wedge$I 6, 7
9	$\quad\quad \exists y(W(y) \wedge S(y,x))$	$\exists$I 8
10	$\quad M(d) \to \exists y(W(y) \wedge S(y,d))$	
11	$\quad \forall x(M(x) \to \exists y(W(y) \wedge S(y,x)))$	$\forall$I 10
12	$\forall x(M(x) \to \exists y(W(y) \wedge S(y,x)))$	$\exists$E 1, 2-11

X.a.12

X. Proofs for full FOL

b. Proofs with identity

Everybody loves my baby

Everybody loves my baby.
But my baby don't love nobody but me.
∴ My baby is me.

$$
\begin{array}{|l}
\forall x\, L(x, b) \\
\forall x (L(b, x) \to x = i) \\
\hline
b = i
\end{array}
$$

- "Everybody Loves my Baby" on YouTube

My baby is me

1	$\forall x\, L(x, b)$	
2	$\forall x (L(b, x) \rightarrow x = i)$	
3	$L(b, b)$	$\forall$E 1
4	$L(b, b) \rightarrow b = i$	$\forall$E 2
5	$b = i$	$\rightarrow$E 3, 4

I am my baby

1	$\forall x\, L(x, b)$	
2	$\forall x(L(b, x) \to x = i)$	
3	$L(b, b)$	$\forall$E 1
4	$L(b, b) \to b = i$	$\forall$E 2
5	$i = b$	?

Proofs with identity

$$\begin{array}{r|ll} & \mathscr{c} = \mathscr{c} & \text{=I} \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{a} = \mathscr{b} & \\ n & \mathscr{A}(\ldots \mathscr{a} \ldots \mathscr{a} \ldots) & \\ & \mathscr{A}(\ldots \mathscr{b} \ldots \mathscr{a} \ldots) & \text{=E } m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{a} = \mathscr{b} & \\ n & \mathscr{A}(\ldots \mathscr{b} \ldots \mathscr{b} \ldots) & \\ & \mathscr{A}(\ldots \mathscr{a} \ldots \mathscr{b} \ldots) & \text{=E } m, n \end{array}$$

I am my baby

- We symbolized "My baby is me" as $b = i$.
- But it's equivalent to "I am my baby," $i = b$.
- $=$I and $=$E let us prove this:

1	$b = i$	
2	$b = b$	=I
3	$i = b$	=E 1, 2

Different properties, different things

- Two names d, e may name the same thing.
- In that case, $d = e$ would be true.
- And then anything that's true about d is also true about e.
- In other words:

$$d = e, P(d) \models P(e)$$

- So if something is true about d but false about e, then $\neg d = e$.
- In other words:

$$P(d), \neg P(e) \models \neg d = e$$

Different properties, different things

1	$P(d)$	
2	$\neg P(e)$	
3	$\quad d = e$	
4	$\quad P(e)$	=E 1, 3
5	$\quad \bot$	¬E 2, 4
6	$\neg d = e$	¬I 2–5

Uniqueness, again

The two symbolizations of “there is exactly one hero” are equivalent:

$$\exists x(H(x) \land \forall y(H(y) \rightarrow x = y))$$

$$\exists x \forall y(H(y) \leftrightarrow x = y)$$

Uniqueness, again

1	$\exists x(H(x) \land \forall y(H(y) \rightarrow x = y))$	
2	$\quad H(a) \land \forall y(H(y) \rightarrow a = y)$	
3	$\quad\quad H(c)$	
4	$\quad\quad \forall y(H(y) \rightarrow a = y)$	$\land$E 2
5	$\quad\quad H(c) \rightarrow a = c$	$\forall$E 4
6	$\quad\quad a = c$	$\rightarrow$E 3, 5
7	$\quad\quad a = c$	
8	$\quad\quad H(a)$	$\land$E 2
9	$\quad\quad H(c)$	$=$E 7, 8
10	$\quad H(c) \leftrightarrow a = c$	$\leftrightarrow$I 3-6, 7-9
11	$\quad \forall y(H(y) \leftrightarrow a = y)$	$\forall$I 10
12	$\quad \exists x \forall y(H(y) \leftrightarrow x = y)$	$\exists$I 11
13	$\exists x \forall y(H(y) \leftrightarrow x = y)$	$\exists$E 1, 2-12

Uniqueness, again

1	$\exists x \forall y (H(y) \leftrightarrow x = y)$	
2	$\quad \forall y (H(y) \leftrightarrow a = y)$	
3	$\quad H(a) \leftrightarrow a = a$	$\forall$E 2
4	$\quad a = a$	$=$I
5	$\quad H(a)$	$\leftrightarrow$E 3, 4
6	$\quad\quad H(c)$	
7	$\quad\quad H(c) \leftrightarrow a = c$	$\forall$E 2
8	$\quad\quad a = c$	$\leftrightarrow$E 6, 7
9	$\quad H(c) \to a = c$	$\to$I 6-8
10	$\quad \forall y (H(y) \to a = y)$	$\forall$I 9
11	$\quad H(a) \land \forall y (H(y) \to a = y)$	$\land$I 5, 10
12	$\quad \exists x (H(x) \land \forall y (H(y) \to x = y))$	$\exists$I 11
13	$\exists x (H(x) \land \forall y (H(y) \to x = y))$	$\exists$E 1, 2-12

Proofs with numerical claims

$$\begin{array}{|l}
\exists x\, P(x) \\
\forall x \forall y ((P(x) \land P(y)) \to x = y) \\
\hline
\exists x (P(x) \land \forall y (P(y) \to x = y))
\end{array}$$

Proofs with numerical claims

1	$\exists x\, P(x)$	
2	$\forall x \forall y((P(x) \land P(y)) \to x = y)$	
3	$\quad P(a)$	
4	$\quad\quad P(c)$	
5	$\quad\quad \forall y((P(a) \land P(y)) \to a = y)$	$\forall$E 2
6	$\quad\quad (P(a) \land P(c)) \to a = c$	$\forall$E 5
7	$\quad\quad P(a) \land P(c)$	$\land$I 3, 4
8	$\quad\quad a = c$	$\to$E 6, 7
9	$\quad P(c) \to a = c$	$\to$I 4-8
10	$\quad \forall y(P(y) \to a = y)$	
11	$\quad P(a) \land \forall y(P(y) \to a = y)$	$\land$I 3, 11
12	$\quad \exists x(P(x) \land \forall y(P(y) \to x = y))$	$\exists$I 11
13	$\exists x(P(x) \land \forall y(P(y) \to x = y))$	$\exists$E 1, 3-12

XI. Interpretations for full FOL

XI. Interpretations for full FOL

a. Interpretations and truth, revisited

Interpretations

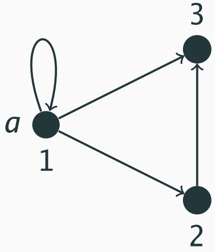
- Domain: collection of objects (not empty)
- **Referents** for each name (which object it names)
- Properties of each object
  - **Extension** of each 1-place predicate symbol: which objects it applies to
- Relations of each pair of objects
  - **Extension** of each 2-place predicate symbol: which pairs of objects standing in the relation
  - Extension of $=$ is all pairs $\langle \alpha, \alpha \rangle$.

Extensions of predicates

Domain: 1, 2, 3

a: 1

$A(x, y)$: $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$



Truth of sentences of FOL

- Given an interpretation I ...
- An **atomic sentence** is true iff the referents of the constants are in the extension of the predicate:
  - $P(a)$ is true iff referent α of a is in extension of P
  - $R(a, b)$ is true iff $\langle \alpha, \beta \rangle$ is in extension of R (where α is referent of a, β is referent of b)
  - $a = b$ true iff a and b are names for one and the the same object.
- $\neg\mathscr{A}$ is true iff $\mathscr{A}$ is false
- $\mathscr{A} \lor \mathscr{B}$ is true iff at least one of $\mathscr{A}$, $\mathscr{B}$ is true
- $\mathscr{A} \land \mathscr{B}$ is true iff both $\mathscr{A}$, $\mathscr{B}$ are true
- $\mathscr{A} \to \mathscr{B}$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true

Satisfaction of formulas

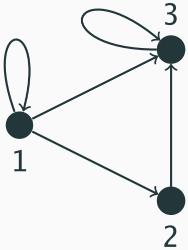
- Suppose $\mathscr{A}(x)$ has only the variable x free.
- Object α in domain **satisfies** $\mathscr{A}(x)$ iff $\mathscr{A}(c)$ is true in interpretation just like I, but with α as referent of c.
- (c is a name that does not occur in $\mathscr{A}(x)$.)

Truth of quantified sentences

- $\exists x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is satisfied by **at least one** object in the domain.
- $\forall x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is satisfied by **every** object in the domain

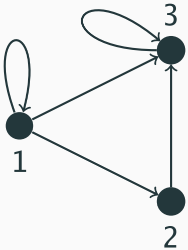
Satisfaction

- 1 satisfies $\exists y\, A(x, y)$...
  - because $\exists y\, A(c, y)$ is true...
  - because 1 (and 2 and 3) satisfies $A(c, y)$.
- 2 satisfies $\exists y\, A(x, y)$...
  - because $\exists y\, A(c, y)$ is true ...
  - because 3 satisfies $A(c, y)$.
- 3 satisfies $\exists y\, A(x, y)$...
  - because $\exists y\, A(c, y)$ is true...
  - because 3 satisfies $A(c, y)$.
- So every object satisfies $\exists y\, A(x, y)$
- So $\forall x \exists y\, A(x, y)$ is true.



Satisfaction with $=$

- 1 satisfies $\exists y\,(\neg x = y \wedge A(x,y))\ldots$
  - because $\exists y\,(\neg c = y \wedge A(c,y))$ is true...
  - because 2 (and 3) satisfies $\neg c = y \wedge A(c,y)$.
- 2 satisfies $\exists y\,(\neg x = y \wedge A(x,y))\ldots$
  - because $\exists y\,(\neg c = y \wedge A(c,y))$ is true ...
  - because 3 satisfies $\neg c = y \wedge A(c,y)$.
- 3 **doesn't** satisfy $\exists y\,(\neg x = y \wedge A(x,y))\ldots$
  - because $\exists y\,(\neg c = y \wedge A(c,y))$ is false...
  - because nothing satisfies $\neg c = y \wedge A(c,y)$.
- So **not** every object satisfies $\exists y\,(\neg x = y \wedge A(x,y))$
- So $\forall x \exists y\,(\neg x = y \wedge A(x,y))$ is **false**.

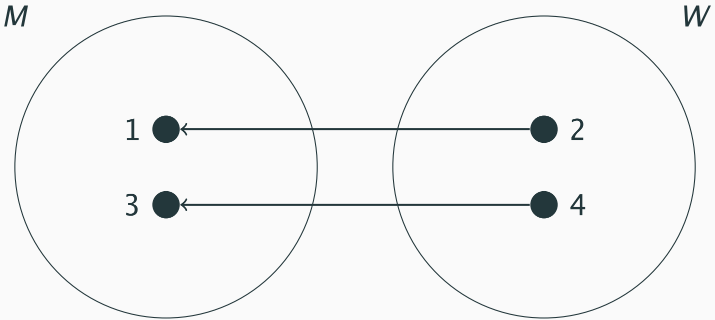


XI. Interpretations for full FOL

b. Constructing (counter)examples

Counterexamples

$$
\begin{array}{rl}
& \exists x\, M(x), \exists y\, W(y), \\
& \forall x(M(x) \to \exists y(W(y) \land S(y,x))) \\
\not\models & \exists y(W(y) \land \forall x(M(x) \to S(y,x)))
\end{array}
$$



Constructing an interpretation

- We want to make $\forall x \exists y\, (\neg x = y \wedge A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
- Add another object instead.
- Add an arrow. How about now?
  - 1 satisfies $\exists y\, (\neg x = y \wedge A(x, y))$.
  - But 2 does not.
- Add another arrow.
- Ok, a different arrow then.
- Now 2 satisfies $\exists y\, (\neg x = y \wedge A(x, y))$ also.
- So $\forall x \exists y\, (\neg x = y \wedge A(x, y))$ is true.

Constructing another interpretation

- Let's make $\exists y \forall x\, (\neg x = y \rightarrow A(x, y))$ true.
- Start with one object. Is it true yet? Yes!
- Add another object. Still true? No!
  - 1 does not satisfy $\forall x\, (\neg x = y \rightarrow A(x, y))$ ...
  - because $\forall x\, (\neg x = c \rightarrow A(x, c))$ is false...
  - because 2 does not satisfy $\neg x = c \rightarrow A(x, c)$
  - 2 also doesn't satisfy $\forall x\, (\neg x = y \rightarrow A(x, y))$.
- Add an arrow. How about now?
  - 2 satisfies $\forall x\, (\neg x = y \rightarrow A(x, y))$ ...
  - because $\forall x\, (\neg x = c \rightarrow A(x, c))$ is true...
  - because 1 satisfies $\neg x = c \rightarrow A(x, c)$,
  - and 2 satisfies $\neg x = c \rightarrow A(x, c)$.
- So $\exists y \forall x\, (\neg x = y \rightarrow A(x, y))$ is true.

Examples and counterexamples

- $\exists y \forall x\, (\neg x = y \rightarrow A(x, y)) \not\models \forall x \exists y\, (\neg x = y \wedge A(x, y))$.

or

- Compare: $\exists y \forall x\, A(x, y) \models \forall x \exists y\, A(x, y)$!
- $\exists y \forall x\, (\neg x = y \rightarrow A(x, y))$, $\forall x \exists y\, (\neg x = y \wedge A(x, y))$ are jointly satisfiable.

- $\forall x \exists y\, (\neg x = y \wedge A(x, y)) \not\models \exists y \forall x\, (\neg x = y \rightarrow A(x, y))$.

XI. Interpretations for full FOL

c. Properties of relations

Relations and extensions

- 2-place predicate symbols express relations.
- Extension of a 2-place predicate symbol is a set of ordered pairs.
- This is exactly how mathematicians think of relations.
- Let's think of properties relations can have, and categorize relations by these properties.

Reflexivity

Definition

A relation R is **reflexive** if every object stands in the relation R to itself.

- x is the same age as y.
- x and y share a parent.
- Not: x and y are siblings.
- $x \leq y$ but not $x < y$.
- $x \mid y$ (x divides y without remainder) .

Expressing reflexivity in FOL

The extension of P in an interpretation is reflexive if and only if $\forall x\, P(x, x)$ is true.

Symmetry

Definition

A relation R is **symmetric** if whenever it holds in one direction, it also holds in the other.

- x is the same age as y.
- x and y share a parent.
- x and y are siblings.
- Not $x \leq y$ or $x < y$.
- Not $x \mid y$.

Expressing symmetry in FOL

The extension of P in an interpretation is symmetric if and only if $\forall x \forall y (P(x, y) \rightarrow P(y, x))$ is true.

Transitivity

Definition

A relation R is **transitive** if whenever it holds between x and y and y and z, it also holds between x and z.

- x is the same age as y.
- Not: x and y share a parent.
- Not: x and y are siblings.
- $x \leq y$ and $x < y$.
- $x \mid y$.

Expressing transitivity in FOL

The extension of P in an interpretation is transitive if and only if $\forall x \forall y \forall z((P(x, y) \wedge P(y, z) \rightarrow P(x, z))$ is true.

Anti-Symmetry

Definition

A relation R is **anti-symmetric** if it never holds in both directions, except possibly for things being R-related to themselves.

- Not: x is the same age as y.
- Not: x and y share a parent.
- Not: x and y are siblings.
- $x \leq y$.
- $x \mid y$ but only on the natural numbers!

Expressing anti-symmetry in FOL

The extension of P in an interpretation is anti-symmetric if and only if $\forall x \forall y((P(x, y) \wedge P(y, x)) \rightarrow x = y)$ is true.

FOL and properties of relations

- A relation is **universal** iff $\forall x \forall y\, P(x, y)$.
- Every universal relation is also:
  - reflexive: $\forall x \forall y\, P(x, y) \models \forall x\, P(x, x)$
  - symmetric: $\forall x \forall y\, P(x, y) \models \forall x \forall y (P(x, y) \to P(y, x))$
  - transitive: $\forall x \forall y\, P(x, y) \models \forall x \forall y \forall z ((P(x, y) \land P(y, z)) \to P(x, z))$.
- But not vice versa!

FOL and properties of relations

- Relations can be symmetric and anti-symmetric at the same time:
- The following are jointly satisfiable:

$$\forall x \forall y (P(x, y) \rightarrow P(y, x))$$
$$\forall x \forall y ((P(x, y) \wedge P(y, x)) \rightarrow x = y)$$

- Relations can be transitive and symmetric without being reflexive:
- We have:

$$\begin{aligned} & \forall x \forall y \forall z ((P(x, y) \wedge P(y, z) \rightarrow P(x, z)) \\ & \forall x \forall y (P(x, y) \rightarrow P(y, x)) \\ \not\models \quad & \forall x \, P(x, x) \end{aligned}$$

XII. Functional completeness and normal forms

XII. Functional completeness and normal forms

a. Functional completeness

Truth functions

Definition

An (n–place) **truth function** t is a mapping of n–tuples of **T** and **F** to either **T** or **F**.

n–place truth functions correspond to truth tables of sentence S with n sentence letters $A_1, \ldots, A_n$.

A_1	A_2	$t_\wedge$	$A_1 \wedge A_2$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

A_1	A_2	$t_\vee$	$A_1 \vee A_2$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Truth functions

Definition

A sentence S containing the sentence letters $A_1, \ldots, A_n$ **expresses** the truth function t iff the truth value of S on the valuation which assigns v_i to A_i is $t(v_1, \ldots, v_n)$.

An n-place truth function is **expressible** if there is a sentence containing sentence letters $A_1, \ldots, A_n$ that expresses it.

Examples

A_1	A_2	t_1	S?
T	T	T	
T	F	T	
F	T	F	
F	F	F	

A_1 or: $A_1 \wedge (A_2 \vee \neg A_2)$

A_1	A_2	t_{xor}	S?
T	T	F	
T	F	T	
F	T	T	
F	F	F	

$(A_1 \vee A_2) \wedge \neg(A_1 \wedge A_2)$ or: $\neg(A_1 \leftrightarrow A_2)$

Functional completeness

Definition

A collection of connectives is **functionally complete** if every truth function is expressible by a sentence containing only those connectives.

Functional completeness: results

- Functionally complete are:
  - Connectives we know:

$$\wedge + \neg \qquad \vee + \neg \qquad \to + \neg \qquad \to + \bot$$

  - Any other set of connectives containing one of those.
  - Two two-place connectives by themselves: neither-not (NOR) and not-both (NAND).
- No other (sets of) one and two-place connectives are functionally complete.
- We'll prove this for $\wedge + \vee$.

XII. Functional completeness and normal forms

b. Proving connectives are functionally complete

$\wedge + \vee + \neg$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \wedge (A_2 \wedge A_3)) \vee$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \wedge (\neg A_2 \wedge \neg A_3)) \vee$
F	T	T	F	
F	T	F	T	$(\neg A_1 \wedge (A_2 \wedge \neg A_3)) \vee$
F	F	T	T	$(\neg A_1 \wedge (\neg A_2 \wedge A_3))$
F	F	F	F	

$\wedge + \vee + \neg$ are functionally complete

- Each line makes one, and only one, conjunction true, e.g.,
- $\neg A_1 \wedge A_2 \wedge \neg A_3$ is true in, and only in, line **F T F**.
- Combine using $\vee$: make S true in all (and only) the lines where it is supposed to be true.

The “neither. . . nor . . . ” connective: $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

$\downarrow$ is functionally complete

- We already know that $\neg + \wedge + \vee$ is functionally complete, i.e.,
- Every truth function can be expressed using only $\vee$, $\wedge$, $\neg$.
- To show $\downarrow$ is functionally complete, suffices to show that **every sentence containing only $\neg$, $\vee$, $\wedge$ is equivalent to one containing only $\downarrow$.**
- For that, it suffices to show that any negated sentence, conjunction, disjunction, can be expressed using only $\downarrow$.

Expressing $\neg$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- Note how $P \downarrow Q$ is **F** in the first line and **T** in the last (when P and Q have same truth value).
- So $P \downarrow P$ is **F** if P is **T**, and **T** if P is **F**, i.e.,

$$\neg P \Leftrightarrow (P \downarrow P).$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither … nor" connective, which can also be expressed as $\neg(P \vee Q)$, i.e.,

$$\neg(P \vee Q) \Leftrightarrow P \downarrow Q$$

- Negate both sides:

$$P \vee Q \Leftrightarrow \neg(P \downarrow Q)$$

- Apply what we figured out in last slide:

$$P \vee Q \Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$$

Expressing $\wedge$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\neg P \wedge \neg Q$, i.e.,

$$(\neg P \wedge \neg Q) \Leftrightarrow P \downarrow Q$$

- Equivalence holds for **all sentences** P, Q, so also if we replace P by $\neg R$ and Q by $\neg S$:

$$\neg\neg R \wedge \neg\neg S \Leftrightarrow (\neg R \downarrow \neg S)$$

- Express $\neg$ using $\downarrow$:

$$R \wedge S \Leftrightarrow (R \downarrow R) \downarrow (S \downarrow S)$$

Functionally complete connectives

- De Morgan's Law: $\wedge$ can be expressed by $\vee$ and $\neg$.
- Similarly: $\vee$ can be expressed by $\wedge$, $\neg$.
- So $\vee, \neg$ and $\wedge, \neg$ are functionally complete.
- $\rightarrow, \bot$ is functionally complete (HW).
- $\rightarrow, \neg$ is functionally complete.
- No other sets of connectives that don't contain one of these sets are functionally complete.
- "Neither ... nor" (NOR) is functionally complete by itself.
- "Not both" (NAND) connective is functionally complete by itself.
- No other 2-place connectives are functionally complete by themselves.

XII. Functional completeness and normal forms

c. Proving connectives aren’t functionally complete

$\wedge + \vee$ not functionally complete

- $\wedge + \vee$ is not functionally complete.
- Remember: To be functionally complete, **every** truth function would have to be expressible using only $\wedge$ and $\vee$.
- Which 2-place truth-functions can be expressed using $\wedge$ and $\vee$?
- Not this one:

		t_{xor}
T	T	F
T	F	T
F	T	T
F	F	F

Proof by induction

- Sometimes need to prove something for **all** sentences.
- E.g., "every sentence containing only $\wedge$ and $\vee$ expresses a truth function **other than** t_{xor}."
- Proof by **induction**:
  - Show that it holds for **sentence letters** (and $\perp$).
  - Suppose sentences $\mathscr{P}$, $\mathbb{Q}$ have the property.
  - Now show that it then also holds for $(\mathscr{P} \wedge \mathbb{Q})$, $(\mathscr{P} \vee \mathbb{Q})$, etc.
- Why does this work?
- This is how we form sentences (involving only $\wedge$, $\vee$).
- Property "S is a sentence expressing a truth function other than t_{xor}" propagates from atomic sentences to all sentences.

Proof by induction: example

Theorem

Every sentence contains an even number of parentheses.

- Every atomic sentence contains an even number of parentheses:

$$B \qquad \perp$$

- If $\mathscr{P}$ contains an even number of parentheses, so does $\neg\mathscr{P}$.
- If $\mathscr{P}$ and $\mathbb{Q}$ both contain an even number of parentheses, so do

$$(\mathscr{P} \land \mathbb{Q}), (\mathscr{P} \lor \mathbb{Q}), (\mathscr{P} \to \mathbb{Q}), (\mathscr{P} \leftrightarrow \mathbb{Q}).$$

$\wedge + \vee$ not functionally complete

Theorem

Any sentence containing only A_1, A_2, $\wedge$, $\vee$ has a **T** *in the first line of its truth table.*

- Sentence letters: truth table of A_i is just copy of column under A_i, so has **T** in first line where valuation assigns **T** to A.
- Suppose $\mathscr{P}$, $\mathbb{Q}$ are sentences which contain only A_1, A_2, $\wedge$, $\vee$ and are true in first line.
- $(\mathscr{P} \wedge \mathbb{Q})$ is true in first line, since **T** $\wedge$ **T** makes **T**.
- $(\mathscr{P} \vee \mathbb{Q})$ is true in first line, since **T** $\vee$ **T** makes **T**.

$\wedge + \vee$ not functionally complete

Theorem

Any sentence containing only A_1, A_2, $\wedge$, $\vee$ expresses a truth function t with $t(\mathbf{T}, \mathbf{T}) = \mathbf{T}$.

- Sentence letters: A_1, A_2: express t_1, t_2.
- Suppose $\mathscr{P}$, $\mathbb{Q}$ are sentences which contain only A_1, A_2, $\wedge$, $\vee$ and express truth functions t, t' with $t(\mathbf{T}, \mathbf{T}) = t'(\mathbf{T}, \mathbf{T}) = \mathbf{T}$
- $(\mathscr{P} \wedge \mathbb{Q})$ expresses truth function s with

$$s(\mathbf{T}, \mathbf{T}) = t_\wedge(t(\mathbf{T}, \mathbf{T}), t'(\mathbf{T}, \mathbf{T})) = \mathbf{T}$$

- $(\mathscr{P} \vee \mathbb{Q})$ expresses truth function s with

$$s(\mathbf{T}, \mathbf{T}) = t_\vee(t(\mathbf{T}, \mathbf{T}), t'(\mathbf{T}, \mathbf{T})) = \mathbf{T}$$

Non-functionally complete connectives

- We've shown that $\wedge + \vee$ are not functionally complete.
- Same idea shows that $\rightarrow$ and $\leftrightarrow$ not functionally complete.
- When we add $\neg$ things get interesting:
  - Functionally complete:

$$\neg + \vee \qquad \neg + \wedge \qquad \neg + \rightarrow$$

  - Not functionally complete:

$$\neg + \leftrightarrow$$

  (harder to prove).

XII. Functional completeness and normal forms

d. Normal forms

Normal forms

- Sometimes interested in sentences that have specific **form**, e.g.,
- Negations apply only to sentence letters.
- Alternation between $\wedge$ and $\vee$ is minimal.
- Useful for applications:
  - Combinational circuits.
  - SAT solvers and theorem provers need inputs in CNF.
  - Complexity theory talks about problems involving sentences in normal form.

Scope of a connective

Definition

The **scope** of an occurrence of a connective in a sentence is that sub–sentence of which the connective is the main connective.

$$(\underbrace{\neg(A \vee B)}_{\text{scope of } \neg} \vee \underbrace{((A \to B) \wedge (B \to C))}_{\text{scope of } \wedge})$$

Disjunctive normal form

DNF

A sentence is in **disjunctive normal form** (DNF) iff it:

- contains only $\wedge$, $\vee$, $\neg$;
- only sentence letters are in scope of $\neg$;
- only sentence letters, $\wedge$, and $\neg$ are in scope of $\wedge$.

In other words: DNF are disjunctions of conjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \wedge \neg B) \vee ((\neg A \wedge C) \vee (B \wedge C))$$

$$\neg A \vee (B \wedge C)$$

$$A \wedge (B \wedge C)$$

$$A \vee (B \vee C)$$

DNF theorem

Theorem

Every sentence is equivalent to one in disjunctive normal form.

Proof.

- Construct truth table.
- Apply method we used to show $\wedge + \vee + \neg$ is functionally complete.
- This gives us a sentence involving only $\wedge$, $\vee$, $\neg$ with same truth table, i.e., is equivalent in TFL.
- That sentence is always in DNF.

$\square$

$\wedge + \vee + \neg$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	**T**	**T**	**T**	$(A_1 \wedge (A_2 \wedge A_3))\vee$
T	**T**	**F**	**F**	
T	**F**	**T**	**F**	
T	**F**	**F**	**T**	$(A_1 \wedge (\neg A_2 \wedge \neg A_3))\vee$
F	**T**	**T**	**F**	
F	**T**	**F**	**T**	$(\neg A_1 \wedge (A_2 \wedge \neg A_3))\vee$
F	**F**	**T**	**T**	$(\neg A_1 \wedge (\neg A_2 \wedge A_3))$
F	**F**	**F**	**F**	

Conjunctive normal form

CNF

A sentence is in **conjunctive normal form** (CNF) if it:

- contains only $\wedge$, $\vee$, $\neg$;
- only sentence letters are in scope of $\neg$;
- only sentence letters, $\vee$, and $\neg$ are in scope of $\vee$.

In other words: CNF are conjunctions of disjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \vee \neg B) \wedge ((\neg A \vee C) \wedge (B \vee C))$$

$$\neg A \wedge (B \vee C)$$

$$A \vee (B \vee C)$$

$$A \wedge (B \wedge C)$$

CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A if A is assigned **F**.
  - Write $\neg A$ if A is assigned **T**.
- Put $\wedge$'s between all of them.
- Resulting is true iff the original sentence is true, and is in CNF.

□

CNF from truth table

A_1	A_2	A_3	S	CNF
T	**T**	**T**	**T**	
T	**T**	**F**	**F**	$(\neg A_1 \vee (\neg A_2 \vee A_3)) \wedge$
T	**F**	**T**	**F**	$(\neg A_1 \vee (A_2 \vee \neg A_3)) \wedge$
T	**F**	**F**	**T**	
F	**T**	**T**	**F**	$(A_1 \vee (\neg A_2 \vee \neg A_3)) \wedge$
F	**T**	**F**	**T**	
F	**F**	**T**	**T**	
F	**F**	**F**	**F**	$(A_1 \vee (A_2 \vee A_3))$

XII. Functional completeness and normal forms

e. Equivalent transformations

Transformation equivalences

Defining $\to$, $\leftrightarrow$ (Cond, Bicond)

$$(\mathscr{P} \to \mathbb{Q}) \Leftrightarrow (\neg\mathscr{P} \lor \mathbb{Q})$$

$$\neg(\mathscr{P} \to \mathbb{Q}) \Leftrightarrow (\mathscr{P} \land \neg\mathbb{Q})$$

$$(\mathscr{P} \leftrightarrow \mathbb{Q}) \Leftrightarrow (\mathscr{P} \to \mathbb{Q}) \land (\mathbb{Q} \to \mathscr{P})$$

Double negation (DN)

$$\neg\neg\mathscr{P} \Leftrightarrow \mathscr{P}$$

Transformation equivalences

De Morgan's Laws (DeM):

$$\neg(\mathscr{P} \vee \mathscr{Q}) \Leftrightarrow (\neg\mathscr{P} \wedge \neg\mathscr{Q})$$

$$\neg(\mathscr{P} \wedge \mathscr{Q}) \Leftrightarrow (\neg\mathscr{P} \vee \neg\mathscr{Q})$$

Commutativity (Comm):

$$\mathscr{P} \vee \mathscr{Q} \Leftrightarrow \mathscr{Q} \vee \mathscr{P}$$

$$\mathscr{P} \wedge \mathscr{Q} \Leftrightarrow \mathscr{Q} \wedge \mathscr{P}$$

Distributivity (Dist):

$$\mathscr{P} \vee (\mathscr{Q} \wedge \mathscr{R}) \Leftrightarrow (\mathscr{P} \vee \mathscr{Q}) \wedge (\mathscr{P} \vee \mathscr{R})$$

$$\mathscr{P} \wedge (\mathscr{Q} \vee \mathscr{R}) \Leftrightarrow (\mathscr{P} \wedge \mathscr{Q}) \vee (\mathscr{P} \wedge \mathscr{R})$$

Transforming sentences into DNF/CNF

- Replace any subsentence of the form $(\mathscr{P} \to \mathbb{Q})$, $(\mathscr{P} \leftrightarrow \mathbb{Q})$ by its equivalent.
- Use De Morgan's laws to place $\neg$'s in front of sentence letters
- Remove double negations.
- Use distributivity and commutativity to ensure
  - DNF: no $\vee$ is in the scope of $\wedge$.
  - CNF: no $\wedge$ is in the scope of $\vee$.

Transforming sentences into CNF/DNF

$\neg[(A \leftrightarrow B) \vee \neg(B \rightarrow C)]$	
$\neg[((A \rightarrow B) \wedge (B \rightarrow A)) \vee \neg(B \rightarrow C)]$	Bicond
$\neg((A \rightarrow B) \wedge (B \rightarrow A)) \wedge \neg\neg(B \rightarrow C)$	DeM
$(\neg(A \rightarrow B) \vee \neg(B \rightarrow A)) \wedge \neg\neg(B \rightarrow C)$	DeM
$((A \wedge \neg B) \vee \neg(B \rightarrow A)) \wedge \neg\neg(B \rightarrow C)$	Cond
$((A \wedge \neg B) \vee (B \wedge \neg A)) \wedge \neg\neg(B \rightarrow C)$	Cond
$((A \wedge \neg B) \vee (B \wedge \neg A)) \wedge (B \rightarrow C)$	DN
$((A \wedge \neg B) \vee (B \wedge \neg A)) \wedge (\neg B \vee C)$	Cond

Transforming sentences into DNF

$[(A \wedge \neg B) \vee (B \wedge \neg A)] \wedge (\neg B \vee C)$

$([(A \wedge \neg B) \vee (B \wedge \neg A)] \wedge \neg B) \vee ([(A \wedge \neg B) \vee (B \wedge \neg A)] \wedge C)$ Dist

$(\neg B \wedge [(A \wedge \neg B) \vee (B \wedge \neg A)]) \wedge ([(A \wedge \neg B) \vee (B \wedge \neg A)] \wedge C)$ Comm

$([\neg B \wedge (A \wedge \neg B)] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \vee (B \wedge \neg A)] \wedge C)$ Dist

$([\neg B \wedge (A \wedge \neg B)] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$ Dist

XII. Functional completeness and normal forms

f. Simplification

Simplification equivalences

Associativity (Assoc):

$$\mathscr{P} \vee (\mathbb{Q} \vee \mathscr{R}) \Leftrightarrow (\mathscr{P} \vee \mathbb{Q}) \vee \mathscr{R} \qquad \mathscr{P} \wedge (\mathbb{Q} \wedge \mathscr{R}) \Leftrightarrow (\mathscr{P} \wedge \mathbb{Q}) \wedge \mathscr{R}$$

Idempotence (Id):

$$(\mathscr{P} \vee \mathscr{P}) \Leftrightarrow \mathscr{P} \qquad (\mathscr{P} \wedge \mathscr{P}) \Leftrightarrow \mathscr{P}$$

Absorption (Abs):

$$\mathscr{P} \wedge (\mathscr{P} \vee \mathbb{Q}) \Leftrightarrow \mathscr{P} \qquad \mathscr{P} \vee (\mathscr{P} \wedge \mathbb{Q}) \Leftrightarrow \mathscr{P}$$

Simplification (Simp):

$$\mathscr{P} \wedge (\mathbb{Q} \vee \neg\mathbb{Q}) \Leftrightarrow \mathscr{P} \qquad \mathscr{P} \vee (\mathbb{Q} \wedge \neg\mathbb{Q}) \Leftrightarrow \mathscr{P}$$

$$\mathscr{P} \vee (\mathbb{Q} \vee \neg\mathbb{Q}) \Leftrightarrow (\mathbb{Q} \vee \neg\mathbb{Q}) \qquad \mathscr{P} \wedge (\mathbb{Q} \wedge \neg\mathbb{Q}) \Leftrightarrow (\mathbb{Q} \wedge \neg\mathbb{Q})$$

Simplifying sentences

$([\neg B \wedge (A \wedge \neg B)] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	
$([\neg B \wedge (\neg B \wedge A)] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Comm
$([(\neg B \wedge \neg B) \wedge A] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Assoc
$([\neg B \wedge A] \vee [\neg B \wedge (B \wedge \neg A)]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Id
$([\neg B \wedge A] \vee [(\neg B \wedge B) \wedge \neg A]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Assoc
$([\neg B \wedge A] \vee [\neg B \wedge B]) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Simp
$(\neg B \wedge A) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Simp
$(A \wedge \neg B) \vee ([(A \wedge \neg B) \wedge C] \vee [(B \wedge \neg A) \wedge C])$	Comm
$((A \wedge \neg B) \vee [(A \wedge \neg B) \wedge C]) \vee [(B \wedge \neg A) \wedge C]$	Assoc
$(A \wedge \neg B) \vee [(B \wedge \neg A) \wedge C]$	Abs

XIII. Further topics

XIII. Further topics

a. History of logic

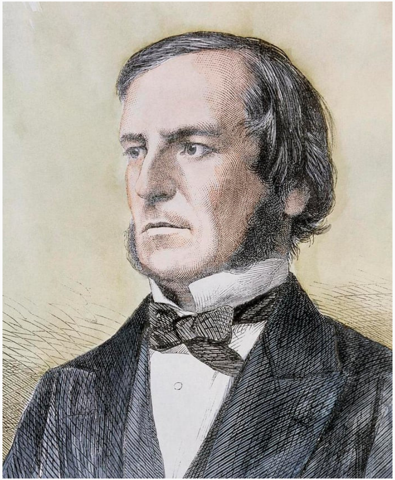
The beginnings

- Rules of debate & rhetoric
- Ancient India: Gautama, **Nyayasutra** (600 BCE–200 CE)
- Ancient Greece: Aristotle (384-322 BCE)
- Cataloged valid arguments ("syllogisms"), e.g.,
- All ungulates have hooves.
  No fish have hooves.
  ∴ No fish are ungulates.

The middle ages

- Ibn Sīnā (Avicenna)
- Pierre Abelard
- William Ockham
- Jean Buridan

Mathematical logic

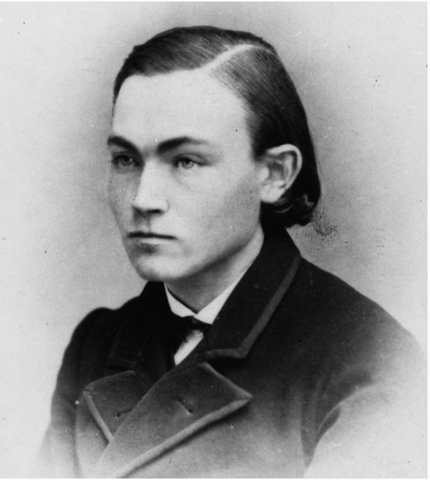


- George Boole
- John Venn
- Augustus De Morgan
- Charles Lutwidge Dodgson
  (aka Lewis Caroll)

Modern logic: Peirce at al

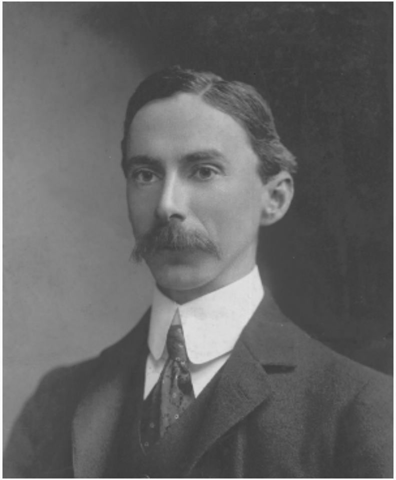
- Charles Sanders Peirce
- Christine Ladd Franklin
- Ernst Schröder

Modern logic: Gottlob Frege



- 1848-1925
- Predicates and quantifiers
- Plan to turn all of math into theorems of logic alone

Modern logic: Bertrand Russell



- 1870-1972
- Showed Frege’s system contradictory (1902)
- Fixed it (*Principia mathematica* 1910-13, 3 vols.)
- Plan to turn all of math into theorems of logic alone

Modern logic: David Hilbert

- 1862-1943
- Combined Russell's and Schröder's systems
- First modern logic textbook
- Plan to turn all of math into consequences of a single set of premises

Modern logic: Kurt Gödel

- 1906-1978
- Showed that every valid argument has a proof
- Showed that Frege/Russell's and Hilbert's plans can't work

Modern logic: Alan Turing

- 1912–1954
- Showed that unlike TFL, FOL has no decision procedure
- Invented Turing machines ("father of computer science")

Modern logic: Gerhard Gentzen

- 1909-1945
- Invented natural deduction
- Founded theory of proofs

Modern logic: modal logic

- Extend logic with operators for "possible" and "necessary"
- Pioneered by philosophers, now used by computer scientists
- Rudolf Carnap, Saul Kripke, Ruth Barcan Marcus

XIII. Further topics

b. Philosophy and nonstandard logics

Validity and validity in FOL

- Philosophers interested in **valid arguments**
- Definition: There is no case where the premises are true and the conclusion is false
  - **Important:** It does not say “it **isn’t in fact the case** that the premises are true and the conclusion is false”
  - That would make every argument with
    - true premises, true conclusion
    - false premises, true conclusion
    - false premises, false conclusion

    valid. But that’s not the case.
  - It says “it is **impossible** that the premises could be true and the conclusion false!”
- Difficulty: What logically possible circumstances are there?

What logic does for validity

- Truth-tables, interpretations, proofs give **sufficient conditions** for validity, i.e.,
  - Every argument valid in TFL is valid
  - Every argument valid in FOL is valid
  - Every argument with a formal proof is valid (soundness!)

Nonstandard logics

- Formal models of logical consequence make a number of simplifying assumptions:
  - Only **determinate** properties allowed, e.g, no vague properties
  - Every (atomic) sentence either **T** or **F**; not both and nothing in between
  - Every name must refer, i.e., no empty names
  - Only truth-functional connectives, e.g., no subjunctive contionals, "because", or tenses
- Non-standard logics: expand TFL, FOL to deal with these

Many-valued logic

- Add to the truth-values **T** and **F**, e.g.,
  - "Undetermined": neither true nor false

P	$\neg P$
T	**F**
U	**U**
F	**T**

P	Q	$(P \wedge Q)$
T	**T**	**T**
T	**U**	**U**
T	**F**	**F**
U	**T**	**U**
U	**U**	**U**
U	**F**	**F**
F	**T**	**F**
F	**U**	**F**
F	**F**	**F**

P	Q	$(P \vee Q)$
T	**T**	**T**
T	**U**	**T**
T	**F**	**T**
U	**T**	**T**
U	**U**	**U**
U	**F**	**U**
F	**T**	**T**
F	**U**	**U**
F	**F**	**F**

  - "Inconsistent": both true and false
  - Fuzzy truth values: any number between 0 and 1

Truth-functional connectives

Definition

A connective $*$ is **truth functional** iff the truth value of $*A$ depends only on the truth value of A.

- "It is not the case that" is truth functional.
- So are "and", "or", "neither nor".
- "If ... then": iffy.

Non-truth-functional connectives

- “Possibly”, “Necessarily”
- Subjunctive conditionals
- Tenses: “Is always true,” “Will be true,” “Was true”
- “Richard believes that”, “Richard knows that”

Possibly

- "It is possible that ...", "Possibly, ..."
- Consider:
  1. It is possible that I will live forever.
  2. It is possible that $2 + 2 = 5$.
- (1) is true and (2) is false.
- But A_1 = "I will live forever" and A_2 = "$2 + 2 = 5$" are both false.
- So "It is possible that A" can't just depend on the truth value of A
- Otherwise (1) and (2) would have to have the same truth value.

Subjunctive Conditionals

- Subjunctive conditionals = if—then statements in **subjunctive** mood
- “If P were true, then Q would be true.”
- Indicative conditional is (plausibly) **truth-functional**: truth value of “If P, then Q” depends **only on truth values** of P and Q.

Subjunctive Conditionals

- Subjunctive conditional is not truth functional
- E.g., consider:
  1. If the world were just, no evil deed would go unpunished.
     P_1 = the world is just
     Q_1 = no evil deed goes unpunished
  2. If the world were flat, no evil deed would go unpunished.
     P_2 = the world is flat
     Q_2 = no evil deed goes unpunished
- P_1, Q_1 both false; P_2, Q_2 both false, but
- (1) is true, but (2) is false

Modal logic

- Alethic logic: “It is possible that” ($\Diamond$), “it is necessary that” ($\Box$)

$$\Box A \to A \qquad \Diamond\Box A \to \Box A$$

- Epistemic logic: “Richard knows that” (K)

$$\mathbf{K}\,A \to A \qquad \mathbf{K}\,A \to \mathbf{K}\mathbf{K}\,A$$

- Conditional logic

  Subjunctive conditionals, “if it were true that . . . , then it would be true that — —” ($\Box\!\!\rightarrow$)

$$(A \mathrel{\Box\!\!\rightarrow} B) \to (A \to B)$$

- Temporal logic

  “It was true that” (P), “It will be true that” (F)

$$\mathbf{F}\,\mathbf{P}\,A \to (\mathbf{P}\,A \vee A \vee \mathbf{F}\,A)$$

Temporal logic

- "Always A": $\Box A$
- "Sometimes A": $\Diamond A$
- If always A, then A (now): $\Box A \to A$
- If A (now), then sometimes A: $A \to \Diamond A$
- Always A iff not sometimes not A
  $\Box A \leftrightarrow \neg\Diamond\neg A$.
- If always A and B, then always A or always B:
  $\Box(A \land B) \to (\Box A \land \Box B)$
- If always A or B, then always A or always B: $\Box(A \lor B) \to (\Box A \lor \Box B)$

XIII. Further topics

c. Metalogic and applications

Semantics

- A **truth-value assignment** is an assignment of **T** or **F** to the sentence letters (schematic letters in the truth-functional form)
- An **interpretation** is a non-empty domain together with
  - extensions for each predicate symbol
  - objects in the domain for each name
- A tautology is a sentence which is true in all truth-value assignments
- A validity is a sentence that's true in all interpretations

Soundness and completeness

- Soundness
  Arguments have formal proofs **only if** they are valid
  If there is a proof of B from premises $A_1, \ldots A_n$, then $A_1, \ldots A_n$ entail B in FOL.
- Completeness
  Arguments have formal proofs **if** they are valid
  If $A_1, \ldots A_n$ entail B in FOL, then there is a proof of B from premises $A_1, \ldots A_n$

Proved by Kurt Gödel (1929)

Church–Turing Theorem

Instance: Sentence A of FOL
Problem: Is A a validity/provable?

- Undecidable: no computer program can answer this question correctly for all A.
- Proved independently by Alonzo Church and Alan Turing in 1935

Cook's Theorem

Instance: Sentence A of TFL
Problem: Is A a tautology?

- Decidable: write a computer program that checks all valuations for A.
- But: it's hard: "co-NP complete"
- Proved independently by Stephen Cook (1971) and Leonid Levin (1973)

Decidable classes

- The decision problem **in general** is undecidable
- But special cases **can** be decided, e.g.:

Instance: Sentence A with only 1-place predicate symbols
Problem: Is A a validity?

- Decidable
- Proved by Leopold Löwenheim (1915)
- Complexity is NEXPTIME-complete.

Theories

- A set of sentences of FOL also called a **theory**, and the sentences in it **axioms**
- Some (types of) interpretations can be characterized as those interpretations in which every sentence in the theory is true
- Examples:
  - Mathematical theories (theory of orders, group theory, arithmetic)
  - KR classification systems, e.g., SNOMED-CT
  - Mereology, theories of truth, scientific theories

The axiomatic method

- Theories + logic: what follows from axioms?
- Axiomatic method: do science by investigating what follows from the axioms of a theory
- Logic can also determine:
  - Are axioms (in)consistent?
  - Are axioms independent, or is one superfluous?
- Paradigm of axiomatic method: geometry (Euclid)

Examples of theories: linear orders

A relation $\preceq$ on a set O is a **linear order** iff it makes following axioms true:

$$\forall x \forall y ((x \preceq y \land y \preceq x) \to x = y) \qquad \text{Antisymmetry}$$

$$\forall x \forall y \forall z ((x \preceq y \land y \preceq z) \to x \preceq z) \qquad \text{Transitivity}$$

$$\forall x \forall y (x \preceq y \lor y \preceq x) \qquad \text{Totality}$$

Every total relation is reflexive:

$$LO \models \forall x\, x \preceq x$$

Examples of theories: Robinson's Q

Theories of arithmetic, such as Robinson's theory Q:

$$\neg\exists x\,(x+1)=0$$
$$\forall x(x=0 \vee \exists y\,(y+1)=x)$$
$$\forall x\forall y((x+1)=(y+1)\to x=y)$$
$$\forall x\,(x+0)=x$$
$$\forall x\forall y\,(x+(y+1))=((x+y)+1)$$
$$\forall x\,(x\times 0)=0$$
$$\forall x\forall y\,(x\times(y+1))=((x\times y)+x)$$

Examples of theories: SNOMED-CT

```
bacterial pneumonia =
         is-a|bacterial infectious disease
         is-a|infective pneumonia
         causative agent|bacteria
         finding site|lung structure
```

$$
\begin{aligned}
&\forall x(\mathit{BacterialPneumonia}(x) \leftrightarrow \\
&\quad \mathit{BacterialInfectiousDisease}(x) \wedge \\
&\quad \mathit{InfectivePneumonia}(x) \wedge \\
&\quad \exists y(\mathit{HasCausativeAgent}(x, y) \wedge \mathit{Bacteria}(y)) \wedge \\
&\quad \exists y(\mathit{HasFindingSite}(x, y) \wedge \mathit{LungStructure}(y)))
\end{aligned}
$$

Examples of theories: SNOMED-CT

- Over 300,000 concepts (predicate symbols), e.g.,
  - 1-place predicates:
    parts of body, findings, organisms, physical objects, procedures, substances, diseases, ...
  - 2-place predicates:
    has finding site, has causative agent, with method, has active ingredient, laterality is, using device, ...
- About 1,000,000 descriptions (axioms)
- SNOMED-CT is decidable

Examples of Theories: Mereology

- Mereology: the theory of the part–whole relation (**metaphysics**)
- Primitive relation: $Pt(x, y)$, "x is a part of y"
- Some axioms:

$$\forall x\, Pt(x, x) \qquad \text{Reflexivity}$$

$$\forall x \forall y \forall z((Pt(x, y) \wedge Pt(y, z)) \rightarrow Pt(x, z)) \qquad \text{Transitivity}$$

$$\forall x \forall y((Pt(x, y) \wedge Pt(y, x)) \rightarrow x = y) \qquad \text{Antisymmetry}$$

Examples of Theories: Mereology

- Defined properties and relations

$$PP(x, y) \leftrightarrow (Pt(x, y) \wedge \neg x = y)$$
$$At(x) \leftrightarrow \neg \exists y\, PP(y, x)$$

- Different theories settle questions differently, e.g.,
  - Are there atoms?
  - Does everything comprise at least one atom?
  - Is everything made of atomless "gunk"?

Property theories and Grelling's paradox

- Primitive relation: $Ap(x, y)$, "x applies to y"
- Proposed axiom ("comprehension"): For any formula $P(y)$,

$$\exists x \forall y (Ap(x, y) \leftrightarrow P(y))$$

- Axiom is **inconsistent** (contradictory)
- A property is **heterological** if it does not apply to itself, i.e., $\neg Ap(x, x)$
- Is the property of being heterological itself heterological? Yes and no!

$$\begin{array}{|l}
\exists x \forall y (Ap(x, y) \leftrightarrow \neg Ap(y, y)) \\
\hline
\bot
\end{array}$$

Completeness of theories

- A theory T is **complete** if for every sentence A in its language, either $T \models A$ or $T \models \neg A$
- Every complete theory is decidable!
- Some incomplete theories are still decidable (e.g., LO)
- Some incomplete theories are incomplete**able**: no consistent extension is complete
- **Gödel's Incompleteness Theorem (1930)**
  Arithmetic, set theory, mereology are incompleteable
- Philosophical upshot of this: truth in the intended interpretation(s) of the theory outstrips provability from the theory