10. Proofs in QL

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a. Some Equivalences in QL

Motivation: Proving Equivalences

- We'd like to have a method for proving that two sentences are equivalent, or that an argument is valid
- Models only provide counterexamples to equivalence or validity
- By extending our natural deduction system to cover QL, we'll be able to prove two sentences are equivalent!
  - Derive one from the other and vice versa
  - Basically: show that the biconditional of the two sentences is a tautology

Some Equivalences: Quantifiers commuting over connectives

1. $Fa \mathbin{\&} (\exists x)Gx$ is equivalent to $(\exists x)(Fa \mathbin{\&} Gx)$
2. $Fa \mathbin{\&} (\forall x)Gx$ is equivalent to $(\forall x)(Fa \mathbin{\&} Gx)$
3. $Fa \vee (\exists x)Gx$ is equivalent to $(\exists x)(Fa \vee Gx)$
4. $Fa \vee (\forall x)Gx$ is equivalent to $(\forall x)(Fa \vee Gx)$
5. $Fa \supset (\exists x)Gx$ is equivalent to $(\exists x)(Fa \supset Gx)$
6. $Fa \supset (\forall x)Gx$ is equivalent to $(\forall x)(Fa \supset Gx)$
   - But note that '$\supset$' is not symmetric, so we have to examine the converse: e.g. $(\exists x)Gx \supset Fa$!
   - And there are no analogous rules for biconditionals

Quantifiers NOT commuting over some conditionals

'$\supset$' is not symmetric, so we have to examine the converse

bad: $(\exists x)Gx \supset Fa$ is NOT equivalent to $(\exists x)(Gx \supset Fa)$

Instead, the existential converts to a universal when we scope over the consequent:

7. $(\exists x)Gx \supset Fa$ is equivalent to $(\forall x)(Gx \supset Fa)$

bad: $(\forall x)Gx \supset Fa$ is NOT equivalent to $(\forall x)(Gx \supset Fa)$

Instead, the universal converts to an existential when we scope over the consequent:

8. $(\forall x)Gx \supset Fa$ is equivalent to $(\exists x)(Gx \supset Fa)$

Informal Argument: If anyone G's, then c F's

7. $(\exists x)Gx \supset Fc$ is equivalent to $(\forall x)(Gx \supset Fc)$

- Recall: "Any" in antecedents but **without** pronouns referring back to them are **existential**:

  If **anyone** is a hero, Greta is.

  Roughly: if there are heroes (at all), Greta is a hero.

  $(\exists x)\, Hx \supset Hg$

- Intuitively: for everybody, if they are a hero, then Greta is a hero

  $(\forall x)(Hx \supset Hg)$

- We would like to show that if we can derive one of these sentences, then we can derive the other.

  Our proof system will show this *naturally*!

Informal Argument: If everyone G's, then c F's

8. $(\forall x)Gx \supset Fc$ is equivalent to $(\exists x)(Gx \supset Fc)$

- By truth conditions for '$\supset$', first sentence is equivalent to:

  $\sim(\forall x)Gx \vee Fc$

  Apply de Morgan's: $(\exists x){\sim}Gx \vee Fc$

- Now apply our rule 3: existential commutes over disjunction (provided the disjunct doesn't contain the bound variable x)

  $(\exists x)({\sim}Gx \vee Fc)$, and apply '$\supset$' truth conditions again:

- Yields $(\exists x)(Gx \supset Fc)$

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b. Rules for ∀

Rules for formal proofs

- Need rules for $\forall$ and $\exists$ for formal proofs
- Formal proofs now more important, because no alternative (truth–table method)
- Intro and Elim rules should be
  - simple
  - elegant (not involve other connectives or quantifiers)
  - yield only valid arguments

Candidates for rules

- Only simple sentence close to $(\forall x)\,\mathcal{A}x$ is $\mathcal{A}c$
- Gives simple, elegant $\forall$E rule:

$$\begin{array}{r|ll} k & (\forall x)\,\mathcal{A}x & \\ & & \\ & \mathcal{A}c & :k\ \forall\mathrm{E} \end{array}$$

- This is a good rule: $(\forall x)\,\mathcal{A}x \models \mathcal{A}c$.

Candidates for rules

- Problem: corresponding “intro rule” isn’t valid:

$$\begin{array}{r|l} k & \mathcal{A}c \\ & \\ & (\forall x)\,\mathcal{A}x \qquad :k \end{array}$$
(doesn’t follow)

- Diagnosis: the c in $\mathcal{A}c$ is a name for a **specific object**.
- We need a name for an **arbitrary, unspecified object**.
- If $\mathcal{A}c$ is true for whatever c **could** name, then $\mathcal{A}x$ is satisfied by **every** object.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.

  Only people who wear capes admire Greta.

  ∴ All heroes wear capes.

  Proof: Let Carl be any hero.
  Since all heroes admire Greta, Carl admires Greta.
  Since only people who wear capes admire Greta, Carl wears a cape. But “Carl” stands for **any** hero.
  So all heroes wear capes.

Universal generalization

$$\begin{array}{r|ll} k & \mathcal{A}c & \\ & & \\ & (\forall \chi)\, \mathcal{A}\chi & :k\ \forall\mathrm{I} \end{array}$$

- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathcal{A}\chi$ is obtained from $\mathcal{A}c$ by replacing **all** occurrences of c by χ.
- In other words, c must also not occur in $\forall\chi\, \mathcal{A}\chi$.

General conditional proof

Proving "All As are Bs"

k	$\quad Ac$	:AS for $\supset$I
l	$\quad Bc$	
$l+1$	$Ac \supset Bc$	:k-l $\supset$I
$l+2$	$(\forall x)(Ax \supset Bx)$	:$l+1$ $\forall$I

Example

All heroes admire Greta.

Only people who wear capes admire Greta.

$\therefore$ All heroes wear capes.

$(\forall x)(Hx \supset Axg)$

$(\forall x)(Axg \supset Cx)$

$\therefore$ $(\forall x)(Hx \supset Cx)$

Let's do it on Carnap (PP10.2)!

Example

1	$(\forall x)(Hx \supset Axg)$	:PR
2	$(\forall x)(Axg \supset Cx)$	:PR
3	Hc	:AS for $\supset$I
4	$Hc \supset Acg$	:1 $\forall$E
5	Acg	:4, 3 $\supset$E
6	$Acg \supset Cc$	:2 $\forall$E
7	Cc	:6, 5 $\supset$E
8	$Hc \supset Cc$	:3-7 $\supset$I
9	$(\forall x)(Hx \supset Cx)$	:8 $\forall$I

Example

1	$(\forall x)\, Ax \vee (\forall x)\, Bx$	:PR
2	$\quad (\forall x)\, Ax$	:AS for $\vee$E
3	$\quad Ac$	:2 $\forall$E
4	$\quad Ac \vee Bc$	:3 $\vee$I
5	$\quad (\forall x)\, Bx$	:AS for $\vee$E
6	$\quad Bc$	:5 $\forall$E
7	$\quad Ac \vee Bc$	:6 $\vee$I
8	$Ac \vee Bc$	:1, 2–4, 5–7 $\vee$E
9	$(\forall x)(Ax \vee Bx)$	:8 $\forall$I

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c. Rules for ∃

Rules for $\exists$

- If we know of a specific object that it satisfies $\mathcal{A}x$, we know that at least one object satisfies $\mathcal{A}x$.
- So this rule is valid:

$$
\begin{array}{r|ll}
k & \mathcal{A}c & \\
\hline
 & (\exists \chi)\,\mathcal{A}\chi & : k\ \exists\mathrm{I}
\end{array}
$$

Arbitrary objects again

- Problem: corresponding "elimination rule" isn't valid:

$$\begin{array}{r|ll} k & (\exists x)\,\mathcal{A}x & \\ & \mathcal{A}c & :k \end{array}$$ **doesn't follow from**

- If we know that $(\exists x)\,\mathcal{A}x$ is true, we know that **at least one** object satisfies $\mathcal{A}x$, but not which one(s).
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathcal{A}(x)$.

Reasoning from existential information

- To use $(\exists x)\mathcal{A}x$, pretend the x has a name c, and reason from $\mathcal{A}(c)$.
- This is what we do to reason informally from existential information, e.g.,

  There are heroes who wear capes.

  Anyone who wears a cape admires Greta.

  $\therefore$ Some heroes admire Greta.

  Proof: We know there are heroes who wear capes.
  Let Cate be an arbitrary one of them.
  So Cate wears a cape. Since anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

Existential elimination (mind the restriction!)

- If
  - we know that some object satisfies $\mathcal{A}x$,
  - we hypothetically assume that c is one of them (i.e., assume $\mathcal{A}c$),
  - and we can prove that $\mathcal{B}$ follows from this assumption,

  then $\mathcal{B}$ follows already from $(\exists x)\,\mathcal{A}x$.
- Rule for existential elimination:

k	$(\exists x)\,\mathcal{A}x$	
m	$\quad\vert\ \mathcal{A}c$	:AS for $\exists$E
n	$\quad\vert\ \mathcal{B}$	
	$\mathcal{B}$	:k, m-n $\exists$E

- c is special: c **must NOT appear outside subproof**

Example

There are heroes who wear capes.

Anyone who wears a cape admires Greta.

∴ Some heroes admire Greta.

$(\exists x)(Hx \,\&\, Cx)$

$(\forall x)(Cx \supset Axg)$

$\therefore (\exists x)(Hx \,\&\, Axg)$

Example (PP10.5)

1	$(\exists x)(Hx \,\&\, Cx)$	:PR
2	$(\forall x)(Cx \supset Axg$	:PR
3	$\quad Hc \,\&\, Cc$	:AS for $\exists$E
4	$\quad Cc$	:3 & E
5	$\quad Cc \supset Acg$	:2 $\forall$E
6	$\quad Acg$	:4, 5 $\supset$E
7	$\quad Hc$	:3 & E
8	$\quad Hc \,\&\, Acg$	:4, 7 & I
9	$\quad (\exists x)(Hx \,\&\, Axg)$	:8 $\exists$I
10	$(\exists x)(Hx \,\&\, Axg)$	:1, 3-9 $\exists$I