13. Metalogic for QL

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13. Metalogic for QL

a. Truth and Satisfaction in QL

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- Our text uses 'models' and 'interpretations' interchangeably, but the above disambiguation is convenient

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- Rab is true in $\mathfrak{M}$ provided that the objects $I(a)$ and $I(b)$ stand in relation R. In this case, we'll write $\mathfrak{M} \vDash Rab$

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- *Shorthand*:if Pc is true in $\mathfrak{M}$, then $I(c)$ satisfies Px in $\mathfrak{M}$
- *Longhand*: define a variable assignment $\mathbf{d}_I$ that maps variables to objects. Then $\mathbf{d}_I$ **satisfies** Px provided that $\mathbf{d}_I(x)$ has property I(P). We can write $\mathfrak{M}_{\mathbf{d}_I} \vDash Px$

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- d_I satisfies $\mathcal{Q}$ provided that the k-tuple of these objects $\langle t_1^D, \dots, t_k^D \rangle$ lies in the extension of $\mathcal{Q}$, i.e. in $I(\mathcal{Q})$

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  - e.g. everything in D is an $\mathcal{A}$
  - Formally, there is a variable assignment d_I such that for each $r \in D$, each variant $d_I[r/x]$ satisfies $\mathcal{A}x$

Examples of Shorthand

- $(\exists x)(\mathcal{A}x \,\&\, \mathcal{B}x)$ is true iff some object satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$' iff it satisfies both $\mathcal{A}x$ and $\mathcal{B}x$
- $(\forall x)(\mathcal{A}x \supset \mathcal{B}x)$ is true iff **every** object satisfies '$\mathcal{A}x \supset \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \supset \mathcal{B}x$' iff either
    - o does not satisfy $\mathcal{A}x$ (vacuously true conditional)

      or
    - o does satisfy $\mathcal{B}x$

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- **Satisfiability**: we say that a set of sentences Γ is jointly **satisfiable** (aka QL-consistent) provided that there exists at least one QL-model $\mathfrak{M}$ where each sentence in Γ is true

13. Metalogic for QL

b. Recap: Substitution Instances

(Full) Substitution Instances

- "$\mathcal{Q}[c/x]$" is the sentence you get from $(\forall x)\mathcal{Q}$ or $(\exists x)\mathcal{Q}$ by dropping the quantifier and putting c in place of **every** x in $\mathcal{Q}$

- The other variables are untouched!

- Read "$[c/x]$" as saying "substitute c for every x", i.e. all the x's are replaced by c's!

Some Examples of Substitution Instances

- Instances of $(\forall y)Hy$:
  - Ha, Hb, Hm_{11}
- Instances of $(\exists z)Haz$:
  - Haa, Hab, Haj_3
- Instances of $(\exists z)(Hz \,\&\, Fzz)$:
  - Remember to replace **EVERY** occurance of z with the chosen constant:
  - $(Ha \,\&\, Faa)$, $(Hc \,\&\, Fcc)$
  - The following are **NOT** substitution instances:
  - $(Ha \,\&\, Faz)$, $(Hy \,\&\, Faa)$, $(Ha \,\&\, Fab)$

Partial Substitution Instances

- For Existential Introduction, we can use a partial substitution instance of the wff $\mathcal{Q}$:
- "$\mathcal{Q}\lceil x/c \rceil$" indicates that the variable x replaces some but not necessarily all occurrences of the constant c in $\mathcal{Q}$.
- You can decide which occurrences of c to replace and which to leave in place

Examples of Partial Substitution Instances!

- '$\mathcal{Q}[\chi/c]$' indicates that the variable χ does not need to replace all occurrences of the constant c in $\mathcal{Q}$

1	Rdd	
2	$(\exists x)Rxx$	:1 ∃I
3	$(\exists x)Rxd$	:1 ∃I
4	$(\exists z)Rdz$	:1 ∃I
5	$(\exists y)(\exists z)Ryz$	:4 ∃I

Existential Introduction (∃I)

m	$\mathcal{Q}$	
$\vdots$	$\vdots$	
s	$(\exists \chi)\mathcal{Q}[\chi/c]$	:m ∃I

– Note: since $\mathcal{Q}$ is a sentence, and by our recursion clause for wff, χ cannot occur in $\mathcal{Q}$.

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- "d_I satisfies $\mathcal{Q}[c/x]$" is equivalent to "$d_I[I(c)/x]$ satisfies $\mathcal{Q}$"
- **Substitution Lemma**: let $\mathcal{Q}$ be a wff of QL. The variable assignment d_I satisfies $\mathcal{Q}[c/\chi]$ if and only if $d_I[I(c)/\chi]$ satisfies $\mathcal{Q}$

13. Metalogic for QL

c. QL rules recap

Rules for the Universal Quantifier

Universal Elimination (∀E)

$$\begin{array}{r|ll} m & (\forall \mathcal{X})\mathcal{Q} & \\ \vdots & \quad\vdots & \\ s & \mathcal{Q}[c/\mathcal{X}] & :m\ \forall\mathrm{E} \end{array}$$

- Note that you replace **EVERY** instance of $\mathcal{X}$ with c

- Notation: $\mathcal{Q}[c/\mathcal{X}]$

- read "c for $\mathcal{X}$"

Universal Introduction (∀I)

$$\begin{array}{r|ll} m & \mathcal{Q} & \\ \vdots & \quad\vdots & \\ s & (\forall \mathcal{X})\mathcal{Q}[\mathcal{X}/c] & :m\ \forall\mathrm{I} \end{array}$$

Provided that both

(i) c does not occur in any other undischarged assumptions that $\mathcal{Q}$ is in the scope of.

(ii) $\mathcal{X}$ does not occur already in $\mathcal{Q}$.

Rules for the Existential Quantifier

Existential Introduction (∃I)

m	$\mathcal{Q}$
$\vdots$	$\vdots$
s	$(\exists \chi)\mathcal{Q}\lceil \chi / c \rceil$:m ∃I

- **Provided that** χ does not occur already in $\mathcal{Q}$.
- As indicated by $\lceil \chi / c \rceil$, χ **may** replace **just some** occurrences of c

Existential Elimination (∃E)

m	$(\exists \chi)\mathcal{Q}$	
	$\vdots$	
n	$\quad\mathcal{Q}[c/\chi]$	:AS for ∃E
	$\quad\vdots$	
s	$\quad\Psi$	
$s+1$	Ψ	:m, n–s ∃E

Simplified: **provided that** c doesn't occur **anywhere else outside** the subproof

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- Without these restrictions, earlier sentences in the derivation would not semantically entail later sentences
- For QND to be sound, we need $\Gamma \vdash_{QND} \mathcal{P}$ to be sufficient for $\Gamma \vDash \mathcal{P}$.
- As with SND, we will prove this by showing that the set of open assumptions Γ_k on line $\#k$ semantically entail the sentence $\mathcal{P}_k$ on that line, for all lines k in any QND derivation

13. Metalogic for QL

d. Soundness of System QND

Semantic entailment for infinitely-many premises

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- In this case we write $\Gamma \nvDash \Theta$
- If there is no such QL-model, then $\Gamma \vDash \Theta$, i.e. if whenever we have $\mathfrak{M} \vDash \Gamma$ we also have $\mathfrak{M} \vDash \Theta$

QND derivability for infinitely-many premises

- Θ is **QND-derivable** from Γ provided there is an QND derivation:
  1.) whose starting premises Δ are a finite subset of Γ
  2.) in which Θ appears on its own in the final line
  3.) where Θ is directly next to the main scope line, i.e. only in the scope of the Δ-premises
- In this case, we write $\Gamma \vdash_{QND} \Theta$ (also: $\Delta \vdash_{QND} \Theta$)
- If no such derivation exists, then we say that Θ is NOT QND-derivable from Γ, and we write $\Gamma \nvdash_{QND} \Theta$

Soundness: Proof Idea and notation

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- **Subgoal**: given a wff P_k on line k, show that $\Gamma_k \vDash P_k$

Soundness: Proof Strategy

- Recall that QND derivations are defined recursively: from a (possibly empty) set of premises, we have a finite number of rules to add a line
  - These ways include all our SND rules plus an intro and elimination rule for our quantifiers $\forall$ and $\exists$
- Hence: do induction on the number of lines in an QND derivation
- Show that the base case has the property (line #1)
- Induction hypothesis: assume the property holds for all lines $\leq k$.
- Induction step: show the property holds for line #k+1 (by considering all possible ways line #k+1 could arise)

Let's remain Righteous!

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- We will extend our induction for SND to cover our four new rules!

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- So we will have shown **Soundness**: If $\Gamma \vdash_{QND} \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$

Base Case

- **Base case**: for any QND derivation, show that $\Gamma_1 \vDash \mathcal{P}_1$.
- Proof: Γ_1 is the set of premises accessible at line #1, which comprises exactly the QL–sentence $\mathcal{P}_1$
- (recall that every premise of a derivation lies in its own scope)
- Clearly, $\mathcal{P}_1 \vDash \mathcal{P}_1$, so $\{\mathcal{P}_1\} \vDash \mathcal{P}_1$
- So line #1 is righteous (i.e. $\Gamma_1 \vDash \mathcal{P}_1$)

Stating the Induction Step

- **Induction Hypothesis**: Assume that every line i for $1 < i \leq k$ is righteous (i.e. that $\Gamma_i \vDash \mathcal{P}_i$)
- Induction step: Consider line #k+1; show that $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- We have 16 cases to consider! We have essentially already considered 12 of these from our soundness proof for SND
- We have four new cases: our intro. and elimin. rules for $\forall$ and $\exists$

Cases 1–12: modifying our soundness proof for SND

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- Equivalently: $\Gamma_{k+1} \cup \{\sim\mathcal{P}_{k+1}\}$ is unsatisfiable in QL

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- So, for each constant c, $\mathfrak{M} \vDash \mathcal{Q}[c/\chi]$, i.e. is true in the model

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- Then $d_I[r/\chi]$ satisfies $\mathcal{Q}\lceil\chi/c\rceil$ (i.e. the open sentence we get by replacing some c's with χ is satisfied by object r)
- Recall: d_I satisfies $(\exists\chi)\mathcal{Q}\lceil\chi/c\rceil$ provided there is some object $r \in D$ s.t. $d_I[r/\chi]$ satisfies $\mathcal{Q}\lceil\chi/c\rceil$

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- Then $d_I[r/\chi]$ satisfies $\mathcal{Q}\lceil \chi/c \rceil$ (i.e. the open sentence we get by replacing some c's with χ is satisfied by object r)
- Recall: d_I satisfies $(\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$ provided there is some object $r \in D$ s.t. $d_I[r/\chi]$ satisfies $\mathcal{Q}\lceil \chi/c \rceil$
- Hence, by the defN of true-in-QL, $(\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$ is true in $\mathfrak{M}$

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- Note that every member of Γ_m is accessible at #k+1 except assumption $\mathcal{Q}[c/\chi]$ on line #j. So $\Gamma_m \subseteq \Gamma_{k+1} \cup \{\mathcal{Q}[c/\chi]\}$

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- Hence, $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$, so line #k+1 is righteous!

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- We will use this lemma for the cases of $\forall I$ and $\exists E$

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- Hence, by Locality, $\mathfrak{M} \vDash \mathcal{P}$, so $\Gamma \vDash \mathcal{P}$

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- Hence, $\Gamma_{k+1} \vDash \mathcal{Q}$
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- Hence, $\Gamma_{k+1} \vDash \mathcal{Q}$
- **Lemma**: if c does not appear in any member of set Γ, then if $\Gamma \vDash \mathcal{Q}$, we have $\Gamma \vDash (\forall \chi)\mathcal{Q}[\chi/c]$
- Our rule $\forall I$ requires that c does not appear in Γ_{k+1}, so by the lemma, $\Gamma_{k+1} \vDash (\forall \chi)\mathcal{Q}[\chi/c]$

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- Goal: show that there exists a variable assignment d_I' that satisfies $(\forall x)\mathcal{Q}[x/c]$
    - i.e. a d_I' s.t. $d_I'[r/x]$ satisfies $\mathcal{Q}[x/c]$ for each object $r \in D$

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  - i.e. a d'_I s.t. $d'_I[r/x]$ satisfies $\mathcal{Q}[x/c]$ for each object $r \in D$
  - We will actually show that the given d_I does the trick!

Lemma for Case 15 continued

- Notice that for $(\forall \mathcal{x})\mathcal{Q}[\mathcal{x}/c]$ to be a wff, $\mathcal{x}$ must not already occur in sentence $\mathcal{Q}$, so $\mathcal{Q}$ can't already have a $\mathcal{x}$–quantifier.

Lemma for Case 15 continued

- Notice that for $(\forall x)\mathcal{Q}[x/c]$ to be a wff, x must not already occur in sentence $\mathcal{Q}$, so $\mathcal{Q}$ can't already have a x−quantifier.
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- Hence, a variable assignment d_I of free variables in $\mathcal{Q}[x/c]$ to objects in D amounts to a choice of object $r \in D$ to assign x

Lemma for Case 15 continued

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- So x occurs freely in $\mathcal{Q}[x/c]$ as the **only** free variable (since $(\forall x)\mathcal{Q}[x/c]$ is, by assumption, a sentence)
- Hence, a variable assignment d_I of free variables in $\mathcal{Q}[x/c]$ to objects in D amounts to a choice of object $r \in D$ to assign x
- So d_I must make some choice $r := I(c)$ of object to assign x

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- Hence d_I simply equals $d_I[r/x]$.
- By 11.1.1, d_I satisfies $\mathcal{Q}[x/c]$ iff $d_I[I(c)/x]$ satisfies $\mathcal{Q}$

Lemma for Case 15 continued

- Notice that for $(\forall \mathcal{x})\mathcal{Q}[\mathcal{x}/c]$ to be a wff, $\mathcal{x}$ must not already occur in sentence $\mathcal{Q}$, so $\mathcal{Q}$ can't already have a $\mathcal{x}$–quantifier.
- So $\mathcal{x}$ occurs freely in $\mathcal{Q}[\mathcal{x}/c]$ as the **only** free variable (since $(\forall \mathcal{x})\mathcal{Q}[\mathcal{x}/c]$ is, by assumption, a sentence)
- Hence, a variable assignment d_I of free variables in $\mathcal{Q}[\mathcal{x}/c]$ to objects in D amounts to a choice of object $r \in D$ to assign $\mathcal{x}$
- So d_I must make some choice $r := I(c)$ of object to assign $\mathcal{x}$
- Hence d_I simply equals $d_I[r/\mathcal{x}]$.
- By 11.1.1, d_I satisfies $\mathcal{Q}[\mathcal{x}/c]$ iff $d_I[I(c)/\mathcal{x}]$ satisfies $\mathcal{Q}$
- So d_I and hence $d_I[r/\mathcal{x}]$ satisfies $\mathcal{Q}[\mathcal{x}/c]$

Lemma for Case 15 continued *more*

- For each $r \in D$, we define a new model $\mathfrak{M}_r$ whose interpretation function I_r is just like I except that it assigns the constant c to r

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- Now, $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{M}_r$ differs by $\mathfrak{M}$ only in how it interprets a symbol that does not occur in Γ.
- So by Locality, we have $\mathfrak{M}_r \vDash \Gamma$ and hence $\mathfrak{M}_r \vDash \mathcal{Q}$

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- So for each object $r \in D$, we have a $d_{I_r}[r/\chi]$ that satisfies $\mathcal{Q}[\chi/c]$.

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- This is equivalent to saying that for each $r \in D$, $d_I[r/\chi]$ satisfies $\mathcal{Q}[\chi/c]$, since d_I and each d_{I_r} agree on what to assign every other variable besides possibly χ

Lemma for Case 15 continued *more*

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- Now, $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{M}_r$ differs by $\mathfrak{M}$ only in how it interprets a symbol that does not occur in Γ.
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- So for each object $r \in D$, we have a $d_{I_r}[r/\chi]$ that satisfies $\mathcal{Q}[\chi/c]$.
- This is equivalent to saying that for each $r \in D$, $d_I[r/\chi]$ satisfies $\mathcal{Q}[\chi/c]$, since d_I and each d_{I_r} agree on what to assign every other variable besides possibly χ
- And by defN, this means that d_I satisfies $(\forall\chi)\mathcal{Q}[\chi/c]$, and hence this sentence is true in $\mathfrak{M}$