What is Logic?

LOGIC I
Benjamin Brast-McKie
September 9, 2024

Motivations

Reasoning: Logic is the study of formal reasoning.

- By 'formal' we don't mean that it uses mathematical symbols.
- Rather, what follows from what *in virtue of logical form*.
- Abstracting from specific subject-matters, logic describes general patterns of reasoning that apply across the disciplines.

Normativity: Logic is not a *descriptive* science studying how human beings in fact reason across the various disciplines.

- Logic is a *normative* science, describing an especially strong form of reasoning that may serve as an ideal.

Artifical: We will primarily work in artificial languages where we will stipulate how to reason in these languages.

- Regimenting English will expose and remove ambiguities.
- We will provide proof systems for our artificial languages by which to compute what follows from what in a manner that vastly extends our natural cognitive capacities.

Interpretations

Proposition: We will begin with propositional logic where a PROPOSITION is a way for things to be which either obtains or does not.

Declarative Sentence: Given an interpretation of the language, an English sentence is DECLARATIVE just in case it expresses a proposition.

- Interrogative, imperative, and exclamatory sentences are not declarative sentences and typically do not have truth-values.
- We will restrict to declarative sentences throughout.

Truth-Values: A declarative sentence is TRUE in an interpretation if, given that interpretation, it expresses a proposition that obtains and FALSE in that interpretation otherwise.

Interpretations: We will only be concerned with the truth-values of sentences in this course, and so it is enough to take an INTERPRETATION to be an assignment of truth-values to sentences.

- This amounts to taking there to be just two propositions.

Examples

Deductive Argument: A DEDUCTIVE ARGUMENT in English is a nonempty sequence of declarative sentences where a single sentence is designated as the CONCLUSION (typically the last line) and all of the other sentences (if any) are the PREMISES.

Snow*: This argument may be compelling, but it is not certain.*

A1. It's snowing.
A2. John drove to work.

Red*: This argument provides certainty, but not on all interpretations.*

B1. The ball is crimson.
B2. The ball is red.

Museum*: This argument's certainty is independent of the interpretation.*

C1. Kate is either at home or at the Museum.
C2. Kate is not at home.
C3. Kate is at the Museum.

Informal Validity

Question 1: What goes wrong if we assume the premises but deny the conclusion in *Snow*, *Red*, and *Museum*?

Snow: Improbable but possible.

Red: Impossible on the intended interpretation.

Museum: Impossible on all interpretations so long as we hold the meanings of logical terms 'not' and 'or' fixed.

Task 1: Clarify what it is to hold the logical terms fixed.

Informal Interpretation: An INFORMAL INTERPRETATION assigns every declarative sentence of English to exactly one TRUTH-VALUE without offending the following informal semantic clauses:

- A *negation* is true just in case the negand is false.
- A *disjunction* is true just in case either disjunct is true.

Informal Validity: An argument in English is INFORMALLY VALID just in case its conclusion is true in every informal interpretation in which all of its premises are true.

Formal Languages

Problem 1: There is no set of all declarative sentences of English, and so no clear notion of an informal interpretation of English.

Suggestion: Could choose some large set of atomic English sentences, but this would be arbitrary and hard to specify precisely.

Solution 1: We will *regiment* English arguments in artificial languages that are both general and easy to specify precisely.

Propositional Language: The SENTENCES of $\mathcal{L}^{\text{PL}}$ are composed of SENTENCE LETTERS $A, B, C, \ldots$ and sentential operators $\neg$ and $\vee$.

Task 2: Regiment *Museum* in $\mathcal{L}^{\text{PL}}$: $H \vee M, \neg H \vDash M$.

- H = 'Kat is at home'.
- M = 'Kat is at the Museum'.

Task 3: Provide a way to interpret the sentences of $\mathcal{L}^{\text{PL}}$.

Schematic Variables: Let $\varphi, \psi, \ldots$ be variables with sentences of $\mathcal{L}^{\text{PL}}$ as values, and let $\Gamma, \Sigma, \ldots$ be variables for sets of sentences of $\mathcal{L}^{\text{PL}}$.

Interpretation: An INTERPRETATION $\mathcal{V}$ of $\mathcal{L}^{\text{PL}}$ assigns exactly one truth-value (1 or 0) to all sentences of $\mathcal{L}^{\text{PL}}$ where for any φ and ψ:

- $\mathcal{V}(\neg\varphi) = 1$ just in case $\mathcal{V}(\varphi) = 0$.
- $\mathcal{V}(\varphi \vee \psi) = 1$ just in case $\mathcal{V}(\varphi) = 1$ or $\mathcal{V}(\psi) = 1$ (or both).

Logical Consequence: $\Gamma \vDash \varphi$ just in case $\mathcal{V}(\varphi) = 1$ for any interpretation $\mathcal{V}$ of $\mathcal{L}^{\text{PL}}$ where $\mathcal{V}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Logical Validity: An argument is LOGICALLY VALID just in case its conclusion φ is a logical consequence of its set of premises Γ, i.e. $\Gamma \vDash \varphi$.

Task 4: Show that *Museum* is logically valid.

Logic

Model Theory: We have characterized logical reasoning as truth-preservation across a space of interpretations for an artificial language.

Proof Theory: Another approach focuses entirely on syntactic rules that specify which inferences in a language are logically valid.

- A system of basic rules for reasoning in an artificial language is referred to as a LOGIC for that language.

- By composing basic rules, we will define what counts as a PROOF in each of the logics that we will study.

Metalogic: Despite their differences, these two strategies will be shown to coincide for the languages that we will study in this book.

Logical Form

Picasso

D1. The painting is either a Picasso or a counterfeit and illegally traded.
D2. The painting is not a Picasso.
D3. The painting is a counterfeit and illegally traded.

Task 5: Regiment *Picasso* in $\mathcal{L}^{\text{PL}}$: $P \vee (Q \wedge R), \neg P \vDash Q \wedge R$.

- P = 'The painting is a Picasso'.
- Q = 'The painting is a counterfeit'.
- R = 'The painting is illegally traded'.

Question 2: How does this argument relate to *Museum*?

Logical Form: Both arguments are instances of $\varphi \vee \psi, \neg\varphi \vDash \psi$ which is a logically valid argument schema, i.e., all instances are valid.

Question 3: How many logically valid argument schemata are there, and how could we hope to describe this space?

Suggestion: The logical consequence relation $\vDash$ for $\mathcal{L}^{\text{PL}}$ describes the space of logically valid arguments, where the logically valid argument schemata are patterns in this space.

Problem 2: $\mathcal{L}^{\text{PL}}$ cannot regiment all logically valid arguments.

Socrates: Every man is mortal, Socrates is a man $\vDash$ Socrates is mortal.

- Our intuitive grasp on logical validity is not exhaustively captured by what we can regiment in $\mathcal{L}^{\text{PL}}$.

Solution 2: Rather, logical validity in $\mathcal{L}^{\text{PL}}$ provides a partial answer, where we may extend the language to provide a broader description of logical validity, e.g., $\mathcal{L}^{\text{FOL}}$.

- We will consider further extensions to $\mathcal{L}^{\text{FOL}}$ in later chapters.

Syntax for $\mathcal{L}^{\text{LP}}$

LOGIC I
Benjamin Brast-McKie
September 10, 2024

Object Language and Metalanguage

Object Language: $\mathcal{L}^{\text{PL}}$ is the OBJECT LANGUAGE under study.

Metalanguage: Mathematical English is the METALANGAUGE with which we will conduct our study.

Quotation: To talk about $\mathcal{L}^{\text{PL}}$ we will take a quoted expression to be the CANONICAL NAME for the expression quoted.

Use/Mention: We MENTION expressions by putting them in quotes, whereas otherwise they are USED.

- 'Sue' is a nickname for Susanna.
- The complex sentence '$A \to B$' includes the sentence letters 'A' and 'B'.
- 'A' belongs to $\mathcal{L}^{\text{PL}}$, but " 'A' " and A do not.

The Expressions of $\mathcal{L}^{\text{PL}}$

Sentential Operators: '$\neg$','$\wedge$','$\vee$','$\to$', and '$\leftrightarrow$'.

- '$\sim$', '&', '.', '$|$', '$\supset$', and '$\equiv$' are also sometimes used.

Punctuation: '(' and ')'.

Sentence Letter: 'A_0', 'A_1', ..., 'B_0', 'B_1', ..., 'Z_0', 'Z_1', ...

Question: How can we specify all sentence letter explicitly?

- A SENTENCE LETTER is the result of subscripting a capital English letter with a numeral.

Corner Quotes: Let $\ulcorner \varphi_x \urcorner$ refer to the result of concatenating φ with $_x$.

- $\ulcorner \varphi_x \urcorner$ is a SENTENCE LETTER for any capital letter φ and numeral for a natural number x.

Primitive Symbols: The sentential operators, punctuation, and sentence letters are the PRIMITIVE SYMBOLS of $\mathcal{L}^{\text{PL}}$.

Expressions: The EXPRESSIONS of $\mathcal{L}^{\text{PL}}$ are defined recursively:

- The primitive symbol of $\mathcal{L}^{\text{PL}}$ are expression of $\mathcal{L}^{\text{PL}}$.
- If φ and ψ are expressions of $\mathcal{L}^{\text{PL}}$, then so is $\ulcorner \varphi\psi \urcorner$.
- Nothing else is an expression of $\mathcal{L}^{\text{PL}}$.

The Sentences of $\mathcal{L}^{\text{PL}}$

Uninterpretable: The expressions '$\neg\neg\neg\neg$', 'B_3A_0', '$))\leftrightarrow$', and '$A_4\vee$' cannot be assigned truth-values in a meaningful way.

- Compare 'MIT is in session' and '$A_4 \wedge P_1$'.

Well-Formed Sentences: Letting $\varphi, \psi, \chi, \ldots$ be variables with expressions for values, we may define the WFSS of $\mathcal{L}^{\text{PL}}$ as follows:

- Every sentence letter of $\mathcal{L}^{\text{PL}}$ is a wfs of $\mathcal{L}^{\text{PL}}$.
- If the expressions φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$, then:
  1. $\ulcorner\neg\varphi\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  2. $\ulcorner(\varphi \wedge \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  3. $\ulcorner(\varphi \vee \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  4. $\ulcorner(\varphi \rightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$; and
  5. $\ulcorner(\varphi \leftrightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$.
- Nothing else is a wff of $\mathcal{L}^{\text{PL}}$.

Sentential Variables: We will often restrict 'φ', 'ψ', 'χ',... to the wfs of $\mathcal{L}^{\text{PL}}$.

Main Operator: The MAIN OPERATOR is the last operator used in the construction of a sentence.

Arguments: The inputs to a main operator are its ARGUMENTS.

Scope: The main operator has SCOPE over its arguments.

Metalinguistic Conventions

Subscripts: We will suppress the subscript '$_0$' to ease exposition.

Task: Build increasingly complex sentences from just A.

Naming: We will refer to the NEGAND in a NEGATION, the CONJUNCTS in a CONJUNCTION, the DISJUNCTS in a DISJUNCTION, the ANTECEDENT and CONSEQUENT in a MATERIAL CONDITIONAL, and the ARGUMENTS in a MATERIAL BICONDITIONAL.

Quotation: We will sometimes drop quotes and corner quotes when the intended meaning is clear from the context.

- We will only do so when this improves readability.

Punctuation: We will drop outermost parentheses for ease.

- Compare $A \wedge B$, $A \vee B \vee C$, and $A \vee B \wedge C$.

Therefore: We will use '$\therefore$' for inline arguments.

Metalinguistic: These abbreviations all happen in the metalanguage.

Truth Functionality

Interpretations: Improving on last time, an INTERPRETATION $\mathcal{I}$ is an assignment of truth-values to sentence letters of $\mathcal{L}^{\text{PL}}$.

Valuation: We may then define a VALUATION function $\mathcal{V}_{\mathcal{I}}$ which assigns truth-values to every sentence of $\mathcal{L}^{\text{PL}}$ by way of the following semantic clauses:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Observe: These clauses resemble the composition rules for $\mathcal{L}^{\text{PL}}$.

Homophonic Semantics: The clauses for $\neg$, $\wedge$, and $\vee$ use analogous operators in the metalanguage, but not so for $\rightarrow$ and $\leftrightarrow$.

Truth Tables: Use the semantics to fill out the TRUTH TABLES below:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Truth Functions: The sentential operators express truth-functions, and so are often called TRUTH-FUNCTIONAL OPERATORS.

Question: How many unary/binary truth-functions are there?

Adequacy: Given these limitations, what should we hope to be able to adequately regiment in $\mathcal{L}^{\text{PL}}$?

Logical Truths: φ is a LOGICAL TRUTH of $\mathcal{L}^{\text{PL}}$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.

Regimentation

LOGIC I
Benjamin Brast-McKie
September 16, 2024

From Last Time…

Definitions: Here is slightly different take on the same definitions:

Well-Formed Sentences: The set WFSS of $\mathcal{L}^{\text{PL}}$ is the smallest set to satisfy:

- φ is a wfs of $\mathcal{L}^{\text{PL}}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$;
- $\neg\varphi$ is a wfs of $\mathcal{L}^{\text{PL}}$ if φ is a wfs of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \wedge \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \vee \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \rightarrow \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \leftrightarrow \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$.

Semantics: For an interpretation $\mathcal{I}$, a VALUATION function $\mathcal{V}_{\mathcal{I}}$ is the smallest function to assign truth-values to every sentence of SL that satisfies the semantic clauses:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Observe: Observe the symmetry between the above.

Recall: The hierarchy of sentences from before…

Complexity

Complexity: $\texttt{Comp}(\varphi)$ is the smallest function to satisfy all of the following conditions for all wfss φ and ψ of $\mathcal{L}^{\text{PL}}$:

- $\texttt{Comp}(\varphi) = 0$ if φ is a sentence letter;
- $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$;
- $\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;
- …

Question: Do we need to include corner quotes?

Validity

$\mathcal{L}^{\text{PL}}$ Validity: An argument in $\mathcal{L}^{\text{PL}}$ is *valid* iff its conclusion is a logical consequence of its premises.

English Validity: An argument in English is *valid* iff it has a (faithful) regimentation (in some language) that is valid.

- Note the imprecision here; there is no avoiding this.

Soundness: An argument is *sound* iff it is valid and has true premises (on an interpretation we care about, probably the intended interpretation).

Examples

Rain

1. If it is raining on a week day, Sam took his car.
2. Kate borrowed Sam's car only if Sam did not take it.
3. Kate borrowed Sam's car just in case she visited her parents.
4. It is raining and Kate visited her parents.
5. Either it is not a week day or it is not raining.

Task 2: Regiment this argument and construct its truth table.

Observe: This argument can be adequately regimented and evaluate in SL.

Negation

Uninitiated

A1. If Sam attended the gathering, then he has been initiated.
A2. Sam is uninitiated.
A3. Sam did not attend the gathering.

Observe: Being uninitiated is the same as not being initiated.

Uninvited

B1. Arden is not invited.
B2. Arden is uninvited.

Observe: Arden can fail to be invited without being uninvited.

Question: What about the converse?

Disjunction

Party

C1. If Adi or James make it to the party, Isa will be happy.
C2. If Adi and James make it to the party, Isa will be happy.

Observe: This argument suggests an inclusive reading of 'or'.

Race

D1. Sasha won the 100 meter dash.
D2. Josh won the high jump.
D3. Either Sasha won the 100 meter dash or Josh won the high jump

Observe: We could strengthen the conclusion.

Vault

E1. If Kin uses the remote, the trunk will open.
E2. If Yu tries the handle, the trunk will open.
E3. If Kin uses the remote and Yu tries the handle, the trunk won't open.
E4. If Kin uses the remote or Yu tries the handle, the trunk will open.

Observe: We cannot regiment the conclusion with inclusive-'or'.

Question: Can we salvage the validity of this argument?

Conjunction

Exam

F1. Henry failed and Megan passed.
F2. Megan passed and Henry failed.

Observe: Perfectly adequate and valid regimentation.

Gym

G1. Kate took a shower and went to the gym.
G2. Kate went to the gym and took a shower.

Observe: Conjunction in English can track temporal order.

Question: How can we capture the invalidity of this argument in $\mathcal{L}^{\mathrm{PL}}$?

Logical Consequence

LOGIC I
Benjamin Brast-McKie
September 17, 2024

From Last Time...

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Characteristic Truth Tables: As drawn in the textbook...

Complete Truth Tables

Setup: Write the sentence on the top right, add the constituent sentence letters on the left, and use the characteristic truth tables.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ:

- $[\varphi] = \{\varphi\}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- For any wfss φ and ψ of $\mathcal{L}^{\text{PL}}$, and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$:

  $(\neg)$ $[\neg\varphi] = [\varphi]$;
  $(\star)$ $[\varphi \star \psi] = [\varphi] \cup [\psi]$;

Rows: Add 2^n rows for n constituent sentence letters.

Examples: $[A \wedge (B \vee A)] \rightarrow A$, $C \leftrightarrow \neg C$, D.

Tautology: Only 1s under its main connective in its complete truth table.

Contradiction: Only 0s under its main connective in its complete truth table.

Logically Contingent: A 1 and a 0 under its main connective in its complete truth table.

Logical Entailment: On any row of a complete truth table, the consequent has a 1 under its main connective whenever the antecedent does.

Logical equivalence: Identical columns under the main connectives for the sentences.

Satisfiable: There is a row where all wfss have a 1 under all main connectives.

Logical Consequence: The conclusion has a 1 under its main connective in every row in which every premise has a 1 under its main connectives.

Decidability

Effective Procedure: A finitely describable and (in principle) usable procedure that always finishes and produces a correct answer to the question asked, requiring only that the instructions be followed accurately.

Question: How to define the main operators and distribute truth-values?

- Recursively, like the formation rules for the wfs of $\mathcal{L}^{\text{PL}}$.

Question: Is it always possible to construct a complete truth table for a wfs?

- Sentences have a finite number of constituent sentence letters.

Decidable: If there is an effective procedure for determining the answer to a question, that question is *decidable*.

- It is decidable whether a wfs of $\mathcal{L}^{\text{PL}}$ is a tautology, etc.

Question: What about a complete truth table for a set of sentences?

- Could require infinitely many sentence letters.
- We might be able to define an infinite table, but we can't use it.

Question: If one procedure is not effective, couldn't there be another one?

- It turns out that there is no effective procedure...
- There is always an effective procedure for finite sets of sentences.

Validity: So the validity of finite arguments is decidable.

Partial Truth Tables

Worry 1: It is not *that* effective... in practice it is daunting for $n > 4$.

Partial Truth Tables: Sometimes only one or two lines are needed.

- $A \to \neg(A \vee B)$: not a tautology or contradiction, so contingent.
- $B \leftrightarrow \neg(A \vee B)$ is a contradiction, so we need a complete table.
- $C \vee (A \to A)$ is a tautology, so we need a complete table.

Complete: To affirm equivalence, entailment, and logical consequence.

Partial: To affirm that a set is satisfiable.

Worry 2: Still daunting sometimes.

Worry 3: Definitions all refer to complete truth tables.

- Definition of a complete truth table has some minor ambiguities.
- These could be fixed, but the result is cumbersome.

Heuristic: The truth table definitions are best taken to be a heuristic guide for grasping the abstract definitions we may now provide.

Semantic Proofs

LOGIC I
Benjamin Brast-McKie
September 19, 2024

From Before...

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Formal Definitions

Interpretation: $\mathcal{I}$ is an *interpretation* of $\mathcal{L}^{\text{PL}}$ *iff* $\mathcal{I}(\varphi) \in \{1, 0\}$ for every sentence letter φ of $\mathcal{L}^{\text{PL}}$.

Tautology: φ is a *tautology iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.

Contradiction: φ is a *contradiction iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ for all $\mathcal{I}$.

Logically Contingent: φ is *contingent iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \neq \mathcal{V}_{\mathcal{J}}(\varphi)$ for some $\mathcal{I}$ and $\mathcal{J}$.

Logical Entailment: φ *entails* ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \leq \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Logical Equivalence: φ is *equivalent* to ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Satisfiable: Γ is *satisfiable iff* $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$ for some $\mathcal{I}$.

Logical Consequence: $\Gamma \vDash \varphi$ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ whenever $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Satisfiability

Which sets of sentences are satisfiable?

Taller

(1) Liza is taller than Sue.

(2) Sue is taller than Paul.

(3) Paul is taller than Liza.

Lost

(4) Kim is either in Somerville or Cambridge.

(5) If Kim is in Somerville, then she is not far from home.

(6) If Kim is not far from home, then she is in Cambridge.

(7) Kim is not in Cambridge.

Validity

Arguments: Sequences of wfss of $\mathcal{L}^{\text{PL}}$, not sets.

Valid: For any argument, it is valid *iff* its conclusion is a logical consequence of its set of premises.

- Many arguments may have the same set of premises.
- An argument is valid *iff* its conclusion is true in every interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ to satisfy the set of premises.

Tautology: A wfs φ of $\mathcal{L}^{\text{PL}}$ is a *tautology* just in case $\vDash \varphi$.

- Every $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ satisfies the empty set.
- Each premise constrains the set of interpretations the conclusion must be true in where the empty set has no constraints.

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

- Each wfs of $\mathcal{L}^{\text{PL}}$ corresponds to a set of all interpretations which make that sentence true: $|\varphi| := \{\mathcal{I} : \mathcal{V}_{\mathcal{I}}(\varphi) = 1\}$.
- Is the interpretation set for the conclusion a subset of the intersection of the premise interpretation sets?

Examples

1. Show that $\neg R \to \neg Q,\ P \wedge Q \vDash P \wedge R$.
2. Show that $A \vee B,\ B \to C,\ A \leftrightarrow C \vDash C$.
3. Show that $P,\ P \to Q,\ \neg Q \vDash A$.
4. Show that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology.
5. Show that $A \leftrightarrow \neg A$ is a contradiction.
6. Show that $\{P,\ P \to Q,\ Q \to \neg P\}$ is unsatisfiable.
7. Show that $\{P \to Q,\ \neg P \vee \neg Q,\ Q \to P\}$ is satisfiable.

Observe: There seem to be patterns.

Question: How could we systematize these proofs?

Methods

Truth Tables:	Mechanical but tedious.
•	Bad if there are lots of sentence letters.
•	Good for counterexamples. $A \leftrightarrow (B \to C),\ A \wedge \neg B,\ D \vee \neg A \ \vDash C$.
Semantic Arguments:	Good if there are lots of sentence letters. $(A \vee B) \to (C \wedge D),\ \neg C \wedge \neg E \ \vDash \neg A$.

The Material Conditional

Roses

A1. Sugar is sweet.

A2. The roses are only red if sugar is sweet.

Observe: First paradox of the material conditional.

Vacation

B1. Casey is not on vacation.

B2. If Casey is on vacation, then he is in Paris.

Observe: Second paradox of the material conditional.

Crimson

C1. Mary doesn't like the ball unless it is crimson.

C2. Mary likes the ball.

C3. If the ball is blue, then Mary likes it.

The Biconditional

Rectangle

D1. The room is a square.

D2. The room is a rectangle.

D3. The room is a square if and only if it is a rectangle.

Work

E1. Kin isn't a professor.

E2. Sue isn't a chef.

E3. Kin is a professor just in case Sue is a chef.

Natural Deduction in PL: Part I

LOGIC I
Benjamin Brast-McKie
October 1, 2024

Review from Last Time...

1. Show that $A \vee B,\ B \rightarrow C,\ A \leftrightarrow C \vDash C$.
2. Show that $\{P,\ P \rightarrow Q,\ Q \rightarrow \neg P\}$ is unsatisfiable.
3. Show that $\{P \rightarrow Q,\ \neg P \vee \neg Q,\ Q \rightarrow P\}$ is satisfiable.

Motivation

Homophonic: Prove that $P \vee Q,\ \neg P \vDash Q$.

- The semantic proof makes the same inference.
- So why not just draw this inference directly in $\mathcal{L}^{\text{PL}}$?
- What are the basic steps we are allowed to make in a proof?

Semantic Proofs: Provide a reasonably efficient way to evaluate validity.

- But they can be cumbersome to write.
- They explain why a logical property or relation holds.
- Doesn't say how to reason from some premises to a conclusion.
- Thus semantic proofs are not persuasive to the uninitiated.
- Not so for semantic proofs of invalidity, satisfiability, etc.

Logical Consequence: How do we describe the extension of $\vDash$?

Natural Deduction: How should we describe the patterns of natural deduction?

- What moves can we make in a proof, *viz.* semantic proofs?
- Want to describe inference itself, starting with the most basic.
- Such inferences hold in virtue of the meanings of the operators.
- Define a proof to be any composition of basic inferences.

Rules: Each operator will have an introduction and elimination rule.

- These rules will describe how to reason with the connectives.
- Want these rules to be valid.
- Also want these rules to be natural.

Metalogic:

- This is a completely different approach to formal reasoning.
- Nevertheless, these two approaches have the same extension.
- Our proof system will help us relate to logical consequence.

Basic Anatomy of a Proof

List: Finite list of lines.

Numbers: Every line is numbered.

Sentences: Each line contains exactly one wfs of $\mathcal{L}^{\mathrm{PL}}$.

Justification: Each line includes a justification.

Assumptions: The justification for a premise is ':PR'.

Bars: A horizontal bar separates the premises from the steps in the proof.

Conclusion: The last line is the conclusion.

Conditional

Elimination: $A,\ A \to B,\ B \to C\ \vdash C.$

- Easy to derive C using $\to$E.
- What if A was excluded from the premises?

Introduction: $A \to B,\ B \to C\ \vdash A \to C.$

- Need something to work with.
- Want to conclude with a conditional claim.
- Assumption of A justified by ':AS'.

Subproofs: Lines in a closed subproof are dead and all else are live.

- $\to$E can only cite to live lines.
- $\to$I can only cite an appropriate subproof.

Assumption

Example: $A\ \vdash D \to [C \to (B \to A)].$

Conjunction

Elimination: $A \to (B \wedge C),\ B \to D\ \vdash A \to D.$

Introduction: $A \wedge B,\ B \to C\ \vdash A \wedge C.$

Disjunction

Introduction: $A \vdash B \vee ((A \vee C) \vee D).$

Elimination: $A \vee (B \wedge C)\ \vdash (A \vee B) \wedge (A \vee C).$

Natural Deduction in PL: Part II

LOGIC I
Benjamin Brast-McKie
October 1, 2024

Biconditional

Elimination: $A \leftrightarrow (B \to [(A \wedge C) \leftrightarrow D]) \vdash (A \wedge B) \to (D \to C)$.

Introduction: $A \to (B \wedge C),\ C \to (B \wedge A) \vdash A \leftrightarrow C$.

Negation and Reiteration

Elimination Rule: $\neg\neg A \vdash A$. (*Double Negation Elimination*)

1. $A \vee \neg A$. (*Law of Excluded Middle*)
2. $A,\ \neg A \vdash B$. (*Ex Falso Quodlibet*)

Introduction Rule: $\neg(A \wedge \neg A)$. (*Law of Non-Contradiction*)

3. $A \vdash \neg\neg A$. (*Double Negation Introduction*)

Proof

Proof: A natural deduction DERIVATION (or PROOF) of a conclusion φ from a set of premises Γ in PL is any finite sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) a premise in Γ;
(2) a discharged assumption; or
(3) follows from previous lines by the rules for PL.

Provable: An wfs φ of $\mathcal{L}^{\text{PL}}$ is DERIVABLE (or PROVABLE) from Γ in PL (i.e., $\Gamma \vdash \varphi$) *iff* there is a natural deduction derivation (proof) of φ from Γ in PL.

Theorem: A wfs φ is a *theorem* of PL (often written $\varphi \in$ PL) *iff* $\vdash \varphi$.

Interderivable: Two wfss φ and ψ of $\mathcal{L}^{\text{PL}}$ are INTERDERIVABLE (i.e., $\varphi \dashv\vdash \psi$) *iff* both $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Bottom: We take $\bot := A \wedge \neg A$ to abbreviate an arbitrarily chosen contradiction.

Inconsistent: A set of sentences Γ is INCONSISTENT if and only if $\Gamma \vdash \bot$.

Logical Analysis

Sound and Complete: $\Gamma \vdash \varphi$ *iff* $\Gamma \vDash \varphi$.

- $\vdash \varphi$ *iff* $\vDash \varphi$.
- $\Gamma \vdash \bot$ *iff* $\Gamma \vDash \bot$.

Question: How can we tell if an argument is valid?

- Construct a truth table.
- Write a semantic proof.
- Derive the conclusion from the premises.

Question: What if we can mange to find a derivation?

- Natural deduction won't tell you if there is no proof.
- A semantic proof will yield a counterexample.

Question: How can we tell what the logical properties are for a wfs of $\mathcal{L}^{\text{PL}}$?

Tautology?	If YES, prove $\vdash \varphi$.	If NO, provide a countermodel.
Contradiction?	If YES, prove $\vdash \neg\varphi$.	If NO, provide a model.
Contingent?	If YES, provide two models.	If NO, prove $\vdash \varphi$ or $\vdash \neg\varphi$.
Equivalent?	If YES, prove $\varphi \dashv\vdash \psi$.	If NO, provide a countermodel.

Rule Schemata

Task: Compare the rules of inference for PL to their instances.

- Whereas the rules are general, PL proofs are particular.
- But nothing in our PL proofs depend on the particulars.

Question: How might we generalize our proofs beyond any instance?

Rule Schemata: Replace sentence letters in PL proofs with schematic variables.

- Premises are replaced with the lines cited by that rule.
- New rules require new names if we are to use them.

Question: Can we also generalize proofs of theorems?

- These amount to lines that can be added without citing lines.

Derived Schemata: To speed up proofs, we want to derive rule schemata.

- These can then be employed just like our basic rules.
- This avoids having to rewrite the same types of proofs over and over.

Derivable Schemata

Law of Excluded Middle: $\vdash \varphi \vee \neg\varphi$.

Law of Non-Contradiction: $\vdash \neg(\varphi \wedge \neg\varphi)$.

Ex Falso Quodlibet: $\varphi,\ \neg\varphi\ \vdash \psi$.

Hypothetical Syllogism: $\varphi \to \psi,\ \psi \to \chi\ \vdash \varphi \to \chi$.

Modus Tollens: $\varphi \to \psi,\ \neg\psi\ \vdash \neg\varphi$.

Contraposition: $\varphi \to \psi\ \vdash \neg\psi \to \neg\varphi$.

Dilemma: $\varphi \vee \psi,\ \varphi \to \chi,\ \psi \to \chi\ \vdash \chi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

$\vee$*-Commutativity:* $\varphi \vee \psi\ \vdash \psi \vee \varphi$.

$\wedge$*-Commutativity:* $\varphi \wedge \psi\ \vdash \psi \wedge \varphi$.

Biconditional MP: $\varphi \leftrightarrow \psi,\ \neg\varphi\ \vdash \neg\psi$.

$\leftrightarrow$*-Commutativity:* $\varphi \leftrightarrow \psi\ \vdash \psi \leftrightarrow \varphi$.

Double Negation: $\neg\neg\varphi\ \dashv\vdash\ \varphi$.

$\wedge$*-De Morgan's:* $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.

$\vee$*-De Morgan's:* $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.

$\vee\wedge$*-Distribution:* $\varphi \vee (\psi \wedge \chi) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$.

$\wedge\vee$*-Distribution:* $\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

$\vee\wedge$*-Absorption:* $\varphi \vee (\varphi \wedge \psi) \dashv\vdash \varphi$.

$\wedge\vee$*-Absorption:* $\varphi \wedge (\varphi \vee \psi) \dashv\vdash \varphi$.

$\wedge$*-Associativity:* $\varphi \wedge (\psi \wedge \chi) \dashv\vdash (\varphi \wedge \psi) \wedge \chi$.

$\vee$*-Associativity:* $\varphi \vee (\psi \vee \chi) \dashv\vdash (\varphi \vee \psi) \vee \chi$.

Mathematical Induction

LOGIC I
Benjamin Brast-McKie
October 7, 2024

From Last Time...

Bottom: We take $\bot := A \wedge \neg A$ to abbreviate an arbitrarily chosen contradiction.

Inconsistent: A set of wfss Γ of $\mathcal{L}^{\text{PL}}$ is INCONSISTENT if and only if $\Gamma \vdash \bot$.

Ex Falso Quodlibet: $\varphi, \neg\varphi \vdash \psi$.

Recursive Definitions

Expressions: The expressions of $\mathcal{L}^{\text{PL}}$ are defined recursively:

- The primitive symbol of $\mathcal{L}^{\text{PL}}$ are expression of $\mathcal{L}^{\text{PL}}$.
- If φ and ψ are expressions of $\mathcal{L}^{\text{PL}}$, then so is $\ulcorner\varphi\psi\urcorner$.
- Nothing else is an expression of $\mathcal{L}^{\text{PL}}$.

Complexity: $\texttt{Comp}(\varphi)$ is the number of operator instances that occur in φ:

- $\texttt{Comp}(\varphi) = 0$ if φ is a sentence letter;
- $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$; and
- $\texttt{Comp}(\varphi \star \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$ for $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Constituents: $[\varphi]$ is the set of sentence letters that occur in φ:

- $[\varphi] = \{\varphi\}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $[\neg\varphi] = [\varphi]$; and
- $[\varphi \star \psi] = [\varphi] \cup [\psi]$ if $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Simplicity: $\texttt{Simple}(\varphi)$ just in case the φ has at most one sentence letter in $\mathcal{L}^{\text{PL}}$:

- $\texttt{Simple}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$; and
- $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$.

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If φ is a sentence letter, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$; and
- $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$ if $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Induction Guide

Step 1: Identify the set of elements and the property in question.

Step 2: Partition the set into a sequence of stages to run induction on.

Step 3: Establish that every element in the base stage has the property.

Step 4: Assume every element in stage n (and below) have the property.

Step 5: Show that every element in stage $n + 1$ have the property.

Examples

Task 1: Every wfs of $\mathcal{L}^{\text{PL}}$ has an even number of parentheses.

Task 2: All expressions of $\mathcal{L}^{\text{PL}}$ are finite length.

Task 3: If $\mathcal{I}(\varphi) = \mathcal{J}(\varphi)$ for all $\varphi \in [\psi]$, then $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{J}}(\psi)$.

Task 4: For every wfs φ of $\mathcal{L}^{\text{PL}}$, if $\texttt{Simple}(\varphi)$, then $\not\vDash \varphi$.

Task 5: For any wfss φ, ψ, χ and sentence letter α of $\mathcal{L}^{\text{PL}}$, if $\vDash \varphi \leftrightarrow \psi$, then $\vDash \chi_{[\varphi/\alpha]} \leftrightarrow \chi_{[\psi/\alpha]}$.

PL Soundness

- Assume $\Gamma \vdash \varphi$ for an arbitrary set wfss Γ and wfs φ of $\mathcal{L}^{\text{PL}}$.
- There is some PL derivation X of φ from Γ.
- Let φ_i be the wfs on the i-th line of the derivation X.
- Let Γ_i be the set of premises and undischarged assumptions on $j \leq i$.

Base Case: $\Gamma_1 \vDash \varphi_1$.

- φ_1 is either a premise or undischarged assumption.
- Either way, $\Gamma_1 = \{\varphi_1\}$ since φ_1 is not discharged at the first line.
- $\Gamma_1 \vDash \varphi_1$ is immediate.

Induction Step: $\Gamma_{n+1} \vDash \varphi_{n+1}$ if $\Gamma_k \vDash \varphi_k$ for every $k \leq n$. (To be proven separately.)

- By strong induction, $\Gamma_n \vDash \varphi_n$ for all n.
- Since every proof is finite in length, there is a last line m of X where $\varphi_m = \varphi$ is the conclusion.
- Since every assumption in X is eventually discharged, $\Gamma_m = \Gamma$ is the set of premises.
- Thus $\Gamma \vDash \varphi$.

Lemmas

(AS) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by AS.

- Assume that φ_{n+1} is justified by AS.
- So φ_{n+1} is an undischarged assumption at line $n+1$.
- So $\varphi_{n+1} \in \Gamma_{n+1}$ by the definition of Γ_{n+1}.
- $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows immediately.

Inheritance: If φ_k is live at line n of a PL derivation where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

- Let X be a PL derivation.
- Assume there is some $\psi \in \Gamma_k$ where $\psi \notin \Gamma_n$ for $n > k$.
- So ψ has been discharged at a line $j > k$ where $j \leq n$.
- So φ_k is dead at n.
- By contraposition, if φ_k is live at line $n > k$, then $\Gamma_k \subseteq \Gamma_n$ as desired.

(R) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by R.

- Assume that φ_{n+1} is justified by R.
- So $\varphi_{n+1} = \varphi_k$ for some $k \leq n$.
- By hypothesis, $\Gamma_k \vDash \varphi_k$.
- Since φ_k is live at line $n+1$, $\Gamma_k \subseteq \Gamma_{n+1}$ by *Inheritance* (**Lemma 4.3**).
- So $\Gamma_{n+1} \vDash \varphi_k$ by *Weakening* (**Lemma 2.1**).
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$.

PL Soundness

LOGIC I
Benjamin Brast-McKie
October 3, 2024

Lemmas

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

Inheritance: If φ_k is live at line n of a PL derivation where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

Interpretation: If $\mathcal{I}$ is a $\mathcal{L}^{\text{PL}}$ interpretation, then $\mathcal{V}_{\mathcal{I}}(\varphi) \in \{1, 0\}$ for all wfss φ of $\mathcal{L}^{\text{PL}}$.

Contradiction: If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

- Assume $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$.
- Assume for contradiction that Γ is satisfiable.
- There is some $\mathcal{L}^{\text{PL}}$ interpretation $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- By assumption, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$.
- By the semantics for negation, $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$, contradicting the above.
- Thus Γ is unsatisfiable.

Unsatisfiable: If $\Gamma \cup \{\varphi\}$ is unsatisfiable, then $\Gamma \vDash \neg\varphi$.

- Assume $\Gamma \cup \{\varphi\}$ is unsatisfiable.
- Let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- Assume for contradiction that $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 0$.
- So $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, and so $\Gamma \cup \{\varphi\}$ is satisfiable contrary to assumption.
- Thus for any $\mathcal{I}$, $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ if $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- By definition, $\Gamma \vDash \neg\varphi$.

Conditional: If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \to \psi$.

- Assume $\Gamma \cup \{\varphi\} \vDash \psi$.
- Let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- Since $\mathcal{V}_{\mathcal{I}}(\varphi) \in \{1, 0\}$ by *Interpretation*, there are two cases to consider.

  Case 1: Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$.
  - So $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma \cup \{\varphi\}$.
  - So $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ by the starting assumption.
  - Thus $\mathcal{V}_{\mathcal{I}}(\varphi \to \psi) = 1$ by the semantics for the conditional.

  Case 2: Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$.
  - So $\mathcal{V}_{\mathcal{I}}(\varphi \to \psi) = 1$ by the semantics for the conditional.

- So $\mathcal{V}_{\mathcal{I}}(\varphi \to \psi) = 1$ in both cases.
- Thus $\Gamma \vDash \varphi \to \psi$ follows by generalizing on $\mathcal{I}$.

PL Deduction Rules

Induction Hypothesis: Recall the assumption that $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

($\neg$I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\neg$I.

- There is a subproof from φ on line i with ψ at line j and $\neg\psi$ at line k.
- By hypothesis $\Gamma_j \vDash \psi$ and $\Gamma_k \vDash \neg\psi$, where $\Gamma_j, \Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By *Weakening*, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$.
- So $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable by *Contradiction*.
- So $\Gamma_{n+1} \vDash \varphi_{n+1}$ by *Unsatisfiable*.

($\wedge$I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\wedge$I.

- $\varphi_{n+1} = \varphi_i \wedge \varphi_j$ where lines $i, j \leq n$ are live at $n+1$.
- By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$.
- By *Inheritance*, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$.
- By *Weakening*, $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$.
- Let $\mathcal{I}$ be a $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$.
- So $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_i \wedge \varphi_j) = 1$ by the semantics.
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ by generalizing on $\mathcal{I}$.

($\rightarrow$I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$I.

- So $\varphi_{n+1} = \varphi_i \rightarrow \varphi_j$, where there is a subproof of φ_j from φ_i.
- By hypothesis $\Gamma_j \vDash \varphi_j$, where $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By *Weakening*, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$.
- By *Conditional*, $\Gamma_{n+1} \vDash \varphi_i \rightarrow \varphi_j$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\rightarrow$E) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$E.

- So $\varphi_i = \varphi_j \rightarrow \varphi_{n+1}$ where the lines $i, j \leq n+1$ are live at $n+1$.
- By hypothesis $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$.
- By *Inheritance*, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$.
- By *Weakening*, $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$, and so $\Gamma_{n+1} \vDash \varphi_j \rightarrow \varphi_{n+1}$.
- Let $\mathcal{I}$ be a $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ and $\mathcal{V}_{\mathcal{I}}(\varphi_j \rightarrow \varphi_{n+1}) = 1$.
- By the semantics, $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 0$ or $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$.
- To avoid contradiction, $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$.
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows from by generalizing on $\mathcal{I}$.

Consistency

Corollary: If Γ is inconsistent, then Γ is unsatisfiable.

- Assume Γ is inconsistent, so $\Gamma \vdash \bot$.
- Thus $\Gamma \vDash \bot$ follows by PL SOUNDNESS.
- Assume for *reductio* that Γ is satisfiable.
- So $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- So $\mathcal{V}_{\mathcal{I}}(\bot) = 1$, i.e., $\mathcal{V}_{\mathcal{I}}(A \wedge \neg A) = 1$.
- By the semantics, $\mathcal{V}_{\mathcal{I}}(A) = 1$ and $\mathcal{V}_{\mathcal{I}}(\neg A) = 1$, so $\mathcal{V}_{\mathcal{I}}(A) \neq 1$.
- By *reductio*, Γ is unsatisfiable.

Contrapositive: If Γ is satisfiable, then Γ is consistent.

- The inconsistency of Γ may be witnessed by a derivation of $\bot$ from Γ.
- There are no witnesses that $\bot$ can't be derived from a consistent set.
- We would somehow need to survey the space of all derivations.
- Could try a reductio, but this is hardly promising.
- Rather, we need only find an interpretation to witness satisfiability.

Theorems: How do we know that the theorems of PL are consistent?

- Because every theorem is a tautology by PL SOUNDNESS.
- So every interpretation witnesses the truth of all of the theorems.
- So the set of theorems are indeed consistent.
- Otherwise we could derive anything from nothing.

Strength: Let $(\varphi) := \{\chi : \varphi \vdash \chi\}$ be the wfs of $\mathcal{L}^{\text{PL}}$ derivable from φ.

- We may show that $(\psi) \subseteq (\varphi)$ if $\varphi \vdash \psi$.
- So (φ) provides a way to think about the STRENGTH of φ.
- Observe that $\varphi \in (\bot)$ for every wfs φ of $\mathcal{L}^{\text{PL}}$.
- Strength is good, but not if it explodes into inconsistency.

More Derivations

Hypothetical Syllogism: $\varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi$.

Modus Tollens: $\varphi \to \psi,\ \neg\psi \vdash \neg\varphi$.

Contraposition: $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

Biconditional MP: $\varphi \leftrightarrow \psi,\ \neg\varphi \vdash \neg\psi$.

PL Completeness: Part I

LOGIC I
Benjamin Brast-McKie
October 10, 2024

Recall from Last Time…

Corollary 4.2 If Γ is satisfiable, then Γ is consistent.

- This followed from PL SOUNDNESS.
- We will now establish the converse of **Corollary 4.2** as a theorem.
- PL COMPLETENESS will follow as a corollary.

Completeness Proof

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

Lemma 2.3 $\Gamma \vDash \varphi$ just in case $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.

Corollary 5.3 (*PL Completeness*) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assume $\Gamma \vDash \varphi$.
- $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by **Lemma 2.3**.
- $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **Theorem 5.1**.
- $\Gamma \vdash \neg\neg\varphi$ by **Lemma 5.1**, so there is a PL derivation X of $\neg\neg\varphi$ from Γ.
- $\Gamma \vdash \varphi$ by an additional application of DN to X.

Basic Lemmas

Lemma 5.1 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

- Assume $\Lambda \cup \{\varphi\}$ is inconsistent.
- So $\Lambda \cup \{\varphi\} \vdash \bot$, so X is a derivation of $A \wedge \neg A$ from Λ.
- Let X' prefix X with φ as an assumption replacing φ as a premise.
- Append lines for A and $\neg A$ by $\wedge$E.
- Discharge φ, concluding $\neg\varphi$ by $\neg$I, so $\Lambda \vdash \varphi$.

Lemma 5.2 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

- Assume $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$.
- X derives φ from Λ, and Y derives $\neg\varphi$ from Λ.
- Let Z append Y to X, renumbering lines.
- Use EFQ on the last lines of X and Y to derive $\bot$ from Λ.
- By definition, Λ is inconsistent.

Lemma 5.3 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

- Assume $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent.
- $\Lambda \vdash \neg\varphi$ and $\Lambda \vdash \neg\neg\varphi$ by **Lemma 5.1**.
- Thus Λ is inconsistent by **Lemma 5.4**.

Henkin Interpretation

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{\text{PL}}$ just in case for every wfs ψ in $\mathcal{L}^{\text{PL}}$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{\text{PL}}$.

Maximization: We may now extend Γ to a maximal set as follows:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$
- $\Delta_\Gamma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

Henkin Interpretation: For all sentence letters φ of $\mathcal{L}^{\text{PL}}$, let: $\mathcal{I}_\Delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta_\Gamma \\ 0 & \text{otherwise.} \end{cases}$

Satisfiable: It remains to show that $\mathcal{V}_{\mathcal{I}_\Delta}(\gamma) = 1$ for all $\gamma \in \Gamma$.

- This will allow us to conclude that Γ is satisfiable.

Lindenbaum's Lemmas

Lemma 5.4 If Γ is consistent in $\mathcal{L}^{\text{PL}}$, then Δ_Γ is maximal consistent.

- Assume Γ is consistent and let φ be any wfs of $\mathcal{L}^{\text{PL}}$.
- $\varphi = \psi_i$ for some $i \in \mathbb{N}$ given the enumeration above.
- Either $\psi_i \in \Delta_{i+1}$ or $\neg\psi_i \in \Delta_{i+1}$.
- Since $\Delta_{i+1} \subseteq \Delta_\Gamma$, either $\varphi \in \Delta_\Gamma$ or $\neg\varphi \in \Delta_\Gamma$, and so Δ_Γ is maximal.

Base Case: Immediate by the assumption that $\Delta_0 = \Gamma$ is consistent.

Induction Step: Assume for weak induction that Δ_n is consistent.

- $\Delta_n \cup \{\psi_n\}$ is either consistent or not.

Case 1: If $\Delta_n \cup \{\psi_n\}$ is consistent, then $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: If $\Delta_n \cup \{\psi_n\}$ is not consistent, then $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

- Assume for contradiction that $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent.
- So Δ_n is inconsistent by **Lemma 5.2**, contradicting the above.
- So Δ_{n+1} is consistent in both cases, and so Δ_k is consistent for all $k \in \mathbb{N}$.

Limit: Assume for contradiction that Δ_Γ is inconsistent.

- X is a PL derivation of $\bot$ from Δ_Γ in a finite number of lines.
- Let $m \in \mathbb{N}$ be the first number where Δ_m includes all premises in X.
- So $\Delta_m \vdash \bot$, and so Δ_k is inconsistent for some $k \in \mathbb{N}$.
- Since this contradicts the above, Δ_Γ is consistent.

Deductive Closure

Deductive Closure: A set Δ of wfss of $\mathcal{L}^{\text{PL}}$ is DEDUCTIVELY CLOSED in PL just in case for any wfs φ of $\mathcal{L}^{\text{PL}}$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$.

Lemma 5.5 If Δ is maximal consistent, then Δ is deductively closed.

- Assume Δ is maximal consistent.
- Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\Delta \vdash \varphi$.
- Assume for contradiction that $\neg\varphi \in \Delta$.
- X derives $\neg\varphi$ from Δ by R, so $\Delta \vdash \neg\varphi$.
- By **Lemma 5.4**, Δ is inconsistent, contradicting the above.
- So $\neg\varphi \notin \Delta$, and so $\varphi \in \Delta$ by maximality.
- Generalizing on φ, we may conclude that Δ is deductively closed.

PL Completeness: Part II

LOGIC I
Benjamin Brast-McKie
October 10, 2024

From Last Time…

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

Corollary 5.3 (*PL Completeness*) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Basic Lemmas

Lindenbaum's Lemma: If Γ is consistent in $\mathcal{L}^{\text{PL}}$, then Δ_Γ is maximal consistent.

Deductive Closure: A set Δ of wfss of $\mathcal{L}^{\text{PL}}$ is DEDUCTIVELY CLOSED in PL just in case for any wfs φ of $\mathcal{L}^{\text{PL}}$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$.

Lemma 5.5 If Δ is maximal consistent, then Δ is deductively closed.

Lemma 5.6 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

- Assuming that $\Lambda \vdash \varphi$, there is a derivation X of φ from Λ in PL.
- Since $\Lambda \subseteq \Lambda \cup \Pi$, X is also a derivation of φ from $\Lambda \cup \Pi$ in PL.
- Thus $\Lambda \cup \Pi \vdash \varphi$.

Henkin Interpretation

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{\text{PL}}$ just in case for every wfs ψ in $\mathcal{L}^{\text{PL}}$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{\text{PL}}$.

Maximization: We may now extend Γ to a maximal set as follows:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$
- $\Delta_\Gamma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

Henkin Interpretation: For all sentence letters φ of $\mathcal{L}^{\text{PL}}$, let: $\mathcal{I}_\Delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta_\Gamma \\ 0 & \text{otherwise.} \end{cases}$

Satisfiable: It remains to show that $\mathcal{V}_{\mathcal{I}_\Delta}(\gamma) = 1$ for all $\gamma \in \Gamma$.

- This will allow us to conclude that Γ is satisfiable.

Henkin Lemmas Continued

Lemma 5.7 If Δ is a maximal consistent set of wfss of $\mathcal{L}^{\text{PL}}$, then every wfs φ of $\mathcal{L}^{\text{PL}}$ is such that $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

- Assume Δ is a maximal consistent set of $\mathcal{L}^{\text{PL}}$ wfss.
- The proof goes by induction on complexity.

Base: Assume $\texttt{Comp}(\varphi) = 0$, so φ is a sentence letter.

- $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\mathcal{I}_\Delta(\varphi) = 1$ by the semantics.
  iff $\varphi \in \Delta$ by the definition of $\mathcal{I}_\Delta$.
- Thus whenever $\texttt{Comp}(\varphi) = 0$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

Induction: Assume that whenever $\texttt{Comp}(\varphi) \leq n$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

- Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\texttt{Comp}(\varphi) = n + 1$.
- There are five cases to consider, one for each operator.

Case 1: $\mathcal{V}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0$ by the semantics.
iff $\psi \notin \Delta$ by hypothesis since $\texttt{Comp}(\psi) \leq n$.
iff $\neg\psi \in \Delta$ by maximal consistency.

Case 2: $\mathcal{V}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1$ by the semantics.
iff $\psi, \chi \in \Delta$ by hypothesis since $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$.
iff $\psi \wedge \chi \in \Delta$ by $(*)$.

$(*)$ If $\psi \wedge \chi \in \Delta$, then $\Delta \vdash \psi$ and $\Delta \vdash \chi$ by $\wedge$E.

- So $\psi, \chi \in \Delta$ by **Lemma 5.5**.
- If $\psi, \chi \in \Delta$, then $\Delta \vdash \psi \wedge \chi$ by $\wedge$I.
- So $\psi \wedge \chi \in \Delta$ by **Lemma 5.5**.

Case 3: Exercise for this weeks PSet.

Case 4: $\mathcal{V}_{\mathcal{I}_\Delta}(\psi \to \chi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0$ or $\mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1$ by the semantics.
iff $\psi \notin \Delta$ or $\chi \in \Delta$ hypothesis since $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$.
iff $\psi \to \chi \in \Delta$ by (†) and (‡).

(†) If $\psi \notin \Delta$, then $\neg\psi \in \Delta$ by maximality.

- Since $\neg\psi \vdash \psi \to \chi$ and $\neg\psi \in \Delta$, we know $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**.
- Thus $\psi \to \chi \in \Delta$ by **Lemma 5.5**.
- If $\chi \in \Delta$, then since $\chi \vdash \psi \to \chi$, we know $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**.
- So if either $\psi \notin \Delta$ or $\chi \in \Delta$, then $\psi \to \chi \in \Delta$.

(‡) Assume instead that $\psi \to \chi \in \Delta$.

- If $\psi \notin \Delta$, then $\psi \notin \Delta$ or $\chi \in \Delta$.

- If $\psi \in \Delta$, then $\Delta \vdash \chi$ by the rule $\to$E, and so $\chi \in \Delta$ by **Lemma 5.5**.
- So if $\psi \to \chi \in \Delta$, then $\psi \notin \Delta$ or $\chi \in \Delta$.

Case 5: Exercise for this weeks PSet.

Conclusion: So whenever $\texttt{Comp}(\varphi) = n + 1$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

- Thus for all wfss φ of $\mathcal{L}^{\text{PL}}$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

Satisfiability

Lemma 5.8 $\Gamma \subseteq \Delta_\Gamma$.

- Immediate from the definition.

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

- Let Γ be a consistent set of wfss of $\mathcal{L}^{\text{PL}}$.
- Δ_Γ is a maximal consistent by **Lemma 5.5**.
- Let $\Delta = \Delta_\Gamma$ and $\mathcal{I}_\Delta$ be the Henkin interpretation of $\mathcal{L}^{\text{PL}}$ defined above.
- By **Lemma 5.7**, for every wfs φ of $\mathcal{L}^{\text{PL}}$: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\varphi \in \Delta$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Delta$.
- Since $\Gamma \subseteq \Delta$ by **Lemma 5.8**, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Gamma$.
- By definition, Γ is satisfiable.

Compactness

Corollary 5.4 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

- Assume $\Gamma \vDash \varphi$.
- $\Gamma \vdash \varphi$ by completeness, and so X derives φ from Γ.
- $\Gamma_X \vdash \varphi$ where Γ_X is the set of premises in X.
- $\Gamma_X \vDash \varphi$ by soundness.
- Since X is finite, Γ_X is also finite.

Corolary 5.5 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

- Assume for contraposition that Γ is unsatisfiable.
- $\Gamma \vDash \bot$ follows vacuously.
- $\Lambda \vDash \bot$ by **Corollary 5.4** for some finite subset $\Lambda \subseteq \Gamma$.
- So some finite subset $\Lambda \subseteq \Gamma$ is unsatisfiable.
- By contraposition, QED.

Midterm Review

LOGIC I
Benjamin Brast-McKie
October 21, 2024

Derivable Schemata

Contraposition: $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$.

Hypothetical Syllogism: $\varphi \supset \psi,\ \psi \supset \chi \vdash \varphi \supset \chi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

$\vee$*-Conditional:* $\varphi \supset \psi \dashv\vdash \neg\varphi \vee \psi$.

$\neg$*-Conditional:* $\neg(\varphi \supset \psi) \dashv\vdash \varphi \wedge \neg\psi$.

Conditional Weakening: $\psi \vdash \varphi \supset \psi$.

Double Negation: $\neg\neg\varphi \dashv\vdash \varphi$.

$\wedge$*-De Morgan's:* $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.

$\vee$*-De Morgan's:* $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.

Modus Tollens: $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi$.

Regimentation

Complete the following tasks for arguments (A) and (B):

Task 1: Write a symbolization key and regiment the argument.

Task 2: Determine if the argument is valid.

Task 3: Provide a derivation in PL if valid, and a countermodel otherwise.

(A) If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

(B) If Cam remembered to do his chores, then things are clean but not neat. Cam forgot only if things are neat but not clean. Therefore, things are clean just in case they are not neat.

Regimentation and Relations

LOGIC I
Benjamin Brast-McKie
October 23, 2024

Polyadic Predicates

Triadic: 'x is between y and z',
'x is more similar to y than to z', ...

Polyadic: We may refer to predicates as n-place or n-adic.

Properties: n-place predicates express n-place properties.

Note: We will typically consider at most binary predicates.

Binary Predicates

Height: Kin is taller than Prema.
Prema is shorter than Kin.

Task: Regiment the argument above.

Predicates: 'is taller than', 'is shorter than', and 'is the same height as'.

Relations: Binary predicates express 2-place properties, i.e., *relations*.

Question: Is the argument above valid? How about the following arguments?

- $Tkp \therefore Spk$.
- $Tkp \therefore \neg Tpk$.
- $Tkp \therefore \neg Tpk \wedge \neg Epk$.

Question: What can we add to make the arguments valid?

- $Tkp,\ Tkp \to Spk \therefore Spk$.
- $Tkp,\ \forall x \forall y(Txy \to Syx) \therefore Spk$.

Task: Regiment the following argument.

Age: Jon is older than Sara.
Sara is older than Ethan.
Jon is older than Ethan.

Predicates: 'is older than'.

- $Ojs,\ Ose \therefore Oje$.

Question: Is this argument valid, and if not how can we make it valid?

- $Ojs,\ Ose,\ (Ojs \wedge Ose) \to Oje \therefore Oje$.
- $Ojs,\ Ose,\ \forall x \forall y \forall z((Oxy \wedge Oyz) \to Oxz) \therefore Oje$.

Restricting Quantifiers

Universals Quantifiers: Regiment the following sentences:

- All dogs go to heaven.
- Jim took every chance he got.
- Everyone who trained hard or got lucky made it to the top or else didn't compete.

Hidden Quantifiers: Regiment the following sentences:

- At least the guests that remained were pleased with the party.
- I haven't met a cat that likes Merra.
- Kiko's only friends are animals.

Existential Quantifiers: Regiment the following sentences:

- Something great is around the corner.
- One of Ken's statues is very old.
- Kate found a job that she loved.

Mixed Quantifiers

1. Nothing is without imperfections.
2. Every dog has its day.
3. Everyone loves someone.
4. Nobody knows everybody.
5. Everybody everybody loves loves somebody.
6. No set is a member of itself.
7. There is a set with no members.

Arguments

Love: Regiment the following argument:

1. Cam doesn't love anyone who loves him back.
2. May loves everyone who loves themselves.
3. If Cam loves himself, he doesn't love May.

Bigger: Regiment the following argument:

1. Whenever something is bigger than another, the latter is not bigger than the former.
2. Nothing is bigger than itself.

Relations

Domain: Let the *domain* D be any set.

Relation: A *relation* R on D is any subset of D^2.

Reflexive: A relation R is *reflexive* on D *iff* $\langle x, x\rangle \in R$ for all $x \in D$.

Non-Reflexive: A relation R is *non-reflexive* on D *iff* R is not reflexive on D.

Question 1: What is it to be *irreflexive*?

Irreflexive: A relation R is *irreflexive* on D *iff* $\langle x, x\rangle \notin R$ for all $x \in D$.

Symmetric: A relation R is *symmetric iff* $\langle y, x\rangle \in R$ whenever $x, y \in R$.

Question 2: Why don't we need to specify a domain?

Question 3: Why is a relation reflexive or irreflexive with respect to a domain?

Asymmetric: A relation R is *asymmetric iff* $\langle y, x\rangle \notin R$ whenever $\langle x, y\rangle \in R$.

Question 4: What is it to be non-symmetric? How about non-asymmetric?

Task 1: Show that every asymmetric relation is irreflexive.

Transitive: A relation R is *transitive iff* $\langle x, z\rangle \in R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Intransitive: A relation R is *intransitive iff* $\langle x, z\rangle \notin R$ whenever $\langle x, y\rangle, \langle y, z\rangle \in R$.

Question 5: Is every symmetric transitive relation reflexive? (No: $R = \varnothing$)

Task 2: Show that every transitive irreflexive relation asymmetric?

Euclidean: A relation R is *euclidean iff* $\langle y, z\rangle \in R$ whenever $\langle x, y\rangle, \langle x, z\rangle \in R$.

Task 3: Show that every transitive symmetric relation is euclidean.

The Semantics for $\mathcal{L}^{\text{FOL}}$

LOGIC I
Benjamin Brast-McKie
October 29, 2024

Examples

Monadic: Casey is dancing.

Dyadic: Al loves Max.

Triadic: Kim is between Boston and New York.

Constants and Referents

Constants: Constants are interpreted as referring to individuals.

Existence: Thus we need to know what things there are.

Domain: A *domain* is any nonempty set $\mathbb{D}$.

Referents: Interpretations assign constants to elements of $\mathbb{D}$.

Question 1: How are we going to interpret predicates?

Predicates and Extensions

Example: 'Al loves Max' is true *iff* Al bears the loves-relation to Max.

Dyadic Predicates: Dyadic predicates are interpreted by sets of *ordered pairs* in $\mathbb{D}^2$.

Question 2: How are we to interpret n-place predicates?

Cartesian Power: $\mathbb{D}^n = \{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle : \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{D}\}$.

Extensions: n-place predicates are interpreted by subsets of $\mathbb{D}^n$.

Singletons: 1-place predicates are interpreted by subsets of $\mathbb{D}^1 = \{\langle \mathbf{x} \rangle : \mathbf{x} \in \mathbb{D}\}$.

Question 3: How are we to interpret 0-place predicates? What is $\mathbb{D}^0$?

n-Tuples: Let $\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle = \{\langle 1, \mathbf{x}_1 \rangle, \ldots, \langle n, \mathbf{x}_n \rangle\}$.

0-*Tuple:* $\langle\rangle = \varnothing$.

Truth-Values: 0-place predicates are interpreted by subsets of $\mathbb{D}^0 = \{\varnothing\}$.

Ordinals: Note that $0 = \varnothing$ and $1 = \{\varnothing\}$ are the first two von Neumann ordinals.

$\mathcal{L}^{\text{FOL}}$ Models

Interpretations: $\mathcal{I}$ is an $\mathcal{L}^{\text{FOL}}$ interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in $\mathcal{L}^{\text{FOL}}$.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Question 4: What happens if $\mathbb{D} = \varnothing$?

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of $\mathcal{L}^{\text{FOL}}$ *iff* $\mathcal{I}$ is an interpretation over $\mathbb{D} \neq \varnothing$.

Task 1: Regiment and interpret the sentences above.

- Dc, Lam, $Bkbn$.
- $\mathbb{D} = \{c, a, m, k, b, n\}$.
- $\mathcal{I}(D) = \{\langle c \rangle\}$.
- $\mathcal{I}(L) = \{\langle a, m \rangle\}$.
- $\mathcal{I}(B) = \{\langle k, b, n \rangle\}$.
- $\mathcal{I}(c) = c$, $\mathcal{I}(a) = a$, $\ldots$

Lagadonian: We often take constants to name themselves.

Question 5: Do models give us truth-values?

Variable Assignments

Assignments: A variable assignment (v.a.) $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in $\mathcal{L}^{\text{FOL}}$.

Singular Terms: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A v.a. $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Example: Let $\mathbb{D} = \{1, 2, 3, 4, 5\}$ where $\hat{a}(x) = 1$, $\hat{a}(y) = 2$, and $\hat{a}(z) = 3$.

Task 2: If $\hat{c}$ is a y-variant of $\hat{a}$, what is $\hat{c}(x)$, $\hat{c}(y)$, and $\hat{c}(z)$?

Example

Universal: Al loves everything, i.e., $\forall x Lax$.

Existential: Someone is dancing, i.e., $\exists x(Px \wedge Dx)$.

Mixed: Everyone loves someone, i.e., $\forall x(Px \rightarrow \exists y Lxy)$.

Complex: Everything everything loves loves something, i.e., $\forall x(\forall y Lyx \rightarrow \exists z Lxz)$.

Semantics for $\mathcal{L}^{\text{FOL}}$

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\alpha_1,\ldots,\alpha_n) = 1$ *iff* $\langle \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_1),\ldots,\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 0$.
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$.
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \rightarrow \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).
$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \leftrightarrow \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi)$.

Truth and Entailment

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for every v.a. $\hat{a}$ where φ is a wfs of $\mathcal{L}^{\text{FOL}}$.
Logical Consequence: $\Gamma \vDash \varphi$ just in case $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ whenever $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
Tautology: φ is a tautology *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all every model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I}\rangle$.
Contradiction: φ is a contradiction *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ for all every model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I}\rangle$.
Contingent: φ is contingent *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \neq \mathcal{V}_{\mathcal{J}}(\varphi)$ for some $\mathcal{I}$ and $\mathcal{J}$.
Satisfiable: Γ is satisfiable *iff* there some $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Minimal Models

LOGIC I
Benjamin Brast-McKie
October 31, 2024

From Last Time...

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of $\mathcal{L}^{\text{FOL}}$ *iff* where $\mathbb{D} \neq \varnothing$ and both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in $\mathcal{L}^{\text{FOL}}$.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Assignments: A variable assignment (v.a.) $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in $\mathcal{L}^{\text{FOL}}$.

Referents: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A v.a. $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Semantics: Given a model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and v.a., $\hat{a}$ defined over $\mathbb{D}$, we recursively define $\mathcal{V}^{\hat{a}}_{\mathcal{I}}$ to be a function from wffs of $\mathcal{L}^{\text{FOL}}$ to $\{0, 1\}$ as follows:

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1, \dots, \alpha_n) = 1$ *iff* $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \dots, \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$.

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$.

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi)$.

Truth and Entailment

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for every v.a. $\hat{a}$ where φ is a wfs of $\mathcal{L}^{\text{FOL}}$.

Logical Consequence: $\Gamma \vDash \varphi$ just in case $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ whenever $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Tautology: φ is a tautology *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all every model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$.

Contradiction: φ is a contradiction *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ for all every model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$.

Contingent: φ is contingent *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \neq \mathcal{V}_{\mathcal{J}}(\varphi)$ for some $\mathcal{I}$ and $\mathcal{J}$.

Satisfiable: Γ is satisfiable *iff* there some $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Assignment Lemmas

Lemma 1: If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

Lemma 2: If φ is a wfs of $\mathcal{L}^{\text{FOL}}$: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

LTR: Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for all v.a. $\hat{a}$ over $\mathbb{D}$.

- Since $\mathbb{D}$ is nonempty, there is some $d \in \mathbb{D}$.
- Let $\hat{a}(\alpha) = d$ for every variable α of $\mathcal{L}^{\text{FOL}}$.
- Thus $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$.

RTL: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$.

- Let $\hat{c}$ be any v.a. defined over $\mathbb{D}$.
- Since φ has no free variables, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$ by *Lemma 1*.
- Since $\hat{c}$ was arbitrary, $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for all v.a. $\hat{c}$ over $\mathbb{D}$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$.

Minimal Models

Example 1: Al loves everything, i.e., $\forall x Lax$.

True: Let $\mathbb{D} = \{a\}$ and $\hat{a}$ be any v.a. defined over $\mathbb{D}$.

- Let $\hat{c}$ be any x-variant of $\hat{a}$.
- So $\hat{c}(x) = a$ and $\mathcal{I}(a) = a$.
- Letting $\mathcal{I}(L) = \{\langle a, a\rangle\}$, we know $\langle \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(a), \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(L)$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(Lax) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x Lax) = 1$ since $\hat{c}$ was arbitrary.
- Thus $\mathcal{V}_{\mathcal{I}}(Lax) = 1$ by since $\hat{a}$ was arbitrary.

False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(L) = \varnothing$.

- Assume $\mathcal{V}_{\mathcal{I}}(\forall x Lax) = 1$ for contradiction.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x Lax) = 1$ for some v.a. $\hat{a}$ by *Lemma 2*.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Lax) = 1$ since $\hat{a}$ is an x-variant of itself.
- So $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(a), \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(L)$, and so $\mathcal{I}(L) \neq \varnothing$.

Example 2: Someone is dancing, i.e., $\exists x(Px \wedge Dx)$.

True: Let $\hat{a}$ be a v.a. over $\mathbb{D} = \{a\}$, and so $\hat{a}(x) = a$.

- Letting $\mathcal{I}(P) = \mathcal{I}(D) = \{\langle a\rangle\}$, we know $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(x)\rangle \in \mathcal{I}(P) = \mathcal{I}(D)$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Px) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(Dx) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(Px \wedge Dx) = 1$.
- Since $\hat{a}$ is a x-variant of itself, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$.
- Thus $\mathcal{V}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$ by *Lemma 1*.

False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(P) = \varnothing$.

- Assume $\mathcal{V}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$ for contradiction.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x(Px \wedge Dx)) = 1$ for some v.a. $\hat{a}$ by *Lemma 2*.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Px \wedge Dx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Px) = 1$, and so $\langle \mathfrak{v}_{\mathcal{I}}^{\hat{c}}(x) \rangle \in \mathcal{I}(P)$.
- Thus $\mathcal{I}(P) \neq \varnothing$.

- No set is a member of itself. `[contingent]`
  $\neg\exists x(Sx \wedge x \in x)$
- There is a set with no members. `[contingent]`
  $\exists x(Sx \wedge \forall y(y \notin x))$
- Everyone loves someone. `[contingent]`
  $\forall x(Px \to \exists yLxy)$.
- The guests that remained were pleased with the party. `[contingent]`
  $\forall x(Rxp \to Px)$.
- I haven't met a cat that likes Merra. `[contingent]`
  $\neg\exists x(Mbx \wedge Cx \wedge Lmx)$
- Kate found a job that she loved. `[contingent]`
  $\exists x(Fkx \wedge Jx \wedge Lkx)$
- Everything everything loves loves something. `[contingent]`
  $\forall x(\forall yLyx \to \exists zLxz)$.

Quantifier Exchange

($\neg\forall$) $\neg\forall x\varphi \dashv\vDash \exists x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model of $\mathcal{L}^{\text{FOL}}$ where $\mathcal{V}_{\mathcal{I}}(\neg\forall x\varphi) = 1$.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\forall x\varphi) = 1$ for some v.a. $\hat{a}$ by *Lemma 2*.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x\varphi) = 0$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 0$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\neg\varphi) = 1$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\exists x\neg\varphi) = 1$ by *Lemma 2*.

($\neg\exists$) $\neg\exists x\varphi \dashv\vDash \forall x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model of $\mathcal{L}^{\text{FOL}}$ where $\mathcal{V}_{\mathcal{I}}(\neg\exists x\varphi) = 1$.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\exists x\varphi) = 1$ for some v.a. $\hat{a}$ by *Lemma 2*.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x\varphi) = 0$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 0$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\neg\varphi) = 1$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\forall x\neg\varphi) = 1$ by *Lemma 2*.

Arguments

Bigger: Regiment the following argument:

(a) Whenever something is bigger than another, the latter is not bigger than the former.
$\forall x \forall y(Bxy \rightarrow \neg Byx)$.

(b) Nothing is bigger than itself.
$\neg \exists x Bxx$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\forall x \forall y(Bxy \rightarrow \neg Byx)) = 1$.

- Assume $\mathcal{V}_{\mathcal{I}}(\neg \exists x Bxx) = 0$ for contradiction.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg \exists x Bxx) = 0$ for $\hat{a}$ in particular.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x Bxx) = 1$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(Bxx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\langle \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(x), \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(x) \rangle \in \mathcal{I}(B)$, and so $\langle \hat{c}(x), \hat{c}(x) \rangle \in \mathcal{I}(B)$.
- From the assumption, $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\forall x \forall y(Bxy \rightarrow \neg Byx)) = 1$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\forall y(Bxy \rightarrow \neg Byx)) = 1$ since $\hat{c}$ is a x-variant of itself.
- So $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(Bxy \rightarrow \neg Byx) = 1$ for y-variant $\hat{e}$ of $\hat{c}$ where $\hat{e}(y) = \hat{c}(x)$.
- So $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(Bxy) = 0$ or $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\neg Byx) = 1$.
- So $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(Bxy) = 0$ or $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(Byx) = 0$.
- So $\langle \hat{e}(x), \hat{e}(y) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{e}(y), \hat{e}(x) \rangle \notin \mathcal{I}(B)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ since $\hat{e}(x) = \hat{c}(x)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$, contradicting the above.

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
  $\forall x(Lxc \rightarrow \neg Lcx)$.
- May loves everyone who loves themselves.
  $\forall y(Lyy \rightarrow Lmy)$.
- If Cam loves himself, he doesn't love May.
  $Lcc \rightarrow \neg Lcm$.

Taller: Regiment the following argument:

- If a first is taller than a second who is taller than a third, then the first is taller than the third.
  $\forall x \forall y \forall z((Txy \wedge Tyz) \rightarrow Txz)$.
- Nothing is taller than itself.
  $\neg \exists x Txx$.
- If a first is taller than a second, the second isn't taller than the first.
  $\forall x \forall y(Txy \rightarrow \neg Tyx)$.

Identity

LOGIC I

Benjamin Brast-McKie

November 5, 2024

Examples

Transitive: Is the following argument valid when regimented in $\mathcal{L}^{\text{FOL}}$?

1. Hesperus is Venus.
2. Venus is Phosphorus.
3. Hesperus is Phosphorus.

Question: What can do to make this argument valid?

Rising Star: Is the following argument valid when regimented in $\mathcal{L}^{\text{FOL}}$?

1. Hesperus is rising.
2. Hesperus is Phosphorus.
3. Phosphorus is rising.

Question: What can do to make this argument valid?

LL: $(\varphi \wedge \alpha = \beta) \rightarrow \varphi[\beta/\alpha]$.

Modus Ponens: Compare the following argument:

1. Lucy is lucky.
2. If Lucy is lucky, then the letter will arrive in time.
3. The letter will arrive in time.

Question: Why don't we need to add anything to make this argument valid?

Answer: 'If. . . , then. . . ' is regimented by $\rightarrow$ which has a semantics.

Logic: The conditional $\rightarrow$ is a *logical term* of both $\mathcal{L}^{\text{PL}}$ and $\mathcal{L}^{\text{FOL}}$.

Extensions: FOL extends PL, but we needn't stop there.

Question: What can we add to $\mathcal{L}^{\text{FOL}}$ to make *Transitive* and *Rising Star* valid?

Logical Terms

Logicality: The primitive symbols of PL and FOL can be divided in three:

`Logical Terms:` $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall\alpha, \exists\alpha, x_n, y_n, z_n \ldots$ for $n \geqslant 0$.

`Non-Logical Terms:` $a_n, b_n, c_n, \ldots$ and $A^n, B^n, \ldots$ for $n \geqslant 0$.

`Punctuation:` $($, $)$

Extensions: The "meanings" of the non-logical terms are fixed by a model.

Semantics: The "meanings" of the logical terms are fixed by the semantics.

Question: How many logical terms are there?

Identity: At least one more, namely identity which we symbolize by '$=$'.

- We mean *identity*, not *duplication*.

Taller: Can we make 'is taller than' a logical term to validate the following?

- Lu is taller than Kin.
- Kin is taller than Sara.
- Lu is taller than Sara.

Question: Why is '=' a logical term but 'is taller than' isn't?

Syntax for $\mathcal{L}^{=}$

Identity: We include '$=$' in the primitive symbols of the language.

Well-Formed Formulas: We may define the well-formed formulas (wffs) of $\mathcal{L}^{=}$ as follows:

1. $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ is a wff of $\mathcal{L}^{=}$ if $\mathcal{F}^n$ is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. $\alpha = \beta$ is a wff of $\mathcal{L}^{=}$ if α and β are singular terms.
3. If φ and ψ are wffs and α is a variable, then:
   - $\exists\alpha\varphi$ is a wff of $\mathcal{L}^{=}$;
   - $\forall\alpha\varphi$ is a wff of $\mathcal{L}^{=}$;
   - $\neg\varphi$ is a wff of $\mathcal{L}^{=}$;
   - $(\varphi \wedge \psi)$ is a wff of $\mathcal{L}^{=}$;
   - $(\varphi \vee \psi)$ is a wff of $\mathcal{L}^{=}$;
   - $(\varphi \rightarrow \psi)$ is a wff of $\mathcal{L}^{=}$; and
   - $(\varphi \leftrightarrow \psi)$ is a wff of $\mathcal{L}^{=}$.
4. Nothing else is a wff of $\mathcal{L}^{=}$.

Atomic Formulas: The wffs of $\mathcal{L}^{=}$ defined by (1) and (2) are *atomic*.

Complexity: $\texttt{Comp}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = \texttt{Comp}(\alpha = \beta) = 0$.
$\texttt{Comp}(\exists\alpha\varphi) = \texttt{Comp}(\forall\alpha\varphi) = \texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$.
$\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi \vee \psi) = \ldots = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$.

Free Variables

Free Variables: We define the *free variables* recursively:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leqslant i \leqslant n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. α is free in $\beta = \gamma$ if $\alpha = \beta$ or $\alpha = \gamma$ where α is a variable.
3. If φ and ψ are wffs and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;
   $\vdots$
4. Nothing else is a free variable.

Sentences of $\mathcal{L}^=$

Sentences: A *wfs* of $\mathcal{L}^=$ is any wff of $\mathcal{L}^=$ without free variables.

Interpretation: The truth-values of the wfss of $\mathcal{L}^=$ are determined by the models of $\mathcal{L}^=$ independent of a variable assignment.

$\mathcal{L}^=$ Models

Question: What in the semantics will have to change?

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of $\mathcal{L}^=$ *iff* $\mathbb{D} \neq \varnothing$ and $\mathcal{I}$ satisfies both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in $\mathcal{L}^=$.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Answer: Nothing changes in the definition of a model.

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in $\mathcal{L}^=$.

Referents: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Note: Nothing changes in these definitions.

Semantics for $\mathcal{L}^=$

Semantics: Given a model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and v.a., $\hat{a}$ defined over $\mathbb{D}$, we recursively define $\mathcal{V}^{\hat{a}}_{\mathcal{I}}$ to be a function from wffs of $\mathcal{L}^=$ to $\{0, 1\}$ as follows:

- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha = \beta) = 1$ *iff* $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$.

$\vdots$

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for all $\hat{a}$ where φ is a sentence of $\mathcal{L}^=$.

Logical Predicates

Taller-Than: Suppose we were to take 'taller than' (T) to be logical.

Question: Could we provide its semantics?

- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(T\alpha\beta) = 1$ *iff* $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha)$ is taller than $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$.

Theory: The semantics would have to rely on a theory of being taller than.

- Providing such a theory lies outside the subject-matter of logic.
- By contrast, identity is something we already grasp.
- Compare our pre-theoretic grasp of negation, conjunction, and the quantifiers.

Question: Could we take set-membership $\in$ to be a logical term?

Question: What is it to be a logical term?

Question: Could we take a term in sentence position to be logical?

($\bot$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\bot) = 1$.
($\top$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\top) = 0$.

Existence

Existence: Observe that $\exists x(x = x)$ is a tautology.

Question: What could we do to prevent this?

- Empty domain? But how would we interpret the constants?
- Free logics are non-classical and weak.

Question: Should logic be a neural arbiter?

Uniqueness and Quantity

LOGIC I
Benjamin Brast-McKie
November 7, 2024

Opacity

Believes: Regiment the following argument:

1. Lois Lane believes that Superman can fly.
2. Superman is Clark Kent.
3. Lois Lane believes that Clark Kent can fly.

Question: Are these arguments intuitively valid?

Opacity: Whereas *Rising* admits substitution, *Believes* does not.

Transparency: We may say that 'is rising' is transparent and that 'believes' is opaque.

Mathematics: Mathematics does not include any opaque contexts.

Substitution

Leibniz's Law: $\alpha = \beta, \varphi \vDash \varphi[\beta/\alpha]$.

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

- How would we define this recursively?

Constants: If β is a constant, then β is free for any α and φ.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Examples: Consider the following cases:

- z is free for x in $\forall y(Fxy \to Fyx)$
- y is not free for x in $\forall y(Fxy \to Fyx)$

Uniqueness

Uniqueness: Ingmar trusts Albert, but no one else.

Only: Regiment the following argument:

1. Lois Lane only loves Clark Kent.
2. Only Clark Kent is Superman.
3. Lois Lane loves Superman.

Definite Descriptions

Task: Regiment the following sentences.

- The king of France is bald.
- The king of France is not bald.
- The king of France is bald or not.
- Everything is bald or not, so the king of France is bald or not.
- If the king of France is bald or not, then there is a king of France.

Question: What is the difference between these two cases?

Existence: If the king of France is bald, then the king of France exists.

Definite Article: 'The king of France' can't be a name.

Regimentation: Russell offered the following analysis:

- $\exists x(Kxf \wedge \forall y(Kyf \rightarrow x = y) \wedge Bx)$.
- $\exists x(\forall y(Kyf \leftrightarrow x = y) \wedge Bx)$.

Negation: Negation applies to the predicate, not the sentence.

Task: Compare the following case:

- Everything can or cannot fly.
- Pegasus can or cannot fly.
- If Pegasus can or cannot fly, then Pegasus exists.

Task: Regiment the following:

- Superman is keeping something from his lover.
- The man with the axe is not Jack.
- The Ace of diamonds is not the man with the axe.
- One-eyed jacks and the man with the axe are wild.
- No spy knows the combination to the safe.
- The one Ingmar trusts is lying.
- The person who knows the combination to the safe is not a spy.

At Least:

Task: Regiment the following claims.

1. There is at least one wild card.
2. There are at least two clubs.
3. There are at least three hearts on the table.

Question: How can we define these quantifiers in general?

Inequality Quantifiers Defined

Definition: We may define the following abbreviations recursively:

Base: $\exists_{\geqslant 1}\alpha\varphi \coloneqq \exists\alpha\varphi(\alpha)$.

Recursive: $\exists_{\geqslant n+1}\alpha\varphi \coloneqq \exists\alpha(\varphi(\alpha) \wedge \exists_{\geqslant n}\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$ β is free for α in φ.

Infinite: $\Gamma_\infty \coloneqq \{\exists_{\geqslant n}x(x = x) : n \in \mathbb{N}\}$.

Question: What is the smallest model to satisfy Γ_∞?

At Most: Regiment the following claims.

1. There is at most one wild card.
2. There are at most two one-eyed jacks.
3. There are at most three black jacks.

At Most: $\exists_{\leqslant n}\alpha\varphi \coloneqq \neg\exists_{\geqslant n+1}\alpha\varphi$.

Between: $\exists_{[n,m]}\alpha\varphi \coloneqq \exists_{\geqslant n}\alpha\varphi \wedge \exists_{\leqslant m}\alpha\varphi$ where $n \leqslant m$.

Cardinality Quantifiers

Task: Regiment the following.

1. There is one wild card.
2. There are two winning hands.
3. There are three hearts on the table.

Question: How can we define the cardinality quantifiers in general?

Base: $\exists_0\alpha\varphi \coloneqq \forall\alpha\neg\varphi(\alpha)$.

Recursive: $\exists_{n+1}\alpha\varphi \coloneqq \exists\alpha(\varphi(\alpha) \wedge \exists_n\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$ where β is free for α in φ.

Question: How do the cardinality quantifiers relate to the inequality quantifiers?

Exact: $\exists_n\alpha\varphi \dashv\vDash \exists_{[n,n]}\alpha\varphi$.

4. Proving this is too hard for a test or problem set, but maybe a fun challenge to think about.

Examples

1. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa)\}$ is satisfiable.
2. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa), \forall x\exists y Rxy\}$ is satisfiable.
3. Show that $\forall x\forall y\; x{=}y \vdash \neg\exists x\; x \neq a$.

Natural Deduction in $\mathcal{L}^=$

LOGIC I

Benjamin Brast-McKie

November 12, 2024

From Last Time…

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Instance: $\varphi[\beta/\alpha]$ is a SUBSTITUTION INSTANCE of $\forall\alpha\varphi$ and $\exists\alpha\varphi$ if β is a constant.

Examples

1. All humans are mortal.	1. $\forall x(Hx \rightarrow Mx)$
2. <u>Socrates is human.</u>	2. $\underline{Hs}$
3. Someone is mortal.	3. $\exists x Mx$

Motivation

Logical Consequence: We have defined logical consequence for $\mathcal{L}^=$.

- We captured logical form by quantifying over all interpretations.
- But semantic proofs are cumbersome to write.

Naturalness: Want a finite and natural description of logical consequence.

Soundness: Our description should be accurate.

Completeness: We also want our description to be complete.

Question: What rules do we need to derive the following?

- Sid loves everything.
- Sid loves Bina.
- Sid loves something.

Universal Elimination and Existential Introduction

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

Task: Derive the Socrates argument above.

Universal: Everyone is rested or beleaguered $\forall x(Rx \lor Bx)$.

Instantial: Therefore Tom is rested or beleaguered $Rt \lor Bt$.

Existential: So something is rested or beleaguered $\exists x(Rx \lor Bx)$, $\exists x(Rt \lor Bx)$, $\ldots$.

Universal Introduction

Invalid: The following argument is invalid and should not be derivable.

1. Socrates is mortal. (Ms)
2. Everything is mortal. $(\forall x Mx)$

Valid: Compare the following valid argument which should be derivable:

1. $\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz)$.
2. $\forall x \neg Rxx$.
3. $\forall x \forall y(Rxy \to \neg Ryx)$.

Task: Use the rules we have to derive as much as we can.

1	$\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz)$	:PR
2	$\forall x \neg Rxx$	:PR
3	$\forall y \forall z((Ray \land Ryz) \to Raz)$	:1 ∀E
4	$\forall z((Rab \land Rbz) \to Raz)$	:3 ∀E
5	$(Rab \land Rba) \to Raa$	:4 ∀E
6	$\neg Raa$	:2 ∀E
7	$\quad Rab$	:AS for →I
8	$\quad\quad Rba$	:AS for ¬I
9	$\quad\quad Rab \land Rba$	:7, 8 ∧I
10	$\quad\quad Raa$	:5, 9 →E
11	$\quad\quad \neg Raa$	:6 R
12	$\quad \neg Rba$	:8–11 ¬I
13	$Rab \to Rba$	:7–12 →I
14	$\forall y(Ray \to Rya)$	:13 ∀I
15	$\forall x \forall y(Rxy \to Ryx)$	:14 ∀I

Question: How are we going to introduce universal quantifiers without making the invalid argument above derivable?

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in a premise or any undischarged assumption.

Arbitrary: The constraints on ($\forall$I) require β to be arbitrary.

Review: Bad inference above is blocked.

In Premise: Anu loves every dog.
$\forall x(Dx \to Lax) \;\vdash\; Da \to Laa \;\nvdash\; \forall x(Dx \to Lxx)$.

In Conclusion: All dogs love themselves.
$\forall x(Dx \to Lxx) \;\vdash\; Da \to Laa \;\nvdash\; \forall x(Dx \to Lax)$.

Existential Elimination

Task: Compare the following invalid inference.

1. Someone is mortal.
2. Zeus is mortal.

Question: How are we going to eliminate existential quantifiers without making the argument above derivable?

Example: Consider the following argument:

1. Everyone who applied found a position $\forall x(Ax \to \exists yFxy)$.
2. Someone applied $\exists xAx$.
3. Someone found a position $\exists x\exists yFxy$.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in a premise or undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Derivation: We can derive the example without deriving the invalid inference.

Relations

Question: Is the following argument valid?

1. $\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$.
2. $\forall x\forall y(Rxy \to Ryx)$.
3. $\forall xRxx$.

Question: Is the following argument valid?

1. $\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$.
2. $\forall x\neg Rxx$.
3. $\forall x\forall y(Rxy \to \neg Ryx)$.

Natural Deduction in FOL$^{=}$

LOGIC I

Benjamin Brast-McKie

November 14, 2024

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Quantifier Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Unregulated Existential Elimination

Task: $\exists xLbx,\ \exists xLsx \nvdash \exists x\exists y(Lxy \wedge Lyx)$.

1	$\exists xLbx$	:PR
2	$\exists xLsx$	:PR
3	$\quad Lbs$	:AS for $\exists$E
4	$\quad\quad Lsb$	:AS for $\exists$E
5	$\quad\quad Lbs \wedge Lsb$	:3, 4 $\wedge$I
6	$\quad\quad \exists y(Lby \wedge Lyb)$	:5 $\exists$I
7	$\quad\quad \exists x\exists y(Lxy \wedge Lyx)$	:6 $\exists$I
8	$\quad \exists x\exists y(Lxy \wedge Lyx)$	:2, 4–7 $\exists$E (INCORRECT)
9	$\exists x\exists y(Lxy \wedge Lyx)$	:1, 3–8 $\exists$E (INCORRECT)

- Can't instantiate with constants that occur in the premises

Identity Rules

(=I) $\vdash \alpha = \alpha$ for any constant α.

Axiom: This rule is better referred to as an axiom schema.

Note: Easy to use, but not always obvious when to use.

Task: Derive the following in FOL$^=$:

- $\forall x(x = x \to \exists yFyx) \nvdash \exists y(Fyy)$.
- $\forall x(x = x \to \exists yFyx) \vdash \exists x \exists y(Fyx)$.
- Everything is something.
- Something exists.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Note: Also easy to use, but not always obvious how to use.

Task: Derive the following in FOL$^=$:

- $m = n \vee n = o,\ An\ \vdash Am \vee Ao$

Lonely: $\forall x \forall y(Rxy \to x = y)$.

- Every symmetric antisymmetric relation is lonely.
- Every irreflexive antisymmetric relation is asymmetric.

Identity Examples

Task: Regiment and derive the following in FOL$^=$.

1. $\forall x(x = m), Rma \vdash \exists xRxx$
2. $\forall x(x{=}n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$
3. $\exists x(Kx \wedge \forall y(Ky \to x{=}y) \wedge Bx), Kd \vdash Bd$
4. $\vdash Pa \to \forall x(Px \vee x{\neq}a)$

Derivability

Derivation: A DERIVATION (or PROOF) of φ from Γ in FOL$^=$ is any finite sequence of wfs of $\mathcal{L}^=$ ending in φ where every wfs in the sequence is either:

1. A premise in Γ;
2. An assumption which is eventually discharged; or
3. Follows from previous lines by a natural deduction rule for FOL$^=$ besides AS.

Note: This is exactly the same as before.

Quantifier Exchange Rules

($\forall\neg$) $\forall\alpha\neg\varphi \vdash \neg\exists\alpha\varphi$.

1	$\forall\alpha\neg\varphi$	:PR
2	$\mid\ \exists\alpha\varphi$	:AS for $\neg$I
3	$\mid\ \mid\ \varphi[\beta/\alpha]$	:AS for $\exists$E where β is a new constant
4	$\mid\ \mid\ \mid\ \exists\alpha\varphi$	:AS for $\neg$I
5	$\mid\ \mid\ \mid\ \varphi[\beta/\alpha]$	:3 R
6	$\mid\ \mid\ \mid\ \neg\varphi[\beta/\alpha]$	:1 $\forall$E
7	$\mid\ \mid\ \neg\exists\alpha\varphi$	:4–6 $\neg$I
8	$\mid\ \neg\exists\alpha\varphi$	:2, 3–7 $\exists$E
9	$\neg\exists\alpha\varphi$	:2–8 $\neg$I

($\neg\exists$) $\neg\exists\alpha\varphi \vdash \forall\alpha\neg\varphi$.

1	$\neg\exists\alpha\varphi$	:PR
2	$\mid\ \varphi[\beta/\alpha]$	:AS for $\neg$I where β is a new constant
3	$\mid\ \exists\alpha\varphi$	:2 $\exists$I
4	$\mid\ \neg\exists\alpha\varphi$	:1 R
5	$\neg\varphi[\beta/\alpha]$	:2–4 $\neg$I
6	$\forall\alpha\neg\varphi$	:5 $\forall$I

Task: Prove the rules below:

(MCP) If $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

($\forall$DN) $\forall\alpha\neg\neg\varphi \dashv\vdash \forall\alpha\varphi$.

($\exists$DN) $\exists\alpha\neg\neg\varphi \dashv\vdash \exists\alpha\varphi$.

($\exists\neg$) $\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$.

1. Replace φ with $\neg\varphi$ in ($\forall\neg$) above: $\forall\alpha\neg\neg\varphi \vdash \neg\exists\alpha\neg\varphi$.
2. So $\forall\alpha\varphi \vdash \neg\exists\alpha\neg\varphi$ by $\forall$DN, and so $\neg\neg\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$ by MCP.

($\neg\forall$) $\neg\forall\alpha\varphi \vdash \exists\alpha\neg\varphi$.

1. Replace φ with $\neg\varphi$ in ($\neg\exists$) above: $\neg\exists\alpha\neg\varphi \vdash \forall\alpha\neg\neg\varphi$.
2. So $\neg\exists\alpha\neg\varphi \vdash \forall\alpha\varphi$ by $\forall$DN, and so $\neg\forall\alpha\varphi \vdash \neg\neg\exists\alpha\neg\varphi$ by MCP.

FOL$^=$ Soundness

LOGIC I
Benjamin Brast-McKie
November 19, 2024

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

FOL$^=$ Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

(=I) $\vdash \alpha = \alpha$ for any constant α.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

From Natural to Normative

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

- Soundness shows that we can trust FOL$^=$ to establish validity.
- Easier to derive a conclusion that to provide a semantic argument.
- Same story as before...

Natural: FOL$^=$ describes (approximately) how we in fact reason.

- The quantifier rules are somewhat natural...
- Can we understand the quantifiers only by their rules?
- Semantics for the quantifiers is more natural.
- Soundness explains why we ought to use FOL$^=$ to reason.

Soundness of FOL$^=$

Assume: $\Gamma \vdash \varphi$, so there is a FOL$^=$ proof X of φ from Γ.

Lines: Let φ_i be the wfs on line i of X.

Dependencies: Let Γ_i be the premises and undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

L11.1: (Base Step) $\Gamma_1 \vDash \varphi_i$.

L11.13: (Induction Step) If $\Gamma_k \vDash \varphi_k$ for all $k \leqslant n$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\varphi_m = \varphi$ and $\Gamma_m = \Gamma$, so $\Gamma \vDash \varphi$.

Base Step

Proof: Every line in a FOL$^=$ proof is either a premise or follows by the rules.

- φ_1 is either a premise or follows by AS or =I.
- If φ_1 is a premise or assumption, then $\Gamma_1 = \{\varphi_1\}$, and so $\Gamma_1 \vDash \varphi_1$.
- If φ_1 follows by =I, then φ_1 is $\alpha = \alpha$ for some constant α.
- Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model, $\mathcal{I}(\alpha) \in \mathbb{D}$.
- Letting $\hat{a}$ be a variable assignment, $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha)$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha = \alpha) = 1$, and so $\vDash \alpha = \alpha$, thus $\Gamma_1 \vDash \varphi_1$ since $\Gamma_1 = \varnothing$.

Induction Step

Assume: $\Gamma_k \vDash \varphi_k$ for all $k \leqslant n$.

- If φ_{n+1} is a premise, assumption, or =I, then the same as above.
- Assume φ_{n+1} follows by the FOL$^=$ rules, to show $\Gamma_{n+1} \vDash \varphi_{n+1}$.
- There are 11 more rules in PL and an additional 5 from FOL$^=$.

Question: Can't we just appeal to the proofs from before for the PL rules?

- Models and variable assignments instead of interpretations.
- Differences are superficial and provided in the book.

Updated: Consider how the proofs of the following lemmas must change:

L2.1 If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

L4.3 For any FOL$^=$ proof X, if φ_k is live at line n where $k \leqslant n$, then $\Gamma_k \subseteq \Gamma_n$.

L11.2 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

L11.3 $\Gamma \cup \{\varphi\}$ is unsatisfiable just in case $\Gamma \vDash \neg\varphi$.

L11.4 If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \rightarrow \psi$.

Unchanged: The following lemmas do not need to change:

L9.1 $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$ if $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ.

L9.2 $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

PL Rules

($\neg$E) There is a proof of ψ at line h and $\neg\psi$ at line j from $\neg\varphi$ on line i.

- By hypothesis $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$, where $\Gamma_h, \Gamma_j \subseteq \Gamma_{n+1} \cup \{\neg\varphi_i\}$.
- By **L2.1**, $\Gamma_{n+1} \cup \{\neg\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\neg\varphi_i\} \vDash \neg\psi$.
- So $\Gamma_{n+1} \cup \{\neg\varphi_i\}$ is unsatisfiable by **L11.2**
- So $\Gamma_{n+1} \vDash \neg\neg\varphi_{n+1}$ follows by **L11.3**, and then appeal to the semantics.

FOL$^{=}$ Lemmas

L11.5 $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ and β is free for α in φ.

Base: Assume φ is $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ or $\alpha_1 = \alpha_2$ where $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$.

- Let $\gamma_i = \begin{cases} \beta & \text{if } \alpha_i = \alpha \\ \alpha_i & \text{otherwise.} \end{cases}$
- $\langle \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\gamma_1), \ldots, \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\gamma_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.
- $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_1) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha_n)$ *iff* $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\gamma_1) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\gamma_n)$.

Induction: If $\mathtt{Comp}(\varphi) \leqslant n$, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ whenever $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$.

Case 2: Assume $\varphi = \psi \wedge \chi$ where $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\chi) = 1$ *iff* ...

Case 6: Assume $\varphi = \forall\gamma\psi$ where $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$.

- If $\gamma = \alpha$, then $\varphi = \varphi[\beta/\alpha]$.
- If $\gamma \neq \alpha$, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\gamma\psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff*...
- Let $\hat{e}$ be an arbitrary γ-variant of $\hat{a}$.
- Since $\gamma \neq \alpha$, $\hat{e}(\alpha) = \hat{a}(\alpha)$ if α is a variable, so $\mathfrak{v}_{\mathcal{I}}^{\hat{e}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha)$.
- Thus $\mathfrak{v}_{\mathcal{I}}^{\hat{e}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ follows from the assumption.
- Since β is free for α in $\forall\gamma\psi$, we know that $\gamma \neq \beta$.
- If β is a variable, then $\hat{e}(\beta) = \hat{a}(\beta)$ since $\hat{e}$ is a γ-variant of $\hat{a}$.
- Thus $\mathfrak{v}_{\mathcal{I}}^{\hat{e}}(\beta) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$, and so $\mathfrak{v}_{\mathcal{I}}^{\hat{e}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{e}}(\beta)$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi[\beta/\alpha])$, where $\hat{e}$ was arbitrary.
- ...*iff* $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi[\beta/\alpha]) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha]) = 1$.

L11.6 If $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}' \rangle$ where $\mathcal{I}$ and $\mathcal{I}'$ agree about every constant α and n-place predicate $\mathcal{F}^n$ that occurs in φ, it follows that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$.

Base: $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathfrak{v}^{\hat{a}}_{\mathcal{I}'}(\alpha_1), \ldots, \mathfrak{v}^{\hat{a}}_{\mathcal{I}'}(\alpha_n) \rangle \in \mathcal{I}'(\mathcal{F}^n)$.

- $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ is immediate from the assumption.
- $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \mathcal{I}(\alpha_i) = \mathcal{I}'(\alpha_i) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a constant.
- $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \hat{a}(\alpha_i) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a variable.

L11.7 For a constant β not in $\forall\alpha\varphi$ or $\psi \in \Gamma$, if $\Gamma \vDash \varphi[\beta/\alpha]$, then $\Gamma \vDash \forall\alpha\varphi$.

- Assume $\Gamma \vDash \varphi[\beta/\alpha]$ for constant β not in $\forall\alpha\varphi$ or Γ.
- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$, and let $\hat{c}$ be arbitrary.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$ for all v.a. $\hat{a}$, and so for $\hat{c}$ in particular.
- Let $\mathcal{M}'$ be like $\mathcal{M}$ but for $\mathcal{I}'(\beta) = \hat{c}(\alpha)$, and so $\mathfrak{v}^{\hat{c}}_{\mathcal{I}'}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}'}(\beta)$.
- Since β does not occur in Γ, we know $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$ by **L11.6**.
- So $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$ by **L9.2**, so $\mathcal{V}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ by assumption.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ for all $\hat{a}$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ for $\hat{c}$ in particular.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi) = 1$ by **L11.5** given $\mathfrak{v}^{\hat{c}}_{\mathcal{I}'}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}'}(\beta)$ above.
- Since β is not in $\forall\alpha\varphi$, we know β is not in φ, so $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ by **L11.6**.
- Since $\hat{c}$ was arbitrary, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ for any $\hat{a}$, so $\mathcal{V}_{\mathcal{I}}(\forall\alpha\varphi) = 1$.
- By generalizing on $\mathcal{M}$, we may conclude that $\Gamma \vDash \forall\alpha\varphi$.

Soundness: Part II

LOGIC I
Benjamin Brast-McKie
November 21, 2024

FOL$^=$ Rules

(∀E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

(∃I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

(∀I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

(∃E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

(=I) $\vdash \alpha = \alpha$ for any constant α.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Lemmas From Last Time...

L2.1 If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

L4.3 For any FOL$^=$ proof X, if φ_k is live at line n where $k \leqslant n$, then $\Gamma_k \subseteq \Gamma_n$.

L9.1 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$ if $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ.

L9.2 $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

L11.2 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

L11.3 $\Gamma \cup \{\varphi\}$ is unsatisfiable just in case $\Gamma \vDash \neg\varphi$.

L11.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

L11.6 If $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}' \rangle$ where $\mathcal{I}$ and $\mathcal{I}'$ agree about every constant α and n-place predicate $\mathcal{F}^n$ that occurs in φ, it follows that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$.

Soundness of FOL$^=$

Assume: $\Gamma \vdash \varphi$, so there is a FOL$^=$ proof X of φ from Γ.

Lines: Let φ_i be the wfs on line i of X and Γ_i be the premises and undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

L11.1: (Base Step) $\Gamma_1 \vDash \varphi_i$.

L11.13: (Induction Step) If $\Gamma_k \vDash \varphi_k$ for all $k \leqslant n$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\varphi_m = \varphi$ and $\Gamma_m = \Gamma$, so $\Gamma \vDash \varphi$.

Induction Step

Assume: $\Gamma_k \vDash \varphi_k$ for all $k \leqslant n$ where φ_{n+1} follows by a FOL$^=$ rule.

- Want to show $\Gamma_{n+1} \vDash \varphi_{n+1}$ by each rule.

Universal Rules

L11.7 For a constant β not in $\forall\alpha\varphi$ or $\psi \in \Gamma$, if $\Gamma \vDash \varphi[\beta/\alpha]$, then $\Gamma \vDash \forall\alpha\varphi$.

- Assume $\Gamma \vDash \varphi[\beta/\alpha]$ for constant β not in $\forall\alpha\varphi$ or Γ.
- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$, and let $\hat{c}$ be arbitrary.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\chi) = 1$ for all $\chi \in \Gamma$ for all v.a. $\hat{a}$, and so for $\hat{c}$ in particular.
- Let $\mathcal{M}'$ be like $\mathcal{M}$ but for $\mathcal{I}'(\beta) = \hat{c}(\alpha)$, and so $\mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\alpha) = \mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\beta)$.
- Since β does not occur in Γ, we know $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\chi) = 1$ for all $\chi \in \Gamma$ by **L11.6**.
- So $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$ by **L9.2**, so $\mathcal{V}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ by assumption.
- So $\mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi[\beta/\alpha]) = 1$ for all $\hat{a}$, and so $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$ for $\hat{c}$ in particular.
- So $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) = 1$ by **L11.5** given $\mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\alpha) = \mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\beta)$ above.
- Since β is not in $\forall\alpha\varphi$, we know β is not in φ, so $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ by **L11.6**.
- Since $\hat{c}$ was arbitrary, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ for any $\hat{a}$, so $\mathcal{V}_{\mathcal{I}}(\forall\alpha\varphi) = 1$.

L11.8 $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$ where α is a variable and $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$.

- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\forall\alpha\varphi) = 1$, so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ for arbitrary $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for an α-variant $\hat{c}$ of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}(\beta)$.
- Since $\mathfrak{v}_{\mathcal{I}}^{\hat{c}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{c}}(\beta)$, we know $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$ by **L11.5**.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi[\beta/\alpha]) = 1$ by **L9.2**, so $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$ generalizing on $\mathcal{M}$.

L11.9 If $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then $\Gamma \cup \Sigma \vDash \psi$.

($\forall$I) $\varphi_i = \varphi[\beta/\alpha]$ for $i \leqslant n$ live at $n+1$ where β is not in φ_{n+1} or Γ_{n+1}.

- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L4.3**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L2.1**, so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$.
- So $\Gamma_{n+1} \vDash \forall\alpha\varphi$ by **L11.7** since β not in $\forall\alpha\varphi$ or Γ_{n+1}.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\forall$E) $\varphi_i = \forall\alpha\varphi$ for $i \leqslant n$ live at $n+1$ where $\varphi_{n+1} = \varphi[\beta/\alpha]$.

- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L4.3**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L2.1**, so $\Gamma_{n+1} \vDash \forall\alpha\varphi$.
- By **L11.8** $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$, and so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$ by **L11.9**.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Existential and Identity Lemmas

L11.10 $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ where α is a variable and $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$.

- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_\mathcal{I}(\varphi[\beta/\alpha]) = 1$, so $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi[\beta/\alpha]) = 1$ for all $\hat{a}$.
- Let $\hat{c}$ be s.t. $\hat{c}(\alpha) = \mathcal{I}(\beta)$, so $\mathfrak{v}_\mathcal{I}^{\hat{c}}(\alpha) = \mathfrak{v}_\mathcal{I}^{\hat{c}}(\beta)$ where β is free for α in φ.
- So $\mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi[\beta/\alpha]) = \mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi)$ by **L11.5**.
- Since $\hat{c}$ is an α-variant of itself, $\mathcal{V}_\mathcal{I}^{\hat{c}}(\exists\alpha\varphi) = 1$ by semantics.
- Since $\varphi[\beta/\alpha]$ is a wfs $\mathcal{L}^=$, at most α is free in φ, and so $\exists\alpha\varphi$ is a wfs.
- Hence $\mathcal{V}_\mathcal{I}(\exists\alpha\varphi) = 1$ by **L9.2**, and so $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$.

L11.11 For any constant β that does not occur in $\exists\alpha\varphi$, ψ, or in any sentence $\chi \in \Gamma$, if $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$, then $\Gamma \vDash \psi$.

- Let $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$ for a constant β not in $\exists\alpha\varphi$, ψ, or Γ.
- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_\mathcal{I}(\chi) = 1$ for all $\chi \in \Gamma$.
- So $\mathcal{V}_\mathcal{I}(\exists\alpha\varphi) = 1$, and so $\mathcal{V}_\mathcal{I}^{\hat{a}}(\exists\alpha\varphi) = 1$ for some v.a. $\hat{a}$ by **L9.2**.
- So $\mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi) = 1$ for a α-variant $\hat{c}$ of $\hat{a}$ by the semantics.
- Let $\mathcal{M}'$ be like $\mathcal{M}$ but for $\mathcal{I}'(\beta) = \hat{c}(\alpha)$, and so $\mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\beta) = \mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\alpha)$.
- Since β is not in $\exists\alpha\varphi$, it's not in φ, and so $\mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi)$ by **L11.6**.
- Since $\mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\beta) = \mathfrak{v}_{\mathcal{I}'}^{\hat{c}}(\alpha)$, we know $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha])$ by **L11.5**.
- Thus $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ by **L9.2**.
- Given any $\hat{e}$, we know $\mathcal{V}_\mathcal{I}(\chi) = 1$ and so $\mathcal{V}_\mathcal{I}^{\hat{e}}(\chi) = 1$ for any $\chi \in \Gamma$.
- Given the assumptions about β, by **L11.6** $\mathcal{V}_{\mathcal{I}'}^{\hat{e}}(\chi) = 1$, so $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$.
- Generalizing on χ, it follows that $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$.
- Hence $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma \cup \{\varphi[\beta/\alpha]\}$.
- So $\mathcal{V}_{\mathcal{I}'}(\psi) = 1$ by assumption, so $\mathcal{V}_{\mathcal{I}'}^{\hat{g}}(\psi) = 1$ for some $\hat{g}$ by **L9.2**.
- Since β does not occur in ψ, $\mathcal{V}_\mathcal{I}^{\hat{g}}(\psi) = 1$ for all $\hat{g}$ by **L11.6**.
- Thus $\mathcal{V}_\mathcal{I}(\psi) = 1$ by **L9.2**, so $\Gamma \vDash \psi$ by generalizing on $\mathcal{M}$.

L11.12 If α and β are constants, then $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$.

- Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_\mathcal{I}(\varphi[\alpha/\gamma]) = \mathcal{V}_\mathcal{I}(\alpha = \beta) = 1$.
- By **L9.2**, $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi[\alpha/\gamma]) = 1$ for some $\hat{a}$, where $\mathcal{V}_\mathcal{I}^{\hat{c}}(\alpha = \beta) = 1$ for all $\hat{c}$.
- So $\mathcal{V}_\mathcal{I}^{\hat{a}}(\alpha = \beta) = 1$ in particular, and so $\mathfrak{v}_\mathcal{I}^{\hat{a}}(\alpha) = \mathfrak{v}_\mathcal{I}^{\hat{a}}(\beta)$ by semantics.
- Since β is a constant, $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi[\alpha/\gamma]) = \mathcal{V}_\mathcal{I}^{\hat{a}}((\varphi[\alpha/\gamma])[\beta/\alpha])$ by **L11.5**.
- However, $(\varphi[\alpha/\gamma])[\beta/\alpha] = \varphi[\beta/\gamma]$, and so $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi[\beta/\gamma]) = 1$.
- Since $\varphi[\beta/\gamma]$ is a wfs of $\mathcal{L}^=$, we know $\mathcal{V}_\mathcal{I}(\varphi[\beta/\gamma]) = 1$.
- By generalizing on $\mathcal{M}$ we may conclude that $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$.

Existential and Identity Rules

($\exists$I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\exists$I.

- Assume that φ_{n+1} follows from Γ_{n+1} by existential introduction $\exists$I.
- Thus $\varphi_i = \varphi[\beta/\alpha]$ is live at $n+1$ for some $i \leqslant n$ where $\varphi_{n+1} = \exists\alpha\varphi$.
- By **L4.3**, we know $\Gamma_i \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ by hypotheses.
- So $\Gamma_{n+1} \vDash \varphi_i$ by **L2.1**, and so equivalently, $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$.
- Since $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ by **L11.10**, we know $\Gamma_{n+1} \vDash \varphi_{n+1}$ by **L11.9**.

($\exists$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\exists$E.

- Assume that φ_{n+1} follows from Γ_{n+1} by existential elimination $\exists$E.
- For some $i < j < k \leqslant n$ where $\varphi_i = \exists\alpha\varphi$ is live at $n+1$, $\varphi_j = \varphi[\beta/\alpha]$ for some constant β that does not occur in φ_i, φ_k, or any $\psi \in \Gamma_i$.

$$
\begin{array}{l|l l}
i & \exists\alpha\varphi & \\
j & \quad \begin{array}{|l} \varphi[\beta/\alpha] \\ \hline \vdots \end{array} & :\text{AS for } \exists\text{E} \\
k & \quad \begin{array}{|l} \psi \end{array} & \\
n+1 & \psi & :i,\ j\text{–}k\ \exists\text{E}
\end{array}
$$

- By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_k \vDash \varphi_k$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **L4.3**.
- With the exception of φ_j, every assumption that is undischarged at line k is also undischarged at line $n+1$, and so $\Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_j\}$.
- By **L2.1** we know $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_k$.
- Equivalently, $\Gamma_{n+1} \vDash \exists\alpha\varphi$ and $\Gamma_{n+1} \cup \{\varphi[\beta/\alpha]\} \vDash \psi$.
- Thus $\Gamma_{n+1} \vDash \psi$ by **L11.11**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($=$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $=$E.

- Assume that φ_{n+1} follows from Γ_{n+1} by existential elimination $=$E.
- By parity, φ_i is $\alpha = \beta$, $\varphi_j = \varphi[\alpha/\gamma]$, and $\varphi_{n+1} = \varphi[\beta/\gamma]$.
- By **L4.3**, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$ by hypotheses.
- So $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$ by **L2.1**.
- Equivalently, $\Gamma_{n+1} \vDash \alpha = \beta$ and $\Gamma_{n+1} \vDash \varphi[\alpha/\gamma]$.
- So $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$ by **L11.12** since α and β are constants.
- By two applications of **L11.9**, $\Gamma_{n+1} \vDash \varphi[\beta/\gamma]$, so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Completeness of FOL$^=$: Part I

LOGIC I
Benjamin Brast-McKie
December 2, 2024

Basic Lemmas

L9.1 If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

L11.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

L11.6 If $\mathcal{M}$ and $\mathcal{M}'$ have the same domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any v.a. $\hat{a}$ defined over $\mathbb{D}$.

L12.1 If α is a constant and X is an FOL$^=$ derivation in which the constant β does not occur, then $X[\beta/\alpha]$ is also an FOL$^=$ derivation.

L12.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L12.4 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L12.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L12.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L12.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Completeness

T12.1 Every consistent set of $\mathcal{L}^=$ wfss Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable (check).
- So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T12.1**.
- So $\Gamma \vdash \neg\neg\varphi$ by **L12.3**, and so $\Gamma \vdash \varphi$ by DN and **L12.4**.

Assume: Let Γ be a set of $\mathcal{L}^=$ wfss that is consistent in FOL$^=$.

Saturation

Extension: Let $\mathcal{L}^{=}_{\mathbb{N}}$ include the extra constants $\mathbb{N}$.

Constants: Let $\mathbb{C}$ be the set of all constants in $\mathcal{L}^{=}_{\mathbb{N}}$.

Free: Let $\varphi(\alpha)$ be a wff of $\mathcal{L}^{=}_{\mathbb{N}}$ with at most one free variable α.

Saturated: A set of wfss Σ is SATURATED in $\mathcal{L}^{=}_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathcal{L}^{=}_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \to \varphi[\beta/\alpha]) \in \Sigma$.

L12.2 Γ is consistent in $\text{FOL}^{=}_{\mathbb{N}}$.

- We will hence forth take 'consistent' to mean 'consistent in $\text{FOL}^{=}_{\mathbb{N}}$'.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathcal{L}^{=}_{\mathbb{N}}$ with at most one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \to \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.
$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \to \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in φ_{k+1} or θ_j for any $j \leqslant k$.

Saturation: $\Sigma_0 = \Gamma$,
$\Sigma_{n+1} = \Sigma_n \cup \{\theta_{n+1}\}$, and
$\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

L12.5 Σ_Γ is consistent and saturated in $\mathcal{L}^{=}_{\mathbb{N}}$.

Base: $\Sigma_0 = \Gamma$ is consistent.

- Immediate from **L12.2**.

Induction: Assume Σ_m is consistent.

- Assume $\Sigma_{m+1} = \Sigma_m \cup \{\theta_{m+1}\}$ is inconsistent for contradiction.
- So $\Sigma_m \vdash \neg\theta_{m+1}$ by **L12.3**, and so $\Sigma_m \vdash \neg(\exists\alpha_{m+1}\varphi_{m+1} \to \varphi_{m+1}[n_{m+1}/\alpha_{m+1}])$.
- So $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$ by derived PL rules.
- So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I since n_{m+1} is not in $\forall\alpha_{m+1}\neg\varphi_{m+1}$ or Σ_m.
- So $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$, and so Σ_m is inconstant by **L12.9**.
- It follows by *reductio* that Σ_{m+1} is consistent.
- By weak induction, we know that Σ_k is consistent for all $k \in \mathbb{N}$.

Limit: If Σ_Γ is inconsistent, then X derives $\bot$ from Σ_Γ in $\text{FOL}^{=}_{\mathbb{N}}$.

- Since X is finite, $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$ including all premises in X.
- So Σ_m is inconsistent, contradicting the above.
- By *reductio*, Σ_Γ is consistent.

Maximization

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{=}_{\mathbb{N}}$ just in case either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every wfs ψ in $\mathcal{L}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{=}_{\mathbb{N}}$.

Maximization: $\Delta_0 = \Sigma_\Gamma$,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases},$$

$\Delta_\Sigma = \bigcup_{i\in\mathbb{N}} \Delta_n$.

L12.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal in $\mathcal{L}^{=}_{\mathbb{N}}$ and consistent.

Base: $\Delta_0 = \Sigma_\Gamma$ is consistent by **L12.5**.

Induction: Assume Δ_n is consistent.

- Want to show that Δ_{n+1} is consistent.

Case 1: If $\Delta_n \cup \{\psi_n\}$ is consistent, then $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: If $\Delta_n \cup \{\psi_n\}$ is inconsistent, then $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

- Assume $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent for contradiction.
- So Δ_n is inconsistent by **L12.6**.
- So Δ_{n+1} is consistent in both cases.

Limit If Δ is inconsistent, then Y derives $\bot$ from Δ in $\text{FOL}^{=}_{\mathbb{N}}$.

- Since Y is finite, $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$ including all premises in Y.
- This contradicts the above, and so Δ is consistent by *reductio*.

Maximal: Let ψ be any wfs of $\mathcal{L}^{=}_{\mathbb{N}}$, and so $\psi = \psi_k$ for some $k \in \mathbb{N}$.

- By construction, $\psi_k \in \Delta_{k+1}$ or $\neg\psi_k \in \Delta_{k+1}$.
- Generalizing on ψ shows that Δ is maximal.

L12.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

- Immediate from the definitions.

L12.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

- Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L12.9**.
- So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
- Thus $\varphi \in \Delta$ by maximality.

Completeness of FOL$^=$: Part II

LOGIC I
Benjamin Brast-McKie
December 2, 2024

Definitions

Extension: Let $\mathcal{L}^=_{\mathbb{N}}$ include the extra constants $\mathbb{N}$.

Constants: Let $\mathbb{C}$ be the set of all constants in $\mathcal{L}^=_{\mathbb{N}}$.

Free: Let $\varphi(\alpha)$ be a wff of $\mathcal{L}^=_{\mathbb{N}}$ with at most one free variable α.

Saturated: A set of wfss Σ is SATURATED in $\mathcal{L}^=_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathcal{L}^=_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \to \varphi[\beta/\alpha]) \in \Sigma$.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathcal{L}^=_{\mathbb{N}}$ with at most one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \to \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.
$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \to \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in φ_{k+1} or θ_j for any $j \leqslant k$.

Saturation: $\Sigma_0 = \Gamma$,
$\Sigma_{n+1} = \Sigma_n \cup \{\theta_{n+1}\}$, and
$\Sigma_\Gamma = \bigcup\limits_{i\in\mathbb{N}} \Sigma_n$.

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^=_{\mathbb{N}}$ just in case either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every wfs ψ in $\mathcal{L}^=_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^=_{\mathbb{N}}$.

Maximization: $\Delta_0 = \Sigma_\Gamma$,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases},$$

$\Delta_\Sigma = \bigcup_{i\in\mathbb{N}} \Delta_n$.

Lemmas

L9.1 If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

L11.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

L11.6 If $\mathcal{M}$ and $\mathcal{M}'$ have the same domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any v.a. $\hat{a}$ defined over $\mathbb{D}$.

L12.1 If α is a constant and X is an FOL$^=$ derivation in which the constant β does not occur, then $X[\beta/\alpha]$ is also an FOL$^=$ derivation.

L12.2 Γ is consistent in FOL$^=_{\mathbb{N}}$.

L12.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L12.4 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L12.5 Σ_Γ is consistent and saturated in $\mathcal{L}^{=}_{\mathbb{N}}$.

L12.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L12.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal in $\mathcal{L}^{=}_{\mathbb{N}}$ and consistent.

L12.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

L12.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L12.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

L12.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Completeness

T12.1 Every consistent set of $\mathcal{L}^{=}$ wfss Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable (check).
- So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T12.1**.
- So $\Gamma \vdash \neg\neg\varphi$ by **L12.3**, and so $\Gamma \vdash \varphi$ by DN and **L12.4**.

Assume: Let Γ be a set of $\mathcal{L}^{=}$ wfss that is consistent in FOL$^{=}$.

Henkin Model

Element: $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

Domain: $\mathbb{D}_\Delta = \{[\alpha]_\Delta \subseteq \mathbb{C} : \alpha \in \mathbb{C}\}$.

L12.12 $\alpha \in [\alpha]_\Delta$ for any constant $\alpha \in \mathbb{C}$.

- Still need to check that elements in $\mathbb{D}_\Delta$ are well defined.

L12.13 If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

- Assuming $\alpha = \beta \in \Delta$ where $\gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
- So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L12.11**.
- Thus $\beta = \gamma \in \Delta$ by **L12.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

Constants: $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

Predicates: $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}^n_\Delta : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

L12.14 If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

- Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
- $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L12.10**.
- Parity of reasoning completes the proof.

Henkin Lemmas

L12.15 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

LTR: Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$, we know $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

- So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
- Thus $\mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L11.5**.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L9.1**.

RTL: Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.

- Let $\hat{c}$ be the α-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\beta)$.
- Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L11.5**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

L12.16 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

- Similar to **L12.15**.

L12.17 $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

- Let $\hat{a}$ be arbitrary.

Base: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

- If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L12.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
- Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L12.13**.
- The predicate case is immediate from *Predicates* definition.

Induction: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leqslant n$.

- Let φ be a wfs of $\mathcal{L}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

Case 2: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

Case 6: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L12.15**.

... *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.

$(\exists)$ ... *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L12.10** given saturation.

- If $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$, then $\Delta \vdash \exists\alpha\psi$ by $\exists$I, so $\exists\alpha\psi \in \Delta$ by **L12.10**.
- If $\exists\alpha\psi \in \Delta$ instead, then $\psi = \varphi_i(\alpha_i)$ for some $i \in \mathbb{N}$ where $\alpha_i = \alpha$.
- Thus $\exists\alpha_i\varphi_i \rightarrow \varphi_i[n_i/\alpha_i] \in \Delta$ by the saturation assumed of Δ.
- Since $n_i \in \mathbb{C}$, it follows that $\exists\alpha\psi \rightarrow \psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$.
- So $\Delta \vdash \psi[\beta/\alpha]$ by conditional elimination $\rightarrow$E.
- Thus $\psi[\beta/\alpha] \in \Delta$ by **L12.10**, thereby establishing $(\exists)$.

Conclusion: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

Restriction

Restriction: $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant α in $\mathcal{L}^=_{\mathbb{N}}$.
$\mathcal{I}'_\Delta(\mathcal{F}^n) = \mathcal{I}_\Delta(\mathcal{F}^n)$ for all n-place predicates $\mathcal{F}^n$.

L12.18 $\mathcal{V}_{\mathcal{I}'_\Delta}(\varphi) = \mathcal{V}_{\mathcal{I}'_\Delta}(\varphi)$ for any wfs φ of $\mathcal{L}^=$.

- Let $\hat{a}$ be an arbitrary v.a. defined over $\mathbb{D}_\Delta$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}'_\Delta}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi)$ for any variable assignment $\hat{a}$ by **L11.6**.

T12.1 Every consistent set Γ of wfss in $\mathcal{L}^=$ is satisfiable.

- Let Γ be a consistent set of $\mathcal{L}^=$ sentences in FOL$^=$.
- By **L12.2**, Γ is a set of $\mathcal{L}^=_{\mathbb{N}}$ sentences that is consistent in FOL$^=_{\mathbb{N}}$.
- So Σ_Γ is consistent and saturated in $\mathcal{L}^=_{\mathbb{N}}$ by **L12.5**.
- Given **L12.7** and **L12.8**, Δ_{Σ_Γ} is a saturated maximal consistent set of sentences in $\mathcal{L}^=_{\mathbb{N}}$ where $\Gamma \subseteq \Delta_{\Sigma_\Gamma}$.
- Letting $\Delta = \Delta_{\Sigma_\Gamma}$, **L12.7** shows that the Henkin model $\mathcal{M}_\Delta$ is such that $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, and so $\mathcal{M}_\Delta$ satisfies Δ.
- Having shown that $\Gamma \subseteq \Delta$, we know that $\mathcal{M}_\Delta$ satisfies Γ.
- Since Γ is a set of $\mathcal{L}^=$ sentences, it follows by **L12.18** that there is a model $\mathcal{M}'_\Delta$ of $\mathcal{L}^=$ that satisfies Γ, so Γ is satisfiable.

Compactness

C12.2 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

- Assuming $\Gamma \vDash \varphi$, we know $\Gamma \vdash \varphi$ by FOL$^=$ Completeness.
- So there is a derivation X of φ from Γ.
- Let Γ_X be the set of premises which appear in X.
- It follows that $\Gamma_X \vdash \varphi$, and so $\Gamma_X \vDash \varphi$ by FOL$^=$ Soundness.
- Since X is finite, Γ_X is finite, and so $\Lambda \vDash \varphi$ for a finite subset $\Lambda \subseteq \Gamma$.

C12.3 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

- Assume for contraposition that Γ is unsatisfiable.
- Vacuously, $\Gamma \vDash \bot$, and so $\Lambda \vDash \bot$ by **C12.2** for a finite subset $\Lambda \subseteq \Gamma$.
- Hence Λ must also be unsatisfiable.
- Thus there is some finite subset $\Lambda \subseteq \Gamma$ that is unsatisfiable.
- By contraposition, QED.

Question: Is it possible to express 'At most finitely many xs are F' in $\mathcal{L}^=$?

- But $\Gamma_\infty = \{\exists_{\geqslant n} xFx : n \in \mathbb{N}\} \vDash \neg\exists^\star \alpha Fx$.
- So $\Lambda_\infty \vDash \neg\exists^\star \alpha Fx$ for finite $\Lambda_\infty \subseteq \{\exists_{\geqslant n} xFx : n \in \mathbb{N}\}$.