Quantified Logic with Identity

LOGIC I
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Logical Terms

Extensions: QL extends SL, but we needn't stop there.

Question 1: How far could we go? What terms could we include?

Logicality: The primitive symbols of SL and QL can be divided in three:

`Logical Terms:` $\neg, \wedge, \vee, \supset, \equiv, \forall\alpha, \exists\alpha, x_n, y_n, z_n \ldots$ for $n \geq 0$.
`Non-Logical Terms:` $a_n, b_n, c_n, \ldots$ and $A^n, B^n, \ldots$ for $n \geq 0$.
`Punctuation:` $(,)$

Extensions: The "meanings" of the non-logical terms are fixed by an interpretation.

Semantics: The "meanings" of the logical terms are fixed by the semantics.

Question 2: How many logical terms are there?

Identity: At least one more, namely identity which we symbolize by '$=$'.

Syntax for QL$^=$

Identity: We include '$=$' in the primitive symbols of the language.

Well-Formed Formulas: We may define the well-formed formulas (wffs) of QL$^=$ as follows:

1. $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ is a wff if $\mathcal{F}^n$ is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. $\alpha = \beta$ is a wff if α and β are singular terms.
3. If φ and ψ are wffs and α is a variable, then:
   (a) $\exists\alpha\varphi$ is a wff;
   (b) $\forall\alpha\varphi$ is a wff;
   (c) $\neg\varphi$ is a wff;
   (d) $(\varphi \wedge \psi)$ is a wff;
   (e) $(\varphi \vee \psi)$ is a wff;
   (f) $(\varphi \supset \psi)$ is a wff; and
   (g) $(\varphi \equiv \psi)$ is a wff.
4. Nothing else is a wff.

Atomic Formulas: The wffs defined by (1) and (2) are *atomic*.

Complexity: $\texttt{Comp}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = \texttt{Comp}(\alpha = \beta) = 0$.
$\texttt{Comp}(\exists\alpha\varphi) = \texttt{Comp}(\forall\alpha\varphi) = \texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$.
$\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi \vee \psi) = \ldots = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$.

Free Variables

Free Variables: We define the *free variables* recursively:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leq i \leq n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. α is free in $\beta = \gamma$ if $\alpha = \beta$ or $\alpha = \gamma$ where α is a variable.
3. If φ and ψ are wffs and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;

   $\vdots$
4. Nothing else is a free variable.

Sentences of QL$^=$

Sentences: A *sentence* of QL$^=$ is any wff without free variables.

Interpretation: Only the sentences of QL$^=$ will have truth-values on an interpretation independent of an assignment function.

QL$^=$ Models

Question 3: What in the semantics will have to change?

Interpretations: $\mathcal{I}$ is an QL$^=$ interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL$^=$.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL$^=$ *iff* $\mathcal{I}$ is a QL$^=$ interpretation on $\mathbb{D} \neq \varnothing$.

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL$^=$.

Referents: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Semantics for QL$^=$

(A) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1,\ldots,\alpha_n) = 1$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1),\ldots,\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.

$(=)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha = \beta) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

$(\forall)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

$(\exists)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

$(\neg)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$.

$\vdots$

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL$^=$.

Example

Task 1: Prove that the following argument is valid.

(1) Hesperus is Phosphorus.
(2) Phosphorus is Venus.
$\therefore$ Hesperus is Venus.

Task 2: Prove that $\forall x\forall y\forall z((x = y \wedge y = z) \supset x = z)$ is a tautology.

Logical Predicates

Taller-Than: Suppose we were to take 'taller than' (T) to be logical.

Question 4: Could we provide its semantics?

(T) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(T\alpha\beta) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha)$ is taller than $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

Theory: The semantics would have to rely on a theory of being taller than.

- Providing such a theory lies outside the subject-matter of logic.
- By contrast, identity is something we already grasp.
- Compare our pre-theoretic grasp of negation, conjunction, and the quantifiers.

Question 5: Could we take set-membership $\in$ to be a logical term?

Question 6: What is it to be a logical term?

Existence: Observe that $\exists x(x = x)$ is a tautology.

Question 7: Could we take a term in sentence position to be logical?

$(\bot)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\bot) = 1$ *iff* $1 \neq 1$.

$(\top)$ $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\top) = 1$ *iff* $1 = 1$.

Assignment Lemmas

Lemma 1: If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$.

Base: Assume $\texttt{Comp}(\varphi) = 0$, so $\varphi = (\alpha = \beta)$ or $\varphi = \mathcal{F}^n\alpha_1, \ldots, \alpha_n$.

$(\alpha = \beta)$: So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha = \beta) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\beta)$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\alpha) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\beta) \ldots$

$(\mathcal{F}^n\alpha_1, \ldots, \alpha_n)$: So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}(F^n) \ldots$

Lemma 2: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

Lemma 3: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Leibniz's Law

Believes: Regiment the following argument:

(1) Lois Lane believes that Superman can fly.
(2) Superman is Clark Kent.
∴ Lois Lane believes that Clark Kent can fly.

Sees: Regiment the following argument:

(1) Lois Lane sees Superman.
(2) Superman is Clark Kent.
∴ Lois Lane sees Clark Kent.

Question 8: Are these arguments intuitively valid?

Opacity: Whereas 'sees' admits substitution, 'believes' does not.

Transparency: We may say that 'sees' is transparent and that 'believes' is opaque.

Mathematics: Importantly, mathematics is transparent insofar as it does not include any opaque contexts.