XIII. Further topics

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a. History of logic

The beginnings

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- Ancient Greece: Aristotle (384-322 BCE)
- Cataloged valid arguments ("syllogisms"), e.g.,
- All ungulates have hooves.
  No fish have hooves.
  ∴ No fish are ungulates.

The middle ages

- Ibn Sīnā (Avicenna)

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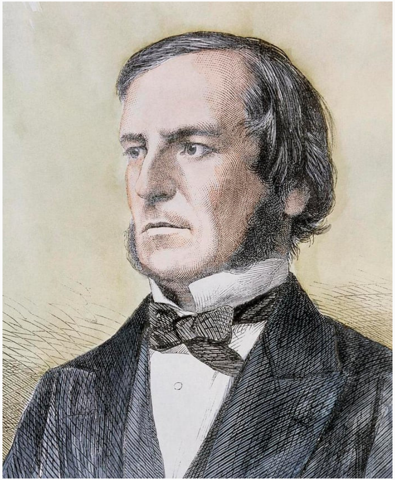

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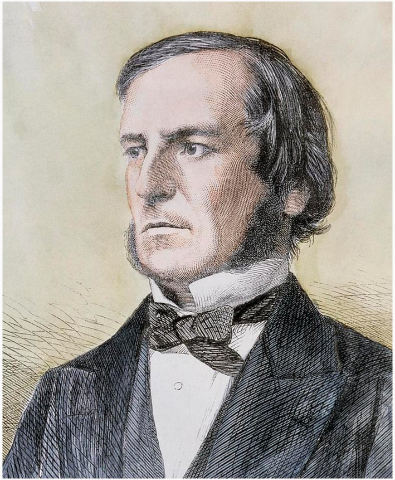

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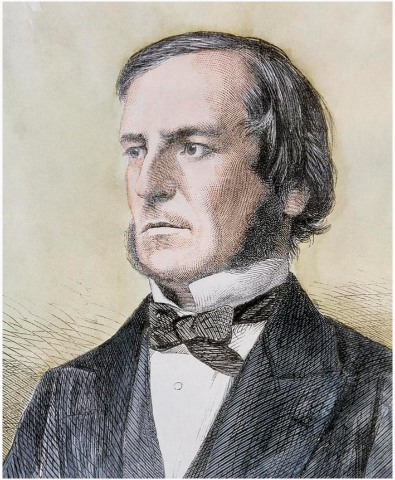

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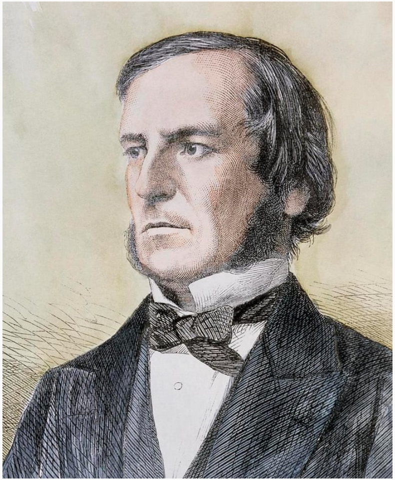

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- Charles Lutwidge Dodgson (aka Lewis Caroll)

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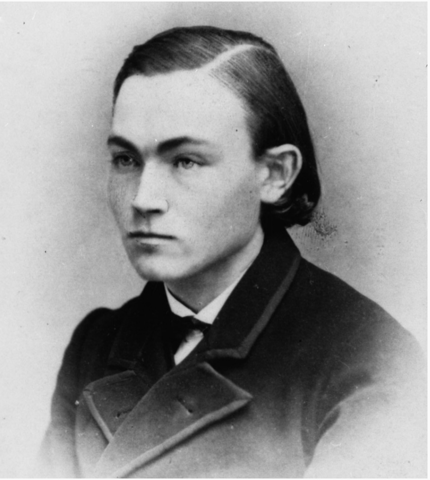

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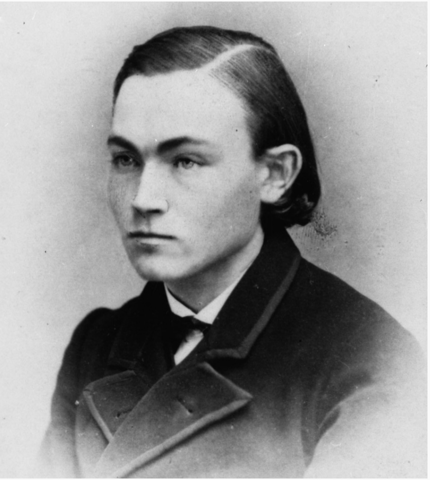

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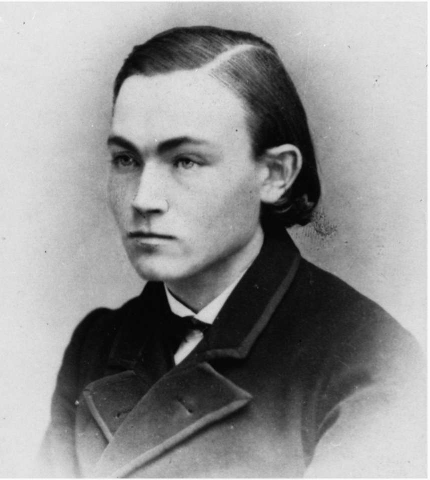

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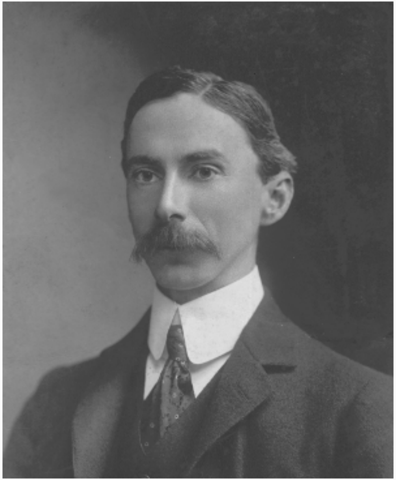

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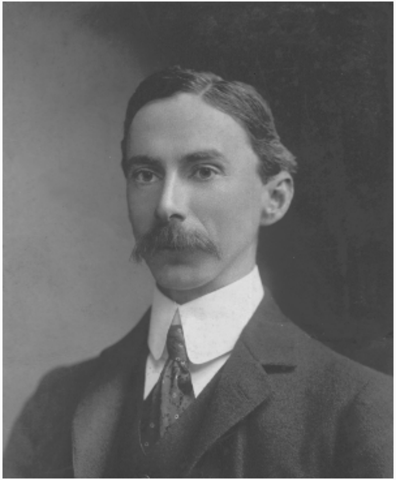

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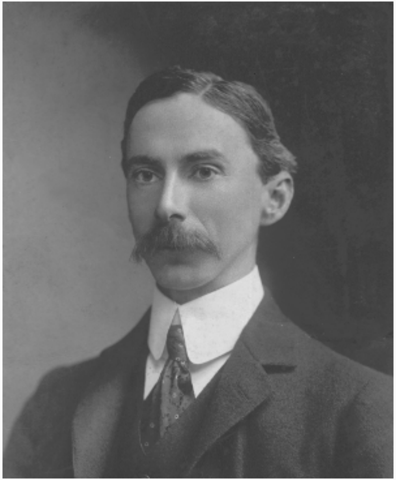

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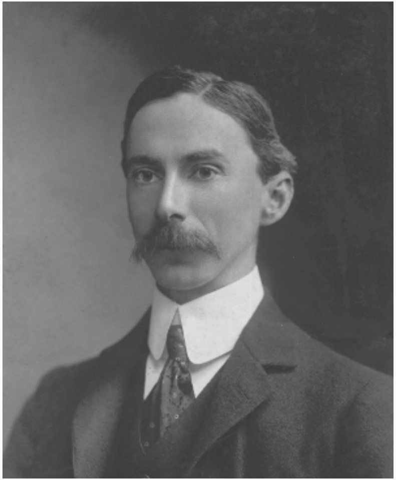

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XIII. Further topics

b. Philosophy and nonstandard logics

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- Difficulty: What logically possible circumstances are there?

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  - Every argument valid in QL is valid
  - Every argument with a formal proof is valid (soundness!)

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- Non-standard logics: expand SL, QL to deal with these

Many-valued logic

- Add to the truth-values **T** and **F**, e.g.,
  - "Undetermined": neither true nor false

P	$\sim P$
T	F
U	U
F	T

P	Q	$(P \,\&\, Q)$
T	T	T
T	U	U
T	F	F
U	T	U
U	U	U
U	F	F
F	T	F
F	U	F
F	F	F

P	Q	$(P \vee Q)$
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T	U	T
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F	T	T
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  - "Inconsistent": both true and false
  - Fuzzy truth values: any number between 0 and 1

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A connective $*$ is **truth functional** iff the truth value of $*A$ depends only on the truth value of A.

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- “Richard believes that”, “Richard knows that”

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- Indicative conditional is (plausibly) **truth–functional**: truth value of “If P, then Q” depends **only on truth values** of P and Q.

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- Temporal logic
  "It was true that" (P), "It will be true that" (F)

$$\mathbf{F}\,\mathbf{P}\,A \supset (\mathbf{P}\,A \vee A \vee \mathbf{F}\,A)$$

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XIII. Further topics

c. Metalogic and applications

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- A validity is a sentence that's true in all interpretations

Soundness and completeness

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  Arguments have formal proofs **only if** they are valid
  If there is a proof of B from premises $A_1, \ldots A_n$, then $A_1, \ldots A_n$ entail B in QL.

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Proved by Kurt Gödel (1929)

Church–Turing Theorem

Instance: Sentence *A* of QL
Problem: Is *A* a validity/provable?

- Undecidable: no computer program can answer this question correctly for all *A*.
- Proved independently by Alonzo Church and Alan Turing in 1935

Cook's Theorem

Instance: Sentence A of SL
Problem: Is A a tautology?

- Decidable: write a computer program that checks all valuations for A.
- But: it's hard: "co−NP complete"
- Proved independently by Stephen Cook (1971) and Leonid Levin (1973)

Decidable classes

- The decision problem **in general** is undecidable
- But special cases **can** be decided, e.g.:

Instance: Sentence A with only 1-place predicate symbols
Problem: Is A a validity?

- Decidable
- Proved by Leopold Löwenheim (1915)
- Complexity is NEXPTIME-complete.

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  - KR classification systems, e.g., SNOMED-CT

Theories

- A set of sentences of QL also called a **theory**, and the sentences in it **axioms**
- Some (types of) interpretations can be characterized as those interpretations in which every sentence in the theory is true
- Examples:
  - Mathematical theories (theory of orders, group theory, arithmetic)
  - KR classification systems, e.g., SNOMED-CT
  - Mereology, theories of truth, scientific theories

The axiomatic method

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- Axiomatic method: do science by investigating what follows from the axioms of a theory
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  - Are axioms independent, or is one superfluous?
- Paradigm of axiomatic method: geometry (Euclid)

Examples of theories: linear orders

A relation $\preceq$ on a set O is a **linear order** iff it makes following axioms true:

$$(\forall x)(\forall y)((x \preceq y \;\&\; y \preceq x) \supset x = y) \qquad \text{Antisymmetry}$$

$$(\forall x)(\forall y)(\forall z)((x \preceq y \;\&\; y \preceq z) \supset x \preceq z) \qquad \text{Transitivity}$$

$$(\forall x)(\forall y)(x \preceq y \lor y \preceq x) \qquad \text{Totality}$$

Every total relation is reflexive:

$$LO \models (\forall x)\, x \preceq x$$

Examples of theories: Robinson's Q

Theories of arithmetic, such as Robinson's theory Q:

$$
\begin{aligned}
\sim(\exists x)\,(x+1) &= 0 \\
(\forall x)(x = 0 \vee (\exists y)\,(y+1) &= x) \\
(\forall x)(\forall y)((x+1) = (y+1) \supset x &= y) \\
(\forall x)\,(x+0) &= x \\
(\forall x)(\forall y)\,(x + (y+1)) &= ((x+y)+1) \\
(\forall x)\,(x \times 0) &= 0 \\
(\forall x)(\forall y)\,(x \times (y+1)) &= ((x \times y) + x)
\end{aligned}
$$

Examples of theories: SNOMED-CT

```
bacterial pneumonia =
         is-a|bacterial infectious disease
         is-a|infective pneumonia
         causative agent|bacteria
         finding site|lung structure
```

$$
\begin{aligned}
(\forall x)(& BacterialPneumonia(x) \equiv \\
& BacterialInfectiousDisease(x)\ \& \\
& InfectivePneumonia(x)\ \& \\
& (\exists y)(HasCausativeAgent(x,y)\ \&\ Bacteria(y))\ \& \\
& (\exists y)(HasFindingSite(x,y)\ \&\ LungStructure(y)))
\end{aligned}
$$

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- SNOMED-CT is decidable

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- Some axioms:

$(\forall x)\, Pt(x, x)$	Reflexivity
$(\forall x)(\forall y)(\forall z)((Pt(x, y) \,\&\, Pt(y, z)) \supset Pt(x, z))$	Transitivity
$(\forall x)(\forall y)((Pt(x, y) \,\&\, Pt(y, x)) \supset x = y)$	Antisymmetry

Examples of Theories: Mereology

- Defined properties and relations

$$PP(x, y) \equiv (Pt(x, y) \,\&\, \sim x = y)$$
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  - Does everything comprise at least one atom?
  - Is everything made of atomless “gunk”?

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- **Gödel's Incompleteness Theorem (1930)**
  Arithmetic, set theory, mereology are incompleteable
- Philosophical upshot of this: truth in the intended interpretation(s) of the theory outstrips provability from the theory

XIII. Further topics

d. A logical party trick

Santa Claus party trick

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5	The first sentence on this slide is true.	$S \vdash S$ is true
6	Santa Claus exists.	:4, 5 $\supset$E