14. Final Review!

14. Final Review!

a. Checking Soundness for Alt. Rules

Alternative Natural Deduction Rules

- As we did with trees (system STD), we can consider whether modifying SND with a new rule preserves soundness

- Method for generating new cases: take a case in the book and add a negation symbol(s) somewhere;

  then figure out what a sound rule would give you.

Negated Conjunction Introduction

- Consider a system SND* just like SND except that we add the following rule:
- **Negated Conjunction Introduction**: from $\sim\mathcal{Q}$ derive $\sim(\mathcal{Q} \,\&\, \mathcal{R})$
- Does this rule preserve soundness? If so, extend our proof by adding a case to the induction (showing that the new line is righteous); If not, provide a concrete counterexample to soundness of SND*
- Strategy: first do a heuristic: do the earlier accessible sentences semantically entail the final sentence?
  - If yes, then the new rule preserves soundness (proceed to formally extend the proof!)
  - If no, then you should be able to construct a concrete counterexample to soundness (i.e. case where $\Gamma \vdash_{SND^*} P$ but $\Gamma \nvDash P$ for a concrete set of SL sentences Γ

Notation for Soundness Cases

- Γ_i stands for the set of assumptions that are open at the i–th line, i.e. these are the accessible premises/assumptions at line i. They are every premise/assumption (sentence sitting on a horizontal line) such that its scope line (vertical line) travels all the way down to line i, and line i is to the right of this vertical line.
- $\mathbf{P_i}$ stands for the sentence that is on the i–th line.
- $\Delta \subseteq \Gamma$ means that the set Δ is a subset of Γ.
- $\Gamma \cup \{\mathbf{Q}\}$ means that we have added the sentence $\mathbf{Q}$ to the set of sentences Γ (we have taken their union).

Induction Hypothesis and Key Fact

- *Induction hypothesis* for Soundness: assume that the soundness/righteousness property holds for all lines i less than the $k+1$–st line, i.e. if $i \leq k$ and if $\Gamma_i \vdash \mathbf{P_i}$, then $\Gamma_i \vDash \mathbf{P_i}$.
- In words: we are assuming that if we can derive a sentence $\mathbf{P_i}$ from a set of assumptions Γ_i, then those assumptions semantically entail that sentence.
- Lemma 6.3.2 (a.k.a. Useful Fact 1): if $\Gamma \vDash \mathbf{P}$ and Γ is a subset of a larger set Γ', then the larger set semantically entails the sentence $\mathbf{P}$ as well, i.e. $\Gamma' \vDash \mathbf{P}$.

Negated Conjunction Introduction

- **Negated Conjunction Introduction**: from $\sim\mathcal{Q}$ derive $\sim(\mathcal{Q} \& \mathcal{R})$
- Draw the deduction with the final sentence on line #k+1; label everything schematically so that you can refer to earlier line numbers and their open premise sets Γ_m
- Extend the proof of soundness by showing that a line generated by this rule is righteous, i.e. $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- Rely on the relevant subset relations between the various Γ premise sets
- Reason about relevant semantic entailment claims by using the truthtables for the connectives

Schematic Solution Steps (if you're totally lost)

1. Label the lines in your diagram with lowercase letters (e.g. j, ℓ, m, n, etc.) so that you can refer to them. Label the LAST LINE as $k+1$.
2. Reexpress the derivation diagram in terms of single turnstiles, i.e. if you have P on line j, then $\Gamma_j \vdash P$ (i.e. the set of open assumptions at line j provides a derivation for P).
3. Apply the induction hypothesis to any lines that are less than the $k+1$–th line. This lets you convert these single turnstiles into double turnstiles, e.g. $\Gamma_j \vDash P$, provided that $j < (k+1)$.
4. Relate the set of assumptions open at various lines (your Γ's) to the set of assumptions open at the last line, Γ_{k+1}. This will involve the subset relation $\subseteq$, e.g. $\Gamma_j \subseteq \Gamma_{k+1}$.
   - If the sentence P_j at line j is an additional open assumption that is not open at line $k+1$, then you need to tack this on, using the union operation: $\Gamma_j \subseteq (\Gamma_{k+1} \cup P_j)$.

Schematic Solution Steps continued

5. Apply useful fact 1 (i.e. lemma 6.3.2), using the relation(s) in the previous step. E.g., if you have $\Gamma_j \vDash P$ (from step 3) and $\Gamma_j \subseteq \Gamma_{k+1}$ (from step 4), then useful fact 1 entails that $\Gamma_{k+1} \vDash P$.

6. Next, consider an arbitrary truth value assignment that makes all of the sentences in Γ_{k+1} true. Use whatever double turnstiles are at your disposal to infer that some other sentence(s) is true.

6. Then, use a truth table to argue why the sentence on the last line (line $k+1$) must be true as well under this truth value assignment.

8. Pat yourself on the back (soundly)!

For additional guidance on Soundness, see...

- Section 6.3 of *The Logic Book* (reading for Week 12)
- pages 246–250 contain most of the cases for our system SND
- PS12 #1 handles negation elimination (case 10)
- §6.3 Exercises on page 250–251, problem #4 parts a thru d
- Think of your own cases by throwing in negation symbols, thinking about de Morgan's or other semantically equivalent sentences, etc.!

14. Final Review!

b. Applications of soundness & completeness

Applying soundness and/or completeness theorems

- PS12, problems #4–7 illustrate simple applications of the soundness and/or completeness theorems

- The final might contain problems of a similar flavor

A Key Fact to Remember, Understand, Retain

- If $\Gamma \cup \{\sim\mathcal{P}\}$ is unsatisfiable, what else can we say?
- Answer: $\Gamma \vDash \mathcal{P}$ (and vice–versa)
- If $\Gamma \vDash \sim\mathcal{Q}$, what else can we say?
- Answer: $\Gamma \cup \{\mathcal{Q}\}$ is unsatisfiable (and vice–versa)
- See p. 245 if you don't believe this; but should be able to give valid arguments for these claims verbally!

Practice w/ Applying Soundness & Completeness

To avoid ambiguity, let the sentences and sets of sentences be from QL, and let '$\vdash$' denote $\vdash_{QND}$

1. Prove or provide a counterexample to the following statement: If $\Gamma \vDash \mathcal{P}$ and $\Delta \vdash \mathcal{Q}$, then $\Gamma \cup \Delta \vdash \mathcal{P} \,\&\, \mathcal{Q}$
2. If $\Gamma \vdash (S \vee R)$ and $\Gamma \vdash \sim(S \vee R)$, prove or provide a counterexample that $\Gamma \vDash S$
3. If $\Gamma \cup \{\sim\mathbf{P}\}$ is unsatisfiable and $\Delta \vdash \mathbf{R}$, prove or provide a counterexample to $(\Gamma \cup \Delta) \vdash (\sim\mathbf{P} \equiv \mathbf{R})$.
4. Prove or give a counterexample to the following statement: If Γ is satisfiable, then $\{\sim S \mid S \in \Gamma\}$ is satisfiable.

Concept Review (if totally lost)

- Soundness theorem for SND: if you have a single turnstile (in SND), then you have a double turnstile. In words: if a set of assumptions gives you a derivation (in SND) for a sentence S, then those assumptions semantically entail that sentence S. In symbols: if $\Gamma \vdash_{SND} \mathbf{S}$, then $\Gamma \vDash \mathbf{S}$.
- Completeness theorem for SND: if you have a double turnstile, then you have a single turnstile (in SND). In words: if a set of assumptions semantically entails a sentence S, then those assumptions gives you a derivation (in SND) for that sentence S. In symbols: if $\Gamma \vDash \mathbf{S}$, then $\Gamma \vdash_{SND} \mathbf{S}$.
- Likewise for QL and QND

Solution Tips for Logically Complete Students

1. Use the soundness theorem to convert any single turnstiles you have (from system SND) into double turnstiles.
2. Convert claims about unsatisfiability into double turnstile relations
3. Use the completeness theorem to convert any double turnstiles you have into single turnstiles.
4. If you get stuck, write out the definitions of any key terms involved. These will guide you on your path to victory.
5. If you have to provide a counterexample, think about the simplest counterexample that gets the job done. Your counterexample must involve ACTUAL sentences; not metavariables
6. Pray for a stroke of insight! (Jk! Try reasoning backwards to figure out what you need!)

14. Final Review!

c. Alt. Cases of Membership Lemma

Alternative Cases of Membership Lemma

- In the completeness proofs, recall that the five SND membership lemma cases are motivated by truth-functional considerations

- We can prove variants of these cases, e.g. the following: modified version of case (e): $\sim\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either i) both $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$ or ii) both $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$.

- Alternately, one can be given an alternative SND rule (replacing one of our 11 sanctioned rules) from which to reprove a given case of the membership lemma (using the Door lemma)

Mega Reminder: TWO directions to show!

- Note that all of these cases have TWO directions, and you need to prove BOTH directions to complete the problem.

  First, you want to assume the thing on the left and derive the thing on the right (forward direction).

  Second, you want to assume the thing on the right and derive the thing on the left (backwards direction).
- Sometimes a case involves subcases, each of which can require its own non-trivial SND deduction (e.g. cases (c) and (d) for disjunction and conditional)
- Finally, remember that the membership lemma is purely syntactic! No mention of truth-value assignments here!

Membership Lemma (not that I'm a bouncer!)

- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:

  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$

  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$

  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$

  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$

  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- Notice how these syntactic constraints mirror truth-conditions!

Two Examples of Membership Lemma

- Case (a) is very useful: $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
- (modified version of case b):
  $\mathbf{P}\&\sim\mathbf{Q} \in \Gamma^*$ if and only if $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$
- Additional practice problem (modified version of case c):
  prove that $\mathbf{P} \vee \sim\mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$.
- Study case (d) for the conditional (bottom of p. 258)!
- Note that the 7-line derivation for case d) has a serious typo on line 2: the justification should be ":A / $\supset$ I", i.e. :AS for conditional intro.

Maximally Consistent-in-SND

- Γ^* is maximally-SND-consistent provided that both (i) Γ^* is consistent in SND (i.e. can't derive any contradictions) and (ii) if **P** is not in Γ^*, then $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in SND.
- In other words: you can't derive a contradiction from assumptions in Γ^*. And if $\mathbf{P} \notin \Gamma^*$, then $\Gamma^* \cup \{\mathbf{P}\}$ lets you derive a contradictory pair (i.e. you can derive both **R** and $\sim\mathbf{R}$).
- Assuming that you are not asked to prove a variant of case (a), you can help yourself to this result. Hence, if a sentence $\mathbf{P} \notin \Gamma^*$, then case (a) lets you conclude that $\sim\mathbf{P} \in \Gamma^*$, and vice versa: if $\sim\mathbf{P} \in \Gamma^*$, then you can conclude that $\mathbf{P} \notin \Gamma^*$.

The Door Lemma

- Lemma 6.4.9 (a.k.a. 'The Door'): this lemma helps you show that a sentence S is a member of a maximally SND-consistent set Γ^*:
  - if you can derive S from a subset Γ of a maximally SND-consistent set Γ^*, then S is a member of Γ^*.
  - In symbols: if $\Gamma \vdash S$ and $\Gamma \subseteq \Gamma^*$, then $S \in \Gamma^*$. In particular, if $\Gamma^* \vdash S$, then $S \in \Gamma^*$.
  - Hence the strategy: if you are trying to show that $S \in \Gamma^*$, figure out how to derive S in SND from sentences you have assumed are in Γ^*. Then, apply The Door.

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d. Translations in QL, with Identity

Some Structures to remember from SL

- **P only if Q**: $P \supset Q$ (order preserved) (equiv: $\sim Q \supset \sim P$)
- **Unless B, C** or **C unless B**: use OR: $B \vee C$
- **J just in case K**: $J \equiv K$
- Q if P; Q provided that P; Q given that P; if P, then Q: $P \supset Q$

Some Simple Examples not involving identity

Domain: all people;

Predicates: **Dx**: x went to Disneyland; **Kxy**: x knows y;

Constants/Names: j for John; m for Mary

- Schematize "Everyone who went to Disneyland knows someone who didn't go there":
  Answer: $(\forall x)(Dx \supset (\exists y)(Kxy \,\&\, {\sim}Dy))$
- "There is someone who knows both Mary and John but doesn't know themself":
  Answer: $(\exists x)(Kxm \,\&\, Kxj \,\&\, {\sim}Kxx)$
- "Everyone who knows John also knows Mary":
  Answer: $(\forall x)(Kxj \supset Kxm)$

Singular “only”

- “**Only Greta** is a hero”:
- Content: No–one other than Greta is a hero, **AND** Greta is a hero:

$$(\forall x)(Hx \supset x{=}g) \,\&\, Hg$$

$$(\forall x)(Hx \equiv x{=}g)$$

Schematizing ‘Neither’

- “Neither hero inspires”: this means that

  There are **exactly 2** heroes, and neither of them inspires:

$$(\exists x)(\exists y)\Big(\big((\sim x{=}y \;\&\; (Hx \;\&\; Hy)) \;\&\; (\forall z)(Hz \supset (z = x \vee z = y))\big) \;\&\; (\sim Ix \;\&\; \sim Iy)\Big)$$

At least *n*

- Remember: we interpret "three heros are inspiring" to mean "**at least** three heros are inspiring"
- At least 1 hero is inspiring:

$$(\exists x)(Hx \,\&\, Ix)$$

- At least 2 heroes are inspiring:

$$(\exists x)(\exists y)(\sim x{=}y \,\&\, ((Hx \,\&\, Ix) \,\&\, (Hy \,\&\, Iy)))$$

- At least 3 heroes are inspiring:

$$(\exists x)(\exists y)(\exists z)\Big((\sim x{=}y \,\&\, (\sim y = z \,\&\, \sim x = z)) \,\&\, \\ ((Hx \,\&\, Ix) \,\&\, ((Hy \,\&\, Iy) \,\&\, (Hz \,\&\, Iz)))\Big)$$

At least n

- Note: must state that **every pair** of variables is different, e.g.,

$$(\exists x_1)(\exists x_2)(\exists x_3)((\sim x_1 = x_2 \,\&\, \sim x_2 = x_3) \,\&\, (Hx_1 \,\&\, (Hx_2 \,\&\, Hx_3)))$$

  only says "There are at least two heroes"!
  - Take extension of Hx to be: 1, 2
  - Then 1 can play role of x_1 and x_3, 2 role of x_2.
  - Both "$\sim 1 = 2$" and "$\sim 2 = 3$" are true.
- UD: People. Predicates: **Dx**: x went to Disneyland; **Kxy**: x knows y
- "There is somebody who went to Disneyland and knows at least two people who didn't go there"
  Answer: $(\exists x)(Dx \,\&\, (\exists y)(\exists z)(\sim y = z \,\&\, Kxy \,\&\, Kxz \,\&\, \sim Dy \,\&\, \sim Dz))$

At most n

- There are **at most n** As $\Leftrightarrow$ There are **not at least $n+1$** As

$$(\exists^{\leq n} x)\, Ax \Leftrightarrow \sim(\exists^{\geq (n+1)} x)\, Ax$$

- For instance: There are at most two heroes:

$$\sim(\exists x)(\exists y)(\exists z)((Hx \,\&\, (Hy \,\&\, Hz)) \,\&\, (\sim x = y \,\&\, (\sim x = z \,\&\, \sim y = z)))$$
$$(\forall x)(\forall y)(\forall z)((Hx \,\&\, (Hy \,\&\, Hz)) \supset (x = y \vee (x = z \vee y = z)))$$

- "At most one person who knows Mary doesn't know John"

  Answer: $\sim(\exists x)(\exists y)(\sim x = y \,\&\, Kxm \,\&\, Kym \,\&\, \sim Kxj \,\&\, \sim Kyj)$

Definite descriptions

- Reminder: singular possessives like "Earth's moon" can be interpreted like the definite description "the moon of Earth." But plural possessives like "Mars's moons" aren't definite descriptions.
- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say

  "The *A* is B"

  is to say:
- There is something, which:
  - is *A*,
  - is the **only** *A* (i.e. the unique thing that is *A*),
  - is *B*.
- In QL:

$$(\exists x)(Ax \,\&\, (\forall y)(Ay \supset x{=}y) \,\&\, Bx)$$

Example: ‘The author of Waverley is blah’

- Schematize “The author of *Waverley* is Scottish”:
- Use the following symbolization key:
- Ax: x is an author; Wxz: x wrote z; Sx: x is Scottish; ℓ: *Waverley*

$$(\exists x)(Ax \,\&\, Wx\ell \,\&\, (\forall y)((Ay \,\&\, Wy\ell) \supset x{=}y) \,\&\, Sx)$$

Singular possessive (a definite description)

- Singular possessives form noun phrases, e.g., "Joe's cape"
- They work like definite descriptions:
  Joe's cape is the cape Joe owns. E.g.:
  - "Autumn wears Joe's cape" symbolizes the same as:
    "Autumn wears the cape that Joe owns":

$$(\exists x)\Big(\big((Ex \,\&\, Ojx) \,\&\, (\forall y)((Ey \,\&\, Ojy) \supset x{=}y)\big) \,\&\, Wax\Big)$$

Singular vs. plural possessive

- Compare **plural** possessives: those are '$\forall$'s':
  - "Autumn wears Joe's cape**s**" symbolizes the same as:

    "Autumn wears every cape that Joe owns":

$$(\forall x)((Ex \,\&\, Ojx) \supset Wax)$$

- So plural possessives are NOT definite descriptions.

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e. Interpretations/Models for QL

Interpretations/Models for QL

- Study PS9, especially problems like 9, 10, 13, 15, 17, and 19
- Understand how to input this stuff into *Carnap*!
- Understand what it takes to make an existential statement true (at least one object in the domain must satisfy the statement)
- Understand what it takes to make a universal statement true (every object in the domain must satisfy the statement)
- Watch out for conditionals, which are trivially satisfied if the antecedent is false

Truth of sentences of QL

- Given an interpretation I ...
- An **atomic sentence** is true iff the referents of the constants are in the extension of the predicate:
  - Pa is true iff referent 'r' of a is in extension of P
  - Rab is true iff $\langle r, p \rangle$ is in extension of R
    (where r is referent of a, and p is referent of b)
- $\sim\mathcal{A}$ is true iff $\mathcal{A}$ is false
- $\mathcal{A} \vee \mathcal{B}$ is true iff at least one of $\mathcal{A}$, $\mathcal{B}$ is true
- $\mathcal{A} \,\&\, \mathcal{B}$ is true iff both $\mathcal{A}$, $\mathcal{B}$ are true
- $\mathcal{A} \supset \mathcal{B}$ is true iff $\mathcal{A}$ is false or $\mathcal{B}$ is true

Truth of quantified sentences

- $(\exists x)\,\mathcal{A}x$ is true iff $\mathcal{A}x$ is **satisfied** by **at least one** object in the domain
    - r satisfies $\mathcal{A}x$ in I iff $\mathcal{A}r$ is true in the interpretation

- $(\forall x)\,\mathcal{A}x$ is true iff $\mathcal{A}x$ is **satisfied** by **every** object in the domain

Truth of quantified sentences

- $(\exists x)(\mathcal{A}x \,\&\, \mathcal{B}x)$ is true iff some object satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$' iff it satisfies both $\mathcal{A}x$ and $\mathcal{B}x$
- $(\forall x)(\mathcal{A}x \supset \mathcal{B}x)$ is true iff **every** object satisfies '$\mathcal{A}x \supset \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \supset \mathcal{B}x$' iff either
    - o does not satisfy $\mathcal{A}x$ (vacuously true conditional)

      or
    - o does satisfy $\mathcal{B}x$

14. Final Review!

f. Derivations in QND

Quick Tips and a Practice Problem

- If you can do the derivations on PS10, you are probably in great shape!
- Focus in particular on the rules/syntax surrounding Existential Elimination, conditional introduction, and Universal Instantiation
- If you are going to do $\exists$E, it's typically best to start your proof with that and work within the $\exists$E subproof until you get what you need (which may not be what you want)
- Construct a deduction showing the following: $(\exists x)Qx, (\forall y)(Qy \supset Py) \vdash_{QND} (\exists z)Pz$
- Another practice problem!: $(\exists x)Gx \supset Fa \;\; \vdash_{QND} (\forall x)(Gx \supset Fa)$
- The other direction is MUCH trickier (but can be done in 10 lines)! $(\forall x)(Gx \supset Fa) \vdash_{QND} (\exists x)Gx \supset Fa$

Some General Advice (locate the MAIN quantifier)

- Make sure you have a firm grip on the four rules of QND, and the rules of SND (see PS6), and the rule sheet
- Make sure especially that you know how to correctly apply existential elimination (and check those three conditions) and universal introduction (and check those two conditions).
- There are no special conditions for universal elimination, and the one for existential introduction is technically enforced by our recursive defN of QL wffs
- Note that you CANNOT apply any of the SND rules *within* the scope of a quantifier! You must first eliminate the quantifiers to apply any SND rules to the stuff inside.
- In general, each rule applies only to the WHOLE sentence, not a part. So you CANNOT apply a rule to just part of a sentence.

Some more Specific Advice

- Make sure you understand how to build up a conditional by using conditional introduction!
  - Assume the conditional's antecedent, justified by :AS for $> I$
  - Derive the consequent in the scope of this assumption (possibly starting further subproofs to get to the consequent, like negation elimination)
  - Then discharge the assumption and write the conditional!
- You may then apply a quantifier rule to the conditional, e.g. Existential Introduction, to which you could then finish an existential elimination if you were in the scope of an EE subproof
- If you get stuck, try to work from the bottom up. Think about what you would first have to derive to build your ultimate goal.
- If you get stuck on a subgoal, assume the opposite of your subgoal to try using either negation introduction or negation elimination to keep going.