14. Compactness of SL & QL

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):
- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- **Completeness**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$
  - reasoning about arbitrary models is not needed to demonstrate validity: QND derivations suffice
  - (logical entailment is fully covered by our syntactic rules)

14. Compactness of SL & QL

a. Compactness of SL

Compactness of SL

- **Compactness of SL**: for any set Γ of SL-sentences (possibly infinite), Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. for each Δ, there is a truth-value assignment that makes all sentences in Δ true).
- Relying on our valiant labors in proving the soundness and completeness of SND, we gain an elementary proof of compactness
- This proof is "impure" because it relies on syntactic notions, whereas the statement of compactness is purely semantic.

An "impure" proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

- This TVA satisfies every finite subset $\Delta \subseteq \Gamma$.

$\Leftarrow$ (nontrivial direction): Assume that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

- Assume for *reductio* that Γ is unsatisfiable.
Then there is no TVA that makes true every sentence in Γ.

- Hence, for any contradiction $\mathcal{C}$ (e.g. $P \,\&\, {\sim}P$), we have $\Gamma \vDash \mathcal{C}$

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.
- Since derivations are finite, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{SND} \mathcal{C}$
- Then, by soundness of SND, $\Delta \vDash \mathcal{C}$. Since $\mathcal{C}$ is unsatisfiable, this means that Δ must be unsatisfiable as well.
- But that contradicts our starting assumption that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.
- So Γ must be satisfiable (proving compactness)

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

  - So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
  - Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$
- *Upshot*: every valid argument relies on finitely-many premises
- Contrast proof here with PS12 #4, which shows same result using completeness *and soundness*, relying on syntactic $\vdash_{SND}$
- Whereas our argument above proceeds entirely semantically, using compactness and semantic entailment $\vDash$
- If only we could prove compactness purely semantically?!

14. Compactness of SL & QL

b. A ‘Pure’ proof of SL compactness

A 'Pure' proof of the Compactness of SL

- Using a very similar idea to our construction of the maximally–SND–consistent set Γ^*, we can provide a purely semantic and yet still elementary proof of SL compactness
- Proof sketch: assuming that every finite subset of Γ is satisfiable, we will construct a superset $\Gamma^* \supset \Gamma$ for which it is easy to define a truth–value assignment that satisfies every sentence in Γ^*, and hence in Γ.
- As with our earlier completeness proof, Γ^* comes along with a membership lemma, which we use for our induction over SL.

Beginning the Proof

- $\Rightarrow$ (easy direction): assume that the (possibly infinite) set of SL-wffs Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ, and this TVA satisfies every finite subset of Γ.
- $\Leftarrow$ (harder direction): Assume that every finite subset $\Delta \subset \Gamma$ is satisfiable. Show that Γ is satisfiable (nontrivial if Γ is infinite).
- ► Notice that it suffices to construct a superset Γ^* of Γ that is satisfiable. Then the TVA that makes true everything in Γ^* will make true everything in Γ.
- ► To proceed, we introduce an idea very similar to the notion of a maximally-consistent-in-SND set. But now using only *semantic* notions (so avoiding our proof system).

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
      Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite–satisfiability
      i.e. $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset

- Next we'll show that any MFS set is satisfiable (this mirrors our "maximal consistency lemma" from our completeness proof)
- To do this, we'll prove a membership lemma that facilitates an induction over SL!
- Finally, we'll show how to construct an MFS Γ^* from any finitely–satisfiable Γ
  (i.e. what we assume at the start of the nontrivial–direction)

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- These syntactic constraints mirror truth-conditions, but we will now NOT rely on our proof system to prove this lemma
- (We built an analog of "the Door" into the definition of MFS sets)

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.
  (So we do lots of *reductio* proofs: assume that a membership case fails, apply Case (a), and then show this would result in an unsatisfiable finite subset—contradicting condition (1), i.e. that all finite subsets are satisfiable).
- So imagine we've proven the membership lemma!
- Then define a TVA $\mathcal{I}^*$ that makes true every atomic sentence in Γ^*;
  - show by induction that this TVA satisfies every sentence in Γ^*
  (just as in our proof of completeness of SND!)

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays FS)
- Then, proceed to the second wff in our enumeration.
  If it can be added to Γ_1 without the new set breaking FS, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$
- Γ^* is the result of 'doing' this procedure for every SL wff
- More precisely, $\Gamma^* := \bigcup_{k=1}^{\infty} \Gamma_k$

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

1.) Clearly, Γ^* is finitely satisfiable. If it were not, then some $\Gamma_k \subset \Gamma^*$ would be finitely unsatisfiable, but that contradicts our construction conditions.

2.) Moreover, Γ^* is maximal: if there were a wff $\mathcal{Q}$ that could be added to Γ^* while preserving finite satisfiability, we would have added $\mathcal{Q}$ at its enumeration stage.

   - So if $\mathcal{Q} \notin \Gamma^*$, it must be that $\Gamma^* \cup \{\mathcal{Q}\}$ is *not* finitely satisfiable.

- So we're done! Any finitely satisfiable Γ is a subset of an MFS Γ^*, which we've shown is satisfiable! So Γ is satisfiable!

14. Compactness of SL & QL

c. Compactness of First-order Languages

Compactness of QL

- **Compactness of QL**: for any set Γ of QL-sentences, Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. $(\forall\Delta)$ $\exists$ a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).
- *Mutatis mutandis*, we can provide an analogous impure proof, relying on the soundness and completeness of system QND

  And also a 'pure' proof, constructing a maximally finitely satisfiable and *existentially complete* superset Γ^*.
- To widen the interest of our results, let's generalize compactness to any first-order language $\mathcal{L}$

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses
  - Names: a set of constants (allowing subscripts $n \in \mathbb{N}$)
  - Predicates: a non-empty set of capital letters (allowing subscripts), each with "an invisible label" giving its arity (e.g. 0-place, 1-place, 2-place, etc.)
  - a set of function symbols $f(c)$ (syntax: f maps terms to terms)
- Different FOLs differ in their names, predicates, and functions

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathcal{L}$ to objects and ordered pairs from D as follows:
     - For each constant $c \in \mathfrak{L}$, $I(c)$ is an element of D, called the *referent* or denotation of c
     - For each k-place predicate P of $\mathfrak{L}$, $I(P)$ is a set of ordered k-tuples of objects in D, called the *extension* of P
     - I maps SL atomics to "true" or "false" (i.e. '1' or '0')

FOL with identity and functions

- With some minor modifications, we could extend our soundness and completeness proofs for QND to FOLs and deduction systems that include (1) a privileged identity predicate "$=$" and (2) functions that syntactically map terms to terms (interpreted as mapping the domain D to itself)
- Like our symbol "$\prime$", we add in some new symbol "α" that doesn't occur in our FOL, to give a countable infinity of unused constants
- To construct our maximally-syntactically-consistent, existentially complete superset Γ^*, we focus on equivalence classes of co-referential constants, since now some constants might name the same object in the domain (e.g. $c = d$)
- Using the axiom of choice, we could even handle FOLs that have *uncountably many* predicates or constants!

Compactness for a first-order language

- **Compactness of a FOL $\mathcal{L}$**: for any set Γ of $\mathcal{L}$-sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

- We could prove this either by (i) using a soundness and completeness result for an $\mathcal{L}$-deduction system;
  (ii) generalizing our 'pure' proof for SL; or
  (iii) generalizing the topological proof of SL compactness (relying on results from topology, e.g. Tychonoff's theorem)

14. Compactness of SL & QL

d. The Löwenheim–Skolem theorems

Downwards!

- Terminology: we'll say that a model $\mathfrak{M}$ is *infinite* if its domain D is infinite in size. Likewise for saying that a model is finite, or countably infinite.
- **Downward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model, then it is satisfiable in a countably infinite model.
- *Gloss*: we can always descend from an infinite model to a countably infinite model
- Proof(s): (1) be impure and piggyback on completeness proof or (2) use compactness and satisfiability lemma for MFS sets

Down to be Impure

- **Converse consistency lemma**: if $\mathcal{L}$-set Γ is satisfiable, then Γ is syntactically-consistent (for a given deduction system $\mathcal{L}$ND that we've shown is sound)
- Proof: good exercise!!! Assume for *reductio* that Γ is syntactically-inconsistent and then apply soundness
- So since Γ is satisfiable, it is syntactically consistent.

  Then, appeal to our consistency lemma shown in the course of proving completeness: for any syntactically-consistent set, there is a maximally-consistent (and $\exists$-complete) set that is satisfiable, where we showed this by constructing a countably infinite model
- So Γ has a countably infinite model

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely-satisfiable
- Modify our construction to form a maximally finitely-satisfiable and $\exists$-complete superset Γ^* of Γ
- Prove a satisfiability lemma: any such Γ^* is satisfiable, where we show this by constructing a countably infinite $\mathcal{L}^+$-model

  ($\mathcal{L}^+$ arises from $\mathcal{L}$ by adding a countable-infinity of new constants)
- Then, Γ has a countably infinite $\mathcal{L}$-model

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$
- *Proof Sketch*: extend the set of constants $\mathcal{C}$ of $\mathcal{L}$ with an uncountable set $\mathcal{E}$ that contains $\mathcal{C}$.
  Extend the FOL $\mathcal{L}$ to $\mathcal{L}^+$ with $\mathcal{E}$ as its set of constants and with identity predicate $=$.
- Construct an $\mathcal{L}^+$–set Γ^+ by adjoining to Γ every sentence of the form $\sim c = d$ for every distinct $c, d \in \mathcal{E}$.
- Show that Γ^+ is finitely–satisfiable and hence by compactness satisfiable. Then note that any $\mathcal{L}^+$–model satisfying Γ^+ must have a domain as large $\mathcal{E}$. Restrict the interpretation function to construct an $\mathcal{L}$–model for Γ with domain $|D| = |\mathcal{E}|$

14. Compactness of SL & QL

e. Skolem’s ‘Paradox’

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC): a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs
- In standard models, we interpret the objects as sets
- A list of axioms or axiom schemas, e.g.
  Null set axiom: $(\exists x)(\forall y)y \notin x$ (i.e. there is an empty set $\varnothing$)
  Axiom of Extensionality: $(\forall x)(\forall y)(\forall z)((z \in x \equiv z \in y) \supset x = y)$ (i.e. two sets are identical iff they have the same members)
- **Axiom of Choice**: if x is a set whose members are non-empty sets and no two members of x share a member, then there is a set y that contains exactly one element of each set in x

Skolem's 'Paradox'

- If ZFC has any models, then it has a **countable model** (since by downward LS, an infinite model entails a countably infinite model. Any finite model is already countable—and can be extended to a countably infinite model as well)
- Yet, we can prove within ZFC that there are uncountable sets, e.g. the power set of $\mathbb{N}$ has cardinality of $\mathbb{R}$
- 'Paradox': how can a countably-infinite model make true the claim that there are uncountable sets?

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'
- U: there is an injection but no bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$; $\mathfrak{M} \models U$
- Since $\mathfrak{M}$ is countable, the sets $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ are definitely countable ($\mathfrak{M}$ has only countably many objects in its domain to serve as members of objects in that domain)
- So clearly, there IS a bijection between the sets that correspond to $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ (we can prove this bijection in a metalanguage)
- BUT (resolution), this bijection is not itself an object in $\mathfrak{M}$. So $\mathfrak{M}$ itself represents $2^{\mathbb{N}}$ as uncountable

14. Compactness of SL & QL

f. Problems for finitism

Saying that there are finitely-many things

- As shown on PS13 problems #2, 5, and 6, we have some ISSUES when it comes to saying that there are finitely-many things in quantifier logic

- It seems like we definitely cannot accomplish this putatively possible task through *sentences*

- Is there any other way we might go about enforcing there being finitely-many things (e.g. if we think there probably are only finitely-many things and want a FOL to reflect that)?

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, $\exists\!\!\!\!\exists$, to FOL
- Syntactically, we define Ǝ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then (Ǝx)$\mathcal{P}$ is a wff
- Semantically, we extend satisfiability semantics (oh boy—not that sh** again) so that (Ǝx)$\mathcal{P}$ is true in a model if and only if there are finitely-many $\mathcal{P}$-objects in the model's D, i.e. $|D|$ is finite
- **Question**: what would it take to modify our derivation system QND to make it sound and complete for quantifier logic with a finitely-many quantifier (QL-Ǝ)?
- **Answer**: no derivation system can be sound & complete for QL-Ǝ! - F***!!! INFINITE F***!!!

Finite Hopes & Finite Dreams: dashed upon ∞–many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL–$\exists$
- Then, we could prove that QL–$\exists$ is compact (see slides 3–4)
- Yet, the entailment relation $\vDash_{QL-\exists}$ for this logic is NOT compact:
- Consider the sentence $F := (\exists x)x = x$, which says "there are finitely-many things that equal themselves." This is just a fancy way of saying that there are finitely-many things in the domain (since everything is identical to itself and nothing else).
- Then consider the set $X := \{F, L_1, L_2, \ldots\}$, containing F and each L_k for $k \in \mathbb{N}$, where L_k says "there are at least k–things"
- Set X is finitely-satisfiable, but it is not satisfiable (violating compactness). Any way of making true the infinitely-many L_n's requires an infinite model, which then can't make true sentence F

14. Compactness of SL & QL

g. A topological proof of SL compactness

What does "compactness" normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets
- A set is closed in (X, τ) if its complement is open (NB: sets can be 'clopen', i.e. both open AND closed)
- Compactness in topology: a topological space is **compact** iff every open cover has a finite subcover
- Equivalently: every collection of closed subsets obeying the *finite intersection property* has non-empty intersection
- Finite intersection property (FIP): a set of subsets $\{F_\beta\}_{\beta \in B}$ of a topological space has the FIP if for every finite subset B_0 of our index set B, the intersection of all the sets F_β for $\beta \in B_0$ is non-empty, i.e. provided that $\bigcap_{\beta \in B} F_\beta$ is non-empty

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U_A^0 be the set of TVAs that assign A false, and let U_A^1 be the set of TVAs that assign A true
- Endow the set $\mathcal{E}$ with a topology by stipulating that (i) for each atomic wff A, U_A^0 and U_A^1 are open and (ii) every non-empty open set arises as a union of these U^0s and U^1s
- **Claim**: the compactness of SL is equivalent to the compactness of this topological space $\mathcal{E}$
- Note that if we prove that 1) compactness of $\mathcal{E}$ entails compactness of SL and that 2) $\mathcal{E}$ is compact, then we will have proven compactness without detour through syntax!

Step 1: $\mathcal{E}$ compact entails SL is compact

- Assume that $(\mathcal{E}, \tau)$ is compact. Consider an arbitrary set Γ of SL sentences that is finitely satisfiable.

  NTS: Γ is satisfiable (the other direction is trivial)

- Consider an arbitrary wff $\mathcal{P}$. *Lemma*: the set $U_{\mathcal{P}} \subset \mathcal{E}$ of TVAs that make $\mathcal{P}$ true is open (proof: use disjunctive normal form and take a matching union of finite intersections of the U_A^0s and U_B^1s for atomics that compose $\mathcal{P}$!)

- So $U_{\sim\mathcal{P}}$ is also open. Since the complement of $U_{\mathcal{P}}$ is $U_{\sim\mathcal{P}}$, $U_{\mathcal{P}}$ is both closed and open

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$
- So to say that each finite subset Γ_0 of Γ is satisfiable is equivalent to saying that the family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is a family of closed subsets of $\mathcal{E}$ with the *finite intersection property* (i.e. for any finite subset of this family, the intersection of its members $U_{\mathcal{Q}}$ is non-empty)
- Since we are assuming that $\mathcal{E}$ is compact, the intersection of ALL members of this family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is non-empty
- i.e. this intersection must contain at least one TVA in $\mathcal{E}$
- Hence, there is a TVA that makes true all of the members of Γ

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.
- Since there are countably many SL atomics, $2^{\mathcal{A}}$ is homeomorphic to the Cantor set (comprises ∞–binary sequences of 0s and 1s)
- Note that the Cantor set is compact, since it is a closed subset of a compact set (namely the closed unit interval [0, 1])
- If we allow $\mathcal{A}$ to have arbitrarily many SL atomics, then we could use Tychonoff's theorem (equivalent to the axiom of choice) to show that $2^{\mathcal{A}}$ is compact
- *Tychonoff*: a product of compact spaces is compact in the product topology