The Semantics for QL

LOGIC I
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Examples

Monadic: Casey is dancing.

Dyadic: Al loves Max.

Triadic: Kim is between Boston and New York.

Constants and Referents

Constants: Constants are interpreted as referring to individuals.

Existence: Thus we need to know what things there are.

Domain: A *domain* is any nonempty set $\mathbb{D}$.

Referents: Interpretations assign constants to elements of $\mathbb{D}$.

Question 1: How are we going to interpret predicates?

Predicates and Extensions

Example: 'Al loves Max' is true *iff* Al bears the loves-relation to Max.

Dyadic Predicates: Dyadic predicates are interpreted by sets of *ordered pairs* in $\mathbb{D}^2$.

Question 2: How are we to interpret n-place predicates?

Cartesian Power: $\mathbb{D}^n = \{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle : \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{D}\}$.

Extensions: n-place predicates are interpreted by subsets of $\mathbb{D}^n$.

Singletons: 1-place predicates are interpreted by subsets of $\mathbb{D}^1 = \{\langle \mathbf{x} \rangle : \mathbf{x} \in \mathbb{D}\}$.

Question 3: How are we to interpret 0-place predicates? What is $\mathbb{D}^0$?

> *n-Tuples:* Let $\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle = \{\langle 1, \mathbf{x}_1 \rangle, \ldots, \langle n, \mathbf{x}_n \rangle\}$.
>
> 0-*Tuple:* $\langle \rangle = \varnothing$.

Truth-Values: 0-place predicates are interpreted by subsets of $\mathbb{D}^0 = \{\varnothing\}$.

Ordinals: Let $1 = \{\varnothing\}$ and $0 = \varnothing$ be the first two von Neumann ordinals.

QL Models

Interpretations: $\mathcal{I}$ is an QL interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Question 4: What happens if $\mathbb{D} = \varnothing$?

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL *iff* $\mathcal{I}$ is a QL interpretation over $\mathbb{D} \neq \varnothing$.

Task 1: Regiment and interpret the sentences above.

- Dc, Lam, $Bkbn$.
- $\mathbb{D} = \{c, a, m, k, b, n\}$.
- $\mathcal{I}(D) = \{\langle c \rangle\}$.
- $\mathcal{I}(L) = \{\langle a, m \rangle\}$.
- $\mathcal{I}(B) = \{\langle k, b, n \rangle\}$.
- $\mathcal{I}(c) = c,\ \mathcal{I}(a) = a,\ \ldots$

Lagadonian: We often take constants to name themselves.

Question 5: Do models give us truth-values?

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL.

Singular Terms: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Example: Let $\mathbb{D} = \{1, 2, 3, 4, 5\}$ where $\hat{a}(x) = 1$, $\hat{a}(y) = 2$, and $\hat{a}(z) = 3$.

Task 2: If $\hat{c}$ is a y-variant of $\hat{a}$, what is $\hat{c}(1)$, $\hat{c}(2)$, and $\hat{c}(3)$?

Example

Universal: Al loves everything, i.e., $\forall x Lax$.

Existential: Someone is dancing, i.e., $\exists x(Px \wedge Dx)$.

Mixed: Everyone loves someone, i.e., $\forall x(Px \supset \exists y Lxy)$.

Complex: Everything everything loves loves something, i.e., $\forall x(\forall y Lyx \supset \exists z Lxz)$.

Semantics for QL

(*A*) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1,\ldots,\alpha_n) = 1$ *iff* $\langle\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1),\ldots,\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)\rangle \in \mathcal{I}(\mathcal{F}^n)$.

($\forall$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

($\exists$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

($\neg$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$.

($\vee$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

($\wedge$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$.

($\supset$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \supset \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

($\equiv$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi)$.

Truth and Entailment

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL.

Satisfaction: $\mathcal{M} = \langle\mathbb{D},\mathcal{I}\rangle$ satisfies Γ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every $\varphi \in \Gamma$.

Singletons: As before $\mathcal{M}$ satisfies φ *iff* $\mathcal{M}$ satisfies $\{\varphi\}$.

Entailment: $\Gamma \vDash \varphi$ just in case every model $\mathcal{M}$ that satisfies Γ also satisfies φ.

Tautology: φ is a tautology *iff* $\vDash \varphi$.

Contradiction: φ is a contradiction *iff* $\vDash \neg\varphi$.

Contingent: φ is contingent *iff* $\vDash$ and $\nvDash \neg\varphi$.

Consistent: Γ is consistent *iff* Γ is satisfiable.

Minimal Models

Task 3: Provide minimal models in which the examples above are true/false.

Regimentation

- Every rose has its thorn.
- At least the guests that remained were pleased with the party.
- I haven't met a cat that likes Merra.
- Kate found a job that she loved.
- Everybody everybody loves loves somebody.
- No set is a member of itself.
- There is a set with no members.

Arguments

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
- May loves everyone who loves themselves.

$\therefore$ If Cam loves himself, he doesn't love May.

Bigger: Regiment the following argument:

- Whenever something is bigger than another, the latter is not bigger than the former.

$\therefore$ Nothing is bigger than itself.

Relations

Domain: Let the *domain* D be any set.

Relation: A *relation* R on D is any subset of D^2.

Reflexive: A relation R is *reflexive* on D *iff* $\langle x, x \rangle \in R$ for all $x \in D$.

Non-Reflexive: A relation R is *non-reflexive* on D *iff* R is not reflexive on D.

Question 1: What is it to be *irreflexive*?

Irreflexive: A relation R is *irreflexive* on D *iff* $\langle x, x \rangle \notin R$ for all $x \in D$.

Symmetric: A relation R is *symmetric iff* $\langle y, x \rangle \in R$ whenever $x, y \in R$.

Question 2: Why don't we need to specify a domain?

Question 3: Why is a relation reflexive or irreflexive with respect to a domain?

Asymmetric: A relation R is *asymmetric iff* $\langle y, x \rangle \notin R$ whenever $\langle x, y \rangle \in R$.

Question 4: What is it to be non-symmetric? How about non-asymmetric?

Task 1: Show that every asymmetric relation is irreflexive.

Transitive: A relation R is *transitive iff* $\langle x, z \rangle \in R$ whenever $\langle x, y \rangle, \langle y, z \rangle \in R$.

Intransitive: A relation R is *intransitive iff* $\langle x, z \rangle \notin R$ whenever $\langle x, y \rangle, \langle y, z \rangle \in R$.

Question 5: Is every symmetric transitive relation reflexive? (No: $R = \varnothing$)

Task 2: Show that every transitive irreflexive relation asymmetric?

Euclidean: A relation R is *euclidean iff* $\langle y, z \rangle \in R$ whenever $\langle x, y \rangle, \langle x, z \rangle \in R$.

Task 3: Show that every transitive symmetric relation is euclidean.