Completeness of QD: Part II

LOGIC I
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Basic Lemmas

L13.1 If α is a constant and X is a proof in which the constant β does not occur, then $X[\beta/\alpha]$ is also a proof.

L13.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L13.5 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L13.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L13.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L13.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Satisfiability

T13.1 Every consistent set of $\mathrm{QL}^{=}$ sentences Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

1. Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.
2. So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T13.1**.
3. So $\Gamma \vdash \neg\neg\varphi$ by **L13.3**, and so $\Gamma \vdash \varphi$ by DN and **L13.5**.

Saturation

Free: Let $\varphi(\alpha)$ be a wff of $\mathrm{QL}^{=}$ with at most one free variable α.

Saturated: A set of sentences Σ is saturated in $\mathrm{QL}^{=}_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathrm{QL}^{=}_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \supset \varphi[\beta/\alpha]) \in \Sigma$.

Constants: Let $\mathbb{C}$ be the constants of $\mathrm{QL}^{=}_{\mathbb{N}}$ where $\mathbb{N} \subseteq \mathbb{C}$ are new constants.

L13.2 Assuming Γ is consistent in $\mathrm{QL}^{=}$, we know Γ is consistent in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathrm{QL}^{=}_{\mathbb{N}}$ with one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \supset \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.

$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \supset \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in θ_j for any $j \leqslant k$.

Saturation: Let $\Sigma_1 = \Gamma$, $\Sigma_{n+1} = \Sigma_n \cup \{\theta_n\}$, and $\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

L13.4 Σ_Γ is consistent and saturated in $\mathrm{QL}^{=}_{\mathbb{N}}$.

1. If Σ_{m+1} is inconsistent, then $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$.
2. So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I, and so $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$.
3. If Σ_Γ is inconsistent, then $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$.

Maximization

Maximal: A set of sentences Δ is maximal in $\mathrm{QL}^{=}_{\mathbb{N}}$ just in case as either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every sentence ψ in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all sentences in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Maximization: Let $\Delta_0 = \Sigma$, $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$, and $\Delta_\Sigma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

L13.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal consistent in $\mathrm{QL}^{=}_{\mathbb{N}}$.

Case 1: $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

1. If $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent, then Δ_n is inconsistent by **L13.6**.
2. So Δ_{n+1} is consistent in both cases.
3. If Δ_Σ is inconsistent, then $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$.
4. Maximality is immediate.

L13.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

1. Immediate from the definitions.

L13.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

1. Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L13.9**.
2. So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
3. Thus $\varphi \in \Delta$ by maximality.

Henkin Model

Element: $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

Domain: $\mathbb{D}_\Delta = \{[\alpha]_\Delta : \alpha \in \mathbb{C}\}$.

L13.13 If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

1. Assuming $\alpha = \beta \in \Delta$ where $\gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
2. So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L13.11**.
3. Thus $\beta = \gamma \in \Delta$ by **L13.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

Constants: $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

Predicates: $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}^n_\Delta : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

L13.14 If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

1. Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
2. $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L13.10**.
3. Parity of reasoning completes the proof.

Henkin Lemmas

L13.15 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

1. Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$ for some $\hat{a}$, $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some α-variant $\hat{c}$.
2. So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
3. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.8**.
4. So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L12.6**.
5. Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.
6. Let $\hat{c}$ be the α-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$.
7. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.8**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

L13.16 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

1. Similar to **L13.15**.

L13.17 $\mathcal{M}_\Delta$ satisfies φ just in case $\varphi \in \Delta$.

Base: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

1. If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L13.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
2. Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L13.13**.

Induction: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leqslant n$.

1. Let φ be a sentence of $\mathrm{QL}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

Case 2: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

Case 6: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L13.15**.

1. *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.
2. *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L13.10** given saturation.

Conclusion: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

Restriction

Restriction: $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant α in $\mathrm{QL}^{=}$.

L13.18 For all $\mathrm{QL}^{=}$ sentences φ, $\mathcal{M}'_\Delta$ satisfies φ just in case $\mathcal{M}_\Delta$ satisfies φ.

T13.1 Every consistent set of $\mathrm{QL}^{=}$ sentences Γ is satisfiable.

Compactness

C13.2 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

C13.3 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

Final Exam Review

Regimentation: (a) No two individuals are at least as tall as each other. Sanna is at least as tall as the finalist, and the finalist is at least as tall as Sanna. Thus, Sanna is the finalist.

Models: (a) $Qab, Qba \not\vDash a = b$.

(b) $\forall x \forall y (Px \supset (Py \supset x \neq y)) \not\vDash \exists x \exists y\, x \neq y$.

Equivalence: $\exists x(\forall y(Py \supset x = y) \wedge Px) \dashv\vDash \exists x \forall y(Py \equiv x = y)$.

Relations: (a) R is symmetric and antisymmetric. Therefore R is reflexive.

(b) R is asymmetric. Therefore R is antisymmetric.