6. Proofs in SL

6. Proofs in SL

a. Who ordered *that*???

The Power of a Proof System

- We have already seen how powerful a proof system can be compared to truth tables:
- Consider our running example argument:

$$C \vee E, A \vee M, A \supset \sim C, \sim S \,\&\, \sim M \therefore E$$

  requires 32 lines and 512 truth values
- With trees, we showed validity in 10 lines (13 sentences)
- Why not just tree always and everywhere?

For the Love of Trees

- The advantages of trees create limitations:
  - Mechanical ⇒ not ‘normal’ pattern of inference
  - Mirror partial truth–tables ⇒ implicitly referencing truth–values
  - Always checking satisfiability of the root ⇒ we don’t really ‘derive’ anything at the end of the proof, unlike in normal inference
  - (this is why so many of us keep forgetting that it is the NEGATION of the conclusion that goes in the root)
- We would like to form a better model of human inference patterns
- e.g., common rules such as Modus Ponens, disjunctive syllogism

Idiosyncrasies of Table or Tree Reasoning

- Tables:
  - Construct a truth table
  - verify there is no TVA where premises are true but conclusion is false
- This is NOT how we typically reason through an argument
- Trees: ask whether a set of sentences is satisfiable:
  - Put premises and NEGATION of conclusion in root
  - If tree closes, then unsatisfiable root (valid argument)
  - If tree remains open, then satisfiable root (invalid argument)
- Again, this is NOT how we typically reason through an argument

The Very Idea of ‘Natural Deduction’

- We commonly reason according to certain inference rules
- And we often make assumptions in the *middle* of our reasoning, derive an intermediate conclusion, and ‘discharge’ the assumption (e.g. in proof by contradiction)
- We would like to see if we can *vindicate* these patterns:
  - Show that these rules never get us into trouble: soundness
  - Show that we have enough rules to handle any valid argument (including additional rules we might want to add): completeness
- Perhaps our natural deduction system *explains* the success of our ordinary inference patterns

Ordinary Reasoning by “Proofs”

- Idea: work our way from premises to conclusion using steps we know are entailed by the premises.
- For instance:
  - From “Neither Sarah nor Amir enjoys hiking” we can conclude “Amir doesn’t enjoy hiking.”
  - From “Either Amir lives in Chicago or he enjoys hiking” and “Amir doesn’t enjoy hiking” we can conclude “Amir lives in Chicago” (Disjunctive syllogism DS).
  - etc.
- If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

1. Sarah lives in Chicago or Erie.
2. Amir lives in Chicago unless he enjoys hiking.
3. If Amir lives in Chicago, Sarah doesn't.
4. Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Erie.

5. Amir doesn't enjoy hiking (from 4).
6. Amir lives in Chicago (from 2 and 5).
7. Sarah doesn't live in Chicago (from 3 and 6).
8. Sarah lives in Erie (from 1 and 7).

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \supset {\sim}C$
4. ${\sim}S \;\&\; {\sim}M$

$\therefore E$

5. ${\sim}M$ (from 4, since $\mathcal{P} \;\&\; \mathcal{Q} \vDash \mathcal{Q}$)
6. A (from 2 and 5, since $\{\mathcal{P} \vee \mathcal{Q}, {\sim}\mathcal{Q}\} \vDash \mathcal{P}$)
7. ${\sim}C$ (from 3 and 6, since $\{\mathcal{P} \supset \mathcal{Q}, \mathcal{P}\} \vDash \mathcal{Q}$, i.e. via ‘modus ponens’)
8. E (from 1 and 7, since $\{\mathcal{P} \vee \mathcal{Q}, {\sim}\mathcal{P}\} \vDash \mathcal{Q}$)

Some aspects of our Formal Deductions

- Numbered lines contain sentences of SL
- A line may be a **premise** (:PR).
- A line may be an **assumption** (:AS)
- If neither a premise nor assumption, it must be **justified**
- Justification requires:
  - a **rule** (e.g. '&E'), and
  - prior line(s) invoked by the rule—referenced by line number(s)
  - starting with a colon: e.g. ': 2 &E'
- But: what are the rules? (very different from 'what IS a rule'?)

Aspects of our Rules for Natural Deduction

- Our Rules will (mostly) be . . .
  - **Simple**: cite just a few lines as justification
  - **Obvious**: new line should clearly be entailed by justifications
  - **Schematic**: can be described just by **forms** of sentences involved
  - **Few in number**: want to make do with just a handful
- We'll have two rules per connective:
  an **introduction** and an **elimination** rule
- They'll be used to either:
  - justify (say) $\mathcal{P} \,\&\, \mathcal{Q}$ (i.e. to 'introduce' $\&$), or
  - justify something **using** $\mathcal{P} \,\&\, \mathcal{Q}$ (i.e. to 'eliminate' $\&$).

6. Proofs in SL

b. Conjunction Intro and Elimination (Rules for &)

Eliminating &

- What can we **justify using** $\mathcal{P} \,\&\, \mathcal{Q}$?
- A conjunction entails each conjunct:

$$\mathcal{P} \,\&\, \mathcal{Q} \vDash \mathcal{P}$$
$$\mathcal{P} \,\&\, \mathcal{Q} \vDash \mathcal{Q}$$

- Already used this above to get $\sim M$ from $\sim S \,\&\, \sim M$, i.e., from "Neither Sarah nor Amir enjoys hiking" we concluded "Amir doesn't enjoy hiking".
- (Role of $\mathcal{P}$ played by $\sim S$ and that of $\mathcal{Q}$ played by $\sim M$)

Introducing &

- What do we **need to justify** $\mathcal{P} \,\&\, \mathcal{Q}$?
- We need both $\mathcal{P}$ and $\mathcal{Q}$:

$$\{\mathcal{P}, \mathcal{Q}\} \vDash \mathcal{P} \,\&\, \mathcal{Q}$$

- For instance, if we have "Sarah doesn't enjoy hiking" and also "Amir doesn't enjoy hiking", we can conclude "Neither Sarah nor Amir enjoys hiking"
- (Role of $\mathcal{P}$ played by $\sim S$ and $\mathcal{Q}$ by $\sim M$: $\{\sim S, \sim M\} \vDash \sim S \,\&\, \sim M$)

Rules for &

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
n & \mathcal{Q} & \\
 & \mathcal{P} \,\&\, \mathcal{Q} & :m,\ n\ \&\mathrm{I}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{P} & :m\ \&\mathrm{E}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{Q} & :m\ \&\mathrm{E}
\end{array}
$$

We'll illustrate using exercises in our Week 6 Practice Problems on Carnap.

1	$A \,\&\, B$	:PR
2	A	:1 & E
3	B	:1 & E
4	$B \,\&\, A$	:2, 3 & I

1	$A \,\&\, (B \,\&\, C)$	:PR
2	A	:1 & E
3	$B \,\&\, C$	:1 & E
4	C	:3 & E
5	$A \,\&\, C$	:2, 4 & I

6. Proofs in SL

c. Conditional Intro and Elim. (Rules for ⊃)

Eliminating $\supset$

- What can we **justify using** $\mathcal{P} \supset \mathcal{Q}$?
- We used the conditional "If Amir lives in Chicago, Sarah doesn't" to justify "Sarah doesn't live in Chicago".
- What is the general rule? What can we justify using $\mathcal{P} \supset \mathcal{Q}$? What do we need in addition to $\mathcal{P} \supset \mathcal{Q}$?
- The principle is **modus ponens** (affirming the antecedent):

$$\{\mathcal{P} \supset \mathcal{Q}, \mathcal{P}\} \vDash \mathcal{Q}$$

- (When inferring from $A \supset {\sim}C$ and A to ${\sim}C$, the role of $\mathcal{P}$ is played by A and role of $\mathcal{Q}$ by ${\sim}C$.)

Elimination rule for ⊃

$$
\begin{array}{r|ll}
m & \mathcal{P} \supset \mathcal{Q} & \\
n & \mathcal{P} & \\
 & \mathcal{Q} & : m,\ n \supset\mathrm{E}
\end{array}
$$

Let's illustrate this rule using an exercise in Carnap:
we show that $\{A \,\&\, B, A \supset C, B \supset D\} \vDash C \,\&\, D$.

1	$A \mathbin{\&} B$	:PR
2	$A \supset C$	:PR
3	$B \supset D$	:PR
4	A	:1 & E
5	C	:2, 4 ⊃E
6	B	:1 & E
7	D	:3, 6 ⊃E
8	$C \mathbin{\&} D$	:5, 7 & I

Introducing $\supset$

- How do we justify a conditional? What should we require for a proof of $\mathcal{P} \supset \mathcal{Q}$ (say, from some premise $\mathcal{R}$)?
- We need a proof that shows that $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.
- Idea: show instead that $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$.
- The conditional $\supset$ no longer appears, so this seems easier.
- It's a good move, because if $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.

Justifying $\supset$I

Fact

If $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.

- If $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then every TVA makes $\mathcal{R}$ or $\mathcal{P}$ false or it makes $\mathcal{Q}$ true
- Let's show that no valuation is a counterexample to $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$:
  1. A valuation that makes $\mathcal{R}$ and $\mathcal{P}$ true, but $\mathcal{Q}$ false, is impossible if $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$.
  2. So any valuation must make $\mathcal{R}$ false, $\mathcal{P}$ false, or $\mathcal{Q}$ true.
  3. If it makes $\mathcal{R}$ false, it's not a counterexample to $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.
  4. If it makes $\mathcal{P}$ false, it makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.
  5. If it makes $\mathcal{Q}$ true, it also makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.
- So, there are no counterexamples to $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.
- If $\mathcal{R} = \varnothing$, then from $\mathcal{P} \vDash \mathcal{Q}$ we can infer $\vDash \mathcal{P} \supset \mathcal{Q}$

Subproofs (CRUCIAL CONCEPT)

- We want to justify $\mathcal{P} \supset \mathcal{Q}$ by giving a proof of $\mathcal{Q}$ from *assumption* $\mathcal{P}$ (and possibly other premises Γ, e.g. $\mathcal{R}$)
- How to do this in a proof? We can add something as a premise and *discharge* it later!
- Solution: add $\mathcal{P}$ as an assumption (:AS), and keep track of what depends on that assumption (say, by indenting and a vertical line)
- Once we're done (have proved $\mathcal{Q}$), close this "subproof".
- Justification of $\mathcal{P} \supset \mathcal{Q}$ is the **entire** subproof (use a HYPHEN)
- **Important**: nothing **inside** a subproof is available outside as a justification (since inner lines might depend on the assumption)

Introduction rule for $\supset$

$$
\begin{array}{l|l|ll}
m & & \mathcal{P} & \text{:AS for } \supset\text{I} \\
\cline{3-3}
 & & \vdots & \\
n & & \mathcal{Q} & \\
 & \mathcal{P} \supset \mathcal{Q} & & :m\text{-}n \supset\text{I}
\end{array}
$$

NOTE THE **HYPHEN** IN THE JUSTIFICATION LINE!!!

We'll illustrate using more exercises from Week 6 Practice Problems

- Show: $\{A \supset B, B \supset C\} \vDash A \supset C$.
- Show: $A \supset (B \supset C) \vDash (A \,\&\, B) \supset (A \,\&\, C)$

1	$A \supset B$	:PR
2	$B \supset C$	:PR
3	$\quad A$	:AS for $\supset$I
4	$\quad B$	:1, 3 $\supset$E
5	$\quad C$	:2, 4 $\supset$E
6	$A \supset C$	:3-5 $\supset$I

1	$A \supset (B \supset C)$	:PR
2	$\quad A \,\&\, B$	:AS for $\supset$I
3	$\quad A$	:2 & E
4	$\quad B \supset C$	:1, 3 $\supset$E
5	$\quad B$	:2 & E
6	$\quad C$	:4, 5 $\supset$E
7	$\quad A \,\&\, C$	:3, 6 & I
8	$(A \,\&\, B) \supset (A \,\&\, C)$	:2-7 $\supset$I

6. Proofs in SL

d. Use of subproofs

Reiteration (for the 11th hour!)

$\mathcal{P} \vDash \mathcal{P}$, so "Reiteration" R is a good rule:

$$\begin{array}{r|ll} m & \mathcal{P} & \\ & & \\ k & \mathcal{P} & \quad : m \text{ R} \end{array}$$

Uses of reiteration (to the Carnap!):

- Proof of $A \vDash A$.
- Proof that $A \supset (B \supset A)$ is a tautology.

1	A	:AS for $\supset$I
2	A	:1 R
3	$A \supset A$	:1-2 $\supset$I

Again, note the **HYPHEN**! Even though our subproof is only two lines, we still write ‘:1–2’ and NOT ‘:1, 2’.

1	$\quad \vert\ A$	:AS for $\supset$I
2	$\quad \vert\ \quad \vert\ B$	:AS for $\supset$I
3	$\quad \vert\ \quad \vert\ A$	:1 R
4	$\quad \vert\ B \supset A$	:2–3 $\supset$I
5	$A \supset (B \supset A)$	:1–4 $\supset$I

Rules for justifications and subproofs

- When a rule calls for a subproof, we cite it as ": m–n", hyphenating the first and last line numbers of the subproof.
- Sentences on the Assumption line **and** last line MUST match rule
- After a subproof is done, you can only cite the whole thing, NOT any line in it (you are 'outside the scope' of these lines)
- Subproofs (subproofs can be nested) can be nested
- You also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is completed (since again you'd be 'outside the scope' of those lines)

Reiteration (do’s a don’ts) DOs and DON’Ts!

Which are correct applications of R?

1	A	:AS
2	A	:AS
3	A	:1 ✓ R
4	A	:1 ✓ R
5	A	:2 ✗ R
6	A	:2 ✗ R
7	A	:1 ✗ R

6. Proofs in SL

e. Disjunction Intro and Elim. (Rules for $\vee$)

Introduction rule for $\vee$

We have $\mathcal{P} \vDash \mathcal{P} \vee \mathcal{Q}$. So:

$$\begin{array}{r|ll} m & \mathcal{P} & \\ & \mathcal{P} \vee \mathcal{Q} & :m\ \vee\mathrm{I} \end{array} \qquad \begin{array}{r|ll} m & \mathcal{Q} & \\ & \mathcal{P} \vee \mathcal{Q} & :m\ \vee\mathrm{I} \end{array}$$

- Note that the introduced disjunct can be ANYTHING!
- And you can introduce on the left OR right side!
- Let's do practice problem 6.10 on Carnap!

1	$\quad \vert\ A$	:AS for $\supset$I
2	$\quad \vert\ B \vee A$	:1 $\vee$I
3	$A \supset (B \vee A)$	:1-2 $\supset$I

Eliminating $\vee$ (Proof by Cases)

- What can we justify with disjunction $\mathcal{P} \vee \mathcal{Q}$?
- Not $\mathcal{P}$ and also not $\mathcal{Q}$: neither is entailed by $\mathcal{P} \vee \mathcal{Q}$.
- But: if both $\mathcal{P}$ and $\mathcal{Q}$ separately entail some third sentence $\mathcal{R}$, then we know that $\mathcal{R}$ follows from the disjunction!
- To show this, we need **two** subproofs that show $\mathcal{R}$, but in each proof we are allowed to use only one of $\mathcal{P}$, $\mathcal{Q}$.

Elimination rule for $\vee$ (Proof by Cases)

$$
\begin{array}{l|l|l}
m & \mathcal{P} \vee \mathcal{Q} & \\
i & \quad \mathcal{P} & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & \\
j & \quad \mathcal{R} & \\
 & -- & \\
k & \quad \mathcal{Q} & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & \\
\ell & \quad \mathcal{R} & \\
 & \mathcal{R} & :m,\ i\text{–}j,\ k\text{–}\ell\ \vee\text{E}
\end{array}
$$

- From $\mathcal{P}$ we derive $\mathcal{R}$
- Start a subproof for each disjunct
- The subproofs need not be adjacent, but if they are, **separate with --**
- From $\mathcal{Q}$ we derive $\mathcal{R}$
- You can swap the order of the subproofs
- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction
- Remember to pop out of subproof level at the end!

1	$A \vee B$	:PR
2	$\quad A$	:AS for $\vee$E
3	$\quad B \vee A$	:2 $\vee$I
4	$\quad B$	:AS for $\vee$E
5	$\quad B \vee A$	:4 $\vee$I
6	$B \vee A$	:1, 2-3, 4-5 $\vee$E

- In Carnap: Need -- between the subproofs
- Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself
- Proceed to Carnap PP6.15!

1	$A \vee B$	:PR
2	$A \supset B$	:PR
3	A	:AS for $\vee$E
4	B	:2, 3 $\supset$E
5	B	:AS for $\vee$E
6	B	:5 R
7	B	:1, 3-4, 5-6 $\vee$E

6. Proofs in SL

f. Negation Intro and Elimination

Introducing $\sim$

- An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.
- For instance:
  - $\mathcal{Q} \vDash \mathcal{P}$ iff $\mathcal{Q}$ and $\sim\mathcal{P}$ are jointly unsatisfiable.
  - $\mathcal{Q} \vDash \sim\mathcal{P}$ iff $\mathcal{Q}$ and $\mathcal{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\sim$I rule: To justify $\sim\mathcal{P}$, show that $\mathcal{P}$ (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction follows!

Negation Introduction ($\sim$I)

m	$\quad \mid \Phi$	:AS for $\sim$I
	$\quad \mid \vdots$	
n	$\quad \mid \Psi$	
	$\quad \mid \vdots$	
o	$\quad \mid \sim\Psi$	
	$\sim\Phi$	:m-o $\sim$I

- Assume the **non**–negated wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **introduce** that negativity!
- Remember to cite the WHOLE subproof (hyphen!)
- Let's try exercise PP6.21:

1	$A \supset B$	:AS for $\supset$I
2	$\sim B$	:AS for $\supset$I
3	A	:AS for $\sim$I
4	B	:1, 3 $\supset$E
5	$\sim B$	:2 R
6	$\sim A$	:3-5 $\sim$I
7	$\sim B \supset \sim A$	:2-6 $\supset$I
8	$(A \supset B) \supset (\sim B \supset \sim A)$	:1-7 $\supset$I

Negation Elimination ($\sim$E)

m	$\sim\Phi$	:AS for $\sim$E
	$\vdots$	
n	Ψ	
	$\vdots$	
o	$\sim\Psi$	
	Φ	:m-o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **eliminate** that negativity!
- Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)
- Let's try exercise PP6.22:

1	$\sim A \supset \sim B$	:AS for $\supset$I
2	$\quad B$	:AS for $\supset$I
3	$\quad\quad \sim A$	:AS for $\sim$E
4	$\quad\quad \sim B$	:1, 3 $\supset$E
5	$\quad\quad B$	:2 R
6	$\quad A$	:3-5 $\sim$E
7	$B \supset A$	:2-6 $\supset$I
8	$(\sim A \supset \sim B) \supset (B \supset A)$	:1-7 $\supset$I

6. Proofs in SL

g. Biconditional Intro and Elimination (<->)

Biconditional Introduction ($\equiv$I) (Type <-> !!!)

i	$\mathcal{A}$	:AS for $\equiv$I
	$\vdots$	
j	$\mathcal{B}$	
	--	
k	$\mathcal{B}$	:AS for $\equiv$I
	$\vdots$	
l	$\mathcal{A}$	
	$\mathcal{A} \equiv \mathcal{B}$	:i-j, k-l $\equiv$I

- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs
- The subproofs need not be adjacent, but if they are, **separate with --**
- Remember to cite BOTH subproofs (hyphens!)
- Remember to pop out of subproof line!

Biconditional Elimination ($\equiv$E) (Type <-> !!!)

m	$\mathcal{A} \equiv \mathcal{B}$	
n	$\mathcal{A}$	
	$\mathcal{B}$	$: m,\ n \equiv$E

m	$\mathcal{A} \equiv \mathcal{B}$	
n	$\mathcal{B}$	
	$\mathcal{A}$	$: m,\ n \equiv$E

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)
- There can be lines between lines m and n
- Remember to cite the lines of both (i) the biconditional and (ii) the side you have already
- Carnap issue: must type <->E

Issue with Typing $\equiv$ in Carnap

- For Carnap to recognize $\equiv$I or $\equiv$E in the justification column, you sadly must type <-> I or <-> E
- This is a bummer; I hope we can have it fixed (eventually)
- It is still fine to type <> for the biconditional symbol in the sentences
- You can also copy/paste the $\equiv$symbol from elsewhere on the page!

6. Proofs in SL

h. Strategies and examples

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)
  - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\vee$I
  - Try Negation Intro or Elimination, working toward a contradiction
- Repeat for each new goal from top

6. Proofs in SL

i. The Rules, Reiterated

The rules, one more time: Reiteration

$$\begin{array}{r|ll} m & \mathcal{P} & \\ & \vdots & \\ k & \mathcal{P} & :m \text{ R} \end{array}$$

- Remember that you must be in the scope of the line you're reiterating
- e.g. if you're outside a subproof, you can't reiterate anything wholly within the subproof

The rules: Conjunction Intro (&I) and Elimination (&E)

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
n & \mathcal{Q} & \\
 & \mathcal{P} \,\&\, \mathcal{Q} & :m,\ n\ \&\mathrm{I}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{P} & :m\ \&\mathrm{E}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{Q} & :m\ \&\mathrm{E}
\end{array}
$$

The rules: Conditional Intro (⊃I) and Elim (⊃E)

$$
\begin{array}{l|l|ll}
m & & \mathcal{P} & \text{:AS for } \supset\text{I} \\ \cline{3-3}
 & & \vdots & \\
n & & \mathcal{Q} & \\
 & \multicolumn{2}{l}{\mathcal{P} \supset \mathcal{Q}} & :m\text{–}n \supset\text{I}
\end{array}
$$

$$
\begin{array}{l|ll}
m & \mathcal{P} \supset \mathcal{Q} & \\
n & \mathcal{P} & \\
 & \mathcal{Q} & :m,\ n \supset\text{E}
\end{array}
$$

The rules: Disjunction Intro ($\vee$I) and Elimination ($\vee$E)

$$
\begin{array}{r|l|l}
m & \mathcal{P} \vee \mathcal{Q} & \\
i & \quad \mathcal{P} & \text{:AS for } \vee\text{E} \\
 & \quad \vdots & \\
j & \quad \mathcal{R} & \\
 & -- & \\
k & \quad \mathcal{Q} & \text{:AS for } \vee\text{E} \\
 & \quad \vdots & \\
\ell & \quad \mathcal{R} & \\
 & \mathcal{R} & :m,\ i\text{-}j,\ k\text{-}\ell\ \vee\text{E}
\end{array}
$$

$$
\begin{array}{r|l|l}
m & \mathcal{P} & \\
 & \mathcal{P} \vee \mathcal{Q} & :m\ \vee\text{I}
\end{array}
$$

$$
\begin{array}{r|l|l}
m & \mathcal{Q} & \\
 & \mathcal{P} \vee \mathcal{Q} & :m\ \vee\text{I}
\end{array}
$$

Remember that $\mathcal{P}$ can be the same wff as $\mathcal{Q}$

so can introduce $\mathcal{P} \vee \mathcal{P}$ from $\mathcal{P}$

The rules: Negation Intro and Elimination

Negation Intro ($\sim$I)

$$
\begin{array}{l|l|ll}
m & & \underline{\Phi} & \text{:AS for } \sim\text{I} \\
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \multicolumn{2}{l}{\sim\Phi} & :m\text{-}o \sim\text{I}
\end{array}
$$

Neg. Elimination ($\sim$E)

$$
\begin{array}{l|l|ll}
m & & \underline{\sim\Phi} & \text{:AS for } \sim\text{E} \\
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \multicolumn{2}{l}{\Phi} & :m\text{-}o \sim\text{E}
\end{array}
$$

Note that you can swap the order of Ψ and $\sim\Psi$ in the subproofs!

The rules: Biconditional Intro and Elimination (<->)

$$
\begin{array}{l|l|ll}
i & \mathcal{A} & & \text{:AS for } \equiv\text{I} \\
 & \vdots & & \\
j & \mathcal{B} & & \\
 & -- & & \\
k & \mathcal{B} & & \text{:AS for } \equiv\text{I} \\
 & \vdots & & \\
l & \mathcal{A} & & \\
 & \mathcal{A} \equiv \mathcal{B} & & :i\text{-}j,\ k\text{-}l \equiv\text{I}
\end{array}
$$

Biconditional Elimination ($\equiv$E)

$$
\begin{array}{l|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{A} & \\
 & \mathcal{B} & :m,\ n \equiv\text{E}
\end{array}
$$

$$
\begin{array}{l|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{B} & \\
 & \mathcal{A} & :m,\ n \equiv\text{E}
\end{array}
$$