13. Completeness of QND

Completeness of QND

- **QND is Complete**: For any set Γ of QL–sentences and any QL–sentence $\mathcal{P}$, if Γ semantically entails $\mathcal{P}$, then there exists a derivation of $\mathcal{P}$ from Γ in our natural deduction system QND
  - In symbols: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$
  - Note that Γ can be countably infinite

- Completeness guarantees that for any valid QL–argument, there is at least one corresponding deduction in QND.
- So we need not reason about arbitrary models to determine if a QL–argument is valid; reasoning in QND suffices! WOW COOL

"$\vDash$": our Semantic Double Turnstile

- "$\Gamma \vDash \mathcal{P}$" means that Γ logically entails $\mathcal{P}$
  In any QL-model $\mathfrak{M}$ where the premises in Γ are true, the conclusion $\mathcal{P}$ is true
- Equivalently: there is no QL-model $\mathfrak{M}$ such that Γ is satisfied while $\mathcal{P}$ is false
- Equivalently, this means that $\Gamma \cup \{\sim\mathcal{P}\}$ **is unsatisfiable**: no QL-model makes-true the premises and negated conclusion
- We'll use this last fact A LOT in our proof that QND is complete!

13. Completeness of QND

a. Semantic vs. Syntactic Consistency

Semantic vs. Syntactic Consistency

- As with SND, we appeal to two distinct notions of consistency
- One is **semantic**:

  there is a QL-model that **satisfies** every sentence in the set
- We introduce a new **syntactic** notion of consistency relative to QND:

  - a set of QL sentences is **QND-consistent** provided that you can't derive contradictory sentences from it in QND
- Core proof idea: we'll show that if a set of sentences is **QND-consistent**, then it is also semantically consistent (i.e. **satisfiable**). So by the contrapositive: if a set is **un**satisfiable, then it is **in**consistent-in-QND.

Semantic: Satisfiable (quantificationally consistent)

- Recall: a set of QL sentences is **satisfiable** provided there is at least one QL-model $\mathfrak{M}$ that makes all of them true
- This is a *semantic* notion of consistency (aka "quantificational consistency")
- Contrast this with the syntactic notion of **consistency in QND**:

Syntactic: (In)consistent-in-QND (derivationally consistent)

- Let Γ be a (possibly infinite) set of QL sentences
- **Inconsistent-in-QND**: from premises in Γ, we can derive contradictory formulas R and $\sim R$ in the scope of the main scope line (i.e. in the scope of these premises)
- **Consistent-in-QND**: Γ is not QND-inconsistent, i.e. there is no derivation from premises in Γ resulting in contradictory formulas within the main scope
- Other words we might use for these concepts: QND-inconsistent, derivationally-inconsistent, QND-consistent, etc.
- Just remember: this syntactic notion has nothing to do with models or interpretations!

13. Completeness of QND

b. Proof Sketch

Proof Sketch: Just like what we did for SL!

- Goal: prove the completeness of QL: for every QL sentence $\mathcal{P}$ and every set Γ of QL sentences, if $\Gamma \vDash \mathcal{P}$ then $\Gamma \vdash_{QND} \mathcal{P}$
- So assume that $\Gamma \vDash \mathcal{P}$.
- This means that $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**:
  no QL–model satisfies the premises and negated conclusion
  (i.e. $\Gamma \cup \{\sim\mathcal{P}\}$ is *semantically* inconsistent)
- We now appeal to a **Consistency lemma** that remains the heart of the enterprise: any QND–consistent set of QL sentences is satisfiable (i.e. semantically consistent)

Proof Sketch: Using the consistency lemma

- **Consistency lemma** (CL): any QND-consistent set of QL sentences is satisfiable, i.e. true in some QL-model $\mathfrak{M}$
- **Contrapositive** of CL: any set of QL sentences that is Unsatisfiable is QND-Inconsistent
- From $\Gamma \vDash \mathcal{P}$ we know that $\Gamma \cup \{\sim\mathcal{P}\}$ is unsatisfiable
- So by the contrapositive of CL, we see that $\Gamma \cup \{\sim\mathcal{P}\}$ is QND-inconsistent
- This means that we can derive a pair of contradictory sentences R and $\sim R$ from $\Gamma \cup \{\sim\mathcal{P}\}$! So using the power of negation elimination, we can derive $\mathcal{P}$ from Γ, i.e. $\Gamma \vdash_{QND} \mathcal{P}$. So we are 'done'!

Negation Elimination Refresher (book's Exercise 11.4.2)

- Claim: if $\Gamma \cup \{\sim\mathcal{P}\}$ is **QND-inconsistent**, then $\Gamma \vdash_{QND} \mathcal{P}$
- Proof: starting with (finitely-many) premises Δ from Γ, introduce $\sim\mathcal{P}$ as a subproof assumption for negation elimination
- Since $\Gamma \cup \{\sim\mathcal{P}\}$ is QND-inconsistent, we can derive a contradictory pair R and $\sim R$ within the scope of sentences in $\Delta \cup \{\sim\mathcal{P}\}$
- Then discharge this assumption $\sim\mathcal{P}$ by negation elimination, writing $\mathcal{P}$, now in the scope of Δ. So $\Delta \vdash_{QND} \mathcal{P}$
- Since $\Delta \subseteq \Gamma$, we have $\Gamma \vdash_{QND} \mathcal{P}$

Core subgoal: Prove consistency lemma (book's 11.4.2)

- So all we have to do is prove the **consistency lemma**: any QND–consistent set of QL sentences is satisfiable
- As with SL, we'll prove this lemma in several 'stages':
- The first two are straightforward: given a QND–consistent set Γ, we construct a **superset** Γ^* that is *maximally QND–consistent* and *existentially complete* ($\exists$–complete)
- In the third stage, we show that any $\exists$–complete, maximally QND–consistent set is satisfiable: we use maximal consistency and $\exists$–completeness to construct a model that satisfies every sentence in Γ^*. *Wrinkle*: we work in an extended language QL$'$!
- Since by construction $\Gamma \subseteq \Gamma^*$, this QL$'$–model satisfies Γ (in QL$'$)
- From our QL$'$–model, we generate a QL–model that satisfies Γ

13. Completeness of QND

c. Stage 0: $\exists$-Completeness and QL$'$

Maximally QND-consistent (no longer enough!)

- A set Γ^* of QL or QL′ sentences is **maximally QND-consistent** provided that:
  1.) Γ^* is QND-consistent (i.e. can't derive contradictory sentences)
  2.) adding **any** additional sentence to Γ^* would result in an QND-inconsistent set
- i.e. for any $P \notin \Gamma^*$, $\{P\} \cup \Gamma^*$ is QND-inconsistent
- Unlike with SL, maximal derivational consistency is no longer enough to ensure satisfiability
- Recall that our purely syntactic membership lemma is motivated by the truth-conditions for QL sentences: sentences belong to Γ^* iff the relevant "truth-condition pieces" belong to Γ^* as well
- To extend our membership lemma to quantified sentences, we require that every existential sentence in Γ^* has a substitution instance also in Γ^*. So we introduce a new property:

Existential-completeness: definition and motivation

- **∃-completeness**: a set Γ of QL or QL$'$ sentences is *existentially-complete* just in case for every sentence in Γ of the form $(\exists \chi)\mathcal{P}$, at least one substitution instance $\mathcal{P}[c/\chi]$ is in Γ

- Motivation: $(\exists \chi)\mathcal{P}$ is true in a model iff some object $r \in D$ is a $\mathcal{P}$
- To construct an ∃-complete set Γ^*, we need recourse to a countable infinity of unused constants.
- Otherwise, new substitution instances that we add could "contradict" sentences already in Γ, spoiling QND-consistency
- *Problem*: our starting Γ might be infinite and so already use infinitely-many constants from QL. What are we to do?

It's a bird! It's a plane! It's ... Language QL′???

- QL′ is exactly like QL except that we allow subscripted *constants* to have **primed-indices**
- e.g. $c_{11'}$, $b_{234'}$, $g_{2'}$ (‘$'$’-symbol always at the end)
- Unsubscripted constants remain the same: a thru v
- So QL′ just adds one new symbol ‘$'$’, allowed to occur only at the end of indices for constants
- The recursive structure of truth-in-QL′ is defined exactly the same as for QL (using our good friend, satisfaction semantics!)
- Note that we do NOT allow primed indices on Predicates
- Moral: reach *for the stars*, **not** drugs

13. Completeness of QND

d. Stage 1: Constructing Γ^*

Stage 1(i): first enumerate the sentences of QL′!

- Let Γ be a QND-consistent set of QL sentences (possibly infinite)
- To construct Γ^*, we first **enumerate** the QL′ sentences, so that every QL′ sentence is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- As with SL, stipulate an ‘alphabetical order’ for QL′ symbols
- $\sim, \vee, \&, \supset, \equiv, (,), 0, 1, \ldots, 9, A, B, \ldots, Z, a, \ldots, v, w, x, y, z, \forall, \exists, '$
- Assign each symbol an **index** between ‘10’ and ‘84’ (skip 17-19)
- Then each QL′ sentence corresponds to a unique positive integer, constructed by replacing each symbol in the sentence with its index, from left to right.
- So with our ordering, ‘A’ is the first sentence; ‘B’ the second …up to Z, and then we hit $\sim A$ ($\mapsto$ 1030), then $\sim B$ ($\mapsto$ 1031), etc.

Recall what we did in SL to form Γ^* Max.–SND–Consist.

- We considered the first sentence 'A' in our enumeration. If A could be added to Γ without the resulting set being SND–inconsistent, then we let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays SND–consistent)
- We proceeded to the 2nd sentence in our enumeration. If it could be added to Γ_1 without the new set being SND–inconsistent, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$
- Γ^* was the result of 'doing' this procedure for every SL sentence
- Now we need to complicate matters a bit, to handle sentences of the form $(\exists\chi)\mathcal{P}$ and ensure we add a suitable substitution instance whenever we can add $(\exists\chi)\mathcal{P}$ while preserving QND–consistency

Building up Γ^*

- Given a QND–consistent set of QL sentences Γ, let $\Gamma_0 := \Gamma$
- Consider the k–th sentence P_k in our enumeration of QL$'$
- Define Γ_{k+1} as follows:
  i.) Γ_k if the set $\Gamma_k \cup \{P_k\}$ is QND–**INconsistent**
  ii.) $\Gamma_k \cup \{P_k\}$ if P_k does NOT have the form $(\exists \mathcal{X})\mathcal{Q}$, and $\Gamma_k \cup \{P_k\}$ is QND–consistent
  iii.) $\Gamma_k \cup \{P_k, P_k^\dagger\}$ if $\Gamma_k \cup \{P_k\}$ is QND–consistent AND P_k DOES have the form $(\exists \mathcal{X})\mathcal{Q}$, where $P_k^\dagger$ is a substitution instance $\mathcal{Q}[c/\mathcal{X}]$, and c is the alphabetically earliest constant not in P_k or any sentence in Γ_k
  - Such a c is guaranteed to exist because Γ_0 belongs to QL.
  - So the countable–infinity of primed subscripted constants from QL$'$ are available at each stage if needed.
- Then $\Gamma^* := \bigcup_{k=0}^{\infty} \Gamma_k$

13. Completeness of QND

e. Stage 2: Γ^* is M-QND-C & $\exists$-complete

Stage 2: Γ^* is maximally QND-consistent & $\exists$-complete

- This requires proving three claims (from the definitions):

  1.) Γ^* is consistent in QND

  2.) Adding any additional sentence to Γ^* would result in a **QND-inconsistent** set

  3.) For every QL′ sentence of the form $(\exists \chi)\mathcal{Q}$ in Γ^*, at least one substitution instance $\mathcal{Q}[c/\chi]$ belongs to Γ^*

- We prove these in turn

Stage 2 (i): Γ^* is QND-consistent

- Assume for *reductio* that Γ^* is inconsistent in QND
- Then there would be a QND derivation with finite premise set $\Delta \subset \Gamma^*$ that derives a contradictory pair R and $\sim R$
- Since Δ is finite, there exists some $k+1 \in \mathbb{N}$ s.t. $\Delta \subset \Gamma_{k+1}$. So then this Γ_{k+1} would be QND-inconsistent.
- Yet, each Γ_{k+1} is necessarily QND-consistent:
  - If P_k is not existential, it joins Γ_{k+1} only if $\Gamma_k \cup \{P_k\}$ is QND-consistent—by condition (ii)
  - If P_k is of the form $(\exists \mathcal{X})\mathcal{Q}$, it joins only if $\Gamma_k \cup \{P_k\}$ is QND-consistent.
    - It remains to show that $\Gamma_k \cup \{(\exists \mathcal{X})\mathcal{Q}, \mathcal{Q}[c/\mathcal{X}]\}$ is QND-consistent
  - **Lemma**: if c does not occur in a QND-C set $\Gamma_k \cup \{(\exists \mathcal{X})\mathcal{Q}\}$, then $\Gamma_k \cup \{(\exists \mathcal{X})\mathcal{Q}, \mathcal{Q}[c/\mathcal{X}]\}$ is QND-consistent
- Hence, Γ^* must be QND-consistent, on pain of *reductio*

Stage 2 (ii): Γ^* is maximally QND-consistent

- Assume for *reductio* that Γ^* weren't maximally QND-consistent, despite being QND-consistent
- i.e. assume *it is not the case that* for all other sentences, adding it to Γ^* would result in a QND-inconsistent set
  $\Rightarrow$ there exists a sentence $\mathcal{Q}$ that we could add to Γ^* while preserving QND-consistency (i.e. there is some sentence we neglected that could make Γ^* a 'bigger' QND-consistent set)
- Yet, $\mathcal{Q}$ would appear in our enumeration as some sentence P_k, 'considered' at the k-th stage of our construction of Γ^*.
- So if $\mathcal{Q}$ isn't in Γ^*, then this is because adding it 'would have' made $\Gamma_k \subset \Gamma^*$ QND-inconsistent.
  So $\{\mathcal{Q}\} \cup \Gamma^*$ must be QND-inconsistent (*reductio*!)
- So we can't add any $\mathcal{Q}$ to Γ^* while preserving QND-consistency

Stage 2 (iii): Γ^* is $\exists$–complete

- We simply need to show that for each sentence of the form $(\exists \chi)\mathcal{Q} \in \Gamma^*$, a substitution instance $\mathcal{Q}[c/\chi]$ also belongs to Γ^*
- Note that this is true by construction: each sentence of the form $(\exists \chi)\mathcal{Q}$ occurs in our QL′–enumeration:
- If we could have "added" $(\exists \chi)\mathcal{Q}$ at the k–th stage while preserving QND–consistency, then we also added a substitution instance.
- This is so even if $(\exists \chi)\mathcal{Q}$ is already in $\Gamma_0 := \Gamma$, since by condition (iii) $\Gamma_{k+1} := \Gamma_k \cup \{(\exists \chi)\mathcal{Q}, \mathcal{Q}[c/\chi]\}$ which in this case would equal $\Gamma_k \cup \{\mathcal{Q}[c/\chi]\}$ (since in this case, $(\exists \chi)\mathcal{Q} \in \Gamma_k$)

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f. Stage 3: Model Construction

Stage 3: The Maximal Consistency Lemma ($\approx$ book's 11.4.7)

- $\exists$–C **Maximal Consistency Lemma**: every QL$'$ set that is maximally–QND–consistent and $\exists$–complete is satisfiable
- (there exists a QL$'$–model that makes–true every sentence in Γ^*)
  We construct this model, calling it "$\mathfrak{M}^*$" ($\approx$book's "$\mathbf{I}^*$")
- Proof idea: since Γ^* is M–QND–C, for any sentence $\mathcal{P}$, either $\mathcal{P} \in \Gamma^*$ or $\sim\mathcal{P} \in \Gamma^*$ (you're either in the club or your 'nemesis' is!)

  This holds in particular for each QL$'$–atomic sentence
- Construct a QL$'$–model $\mathfrak{M}^*$ such that for each atomic QL$'$–sentence $\mathcal{A}$, $\mathfrak{M}^* \vDash \mathcal{A}$ iff $\mathcal{A} \in \Gamma^*$
- Then by the recursive structure of QL$'$ sentences, $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$

Stage 3 (i): the Membership Lemma (book's 11.4.6)

- To induct on QL$'$, we first constrain Γ^* membership
- Basically, Γ^* is *THE* club with the MOST ANGELIC bouncer you've eva seen, who enforces maximal consistency. Before this $\ast$angel$\ast$ lets a sentence into Γ^*, he checks who else is GOOD. You hear?
- **Membership Lemma** for club: if $\mathcal{P}$ and $\mathcal{Q}$ are QL$'$ sentences, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
  f.) $(\forall \chi)\mathcal{P} \in \Gamma^*$ iff for each constant c, $\mathcal{P}[c/\chi] \in \Gamma^*$
  g.) $(\exists \chi)\mathcal{P} \in \Gamma^*$ iff for at least one constant c, $\mathcal{P}[c/\chi] \in \Gamma^*$

Stage 3 (i): The Stairway to heaven (book's 11.4.5)

- To prove the membership lemma's cases (a)-(g), we'll use another lemma (*NB*: and she's buying, a lemma, to heavennnnnnnn!):
- **The Stairway**: if $\Gamma \vdash P$, and Γ^* is a maximally QND-consistent superset of Γ, then $P \in \Gamma^*$
  (mnemonic: "$\Gamma \vdash P$" pushes P up to QL′-heaven!)
- Proof: first, assume that $\Gamma \vdash P$ (we'll use this fact below)
  - Next, assume for *reductio* that $P \notin \Gamma^*$. Then since Γ^* is maximally QND-consistent, $\Gamma^* \cup \{P\}$ must be inconsistent in QND.
  - Hence, by negation introduction, $\Gamma^* \vdash {\sim}P$
  - By assumption, $\Gamma \vdash P$, so also $\Gamma^* \vdash P$, since $\Gamma \subseteq \Gamma^*$
  - So Γ^* derives both P and ${\sim}P$. *Reductio*! (since Γ^* is M-QND-C)
  - Hence, if $\Gamma \vdash P$ and $\Gamma \subseteq \Gamma^*$, then P must belong to Γ^*

Membership Lemma: Cases (a)–(e)

- I have a feeling that. . .
- WE'VE SEEN THIS incredible content BEFORE! (for SL)
- see the next slide for a refresher
- *Long story short*: There's a feeling I get

  When I look to the west

  And my spirit is crying for leaving

Membership Lemma: Case (a)

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
- Two directions to prove:

  $\Rightarrow$: Assume $\sim\mathcal{P} \in \Gamma^*$. Then if $\mathcal{P}$ were in Γ^*, we could derive contradictory sentences.

  So since Γ^* is QND–consistent, we must have $\mathcal{P} \notin \Gamma^*$

  $\Leftarrow$: Assume $\mathcal{P} \notin \Gamma^*$. Then adding $\mathcal{P}$ to Γ^* results in an QND–inconsistent set. Hence, there is some finite subset $\Delta \subset \Gamma^*$ s.t. $\Delta \cup \{\mathcal{P}\}$ is QND–inconsistent (i.e. derives contradictory sentence pair).
- So by negation introduction, $\Delta \vdash \sim\mathcal{P}$
- So by **The Stairway**, $\sim\mathcal{P} \in \Gamma^*$

Membership Lemma: Case (f) (something Universally new)

- **Case (f)**: $(\forall x)\mathcal{P} \in \Gamma^*$ iff for each constant c, $\mathcal{P}[c/x] \in \Gamma^*$
- Two directions to prove:

  $\Rightarrow$: Assume $(\forall x)\mathcal{P} \in \Gamma^*$

  - Then for any substitution instance $\mathcal{P}[c/x]$, we note that $(\forall x)\mathcal{P} \vdash_{QND} \mathcal{P}[c/x]$ by $\forall E$. So by the Stairway, $\mathcal{P}[c/x] \in \Gamma^*$

  $\Leftarrow$: Assume $(\forall x)\mathcal{P} \notin \Gamma^*$. Show that for some constant c, $\mathcal{P}[c/x] \notin \Gamma^*$

  - Then $\sim(\forall x)\mathcal{P} \in \Gamma^*$ by membership clause (a)

  - Then the derivation on p. 573 or—if I have no life—the derivation on the next slide, shows by the Stairway that $(\exists x)\sim\mathcal{P} \in \Gamma^*$, i.e. $\sim(\forall x)\mathcal{P} \vdash_{QND} (\exists x)\sim\mathcal{P}$

  - Then since Γ^* is $\exists$-complete, there is at least one substitution instance $\sim\mathcal{P}[b/x] \in \Gamma^*$. So by (a), $\mathcal{P}[b/x] \notin \Gamma^*$, which is what we needed to show.

Membership Lemma: Case (g) (it's getting existential)

- **Case (g)**: $(\exists x)\mathcal{P} \in \Gamma^*$ iff for at least one constant c, $\mathcal{P}[c/x] \in \Gamma^*$
- $\Rightarrow$: Assume $(\exists x)\mathcal{P} \in \Gamma^*$

  Then since Γ^* is $\exists$–complete, there is at least one substitution instance $\mathcal{P}[c/x] \in \Gamma^*$
- $\Leftarrow$: assume that $\mathcal{P}[c/x] \in \Gamma^*$.

  Note that $\mathcal{P}[c/x] \vdash_{QND} (\exists x)\mathcal{P}$ by Existential introduction

  So by the Stairway, $(\exists x)\mathcal{P} \in \Gamma^*$
- This completes the Membership Lemma, so we proceed to construct a model that satisfies Γ^* (in virtue of being maximally–QND–consistent and $\exists$–complete)!

Stage 3 (ii): Model construction (smart choices=lazy choices)

- A model's domain can be *any* set of objects. Note that, conveniently, symbols *are* objects ("words are labels on boxes")
- We define $\mathfrak{M}^* := (D, I^*)$ as follows:
  1. Let $D =$ the set of constant symbols in QL′, which includes all QL–constants (e.g. unprimed subscripted constants like j_{22})
  2. For the 0–th place predicates, i.e. the sentence letters B, $I^*(B) = \mathit{true}$ iff $B \in \Gamma^*$
  3. For each QL′–constant c, define $I^*(c) = c$ (each names itself)
  4. For each k–place predicate P, $I^*(P) := \mathit{Ext}(P)$ includes all and only those k–tuples $\langle c_1, \ldots, c_k \rangle$ such that $P c_1 \ldots c_k \in \Gamma^*$

Some important properties of our Model $\mathfrak{M}^*$

- By condition 3, each individual constant refers to a *unique* member of the domain, namely 'itself' (now 'objectified' in D!)
- For each atomic sentence $\mathcal{A}$ of QL′, $\mathfrak{M}^* \vDash \mathcal{A}$ iff $\mathcal{A} \in \Gamma^*$ (follows from conditions 2–4)
- By condition 3, every member of the domain is named by a constant, namely itself
- We will occasionally rely on these properties in our induction

Stage 3 (iii): Induction on QL′ (i.e. we still be clubbin')

- Goal: construct a QL′–model $\mathfrak{M}^*$ that satisfies the ∃–C M–QND–C set Γ^*, i.e. that makes true everything in Γ^* ($\mathfrak{M}^* \vDash \Gamma^*$)
  - Suffices to construct $\mathfrak{M}^*$ s.t. $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$
  Say that a sentence is "**clubbin'** " whenever it meets this property
- We induct on the number of logical operators in a QL′ sentence: i.e. the five connectives and two quantifiers ("conquans")
- **Base case**: show that each QL′–sentence with zero logical operators is clubbin' (i.e. the QL′–atomics be clubbin')
- (Strong) **Induction hypothesis**: assume every QL′ sentence with 1 to k–many operators is clubbin'
- Induction step: show that an arbitrary QL′ sentence with k+1–many operators is clubbin'

Base Case (true by construction)

- Consider an arbitrary QL′–sentence $\mathcal{A}$ that has zero logical operators. (Two directions to show! "iff")
- Then $\mathcal{A}$ is either an atomic sentence letter B or of the form $P c_1 \ldots c_n$ for n–place predicate P.
- If a sentence letter, then by part 2 of our definition of $\mathfrak{M}^*$, $I^*(B) = \mathit{true}$ iff $B \in \Gamma^*$ (i.e. $\mathfrak{M}^* \vDash B$ iff $B \in \Gamma^*$)
- If $\mathcal{A}$ is of form $P c_1 \ldots c_n$, then by definition $\mathfrak{M}^* \vDash P c_1 \ldots c_n$ iff $\langle c_1^D, \ldots, c_n^D \rangle \in Ext(P)$.
  By part 4, $\langle c_1^D, \ldots, c_n^D \rangle = \langle c_1, \ldots, c_n \rangle \in Ext(P)$ iff $P c_1 \ldots c_n \in \Gamma^*$
- We proceed to do induction using our QL′ induction schema: an arbitrary sentence $\mathcal{P}$ with k+1–many connectives has one of seven forms, coming from our seven operators

Induction on QL$'$: Cases 1–5

- Cases 1-5 are just like what did to prove the completeness of SND
- See the next slide for a refresher (*mutatis mutandis*)!
- Need to show: $\mathcal{P}$ be clubbin', i.e. $\mathcal{P}$ is true on $\mathfrak{M}^*$ iff $\mathcal{P} \in \Gamma^*$, where $\mathcal{P}$ is arbitrary QL$'$ sentence with k+1–many operators
- Induction hypothesis: assume every QL sentence with 1 to k–many operators is clubbin'
- Case 1: $\mathcal{P}$ has the form $\sim\mathcal{Q}$
- Case 2: $\mathcal{P}$ has the form $\mathcal{Q} \,\&\, \mathcal{R}$
- Case 3: $\mathcal{P}$ has the form $\mathcal{Q} \vee \mathcal{R}$
- Case 4: $\mathcal{P}$ has the form $\mathcal{Q} \supset \mathcal{R}$
- Case 5: $\mathcal{P}$ has the form $\mathcal{Q} \equiv \mathcal{R}$

Induction on QL′: Case 1

- **Case 1**: $\mathcal{P}$ has the form $\sim\mathcal{Q}$, where since $\mathcal{Q}$ has k−operators, it is clubbin by the IH (i.e. $\mathfrak{M}^* \vDash \mathcal{Q}$ if and only if $\mathcal{Q} \in \Gamma^*$)
- NTS: (i) (the $\Rightarrow$direction) if $\mathfrak{M}^* \vDash \mathcal{P}$ then $\mathcal{P} \in \Gamma^*$ and
  (ii) (the $\Leftarrow$direction) if $\mathcal{P} \in \Gamma^*$, then $\mathfrak{M}^* \vDash \mathcal{P}$

  (*Alternative (ii)*: show contrapositive: if $\mathfrak{M}^* \nvDash \mathcal{P}$, then $\mathcal{P} \notin \Gamma^*$)

$\Rightarrow$ if $\mathfrak{M}^* \vDash \mathcal{P}$, then $\mathfrak{M}^* \nvDash \mathcal{Q}$. Since $\mathcal{Q}$ is clubbin', we have $\mathcal{Q} \notin \Gamma^*$

By Membership lemma (a), $\sim\mathcal{Q} \in \Gamma^*$, so $\mathcal{P} \in \Gamma^*$

$\Leftarrow$ if $\mathcal{P} \in \Gamma^*$, then $\sim\mathcal{Q} \in \Gamma^*$. So by Membership lemma (a), $\mathcal{Q} \notin \Gamma^*$.

Since $\mathcal{Q}$ is clubbin', we have $\mathfrak{M}^* \nvDash \mathcal{Q}$. (i.e. $\mathcal{Q}$ is false in $\mathfrak{M}^*$)

So by the truth conditions for negation, $\mathcal{P}$ is true in $\mathfrak{M}^*$, i.e. $\mathfrak{M}^* \vDash \mathcal{P}$

Induction on QL′: Case 7 (existential quantifier)

- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$
  (warning: "$\mathcal{Q}$" is not a sentence, so sadly it can't be clubbin')
- We will use Membership Lemma **Case (g)**:
  $(\exists \chi)\mathcal{Q} \in \Gamma^*$ iff for at least one constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

$\Rightarrow$ Assume $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$. (need to show that $(\exists \chi)\mathcal{Q} \in \Gamma^*$)
- Then by the truth–conditions for existential, there is some object $r \in D$ that satisfies $\mathcal{Q}$.
- 'r' names object r, so substitution instance $\mathcal{Q}[r/\chi]$ is true in $\mathfrak{M}^*$
- This substitution instance has less than $k+1$–operators, so it is clubbin'. Hence, by the IH, $\mathcal{Q}[r/\chi] \in \Gamma^*$ (since $\mathfrak{M}^* \vDash \mathcal{Q}[r/\chi]$)
- So by membership case (g), $(\exists \chi)\mathcal{Q} \in \Gamma^*$

Induction on QL′: Case 7 backwards direction

- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$
- Use Membership Lemma **Case (g)**:
  $(\exists \chi)\mathcal{Q} \in \Gamma^*$ iff for at least one constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

$\Leftarrow$ Assume $(\exists \chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$

- Then by membership case (g), there is at least one substitution instance $\mathcal{Q}[c/\chi] \in \Gamma^*$, for some constant c
- Since $\mathcal{Q}[c/\chi]$ has fewer than $k+1$–operators, it is clubbin'.
- So by the Induction Hypothesis, $\mathfrak{M}^* \vDash \mathcal{Q}[c/\chi]$.
- Since 'c' names object c, we see that c satisfies $\mathcal{Q}$ in $\mathfrak{M}^*$
- So by the truth–conditions for existentials, $(\exists \chi)\mathcal{Q}$ is true in $\mathfrak{M}^*$

Induction on QL′: Case 6 (universal quantifier)

- **Case 6**: $\mathcal{P}$ has the form $(\forall\chi)\mathcal{Q}$
  (warning: "$\mathcal{Q}$" is not a sentence, so it can't be clubbin')
- We will use Membership Lemma **Case (f)**:
  $(\forall\chi)\mathcal{Q} \in \Gamma^*$ iff for each constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

$\Rightarrow$ Assume $\mathfrak{M}^* \vDash (\forall\chi)\mathcal{Q}$. Show that $(\forall\chi)\mathcal{Q} \in \Gamma^*$

- Then every object satisfies $\mathcal{Q}$, so every substitution instance for every constant is true in $\mathfrak{M}^*$ (since each object is named by itself)
- These $\mathcal{Q}[c/\chi]$ are clubbin' by the IH, so they all belong to Γ^*. So then by Membership Lemma case (f), $(\forall\chi)\mathcal{Q} \in \Gamma^*$

$\Leftarrow$ Assume $(\forall\chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\forall\chi)\mathcal{Q}$

- Practice this yourself!

Upshots of our Induction

- Having handled every case (in spirit), we conclude that every sentence of QL′ is clubbin’:
- For all QL′–sentences $\mathcal{P}$, $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$
- Hence, the QL′–model $\mathfrak{M}^*$ makes–true every sentence in Γ^*, showing that this set is satisfiable
- Hence, we have proven the **∃–C Maximal Consistency Lemma**: every QL′ set that is maximally–QND–consistent and ∃–complete is satisfiable in QL′
- It remains to prove the **Consistency Lemma**, i.e. that any QND–consistent QL–set (like our O.G. Γ) is satisfiable **in QL**!

From satisfiability of Γ^* to satisfiability of Γ

- We have shown that the maximally-QND-consistent and existentially complete Γ^* is satisfiable in QL′
- It remains to show that QND-consistent Γ is satisfiable **in QL**
- i.e. we need a QL-model $\mathfrak{M}$ s.t. $\mathfrak{M} \vDash \Gamma$
- **Hopes and dreams**: by construction $\Gamma \subset \Gamma^*$, so $\mathfrak{M}^* \vDash \Gamma$ in QL′. But how are we to get a QL-model for Γ from this??????
- **Salvation**: note that the model $\mathfrak{M}^*$ we constructed is *not only* a QL′ model for Γ^* *BUT ALSO* a QL-model for Γ!
- Since the language of QL is contained in QL′, $\mathfrak{M}^* := (D, I^*)$ maps all symbols of QL to objects in D
- If you like, you can define a QL-model $\mathfrak{M} := (D, I)$ s.t. I is the restriction of I^* to unprimed constants in QL. Then $\mathfrak{M} \vDash \Gamma$.
- $\square$ Q.E.D. MOST BLESSED STUDENTS!!! (i.e. *quod erat demonstrandum*)

Did we need to manually enforce ∃-completeness?

- In our condition (iii) for building up Γ^*, we manually enforced adding a substitution instance to our growing Γ_{k+1} whenever we add an existential sentence.
- Some have wondered: shouldn't condition (ii) take care of this? Substitution instances are QL′ sentences, so they arise at some k in our enumeration as well
- Really the issue is the following: are there maximally QND-consistent sets that are NOT existentially-complete? If so, then our condition (iii) is not idle
- So to show the necessity of our condition (iii) (or something like it), it suffices to construct a maximally QND-consistent set that has an existential sentence but no substitution instances for it.

A maximally QND-consistent but existentially INCOMPLETE set

- Let Γ_0 be the set $\{(\exists x){\sim}Fx\}$
- Its substitution instances have the form ${\sim}F[c/x]$, e.g. ${\sim}Fc$.
- Notice that the 'enemies' of these substitution instances always occur earlier in our enumeration, e.g. Fc occurs before ${\sim}Fc$, Fj'_{22} occurs before ${\sim}Fj'_{22}$ (the enemies always have one less symbol, so their index has two fewer digits)
- So imagine that we dropped condition (iii) and built up Γ^* using only conditions (i) and (ii).
- then Γ^* would contain $(\exists x){\sim}Fx$ and every instance of an 'enemy' substitution instance: $F\mathcal{c}$ for all constants $\mathcal{c}$
- Γ^* would NOT contain a single substitution instance of $(\exists x){\sim}Fx$ because every time we hit a ${\sim}F\mathcal{c}$ at its stage, Γ_k would already contain its enemy $F\mathcal{c}$, so that adding ${\sim}F\mathcal{c}$ would result in a QND-inconsistent set.

Some remaining concerns about this construction

- Intuitively, you might think that a set containing $(\exists x)\sim Fx$ and all these enemies $F\mathcal{c}$ would be QND–inconsistent.
- But it is not! Note that from existential elimination, we cannot start our subproof with a constant occuring in a premise, and these 'enemies' would be premises. So there is no way to derive a contradiction
- Similarly, we can NOT go from an enemy to a contradictory universal $(\forall x)Fx$ because the constant can't occur in a premise
- Notice as well that the membership lemma would fail. We would have every instance of $F\mathcal{c}$ but Γ^* would NOT contain $(\forall x)Fx$ because this sentence is QND–inconsistent with $(\exists x)\sim Fx$ (as an 8–line deduction shows)