Completeness of FOL$^=$

LOGIC I
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Basic Lemmas

L9.1 If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

L11.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

L11.6 If $\mathcal{M}$ and $\mathcal{M}'$ have the same domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any v.a. $\hat{a}$ defined over $\mathbb{D}$.

L12.1 If α is a constant and X is an FOL$^=$ derivation in which the constant β does not occur, then $X[\beta/\alpha]$ is also an FOL$^=$ derivation.

L12.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L12.4 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L12.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L12.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L12.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Completeness

T12.1 Every consistent set of $\mathcal{L}^=$ wfss Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable (check).
- So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T12.1**.
- So $\Gamma \vdash \neg\neg\varphi$ by **L12.3**, and so $\Gamma \vdash \varphi$ by DN and **L12.4**.

Assume: Let Γ be a set of $\mathcal{L}^=$ wfss that is consistent in FOL$^=$.

Saturation

Extension: Let $\mathcal{L}^{=}_{\mathbb{N}}$ include the extra constants $\mathbb{N}$.

- Let $\mathbb{C}$ be the set of all constants in $\mathcal{L}^{=}_{\mathbb{N}}$.

Free: Let $\varphi(\alpha)$ be a wff of $\mathcal{L}^{=}_{\mathbb{N}}$ with at most one free variable α.

Saturated: A set of wfss Σ is SATURATED in $\mathcal{L}^{=}_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathcal{L}^{=}_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \rightarrow \varphi[\beta/\alpha]) \in \Sigma$.

L12.2 Γ is consistent in $\text{FOL}^{=}_{\mathbb{N}}$.

- We will hence forth take 'consistent' to mean 'consistent in $\text{FOL}^{=}_{\mathbb{N}}$'.

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathcal{L}^{=}_{\mathbb{N}}$ with at most one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \rightarrow \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.
$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \rightarrow \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in φ_{k+1} or θ_j for any $j \leqslant k$.

Saturation: $\Sigma_0 = \Gamma$,
$\Sigma_{n+1} = \Sigma_n \cup \{\theta_{n+1}\}$, and
$\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

L12.5 Σ_Γ is consistent and saturated in $\mathcal{L}^{=}_{\mathbb{N}}$.

Base: $\Sigma_0 = \Gamma$ is consistent.

- Immediate from **L12.2**.

Induction: Assume Σ_m is consistent.

- Assume $\Sigma_{m+1} = \Sigma_m \cup \{\theta_{m+1}\}$ is inconsistent for contradiction.
- So $\Sigma_m \vdash \neg\theta_{m+1}$ by **L12.3**, and so $\Sigma_m \vdash \neg(\exists\alpha_{m+1}\varphi_{m+1} \rightarrow \varphi_{m+1}[n_{m+1}/\alpha_{m+1}])$.
- So $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$ by derived PL rules.
- So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I since n_{m+1} is not in $\forall\alpha_{m+1}\neg\varphi_{m+1}$ or Σ_m.
- So $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$, and so Σ_m is inconstant by **L12.9**.
- It follows by *reductio* that Σ_{m+1} is consistent.
- By weak induction, we know that Σ_k is consistent for all $k \in \mathbb{N}$.

Limit: If Σ_Γ is inconsistent, then X derives $\perp$ from Σ_Γ in $\text{FOL}^{=}_{\mathbb{N}}$.

- Since X is finite, $\Sigma_m \vdash \perp$ for some $m \in \mathbb{N}$ including all premises in X.
- So Σ_m is inconsistent, contradicting the above.
- By *reductio*, Σ_Γ is consistent.

Maximization

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{=}_{\mathbb{N}}$ just in case either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every wfs ψ in $\mathcal{L}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{=}_{\mathbb{N}}$.

Maximization: $\Delta_0 = \Sigma_\Gamma$,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases},$$

$\Delta_\Sigma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

L12.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal in $\mathcal{L}^{=}_{\mathbb{N}}$ and consistent.

Base: $\Delta_0 = \Sigma_\Gamma$ is consistent by **L12.5**.

Induction: Assume Δ_n is consistent.

- Want to show that Δ_{n+1} is consistent.

Case 1: If $\Delta_n \cup \{\psi_n\}$ is consistent, then $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: If $\Delta_n \cup \{\psi_n\}$ is inconsistent, then $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

- Assume $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent for contradiction.
- So Δ_n is inconsistent by **L12.6**.
- So Δ_{n+1} is consistent in both cases.

Limit If Δ is inconsistent, then Υ derives $\perp$ from Δ in $\mathrm{FOL}^{=}_{\mathbb{N}}$.

- Since Υ is finite, $\Delta_m \vdash \perp$ for some $m \in \mathbb{N}$ including all premises in Υ.
- This contradicts the above, and so Δ is consistent by *reductio*.

Maximal: Let ψ be any wfs of $\mathcal{L}^{=}_{\mathbb{N}}$, and so $\psi = \psi_k$ for some $k \in \mathbb{N}$.

- By construction, $\psi_k \in \Delta_{k+1}$ or $\neg\psi_k \in \Delta_{k+1}$.
- Generalizing on ψ shows that Δ is maximal.

L12.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

- Immediate from the definitions.

L12.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

- Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L12.9**.
- So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
- Thus $\varphi \in \Delta$ by maximality.