# Completeness of QD

LOGIC I
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## Basic Lemmas

**L13.1** If $\alpha$ is a constant and $X$ is a proof in which the constant $\beta$ does not occur, then $X[\beta/\alpha]$ is also a proof.

**L13.3** If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

**L13.5** If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

**L13.6** If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then $\Lambda$ is inconsistent.

**L13.9** If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then $\Lambda$ is inconsistent.

**L13.11** If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

## Satisfiability

**T13.1** Every consistent set of $\mathrm{QL}^=$ sentences $\Gamma$ is satisfiable.

*Completeness:* If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

1. Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.
2. So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T13.1**.
3. So $\Gamma \vdash \neg\neg\varphi$ by **L13.3**, and so $\Gamma \vdash \varphi$ by DN and **L13.5**.

## Saturation

*Free:* Let $\varphi(\alpha)$ be a wff of $\mathrm{QL}^=$ with at most one free variable $\alpha$.

*Saturated:* A set of sentences $\Sigma$ is saturated in $\mathrm{QL}^=_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathrm{QL}^=_{\mathbb{N}}$, there is a constant $\beta$ where $(\exists\alpha\varphi \supset \varphi[\beta/\alpha]) \in \Sigma$.

*Constants:* Let $\mathbb{C}$ be the constants of $\mathrm{QL}^=_{\mathbb{N}}$ where $\mathbb{N} \subseteq \mathbb{C}$ are new constants.

**L13.2** Assuming $\Gamma$ is consistent in $\mathrm{QL}^=$, we know $\Gamma$ is consistent in $\mathrm{QL}^=_{\mathbb{N}}$.

*Free Enumeration:* Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathrm{QL}^=_{\mathbb{N}}$ with one free variable.

*Witnesses:* $\theta_1 = (\exists\alpha_1\varphi_1 \supset \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in $\varphi_1$.

$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \supset \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in $\theta_j$ for any $j \leq k$.

*Saturation:* Let $\Sigma_1 = \Gamma$, $\Sigma_{n+1} = \Sigma_n \cup \{\theta_n\}$, and $\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

**L13.4** $\Sigma_\Gamma$ is consistent and saturated in $\mathrm{QL}^=_{\mathbb{N}}$.

1. If $\Sigma_{m+1}$ is inconsistent, then $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$.
2. So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I, and so $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$.
3. If $\Sigma_\Gamma$ is inconsistent, then $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$.

## Maximization

*Maximal:* A set of sentences $\Delta$ is maximal in $\mathrm{QL}^{=}_{\mathbb{N}}$ just in case as either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every sentence $\psi$ in $\mathrm{QL}^{=}_{\mathbb{N}}$.

*Full Enumeration:* Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all sentences in $\mathrm{QL}^{=}_{\mathbb{N}}$.

*Maximization:* Let $\Delta_0 = \Sigma$, $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$, and $\Delta_\Sigma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

**L13.7** $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal consistent in $\mathrm{QL}^{=}_{\mathbb{N}}$.

*Case 1:* $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

*Case 2:* $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

1. If $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent, then $\Delta_n$ is inconsistent by **L13.6**.
2. So $\Delta_{n+1}$ is consistent in both cases.
3. If $\Delta_\Sigma$ is inconsistent, then $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$.
4. Maximality is immediate.

**L13.8** $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where $\Delta$ is saturated.

1. Immediate from the definitions.

**L13.10** $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

1. Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L13.9**.
2. So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
3. Thus $\varphi \in \Delta$ by maximality.

## Henkin Model

*Element:* $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

*Domain:* $\mathbb{D}_\Delta = \{[\alpha]_\Delta : \alpha \in \mathbb{C}\}$.

**L13.13** If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

1. Assuming $\alpha = \beta \in \Delta$ where $\Gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
2. So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L13.11**.
3. Thus $\beta = \gamma \in \Delta$ by **L13.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

*Constants:* $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

*Predicates:* $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}^n_\Delta : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

**L13.14** If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

1. Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
2. $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L13.10**.
3. Parity of reasoning completes the proof.

## Henkin Lemmas

**L13.15** $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

1. Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$ for some $\hat{a}$, $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some $\alpha$-variant $\hat{c}$.
2. So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
3. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.9**.
4. So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L12.6**.
5. Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.
6. Let $\hat{c}$ be the $\alpha$-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\beta)$.
7. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L12.9**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

**L13.16** $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

1. Similar to **L13.15**.

**L13.17** $\mathcal{M}_\Delta$ satisfies $\varphi$ just in case $\varphi \in \Delta$.

*Base:* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

1. If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L13.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
2. Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L13.13**.

*Induction:* Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leq n$.

1. Let $\varphi$ be a sentence of $\text{QL}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

*Case 1:* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

*Case 2:* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

*Case 6:* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L13.15**.

1. *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.
2. *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L13.10** given saturation.

*Conclusion:* So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

## Restriction

*Restriction:* $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant $\alpha$ in $\text{QL}^{=}$.

**L13.18** For all $\text{QL}^{=}$ sentences $\varphi$, $\mathcal{M}'_\Delta$ satisfies $\varphi$ just in case $\mathcal{M}_\Delta$ satisfies $\varphi$.

**T13.1** Every consistent set of $\text{QL}^{=}$ sentences $\Gamma$ is satisfiable.

## Compactness

**C13.2** If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

**C13.3** $\Gamma$ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.