Uniqueness and Quantity

LOGIC I
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Uniqueness

Uniqueness: Ingmar trusts Albert, but no one else.

Only: Regiment the following argument:

(1) Lois Lane only loves Clark Kent.
(2) Only Clark Kent is Superman.
$\therefore$ Lois Lane loves Superman.

Definite Descriptions

Question 1: Regiment the following sentences.

- Socrates is guilty.
- Socrates is not guilty.
- Socrates is guilty or not.

Question 2: Regiment the following sentences.

- The king of France is bald.
- The king of France is not bald.
- The king of France is bald or not.

Question 3: What is the difference between these two cases?

Existence: If the king of France is Bald, then the king of France exists.

Definite Article: 'The king of France' can't be a name.

Regimentation: Russell offered the following analysis:

- $\exists x(Kxf \wedge \forall y(Kyf \supset x = y) \wedge Bx)$.
- $\exists x(\forall y(Kyf \equiv x = y) \wedge Bx)$.

Negation: Negation applies to the predicate, not the sentence.

Task 1: Regiment the following:

1. Superman is keeping something from his lover.
2. The man with the axe is not Jack.
3. The Ace of diamonds is not the man with the axe.
4. One-eyed jacks and the man with the axe are wild.
5. No spy knows the combination to the safe.
6. The one Ingmar trusts is lying.
7. The person who knows the combination to the safe is not a spy.

At Least:

Task 2: Regiment the following claims.

1. There is at least one wild card.
2. There are at least two clubs.
3. There are at least three hearts on the table.

Question 4: How can we define these quantifiers in general?

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Constants: If β is a constant, then β is free for any α and φ.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Examples: Consider the following cases:

(a) z is free for x in $\forall y(Fxy \supset Fyx)$

(b) y is not free for x in $\forall y(Fxy \supset Fyx)$

Inequality Quantifiers Defined

Definition: We may define the following abbreviations recursively:

Base: $\exists_{\geq 1}\alpha\varphi := \exists\alpha\varphi$.

Recursive: $\exists_{\geq n+1}\alpha\varphi := \exists\alpha(\varphi \wedge \exists_{\geq n}\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$ where β is free for α.

Infinite: $\Gamma_\infty := \{\exists_{\geq n}x(x = x) : n \in \mathbb{N}\}$.

Question 5: What is the smallest model to satisfy Γ_∞?

At Most: Regiment the following claims.

1. There is at most one wild card.
2. There are at most two one-eyed jacks.
3. There are at most three black jacks.

Definition: $\exists_{\leq n}\alpha\varphi := \neg\exists_{\geq n+1}\alpha\varphi$.

Cardinality Quantifiers

Task 3: Regiment the following.

1. There is one wild card.
2. There are two winning hands.
3. There are three hearts on the table.

Question 6: How can we define the cardinality quantifiers in general?

Base: $\exists_0\alpha\varphi := \forall\alpha\neg\varphi$.

Recursive: $\exists_{n+1}\alpha\varphi := \exists\alpha(\varphi \wedge \exists_n\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha]))$.

Question 7: How do the cardinality quantifiers relate to the inequality quantifiers?

Between: $\exists_{(n,m)}\alpha\varphi := \exists_{\geq n}\alpha\varphi \wedge \exists_{\leq m}\alpha\varphi$ where $n \leq m$.

Exact: $\exists_n\alpha\varphi := \exists_{(n,n)}\alpha\varphi$.

Examples

1. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa)\}$ is satisfiable.
2. Show that $\{\neg Raa, \forall x(x{=}a \vee Rxa), \forall x\exists y Rxy\}$ is satisfiable.
3. Show that $\forall x\forall y\ x{=}y \vdash \neg\exists x\ x \neq a$.

Relations

Task 4: Is the following argument valid?

- $\forall x\forall y(Rxy \supset Ryx)$.
- $\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$.

$\therefore$ $\forall x Rxx$.

Task 5: Is the following argument valid?

- $\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$.
- $\forall x\neg Rxx$.

$\therefore$ $\forall x\forall y(Rxy \supset \neg Ryx)$.