3. Mathematical Induction & Recursive Definitions

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a. Intro to Math. Induction

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- How long should you expect to spend in this class?

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A Simple Example

- Prove by induction that every well-formed formulae (wff) of SL has exactly as many left parentheses as it has right parentheses

- Don't forget to explicitly state the **base case**

- Don't forget to explicitly state the **Induction Step**!

Three More Examples!

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2. No wff contains two consecutive binary connectives (i.e. with no symbols between them)
3. If a wff doesn’t contain any binary connectives, then it is contingent. (hint: say that a wff is *baller* if it either contains a binary connective or is contingent. Use induction to show that every wff is baller.)

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  - Therefore, the next raven I see will be black
  - Notice how your conclusion could be wrong! Albino ravens!

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b. Recursive Definitions

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- Recursive definitions are one instance of applying this rule

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- But this is entirely uncreative. You can only say/understand things you have seen before using this technique.

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- What makes this possible is that our language has a recursive structure – we have basic components and then rules for making more complex, meaningful expressions out of them.

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- Definition by Recursion is the canonical way to define objects or structures that depend on iterated rules applied to a given basis
- Proof by Induction is a powerful way to prove things about structures that are defined in this way.

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3. **Closure clause**: Nothing else is a wff of SL

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- Any whole number greater than 0 can be reached by starting with 0 and iterating this operation.
- (N.B.: Sometimes $\mathbb{N}$ is defined excluding '0';
  but remember: we are inclusive!!!)

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- This definition generates the entire set $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$, starting with 0 and repeating the operation of "$+1$".

Strings! (of letters—not of physics!)

- It can be useful to treat the set of *all strings of letters* in a given alphabet recursively

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- Any string can be built up from a single letter by iterating this operation—in effect, simply spelling out 'x' left-to-right.

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- Example: '$aaba * cca$' is just the string 'aabacca'

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3. Mathematical Induction & Recursive Definitions

c. Mathematical Induction

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- We have inductive evidence that you'll be doing a lot of induction!
- Mathematical induction can be used in any domain where some objects can be singled out as basic, and where complex objects are built up out of simpler ones by iterating an operation.

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- ALWAYS REMEMBER TO EXPLICITLY NOTE BOTH THE **BASE CASE(s)** AND THE **INDUCTION STEP**!

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- So just write: $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$

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- When you try the formula in simple cases, it works. You might conjecture that it holds generally – but that isn't a *proof*.

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- Simplifies to $10 + 5 = 15$, given what you already know.

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- In the IH, we could have even used 'n' rather than 'k'!

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  [**Note:** We assume the formula is true for some number k, and we prove, *given this assumption*, that it must be true for $k+1$ as well.]

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$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} = \frac{1}{2}(k+1)(k+2)$
- And that's what we wanted to prove!

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- These two facts entail that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ when $n =$ **any** $k \in \mathbb{N}$, i.e. for all $n \in \mathbb{N}$

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  - Feel that? That's the power of induction.

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- But notice how—in contrast to kid Gauss—induction doesn't require any special insight or inspiration!

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- Just carry out the steps, noting each explicitly:

1) What is the **Base Case** we need to show?
2) What is our **Induction step**?
   i) Induction hypothesis?; ii) What do we need to show?
3) Just do it! (Note what you've proven)

An Example to Skip in Lecture!

- Another numerical example with the same flavor (i.e. *tasty*!)
- $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$
- The general strategy for attacking this problem inductively:

  1) (**Base Case**) prove $\sum_{i=1}^{1} i^2 = \frac{1}{6}(1)(1+1)(2(1)+1)$
- Just elementary arithmetic (typically base cases are *chill*)

Another example continued

2) (**Induction Step**): assume that for a given k,

$$\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$$

- Then set out to show that

$$\sum_{i=1}^{k+1} i^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1)$$

- The trick is to find a way to *use* the information you are given in the induction hypothesis
  - Break down the $n = k+1$ case so it consists of the $n = k$ case plus some comparatively simple other stuff.
  - This is usually the step that requires the most thought

Working through the Deets

- Happily, when we are dealing with sums, it is easy to simplify the $n = k + 1$ case by appealing to the $n = k$ case:

$$\sum_{i=1}^{k+1} i^2 = \underbrace{1 + 2 + 3 + \ldots k}_{\text{This is } \sum_{i=1}^{k} i^2} + (k+1)^2$$

$$= [\textstyle\sum_{i=1}^{k} i^2] + (k+1)^2$$

$$= \underbrace{\frac{1}{6}k(k+1)(2k+1)}_{\text{Substituting } \frac{1}{6}k(k+1)(2k+1) \text{ for } \sum_{i=1}^{k+1} i^2} + (k+1)^2.$$

Now we have an equation that doesn't have the sum operator in it at all.

Working through more Deets

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- We now have a *much* simpler problem in front of us:
- Consider whether we can algebraically reduce $\frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ to

  $\frac{1}{6}(k+1)(k+2)(2(k+1)+1)$.
- Spoiler alert!
- It is possible. (But I would not pay $$$ for this show)

3. Mathematical Induction & Recursive Definitions

d. Ordinary vs. Complete Induction

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- These argument patterns are equally rigorous, but they differ a bit in their logical structure, so it is worth pointing out the difference in form explicitly.
- In the cases we have just considered, we first prove some property is true of 0 (or 1, or whatever else is the first(s) in the series)
- Second, we prove that if we assume that property is true of an arbitrary n, we can prove it holds for the $(n+1)$–th case

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- Each method is equally rigorous: it just happens that one form is convenient for some problems, the other for others

Ordinary (a.k.a Weak) vs. Complete (a.k.a Strong) Induction

Ordinary mathematical induction:

Base Case: $C(x)$ holds of the stuff in base clause, e.g. `0`

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- In both cases, if you can prove the Base Case and the Induction Step, you can conclude that $C(x)$ is true for every x (yee haw!)

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- Language *SL* says "stay strong!"

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e. Recursion and Induction for Palindromes

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- This is useful for providing a simple example of complete induction

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  (We sometimes wonder: what if the reader said at this juncture, “No! Thou shall not turn now!” ?)

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Recursively Defining Palindromes

- This mind-blowing fact lets us recursively define "palindrome"
- We can use the structure we've discovered to incorporate useful info about the set of palindromes into a recursive definition.
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  -But if s is an R–palindrome by the recursion clause, then it must contain at least three letters.)

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- You WON’T need to prove both directions on your problem set!

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- So, by the recursion clause of the definition of R–palindrome, s is an R–palindrome.

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- (i) Give a recursive definition of “a – palindrome”, and
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- (In principle, you would also have to prove that the two definitions pick out the same set. That is, that a–palindromes are exactly the recursive a–palindromes. But I’m NOT asking you to do this as part of PS 3)

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- Then use induction to prove the claim about a–palindromes having an even number of "b"s

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3. Mathematical Induction & Recursive Definitions

f. Recasting Induction in SL as Complete Induction

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  Then $\mathcal{S}$ is an atomic sentence, so it has no parentheses. Thus there are the same number of right and left parentheses.

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- $\mathcal{S}$ is not a sentence letter, so it must be of the form $(\mathcal{S}_1 \& \mathcal{S}_2)$, with $\mathcal{S}_1$ and $\mathcal{S}_2$ sentences of SL (since we're focusing on the conjunctive fragment of SL)
- Note that $\mathcal{S}_1$ and $\mathcal{S}_2$ have fewer than k symbols
- So by the induction hypothesis, both $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same number of left and right parentheses
- (Note that we can't assign a specific length to $\mathcal{S}_1$ or $\mathcal{S}_2$. We just know that the length of each is less than k. That's why we use complete induction rather than ordinary induction in this case)

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- This completes the induction step, and so the proof.

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- The point is not that this is a hard problem but rather that it is so simple that it helps make clear how common-sensical the inductive pattern of reasoning is, when it isn't bound up with other complications

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- Sensing strong Descartes-energy here

3. Mathematical Induction & Recursive Definitions

g. More induction and recursion on strings

aab strings: a common sense version

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- So aab, $aabaab$, $aabaabaab$, $aabaabaabaab$, ... are aab–strings.

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- When you say this, you are implicitly using recursive/inductive reasoning

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- Each time you stick on a new aab you add one b and two a's. And you start out with more a's than b's in aab, the shortest aab–string.
- There's no way to get more b's than a's with that process, no matter how many times you repeat it.

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- Next, use the fact that the aab–strings are built up this way to argue **by complete induction on string length** that they must have more a's than b's

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- Try to finish the proof yourself before flipping to next slide! Hint: consider an arbitrary aab–string t with k letters.

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3. Mathematical Induction & Recursive Definitions

h. Extra ‘Big Picture’ stuff for Recursion

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A Guideline to Avoid Circularity

- This looks like a good guideline: set the word you want to define on a specific side of the definition (usually it's the left hand side, but our recursive definition has the word to be defined on the right). Then if the word also occurs on the *other* side, the definition is circular.

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- I'll continue with the definition of our SL wffs and then revisit the point.

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- But "atomic sentence " is a defined expression: we defined it by listing the capital letters of the English alphabet, allowing subscripts from the natural numbers
- You don't need to know what the word "sentence" means to understand what an atomic sentence is from this definition.

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- You don't have to know what "sentence" means to understand the stipulation: The *atomic sentence letters* are the capital Roman letters in the English alphabet, with or without number subscripts.

3. Mathematical Induction & Recursive Definitions

i. Towers of Hanoi Example

Towers of Hanoi

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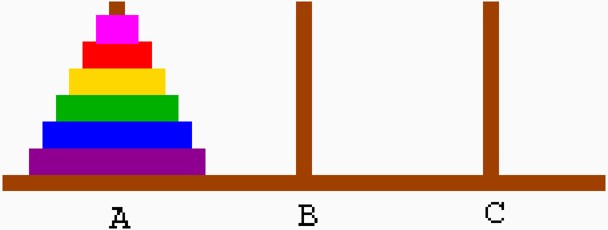
- You are given three pegs and n discs. No two discs are the same size.
- At the beginning of the game, all n discs are all on the leftmost peg, with the largest disc on the bottom, and then the remaining discs in decreasing order of size piled on top of the first, with the smallest on top.

Towers of Hanoi: Initial State

- Given three pegs and *n* discs. No two discs are the same size.



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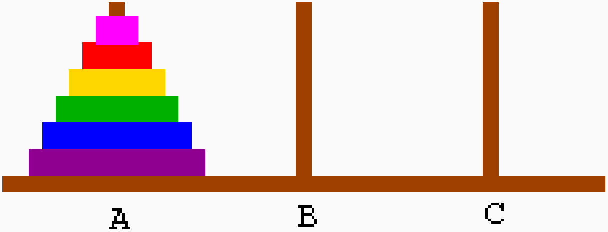

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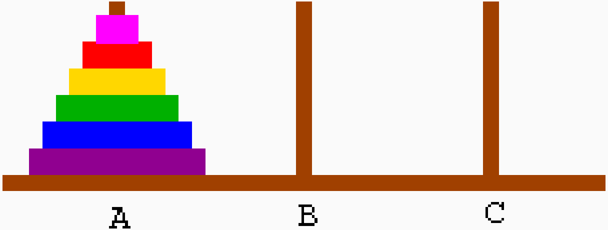
- Here is a drawing of the opening setup in the case of 6 discs



Towers of Hanoi: Goals and Rules

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Towers of Hanoi: Goals and Rules

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- There is an online version at Math is Fun!