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a. Big Picture Stuff

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- But our shortcuts are justified only if system STD is *sound*
- And guaranteed to have a shortcut only if system STD is *complete*

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- Throughout, 'Γ' is a *finite* set of SL sentences

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- Equivalently, this means that $\Gamma \cup \{\sim\Theta\}$ is logically inconsistent: no TVA satisfies the premises and negated conclusion

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  - (Means: we wrote down *enough* rules!)

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- Notice that both soundness and completeness are if–then statements (i.e. of the form $P \supset Q$):
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  - **Complete**: If $\Gamma \models \Theta$, then $\Gamma \vdash_{STD} \Theta$
- Hence, they are logically equivalent to their contrapositives:
  - Contrapositive of $(P \supset Q)$ is $(\sim Q \supset \sim P)$
  - **Soundness**: If $\Gamma \not\models \Theta$, then $\Gamma \nvdash_{STD} \Theta$
  - **Completeness**: If $\Gamma \nvdash_{STD} \Theta$, then $\Gamma \not\models \Theta$
- We will prove the contrapositives, using induction! Wooooo!

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b. Soundness of System STD

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- "$\Gamma \nvDash \Theta$" means that *it is not the case that* the set Γ entails Θ, i.e. there is a TVA where Γ is true but Θ is false.
  - So on this TVA, $\sim\Theta$ is true $\Rightarrow \Gamma \cup \{\sim\Theta\}$ is CONSISTENT

Analyzing Key Definitions Continued

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    - Equivalently: ANY tree with root $\Gamma \cup \{\sim\Theta\}$ possesses **at least one complete open branch**
    - (Aside: this is NOT the same as saying that the argument is **tree-invalid**, since that only requires the existence of a single tree with a complete open branch)

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- (i.e. whenever the root is satisfiable, the tree will not close)

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- Hence, $\mathcal{I}$ determines a truth value for any wff built from these atomic sentences
- for any such wff Ψ, $\mathcal{I}$ makes Ψ true if and only if it makes $\sim\Psi$ false

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- **Proof plan**: show that every **good** tree contains an **$\mathcal{I}$–branch**
- This will include all the completed good trees, which will have a *complete* open branch, which is what we want to show

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- Induction Step: show that an arbitrary good tree with k nodes also contains an $\mathcal{I}$–branch (where $k > 1$)

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- Since 'Theo' falls within the scope of our induction hypothesis, 'Theo' has at least one $\mathcal{I}$–branch, called 'Ida'
- Case a) Ruby extends a different open branch than Ida. Then Theodore still has $\mathcal{I}$–branch Ida, and so has an $\mathcal{I}$–branch
- Case b) Ruby extends Ida into branchlet(s). We need to show that for any rule, at least one of the branchlets is an $\mathcal{I}$–branch

The Simple way of Putting Case b)

- We have to show the following: if a good tree Theo contains an $\mathcal{I}$–branch, then so do all the trees that can be built from it by applying a single tree rule
(all the possible Theodore's, from all the possible Ruby's)

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- The book's induction step basically begins here at 'Case b'. We've just justified why it's okay to jump right to Case b.
- This mirrors our lazy induction schema for SL: where we start with two arbitrary wffs Φ and Ψ that have the property, and show that any application of the SL recursion clause preserves the property

Subcase i) Ruby is Double Negation ($\sim$)

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- Hence, Theodore contains at least one $\mathcal{I}$–branch

5. Metalogic for STD

c. Testing Alternative Rules: Soundness

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- If '**no**', then the rule BREAKS soundness:
  proceed to **construct a counter-example** using the rule.

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  3.) Show that the tree **closes** (i.e. NO complete open branches)

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- If our system *were* sound, then whenever a tree closes, it *would* correspond to a semantically valid argument.

Liberal Conditional and Conservative Biconditional

*STD**: replace rule (⊃) with:

Liberal Conditional (L⊃)

m.	$\Phi \supset \Psi$		
⋮	⋮		
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STD†: replace rule ($\equiv$) with:

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j+1.	Ψ	

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- Next, proceed to formally extend our soundness proof!

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- Either way, $\mathcal{I}$ satisfies the new sentences on **at least one** branch.
- Hence, Liberal Conditional does not break soundness

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- Hence, **to the counterexample!**
  (note that the problem REQUIRES this! can't stop won't stop!)

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- But $(P \equiv P) \nvDash P$ because $\{(P \equiv P), \sim P\}$ is consistent! Assign 'P' false! N.B.: your counterexample MUST use actual sentences of SL; not meta-variables!

5. Metalogic for STD

d. Completeness of System STD

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- We will prove the contrapositive, using induction:
- **Complete**: If $\Gamma \nvdash_{STD} \Theta$, then $\Gamma \not\models \Theta$
- "if an argument is not tree-valid, then it's not semantically valid"

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- This will prove: **Completeness**: If $\Gamma \nvdash_{STD} \Theta$, then $\Gamma \nvDash \Theta$

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- (So we see that we could do induction, starting 'from the bottom' of our tree)

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- We'll leave this to an optional HW question and proceed casually

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- Consider an arbitrary wff Δ on Oprah
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- Otherwise, there must exist wff α and β that compose Δ according to our recursive definition of SL sentences

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- Consider an arbitrary wff Δ on Oprah
- If Δ is an atomic sentence or negated atomic sentence, then $\mathcal{I}$ makes it true by definition
- Otherwise, there must exist wff α and β that compose Δ according to our recursive definition of SL sentences
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- Given this, we aim to show that $\mathcal{I}$ makes Δ true, **no matter which** child node lies on Oprah and is satisfied by $\mathcal{I}$

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- We need to show that *for* ***EACH*** *child node*, if $\mathcal{I}$ satisfies it, then $\mathcal{I}$ makes the resolved sentence Δ true
- Effectively, we are showing that each child node separately entails the resolved sentence
- There are nine cases to consider, coming from our nine tree rules

Subcase i) Δ is resolved by Double Negation ($\sim$)

Double Negation ($\sim$)

$$\begin{array}{lcl} \text{m.} & \sim\sim\Phi & \\ & \vdots & \\ \vdots & | & \\ \text{j.} & \Phi & \text{m} \sim \end{array}$$

- Only one child node $\Rightarrow \Phi$ lies on O

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- On to the next one!

Subcase iii) Δ is resolved by Negated Conjunction ($\sim\&$)

Negated Conjunction ($\sim\&$)

m.	$\sim(\Phi \,\&\, \Psi)$		
$\vdots$	$\vdots$		
j.	$\sim\Phi$	$\sim\Psi$	m $\sim\&$

- Exactly one child node lies on O, but we don't know which one (so we must show $\mathcal{I}$ satisfies Δ either way!)

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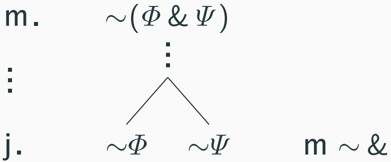

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$$
\begin{array}{llll}
\text{m.} & & \sim(\Phi \,\&\, \Psi) & \\
\vdots & & \vdots & \\
\text{j.} & \sim\Phi & \sim\Psi & \text{m} \sim\&
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- b) Likewise if $\sim\Psi$ lies on O...

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$$
\begin{array}{lccc}
\text{m.} & \multicolumn{2}{c}{\sim(\Phi \,\&\, \Psi)} & \\
\vdots & \multicolumn{2}{c}{\vdots} & \\
\text{j.} & \sim\Phi & \sim\Psi & \text{m} \sim\&
\end{array}
$$

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Subcase v) Δ is resolved by Negated Disjunction ($\sim\vee$)

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$$
\begin{array}{lll}
\text{m.} & \sim(\Phi \vee \Psi) & \\
\vdots & \vdots & \\
 & | & \\
\text{j.} & \sim\Phi & \text{m} \sim\vee \\
\text{j+1.} & \sim\Psi &
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- Upshot: if an argument is tree-invalid (i.e. has at least one complete open tree) in a 'partially complete' system , then there is a truth-value assignment that satisfies the root (so $\Gamma \nvDash \Theta$)
- If the system is also SOUND, one could then conclude that the argument is *not* tree-valid.

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- Note that the system STD^{conj} which has only the rules ($\sim$), (&), and ($\sim$ &) is partially complete in this sense:
- for the sentences that *can be* completely resolved with these limited rules, we can turn any complete open branch into a TVA that satisfies all wff on the branch.

5. Metalogic for STD

e. Testing Alternative Rules: Completeness

Checking Completeness: “bottom-up” reasoning

- **Heuristic for Completeness checking**: ask whether **EACH** child node (below) individually entails the sentence being resolved (i.e. the sentence ‘up top’)

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- If '**no**', then the rule BREAKS completeness: proceed to **construct a counter-example** using the rule.

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  3.) Show that the tree has a **complete open branch**, i.e. does NOT close

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  - '$\Gamma \nvdash_{STD^*} \Theta$' means that it is NOT the case that the argument is tree–valid in STD^* (a claim about ALL trees)
  - From completeness proof, we know that if the system *were* complete, then we could use any **complete open branch** to construct a TVA that **satisfies** the root
  - Hence, if we construct a **complete open tree** with an **unsatisfiable root**, this is a reductio of completeness

Liberal Conditional and Conservative Biconditional

STD*: replace rule ($\supset$) with:

Liberal Conditional (L$\supset$)

m.	$\Phi \supset \Psi$		
$\vdots$	$\vdots$		
j.	$\sim\Phi$	$\Psi \vee \Phi$	m L$\supset$

STD†: replace rule ($\equiv$) with:

Conservative Biconditional (C$\equiv$)

m.	$\Phi \equiv \Psi$	
$\vdots$	$\vdots$	
j.	Φ	m C$\equiv$
j+1.	Ψ	

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- $(\Psi \vee \Phi)$ does NOT entail $\Phi \supset \Psi$, since Φ doesn't entail it
- (Remember: $\Phi \supset \Psi$ is equivalent to $\sim\Phi \vee \Psi$)
- Hence, we **proceed to counterexample!**

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1.	$P \supset Q$ ✓		
2.	P		
3.	$\sim Q$		$\sim$Conclusion
4.	$\sim P$	$Q \vee P$ ✓	1 L$\supset$
	$\times$ 2, 4		
5.		Q P	4 $\vee$
		$\times$ 3, 5 $\uparrow$	

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		$\times$ 3, 5	$\uparrow$	

- Note that $\{P \supset Q,\ P\} \vDash Q$ but **tree-invalid** in STD^*
- If STD^* *were* complete, the complete open branch would lead to a TVA that satisfies the root. Contradiction $\Rightarrow$ not complete

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- So make sure you FULLY RESOLVE every sentence on the open branch, until you can write that coveted up arrow '$\uparrow$'!
- When in doubt, just complete the whole tree

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  - **Formally**: since only one child node, both Φ and Ψ lie on the complete open branch O.

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  - Hence, $\Phi \equiv \Psi$ is true on $\mathcal{I}$, which is what we needed to show.

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