14. Compactness of SL & QL

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):

- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):

- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):

- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)
  - (syntactic to semantic: i.e. we chose 'good' rules!)

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):
- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- **Completeness**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):
- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- **Completeness**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$
  - reasoning about arbitrary models is not needed to demonstrate validity: QND derivations suffice

Soundness and Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- For our two natural deduction systems SND and QND, we have proven the following (where QND extends SND):
- **Soundness**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - QND derivations are 'safe' (they preserve truth)
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- **Completeness**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$
  - reasoning about arbitrary models is not needed to demonstrate validity: QND derivations suffice
  - (logical entailment is fully covered by our syntactic rules)

14. Compactness of SL & QL

a. Compactness of SL

Compactness of SL

- **Compactness of SL**: for any set Γ of SL-sentences (possibly infinite), Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. for each Δ, there is a truth-value assignment that makes all sentences in Δ true).

Compactness of SL

- **Compactness of SL**: for any set Γ of SL-sentences (possibly infinite), Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. for each Δ, there is a truth-value assignment that makes all sentences in Δ true).
- Relying on our valiant labors in proving the soundness and completeness of SND, we gain an elementary proof of compactness

Compactness of SL

- **Compactness of SL**: for any set Γ of SL-sentences (possibly infinite), Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. for each Δ, there is a truth-value assignment that makes all sentences in Δ true).
- Relying on our valiant labors in proving the soundness and completeness of SND, we gain an elementary proof of compactness
- This proof is "impure" because it relies on syntactic notions, whereas the statement of compactness is purely semantic.

An “impure” proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

An "impure" proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

An “impure” proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

- This TVA satisfies every finite subset $\Delta \subseteq \Gamma$.

An “impure” proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

- This TVA satisfies every finite subset $\Delta \subseteq \Gamma$.

$\Leftarrow$ (nontrivial direction): Assume that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

An “impure” proof of Compactness

- **Compactness of SL**: for any set Γ of SL–sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

- This TVA satisfies every finite subset $\Delta \subseteq \Gamma$.

$\Leftarrow$ (nontrivial direction): Assume that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

- Assume for *reductio* that Γ is unsatisfiable.

Then there is no TVA that makes true every sentence in Γ.

An "impure" proof of Compactness

- **Compactness of SL**: for any set Γ of SL-sentences, Γ is satisfiable **if and only if** every finite subset $\Delta \subseteq \Gamma$ is satisfiable

$\Rightarrow$ (trivial direction): Assume that Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ.

- This TVA satisfies every finite subset $\Delta \subseteq \Gamma$.

$\Leftarrow$ (nontrivial direction): Assume that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

- Assume for *reductio* that Γ is unsatisfiable.
Then there is no TVA that makes true every sentence in Γ.

- Hence, for any contradiction $\mathcal{C}$ (e.g. $P \,\&\, {\sim}P$), we have $\Gamma \vDash \mathcal{C}$

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.
- Since derivations are finite, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{SND} \mathcal{C}$

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \models \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.
- Since derivations are finite, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{SND} \mathcal{C}$
- Then, by soundness of SND, $\Delta \models \mathcal{C}$. Since $\mathcal{C}$ is unsatisfiable, this means that Δ must be unsatisfiable as well.

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.
- Since derivations are finite, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{SND} \mathcal{C}$
- Then, by soundness of SND, $\Delta \vDash \mathcal{C}$. Since $\mathcal{C}$ is unsatisfiable, this means that Δ must be unsatisfiable as well.
- But that contradicts our starting assumption that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

Impure proof: non-trivial direction continued

- (From above: Γ unsatisfiable $\Rightarrow \Gamma \vDash \mathcal{C}$, for contradiction $\mathcal{C}$)
- Hence, by completeness of SND, we can derive $\mathcal{C}$ from Γ: $\Gamma \vdash_{SND} \mathcal{C}$.
- Since derivations are finite, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{SND} \mathcal{C}$
- Then, by soundness of SND, $\Delta \vDash \mathcal{C}$. Since $\mathcal{C}$ is unsatisfiable, this means that Δ must be unsatisfiable as well.
- But that contradicts our starting assumption that every finite subset $\Delta \subseteq \Gamma$ is satisfiable.
- So Γ must be satisfiable (proving compactness)

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{{\sim}\mathcal{P}\}$?

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

$\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

- So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

$\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

- So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
- Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely–many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

  - So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
  - Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$
- *Upshot*: every valid argument relies on finitely–many premises

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

  - So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
  - Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$
- *Upshot*: every valid argument relies on finitely-many premises
- Contrast proof here with PS12 #4, which shows same result using completeness *and soundness*, relying on syntactic $\vdash_{SND}$

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

  - So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
  - Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$
- *Upshot*: every valid argument relies on finitely-many premises
- Contrast proof here with PS12 #4, which shows same result using completeness *and soundness*, relying on syntactic $\vdash_{SND}$
- Whereas our argument above proceeds entirely semantically, using compactness and semantic entailment $\vDash$

What does compactness of SL tell us?

- *Question*: Are there any arguments of SL that have infinitely-many premises, where no premise is redundant?
- Assume that $\Gamma \vDash \mathcal{P}$. Then what can we say about $\Gamma \cup \{\sim\mathcal{P}\}$?

  $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**!

  - So by one Contrapositive of Compactness, there exists a finite subset $\Delta \subset \Gamma \cup \{\sim\mathcal{P}\}$ that is unsatisfiable.
  - Easy to show that there is a finite $\Gamma_f \subset \Gamma$ s.t. $\Gamma_f \cup \{\sim\mathcal{P}\}$ is unsatisfiable as well. So $\Gamma_f \vDash \mathcal{P}$
- *Upshot*: every valid argument relies on finitely-many premises
- Contrast proof here with PS12 #4, which shows same result using completeness *and soundness*, relying on syntactic $\vdash_{SND}$
- Whereas our argument above proceeds entirely semantically, using compactness and semantic entailment $\vDash$
- If only we could prove compactness purely semantically?!

14. Compactness of SL & QL

b. A ‘Pure’ proof of SL compactness

A ‘Pure’ proof of the Compactness of SL

- Using a very similar idea to our construction of the maximally-SND-consistent set Γ^*, we can provide a purely semantic and yet still elementary proof of SL compactness

A 'Pure' proof of the Compactness of SL

- Using a very similar idea to our construction of the maximally-SND-consistent set Γ^*, we can provide a purely semantic and yet still elementary proof of SL compactness
- Proof sketch: assuming that every finite subset of Γ is satisfiable, we will construct a superset $\Gamma^* \supset \Gamma$ for which it is easy to define a truth-value assignment that satisfies every sentence in Γ^*, and hence in Γ.

A ‘Pure’ proof of the Compactness of SL

- Using a very similar idea to our construction of the maximally–SND–consistent set Γ^*, we can provide a purely semantic and yet still elementary proof of SL compactness
- Proof sketch: assuming that every finite subset of Γ is satisfiable, we will construct a superset $\Gamma^* \supset \Gamma$ for which it is easy to define a truth–value assignment that satisfies every sentence in Γ^*, and hence in Γ.
- As with our earlier completeness proof, Γ^* comes along with a membership lemma, which we use for our induction over SL.

Beginning the Proof

- $\Rightarrow$ (easy direction): assume that the (possibly infinite) set of SL-wffs Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ, and this TVA satisfies every finite subset of Γ.

Beginning the Proof

- $\Rightarrow$ (easy direction): assume that the (possibly infinite) set of SL–wffs Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ, and this TVA satisfies every finite subset of Γ.
- $\Leftarrow$ (harder direction): Assume that every finite subset $\Delta \subset \Gamma$ is satisfiable. Show that Γ is satisfiable (nontrivial if Γ is infinite).

Beginning the Proof

- $\Rightarrow$ (easy direction): assume that the (possibly infinite) set of SL–wffs Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ, and this TVA satisfies every finite subset of Γ.
- $\Leftarrow$ (harder direction): Assume that every finite subset $\Delta \subset \Gamma$ is satisfiable. Show that Γ is satisfiable (nontrivial if Γ is infinite).
- Notice that it suffices to construct a superset Γ^* of Γ that is satisfiable. Then the TVA that makes true everything in Γ^* will make true everything in Γ.

Beginning the Proof

- $\Rightarrow$ (easy direction): assume that the (possibly infinite) set of SL-wffs Γ is satisfiable. Then there is a TVA that makes true every sentence in Γ, and this TVA satisfies every finite subset of Γ.
- $\Leftarrow$ (harder direction): Assume that every finite subset $\Delta \subset \Gamma$ is satisfiable. Show that Γ is satisfiable (nontrivial if Γ is infinite).
- ▶ Notice that it suffices to construct a superset Γ^* of Γ that is satisfiable. Then the TVA that makes true everything in Γ^* will make true everything in Γ.
- ▶ To proceed, we introduce an idea very similar to the notion of a maximally-consistent-in-SND set. But now using only *semantic* notions (so avoiding our proof system).

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
      Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite–satisfiability

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
  Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite–satisfiability
  i.e. $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
     Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite–satisfiability
     i.e. $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset

- Next we'll show that any MFS set is satisfiable (this mirrors our "maximal consistency lemma" from our completeness proof)

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
  Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite-satisfiability
  i.e. $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- Next we'll show that any MFS set is satisfiable (this mirrors our "maximal consistency lemma" from our completeness proof)
- To do this, we'll prove a membership lemma that facilitates an induction over SL!

Maximally finitely satisfiable sets

- A set Γ^* of SL wffs is **maximally finitely satisfiable** (MFS) provided that:
  1.) Every finite subset of Γ^* is satisfiable (Γ^* is "**finitely satisfiable**")
  2.) For each SL wff $\mathcal{P}$, if $\Gamma^* \cup \{\mathcal{P}\}$ is FS, then $\mathcal{P} \in \Gamma^*$ ("*semantic Door*")
     Otherwise, adding any additional $\mathcal{P}$ to Γ^* breaks finite-satisfiability
     i.e. $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- Next we'll show that any MFS set is satisfiable (this mirrors our "maximal consistency lemma" from our completeness proof)
- To do this, we'll prove a membership lemma that facilitates an induction over SL!
- Finally, we'll show how to construct an MFS Γ^* from any finitely-satisfiable Γ
  (i.e. what we assume at the start of the nontrivial-direction)

Membership Lemma for MFS sets (“complete clubs”)

- To induct on SL, we first show some constraints on Γ^* membership

Membership Lemma for MFS sets (“complete clubs”)

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:

Membership Lemma for MFS sets (“complete clubs”)

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:

  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$

Membership Lemma for MFS sets (“complete clubs”)

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  - a.) ${\sim}\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  - b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  - c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$

Membership Lemma for MFS sets (“complete clubs”)

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- These syntactic constraints mirror truth-conditions, but we will now NOT rely on our proof system to prove this lemma

Membership Lemma for MFS sets ("complete clubs")

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* has a a bouncer who enforces maximal finite satisfiability.
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- These syntactic constraints mirror truth–conditions, but we will now NOT rely on our proof system to prove this lemma
- (We built an analog of "the Door" into the definition of MFS sets)

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.
  (So we do lots of *reductio* proofs: assume that a membership case fails, apply Case (a), and then show this would result in an unsatisfiable finite subset—contradicting condition (1), i.e. that all finite subsets are satisfiable).

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.
  (So we do lots of *reductio* proofs: assume that a membership case fails, apply Case (a), and then show this would result in an unsatisfiable finite subset—contradicting condition (1), i.e. that all finite subsets are satisfiable).
- So imagine we've proven the membership lemma!

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.
  (So we do lots of *reductio* proofs: assume that a membership case fails, apply Case (a), and then show this would result in an unsatisfiable finite subset—contradicting condition (1), i.e. that all finite subsets are satisfiable).
- So imagine we've proven the membership lemma!
- Then define a TVA $\mathcal{I}^*$ that makes true every atomic sentence in Γ^*;

Proof of Membership Lemma for MFS Sets

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ iff $\mathcal{P} \notin \Gamma^*$: use condition 2) ("semantic Door") of MFS sets: $\mathcal{P} \notin \Gamma^*$ iff $\Gamma^* \cup \{\mathcal{P}\}$ has an unsatisfiable finite subset
- For the other cases, we rely on Case (a), the truth tables for the connectives, and the fact that Γ^* is finitely-satisfiable, i.e. every finite subset is satisfiable.
  (So we do lots of *reductio* proofs: assume that a membership case fails, apply Case (a), and then show this would result in an unsatisfiable finite subset—contradicting condition (1), i.e. that all finite subsets are satisfiable).
- So imagine we've proven the membership lemma!
- Then define a TVA $\mathcal{I}^*$ that makes true every atomic sentence in Γ^*;
  - show by induction that this TVA satisfies every sentence in Γ^*
  (just as in our proof of completeness of SND!)

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \dots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \dots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays FS)

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays FS)
- Then, proceed to the second wff in our enumeration.
  If it can be added to Γ_1 without the new set breaking FS, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays FS)
- Then, proceed to the second wff in our enumeration.
  If it can be added to Γ_1 without the new set breaking FS, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$
- Γ^* is the result of 'doing' this procedure for every SL wff

Building an MFS Γ^* from a finitely-satisfiable Γ

- It remains to construct a maximally finitely-satisfiable superset Γ^* of a finitely-satisfiable Γ
- We first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- Consider the first wff 'A' in our enumeration.
  If A can be added to Γ while preserving finite satisfiability, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays FS)
- Then, proceed to the second wff in our enumeration.
  If it can be added to Γ_1 without the new set breaking FS, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$
- Γ^* is the result of 'doing' this procedure for every SL wff
- More precisely, $\Gamma^* := \bigcup_{k=1}^{\infty} \Gamma_k$

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

1.) Clearly, Γ^* is finitely satisfiable. If it were not, then some $\Gamma_k \subset \Gamma^*$ would be finitely unsatisfiable, but that contradicts our construction conditions.

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

1.) Clearly, Γ^* is finitely satisfiable. If it were not, then some $\Gamma_k \subset \Gamma^*$ would be finitely unsatisfiable, but that contradicts our construction conditions.

2.) Moreover, Γ^* is maximal: if there were a wff $\mathcal{Q}$ that could be added to Γ^* while preserving finite satisfiability, we would have added $\mathcal{Q}$ at its enumeration stage.

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

1.) Clearly, Γ^* is finitely satisfiable. If it were not, then some $\Gamma_k \subset \Gamma^*$ would be finitely unsatisfiable, but that contradicts our construction conditions.

2.) Moreover, Γ^* is maximal: if there were a wff $\mathcal{Q}$ that could be added to Γ^* while preserving finite satisfiability, we would have added $\mathcal{Q}$ at its enumeration stage.

   - So if $\mathcal{Q} \notin \Gamma^*$, it must be that $\Gamma^* \cup \{\mathcal{Q}\}$ is *not* finitely satisfiable.

Claim: Γ^* is maximally finitely satisfiable (MFS)

- At this point, it suffices to prove that Γ^* is MFS

1.) Clearly, Γ^* is finitely satisfiable. If it were not, then some $\Gamma_k \subset \Gamma^*$ would be finitely unsatisfiable, but that contradicts our construction conditions.

2.) Moreover, Γ^* is maximal: if there were a wff $\mathcal{Q}$ that could be added to Γ^* while preserving finite satisfiability, we would have added $\mathcal{Q}$ at its enumeration stage.

- So if $\mathcal{Q} \notin \Gamma^*$, it must be that $\Gamma^* \cup \{\mathcal{Q}\}$ is *not* finitely satisfiable.

- So we're done! Any finitely satisfiable Γ is a subset of an MFS Γ^*, which we've shown is satisfiable! So Γ is satisfiable!

14. Compactness of SL & QL

c. Compactness of First-order Languages

Compactness of QL

- **Compactness of QL**: for any set Γ of QL-sentences, Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. $(\forall\Delta)$ $\exists$ a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).

Compactness of QL

- **Compactness of QL**: for any set Γ of QL-sentences, Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. $(\forall\Delta)\ \exists$ a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).
- *Mutatis mutandis*, we can provide an analogous impure proof, relying on the soundness and completeness of system QND

Compactness of QL

- **Compactness of QL**: for any set Γ of QL-sentences, Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. $(\forall\Delta)\ \exists$ a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).
- *Mutatis mutandis*, we can provide an analogous impure proof, relying on the soundness and completeness of system QND

  And also a 'pure' proof, constructing a maximally finitely satisfiable and *existentially complete* superset Γ^*.

Compactness of QL

- **Compactness of QL**: for any set Γ of QL-sentences, Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. $(\forall\Delta)\ \exists$ a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).
- *Mutatis mutandis*, we can provide an analogous impure proof, relying on the soundness and completeness of system QND

  And also a 'pure' proof, constructing a maximally finitely satisfiable and *existentially complete* superset Γ^*.
- To widen the interest of our results, let's generalize compactness to any first-order language $\mathcal{L}$

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses
  - Names: a set of constants (allowing subscripts $n \in \mathbb{N}$)

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses
  - Names: a set of constants (allowing subscripts $n \in \mathbb{N}$)
  - Predicates: a non-empty set of capital letters (allowing subscripts), each with "an invisible label" giving its arity (e.g. 0-place, 1-place, 2-place, etc.)

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses
  - Names: a set of constants (allowing subscripts $n \in \mathbb{N}$)
  - Predicates: a non-empty set of capital letters (allowing subscripts), each with "an invisible label" giving its arity (e.g. 0-place, 1-place, 2-place, etc.)
  - a set of function symbols $f(c)$ (syntax: f maps terms to terms)

First-order Languages (FOLs)

- **First-order language $\mathcal{L}$**: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include:
  - Variables: w, x, y, z (allowing subscripts $n \in \mathbb{N}$)
  - Operators: our five sentential connectives and two quantifiers
  - Punctuation: left and right parentheses
  - Names: a set of constants (allowing subscripts $n \in \mathbb{N}$)
  - Predicates: a non-empty set of capital letters (allowing subscripts), each with "an invisible label" giving its arity (e.g. 0-place, 1-place, 2-place, etc.)
  - a set of function symbols $f(c)$ (syntax: f maps terms to terms)
- Different FOLs differ in their names, predicates, and functions

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathcal{L}$ to objects and ordered pairs from D as follows:

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathcal{L}$ to objects and ordered pairs from D as follows:
     - For each constant $c \in \mathfrak{L}$, $I(c)$ is an element of D, called the *referent* or denotation of c

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathcal{L}$ to objects and ordered pairs from D as follows:
     - For each constant $c \in \mathfrak{L}$, $I(c)$ is an element of D, called the *referent* or denotation of c
     - For each k-place predicate P of $\mathfrak{L}$, $I(P)$ is a set of ordered k-tuples of objects in D, called the *extension* of P

$\mathcal{L}$-models and interpretations

- Let $\mathcal{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathcal{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathcal{L}$ to objects and ordered pairs from D as follows:
     - For each constant $c \in \mathfrak{L}$, $I(c)$ is an element of D, called the *referent* or denotation of c
     - For each k-place predicate P of $\mathfrak{L}$, $I(P)$ is a set of ordered k-tuples of objects in D, called the *extension* of P
     - I maps SL atomics to “true” or “false” (i.e. ‘1’ or ‘0’)

FOL with identity and functions

- With some minor modifications, we could extend our soundness and completeness proofs for QND to FOLs and deduction systems that include (1) a privileged identity predicate "$=$" and (2) functions that syntactically map terms to terms (interpreted as mapping the domain D to itself)

FOL with identity and functions

- With some minor modifications, we could extend our soundness and completeness proofs for QND to FOLs and deduction systems that include (1) a privileged identity predicate "$=$" and (2) functions that syntactically map terms to terms (interpreted as mapping the domain D to itself)
- Like our symbol "$\prime$", we add in some new symbol "α" that doesn't occur in our FOL, to give a countable infinity of unused constants

FOL with identity and functions

- With some minor modifications, we could extend our soundness and completeness proofs for QND to FOLs and deduction systems that include (1) a privileged identity predicate "$=$" and (2) functions that syntactically map terms to terms (interpreted as mapping the domain D to itself)
- Like our symbol "$\prime$", we add in some new symbol "α" that doesn't occur in our FOL, to give a countable infinity of unused constants
- To construct our maximally-syntactically-consistent, existentially complete superset Γ^*, we focus on equivalence classes of co-referential constants, since now some constants might name the same object in the domain (e.g. $c = d$)

FOL with identity and functions

- With some minor modifications, we could extend our soundness and completeness proofs for QND to FOLs and deduction systems that include (1) a privileged identity predicate “$=$” and (2) functions that syntactically map terms to terms (interpreted as mapping the domain D to itself)
- Like our symbol “$'$”, we add in some new symbol “α” that doesn’t occur in our FOL, to give a countable infinity of unused constants
- To construct our maximally-syntactically-consistent, existentially complete superset Γ^*, we focus on equivalence classes of co-referential constants, since now some constants might name the same object in the domain (e.g. $c = d$)
- Using the axiom of choice, we could even handle FOLs that have *uncountably many* predicates or constants!

Compactness for a first-order language

- **Compactness of a FOL $\mathcal{L}$**: for any set Γ of $\mathcal{L}$-sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

Compactness for a first-order language

- **Compactness of a FOL $\mathcal{L}$**: for any set Γ of $\mathcal{L}$–sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

- We could prove this either by (i) using a soundness and completeness result for an $\mathcal{L}$–deduction system;
  (ii) generalizing our 'pure' proof for SL; or
  (iii) generalizing the topological proof of SL compactness (relying on results from topology, e.g. Tychonoff's theorem)

14. Compactness of SL & QL

d. The Löwenheim–Skolem theorems

Downwards!

- Terminology: we'll say that a model $\mathfrak{M}$ is *infinite* if its domain D is infinite in size. Likewise for saying that a model is finite, or countably infinite.

Downwards!

- Terminology: we'll say that a model $\mathfrak{M}$ is *infinite* if its domain D is infinite in size. Likewise for saying that a model is finite, or countably infinite.
- **Downward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model, then it is satisfiable in a countably infinite model.

Downwards!

- Terminology: we'll say that a model $\mathfrak{M}$ is *infinite* if its domain D is infinite in size. Likewise for saying that a model is finite, or countably infinite.
- **Downward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model, then it is satisfiable in a countably infinite model.
- *Gloss*: we can always descend from an infinite model to a countably infinite model

Downwards!

- Terminology: we'll say that a model $\mathfrak{M}$ is *infinite* if its domain D is infinite in size. Likewise for saying that a model is finite, or countably infinite.
- **Downward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model, then it is satisfiable in a countably infinite model.
- *Gloss*: we can always descend from an infinite model to a countably infinite model
- Proof(s): (1) be impure and piggyback on completeness proof or (2) use compactness and satisfiability lemma for MFS sets

- **Converse consistency lemma**: if $\mathcal{L}$-set Γ is satisfiable, then Γ is syntactically-consistent (for a given deduction system $\mathcal{L}$ND that we've shown is sound)

Down to be Impure

- **Converse consistency lemma**: if $\mathcal{L}$–set Γ is satisfiable, then Γ is syntactically–consistent (for a given deduction system $\mathcal{L}$ND that we've shown is sound)
- Proof: good exercise!!! Assume for *reductio* that Γ is syntactically–inconsistent and then apply soundness

Down to be Impure

- **Converse consistency lemma**: if $\mathcal{L}$-set Γ is satisfiable, then Γ is syntactically-consistent (for a given deduction system $\mathcal{L}$ND that we've shown is sound)
- Proof: good exercise!!! Assume for *reductio* that Γ is syntactically-inconsistent and then apply soundness
- So since Γ is satisfiable, it is syntactically consistent.

Down to be Impure

- **Converse consistency lemma**: if $\mathcal{L}$-set Γ is satisfiable, then Γ is syntactically-consistent (for a given deduction system $\mathcal{L}ND$ that we've shown is sound)
- Proof: good exercise!!! Assume for *reductio* that Γ is syntactically-inconsistent and then apply soundness
- So since Γ is satisfiable, it is syntactically consistent.

  Then, appeal to our consistency lemma shown in the course of proving completeness: for any syntactically-consistent set, there is a maximally-consistent (and $\exists$-complete) set that is satisfiable, where we showed this by constructing a countably infinite model

Down to be Impure

- **Converse consistency lemma**: if $\mathcal{L}$-set Γ is satisfiable, then Γ is syntactically-consistent (for a given deduction system $\mathcal{L}$ND that we've shown is sound)
- **Proof: good exercise**!!! Assume for *reductio* that Γ is syntactically-inconsistent and then apply soundness
- So since Γ is satisfiable, it is syntactically consistent.

  Then, appeal to our consistency lemma shown in the course of proving completeness: for any syntactically-consistent set, there is a maximally-consistent (and $\exists$-complete) set that is satisfiable, where we showed this by constructing a countably infinite model
- So Γ has a countably infinite model

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely-satisfiable

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely-satisfiable
- Modify our construction to form a maximally finitely-satisfiable and $\exists$-complete superset Γ^* of Γ

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely-satisfiable
- Modify our construction to form a maximally finitely-satisfiable and $\exists$-complete superset Γ^* of Γ
- Prove a satisfiability lemma: any such Γ^* is satisfiable, where we show this by constructing a countably infinite $\mathcal{L}^+$-model

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely–satisfiable
- Modify our construction to form a maximally finitely–satisfiable and $\exists$–complete superset Γ^* of Γ
- Prove a satisfiability lemma: any such Γ^* is satisfiable, where we show this by constructing a countably infinite $\mathcal{L}^+$–model

  ($\mathcal{L}^+$ arises from $\mathcal{L}$ by adding a countable–infinity of new constants)

Down with impurity: apply compactness

- *Pure proof of Downward LS*: assume that Γ is satisfied in an infinite model. Then it is satisfiable, and so by compactness theorem for FOL, Γ is finitely-satisfiable
- Modify our construction to form a maximally finitely-satisfiable and $\exists$-complete superset Γ^* of Γ
- Prove a satisfiability lemma: any such Γ^* is satisfiable, where we show this by constructing a countably infinite $\mathcal{L}^+$-model

  ($\mathcal{L}^+$ arises from $\mathcal{L}$ by adding a countable-infinity of new constants)
- Then, Γ has a countably infinite $\mathcal{L}$-model

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$
- *Proof Sketch*: extend the set of constants $\mathcal{C}$ of $\mathcal{L}$ with an uncountable set $\mathcal{E}$ that contains $\mathcal{C}$.

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$
- *Proof Sketch*: extend the set of constants $\mathcal{C}$ of $\mathcal{L}$ with an uncountable set $\mathcal{E}$ that contains $\mathcal{C}$.
  Extend the FOL $\mathcal{L}$ to $\mathcal{L}^+$ with $\mathcal{E}$ as its set of constants and with identity predicate $=$.

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$
- *Proof Sketch*: extend the set of constants $\mathcal{C}$ of $\mathcal{L}$ with an uncountable set $\mathcal{E}$ that contains $\mathcal{C}$.
  Extend the FOL $\mathcal{L}$ to $\mathcal{L}^+$ with $\mathcal{E}$ as its set of constants and with identity predicate $=$.
- Construct an $\mathcal{L}^+$–set Γ^+ by adjoining to Γ every sentence of the form $\sim c = d$ for every distinct $c, d \in \mathcal{E}$.

Onwards and Upwards!

- **Upward Löwenheim–Skolem**: let Γ be a set of $\mathcal{L}$–sentences. If Γ is satisfiable in an infinite model $\mathfrak{M} := (D, I)$, then it is satisfiable in models of arbitrary size larger than $|D|$
- *Proof Sketch*: extend the set of constants $\mathcal{C}$ of $\mathcal{L}$ with an uncountable set $\mathcal{E}$ that contains $\mathcal{C}$.
  Extend the FOL $\mathcal{L}$ to $\mathcal{L}^+$ with $\mathcal{E}$ as its set of constants and with identity predicate $=$.
- Construct an $\mathcal{L}^+$–set Γ^+ by adjoining to Γ every sentence of the form $\sim c = d$ for every distinct $c, d \in \mathcal{E}$.
- Show that Γ^+ is finitely–satisfiable and hence by compactness satisfiable. Then note that any $\mathcal{L}^+$–model satisfying Γ^+ must have a domain as large $\mathcal{E}$. Restrict the interpretation function to construct an $\mathcal{L}$–model for Γ with domain $|D| = |\mathcal{E}|$

14. Compactness of SL & QL

e. Skolem's 'Paradox'

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC):
  a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC): a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs
- In standard models, we interpret the objects as sets

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC): a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs
- In standard models, we interpret the objects as sets
- A list of axioms or axiom schemas, e.g. *Null set axiom*: $(\exists x)(\forall y)\, y \notin x$ (i.e. there is an empty set $\varnothing$)

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC): a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs
- In standard models, we interpret the objects as sets
- A list of axioms or axiom schemas, e.g.
  Null set axiom: $(\exists x)(\forall y) y \notin x$ (i.e. there is an empty set $\varnothing$)
  Axiom of Extensionality: $(\forall x)(\forall y)(\forall z)((z \in x \equiv z \in y) \supset x = y)$
  (i.e. two sets are identical iff they have the same members)

ZFC as a first-order language

- **Zermelo–Fraenkel set theory with choice** (ZFC):
  a FOL $\mathcal{ZFC}$ that has identity and a 2-place predicate for set-membership '$\in$', written between (rather than before) terms when forming atomic wffs
- In standard models, we interpret the objects as sets
- A list of axioms or axiom schemas, e.g.
  Null set axiom: $(\exists x)(\forall y)y \notin x$ (i.e. there is an empty set $\varnothing$)
  Axiom of Extensionality: $(\forall x)(\forall y)(\forall z)((z \in x \equiv z \in y) \supset x = y)$
  (i.e. two sets are identical iff they have the same members)
- **Axiom of Choice**: if x is a set whose members are non-empty sets and no two members of x share a member, then there is a set y that contains exactly one element of each set in x

Skolem's 'Paradox'

- If ZFC has any models, then it has a **countable model** (since by downward LS, an infinite model entails a countably infinite model. Any finite model is already countable—and can be extended to a countably infinite model as well)

Skolem's 'Paradox'

- If ZFC has any models, then it has a **countable model** (since by downward LS, an infinite model entails a countably infinite model. Any finite model is already countable—and can be extended to a countably infinite model as well)
- Yet, we can prove within ZFC that there are uncountable sets, e.g. the power set of $\mathbb{N}$ has cardinality of $\mathbb{R}$

Skolem's 'Paradox'

- If ZFC has any models, then it has a **countable model** (since by downward LS, an infinite model entails a countably infinite model. Any finite model is already countable—and can be extended to a countably infinite model as well)
- Yet, we can prove within ZFC that there are uncountable sets, e.g. the power set of $\mathbb{N}$ has cardinality of $\mathbb{R}$
- 'Paradox': how can a countably-infinite model make true the claim that there are uncountable sets?

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'
- U: there is an injection but no bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$; $\mathfrak{M} \vDash U$

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'
- U: there is an injection but no bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$; $\mathfrak{M} \vDash U$
- Since $\mathfrak{M}$ is countable, the sets $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ are definitely countable ($\mathfrak{M}$ has only countably many objects in its domain to serve as members of objects in that domain)

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'
- U: there is an injection but no bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$; $\mathfrak{M} \vDash U$
- Since $\mathfrak{M}$ is countable, the sets $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ are definitely countable ($\mathfrak{M}$ has only countably many objects in its domain to serve as members of objects in that domain)
- So clearly, there IS a bijection between the sets that correspond to $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ (we can prove this bijection in a metalanguage)

Paradox Assuaged! (paradise regained?)

- Suppose that ZFC is satisfiable and so has a countable model $\mathfrak{M}$
- $\mathfrak{M}$ makes true all the axioms of ZFC and hence all the consequences of these axioms, including the claim U that says "the powerset of $\mathbb{N}$ is uncountable". Denote this set as '$2^{\mathbb{N}}$'
- U: there is an injection but no bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$; $\mathfrak{M} \vDash U$
- Since $\mathfrak{M}$ is countable, the sets $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ are definitely countable ($\mathfrak{M}$ has only countably many objects in its domain to serve as members of objects in that domain)
- So clearly, there IS a bijection between the sets that correspond to $\mathbb{N}$ and $2^{\mathbb{N}}$ in $\mathfrak{M}$ (we can prove this bijection in a metalanguage)
- BUT (resolution), this bijection is not itself an object in $\mathfrak{M}$. So $\mathfrak{M}$ itself represents $2^{\mathbb{N}}$ as uncountable

14. Compactness of SL & QL

f. Problems for finitism

Saying that there are finitely-many things

- As shown on PS13 problems #2, 5, and 6, we have some ISSUES when it comes to saying that there are finitely-many things in quantifier logic

Saying that there are finitely-many things

- As shown on PS13 problems #2, 5, and 6, we have some ISSUES when it comes to saying that there are finitely-many things in quantifier logic
- It seems like we definitely cannot accomplish this putatively possible task through *sentences*

Saying that there are finitely-many things

- As shown on PS13 problems #2, 5, and 6, we have some ISSUES when it comes to saying that there are finitely-many things in quantifier logic
- It seems like we definitely cannot accomplish this putatively possible task through *sentences*
- Is there any other way we might go about enforcing there being finitely-many things (e.g. if we think there probably are only finitely-many things and want a FOL to reflect that)?

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, Ǝ, to FOL

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, $\exists$, to FOL
- Syntactically, we define $\exists$ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then $(\exists x)\mathcal{P}$ is a wff

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, Ⅎ, to FOL
- Syntactically, we define Ⅎ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then (Ⅎx)$\mathcal{P}$ is a wff
- Semantically, we extend satisfiability semantics (oh boy—not that sh∗∗ again) so that (Ⅎx)$\mathcal{P}$ is true in a model if and only if there are finitely-many $\mathcal{P}$-objects in the model's D, i.e. $|D|$ is finite

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, Ⅎ, to FOL
- Syntactically, we define Ⅎ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then (Ⅎx)$\mathcal{P}$ is a wff
- Semantically, we extend satisfiability semantics (oh boy—not that sh** again) so that (Ⅎx)$\mathcal{P}$ is true in a model if and only if there are finitely-many $\mathcal{P}$-objects in the model's D, i.e. $|D|$ is finite
- **Question**: what would it take to modify our derivation system QND to make it sound and complete for quantifier logic with a finitely-many quantifier (QL-Ⅎ)?

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, $\exists$, to FOL
- Syntactically, we define $\exists$ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then $(\exists x)\mathcal{P}$ is a wff
- Semantically, we extend satisfiability semantics (oh boy—not that sh** again) so that $(\exists x)\mathcal{P}$ is true in a model if and only if there are finitely-many $\mathcal{P}$-objects in the model's D, i.e. $|D|$ is finite
- **Question**: what would it take to modify our derivation system QND to make it sound and complete for quantifier logic with a finitely-many quantifier (QL-$\exists$)?
- **Answer**: no derivation system can be sound & complete for QL-$\exists$!

Adding a finitely-many Quantifier

- If not through sentences, perhaps through operators, e.g. quantifiers!
- *Idea*: add a 'finitely-many' quantifier, Ⅎ, to FOL
- Syntactically, we define Ⅎ just like a quantifier: if $\mathcal{P}$ is a formula where x does not appear bound, then (Ⅎx)$\mathcal{P}$ is a wff
- Semantically, we extend satisfiability semantics (oh boy—not that sh** again) so that (Ⅎx)$\mathcal{P}$ is true in a model if and only if there are finitely-many $\mathcal{P}$-objects in the model's D, i.e. $|D|$ is finite
- **Question**: what would it take to modify our derivation system QND to make it sound and complete for quantifier logic with a finitely-many quantifier (QL-Ⅎ)?
- **Answer**: no derivation system can be sound & complete for QL-Ⅎ!
  - F***!!! INFINITE F***!!!

Finite Hopes & Finite Dreams: dashed upon ∞–many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL–∃

Finite Hopes & Finite Dreams: dashed upon ∞–many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL–∃
- Then, we could prove that QL–∃ is compact (see slides 3–4)

Finite Hopes & Finite Dreams: dashed upon ∞-many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL−∃
- Then, we could prove that QL−∃ is compact (see slides 3–4)
- Yet, the entailment relation $\vDash_{QL-\exists}$ for this logic is NOT compact:

Finite Hopes & Finite Dreams: dashed upon ∞-many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL-Ǝ
- Then, we could prove that QL-Ǝ is compact (see slides 3-4)
- Yet, the entailment relation $\vDash_{QL-\text{Ǝ}}$ for this logic is NOT compact:
- Consider the sentence $F := (\text{Ǝ}x)x = x$, which says "there are finitely-many things that equal themselves." This is just a fancy way of saying that there are finitely-many things in the domain (since everything is identical to itself and nothing else).

Finite Hopes & Finite Dreams: dashed upon ∞–many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL–Ǝ
- Then, we could prove that QL–Ǝ is compact (see slides 3–4)
- Yet, the entailment relation $\vDash_{QL-Ǝ}$ for this logic is NOT compact:
- Consider the sentence $F := (Ǝx)x = x$, which says "there are finitely–many things that equal themselves." This is just a fancy way of saying that there are finitely–many things in the domain (since everything is identical to itself and nothing else).
- Then consider the set $X := \{F, L_1, L_2, \ldots\}$, containing F and each L_k for $k \in \mathbb{N}$, where L_k says "there are at least k–things"

Finite Hopes & Finite Dreams: dashed upon ∞–many rocks

- Suppose for *reductio* that we had a sound and complete derivation system for QL–∃
- Then, we could prove that QL–∃ is compact (see slides 3–4)
- Yet, the entailment relation $\vDash_{QL-\exists}$ for this logic is NOT compact:
- Consider the sentence $F := (\exists x)x = x$, which says "there are finitely–many things that equal themselves." This is just a fancy way of saying that there are finitely–many things in the domain (since everything is identical to itself and nothing else).
- Then consider the set $X := \{F, L_1, L_2, \ldots\}$, containing F and each L_k for $k \in \mathbb{N}$, where L_k says "there are at least k–things"
- Set X is finitely–satisfiable, but it is not satisfiable (violating compactness). Any way of making true the infinitely–many L_n's requires an infinite model, which then can't make true sentence F

14. Compactness of SL & QL

g. A topological proof of SL compactness

What does “compactness” normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets

What does "compactness" normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets
- A set is closed in (X, τ) if its complement is open
  (NB: sets can be 'clopen', i.e. both open AND closed)

What does “compactness” normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets
- A set is closed in (X, τ) if its complement is open (NB: sets can be ‘clopen’, i.e. both open AND closed)
- Compactness in topology: a topological space is **compact** iff every open cover has a finite subcover

What does "compactness" normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets
- A set is closed in (X, τ) if its complement is open (NB: sets can be 'clopen', i.e. both open AND closed)
- Compactness in topology: a topological space is **compact** iff every open cover has a finite subcover
- Equivalently: every collection of closed subsets obeying the *finite intersection property* has non–empty intersection

What does "compactness" normally mean?

- **Topological space** (X, τ): a topology on a set X is a collection of **open sets** τ s.t. the following sets are open: (i) $\varnothing$ and X; (ii) arbitrary unions of open sets; (iii) finite intersections of open sets
- A set is closed in (X, τ) if its complement is open (NB: sets can be 'clopen', i.e. both open AND closed)
- Compactness in topology: a topological space is **compact** iff every open cover has a finite subcover
- Equivalently: every collection of closed subsets obeying the *finite intersection property* has non-empty intersection
- Finite intersection property (FIP): a set of subsets $\{F_\beta\}_{\beta \in B}$ of a topological space has the FIP if for every finite subset B_0 of our index set B, the intersection of all the sets F_β for $\beta \in B_0$ is non-empty, i.e. provided that $\bigcap_{\beta \in B} F_\beta$ is non-empty

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U_A^0 be the set of TVAs that assign A false, and let U_A^1 be the set of TVAs that assign A true

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U_A^0 be the set of TVAs that assign A false, and let U_A^1 be the set of TVAs that assign A true
- Endow the set $\mathcal{E}$ with a topology by stipulating that (i) for each atomic wff A, U_A^0 and U_A^1 are open and (ii) every non-empty open set arises as a union of these U^0s and U^1s

Why call the logical property "compactness"?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U^0_A be the set of TVAs that assign A false, and let U^1_A be the set of TVAs that assign A true
- Endow the set $\mathcal{E}$ with a topology by stipulating that (i) for each atomic wff A, U^0_A and U^1_A are open and (ii) every non-empty open set arises as a union of these U^0s and U^1s
- **Claim**: the compactness of SL is equivalent to the compactness of this topological space $\mathcal{E}$

Why call the logical property “compactness”?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U_A^0 be the set of TVAs that assign A false, and let U_A^1 be the set of TVAs that assign A true
- Endow the set $\mathcal{E}$ with a topology by stipulating that (i) for each atomic wff A, U_A^0 and U_A^1 are open and (ii) every non-empty open set arises as a union of these U^0s and U^1s
- **Claim**: the compactness of SL is equivalent to the compactness of this topological space $\mathcal{E}$
- Note that if we prove that 1) compactness of $\mathcal{E}$ entails compactness of SL and that 2) $\mathcal{E}$ is compact, then

Why call the logical property “compactness”?

- The compactness of SL is equivalent to the compactness of a particular topological space, namely a topology on the set of truth-value assignments (TVAs)
- Let $\mathcal{A}$ be the set of atomic wffs and let $\mathcal{E}$ be the set of TVAs
- for each atomic wff A, let U_A^0 be the set of TVAs that assign A false, and let U_A^1 be the set of TVAs that assign A true
- Endow the set $\mathcal{E}$ with a topology by stipulating that (i) for each atomic wff A, U_A^0 and U_A^1 are open and (ii) every non-empty open set arises as a union of these U^0s and U^1s
- **Claim**: the compactness of SL is equivalent to the compactness of this topological space $\mathcal{E}$
- Note that if we prove that 1) compactness of $\mathcal{E}$ entails compactness of SL and that 2) $\mathcal{E}$ is compact, then we will have proven compactness without detour through syntax!

Step 1: $\mathcal{E}$ compact entails SL is compact

- Assume that $(\mathcal{E}, \tau)$ is compact. Consider an arbitrary set Γ of SL sentences that is finitely satisfiable.

Step 1: $\mathcal{E}$ compact entails SL is compact

- Assume that $(\mathcal{E}, \tau)$ is compact. Consider an arbitrary set Γ of SL sentences that is finitely satisfiable.

  NTS: Γ is satisfiable (the other direction is trivial)

Step 1: $\mathcal{E}$ compact entails SL is compact

- Assume that $(\mathcal{E}, \tau)$ is compact. Consider an arbitrary set Γ of SL sentences that is finitely satisfiable.

  NTS: Γ is satisfiable (the other direction is trivial)

- Consider an arbitrary wff $\mathcal{P}$. *Lemma*: the set $U_{\mathcal{P}} \subset \mathcal{E}$ of TVAs that make $\mathcal{P}$ true is open (proof: use disjunctive normal form and take a matching union of finite intersections of the U_A^0s and U_B^1s for atomics that compose $\mathcal{P}$!)

Step 1: $\mathcal{E}$ compact entails SL is compact

- Assume that $(\mathcal{E}, \tau)$ is compact. Consider an arbitrary set Γ of SL sentences that is finitely satisfiable.

  NTS: Γ is satisfiable (the other direction is trivial)

- Consider an arbitrary wff $\mathcal{P}$. *Lemma*: the set $U_{\mathcal{P}} \subset \mathcal{E}$ of TVAs that make $\mathcal{P}$ true is open (proof: use disjunctive normal form and take a matching union of finite intersections of the U_A^0s and U_B^1s for atomics that compose $\mathcal{P}$!)

- So $U_{\sim\mathcal{P}}$ is also open. Since the complement of $U_{\mathcal{P}}$ is $U_{\sim\mathcal{P}}$, $U_{\mathcal{P}}$ is both closed and open

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$
- So to say that each finite subset Γ_0 of Γ is satisfiable is equivalent to saying that the family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is a family of closed subsets of $\mathcal{E}$ with the *finite intersection property* (i.e. for any finite subset of this family, the intersection of its members $U_{\mathcal{Q}}$ is non-empty)

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$
- So to say that each finite subset Γ_0 of Γ is satisfiable is equivalent to saying that the family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is a family of closed subsets of $\mathcal{E}$ with the *finite intersection property* (i.e. for any finite subset of this family, the intersection of its members $U_{\mathcal{Q}}$ is non-empty)
- Since we are assuming that $\mathcal{E}$ is compact, the intersection of ALL members of this family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is non-empty

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$
- So to say that each finite subset Γ_0 of Γ is satisfiable is equivalent to saying that the family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is a family of closed subsets of $\mathcal{E}$ with the *finite intersection property* (i.e. for any finite subset of this family, the intersection of its members $U_{\mathcal{Q}}$ is non-empty)
- Since we are assuming that $\mathcal{E}$ is compact, the intersection of ALL members of this family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is non-empty
- i.e. this intersection must contain at least one TVA in $\mathcal{E}$

Step 1 continued: applying topological compactness

- So, for each wff $\mathcal{P}$ in Γ, the set of TVAs $U_{\mathcal{P}}$ that make $\mathcal{P}$ true is a closed subset of $\mathcal{E}$
- So to say that each finite subset Γ_0 of Γ is satisfiable is equivalent to saying that the family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is a family of closed subsets of $\mathcal{E}$ with the *finite intersection property* (i.e. for any finite subset of this family, the intersection of its members $U_{\mathcal{Q}}$ is non-empty)
- Since we are assuming that $\mathcal{E}$ is compact, the intersection of ALL members of this family $\{U_{\mathcal{P}} : \mathcal{P} \in \Gamma\}$ is non-empty
- i.e. this intersection must contain at least one TVA in $\mathcal{E}$
- Hence, there is a TVA that makes true all of the members of Γ

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.
- Since there are countably many SL atomics, $2^{\mathcal{A}}$ is homeomorphic to the Cantor set (comprises ∞–binary sequences of 0s and 1s)

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.
- Since there are countably many SL atomics, $2^{\mathcal{A}}$ is homeomorphic to the Cantor set (comprises ∞–binary sequences of 0s and 1s)
- Note that the Cantor set is compact, since it is a closed subset of a compact set (namely the closed unit interval [0, 1])

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.
- Since there are countably many SL atomics, $2^{\mathcal{A}}$ is homeomorphic to the Cantor set (comprises ∞–binary sequences of 0s and 1s)
- Note that the Cantor set is compact, since it is a closed subset of a compact set (namely the closed unit interval [0, 1])
- If we allow $\mathcal{A}$ to have arbitrarily many SL atomics, then we could use Tychonoff's theorem (equivalent to the axiom of choice) to show that $2^{\mathcal{A}}$ is compact

Step 2: show that $(\mathcal{E}, \tau)$ is compact

- We can think about $\mathcal{E}$ as equalling $2^{\mathcal{A}}$, i.e. the set of maps from the SL atomics $\mathcal{A}$ to the set $\{0, 1\}$
- Equip the set $\{0, 1\}$ with the discrete topology (i.e. every subset is open). Then the product topology on $2^{\mathcal{A}}$ equals the topology $(\mathcal{E}, \tau)$ defined earlier.
- Since there are countably many SL atomics, $2^{\mathcal{A}}$ is homeomorphic to the Cantor set (comprises ∞−binary sequences of 0s and 1s)
- Note that the Cantor set is compact, since it is a closed subset of a compact set (namely the closed unit interval [0, 1])
- If we allow $\mathcal{A}$ to have arbitrarily many SL atomics, then we could use Tychonoff's theorem (equivalent to the axiom of choice) to show that $2^{\mathcal{A}}$ is compact
- *Tychonoff*: a product of compact spaces is compact in the product topology