Philosophy 24.241
Logic 1
Fall 2022

Solutions to Problem Set 3
(please don't post these to the internets or else I will be sad :( )

1. (Recursively define a-palindrome and prove by induction that a-palindromes have even numbers of b's.)

   (i) Recursive Definition:
   Base clause: "a" is an a-palindrome.
   Recursion clause: If s is an a-palindrome, then asa and bsb are a-palindromes.
   Closure Clause: Nothing else is an a-palindrome over $\{a, b\}$

   (ii) Strategy for Proof by Induction: proceed by complete induction on the number n of letters in a string. (Make sure to "break down" an arbitrary string s of length k rather than to "build up" from specific, smaller strings!)

   Base case: The base case has to cover n = 1, since that is the case in the base clause of the recursive definition. If the length of s is 1, then s must be "a". Since "a" has no b's, in it, it trivially has an even number of b's, namely zero.

   Induction step: Assume that any a-palindrome with fewer than k symbols has an even number of b's (with $1 < k$) . (Induction hypothesis for complete induction)

   Next, let s be an arbitrary a-palindrome with k letters. Since $k > 1$ (so that s has at least two letters) s must result from an application of the recursion clause of the definition of a-palindrome (so we see that k actually is ≥ 3)

   Hence s is one of: $as'a$ or $bs'b$, where s' is an a-palindrome with $k - 2$ symbols. (It's important to note that s' is itself an a-palindrome, since otherwise one hasn't justified that s' falls within the scope of the induction hypothesis).

   The induction hypothesis applies, so we know that s' has an even number of b's. Since s has exactly the letters of s' except that it has either exactly two more a's or exactly two more b's, we know that s must also have an even number of b's.

2. Here is the inductive definition of n! (read "n factorial"):
$1! = 1$
$(n+1)! = (n+1) \times n!$

That is, $n! = \underbrace{(n \times (n-1) \times \ldots 3 \times 2 \times 1)}_{n\ times}$

Prove by induction: For every n greater than or equal to 5, $3^{n-1} < n!$

Hint: A simple bit of algebra will be useful here: if a, b, c, d are all natural numbers, with $a < b$ and $c < d$, then $a \cdot c < b \cdot d$.

Base case: (The base case will be $n = 5$ since we are restricting attention to n such that $5 \leq n$.) If $n = 5$ then $3^{n-1} = 3^4 = 81$ and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

$81 < 120$, so the relevant property is true of the base case.

Induction Step:
Assume that for an arbitrary k (such that $5 \leq k$) we have $3^{k-1} < k!$.

By definition of exponent and factorial, $3^{(k+1)-1} = 3 \times 3^{k-1}$ and $(k+1)! = (k+1) \times k!$.

Since $3 < k+1$ (since $5 \leq k$), and $3^{k-1} < k!$ by the induction hypothesis, we can apply the elementary principle that if $a < b$ and $c < d$ with a, b, c, d all positive integers, then $a \cdot c < b \cdot d$ to get the inequality we want to prove:

$3^{(k+1)-1} = 3 \times 3^{k-1} < (k+1) \times k! = (k+1)!$

This completes the induction step, and hence the proof.

3. Prove by induction that if you just have 4 and 11 cent stamps, you can get a combination of stamps for 30 cents, and *any* amount greater than 30. (Hint: The base case you need in this one needs to be crafted carefully. You will need to prove more than one case.)

   Proof:

   Base case: The base case here is a bit unusual, because it is easiest to consider four possibilities:
   $n = 30$ cents, $n = 31$ cents, $n = 32$, and $n = 33$ cents.

   Here is how you can get these numbers from 4 and 11: $30 = 4 \times 2 + 11 \times 2; 31 = 4 \times 5 + 11; 32 = 4 \times 8; 33 = 11 \times 3$.

   (The point is that you need to have a stretch of 4 consecutive numbers that you can make up with the stamps. (30 is the smallest number that will work.) Then for any given k, you know that $k - 4$ will fall in the scope of the induction hypothesis.)

   Induction Step: Consider an arbitrary number $k > 34$.
   Induction hypothesis: assume that for every n, with $30 \leq n < k$, n cents can be made up of a combination of 4 and 11 cent stamps. We want to show that we can make up k cents with a combination of 4 and 11 cent stamps (again: we are 'breaking down' value k into pieces under control by the Induction Hypothesis, NOT "building k up")

   Since $34 < k$, $k - 4$ is in the range $30 \leq n < k$, and hence it falls within the scope of the induction hypothesis.

   (***Note: This is why matters are simplest if we prove four cases in the base case. If we couldn't require k to be greater than 34, then $k - 4$ could be < 30, which would put $k - 4$ outside the range considered in the induction hypothesis.***)

   So, by the induction hypothesis, there are positive natural numbers a and b such that $4 \times a + 11 \times b = (k - 4)$. But then by adding a four-cent stamp, we get exactly k: $4 \times a + 11 \times b + 4 = (k - 4) + 4 = k$.

   Simplifying $4 \times a + 11 \times b + 4 = (k - 4) + 4 = k$, we have that $4 \times (a + 1) + 11 \times b = k$. This completes the induction step, and hence the proof.

4. Prove by induction that the product of n odd numbers (with $2 \leq n$) is odd.

   (You may find this fact useful: Any odd natural number m can be written as $2k + 1$, for some other natural number k.)

   Proof:
   We'll do the induction on the number n of odd numbers multiplied together. (*Not* on the numbers that are multiplied together, or on the size of the number that is the product.)

   Base case: Say that $n = 2$, and let o_1 and o_2 be odd numbers. Then there exist natural numbers k, l such that $o_1 = 2k + 1$ and $o_2 = 2l + 1$.

   $o_1 \times o_2 = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$

   The right-hand side is an even number plus 1, and so it's an odd number.

   Induction Step
   Induction Hypothesis: Assume that for every n, with $2 < n < k$ the product of n odd numbers is odd.

   Consider an arbitrary number c that is the product of exactly k odd numbers. Then $c = m_1 \times m_2 \times \ldots \times m_k$, where each factor m_i is odd.

   Since $2 < k$, we can write c as the product of two factors $c = c_1 \times c_2$,
   where $c_1 = m_1 \times m_2 \times \ldots \times m_i$ and $c_2 = m_{i+1} \times m_{i+2} \times \ldots \times m_k$, with $2 < i < k$.

   Both c_1 and c_2 are products of fewer than k odd numbers, so they are odd by the induction hypothesis.

   Hence $c = c_1 \times c_2$ is the product of two odd numbers. By the argument in the base case, it follows that c is an odd number. This completes the induction step, and hence the proof.

   (Note: it is also straightforward to solve this problem by ordinary induction on the number of odd factors)

5. Prove that no well-formed formula of sentential logic ever contains consecutive atomic formulas [e.g. nothing like '$(PP\&Q)$'.]

Proof (by complete induction on string-length n):

Base Case: from the recursive definition of wffs of SL, the base case includes all sentences of length 1, namely the atomic sentences. Since an atomic sentence contains only a single symbol, it cannot contain consecutive atomic formulae.

Induction step: assume (for complete induction) that every wff of SL of length n—where $1 \leq n < k$—satisfies the property, i.e. does not contain consecutive atomic formulae. Need to show: an arbitrary wff of length k also does not contain consecutive atomic formulae (again: we must 'break down' rather than 'build up').

Let Γ be an arbitrary wff of length k. Since $k > 1$, there must exist wff Φ and Ψ of length less than k, such that Γ equals one of the following five cases (that these are all and only the cases follows from the recursion clause and closure clause of our definition of "wff of SL", but you don't have to note that). Before proceeding, note that since Φ and Ψ are wff of length less than k, they each satisfy the relevant property by the induction hypothesis (i.e. neither contains consecutive atomic formulae).

i. $\sim\Phi$. In this case, Γ merely has an additional negation symbol, compared to Φ. Since Φ does not have consecutive atomic formulae, and since a negation symbol is not an atomic formula, it follows that Γ also does not have consecutive atomic formulae.

ii. $(\Phi \,\&\, \Psi)$. In this case, Γ has an additional left parenthesis, ampersand, and right parenthesis compared to its component wffs. Since none of these symbols is an atomic formula, and since neither Φ nor Ψ contain consecutive atomic formula, it follows that Γ also does not have consecutive atomic formulae.

iii-v. $(\Phi \vee \Psi)$; $(\Phi \supset \Psi)$; $(\Phi \equiv \Psi)$. In each of these three remaining cases, the reasoning is exactly parallel to the case of '$(\Phi \,\&\, \Psi)$', *mutatis mutandis.*

Hence, in any possible case, Γ does not contain consecutive atomic formulae. This completes the induction step and hence the proof.

Alternative proof procedure, allowed for sentences of SL only: As noted in lecture, we are allowing you to be particularly lazy when it comes to proofs involving properties of sentences of SL. After proving the base case, you can simply introduce two arbitrary wffs Φ and Ψ and argue that in each of the five cases coming from the recursion clause of the definition, the relevant property obtains. Hopefully, it is clear why this lazy procedure is justified: it is an implicit case of complete induction on the string length.

Note as well: you can also solve this problem by performing induction on either
i) the number of atomic formulae or ii) the number of connectives.