6. Proofs in SL

1. Proofs in SL

1.1 Who ordered *that*???

1.2 Conjunction Intro and Elimination (Rules for &)

1.3 Conditional Intro and Elim. (Rules for $\supset$)

1.4 Use of subproofs

1.5 Disjunction Intro and Elim. (Rules for $\vee$)

1.6 Negation Intro and Elimination

1.7 Biconditional Intro and Elimination (<->)

1.8 Strategies and examples

1.9 The Rules, Reiterated

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- Why not just tree always and everywhere?

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- e.g., common rules such as Modus Ponens, disjunctive syllogism

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- Perhaps our natural deduction system *explains* the success of our ordinary inference patterns

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  - etc.
- If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

1. Sarah lives in Chicago or Erie.
2. Amir lives in Chicago unless he enjoys hiking.
3. If Amir lives in Chicago, Sarah doesn't.
4. Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Erie.

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7. Sarah doesn't live in Chicago (from 3 and 6).
8. Sarah lives in Erie (from 1 and 7).

A more formal proof

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- But: what are the rules? (very different from 'what IS a rule'?)

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  - justify something **using** $\mathcal{P} \,\&\, \mathcal{Q}$ (i.e. to 'eliminate' $\&$).

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b. Conjunction Intro and Elimination (Rules for &)

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- (Role of $\mathcal{P}$ played by $\sim S$ and that of $\mathcal{Q}$ played by $\sim M$)

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Rules for &

$$
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n & \mathcal{Q} & \\
 & \mathcal{P} \,\&\, \mathcal{Q} & :m,\ n\ \ \&\mathrm{I}
\end{array}
$$

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We'll illustrate using exercises in our Week 6 Practice Problems on Carnap.

1	$A \& B$	:PR
2	A	:1 & E
3	B	:1 & E
4	$B \& A$	:2, 3 & I

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3	$B \,\&\, C$	:1 & E
4	C	:3 & E
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c. Conditional Intro and Elim. (Rules for ⊃)

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- (When inferring from $A \supset {\sim}C$ and A to ${\sim}C$, the role of $\mathcal{P}$ is played by A and role of $\mathcal{Q}$ by ${\sim}C$.)

Elimination rule for $\supset$

$$
\begin{array}{r|ll}
m & \mathcal{P} \supset \mathcal{Q} & \\
n & \mathcal{P} & \\
 & \mathcal{Q} & :m,\ n \supset\mathrm{E}
\end{array}
$$

Let's illustrate this rule using an exercise in Carnap: we show that $\{A \,\&\, B, A \supset C, B \supset D\} \vDash C \,\&\, D$.

1	$A \& B$	:PR
2	$A \supset C$	:PR
3	$B \supset D$	:PR
4	A	:1 & E
5	C	:2, 4 ⊃E
6	B	:1 & E
7	D	:3, 6 ⊃E
8	$C \& D$	:5, 7 & I

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- It's a good move, because if $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.

Justifying $\supset$I

Fact

If $\mathcal{R}, \mathcal{P} \vDash \mathcal{Q}$ then $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.

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  4. If it makes $\mathcal{P}$ false, it makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.

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  5. If it makes $\mathcal{Q}$ true, it also makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.
- So, there are no counterexamples to $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.
- If $\mathcal{R} = \varnothing$, then from $\mathcal{P} \vDash \mathcal{Q}$ we can infer $\vDash \mathcal{P} \supset \mathcal{Q}$

Subproofs (CRUCIAL CONCEPT)

- We want to justify $\mathcal{P} \supset \mathcal{Q}$ by giving a proof of $\mathcal{Q}$ from *assumption* $\mathcal{P}$ (and possibly other premises Γ, e.g. $\mathcal{R}$)

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- Once we're done (have proved $\mathcal{Q}$), close this "subproof".
- Justification of $\mathcal{P} \supset \mathcal{Q}$ is the **entire** subproof (use a HYPHEN)
- **Important**: nothing **inside** a subproof is available outside as a justification (since inner lines might depend on the assumption)

Introduction rule for ⊃

m	$\mathcal{P}$	:AS for ⊃I
	$\vdots$	
n	$\mathcal{Q}$	
	$\mathcal{P} \supset \mathcal{Q}$	:m-n ⊃I

NOTE THE **HYPHEN** IN THE JUSTIFICATION LINE!!!

We'll illustrate using more exercises from Week 6 Practice Problems

- Show: $\{A \supset B, B \supset C\} \vDash A \supset C$.
- Show: $A \supset (B \supset C) \vDash (A \,\&\, B) \supset (A \,\&\, C)$

1	$A \supset B$	:PR
2	$B \supset C$	:PR
3	A	:AS for $\supset$I
4	B	:1, 3 $\supset$E
5	C	:2, 4 $\supset$E
6	$A \supset C$	:3-5 $\supset$I

1	$A \supset (B \supset C)$	:PR
2	$\quad A \,\&\, B$	:AS for $\supset$I
3	$\quad A$	:2 & E
4	$\quad B \supset C$	:1, 3 $\supset$E
5	$\quad B$	:2 & E
6	$\quad C$	:4, 5 $\supset$E
7	$\quad A \,\&\, C$	:3, 6 & I
8	$(A \,\&\, B) \supset (A \,\&\, C)$	:2-7 $\supset$I

6. Proofs in SL

d. Use of subproofs

Reiteration (for the 11th hour!)

$\mathcal{P} \vDash \mathcal{P}$, so "Reiteration" R is a good rule:

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
 & & \\
k & \mathcal{P} & :m \text{ R}
\end{array}
$$

Uses of reiteration (to the Carnap!):

- Proof of $A \vDash A$.

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Uses of reiteration (to the Carnap!):

- Proof of $A \vDash A$.
- Proof that $A \supset (B \supset A)$ is a tautology.

1	$\quad A$	:AS for $\supset$I
2	$\quad A$	:1 R
3	$A \supset A$	:1-2 $\supset$I

Again, note the **HYPHEN**! Even though our subproof is only two lines, we still write ':1–2' and NOT ':1, 2'.

1	$\mid\ \mid\ A$	:AS for $\supset$I
2	$\mid\ \mid\ \mid\ B$	:AS for $\supset$I
3	$\mid\ \mid\ \mid\ A$	:1 R
4	$\mid\ \mid\ B \supset A$	:2–3 $\supset$I
5	$\mid\ A \supset (B \supset A)$	:1–4 $\supset$I

Rules for justifications and subproofs

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- After a subproof is done, you can only cite the whole thing, NOT any line in it (you are 'outside the scope' of these lines)
- Subproofs (subproofs can be nested) can be nested
- You also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is completed (since again you'd be 'outside the scope' of those lines)

Reiteration (do's a don'ts) DOs and DON'Ts!

Which are correct applications of R?

1	A	:AS	
2	A	:AS	
3	A	:1	R
4	A	:1	R
5	A	:2	R
6	A	:2	R
7	A	:1	R

Reiteration (do’s a don’ts) DOs and DON’Ts!

Which are correct applications of R?

1	$\vert\ \vert\ A$	:AS
2	$\vert\ \vert\ \vert\ A$	:AS
3	$\vert\ \vert\ \vert\ A$	:1 ✓ R
4	$\vert\ \vert\ A$	:1 R
5	$\vert\ \vert\ A$	:2 R
6	$\vert\ A$	:2 R
7	$\vert\ A$	:1 R

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2			A	:AS
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6	A			:2 ✗ R
7	A			:1 ✗ R

6. Proofs in SL

e. Disjunction Intro and Elim. (Rules for $\vee$)

Introduction rule for $\vee$

We have $\mathcal{P} \vDash \mathcal{P} \vee \mathcal{Q}$. So:

$$\begin{array}{r|ll} m & \mathcal{P} & \\ & \mathcal{P} \vee \mathcal{Q} & : m\ \vee\mathrm{I} \end{array} \qquad \begin{array}{r|ll} m & \mathcal{Q} & \\ & \mathcal{P} \vee \mathcal{Q} & : m\ \vee\mathrm{I} \end{array}$$

- Note that the introduced disjunct can be ANYTHING!

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- Note that the introduced disjunct can be ANYTHING!
- And you can introduce on the left OR right side!

Introduction rule for $\vee$

We have $\mathcal{P} \vDash \mathcal{P} \vee \mathcal{Q}$. So:

m	$\mathcal{P}$	
	$\mathcal{P} \vee \mathcal{Q}$	$: m \vee \mathrm{I}$

m	$\mathcal{Q}$	
	$\mathcal{P} \vee \mathcal{Q}$	$: m \vee \mathrm{I}$

- Note that the introduced disjunct can be ANYTHING!
- And you can introduce on the left OR right side!
- Let's do practice problem 6.10 on Carnap!

1	$\quad \mid A$	:AS for $\supset$I
2	$\quad \mid B \vee A$	:1 $\vee$I
3	$A \supset (B \vee A)$	:1–2 $\supset$I

Eliminating $\vee$ (Proof by Cases)

- What can we justify with disjunction $\mathcal{P} \vee \mathcal{Q}$?

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- What can we justify with disjunction $\mathcal{P} \vee \mathcal{Q}$?
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- But: if both $\mathcal{P}$ and $\mathcal{Q}$ separately entail some third sentence $\mathcal{R}$, then we know that $\mathcal{R}$ follows from the disjunction!

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- What can we justify with disjunction $\mathcal{P} \vee \mathcal{Q}$?
- Not $\mathcal{P}$ and also not $\mathcal{Q}$: neither is entailed by $\mathcal{P} \vee \mathcal{Q}$.
- But: if both $\mathcal{P}$ and $\mathcal{Q}$ separately entail some third sentence $\mathcal{R}$, then we know that $\mathcal{R}$ follows from the disjunction!
- To show this, we need **two** subproofs that show $\mathcal{R}$, but in each proof we are allowed to use only one of $\mathcal{P}$, $\mathcal{Q}$.

Elimination rule for ∨ (Proof by Cases)

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad \mathcal{P}$	:AS for $\vee$E
	$\quad \vdots$	
j	$\quad \mathcal{R}$	
	$--$	
k	$\quad \mathcal{Q}$	:AS for $\vee$E
	$\quad \vdots$	
ℓ	$\quad \mathcal{R}$	
	$\mathcal{R}$	:m, i–j, k–ℓ $\vee$E

- From $\mathcal{P}$ we derive $\mathcal{R}$

Elimination rule for ∨ (Proof by Cases)

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad \mathcal{P}$	:AS for $\vee$E
	$\quad \vdots$	
j	$\quad \mathcal{R}$	
	$--$	
k	$\quad \mathcal{Q}$	:AS for $\vee$E
	$\quad \vdots$	
ℓ	$\quad \mathcal{R}$	
	$\mathcal{R}$	:m, i–j, k–ℓ $\vee$E

- From $\mathcal{P}$ we derive $\mathcal{R}$
- Start a subproof for each disjunct

Elimination rule for $\vee$ (Proof by Cases)

$$
\begin{array}{l|ll}
m & \mathcal{P} \vee \mathcal{Q} & \\
i & \quad \mathcal{P} & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & \\
j & \quad \mathcal{R} & \\
 & -- & \\
k & \quad \mathcal{Q} & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & \\
\ell & \quad \mathcal{R} & \\
 & \mathcal{R} & :m,\ i\text{-}j,\ k\text{-}\ell\ \vee\text{E}
\end{array}
$$

- From $\mathcal{P}$ we derive $\mathcal{R}$
- Start a subproof for each disjunct
- The subproofs need not be adjacent, but if they are, **separate with --**

Elimination rule for ∨ (Proof by Cases)

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad\mathcal{P}$	:AS for ∨E
	$\quad\vdots$	
j	$\quad\mathcal{R}$	
	--	
k	$\quad\mathcal{Q}$	:AS for ∨E
	$\quad\vdots$	
ℓ	$\quad\mathcal{R}$	
	$\mathcal{R}$	:m, i–j, k–ℓ ∨E

- From $\mathcal{P}$ we derive $\mathcal{R}$
- Start a subproof for each disjunct
- The subproofs need not be adjacent, but if they are, **separate with --**
- From $\mathcal{Q}$ we derive $\mathcal{R}$

Elimination rule for $\vee$ (Proof by Cases)

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad \mathcal{P}$	:AS for $\vee$E
	$\quad \vdots$	
j	$\quad \mathcal{R}$	
	--	
k	$\quad \mathcal{Q}$	:AS for $\vee$E
	$\quad \vdots$	
ℓ	$\quad \mathcal{R}$	
	$\mathcal{R}$	:m, i-j, k-ℓ $\vee$E

- From $\mathcal{P}$ we derive $\mathcal{R}$
- Start a subproof for each disjunct
- The subproofs need not be adjacent, but if they are, **separate with --**
- From $\mathcal{Q}$ we derive $\mathcal{R}$
- You can swap the order of the the subproofs

Elimination rule for ∨ (Proof by Cases)

$$
\begin{array}{l|l|ll}
m & \mathcal{P} \vee \mathcal{Q} & & \\
i & \quad \mathcal{P} & & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & & \\
j & \quad \mathcal{R} & & \\
 & -- & & \\
k & \quad \mathcal{Q} & & :\text{AS for } \vee\text{E} \\
 & \quad \vdots & & \\
\ell & \quad \mathcal{R} & & \\
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- The subproofs need not be adjacent, but if they are, **separate with --**
- From $\mathcal{Q}$ we derive $\mathcal{R}$
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- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction

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\begin{array}{l|l l}
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 & -- & \\
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 & \quad \vdots & \\
\ell & \quad \mathcal{R} & \\
 & \mathcal{R} & :m,\ i\text{–}j,\ k\text{–}\ell\ \vee\text{E}
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- From $\mathcal{Q}$ we derive $\mathcal{R}$
- You can swap the order of the subproofs
- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction
- Remember to pop out of subproof level at the end!

1	$A \vee B$	:PR
2	$\quad A$	:AS for $\vee$E
3	$\quad B \vee A$	:2 $\vee$I
4	$\quad B$	:AS for $\vee$E
5	$\quad B \vee A$	:4 $\vee$I
6	$B \vee A$	:1, 2–3, 4–5 $\vee$E

- In Carnap: Need -- between the subproofs

1	$A \vee B$	:PR
2	$\quad A$	:AS for $\vee$E
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- In Carnap: Need -- between the subproofs
- Note: need the **SAME sentence** as the last line of each subproof

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- In Carnap: Need -- between the subproofs
- Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself

1	$A \vee B$	:PR
2	$\quad A$	:AS for $\vee$E
3	$\quad B \vee A$	:2 $\vee$I
4	$\quad B$	:AS for $\vee$E
5	$\quad B \vee A$	:4 $\vee$I
6	$B \vee A$	:1, 2-3, 4-5 $\vee$E

- In Carnap: Need -- between the subproofs
- Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself
- Proceed to Carnap PP6.15!

1	$A \vee B$	:PR
2	$A \supset B$	:PR
3	$\quad A$	:AS for $\vee$E
4	$\quad B$	:2, 3 $\supset$E
5	$\quad B$	:AS for $\vee$E
6	$\quad B$	:5 R
7	B	:1, 3-4, 5-6 $\vee$E

6. Proofs in SL

f. Negation Intro and Elimination

Introducing $\sim$

- An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.

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- For instance:
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  - $\mathcal{Q} \vDash \sim\mathcal{P}$ iff $\mathcal{Q}$ and $\mathcal{P}$ are jointly unsatisfiable.

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  - $\mathcal{Q} \vDash \mathcal{P}$ iff $\mathcal{Q}$ and $\sim\mathcal{P}$ are jointly unsatisfiable.
  - $\mathcal{Q} \vDash \sim\mathcal{P}$ iff $\mathcal{Q}$ and $\mathcal{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\sim$I rule: To justify $\sim\mathcal{P}$, show that $\mathcal{P}$ (together with all other premises) is unsatisfiable.

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- For instance:
  - $\mathcal{Q} \vDash \mathcal{P}$ iff $\mathcal{Q}$ and $\sim\mathcal{P}$ are jointly unsatisfiable.
  - $\mathcal{Q} \vDash \sim\mathcal{P}$ iff $\mathcal{Q}$ and $\mathcal{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\sim$I rule: To justify $\sim\mathcal{P}$, show that $\mathcal{P}$ (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction follows!

Negation Introduction ($\sim$I)

m	Φ	:AS for $\sim$I
	$\vdots$	
n	Ψ	
	$\vdots$	
o	$\sim\Psi$	
	$\sim\Phi$	: m–o $\sim$I

- Assume the **non**–negated wff!

Negation Introduction ($\sim$I)

m	$\quad \Phi$	:AS for $\sim$I
	$\quad \vdots$	
n	$\quad \Psi$	
	$\quad \vdots$	
o	$\quad \sim\Psi$	
	$\sim\Phi$	:m-o $\sim$I

- Assume the **non**-negated wff!
- Derive a sentence and its negation (could be Φ!)

Negation Introduction ($\sim$I)

$$
\begin{array}{l|l|ll}
m & & \Phi & :\text{AS for } \sim\text{I} \\
\hline
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \sim\Phi & & :m\text{-}o\ \sim\text{I}
\end{array}
$$

- Assume the **non**–negated wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)

Negation Introduction ($\sim$I)

$$
\begin{array}{l|l|ll}
m & & \underline{\Phi} & :\text{AS for } \sim\text{I} \\
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \sim\Phi & & :m\text{-}o \sim\text{I}
\end{array}
$$

- Assume the **non**–negated wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **introduce** that negativity!

Negation Introduction ($\sim$I)

$$
\begin{array}{l|l|ll}
m & & \Phi & \text{:AS for } \sim\text{I} \\
\hline
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \sim\Phi & & :m\text{-}o \sim\text{I}
\end{array}
$$

- Assume the **non**–negated wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **introduce** that negativity!
- Remember to cite the WHOLE subproof (hyphen!)

Negation Introduction (∼I)

m	$\quad\Phi$	:AS for ∼I
	$\quad\vdots$	
n	$\quad\Psi$	
	$\quad\vdots$	
o	$\quad\sim\Psi$	
	$\sim\Phi$	:m-o ∼I

- Assume the **non**–negated wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **introduce** that negativity!
- Remember to cite the WHOLE subproof (hyphen!)
- Let's try exercise PP6.21:

1	$A \supset B$	:AS for $\supset$I
2	$\sim B$	:AS for $\supset$I
3	A	:AS for $\sim$I
4	B	:1, 3 $\supset$E
5	$\sim B$	:2 R
6	$\sim A$	:3-5 $\sim$I
7	$\sim B \supset \sim A$	:2-6 $\supset$I
8	$(A \supset B) \supset (\sim B \supset \sim A)$	:1-7 $\supset$I

Negation Elimination ($\sim$E)

$$
\begin{array}{l|l|ll}
m & & \sim\Phi & :\text{AS for } \sim\text{E} \\
\hline
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \Phi & & :m\text{–}o \sim\text{E}
\end{array}
$$

- Assume the **negated** wff!

Negation Elimination ($\sim$E)

$$
\begin{array}{l|l|ll}
m & & \sim\Phi & :\text{AS for } \sim\text{E} \\
\hline
 & & \vdots & \\
n & & \Psi & \\
 & & \vdots & \\
o & & \sim\Psi & \\
 & \Phi & & :m\text{–}o \sim\text{E}
\end{array}
$$

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)

Negation Elimination ($\sim$E)

m	$\quad\sim\Phi$	:AS for $\sim$E
	$\quad\vdots$	
n	$\quad\Psi$	
	$\quad\vdots$	
o	$\quad\sim\Psi$	
	Φ	: m–o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)

Negation Elimination ($\sim$E)

m		$\sim\Phi$	:AS for $\sim$E
		$\vdots$	
n		Ψ	
		$\vdots$	
o		$\sim\Psi$	
	Φ		:m–o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **eliminate** that negativity!

Negation Elimination ($\sim$E)

m	$\quad\vert\quad \sim\Phi$	:AS for $\sim$E
	$\quad\vert\quad \vdots$	
n	$\quad\vert\quad \Psi$	
	$\quad\vert\quad \vdots$	
o	$\quad\vert\quad \sim\Psi$	
	Φ	:m–o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **eliminate** that negativity!
- Put a smile on!

Negation Elimination ($\sim$E)

m	$\quad \sim\Phi$	:AS for $\sim$E
	$\quad \vdots$	
n	$\quad \Psi$	
	$\quad \vdots$	
o	$\quad \sim\Psi$	
	Φ	: m–o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **eliminate** that negativity!
- Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)

Negation Elimination ($\sim$E)

m	$\sim\Phi$	:AS for $\sim$E
	$\vdots$	
n	Ψ	
	$\vdots$	
o	$\sim\Psi$	
	Φ	:m-o $\sim$E

- Assume the **negated** wff!
- Derive a sentence and its negation (could be Φ!)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and **eliminate** that negativity!
- Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)
- Let's try exercise PP6.22:

1	$\sim A \supset \sim B$	:AS for $\supset$I
2	B	:AS for $\supset$I
3	$\sim A$	:AS for $\sim$E
4	$\sim B$	:1, 3 $\supset$E
5	B	:2 R
6	A	:3-5 $\sim$E
7	$B \supset A$	:2-6 $\supset$I
8	$(\sim A \supset \sim B) \supset (B \supset A)$	:1-7 $\supset$I

6. Proofs in SL

g. Biconditional Intro and Elimination (<->)

Biconditional Introduction (≡I) (Type <-> !!!)

$$
\begin{array}{l|l|ll}
i & & \mathcal{A} & \text{:AS for } \equiv\mathrm{I} \\
 & & \vdots & \\
j & & \mathcal{B} & \\
 & -- & & \\
k & & \mathcal{B} & \text{:AS for } \equiv\mathrm{I} \\
 & & \vdots & \\
l & & \mathcal{A} & \\
 & \mathcal{A} \equiv \mathcal{B} & & : i\text{-}j,\ k\text{-}l \equiv\mathrm{I}
\end{array}
$$

- Like doing conditional intro twice, from both directions

Biconditional Introduction (≡I) (Type <-> !!!)

$$
\begin{array}{r|l|l}
i & \quad \mathcal{A} & \text{:AS for } \equiv\text{I} \\
 & \quad \vdots & \\
j & \quad \mathcal{B} & \\
 & -- & \\
k & \quad \mathcal{B} & \text{:AS for } \equiv\text{I} \\
 & \quad \vdots & \\
l & \quad \mathcal{A} & \\
 & \mathcal{A} \equiv \mathcal{B} & :i\text{-}j,\ k\text{-}l \equiv\text{I}
\end{array}
$$

- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs

Biconditional Introduction (≡I) (Type <-> !!!)

i	$\mathcal{A}$	:AS for $\equiv$I
	$\vdots$	
j	$\mathcal{B}$	
	– –	
k	$\mathcal{B}$	:AS for $\equiv$I
	$\vdots$	
l	$\mathcal{A}$	
	$\mathcal{A} \equiv \mathcal{B}$	:i-j, k-l $\equiv$I

- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs
- The subproofs need not be adjacent, but if they are, **separate with --**

Biconditional Introduction (≡I) (Type <-> !!!)

i	$\quad \mathcal{A}$	:AS for $\equiv$I
	$\quad \vdots$	
j	$\quad \mathcal{B}$	
	--	
k	$\quad \mathcal{B}$	:AS for $\equiv$I
	$\quad \vdots$	
l	$\quad \mathcal{A}$	
	$\mathcal{A} \equiv \mathcal{B}$	:i-j, k-l $\equiv$I

- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs
- The subproofs need not be adjacent, but if they are, **separate with --**
- Remember to cite BOTH subproofs (hyphens!)

Biconditional Introduction (≡I) (Type <-> !!!)

$$
\begin{array}{l|l|ll}
i & & \mathcal{A} & \text{:AS for } \equiv\text{I} \\
 & & \vdots & \\
j & & \mathcal{B} & \\
 & -- & & \\
k & & \mathcal{B} & \text{:AS for } \equiv\text{I} \\
 & & \vdots & \\
l & & \mathcal{A} & \\
 & \mathcal{A} \equiv \mathcal{B} & & :i\text{-}j,\ k\text{-}l \equiv\text{I}
\end{array}
$$

- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs
- The subproofs need not be adjacent, but if they are, **separate with --**
- Remember to cite BOTH subproofs (hyphens!)
- Remember to pop out of subproof line!

Biconditional Elimination ($\equiv$E) (Type <-> !!!)

$$
\begin{array}{r|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{A} & \\
 & \mathcal{B} & : m,\ n \equiv\mathrm{E}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{B} & \\
 & \mathcal{A} & : m,\ n \equiv\mathrm{E}
\end{array}
$$

- Just like conditional elimination!

Biconditional Elimination ($\equiv$E) (Type <-> !!!)

$$
\begin{array}{l|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{A} & \\
 & \mathcal{B} & : m,\ n \equiv\text{E}
\end{array}
$$

$$
\begin{array}{l|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{B} & \\
 & \mathcal{A} & : m,\ n \equiv\text{E}
\end{array}
$$

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)

Biconditional Elimination (≡E) (Type <-> !!!)

m	$\mathcal{A} \equiv \mathcal{B}$	
n	$\mathcal{A}$	
	$\mathcal{B}$	$: m, n \equiv$E

m	$\mathcal{A} \equiv \mathcal{B}$	
n	$\mathcal{B}$	
	$\mathcal{A}$	$: m, n \equiv$E

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)
- There can be lines between lines m and n

Biconditional Elimination ($\equiv$E) (Type <-> !!!)

$$
\begin{array}{r|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{A} & \\
 & \mathcal{B} & : m,\ n \equiv\text{E}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{A} \equiv \mathcal{B} & \\
n & \mathcal{B} & \\
 & \mathcal{A} & : m,\ n \equiv\text{E}
\end{array}
$$

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)
- There can be lines between lines m and n
- Remember to cite the lines of both (i) the biconditional and (ii) the side you have already

Biconditional Elimination (≡E) (Type <-> !!!)

$$\begin{array}{r|ll} m & \mathcal{A} \equiv \mathcal{B} & \\ n & \mathcal{A} & \\ & \mathcal{B} & : m,\ n \equiv\text{E} \end{array}$$

$$\begin{array}{r|ll} m & \mathcal{A} \equiv \mathcal{B} & \\ n & \mathcal{B} & \\ & \mathcal{A} & : m,\ n \equiv\text{E} \end{array}$$

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)
- There can be lines between lines m and n
- Remember to cite the lines of both (i) the biconditional and (ii) the side you have already
- Carnap issue: must type <->E

Issue with Typing $\equiv$ in Carnap

- For Carnap to recognize $\equiv$I or $\equiv$E in the justification column, you sadly must type <-> I or <-> E

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- It is still fine to type $<>$ for the biconditional symbol in the sentences

Issue with Typing $\equiv$ in Carnap

- For Carnap to recognize $\equiv$I or $\equiv$E in the justification column, you sadly must type $<->$ I or $<->$ E
- This is a bummer; I hope we can have it fixed (eventually)
- It is still fine to type $<>$ for the biconditional symbol in the sentences
- You can also copy/paste the $\equiv$symbol from elsewhere on the page!

6. Proofs in SL

h. Strategies and examples

Working forward and backward

- **Working backward** from a conclusion (goal) means:

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence

Working forward and backward

- **Working backward** from a conclusion (goal) means:
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- **Working backward** from a conclusion (goal) means:
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  - Write out (above the goal!) what you'd need to apply that rule

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule

- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule

- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)

Working forward and backward

- **Working backward** from a conclusion (goal) means:
  - Find main connective of goal sentence
  - Match with conclusion of corresponding **I**ntro rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)
  - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E
  - Work forward using $\supset$E, $\equiv$E

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\vee$I

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\vee$I
  - Try Negation Intro or Elimination, working toward a contradiction

Constructing a proof

- Write out premises at the top (if there are any)
- Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I/E, or forward using $\vee$E
  - Work forward using $\&$E
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\vee$I
  - Try Negation Intro or Elimination, working toward a contradiction
- Repeat for each new goal from top

6. Proofs in SL

i. The Rules, Reiterated

The rules, one more time: Reiteration

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
 & \vdots & \\
k & \mathcal{P} & :m\ \mathrm{R}
\end{array}
$$

- Remember that you must be in the scope of the line you're reiterating
- e.g. if you're outside a subproof, you can't reiterate anything wholly within the subproof

The rules: Conjunction Intro (&I) and Elimination (&E)

$$
\begin{array}{r|ll}
m & \mathcal{P} & \\
n & \mathcal{Q} & \\
 & \mathcal{P} \,\&\, \mathcal{Q} & : m,\ n\ \&\mathrm{I}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{P} & : m\ \&\mathrm{E}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \mathcal{P} \,\&\, \mathcal{Q} & \\
 & \mathcal{Q} & : m\ \&\mathrm{E}
\end{array}
$$

The rules: Conditional Intro (⊃I) and Elim (⊃E)

$$
\begin{array}{r|l l}
m & \quad \begin{array}{|l} \underline{\mathcal{P}} \end{array} & :\text{AS for } \supset\text{I} \\
 & \quad \begin{array}{|l} \vdots \end{array} & \\
n & \quad \begin{array}{|l} \mathcal{Q} \end{array} & \\
 & \mathcal{P} \supset \mathcal{Q} & : m\text{–}n \supset\text{I}
\end{array}
$$

$$
\begin{array}{r|l l}
m & \mathcal{P} \supset \mathcal{Q} & \\
n & \mathcal{P} & \\
 & \mathcal{Q} & : m,\ n \supset\text{E}
\end{array}
$$

The rules: Disjunction Intro (∨I) and Elimination (∨E)

m	$\mathcal{P} \vee \mathcal{Q}$	
i	$\quad \mathcal{P}$	:AS for ∨E
	$\quad \vdots$	
j	$\quad \mathcal{R}$	
	$--$	
k	$\quad \mathcal{Q}$	:AS for ∨E
	$\quad \vdots$	
ℓ	$\quad \mathcal{R}$	
	$\mathcal{R}$	:m, i–j, k–ℓ ∨E

m	$\mathcal{P}$	
	$\mathcal{P} \vee \mathcal{Q}$	:m ∨I

m	$\mathcal{Q}$	
	$\mathcal{P} \vee \mathcal{Q}$	:m ∨I

Remember that $\mathcal{P}$ can be the same wff as $\mathcal{Q}$

so can introduce $\mathcal{P} \vee \mathcal{P}$ from $\mathcal{P}$

The rules: Negation Intro and Elimination

Negation Intro ($\sim$I)

m	$\quad\mid\; \Phi$	:AS for $\sim$I
	$\quad\mid\; \vdots$	
n	$\quad\mid\; \Psi$	
	$\quad\mid\; \vdots$	
o	$\quad\mid\; \sim\Psi$	
	$\sim\Phi$	:m-o $\sim$I

Neg. Elimination ($\sim$E)

m	$\quad\mid\; \sim\Phi$	:AS for $\sim$E
	$\quad\mid\; \vdots$	
n	$\quad\mid\; \Psi$	
	$\quad\mid\; \vdots$	
o	$\quad\mid\; \sim\Psi$	
	Φ	:m-o $\sim$E

Note that you can swap the order of Ψ and $\sim\Psi$ in the subproofs!

The rules: Biconditional Intro and Elimination (<->)

$$
\begin{array}{l|l}
i & \quad \mathcal{A} \qquad :\text{AS for } \equiv\text{I} \\
 & \quad \vdots \\
j & \quad \mathcal{B} \\
 & -- \\
k & \quad \mathcal{B} \qquad :\text{AS for } \equiv\text{I} \\
 & \quad \vdots \\
l & \quad \mathcal{A} \\
 & \mathcal{A} \equiv \mathcal{B} \qquad : i\text{-}j,\ k\text{-}l \equiv\text{I}
\end{array}
$$

Biconditional Elimination ($\equiv$E)

$$
\begin{array}{l|l}
m & \mathcal{A} \equiv \mathcal{B} \\
n & \mathcal{A} \\
 & \mathcal{B} \qquad : m,\ n \equiv\text{E}
\end{array}
$$

$$
\begin{array}{l|l}
m & \mathcal{A} \equiv \mathcal{B} \\
n & \mathcal{B} \\
 & \mathcal{A} \qquad : m,\ n \equiv\text{E}
\end{array}
$$