What is Logic?

LOGIC I
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Motivations

Reasoning: Logic is the study of formal reasoning.

- By 'formal' we don't mean that it uses mathematical symbols.
- Rather, what follows from what *in virtue of logical form*.
- Abstracting from specific subject-matters, logic describes general patterns of reasoning that apply across the disciplines.

Normativity: Logic is not a *descriptive* science studying how human beings in fact reason across the various disciplines.

- Logic is a *normative* science, describing an especially strong form of reasoning that may serve as an ideal.

Artifical: We will primarily work in artificial languages where we will stipulate how to reason in these languages.

- Regimenting English will expose and remove ambiguities.
- We will provide proof systems for our artificial languages by which to compute what follows from what in a manner that vastly extends our natural cognitive capacities.

Interpretations

Proposition: We will begin with propositional logic where a PROPOSITION is a way for things to be which either obtains or does not.

Declarative Sentence: Given an interpretation of the language, an English sentence is DECLARATIVE just in case it expresses a proposition.

- Interrogative, imperative, and exclamatory sentences are not declarative sentences and typically do not have truth-values.
- We will restrict to declarative sentences throughout.

Truth-Values: A declarative sentence is TRUE in an interpretation if, given that interpretation, it expresses a proposition that obtains and FALSE in that interpretation otherwise.

Interpretations: We will only be concerned with the truth-values of sentences in this course, and so it is enough to take an INTERPRETATION to be an assignment of truth-values to sentences.

- This amounts to taking there to be just two propositions.

Examples

Deductive Argument: A DEDUCTIVE ARGUMENT in English is a nonempty sequence of declarative sentences where a single sentence is designated as the CONCLUSION (typically the last line) and all of the other sentences (if any) are the PREMISES.

Snow*: This argument may be compelling, but it is not certain.*

A1. It's snowing.
A2. John drove to work.

Red*: This argument provides certainty, but not on all interpretations.*

B1. The ball is crimson.
B2. The ball is red.

Museum*: This argument's certainty is independent of the interpretation.*

C1. Kate is either at home or at the Museum.
C2. Kate is not at home.
C3. Kate is at the Museum.

Informal Validity

Question 1: What goes wrong if we assume the premises but deny the conclusion in *Snow*, *Red*, and *Museum*?

Snow: Improbable but possible.

Red: Impossible on the intended interpretation.

Museum: Impossible on all interpretations so long as we hold the meanings of logical terms 'not' and 'or' fixed.

Task 1: Clarify what it is to hold the logical terms fixed.

Informal Interpretation: An INFORMAL INTERPRETATION assigns every declarative sentence of English to exactly one TRUTH-VALUE without offending the following informal semantic clauses:

- A *negation* is true just in case the negand is false.
- A *disjunction* is true just in case either disjunct is true.

Informal Validity: An argument in English is INFORMALLY VALID just in case its conclusion is true in every informal interpretation in which all of its premises are true.

Formal Languages

Problem 1: There is no set of all declarative sentences of English, and so no clear notion of an informal interpretation of English.

Suggestion: Could choose some large set of atomic English sentences, but this would be arbitrary and hard to specify precisely.

Solution 1: We will *regiment* English arguments in artificial languages that are both general and easy to specify precisely.

Propositional Language: The SENTENCES of $\mathcal{L}^{\text{PL}}$ are composed of SENTENCE LETTERS $A, B, C, \ldots$ and sentential operators $\neg$ and $\vee$.

Task 2: Regiment *Museum* in $\mathcal{L}^{\text{PL}}$: $H \vee M, \neg H \vDash M$.

- H = 'Kat is at home'.
- M = 'Kat is at the Museum'.

Task 3: Provide a way to interpret the sentences of $\mathcal{L}^{\text{PL}}$.

Schematic Variables: Let $\varphi, \psi, \ldots$ be variables with sentences of $\mathcal{L}^{\text{PL}}$ as values, and let $\Gamma, \Sigma, \ldots$ be variables for sets of sentences of $\mathcal{L}^{\text{PL}}$.

Interpretation: An INTERPRETATION $\mathcal{V}$ of $\mathcal{L}^{\text{PL}}$ assigns exactly one truth-value (1 or 0) to all sentences of $\mathcal{L}^{\text{PL}}$ where for any φ and ψ:

- $\mathcal{V}(\neg\varphi) = 1$ just in case $\mathcal{V}(\varphi) = 0$.
- $\mathcal{V}(\varphi \vee \psi) = 1$ just in case $\mathcal{V}(\varphi) = 1$ or $\mathcal{V}(\psi) = 1$ (or both).

Logical Consequence: $\Gamma \vDash \varphi$ just in case $\mathcal{V}(\varphi) = 1$ for any interpretation $\mathcal{V}$ of $\mathcal{L}^{\text{PL}}$ where $\mathcal{V}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Logical Validity: An argument is LOGICALLY VALID just in case its conclusion φ is a logical consequence of its set of premises Γ, i.e. $\Gamma \vDash \varphi$.

Task 4: Show that *Museum* is logically valid.

Logic

Model Theory: We have characterized logical reasoning as truth-preservation across a space of interpretations for an artificial language.

Proof Theory: Another approach focuses entirely on syntactic rules that specify which inferences in a language are logically valid.

- A system of basic rules for reasoning in an artificial language is referred to as a LOGIC for that language.

- By composing basic rules, we will define what counts as a PROOF in each of the logics that we will study.

Metalogic: Despite their differences, these two strategies will be shown to coincide for the languages that we will study in this book.

Logical Form

Picasso

D1. The painting is either a Picasso or a counterfeit and illegally traded.
D2. The painting is not a Picasso.
D3. The painting is a counterfeit and illegally traded.

Task 5: Regiment *Picasso* in $\mathcal{L}^{\text{PL}}$: $P \lor (Q \land R), \neg P \vDash Q \land R$.

- P = 'The painting is a Picasso'.
- Q = 'The painting is a counterfeit'.
- R = 'The painting is illegally traded'.

Question 2: How does this argument relate to *Museum*?

Logical Form: Both arguments are instances of $\varphi \lor \psi, \neg\varphi \vDash \psi$ which is a logically valid argument schema, i.e., all instances are valid.

Question 3: How many logically valid argument schemata are there, and how could we hope to describe this space?

Suggestion: The logical consequence relation $\vDash$ for $\mathcal{L}^{\text{PL}}$ describes the space of logically valid arguments, where the logically valid argument schemata are patterns in this space.

Problem 2: $\mathcal{L}^{\text{PL}}$ cannot regiment all logically valid arguments.

Socrates: Every man is mortal, Socrates is a man $\vDash$ Socrates is mortal.

- Our intuitive grasp on logical validity is not exhaustively captured by what we can regiment in $\mathcal{L}^{\text{PL}}$.

Solution 2: Rather, logical validity in $\mathcal{L}^{\text{PL}}$ provides a partial answer, where we may extend the language to provide a broader description of logical validity, e.g., $\mathcal{L}^{\text{FOL}}$.

- We will consider further extensions to $\mathcal{L}^{\text{FOL}}$ in later chapters.

Syntax for $\mathcal{L}^{\text{LP}}$

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Object Language and Metalanguage

Object Language: $\mathcal{L}^{\text{PL}}$ is the OBJECT LANGUAGE under study.

Metalanguage: Mathematical English is the METALANGAUGE with which we will conduct our study.

Quotation: To talk about $\mathcal{L}^{\text{PL}}$ we will take a quoted expression to be the CANONICAL NAME for the expression quoted.

Use/Mention: We MENTION expressions by putting them in quotes, whereas otherwise they are USED.

- 'Sue' is a nickname for Susanna.
- The complex sentence '$A \to B$' includes the sentence letters 'A' and 'B'.
- 'A' belongs to $\mathcal{L}^{\text{PL}}$, but " 'A' " and A do not.

The Expressions of $\mathcal{L}^{\text{PL}}$

Sentential Operators: '$\neg$','$\wedge$','$\vee$','$\to$', and '$\leftrightarrow$'.

- '$\sim$', '&', '.', '|', '$\supset$', and '$\equiv$' are also sometimes used.

Punctuation: '(' and ')'.

Sentence Letter: 'A_0', 'A_1', ..., 'B_0', 'B_1', ..., 'Z_0', 'Z_1', ...

Question: How can we specify all sentence letter explicitly?

- A SENTENCE LETTER is the result of subscripting a capital English letter with a numeral.

Corner Quotes: Let $\ulcorner \varphi_x \urcorner$ refer to the result of concatenating φ with $_x$.

- $\ulcorner \varphi_x \urcorner$ is a SENTENCE LETTER for any capital letter φ and numeral for a natural number x.

Primitive Symbols: The sentential operators, punctuation, and sentence letters are the PRIMITIVE SYMBOLS of $\mathcal{L}^{\text{PL}}$.

Expressions: The EXPRESSIONS of $\mathcal{L}^{\text{PL}}$ are defined recursively:

- The primitive symbol of $\mathcal{L}^{\text{PL}}$ are expression of $\mathcal{L}^{\text{PL}}$.
- If φ and ψ are expressions of $\mathcal{L}^{\text{PL}}$, then so is $\ulcorner \varphi\psi \urcorner$.
- Nothing else is an expression of $\mathcal{L}^{\text{PL}}$.

The Sentences of $\mathcal{L}^{\text{PL}}$

Uninterpretable: The expressions '$\neg\neg\neg\neg$', 'B_3A_0', '$))\leftrightarrow$', and '$A_4\vee$' cannot be assigned truth-values in a meaningful way.

- Compare 'MIT is in session' and '$A_4 \wedge P_1$'.

Well-Formed Sentences: Letting $\varphi, \psi, \chi, \ldots$ be variables with expressions for values, we may define the WFSS of $\mathcal{L}^{\text{PL}}$ as follows:

- Every sentence letter of $\mathcal{L}^{\text{PL}}$ is a wfs of $\mathcal{L}^{\text{PL}}$.
- If the expressions φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$, then:
  1. $\ulcorner\neg\varphi\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  2. $\ulcorner(\varphi \wedge \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  3. $\ulcorner(\varphi \vee \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$;
  4. $\ulcorner(\varphi \rightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$; and
  5. $\ulcorner(\varphi \leftrightarrow \psi)\urcorner$ is a wff of $\mathcal{L}^{\text{PL}}$.
- Nothing else is a wff of $\mathcal{L}^{\text{PL}}$.

Sentential Variables: We will often restrict 'φ', 'ψ', 'χ',... to the wfs of $\mathcal{L}^{\text{PL}}$.

Main Operator: The MAIN OPERATOR is the last operator used in the construction of a sentence.

Arguments: The inputs to a main operator are its ARGUMENTS.

Scope: The main operator has SCOPE over its arguments.

Metalinguistic Conventions

Subscripts: We will suppress the subscript '$_0$' to ease exposition.

Task: Build increasingly complex sentences from just A.

Naming: We will refer to the NEGAND in a NEGATION, the CONJUNCTS in a CONJUNCTION, the DISJUNCTS in a DISJUNCTION, the ANTECEDENT and CONSEQUENT in a MATERIAL CONDITIONAL, and the ARGUMENTS in a MATERIAL BICONDITIONAL.

Quotation: We will sometimes drop quotes and corner quotes when the intended meaning is clear from the context.

- We will only do so when this improves readability.

Punctuation: We will drop outermost parentheses for ease.

- Compare $A \wedge B$, $A \vee B \vee C$, and $A \vee B \wedge C$.

Therefore: We will use '$\because$' for inline arguments.

Metalinguistic: These abbreviations all happen in the metalanguage.

Truth Functionality

Interpretations: Improving on last time, an INTERPRETATION $\mathcal{I}$ is an assignment of truth-values to sentence letters of $\mathcal{L}^{\text{PL}}$.

Valuation: We may then define a VALUATION function $\mathcal{V}_{\mathcal{I}}$ which assigns truth-values to every sentence of $\mathcal{L}^{\text{PL}}$ by way of the following semantic clauses:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Observe: These clauses resemble the composition rules for $\mathcal{L}^{\text{PL}}$.

Homophonic Semantics: The clauses for $\neg$, $\wedge$, and $\vee$ use analogous operators in the metalanguage, but not so for $\rightarrow$ and $\leftrightarrow$.

Truth Tables: Use the semantics to fill out the TRUTH TABLES below:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Truth Functions: The sentential operators express truth-functions, and so are often called TRUTH-FUNCTIONAL OPERATORS.

Question: How many unary/binary truth-functions are there?

Adequacy: Given these limitations, what should we hope to be able to adequately regiment in $\mathcal{L}^{\text{PL}}$?

Logical Truths: φ is a LOGICAL TRUTH of $\mathcal{L}^{\text{PL}}$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.

Regimentation

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From Last Time...

Definitions: Here is slightly different take on the same definitions:

Well-Formed Sentences: The set WFSS of $\mathcal{L}^{\text{PL}}$ is the smallest set to satisfy:

- φ is a wfs of $\mathcal{L}^{\text{PL}}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$;
- $\neg\varphi$ is a wfs of $\mathcal{L}^{\text{PL}}$ if φ is a wfs of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \wedge \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \vee \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \rightarrow \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$;
- $(\varphi \leftrightarrow \psi)$ is a wff of $\mathcal{L}^{\text{PL}}$ if φ and ψ are wfss of $\mathcal{L}^{\text{PL}}$.

Semantics: For an interpretation $\mathcal{I}$, a VALUATION function $\mathcal{V}_\mathcal{I}$ is the smallest function to assign truth-values to every sentence of SL that satisfies the semantic clauses:

- $\mathcal{V}_\mathcal{I}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_\mathcal{I}(\neg\varphi) = 1$ iff $\mathcal{V}_\mathcal{I}(\varphi) = 0$ (i.e., $\mathcal{V}_\mathcal{I}(\varphi) \neq 1$).
- $\mathcal{V}_\mathcal{I}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_\mathcal{I}(\varphi) = 1$ and $\mathcal{V}_\mathcal{I}(\psi) = 1$.
- $\mathcal{V}_\mathcal{I}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_\mathcal{I}(\varphi) = 1$ or $\mathcal{V}_\mathcal{I}(\psi) = 1$ (or both).
- $\mathcal{V}_\mathcal{I}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_\mathcal{I}(\varphi) = 0$ or $\mathcal{V}_\mathcal{I}(\psi) = 1$ (or both).
- $\mathcal{V}_\mathcal{I}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_\mathcal{I}(\varphi) = \mathcal{V}_\mathcal{I}(\psi)$.

Observe: Observe the symmetry between the above.

Recall: The hierarchy of sentences from before...

Complexity

Complexity: $\texttt{Comp}(\varphi)$ is the smallest function to satisfy all of the following conditions for all wfss φ and ψ of $\mathcal{L}^{\text{PL}}$:

- $\texttt{Comp}(\varphi) = 0$ if φ is a sentence letter;
- $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$;
- $\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;
- ...

Question: Do we need to include corner quotes?

Validity

$\mathcal{L}^{\text{PL}}$ Validity: An argument in $\mathcal{L}^{\text{PL}}$ is *valid* iff its conclusion is a logical consequence of its premises.

English Validity: An argument in English is *valid* iff it has a (faithful) regimentation (in some language) that is valid.

- Note the imprecision here; there is no avoiding this.

Soundness: An argument is *sound* iff it is valid and has true premises (on an interpretation we care about, probably the intended interpretation).

Examples

Rain

1. If it is raining on a week day, Sam took his car.
2. Kate borrowed Sam's car only if Sam did not take it.
3. Kate borrowed Sam's car just in case she visited her parents.
4. It is raining and Kate visited her parents.
5. Either it is not a week day or it is not raining.

Task 2: Regiment this argument and construct its truth table.

Observe: This argument can be adequately regimented and evaluate in SL.

Negation

Uninitiated

A1. If Sam attended the gathering, then he has been initiated.
A2. Sam is uninitiated.
A3. Sam did not attend the gathering.

Observe: Being uninitiated is the same as not being initiated.

Uninvited

B1. Arden is not invited.
B2. Arden is uninvited.

Observe: Arden can fail to be invited without being uninvited.

Question: What about the converse?

Disjunction

Party

C1. If Adi or James make it to the party, Isa will be happy.
C2. If Adi and James make it to the party, Isa will be happy.

Observe: This argument suggests an inclusive reading of 'or'.

Race

D1. Sasha won the 100 meter dash.
D2. Josh won the high jump.
D3. Either Sasha won the 100 meter dash or Josh won the high jump

Observe: We could strengthen the conclusion.

Vault

E1. If Kin uses the remote, the trunk will open.
E2. If Yu tries the handle, the trunk will open.
E3. If Kin uses the remote and Yu tries the handle, the trunk won't open.
E4. If Kin uses the remote or Yu tries the handle, the trunk will open.

Observe: We cannot regiment the conclusion with inclusive-'or'.

Question: Can we salvage the validity of this argument?

Conjunction

Exam

F1. Henry failed and Megan passed.
F2. Megan passed and Henry failed.

Observe: Perfectly adequate and valid regimentation.

Gym

G1. Kate took a shower and went to the gym.
G2. Kate went to the gym and took a shower.

Observe: Conjunction in English can track temporal order.

Question: How can we capture the invalidity of this argument in $\mathcal{L}^{\text{PL}}$?

Logical Consequence

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From Last Time...

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \to \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Characteristic Truth Tables: As drawn in the textbook...

Complete Truth Tables

Setup: Write the sentence on the top right, add the constituent sentence letters on the left, and use the characteristic truth tables.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ:

- $[\varphi] = \{\varphi\}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- For any wfss φ and ψ of $\mathcal{L}^{\text{PL}}$, and $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$;
  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$;

Rows: Add 2^n rows for n constituent sentence letters.

Examples: $[A \wedge (B \vee A)] \to A$, $C \leftrightarrow \neg C$, D.

Tautology: Only 1s under its main connective in its complete truth table.

Contradiction: Only 0s under its main connective in its complete truth table.

Logically Contingent: A 1 and a 0 under its main connective in its complete truth table.

Logical Entailment: On any row of a complete truth table, the consequent has a 1 under its main connective whenever the antecedent does.

Logical equivalence: Identical columns under the main connectives for the sentences.

Satisfiable: There is a row where all wfss have a 1 under all main connectives.

Logical Consequence: The conclusion has a 1 under its main connective in every row in which every premise has a 1 under its main connectives.

Decidability

Effective Procedure: A finitely describable and (in principle) usable procedure that always finishes and produces a correct answer to the question asked, requiring only that the instructions be followed accurately.

Question: How to define the main operators and distribute truth-values?

- Recursively, like the formation rules for the wfs of $\mathcal{L}^{\mathrm{PL}}$.

Question: Is it always possible to construct a complete truth table for a wfs?

- Sentences have a finite number of constituent sentence letters.

Decidable: If there is an effective procedure for determining the answer to a question, that question is *decidable*.

- It is decidable whether a wfs of $\mathcal{L}^{\mathrm{PL}}$ is a tautology, etc.

Question: What about a complete truth table for a set of sentences?

- Could require infinitely many sentence letters.
- We might be able to define an infinite table, but we can't use it.

Question: If one procedure is not effective, couldn't there be another one?

- It turns out that there is no effective procedure…
- There is always an effective procedure for finite sets of sentences.

Validity: So the validity of finite arguments is decidable.

Partial Truth Tables

Worry 1: It is not *that* effective… in practice it is daunting for $n > 4$.

Partial Truth Tables: Sometimes only one or two lines are needed.

- $A \rightarrow \neg(A \vee B)$: not a tautology or contradiction, so contingent.
- $B \leftrightarrow \neg(A \vee B)$ is a contradiction, so we need a complete table.
- $C \vee (A \rightarrow A)$ is a tautology, so we need a complete table.

Complete: To affirm equivalence, entailment, and logical consequence.

Partial: To affirm that a set is satisfiable.

Worry 2: Still daunting sometimes.

Worry 3: Definitions all refer to complete truth tables.

- Definition of a complete truth table has some minor ambiguities.
- These could be fixed, but the result is cumbersome.

Heuristic: The truth table definitions are best taken to be a heuristic guide for grasping the abstract definitions we may now provide.

Semantic Proofs

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From Before...

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Formal Definitions

Interpretation: $\mathcal{I}$ is an *interpretation* of $\mathcal{L}^{\text{PL}}$ *iff* $\mathcal{I}(\varphi) \in \{1, 0\}$ for every sentence letter φ of $\mathcal{L}^{\text{PL}}$.

Tautology: φ is a *tautology iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.

Contradiction: φ is a *contradiction iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ for all $\mathcal{I}$.

Logically Contingent: φ is *contingent iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \neq \mathcal{V}_{\mathcal{J}}(\varphi)$ for some $\mathcal{I}$ and $\mathcal{J}$.

Logical Entailment: φ *entails* ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \leq \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Logical Equivalence: φ is *equivalent* to ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Satisfiable: Γ is *satisfiable iff* $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$ for some $\mathcal{I}$.

Logical Consequence: $\Gamma \vDash \varphi$ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ whenever $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Satisfiability

Which sets of sentences are satisfiable?

Taller

(1) Liza is taller than Sue.

(2) Sue is taller than Paul.

(3) Paul is taller than Liza.

Lost

(4) Kim is either in Somerville or Cambridge.

(5) If Kim is in Somerville, then she is not far from home.

(6) If Kim is not far from home, then she is in Cambridge.

(7) Kim is not in Cambridge.

Validity

Arguments: Sequences of wfss of $\mathcal{L}^{\text{PL}}$, not sets.

Valid: For any argument, it is valid *iff* its conclusion is a logical consequence of its set of premises.

- Many arguments may have the same set of premises.
- An argument is valid *iff* its conclusion is true in every interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ to satisfy the set of premises.

Tautology: A wfs φ of $\mathcal{L}^{\text{PL}}$ is a *tautology* just in case $\vDash \varphi$.

- Every $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ satisfies the empty set.
- Each premise constrains the set of interpretations the conclusion must be true in where the empty set has no constraints.

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

- Each wfs of $\mathcal{L}^{\text{PL}}$ corresponds to a set of all interpretations which make that sentence true: $|\varphi| := \{\mathcal{I} : \mathcal{V}_{\mathcal{I}}(\varphi) = 1\}$.
- Is the interpretation set for the conclusion a subset of the intersection of the premise interpretation sets?

Examples

1. Show that $\neg R \to \neg Q,\ P \wedge Q \vDash P \wedge R$.
2. Show that $A \vee B,\ B \to C,\ A \leftrightarrow C \vDash C$.
3. Show that $P,\ P \to Q,\ \neg Q \vDash A$.
4. Show that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology.
5. Show that $A \leftrightarrow \neg A$ is a contradiction.
6. Show that $\{P,\ P \to Q,\ Q \to \neg P\}$ is unsatisfiable.
7. Show that $\{P \to Q,\ \neg P \vee \neg Q,\ Q \to P\}$ is satisfiable.

Observe: There seem to be patterns.

Question: How could we systematize these proofs?

Methods

Truth Tables: Mechanical but tedious.

- Bad if there are lots of sentence letters.
- Good for counterexamples.
  $A \leftrightarrow (B \to C),\ A \wedge \neg B,\ D \vee \neg A \ \vDash C.$

Semantic Arguments: Good if there are lots of sentence letters.
$(A \vee B) \to (C \wedge D),\ \neg C \wedge \neg E \ \vDash \neg A.$

The Material Conditional

Roses

A1. Sugar is sweet.
A2. The roses are only red if sugar is sweet.

Observe: First paradox of the material conditional.

Vacation

B1. Casey is not on vacation.
B2. If Casey is on vacation, then he is in Paris.

Observe: Second paradox of the material conditional.

Crimson

C1. Mary doesn't like the ball unless it is crimson.
C2. Mary likes the ball.
C3. If the ball is blue, then Mary likes it.

The Biconditional

Rectangle

D1. The room is a square.
D2. The room is a rectangle.
D3. The room is a square if and only if it is a rectangle.

Work

E1. Kin isn't a professor.
E2. Sue isn't a chef.
E3. Kin is a professor just in case Sue is a chef.

Natural Deduction in PL: Part I

LOGIC I
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October 1, 2024

Review from Last Time…

1. Show that $A \vee B,\ B \to C,\ A \leftrightarrow C \vDash C$.
2. Show that $\{P,\ P \to Q,\ Q \to \neg P\}$ is unsatisfiable.
3. Show that $\{P \to Q,\ \neg P \vee \neg Q,\ Q \to P\}$ is satisfiable.

Motivation

Homophonic: Prove that $P \vee Q,\ \neg P \vDash Q$.

- The semantic proof makes the same inference.
- So why not just draw this inference directly in $\mathcal{L}^{\text{PL}}$?
- What are the basic steps we are allowed to make in a proof?

Semantic Proofs: Provide a reasonably efficient way to evaluate validity.

- But they can be cumbersome to write.
- They explain why a logical property or relation holds.
- Doesn't say how to reason from some premises to a conclusion.
- Thus semantic proofs are not persuasive to the uninitiated.
- Not so for semantic proofs of invalidity, satisfiability, etc.

Logical Consequence: How do we describe the extension of $\vDash$?

Natural Deduction: How should we describe the patterns of natural deduction?

- What moves can we make in a proof, *viz.* semantic proofs?
- Want to describe inference itself, starting with the most basic.
- Such inferences hold in virtue of the meanings of the operators.
- Define a proof to be any composition of basic inferences.

Rules: Each operator will have an introduction and elimination rule.

- These rules will describe how to reason with the connectives.
- Want these rules to be valid.
- Also want these rules to be natural.

Metalogic:

- This is a completely different approach to formal reasoning.
- Nevertheless, these two approaches have the same extension.
- Our proof system will help us relate to logical consequence.

Basic Anatomy of a Proof

List: Finite list of lines.

Numbers: Every line is numbered.

Sentences: Each line contains exactly one wfs of $\mathcal{L}^{\text{PL}}$.

Justification: Each line includes a justification.

Assumptions: The justification for a premise is ':PR'.

Bars: A horizontal bar separates the premises from the steps in the proof.

Conclusion: The last line is the conclusion.

Conditional

Elimination: $A,\ A \rightarrow B,\ B \rightarrow C\ \vdash C$.

- Easy to derive C using $\rightarrow$E.
- What if A was excluded from the premises?

Introduction: $A \rightarrow B,\ B \rightarrow C\ \vdash A \rightarrow C$.

- Need something to work with.
- Want to conclude with a conditional claim.
- Assumption of A justified by ':AS'.

Subproofs: Lines in a closed subproof are dead and all else are live.

- $\rightarrow$E can only cite to live lines.
- $\rightarrow$I can only cite an appropriate subproof.

Assumption

Example: $A\ \vdash D \rightarrow [C \rightarrow (B \rightarrow A)]$.

Conjunction

Elimination: $A \rightarrow (B \wedge C),\ B \rightarrow D\ \vdash A \rightarrow D$.

Introduction: $A \wedge B,\ B \rightarrow C\ \vdash A \wedge C$.

Disjunction

Introduction: $A \vdash B \vee ((A \vee C) \vee D)$.

Elimination: $A \vee (B \wedge C)\ \vdash (A \vee B) \wedge (A \vee C)$.

Natural Deduction in PL: Part II

LOGIC I
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October 1, 2024

Biconditional

Elimination: $A \leftrightarrow (B \to [(A \wedge C) \leftrightarrow D]) \vdash (A \wedge B) \to (D \to C)$.

Introduction: $A \to (B \wedge C),\ C \to (B \wedge A) \vdash A \leftrightarrow C$.

Negation and Reiteration

Elimination Rule: $\neg\neg A \vdash A$. (*Double Negation Elimination*)

1. $A \vee \neg A$. (*Law of Excluded Middle*)
2. $A,\ \neg A \vdash B$. (*Ex Falso Quodlibet*)

Introduction Rule: $\neg(A \wedge \neg A)$. (*Law of Non-Contradiction*)

3. $A \vdash \neg\neg A$. (*Double Negation Introduction*)

Proof

Proof: A natural deduction DERIVATION (or PROOF) of a conclusion φ from a set of premises Γ in PL is any finite sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) a premise in Γ;

(2) a discharged assumption; or

(3) follows from previous lines by the rules for PL.

Provable: An wfs φ of $\mathcal{L}^{\text{PL}}$ is DERIVABLE (or PROVABLE) from Γ in PL (i.e., $\Gamma \vdash \varphi$) *iff* there is a natural deduction derivation (proof) of φ from Γ in PL.

Theorem: A wfs φ is a *theorem* of PL (often written $\varphi \in$ PL) *iff* $\vdash \varphi$.

Interderivable: Two wfss φ and ψ of $\mathcal{L}^{\text{PL}}$ are INTERDERIVABLE (i.e., $\varphi \dashv\vdash \psi$) *iff* both $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Bottom: We take $\bot := A \wedge \neg A$ to abbreviate an arbitrarily chosen contradiction.

Inconsistent: A set of sentences Γ is INCONSISTENT if and only if $\Gamma \vdash \bot$.

Logical Analysis

Sound and Complete: $\Gamma \vdash \varphi$ *iff* $\Gamma \vDash \varphi$.

- $\vdash \varphi$ *iff* $\vDash \varphi$.
- $\Gamma \vdash \bot$ *iff* $\Gamma \vDash \bot$.

Question: How can we tell if an argument is valid?

- Construct a truth table.
- Write a semantic proof.
- Derive the conclusion from the premises.

Question: What if we can mange to find a derivation?

- Natural deduction won't tell you if there is no proof.
- A semantic proof will yield a counterexample.

Question: How can we tell what the logical properties are for a wfs of $\mathcal{L}^{\text{PL}}$?

Tautology?	If YES, prove $\vdash \varphi$.	If NO, provide a countermodel.
Contradiction?	If YES, prove $\vdash \neg\varphi$.	If NO, provide a model.
Contingent?	If YES, provide two models.	If NO, prove $\vdash \varphi$ or $\vdash \neg\varphi$.
Equivalent?	If YES, prove $\varphi \dashv\vdash \psi$.	If NO, provide a countermodel.

Rule Schemata

Task: Compare the rules of inference for PL to their instances.

- Whereas the rules are general, PL proofs are particular.
- But nothing in our PL proofs depend on the particulars.

Question: How might we generalize our proofs beyond any instance?

Rule Schemata: Replace sentence letters in PL proofs with schematic variables.

- Premises are replaced with the lines cited by that rule.
- New rules require new names if we are to use them.

Question: Can we also generalize proofs of theorems?

- These amount to lines that can be added without citing lines.

Derived Schemata: To speed up proofs, we want to derive rule schemata.

- These can then be employed just like our basic rules.
- This avoids having to rewrite the same types of proofs over and over.

Derivable Schemata

Law of Excluded Middle: $\vdash \varphi \vee \neg\varphi$.

Law of Non-Contradiction: $\vdash \neg(\varphi \wedge \neg\varphi)$.

Ex Falso Quodlibet: $\varphi, \neg\varphi \vdash \psi$.

Hypothetical Syllogism: $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$.

Modus Tollens: $\varphi \to \psi, \neg\psi \vdash \neg\varphi$.

Contraposition: $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$.

Dilemma: $\varphi \vee \psi, \varphi \to \chi, \psi \to \chi \vdash \chi$.

Disjunctive Syllogism: $\varphi \vee \psi, \neg\varphi \vdash \psi$.

$\vee$*-Commutativity:* $\varphi \vee \psi \vdash \psi \vee \varphi$.

$\wedge$*-Commutativity:* $\varphi \wedge \psi \vdash \psi \wedge \varphi$.

Biconditional MP: $\varphi \leftrightarrow \psi, \neg\varphi \vdash \neg\psi$.

$\leftrightarrow$*-Commutativity:* $\varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi$.

Double Negation: $\neg\neg\varphi \dashv\vdash \varphi$.

$\wedge$*-De Morgan's:* $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.

$\vee$*-De Morgan's:* $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.

$\vee\wedge$*-Distribution:* $\varphi \vee (\psi \wedge \chi) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$.

$\wedge\vee$*-Distribution:* $\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

$\vee\wedge$*-Absorption:* $\varphi \vee (\varphi \wedge \psi) \dashv\vdash \varphi$.

$\wedge\vee$*-Absorption:* $\varphi \wedge (\varphi \vee \psi) \dashv\vdash \varphi$.

$\wedge$*-Associativity:* $\varphi \wedge (\psi \wedge \chi) \dashv\vdash (\varphi \wedge \psi) \wedge \chi$.

$\vee$*-Associativity:* $\varphi \vee (\psi \vee \chi) \dashv\vdash (\varphi \vee \psi) \vee \chi$.

Mathematical Induction

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From Last Time...

Bottom: We take $\bot := A \wedge \neg A$ to abbreviate an arbitrarily chosen contradiction.

Inconsistent: A set of wfss Γ of $\mathcal{L}^{\text{PL}}$ is INCONSISTENT if and only if $\Gamma \vdash \bot$.

Ex Falso Quodlibet: $\varphi, \neg\varphi \vdash \psi$.

Recursive Definitions

Expressions: The expressions of $\mathcal{L}^{\text{PL}}$ are defined recursively:

- The primitive symbol of $\mathcal{L}^{\text{PL}}$ are expression of $\mathcal{L}^{\text{PL}}$.
- If φ and ψ are expressions of $\mathcal{L}^{\text{PL}}$, then so is $\ulcorner\varphi\psi\urcorner$.
- Nothing else is an expression of $\mathcal{L}^{\text{PL}}$.

Complexity: $\texttt{Comp}(\varphi)$ is the number of operator instances that occur in φ:

- $\texttt{Comp}(\varphi) = 0$ if φ is a sentence letter;
- $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$; and
- $\texttt{Comp}(\varphi \star \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$ for $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Constituents: $[\varphi]$ is the set of sentence letters that occur in φ:

- $[\varphi] = \{\varphi\}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $[\neg\varphi] = [\varphi]$; and
- $[\varphi \star \psi] = [\varphi] \cup [\psi]$ if $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Simplicity: $\texttt{Simple}(\varphi)$ just in case the φ has at most one sentence letter in $\mathcal{L}^{\text{PL}}$:

- $\texttt{Simple}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$; and
- $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$.

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If φ is a sentence letter, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$; and
- $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$ if $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Induction Guide

Step 1: Identify the set of elements and the property in question.

Step 2: Partition the set into a sequence of stages to run induction on.

Step 3: Establish that every element in the base stage has the property.

Step 4: Assume every element in stage n (and below) have the property.

Step 5: Show that every element in stage $n+1$ have the property.

Examples

Task 1: Every wfs of $\mathcal{L}^{\text{PL}}$ has an even number of parentheses.

Task 2: All expressions of $\mathcal{L}^{\text{PL}}$ are finite length.

Task 3: If $\mathcal{I}(\varphi) = \mathcal{J}(\varphi)$ for all $\varphi \in [\psi]$, then $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{J}}(\psi)$.

Task 4: For every wfs φ of $\mathcal{L}^{\text{PL}}$, if $\texttt{Simple}(\varphi)$, then $\nvDash \varphi$.

Task 5: For any wfss φ, ψ, χ and sentence letter α of $\mathcal{L}^{\text{PL}}$, if $\vDash \varphi \leftrightarrow \psi$, then $\vDash \chi_{[\varphi/\alpha]} \leftrightarrow \chi_{[\psi/\alpha]}$.

PL Soundness

- Assume $\Gamma \vdash \varphi$ for an arbitrary set wfss Γ and wfs φ of $\mathcal{L}^{\text{PL}}$.
- There is some PL derivation X of φ from Γ.
- Let φ_i be the wfs on the i-th line of the derivation X.
- Let Γ_i be the set of premises and undischarged assumptions on $j \leq i$.

Base Case: $\Gamma_1 \vDash \varphi_1$.

- φ_1 is either a premise or undischarged assumption.
- Either way, $\Gamma_1 = \{\varphi_1\}$ since φ_1 is not discharged at the first line.
- $\Gamma_1 \vDash \varphi_1$ is immediate.

Induction Step: $\Gamma_{n+1} \vDash \varphi_{n+1}$ if $\Gamma_k \vDash \varphi_k$ for every $k \leq n$. (To be proven separately.)

- By strong induction, $\Gamma_n \vDash \varphi_n$ for all n.
- Since every proof is finite in length, there is a last line m of X where $\varphi_m = \varphi$ is the conclusion.
- Since every assumption in X is eventually discharged, $\Gamma_m = \Gamma$ is the set of premises.
- Thus $\Gamma \vDash \varphi$.

Lemmas

(AS) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by AS.

- Assume that φ_{n+1} is justified by AS.
- So φ_{n+1} is an undischarged assumption at line $n+1$.
- So $\varphi_{n+1} \in \Gamma_{n+1}$ by the definition of Γ_{n+1}.
- $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows immediately.

Inheritance: If φ_k is live at line n of a PL derivation where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

- Let X be a PL derivation.
- Assume there is some $\psi \in \Gamma_k$ where $\psi \notin \Gamma_n$ for $n > k$.
- So ψ has been discharged at a line $j > k$ where $j \leq n$.
- So φ_k is dead at n.
- By contraposition, if φ_k is live at line $n > k$, then $\Gamma_k \subseteq \Gamma_n$ as desired.

(R) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by R.

- Assume that φ_{n+1} is justified by R.
- So $\varphi_{n+1} = \varphi_k$ for some $k \leq n$.
- By hypothesis, $\Gamma_k \vDash \varphi_k$.
- Since φ_k is live at line $n+1$, $\Gamma_k \subseteq \Gamma_{n+1}$ by *Inheritance* (**Lemma 4.3**).
- So $\Gamma_{n+1} \vDash \varphi_k$ by *Weakening* (**Lemma 2.1**).
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$.

PL Soundness

LOGIC I
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October 3, 2024

Lemmas

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

Inheritance: If φ_k is live at line n of a PL derivation where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

Interpretation: If $\mathcal{I}$ is a $\mathcal{L}^{\text{PL}}$ interpretation, then $\mathcal{V}_\mathcal{I}(\varphi) \in \{1, 0\}$ for all wfss φ of $\mathcal{L}^{\text{PL}}$.

Contradiction: If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

- Assume $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$.
- Assume for contradiction that Γ is satisfiable.
- There is some $\mathcal{L}^{\text{PL}}$ interpretation $\mathcal{I}$ where $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- By assumption, $\mathcal{V}_\mathcal{I}(\varphi) = 1$ and $\mathcal{V}_\mathcal{I}(\neg\varphi) = 1$.
- By the semantics for negation, $\mathcal{V}_\mathcal{I}(\varphi) \neq 1$, contradicting the above.
- Thus Γ is unsatisfiable.

Unsatisfiable: If $\Gamma \cup \{\varphi\}$ is unsatisfiable, then $\Gamma \vDash \neg\varphi$.

- Assume $\Gamma \cup \{\varphi\}$ is unsatisfiable.
- Let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- Assume for contradiction that $\mathcal{V}_\mathcal{I}(\neg\varphi) = 0$.
- So $\mathcal{V}_\mathcal{I}(\varphi) = 1$, and so $\Gamma \cup \{\varphi\}$ is satisfiable contrary to assumption.
- Thus for any $\mathcal{I}$, $\mathcal{V}_\mathcal{I}(\neg\varphi) = 1$ if $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- By definition, $\Gamma \vDash \neg\varphi$.

Conditional: If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \rightarrow \psi$.

- Assume $\Gamma \cup \{\varphi\} \vDash \psi$.
- Let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- Since $\mathcal{V}_\mathcal{I}(\varphi) \in \{1, 0\}$ by *Interpretation*, there are two cases to consider.

Case 1: Assume $\mathcal{V}_\mathcal{I}(\varphi) = 1$.
  - So $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma \cup \{\varphi\}$.
  - So $\mathcal{V}_\mathcal{I}(\psi) = 1$ by the starting assumption.
  - Thus $\mathcal{V}_\mathcal{I}(\varphi \rightarrow \psi) = 1$ by the semantics for the conditional.

Case 2: Assume $\mathcal{V}_\mathcal{I}(\varphi) = 0$.
  - So $\mathcal{V}_\mathcal{I}(\varphi \rightarrow \psi) = 1$ by the semantics for the conditional.

- So $\mathcal{V}_\mathcal{I}(\varphi \rightarrow \psi) = 1$ in both cases.
- Thus $\Gamma \vDash \varphi \rightarrow \psi$ follows by generalizing on $\mathcal{I}$.

PL Deduction Rules

Induction Hypothesis: Recall the assumption that $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

(¬I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\neg$I.

- There is a subproof from φ on line i with ψ at line j and $\neg\psi$ at line k.
- By hypothesis $\Gamma_j \vDash \psi$ and $\Gamma_k \vDash \neg\psi$, where $\Gamma_j, \Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By *Weakening*, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$.
- So $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable by *Contradiction*.
- So $\Gamma_{n+1} \vDash \varphi_{n+1}$ by *Unsatisfiable*.

(∧I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\wedge$I.

- $\varphi_{n+1} = \varphi_i \wedge \varphi_j$ where lines $i, j \leq n$ are live at $n+1$.
- By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$.
- By *Inheritance*, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$.
- By *Weakening*, $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$.
- Let $\mathcal{I}$ be a $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$.
- So $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_i \wedge \varphi_j) = 1$ by the semantics.
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ by generalizing on $\mathcal{I}$.

(→I) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$I.

- So $\varphi_{n+1} = \varphi_i \rightarrow \varphi_j$, where there is a subproof of φ_j from φ_i.
- By hypothesis $\Gamma_j \vDash \varphi_j$, where $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By *Weakening*, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$.
- By *Conditional*, $\Gamma_{n+1} \vDash \varphi_i \rightarrow \varphi_j$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

(→E) *Proof:* $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$E.

- So $\varphi_i = \varphi_j \rightarrow \varphi_{n+1}$ where the lines $i, j \leq n+1$ are live at $n+1$.
- By hypothesis $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$.
- By *Inheritance*, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$.
- By *Weakening*, $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$, and so $\Gamma_{n+1} \vDash \varphi_j \rightarrow \varphi_{n+1}$.
- Let $\mathcal{I}$ be a $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ and $\mathcal{V}_{\mathcal{I}}(\varphi_j \rightarrow \varphi_{n+1}) = 1$.
- By the semantics, $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 0$ or $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$.
- To avoid contradiction, $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$.
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows from by generalizing on $\mathcal{I}$.

Consistency

Corollary: If Γ is inconsistent, then Γ is unsatisfiable.

- Assume Γ is inconsistent, so $\Gamma \vdash \bot$.
- Thus $\Gamma \vDash \bot$ follows by PL SOUNDNESS.
- Assume for *reductio* that Γ is satisfiable.
- So $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.
- So $\mathcal{V}_{\mathcal{I}}(\bot) = 1$, i.e., $\mathcal{V}_{\mathcal{I}}(A \wedge \neg A) = 1$.
- By the semantics, $\mathcal{V}_{\mathcal{I}}(A) = 1$ and $\mathcal{V}_{\mathcal{I}}(\neg A) = 1$, so $\mathcal{V}_{\mathcal{I}}(A) \neq 1$.
- By *reductio*, Γ is unsatisfiable.

Contrapositive: If Γ is satisfiable, then Γ is consistent.

- The inconsistency of Γ may be witnessed by a derivation of $\bot$ from Γ.
- There are no witnesses that $\bot$ can't be derived from a consistent set.
- We would somehow need to survey the space of all derivations.
- Could try a reductio, but this is hardly promising.
- Rather, we need only find an interpretation to witness satisfiability.

Theorems: How do we know that the theorems of PL are consistent?

- Because every theorem is a tautology by PL SOUNDNESS.
- So every interpretation witnesses the truth of all of the theorems.
- So the set of theorems are indeed consistent.
- Otherwise we could derive anything from nothing.

Strength: Let $(\varphi) := \{\chi : \varphi \vdash \chi\}$ be the wfs of $\mathcal{L}^{\text{PL}}$ derivable from φ.

- We may show that $(\psi) \subseteq (\varphi)$ if $\varphi \vdash \psi$.
- So (φ) provides a way to think about the STRENGTH of φ.
- Observe that $\varphi \in (\bot)$ for every wfs φ of $\mathcal{L}^{\text{PL}}$.
- Strength is good, but not if it explodes into inconsistency.

More Derivations

Hypothetical Syllogism: $\varphi \to \psi,\ \psi \to \chi\ \vdash \varphi \to \chi$.

Modus Tollens: $\varphi \to \psi,\ \neg\psi\ \vdash \neg\varphi$.

Contraposition: $\varphi \to \psi\ \vdash \neg\psi \to \neg\varphi$.

Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.

Biconditional MP: $\varphi \leftrightarrow \psi,\ \neg\varphi\ \vdash \neg\psi$.

PL Completeness: Part I

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October 10, 2024

Recall from Last Time...

Corollary 4.2 If Γ is satisfiable, then Γ is consistent.

- This followed from PL SOUNDNESS.
- We will now establish the converse of **Corollary 4.2** as a theorem.
- PL COMPLETENESS will follow as a corollary.

Completeness Proof

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

Lemma 2.3 $\Gamma \vDash \varphi$ just in case $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.

Corollary 5.3 (*PL Completeness*) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assume $\Gamma \vDash \varphi$.
- $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by **Lemma 2.3**.
- $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **Theorem 5.1**.
- $\Gamma \vdash \neg\neg\varphi$ by **Lemma 5.1**, so there is a PL derivation X of $\neg\neg\varphi$ from Γ.
- $\Gamma \vdash \varphi$ by an additional application of DN to X.

Basic Lemmas

Lemma 5.1 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

- Assume $\Lambda \cup \{\varphi\}$ is inconsistent.
- So $\Lambda \cup \{\varphi\} \vdash \bot$, so X is a derivation of $A \wedge \neg A$ from Λ.
- Let X' prefix X with φ as an assumption replacing φ as a premise.
- Append lines for A and $\neg A$ by $\wedge$E.
- Discharge φ, concluding $\neg\varphi$ by $\neg$I, so $\Lambda \vdash \varphi$.

Lemma 5.2 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

- Assume $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$.
- X derives φ from Λ, and Y derives $\neg\varphi$ from Λ.
- Let Z append Y to X, renumbering lines.
- Use EFQ on the last lines of X and Y to derive $\bot$ from Λ.
- By definition, Λ is inconsistent.

Lemma 5.3 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

- Assume $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent.
- $\Lambda \vdash \neg\varphi$ and $\Lambda \vdash \neg\neg\varphi$ by **Lemma 5.1**.
- Thus Λ is inconsistent by **Lemma 5.4**.

Henkin Interpretation

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{\text{PL}}$ just in case for every wfs ψ in $\mathcal{L}^{\text{PL}}$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{\text{PL}}$.

Maximization: We may now extend Γ to a maximal set as follows:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$
- $\Delta_\Gamma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

Henkin Interpretation: For all sentence letters φ of $\mathcal{L}^{\text{PL}}$, let: $\mathcal{I}_\Delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta_\Gamma \\ 0 & \text{otherwise.} \end{cases}$

Satisfiable: It remains to show that $\mathcal{V}_{\mathcal{I}_\Delta}(\gamma) = 1$ for all $\gamma \in \Gamma$.

- This will allow us to conclude that Γ is satisfiable.

Lindenbaum's Lemmas

Lemma 5.4 If Γ is consistent in $\mathcal{L}^{\text{PL}}$, then Δ_Γ is maximal consistent.

- Assume Γ is consistent and let φ be any wfs of $\mathcal{L}^{\text{PL}}$.
- $\varphi = \psi_i$ for some $i \in \mathbb{N}$ given the enumeration above.
- Either $\psi_i \in \Delta_{i+1}$ or $\neg\psi_i \in \Delta_{i+1}$.
- Since $\Delta_{i+1} \subseteq \Delta_\Gamma$, either $\varphi \in \Delta_\Gamma$ or $\neg\varphi \in \Delta_\Gamma$, and so Δ_Γ is maximal.

Base Case: Immediate by the assumption that $\Delta_0 = \Gamma$ is consistent.

Induction Step: Assume for weak induction that Δ_n is consistent.

- $\Delta_n \cup \{\psi_n\}$ is either consistent or not.

Case 1: If $\Delta_n \cup \{\psi_n\}$ is consistent, then $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: If $\Delta_n \cup \{\psi_n\}$ is not consistent, then $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

- Assume for contradiction that $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent.
- So Δ_n is inconsistent by **Lemma 5.2**, contradicting the above.
- So Δ_{n+1} is consistent in both cases, and so Δ_k is consistent for all $k \in \mathbb{N}$.

Limit: Assume for contradiction that Δ_Γ is inconsistent.

- X is a PL derivation of $\bot$ from Δ_Γ in a finite number of lines.
- Let $m \in \mathbb{N}$ be the first number where Δ_m includes all premises in X.
- So $\Delta_m \vdash \bot$, and so Δ_k is inconsistent for some $k \in \mathbb{N}$.
- Since this contradicts the above, Δ_Γ is consistent.

Deductive Closure

Deductive Closure: A set Δ of wfss of $\mathcal{L}^{\text{PL}}$ is DEDUCTIVELY CLOSED in PL just in case for any wfs φ of $\mathcal{L}^{\text{PL}}$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$.

Lemma 5.5 If Δ is maximal consistent, then Δ is deductively closed.

- Assume Δ is maximal consistent.
- Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\Delta \vdash \varphi$.
- Assume for contradiction that $\neg\varphi \in \Delta$.
- X derives $\neg\varphi$ from Δ by R, so $\Delta \vdash \neg\varphi$.
- By **Lemma 5.4**, Δ is inconsistent, contradicting the above.
- So $\neg\varphi \notin \Delta$, and so $\varphi \in \Delta$ by maximality.
- Generalizing on φ, we may conclude that Δ is deductively closed.

PL Completeness: Part II

LOGIC I
Benjamin Brast-McKie
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From Last Time...

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

Corollary 5.3 (*PL Completeness*) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Basic Lemmas

Lindenbaum's Lemma: If Γ is consistent in $\mathcal{L}^{\text{PL}}$, then Δ_Γ is maximal consistent.

Deductive Closure: A set Δ of wfss of $\mathcal{L}^{\text{PL}}$ is DEDUCTIVELY CLOSED in PL just in case for any wfs φ of $\mathcal{L}^{\text{PL}}$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$.

Lemma 5.5 If Δ is maximal consistent, then Δ is deductively closed.

Lemma 5.6 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

- Assuming that $\Lambda \vdash \varphi$, there is a derivation X of φ from Λ in PL.
- Since $\Lambda \subseteq \Lambda \cup \Pi$, X is also a derivation of φ from $\Lambda \cup \Pi$ in PL.
- Thus $\Lambda \cup \Pi \vdash \varphi$.

Henkin Interpretation

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{\text{PL}}$ just in case for every wfs ψ in $\mathcal{L}^{\text{PL}}$ either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{\text{PL}}$.

Maximization: We may now extend Γ to a maximal set as follows:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$
- $\Delta_\Gamma = \bigcup_{i \in \mathbb{N}} \Delta_n$.

Henkin Interpretation: For all sentence letters φ of $\mathcal{L}^{\text{PL}}$, let: $\mathcal{I}_\Delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta_\Gamma \\ 0 & \text{otherwise.} \end{cases}$

Satisfiable: It remains to show that $\mathcal{V}_{\mathcal{I}_\Delta}(\gamma) = 1$ for all $\gamma \in \Gamma$.

- This will allow us to conclude that Γ is satisfiable.

Henkin Lemmas Continued

Lemma 5.7 If Δ is a maximal consistent set of wfss of $\mathcal{L}^{\text{PL}}$, then every wfs φ of $\mathcal{L}^{\text{PL}}$ is such that $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

- Assume Δ is a maximal consistent set of $\mathcal{L}^{\text{PL}}$ wfss.
- The proof goes by induction on complexity.

Base: Assume $\texttt{Comp}(\varphi) = 0$, so φ is a sentence letter.

- $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\mathcal{I}_\Delta(\varphi) = 1$ by the semantics.
  iff $\varphi \in \Delta$ by the definition of $\mathcal{I}_\Delta$.
- Thus whenever $\texttt{Comp}(\varphi) = 0$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

Induction: Assume that whenever $\texttt{Comp}(\varphi) \leq n$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

- Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\texttt{Comp}(\varphi) = n + 1$.
- There are five cases to consider, one for each operator.

Case 1: $\mathcal{V}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0$ by the semantics.
iff $\psi \notin \Delta$ by hypothesis since $\texttt{Comp}(\psi) \leq n$.
iff $\neg\psi \in \Delta$ by maximal consistency.

Case 2: $\mathcal{V}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1$ by the semantics.
iff $\psi, \chi \in \Delta$ by hypothesis since $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$.
iff $\psi \wedge \chi \in \Delta$ by $(*)$.

$(*)$ If $\psi \wedge \chi \in \Delta$, then $\Delta \vdash \psi$ and $\Delta \vdash \chi$ by $\wedge$E.

- So $\psi, \chi \in \Delta$ by **Lemma 5.5**.
- If $\psi, \chi \in \Delta$, then $\Delta \vdash \psi \wedge \chi$ by $\wedge$I.
- So $\psi \wedge \chi \in \Delta$ by **Lemma 5.5**.

Case 3: Exercise for this weeks PSet.

Case 4: $\mathcal{V}_{\mathcal{I}_\Delta}(\psi \to \chi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0$ or $\mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1$ by the semantics.
iff $\psi \notin \Delta$ or $\chi \in \Delta$ hypothesis since $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$.
iff $\psi \to \chi \in \Delta$ by (†) and (‡).

(†) If $\psi \notin \Delta$, then $\neg\psi \in \Delta$ by maximality.

- Since $\neg\psi \vdash \psi \to \chi$ and $\neg\psi \in \Delta$, we know $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**.
- Thus $\psi \to \chi \in \Delta$ by **Lemma 5.5**.
- If $\chi \in \Delta$, then since $\chi \vdash \psi \to \chi$, we know $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**.
- So if either $\psi \notin \Delta$ or $\chi \in \Delta$, then $\psi \to \chi \in \Delta$.

(‡) Assume instead that $\psi \to \chi \in \Delta$.

- If $\psi \notin \Delta$, then $\psi \notin \Delta$ or $\chi \in \Delta$.

- If $\psi \in \Delta$, then $\Delta \vdash \chi$ by the rule $\rightarrow$E, and so $\chi \in \Delta$ by **Lemma 5.5**.
- So if $\psi \rightarrow \chi \in \Delta$, then $\psi \notin \Delta$ or $\chi \in \Delta$.

Case 5: Exercise for this weeks PSet.

Conclusion: So whenever $\texttt{Comp}(\varphi) = n + 1$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

- Thus for all wfss φ of $\mathcal{L}^{\text{PL}}$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ *iff* $\varphi \in \Delta$.

Satisfiability

Lemma 5.8 $\Gamma \subseteq \Delta_\Gamma$.

- Immediate from the definition.

Theorem 5.1 If Γ is consistent, then Γ is satisfiable.

- Let Γ be a consistent set of wfss of $\mathcal{L}^{\text{PL}}$.
- Δ_Γ is a maximal consistent by **Lemma 5.5**.
- Let $\Delta = \Delta_\Gamma$ and $\mathcal{I}_\Delta$ be the Henkin interpretation of $\mathcal{L}^{\text{PL}}$ defined above.
- By **Lemma 5.7**, for every wfs φ of $\mathcal{L}^{\text{PL}}$: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\varphi \in \Delta$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Delta$.
- Since $\Gamma \subseteq \Delta$ by **Lemma 5.8**, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Gamma$.
- By definition, Γ is satisfiable.

Compactness

Corollary 5.4 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

- Assume $\Gamma \vDash \varphi$.
- $\Gamma \vdash \varphi$ by completeness, and so X derives φ from Γ.
- $\Gamma_X \vdash \varphi$ where Γ_X is the set of premises in X.
- $\Gamma_X \vDash \varphi$ by soundness.
- Since X is finite, Γ_X is also finite.

Corolary 5.5 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

- Assume for contraposition that Γ is unsatisfiable.
- $\Gamma \vDash \bot$ follows vacuously.
- $\Lambda \vDash \bot$ by **Corollary 5.4** for some finite subset $\Lambda \subseteq \Gamma$.
- So some finite subset $\Lambda \subseteq \Gamma$ is unsatisfiable.
- By contraposition, QED.