Problem Set 13

Partial Solutions: We Assume you will never share these!

Apologies for typos! Let me know if you catch any! (Gotta catch-em all!)

1.) Using the new existential intro and elimination rules, we show that $\sim(\forall x)\, P \vdash_{QND^*} (\exists x) \sim P$:

1	$\sim(\forall x)\, P$	:PR
2	$(\forall x) \sim\sim P$	:AS for $\sim$I
3	$\sim(\forall x) \sim P$	:AS for $\sim$E
4	$(\exists x) \sim P$	:3 $\exists$I
5	$\sim(\forall x) \sim\sim P$	:4 $\exists$E
6	$(\forall x) \sim\sim P$	:2 R
7	$(\forall x) \sim P$	:3–6 $\sim$E
8	$\sim P[a/x]$	:7 $\forall$E
9	$\sim\sim P[a/x]$	:2 $\forall$E
10	$\sim(\forall x) \sim\sim P$	:2–9 $\sim$I
11	$(\exists x) \sim P$	:10 $\exists$I

4. In the cases of soundness for SND and QND, we were able to define derivability for infinitely-many premises as follows: Θ is *SND-derivable* from Γ provided there is an SND derivation:

   1.) whose starting premises Δ are a finite subset of Γ

   2.) in which Θ appears on its own in the final line

   3.) where Θ is directly next to the main scope line, i.e. only in the scope of the Δ-premises

   In this case, we write $\Gamma \vdash_{SND} \Theta$ (also: $\Delta \vdash_{SND} \Theta$)

   If no such derivation exists, then we say that Θ is NOT SND-derivable from Γ, and we write $\Gamma \nvdash_{SND} \Theta$. Notice how "not-SND-derivable" is the negation of "SND-derivable."

   Now, going back to trees, recall that in this case, "tree-invalid" was not the negation of "tree-valid."

   Tree-valid: $\Gamma \cup \{\sim\Theta\}$ is *tree-inconsistent*: There is at least one tree with this set as the root such that *all branches close*

Tree-invalid: $\Gamma \cup \{\sim\Theta\}$ is *tree-consistent*: There is at least one tree with this set as the root such that there is a complete open branch (i.e. all wffs have been resolved and no contradictory pairs appear on this branch)

Our proofs of soundness proceeded by proving the contrapositive:
Soundness: If $\Gamma \not\models \Theta$, then $\Gamma \nvdash_{STD} \Theta$. Where here, "$\Gamma \nvdash_{STD} \Theta$" means that *it is not the case that* the argument from Γ to Θ is **tree-valid**.

- i.e., *it is not the case that* there exists a tree with root $\Gamma \cup \{\sim\Theta\}$ that possesses *all closed branches*
- Equivalently: ANY tree with root $\Gamma \cup \{\sim\Theta\}$ possesses **at least one complete open branch**
- (Aside: this is NOT the same as saying that the argument is **tree-invalid**, since that only requires the existence of a single tree with a complete open branch)

So it seems like the obvious way to modify these definitions for infinite premise sets would be the following:

Tree-valid: for infinite Γ, $\Gamma \cup \{\sim\Theta\}$ is *tree-inconsistent* provided there is at least one finite subset $\Delta \subset \Gamma$ such that there is at least one tree with this set Δ as the root such that *all branches close.*

Tree-invalid: for infinite Γ, $\Gamma \cup \{\sim\Theta\}$ is *tree-consistent* provided there is at least one finite subset $\Delta \subset \Gamma$ such that there is at least one tree with this set Δ as the root such that *there is a complete open branch.*

At least one issue we run into: with these definitions, many arguments with infinite premise sets will be both tree-valid and tree-invalid: we can choose different finite subsets Δ that respectively have the properties above.

Following the above idea, for infinite Γ, **not-tree-valid**, i.e. $\Gamma \nvdash_{STD} \Theta$ amounts to the following: there exists at least one finite subset $\Delta \subset \Gamma$ such that *it is not the case that* there exists a tree with root $\Delta \cup \{\sim\Theta\}$ that possesses *all closed branches.* Equivalently, this means that for some finite $\Delta \subset \Gamma$, ANY tree with root $\Delta \cup \{\sim\Theta\}$ possesses **at least one complete open branch**.

So then for infinite Γ, it would suffice to find a single finite subset Δ such that any tree with root $\Delta \cup \{\sim\Theta\}$ possesses a complete open branch. However, it will often be trivial to find such a Δ, simply by taking Δ small enough. Provided there is at least one wff in Γ that is not logically equivalent to the conclusion Θ, we can take that one wff as our Δ and the resulting tree with root $\Delta \cup \{\sim\Theta\}$ will possess a complete open branch.

Hence, we would be led to say that an infinite premise set containing $\{P, P \supset Q\}$ is not-tree-valid for conclusion Q, since we can take finite subset Δ to simply be $\{P \supset Q\}$ and note that all trees with root $\{P \supset Q\} \cup \{\sim Q\}$ have a complete open branch.

Of course, an infinite premise set containing $\{P, P \supset Q\}$ would ALSO count as tree-valid, since the finite subset $\Delta' := \{P, P \supset Q\}$ leads to a root $\Delta \cup \{\sim\Theta\}$ such that all branches close on this tree.

Alternative idea that actually could work: for infinite Γ, define **not-tree-valid**, i.e. $\Gamma \nvdash_{STD} \Theta$ as follows: for ALL finite subsets $\Delta \subset \Gamma$, ANY tree with root $\Delta \cup \{\sim\Theta\}$ possesses a complete open branch.