Quantifier Logic

Logic I
Benjamin Brast-McKie
October 31, 2023

Expressive Limitations

Socrates: Consider the following argument:

(a) Every human is mortal.
(b) Socrates is human.
(c) ∴ Socrates is mortal.

Mammals: Consider the following argument:

(a) All humans are mammals.
(b) All mammals are multi-celled organisms.
(c) ∴ All humans are multi-celled organisms.

SL Regimentation: Neither argument is valid in SL.

Predicates, Variables, and Quantifiers

Mammals (a): Everything is such that if it is human then it is a mammal.

Mammals (b): Everything is such that if it is a mammal then it is a multi-celled organism.

Mammals (c): Everything is such that if it is human then it is a multi-celled organism.

Predicates: 'is human',
'is a mammal', and
'is a multi-celled organism'.

Properties: Predicates express properties.

Variables: 'it'.

Reference: What does 'it' refer to?

Atomic Formulas: 'it is human',
'it is a mammal', and
'it is a multi-celled organism'.

Complex Formulas: 'if it is human then it is a mammal',
'if it is a mammal then it is a multi-celled organism', and
'if it is human then it is a multi-celled organism'.

Quantifiers: 'Everything is such that'.

Constants

Socrates (a): Everything is such that if it is human then it is mortal.

Socrates (b): Socrates is human.

Socrates (c): Socrates is mortal.

Predicates: 'is human' and 'is mortal'.

Variables: 'it'.

Constants: 'Socrates'.

Reference: Constants refer to objects.

Atomic Formulas: 'it is human', 'it is mortal', 'Socrates is human', and 'Socrates is mortal'.

Complex Formulas: 'if it is human then it is mortal'.

Quantifiers: 'Everything is such that'.

Binary Predicates

Height: Kin is taller than Prema.
∴ Prema is shorter than Kin.

Task 1: Regiment the argument above.

Predicates: 'is taller than', 'is shorter than', and 'is the same height as'.

Relations: Binary predicates express 2-place properties, i.e., *relations*.

- $Tkp \therefore Spk$.
- $Tkp \therefore \neg Tpk$.
- $Tkp \therefore \neg Tpk \wedge \neg Epk$.

Question 1: Is this argument valid, and if not how can we make it valid?

- $Tkp, Tkp \supset Spk \therefore Spk$.
- $Tkp, \forall x \forall y (Txy \supset Syx) \therefore Spk$.

Age: Jon is older than Sara.
Sara is older than Ethan.
∴ Jon is older than Ethan.

Task 2: Regiment the argument above.

Predicates: 'is older than'.

- $Ojs, Ose \therefore Oje$.

Question 2: Is this argument valid, and if not how can we make it valid?

- $Ojs, Ose, (Ojs \wedge Ose) \supset Oje \therefore Oje$.
- $Ojs, Ose, \forall x \forall y \forall z ((Oxy \wedge Oyz) \supset Oxz) \therefore Oje$.

Polyadic Predicates

Triadic: 'x is between y and z',
'x is more similar to y than to z',
'x is closer to y than to z', ...

Polyadic: We may refer to predicates as n-place or n-adic.

Properties: n-place predicates express n-place properties.

Primitive Symbols of QL

Predicates: n-place predicates $A^n, \ldots, Z^n$ for $n \geq 0$ possibly with subscripts.

Constants: $a, b, c, \ldots$ possibly with subscripts.

Variables: $x, y, z, \ldots$ possibly with subscripts.

Connectives: $\neg, \wedge, \vee, \supset, \equiv$.

Quantifiers: $\forall, \exists$.

Parentheses: $(,)$.

Well-Formed Formulas of QL

Singular Terms: Constants and variables are called *singular terms*.

Well-Formed Formulas: We may define the well-formed formulas (wffs) of QL as follows:

1. $\mathcal{F}^n \alpha_1, \ldots, \alpha_n$ is a wff if $\mathcal{F}^n$ is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. If φ and ψ are wffs and α is a variable, then:
   (a) $\exists \alpha \varphi$ is a wff;
   (b) $\forall \alpha \varphi$ is a wff;
   (c) $\neg \varphi$ is a wff;
   (d) $(\varphi \wedge \psi)$ is a wff;
   (e) $(\varphi \vee \psi)$ is a wff;
   (f) $(\varphi \supset \psi)$ is a wff; and
   (g) $(\varphi \equiv \psi)$ is a wff.
3. Nothing else is a wff.

Atomic Formulas: The wffs defined by (1) are *atomic*.

Arguments: The singular terms in an atomic wff are the *arguments* of the predicate.

Composition Rules: The clauses in (2) are called *composition rules*.

Scope: φ is the *scope* of the quantifier in $\exists \alpha \varphi$ and $\forall \alpha \varphi$.

- Compare the scope of negation.

Question 3: Does the definition above make sense as stated?

Task 3: How can we fix the definition above to respect use/mention?

The Sentences of QL

Free Variables: We define the *free variables* recursively:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leq i \leq n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. If φ and ψ are wffs and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;
   (d) α is free in $(\varphi \wedge \psi)$ if α is free in φ or α is free in ψ;
   (e) α is free in $(\varphi \vee \psi)$ if α is free in φ or α is free in ψ;
   (f) α is free in $(\varphi \supset \psi)$ if α is free in φ or α is free in ψ;
   (g) α is free in $(\varphi \equiv \psi)$ if α is free in φ or α is free in ψ;
3. Nothing else is a free variable.

Bound Variables: Every free occurrence of α in φ is *bound* in $\exists\alpha\varphi$ and $\forall\alpha\varphi$.

Binding: The variable α is the *binding variable* in $\exists\alpha\varphi$ and $\forall\alpha\varphi$.

Open Sentences: An *open sentence* of QL is any wff with free variables.

Sentences: A *sentence* of QL is any wff without free variables.

Interpretation: Only the sentences of QL will have truth-values on an interpretation independent of an assignment function.