Minimal Models and Variable Assignments

LOGIC I
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November 9, 2023

QL Models

Interpretations: $\mathcal{I}$ is an QL interpretation over $\mathbb{D}$ *iff* both:

- $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α in QL.
- $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$.

Model: $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ is a model of QL *iff* $\mathcal{I}$ is a QL interpretation over $\mathbb{D} \neq \varnothing$.

Variable Assignments

Assignments: A variable assignment $\hat{a}(\alpha) \in \mathbb{D}$ for every variable α in QL.

Singular Terms: We may define the referent of α in $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ as follows:

$$\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

Variants: A $\hat{c}$ is an α-variant of $\hat{a}$ *iff* $\hat{c}(\beta) = \hat{a}(\beta)$ for all $\beta \neq \alpha$.

Semantics for QL

(A) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.

($\forall$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

($\exists$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

($\neg$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) \neq 1$.

($\vee$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

($\wedge$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$.

($\supset$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \supset \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 1$ (or both).

($\equiv$) $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi)$.

Truth: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for some $\hat{a}$ where φ is a sentence of QL.

Assignment Lemmas

Lemma 1: If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$.

- Goes by routine induction on complexity.

Lemma 2: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

LTR: Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{c}$ over $\mathbb{D}$.

- Let $\hat{a}$ be any v.a. over $\mathbb{D}$.
- Since φ has no free variables, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$ by *Lemma 1*.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for all v.a. $\hat{c}$ over $\mathbb{D}$.

RTL: Assume $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for all v.a. $\hat{a}$ over $\mathbb{D}$.

- Since $\mathbb{D}$ is nonempty, there is some v.a. $\hat{a}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$.

Lemma 3: For any sentence φ: $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Minimal Models

Task 1: Provide minimal models in which the following are true/false.

- Al loves everything, i.e., $\forall x Lax$.

  True: Let $\hat{a}$ be a v.a. over $\mathbb{D} = \{a\}$.
  - Let $\hat{c}$ be any x-variant of $\hat{a}$.
  - So $\hat{c}(x) = a$ and $\mathcal{I}(a) = a$.
  - Since $\mathcal{I}(L) = \{\langle a, a\rangle\}$, we know $\langle \mathcal{V}_{\mathcal{I}}^{\hat{c}}(a), \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x)\rangle \in \mathcal{I}(L)$.
  - So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Lax) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x Lax) = 1$.

  False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(L) = \varnothing$.
  - Assume $\mathcal{V}_{\mathcal{I}}(\forall x Lax) = 1$ for contradiction.
  - So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x Lax) = 1$ for some v.a. $\hat{a}$.
  - So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Lax) = 1$ since $\hat{a}$ is an x-variant of itself.
  - So $\langle \mathcal{V}_{\mathcal{I}}^{\hat{a}}(a), \mathcal{V}_{\mathcal{I}}^{\hat{a}}(x)\rangle \in \mathcal{I}(L)$, and so $\mathcal{I}(L) \neq \varnothing$.

- Someone is dancing, i.e., $\exists x(Px \land Dx)$.

  True: Let $\hat{a}$ be a v.a. over $\mathbb{D} = \{a\}$ where $a(x) = a$.
  - Since $\mathcal{I}(P) = \mathcal{I}(D) = \{\langle a\rangle\}$, we know $\langle \mathcal{V}_{\mathcal{I}}^{\hat{a}}(x)\rangle \in \mathcal{I}(P) = \mathcal{I}(D)$.
  - So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Px) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(Dx) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Px \land Dx) = 1$.
  - Since $\hat{a}$ is a x-variant of itself, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x(Px \land Dx)) = 1$.
  - Thus $\mathcal{V}_{\mathcal{I}}(\exists x(Px \land Dx)) = 1$.

False: Let $\mathbb{D} = \{a\}$ and $\mathcal{I}(P) = \varnothing$.

- Assume $\mathcal{V}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$ for contradiction.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x(Px \wedge Dx)) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(Px \wedge Dx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(Px) = 1$, and so $\langle \mathcal{V}^{\hat{c}}_{\mathcal{I}}(x) \rangle \in \mathcal{I}(P)$.
- Thus $\mathcal{I}(P) \neq \varnothing$.

- No set is a member of itself. `[contingent]`
  $\neg\exists x(Sx \wedge x \in x)$
- There is a set with no members. `[contingent]`
  $\exists x(Sx \wedge \forall y(y \notin x))$
- Everyone loves someone. `[contingent]`
  $\forall x(Px \supset \exists y Lxy)$.
- The guests that remained were pleased with the party. `[contingent]`
  $\forall x(Rxp \supset Px)$.
- I haven't met a cat that likes Merra. `[contingent]`
  $\neg\exists x(Mbx \wedge Cx \wedge Lmx)$
- Kate found a job that she loved. `[contingent]`
  $\exists x(Fkx \wedge Jx \wedge Lkx)$
- Everything everything loves loves something. `[contingent]`
  $\forall x(\forall y Lyx \supset \exists z Lxz)$.

Quantifier Exchange

($\neg\forall$) $\neg\forall x\varphi \vDash \exists x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ satisfy $\neg\forall x\varphi$.

- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\forall x\varphi) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x\varphi) \neq 1$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) \neq 1$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\neg\varphi) = 1$ for some x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\forall x\neg\varphi) = 1$.

($\neg\exists$) $\neg\exists x\varphi \vDash \forall x\neg\varphi$.

LTR: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ satisfy $\neg\exists x\varphi$.

- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\exists x\varphi) = 1$ for some v.a. $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\exists x\varphi) \neq 1$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) \neq 1$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\neg\varphi) = 1$ for all x-variants $\hat{c}$ of $\hat{a}$.
- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall x\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\forall x\neg\varphi) = 1$.

Arguments

Bigger: Regiment the following argument:

- Whenever something is bigger than another, the latter is not bigger than the former.
  $\forall x \forall y(Bxy \supset \neg Byx)$.

$\therefore$ Nothing is bigger than itself.
  $\neg \exists x Bxx$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model which satisfies the premise.

- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x \forall y(Bxy \supset \neg Byx)) = 1$ for some v.a. $\hat{a}$.
- Assume $\mathcal{V}_{\mathcal{I}}(\neg \exists x Bxx) \neq 1$ for contradiction.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg \exists x Bxx) \neq 1$ in particular.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists x Bxx) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Bxx) = 1$ for some x-variant $\hat{c}$ of $\hat{a}$.
- So $\langle \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x), \mathcal{V}_{\mathcal{I}}^{\hat{c}}(x) \rangle \in \mathcal{I}(B)$, and so $\langle \hat{c}(x), \hat{c}(x) \rangle \in \mathcal{I}(B)$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\forall y(Bxy \supset \neg Byx)) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy \supset \neg Byx) = 1$ for y-variant $\hat{e}$ where $\hat{e}(y) = \hat{c}(x)$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\neg Byx) = 1$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Bxy) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Byx) \neq 1$.
- So $\langle \hat{e}(x), \hat{e}(y) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{e}(y), \hat{e}(x) \rangle \notin \mathcal{I}(B)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ or $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$ since $\hat{e}(x) = \hat{c}(x)$.
- So $\langle \hat{c}(x), \hat{c}(x) \rangle \notin \mathcal{I}(B)$, contradicting the above.

Love: Regiment the following argument:

- Cam doesn't love anyone who loves him back.
  $\forall x(Lxc \supset \neg Lcx)$.
- May loves everyone who loves themselves.
  $\forall y(Lyy \supset Lmy)$.

$\therefore$ If Cam loves himself, he doesn't love May.
  $Lcc \supset \neg Lcm$.

Taller: Regiment the following argument:

- If a first is taller than a second who is taller than a third, then the first is taller than the third.
  $\forall x \forall y \forall z((Txy \wedge Tyz) \supset Txz)$.
- Nothing is taller than itself.
  $\neg \exists x Txx$.

$\therefore$ If a first is taller than a second, the second isn't taller than the first.
  $\forall x \forall y(Txy \supset \neg Tyx)$.