12. Metalogic for SL

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a. A Meta-refresher

SND as a derivation system, provided that...

- As we have seen, Sentential Natural Deduction allows us to derive a conclusion from a set of premises:
  1.) valid argument: conclusion on last line, in scope of just premises
  2.) tautology: on last line in scope of NO premises
  3.) two logically equivalent sentences: (i) their biconditional is a tautology or (ii) derive one from the other and vice versa (which mirrors biconditional introduction!)
- But our derivations are justified only if system SND is *sound*
- And guaranteed to have a derivation for every valid argument only if system SND is *complete*

A tale of three turnstiles: one semantic; two syntactic

- Double Turnstile $\vDash$: logical entailment (indexed to our choice of semantics, i.e. the truth-tables for our connectives)

- Single Turnstile Tree $\vdash_{STD}$: tree-validity in STD (i.e. premises and negated conclusion as root of a tree whose branches all close—recall that this means that $\Gamma \cup \{\sim\Theta\}$ is **tree-inconsistent**)

- Single Turnstile Natural $\vdash_{SND}$: **derivability** in SND

A Tale of Three Turnstiles $\vDash$ the semantic one

- "$\Gamma \vDash \Theta$" means that Γ logically entails Θ
  Whenever the premises in Γ are true, the conclusion Θ is true
- Equivalently: there is no truth-value assignment (TVA) s.t. Γ is satisfied while Θ is false
- Equivalently, this means that $\Gamma \cup \{\sim\Theta\}$ **is unsatisfiable**:
  no TVA satisfies the premises and negated conclusion
- We'll use this last fact A LOT in our proof that SND is complete!

Soundness vs. Completeness

- By proving that our derivation system is *sound*, we show that SND derivations are 'safe' (they never lead us astray)
  - **Sound**: If $\Gamma \vdash_{SND} \Theta$, then $\Gamma \vDash \Theta$
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- By proving that SND is *complete*, we show truth tables are not needed to demonstrate validity: SND derivations suffice
  - **Complete**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{SND} \Theta$
  - (logical entailment is fully covered by our syntactic rules)
  - (Means: we wrote down *enough* rules!)

Some contrasts with our metalogic proofs for trees (STD)

- Recall that to prove the soundness and completeness of our tree system STD, we proved the *contrapositive* of these statements

  - vs. With SND, we'll proceed directly

- With trees, our premise set Γ was finite

  - vs. Here, we'll let Γ be infinite. Although of course, whenever we talk about an SND derivation, this derivation must have a FINITE premise set $\Delta \subseteq \Gamma$ (i.e. a finite list of SL wffs justified by ':PR')

- Is this finiteness restriction a limitation of trees?

- Not in practice: no valid SL argument ever requires infinitely-many premises to entail its conclusion (PS 12 #4)

Semantic entailment for infinitely-many premises

- Let Γ be a possibly infinite set of premises; Θ a conclusion
- Recall: a TVA assigns 'True' or 'False' to the (infinitely-many) SL atomic wffs
- In the case where Γ is finite, its premises contain finitely-many atomic wffs, so we can restrict a TVA to a row of a truth table
- An argument is **semantically invalid** if there is a TVA that makes each wff in Γ true but which makes Θ false
- In this case we write $\Gamma \not\models \Theta$
- If there is no such TVA, then $\Gamma \models \Theta$

SND derivability for infinitely-many premises

- Θ is **SND-derivable** from Γ provided there is an SND derivation:
  1.) whose starting premises Δ are a finite subset of Γ

  2.) in which Θ appears on its own in the final line

  3.) where Θ is directly next to the main scope line, i.e. only in the scope of the Δ-premises
- In this case, we write $\Gamma \vdash_{SND} \Theta$ (also: $\Delta \vdash_{SND} \Theta$)
- If no such derivation exists, then we say that Θ is NOT SND-derivable from Γ, and we write $\Gamma \nvdash_{SND} \Theta$

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b. Soundness of System SND

Soundness: Proof Idea and notation

- Subgoal: given any line in an SND derivation, show that the well-formed formula (wff) on that line is entailed by the premises or assumptions accessible from that line
- Let "P_k" be the wff on line k, i.e. the k–th wff in our derivation
- Let "Γ_k" be the set of premises/assumptions accessible on line k, i.e. the set of open assumptions/premises in whose scope P_k lies
- **Subgoal**: given a wff P_k on line k, show that $\Gamma_k \vDash P_k$
- (like with soundness for trees, we reason "from the top down")

Soundness: Proof Strategy

- Recall that SND derivations are defined recursively:
  from a (possibly empty) set of premises, we have a finite number of rules to add a line

  – These ways include reiteration and an intro and elimination rule for each of our five connectives
- Hence: do induction on the number of lines in an SND derivation
- Show that the base case has the property (line #1)
- Induction hypothesis: assume the property holds for all lines $\leq k$.
- Induction step: show the property holds for line #k+1
  (by considering all possible ways line #k+1 could arise)

Let's get Righteous!

- Say that a line i of a derivation is **righteous** just in case $\Gamma_i \vDash P_i$, i.e. just in case the set of assumptions/premises accessible from i semantically entail the wff on that line.

- Call a derivation *righteous* if every line in it is righteous

- Our goal is to prove that every derivation in SND is righteous!

Do I sound righteous? (from righteousness to soundness)

- Let Γ be any set of SL wffs (possibly infinite)
- If $\Gamma \vdash_{SND} \mathcal{P}$, then by definition there is a derivation whose (finitely-many) premises Δ belong to Γ, such that $\mathcal{P}$ occurs on the final line and lies in the scope of Δ (i.e. $\Delta \vdash_{SND} \mathcal{P}$)
- Then by righteousness, $\Delta \vDash \mathcal{P}$

  - i.e. any TVA that makes Δ true must make $\mathcal{P}$ true
- So there is no truth-value assignment that makes all the sentences in Γ true while making $\mathcal{P}$ false, so $\Gamma \vDash \mathcal{P}$ as well
- So we will have shown **Soundness**: If $\Gamma \vdash_{SND} \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$

Base Case

- **Base case**: for any SND derivation, show that $\Gamma_1 \vDash \mathcal{P}_1$.
- Proof: Γ_1 is the set of premises accessible at line #1, which comprises exactly the wff $\mathcal{P}_1$
- (recall that every premise of a derivation lies in its own scope—

  i.e. these premises be gettin' high off their own supply)
- Clearly, $\mathcal{P}_1 \vDash \mathcal{P}_1$, so $\{\mathcal{P}_1\} \vDash \mathcal{P}_1$
- So line #1 is righteous (i.e. $\Gamma_1 \vDash \mathcal{P}_1$)

Stating the Induction Step

- **Induction Hypothesis**: Assume that every line i for $1 < i \leq k$ is righteous (i.e. that $\Gamma_i \vDash \mathcal{P}_i$)
- Induction step: Consider line #k+1; show that $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- We have 12 cases to consider! 11 of these arise from our 11 SND-sanctioned rules for extending a derivation.
- What is the 12th case?? (We could say 13, but that is BAD LUCK)

Case 1: Premise or Assumption

- **Case 1**: $\mathcal{P}_{k+1}$ is a premise (:PR) or a subproof assumption (:AS). Show that $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- Either way, $\mathcal{P}_{k+1} \in \Gamma_{k+1}$ (since every premise and assumption lies within its own scope)
- So given a TVA that makes every sentence in Γ_{k+1} true, this TVA must make $\mathcal{P}_{k+1}$ true
- So $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$; so this case be righteous!

Case 2: Reiteration

- **Case 2**: $\mathcal{P}_{k+1}$ arises from an application of rule R, reiteration
- Then wff $\mathcal{P}_{k+1}$ appears on an earlier line #i as the wff $\mathcal{P}_i$
- By the induction hypothesis, line #i is righteous, so $\Gamma_i \vDash \mathcal{P}_i$.

  -Hence, we also have $\Gamma_i \vDash \mathcal{P}_{k+1}$ (since $\mathcal{P}_i = \mathcal{P}_{k+1}$)
- To apply rule R, $\mathcal{P}_{k+1}$ must lie to the right of line #i's rightmost scope line $\Rightarrow \Gamma_i \subseteq \Gamma_{k+1}$ (i.e., all of the premises/assumptions accessible at line #i must also be accessible at line #k+1).
- Since $\Gamma_i \vDash \mathcal{P}_{k+1}$ and $\Gamma_i \subseteq \Gamma_{k+1}$, we have $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- Draw a schematic derivation to better understand $\Gamma_i \subseteq \Gamma_{k+1}$!

Case 3: Conjunction Introduction (Things be heating up—finally!)

- **Case 3**: $\mathcal{P}_{k+1} := (\mathcal{Q} \,\&\, \mathcal{R})$ arises from an application of rule $\&$I
- Then on two earlier lines #h and #j, $\mathcal{Q}$ and $\mathcal{R}$ appear, respectively
- By the IH, both of these lines are righteous, so $\Gamma_h \vDash \mathcal{Q}$ and $\Gamma_j \vDash \mathcal{R}$
- By rule $\&$I, both these lines must be accessible on line #k+1
- So $\Gamma_h \cup \Gamma_j \subseteq \Gamma_{k+1}$ (i.e. both Γ_h and Γ_j are subsets of Γ_{k+1})
- Hence, any TVA that satisfies Γ_{k+1} must satisfy both Γ_h and Γ_j, and hence satisfy $\mathcal{Q}$ and also satisfy $\mathcal{R}$
- Thus, any TVA that satisfies Γ_{k+1} satisfies $(\mathcal{Q} \,\&\, \mathcal{R})$
- So $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$

Case 4: Conjunction Elimination

- **Case 4**: $\mathcal{P}_{k+1}$ arises from an application of rule $\&$E

  I'm about to eliminate this proof, son!

- Then there is an earlier line #h of the form $\mathcal{P}_{k+1} \& \mathcal{Q}$ or $\mathcal{Q} \& \mathcal{P}_{k+1}$
- By the IH, line #h is righteous, so $\Gamma_h \vDash \mathcal{P}_h$
- Since line #h is accessible at line #k+1, $\Gamma_h \subseteq \Gamma_{k+1}$
- So any TVA that satisfies Γ_{k+1} also satisfies Γ_h and thereby makes true $\mathcal{P}_h$
- By the truth conditions for conjunctions, any TVA that satisfies $\mathcal{P}_h$ satisfies both conjuncts, in particular $\mathcal{P}_{k+1}$
- So $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$ and line #k+1 is righteous

Case 8: Conditional Introduction

- **Case 8**: $\mathcal{P}_{k+1}$ arises from rule $\supset$I, which involves a subproof!
- $\mathcal{P}_{k+1}$ must be of the form $\mathcal{Q} \supset \mathcal{R}$ (**draw derivation** to define terms)
- NTS: $\Gamma_{k+1} \vDash \mathcal{Q} \supset \mathcal{R}$ given that $\Gamma_h \vDash \mathcal{Q}$ and $\Gamma_j \vDash \mathcal{R}$, by Ind. Hyp.
- Proceed by cases: either Γ_{k+1} satisfies $\mathcal{Q}$ or it doesn't:
- If Γ_{k+1} does not satisfy $\mathcal{Q}$, then it trivially satisfies $\mathcal{Q} \supset \mathcal{R}$
- Otherwise, Γ_{k+1} satisfies $\mathcal{Q}$. Since $\Gamma_j \subseteq \Gamma_{k+1} \cup \{\mathcal{Q}\}$, this means that Γ_j is satisfied in this case. Then since line #j is righteous, we have $\Gamma_{k+1} \cup \{\mathcal{Q}\} \vDash \mathcal{R}$. So in this case, Γ_{k+1} satisfies $\mathcal{Q} \supset \mathcal{R}$ as well.
- So in either case, $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$

Case 9: Negation Introduction

- **Case 8**: $\mathcal{P}_{k+1}$ arises from rule $\sim$I, using a subproof!
- $\mathcal{P}_{k+1}$ must be of form $\sim\mathcal{Q}$; **draw derivation to define lines**
- NTS: $\Gamma_{k+1} \vDash \sim\mathcal{Q}$ given that $\Gamma_h \vDash \mathcal{Q}$, $\Gamma_j \vDash \mathcal{R}$ **and** $\Gamma_m \vDash \sim\mathcal{R}$ (by IH)
- Notice that Γ_j and Γ_m are both subsets of $\Gamma_{k+1} \cup \{\mathcal{Q}\}$

  Hence, $\Gamma_{k+1} \cup \{\mathcal{Q}\}$ entails both $\mathcal{R}$ and $\sim\mathcal{R}$ as well.

  Thus, any TVA that satisfies $\Gamma_{k+1} \cup \{\mathcal{Q}\}$ must make both $\mathcal{R}$ and $\sim\mathcal{R}$ true, which is impossible (i.e. there can be no such TVA).

  $\Rightarrow \Gamma_{k+1} \cup \{\mathcal{Q}\}$ is unsatisfiable. Hence, $\Gamma_{k+1} \vDash \sim\mathcal{Q}$

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c. Completeness of System SND

Semantic vs. Syntactic Consistency

- We will appeal to two distinct notions of consistency throughout
- One is **semantic**: this is the notion we are already familiar with: there is a TVA that **satisfies** every sentence in the set
- We introduce a new **syntactic** notion of consistency relative to our SND derivation system:

  - a set of SL wffs is **SND-consistent** provided that you can't derive contradictory sentences from it in SND
- Core proof idea: we'll show that if a set of sentences is **consistent-in-SND**, then it is also semantically consistent (i.e. **satisfiable**). So by the contrapositive: if a set is **un**satisfiable, then it is **in**consistent-in-SND.

Semantic: Satisfiable (truth-functionally consistent)

- Recall: a set of SL sentences is **satisfiable** provided there is a TVA that makes all of them true
- This is a *semantic* notion of consistency
- i.e. **truth-functionally consistent**
- Contrast this with the syntactic notion of **consistency in SND**:

Syntactic: (In)consistent-in-SND (derivationally consistent)

- Let Γ be a (possibly infinite) set of SL wffs
- **Inconsistent-in-SND**: from premises in Γ, we can derive contradictory formulas R and $\sim R$ in the scope of the main scope line (i.e. these premises)
- **Consistent-in-SND**: Γ is not SND-inconsistent, i.e. there is no derivation from premises in Γ resulting in contradictory formulas within the main scope
- Other words we might use for these concepts: SND-inconsistent, derivationally-inconsistent, SND-consistent, etc.
- Just remember: this syntactic notion has nothing to do with truth value assignments!

Proof Sketch

- Goal: prove the completeness of SL: for every SL wff $\mathcal{P}$ and every set Γ of SL sentences, if $\Gamma \vDash \mathcal{P}$ then $\Gamma \vdash \mathcal{P}$
- So assume that $\Gamma \vDash \mathcal{P}$.
- Recall from week 5: this means that $\Gamma \cup \{\sim\mathcal{P}\}$ is semantically inconsistent (i.e. **unsatisfiable**):
  no TVA satisfies the premises and negated conclusion
- We now appeal to a **Consistency lemma** that is the heart of the enterprise: any SND-consistent set of SL sentences is satisfiable (i.e. semantically consistent)

Proof Sketch: Using the consistency lemma

- **Consistency lemma**: any SND–consistent set of SL sentences is satisfiable
- **Contrapositive** of CL: any set of SL sentences that is Unsatisfiable is SND–Inconsistent
- From $\Gamma \vDash \mathcal{P}$ we know that $\Gamma \cup \{\sim\mathcal{P}\}$ is unsatisfiable
- So by the contrapositive of CL, we see that $\Gamma \cup \{\sim\mathcal{P}\}$ is SND–inconsistent
- This means that we can derive a pair of contradictory sentences R and $\sim R$ from $\Gamma \cup \{\sim\mathcal{P}\}$! So using the power of negation elimination, we can derive $\mathcal{P}$ from Γ, i.e. $\Gamma \vdash \mathcal{P}$. So we are 'done'!

Negation Elimination Refresher (book's claim 6.4.4)

- Claim: if $\Gamma \cup \{\sim\mathcal{P}\}$ is **SND-inconsistent**, then $\Gamma \vdash \mathcal{P}$
- Proof: starting with (finitely-many) premises Δ from Γ, introduce $\sim\mathcal{P}$ as a subproof assumption for negation elimination
- Since $\Gamma \cup \{\sim\mathcal{P}\}$ is SND-inconsistent, we can derive a contradictory pair R and $\sim R$ within the scope of wffs in $\Delta \cup \{\sim\mathcal{P}\}$
- Then discharge this assumption $\sim\mathcal{P}$ by negation elimination, writing $\mathcal{P}$, now in the scope of Δ. So $\Delta \vdash \mathcal{P}$
- Since $\Delta \subseteq \Gamma$, we have $\Gamma \vdash \mathcal{P}$

Core subgoal: Prove consistency lemma (book's 6.4.2)

- So all we have to do is prove the **consistency lemma**: any SND-consistent set of SL sentences is satisfiable
- We'll prove this lemma in three 'stages':
- The first two are straightforward: given an SND-consistent set Γ, we construct a **superset** Γ^* that is *maximally SND-consistent*
- In the third stage, we show that any maximally SND-consistent set is satisfiable: we use maximal consistency to construct a TVA that satisfies every sentence in Γ^*
- Since by construction $\Gamma \subseteq \Gamma^*$, this TVA satisfies Γ as well.
- (The idea in the third stage is similar to what we did with trees: use a syntactic consistency property to construct a TVA that satisfies a set of wffs: with trees we had 'complete open branches'; here we have maximal–SND–consistency)
- The third stage comprises a tedious lemma and induction! PS12 problems 2 and 3 provide practice with this tedium!

Maximally SND-consistent

- A set Γ^* of SL wffs is **maximally SND-consistent** provided that:
  1.) Γ^* is SND-consistent (i.e. can't derive contradictory sentences)
  2.) adding **any** additional wff to Γ^* would result in an SND-inconsistent set
- i.e. for any $P \notin \Gamma^*$, $\{P\} \cup \Gamma^*$ is SND-inconsistent
- Motivation: it is straightforward (but tedious) to show that a maximally SND-consistent set is semantically consistent

  - Moreover, every SND-consistent set is a subset of a maximally SND-consistent set.

  - So we piggyback on an appropriate Γ^* to show that any SND-consistent set Γ is also satisfiable

Stage 1: Constructing Γ^*

- Let Γ be an SND-consistent set of SL wffs (possibly infinite)
- To construct Γ^*, we first **enumerate** the SL wffs, so that every SL wff is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
- Then consider the first wff 'A' in our enumeration. If A can be added to Γ without the resulting set being SND-inconsistent, then let $\Gamma_1 := \Gamma \cup \{A\}$.
- Otherwise, let $\Gamma_1 := \Gamma$ (so that Γ_1 stays SND-consistent)
- Then, proceed to the second wff in our enumeration. If it can be added to Γ_1 without the new set being SND-inconsistent, let Γ_2 be the result. Otherwise, let $\Gamma_2 := \Gamma_1$
- Γ^* is the result of 'doing' this procedure for every SL wff
- More precisely, $\Gamma^* := \bigcup_{k=1}^{\infty} \Gamma_k$

Enumeration (lexical ordering)

- Analogy: we can enumerate words by length, using their alphabetical order to break ties
- Can do the same for SL wffs by stipulating an 'alphabetical order':
- $\sim, \vee, \&, \supset, \equiv, (,), 0, 1, \ldots, 9, A, B, \ldots, Z$
- Each symbol is assigned an **index** between '10' and '55'
- Then each SL wff corresponds to a unique positive integer, constructed by replacing each symbol in the wff with its index, from left to right.
- So with our ordering, 'A' is the first wff; 'B' the second ... up to Z, and then we hit $\sim A$ ($\mapsto$ 1030), then $\sim B$ ($\mapsto$ 1031), etc.

Stage 2: Γ^* is maximally SND-consistent

- This requires proving two claims (from definition of M-SND-C):

  1.) Γ^* is consistent in SND

  2.) Adding any additional wff to Γ^* would result in an **SND-inconsistent** set

- We prove these in turn

Stage 2 (i): Γ^* is SND-consistent

- Assume for *reductio* that Γ^* is inconsistent in SND
- Then there would be an SND derivation with finite premise set $\Delta \subset \Gamma^*$ that derives a contradictory pair R and $\sim R$
- Since Δ is finite, there exists some $k \in \mathbb{N}$ s.t. $\Delta \subset \Gamma_k$.
  So then this Γ_k would be SND-inconsistent.
- Yet, we constructed each Γ_k such that each is SND-consistent:
  - In general, if P_k is the k-th sentence in our enumeration, then Γ_{k+1} is $\Gamma_k \cup \{P_k\}$ provided that $\Gamma_k \cup \{P_k\}$ is SND-consistent; otherwise, Γ_{k+1} equals Γ_k (so SND-consistent either way)
- Hence, Γ^* must be SND-consistent, on pain of *reductio*
- (note: the book's proof, p. 256, is way more complicated than necessary...)

Stage 2 (ii): Γ^* is maximally SND-consistent

- Assume for *reductio* that Γ^* weren't maximally SND-consistent, despite being SND-consistent
- i.e. assume *it is not the case that* for all additional wff, adding it to Γ^* would result in an SND-inconsistent set
  $\Rightarrow$ there exists a wff $\mathcal{Q}$ that we could add to Γ^* while preserving SND-consistency (i.e. there would be some wff that we neglected that could make Γ^* an even 'bigger' SND-consistent set)
- Yet, $\mathcal{Q}$ would appear in our enumeration as some wff P_k, 'considered' at the k–th stage of our construction of Γ^*.
- So if $\mathcal{Q}$ isn't in Γ^*, then this is because adding it 'would have' made $\Gamma_k \subset \Gamma^*$ SND-inconsistent.
  So $\{\mathcal{Q}\} \cup \Gamma^*$ must be SND-inconsistent (*reductio*!)
- So we can't add any $\mathcal{Q}$ to Γ^* while preserving SND-consistency

Stage 3: The Maximal Consistency Lemma (book's 6.4.8)

- **Maximal Consistency Lemma**: any set that is maximally–SND–consistent is satisfiable
- So there exists a TVA that satisfies every sentence in Γ^*. We construct this TVA, calling it "$\mathcal{I}$" (the book calls it $\mathbf{A}^*$)
- Proof idea: since Γ^* is M–SND–C, for any wff $\mathcal{Q}$, either $\mathcal{Q} \in \Gamma^*$ or $\sim\mathcal{Q} \in \Gamma^*$ (you're either in the club or your 'nemesis' is!)

  This holds in particular for each atomic wff
- Define the TVA $\mathcal{I}$ such that $\mathcal{I}(B) = \mathit{True}$ iff atomic $B \in \Gamma^*$
- Then by the recursive structure of SL wffs, $\mathcal{I}(\mathcal{Q}) = \mathit{True}$ iff $\mathcal{Q} \in \Gamma^*$

Stage 3 (i): the Membership Lemma (book's 6.4.11)

- To induct on SL, we first show some constraints on Γ^* membership
- Basically, Γ^* is like a club with a bouncer who enforces maximal consistency. Before the bouncer lets a wff into Γ^*, he checks who else is in the club
- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:
  - a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
  - b.) $\mathcal{P} \,\&\, \mathcal{Q} \in \Gamma^*$ if and only if both $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$
  - c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- Notice how these syntactic constraints mirror truth–conditions!
- Moral: We all want to belong, but sometimes our enemies get in the way!

Stage 3 (i): Key Fact aka The Door lemma (book's 6.4.9)

- To prove the membership lemma's cases (a)-(e), we'll use another lemma (hint: it's lemmas all the way down):
- **The Door**: if $\Gamma \vdash P$, and Γ^* is a maximally SND–consistent superset of Γ, then $P \in \Gamma^*$
  (mnemonic: "$\Gamma \vdash P$" pushes P through the door!)
- Proof: first, assume that $\Gamma \vdash P$ (we'll use this fact below)
- Next, assume for *reductio* that $P \notin \Gamma^*$. Then since Γ^* is maximally SND–consistent, $\Gamma^* \cup \{P\}$ must be inconsistent in SND.
- Hence, by negation introduction, $\Gamma^* \vdash \sim P$
- By assumption, $\Gamma \vdash P$, so also $\Gamma^* \vdash P$, since $\Gamma \subseteq \Gamma^*$
- So Γ^* derives both P and $\sim P$. *Reductio*! (since Γ^* is M–SND–C)
- Hence, if $\Gamma \vdash P$ and $\Gamma \subseteq \Gamma^*$, then P must belong to Γ^*

Membership Lemma: Case (a)

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
- Two directions to prove:

  $\Rightarrow$: Assume $\sim\mathcal{P} \in \Gamma^*$. Then if $\mathcal{P}$ were in Γ^*, we could derive contradictory sentences.

  So since Γ^* is SND–consistent, we must have $\mathcal{P} \notin \Gamma^*$

  $\Leftarrow$: Assume $\mathcal{P} \notin \Gamma^*$. Then adding $\mathcal{P}$ to Γ^* results in an SND–inconsistent set. Hence, there is some finite subset $\Delta \subset \Gamma^*$ s.t. $\Delta \cup \{\mathcal{P}\}$ is SND–inconsistent (i.e. derives contradictory sentence pair).
- So by negation introduction, $\Delta \vdash \sim\mathcal{P}$
- So by The Door lemma, $\sim\mathcal{P} \in \Gamma^*$

Membership Lemma: Cases (b)–(e)

- See the book for cases (b) ($\mathcal{P} \,\&\, \mathcal{Q}$) and (d) ($\mathcal{P} \supset \mathcal{Q}$)
- Case (c) is PS12 #2: $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
- We skip case (e) ($\mathcal{P} \equiv \mathcal{Q}$) because ... **YOLO**

Stage 3 (ii): Induction on SL (i.e. we be clubbin')

- Goal: construct a TVA $\mathcal{I}$ that satisfies the M-SND-C set Γ^*

  Suffices to construct $\mathcal{I}$ s.t. $\mathcal{I}(\mathcal{Q}) = \mathit{True}$ iff $\mathcal{Q} \in \Gamma^*$, $\forall \mathcal{Q} \in$ SL.
  Say that a wff is "**clubbin'**" whenever it meets this property
- Define $\mathcal{I}$ such that $\mathcal{I}(B) = \mathit{True}$ iff atomic $B \in \Gamma^*$
- **Base case**: each atomic wff is true on $\mathcal{I}$ iff it belongs to Γ^* (i.e. the atomics be clubbin')
- (Strong) **Induction hypothesis**: assume every SL wff with 1 to k–many connectives is clubbin'
- Induction step: show that an arbitrary SL wff with k+1–many connectives is clubbin'

Base Case

- Need to show **TWO** directions!:
- **Base case**: each atomic wff is true on $\mathcal{I}$ **iff** it belongs to Γ^*
- Recall that we defined $\mathcal{I}$ such that $\mathcal{I}(B) = \mathit{True}$ **iff** atomic $B \in \Gamma^*$
- So both directions are met by construction
- We proceed to do induction using our SL induction schema: an arbitrary sentence $\mathcal{P}$ with k+1–many connectives has one of five forms, coming from our five connectives.

Induction on SL: Case 1

- **Case 1**: $\mathcal{P}$ has the form $\sim\mathcal{Q}$, where since $\mathcal{Q}$ has k-connectives, it is clubbin by the IH (i.e. $\mathcal{I}(\mathcal{Q}) = 1$ if and only if $\mathcal{Q} \in \Gamma^*$)
- NTS: (i) (the $\Rightarrow$direction) if $\mathcal{I}(\mathcal{P}) = \mathit{True}$ then $\mathcal{P} \in \Gamma^*$ and (ii) (the $\Leftarrow$direction) if $\mathcal{P} \in \Gamma^*$, then $\mathcal{I}(\mathcal{P}) = \mathit{True}$

  (*Alternative (ii)*: show contrapositive: if $\mathcal{I}(\mathcal{P}) = 0$, then $\mathcal{P} \notin \Gamma^*$)

$\Rightarrow$ if $\mathcal{I}(\mathcal{P}) = 1$, then $\mathcal{I}(\mathcal{Q}) = 0$. Since $\mathcal{Q}$ is clubbin', we have $\mathcal{Q} \notin \Gamma^*$.

By Membership lemma (a), $\sim\mathcal{Q} \in \Gamma^*$, so $\mathcal{P} \in \Gamma^*$

$\Leftarrow$ if $\mathcal{P} \in \Gamma^*$, then $\sim\mathcal{Q} \in \Gamma^*$. So by Membership lemma (a), $\mathcal{Q} \notin \Gamma^*$.

Since $\mathcal{Q}$ is clubbin', we have $\mathcal{I}(\mathcal{Q}) = 0$.

So by the truth conditions for negation, $\mathcal{I}(\mathcal{P}) = 1$

Induction on SL: Cases 2–5

- Need to show: $\mathcal{P}$ be clubbin', i.e. $\mathcal{I}(\mathcal{P}) = \mathit{True}$ iff $\mathcal{P} \in \Gamma^*$, where $\mathcal{P}$ is arbitrary SL wff with k+1–many connectives
- Induction hypothesis: assume every SL wff with 1 to k–many connectives is clubbin'
- **Case 2**: $\mathcal{P}$ has the form $\mathcal{Q} \,\&\, \mathcal{R}$
- **Case 3** is PS12 #3: $\mathcal{P}$ has the form $\mathcal{Q} \vee \mathcal{R}$
- Case 4: $\mathcal{P}$ has the form $\mathcal{Q} \supset \mathcal{R}$ (see book p.260!)
- Case 5: $\mathcal{P}$ has the form $\mathcal{Q} \equiv \mathcal{R}$ (we'll do this case if and only if we accomplish all other goals in our lives)

Reminder for Josh!

- If we actually make it this far, give hints on PS12 completeness question ($P \vee Q$)! or do Case (d), which is most analogous
- If the people don't want these hints, then clearly they're already complete!
- "The customer is always right!"
- (Schematize this sentence in quantifier logic)