13. Metalogic for QL

Soundness vs. Completeness

- Let Γ be any set of *sentences* of QL and Θ any sentence of QL.
- By proving that our derivation system is *sound*, we show that QND derivations are 'safe' (they preserve truth)
  - **Sound**: If $\Gamma \vdash_{QND} \Theta$, then $\Gamma \vDash \Theta$
  - (syntactic to semantic: i.e. we chose 'good' rules!)
- By proving that QND is *complete*, we show that reasoning about arbitrary models is not needed to demonstrate validity: QND derivations suffice
  - **Complete**: If $\Gamma \vDash \Theta$, then $\Gamma \vdash_{QND} \Theta$
  - (logical entailment is fully covered by our syntactic rules)
  - (Means: we wrote down *enough* rules!)

13. Metalogic for QL

a. Truth and Satisfaction in QL

Recap: models and interpretations

- Let $\mathfrak{L}$ be a first-order language, containing constants and k-place predicates (e.g. the language of QL)
  - recall that the atomic sentences of SL are 0th-place predicates
- An $\mathfrak{L}$-model $\mathfrak{M} := (D, I)$ consists of
  1. A non-empty set D of objects, called the domain of $\mathfrak{M}$
  2. A map I (the *interpretation* of $\mathfrak{M}$), which maps the vocabulary of $\mathfrak{L}$ to objects and ordered pairs from D as follows:
     - For each constant $c \in \mathfrak{L}$, $I(c)$ is an element of D, called the *referent* or denotation of c
     - For each k-place predicate P of $\mathfrak{L}$, $I(P)$ is a set of ordered k-tuples of objects in D, called the *extension* of P
- Our text uses 'models' and 'interpretations' interchangeably, but the above disambiguation is convenient

Truth in a Model: simple examples

- Consider a simple language $\mathfrak{L}$ comprising a one-place predicate P, a two-place predicate R, and constants a and b.
- Fix an $\mathfrak{L}$-model $\mathfrak{M} := (D, I)$, e.g. our D could be $\mathbb{N}$.
  I is what we would punch into *Carnap* on PS9
- Pa is true in $\mathfrak{M}$ provided that the object $I(a)$ has the property $I(P)$, i.e. $I(a)$ lies in the extension of P. Then we'll write $\mathfrak{M} \vDash Pa$
- Rab is true in $\mathfrak{M}$ provided that the objects $I(a)$ and $I(b)$ stand in relation R. In this case, we'll write $\mathfrak{M} \vDash Rab$

Satisfaction in a model: simple example

- What to say about something with free variables, such as Px?
- This is a wff of QL but not a sentence (it is neither true nor false in a model)
- Idea: for each object r in D, we know whether Px *would be true* if we replaced r for x, i.e. if x stood for r
(b/c we know the extension of P, i.e. all the objects that are P)
- *Shorthand*:if Pc is true in $\mathfrak{M}$, then $I(c)$ satisfies Px in $\mathfrak{M}$
- *Longhand*: define a variable assignment $\mathbf{d}_I$ that maps variables to objects. Then $\mathbf{d}_I$ **satisfies** Px provided that $\mathbf{d}_I(x)$ has property I(P). We can write $\mathfrak{M}_{\mathbf{d}_I} \vDash Px$

Satisfaction for Atomic Wffs

- **Atomic wffs**: Suppose $\mathcal{Q}$ is atomic. Then $\mathcal{Q}$ is of the form $\mathcal{P}t_1 \dots t_k$ where each t_i is a term, i.e. a constant or a variable.
- I assigns constants to objects t_i^D; d_I maps variables to objects t_i^D (the text calls these denotations under I or d_I "$\text{den}_{I,d_I}(t_i)$")
- d_I satisfies $\mathcal{Q}$ provided that the k–tuple of these objects $\langle t_1^D, \dots, t_k^D \rangle$ lies in the extension of $\mathcal{Q}$, i.e. in $I(\mathcal{Q})$

Satisfaction for Quantified Wffs

- For $r \in D$ we write $d_I[r/x]$ for the assignment that agrees with d_I except necessarily assigning r for x
- **Existentially Quantified**: Suppose we have a wff of the form $(\exists x)\mathcal{Q}$. Then d_I satisfies $(\exists x)\mathcal{Q}$ provided there is SOME object $r \in D$ such that $d_I[r/x]$ satisfies $\mathcal{Q}$
  - Intuition: provided there's at least one thing you can plug in for x such that $\mathcal{Q}$ comes out true
- **Universally Quantified**: Suppose we have a wff of the form $(\forall x)\mathcal{Q}$. Then d_I satisfies $(\forall x)\mathcal{Q}$ provided $d_I[r/x]$ satisfies $\mathcal{Q}$ for EACH object $r \in D$
  - Intuition: no matter what you plug in for x, $\mathcal{Q}$ comes out true

From Satisfaction to Truth

- Focus on the sentences of QL, which have no free variables
- Lemma: given a model $\mathfrak{M} = (D, I)$ and a QL *sentence* $\mathcal{P}$, either all variable assignments d_I satisfy $\mathcal{P}$ or none do.
- Hence, we can define truth in a QL–model as follows:
- A sentence $\mathcal{P}$ of QL is **true** on model $\mathfrak{M}$ iff some variable assignment d_I satisfies $\mathcal{P}$ in $\mathfrak{M}$
- A sentence $\mathcal{P}$ of QL is **false** on model $\mathfrak{M}$ otherwise, i.e. if no variable assignment d_I satisfies $\mathcal{P}$ in $\mathfrak{M}$

Shorthand: Truth of quantified sentences

- $(\exists x)\,\mathcal{A}x$ is true iff $\mathcal{A}x$ is **satisfied** by **at least one** object in D
  - $r = I(c)$ satisfies $\mathcal{A}x$ in $\mathfrak{M}$ iff $\mathcal{A}c$ is true in $\mathfrak{M}$
  - e.g. there is at least one object in the domain that is an $\mathcal{A}$
  - Formally, there is a variable assignment d_I with at least one variant $d_I[r/x]$ s.t. $d_I[r/x]$ satisfies $\mathcal{A}x$
- $(\forall x)\,\mathcal{A}x$ is true iff $\mathcal{A}x$ is **satisfied** by **every** object in the domain
  - e.g. everything in D is an $\mathcal{A}$
  - Formally, there is a variable assignment d_I such that for each $r \in D$, each variant $d_I[r/x]$ satisfies $\mathcal{A}x$

Examples of Shorthand

- $(\exists x)(\mathcal{A}x \,\&\, \mathcal{B}x)$ is true iff some object satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \,\&\, \mathcal{B}x$' iff it satisfies both $\mathcal{A}x$ and $\mathcal{B}x$
- $(\forall x)(\mathcal{A}x \supset \mathcal{B}x)$ is true iff **every** object satisfies '$\mathcal{A}x \supset \mathcal{B}x$'
  - o satisfies '$\mathcal{A}x \supset \mathcal{B}x$' iff either
    - o does not satisfy $\mathcal{A}x$ (vacuously true conditional)

      or
    - o does satisfy $\mathcal{B}x$

Semantic Notions in QL

- Given a premise-set Γ of QL-*sentences* and a conclusion *sentence* $\mathcal{Q}$, we have the following semantic notions:

- **Entailment**: Γ QL-entails $\mathcal{Q}$ provided that there is no QL-model $\mathfrak{M}$ where Γ is true but $\mathcal{Q}$ is false. We write $\Gamma \vDash \mathcal{Q}$
  - we say that the argument from Γ to $\mathcal{Q}$ is **QL-valid**

- **Satisfiability**: we say that a set of sentences Γ is jointly **satisfiable** (aka QL-consistent) provided that there exists at least one QL-model $\mathfrak{M}$ where each sentence in Γ is true

13. Metalogic for QL

b. Recap: Substitution Instances

(Full) Substitution Instances

- "$\mathcal{Q}[c/x]$" is the sentence you get from $(\forall x)\mathcal{Q}$ or $(\exists x)\mathcal{Q}$ by dropping the quantifier and putting c in place of **every** x in $\mathcal{Q}$

- The other variables are untouched!

- Read "$[c/x]$" as saying "substitute c for every x", i.e. all the x's are replaced by c's!

Some Examples of Substitution Instances

- Instances of $(\forall y)Hy$:
  - Ha, Hb, Hm_{11}
- Instances of $(\exists z)Haz$:
  - Haa, Hab, Haj_3
- Instances of $(\exists z)(Hz \,\&\, Fzz)$:
  - Remember to replace **EVERY** occurance of z with the chosen constant:
  - $(Ha \,\&\, Faa)$, $(Hc \,\&\, Fcc)$
  - The following are **NOT** substitution instances:
  - $(Ha \,\&\, Faz)$, $(Hy \,\&\, Faa)$, $(Ha \,\&\, Fab)$

Partial Substitution Instances

- For Existential Introduction, we can use a partial substitution instance of the wff $\mathcal{Q}$:

- "$\mathcal{Q}\lceil \chi / c \rceil$" indicates that the variable χ replaces some but not necessarily all occurrences of the constant c in $\mathcal{Q}$.

- You can decide which occurrences of c to replace and which to leave in place

Examples of Partial Substitution Instances!

- '$\mathcal{Q}\lceil \chi/c \rceil$' indicates that the variable χ does not need to replace all occurrences of the constant c in $\mathcal{Q}$

1	Rdd	
2	$(\exists x)Rxx$	:1 $\exists$I
3	$(\exists x)Rxd$	:1 $\exists$I
4	$(\exists z)Rdz$	:1 $\exists$I
5	$(\exists y)(\exists z)Ryz$	:4 $\exists$I

Existential Introduction ($\exists$I)

m	$\mathcal{Q}$	
$\vdots$	$\vdots$	
s	$(\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$	:m $\exists$I

– Note: since $\mathcal{Q}$ is a sentence, and by our recursion clause for wff, χ cannot occur in $\mathcal{Q}$.

Substitution Lemma (Logic Book 11.1.1)

- Consider $\mathcal{Q} := Fxx$. Then $\mathcal{Q}[c/x] = Fcc$
- "variant $d_I[I(c)/x] = d_I[r/x]$ satisfies Fxx" means that when we assign x to $r = I(c)$, Fcc is true ($Fcc \in Extension(F)$)
- "d_I satisfies $\mathcal{Q}[c/x]$" means roughly that whatever objects d_I assigns variables, the result lies in the Extension of $\mathcal{Q}$
- Note that since x doesn't appear in Fcc, d_I treats Fcc just like $d_I[I(c)/x]$ treats Fxx
- "d_I satisfies $\mathcal{Q}[c/x]$" is equivalent to "$d_I[I(c)/x]$ satisfies $\mathcal{Q}$"
- **Substitution Lemma**: let $\mathcal{Q}$ be a wff of QL. The variable assignment d_I satisfies $\mathcal{Q}[c/\chi]$ if and only if $d_I[I(c)/\chi]$ satisfies $\mathcal{Q}$

13. Metalogic for QL

c. QL rules recap

Rules for the Universal Quantifier

Universal Elimination (∀E)

m	$(\forall \chi)\mathcal{Q}$
$\vdots$	$\vdots$
s	$\mathcal{Q}[c/\chi]$ $: m\ \forall E$

- Note that you replace **EVERY** instance of χ with c

- Notation: $\mathcal{Q}[c/\chi]$

- read "c for χ"

Universal Introduction (∀I)

m	$\mathcal{Q}$
$\vdots$	$\vdots$
s	$(\forall \chi)\mathcal{Q}[\chi/c]$ $: m\ \forall I$

Provided that both

(i) c does not occur in any other undischarged assumptions that $\mathcal{Q}$ is in the scope of.

(ii) χ does not occur already in $\mathcal{Q}$.

Rules for the Existential Quantifier

Existential Introduction (∃I)

$$
\begin{array}{l|ll}
m & \mathcal{Q} & \\
\vdots & \quad\vdots & \\
s & (\exists\chi)\mathcal{Q}\lceil\chi/c\rceil & :m\ \exists\mathrm{I}
\end{array}
$$

- **Provided that** χ does not occur already in $\mathcal{Q}$.
- As indicated by $\lceil\chi/c\rceil$, χ **may** replace **just some** occurrences of c

Existential Elimination (∃E)

$$
\begin{array}{l|lll}
m & (\exists\chi)\mathcal{Q} & & \\
 & \quad\vdots & & \\
n & \quad\vert\ \underline{\mathcal{Q}[c/\chi]} & & :\text{AS for }\exists\mathrm{E} \\
 & \quad\vert\quad\vdots & & \\
s & \quad\vert\ \Psi & & \\
s+1 & \Psi & & :m,\ n\text{–}s\ \exists\mathrm{E}
\end{array}
$$

Simplified: **provided that** c doesn't occur **anywhere else outside** the subproof

Motivating these restrictions on various rules!

- We'll now see why the rules $\forall I$ and $\exists E$ require us to follow the stated, non-trivial restrictions
- Without these restrictions, earlier sentences in the derivation would not semantically entail later sentences
- For QND to be sound, we need $\Gamma \vdash_{QND} \mathcal{P}$ to be sufficient for $\Gamma \vDash \mathcal{P}$.
- As with SND, we will prove this by showing that the set of open assumptions Γ_k on line $\#k$ semantically entail the sentence $\mathcal{P}_k$ on that line, for all lines k in any QND derivation

13. Metalogic for QL

d. Soundness of System QND

Semantic entailment for infinitely-many premises

- Let Γ be a possibly infinite set of QL-**sentences**; Θ a conclusion

- An argument is **semantically invalid** if there is a model $\mathfrak{M}$ that makes true each sentence in Γ but which makes Θ false

- In this case we write $\Gamma \nvDash \Theta$

- If there is no such QL-model, then $\Gamma \vDash \Theta$, i.e. if whenever we have $\mathfrak{M} \vDash \Gamma$ we also have $\mathfrak{M} \vDash \Theta$

QND derivability for infinitely-many premises

- Θ is **QND-derivable** from Γ provided there is an QND derivation:
  1.) whose starting premises Δ are a finite subset of Γ

  2.) in which Θ appears on its own in the final line

  3.) where Θ is directly next to the main scope line, i.e. only in the scope of the Δ-premises
- In this case, we write $\Gamma \vdash_{QND} \Theta$ (also: $\Delta \vdash_{QND} \Theta$)
- If no such derivation exists, then we say that Θ is NOT QND-derivable from Γ, and we write $\Gamma \nvdash_{QND} \Theta$

Soundness: Proof Idea and notation

- Subgoal: given any line in a QND derivation, show that the well-formed formula (wff) on that line is entailed by the premises or assumptions accessible from that line
- Let "P_k" be the wff on line k, i.e. the k–th wff in our derivation
- Let "Γ_k" be the set of premises/assumptions accessible on line k, i.e. the set of open assumptions/premises in whose scope P_k lies
- **Subgoal**: given a wff P_k on line k, show that $\Gamma_k \vDash P_k$

Soundness: Proof Strategy

- Recall that QND derivations are defined recursively:
  from a (possibly empty) set of premises, we have a finite number of rules to add a line

  – These ways include all our SND rules plus an intro and elimination rule for our quantifiers $\forall$ and $\exists$
- Hence: do induction on the number of lines in an QND derivation
- Show that the base case has the property (line #1)
- Induction hypothesis: assume the property holds for all lines $\leq k$.
- Induction step: show the property holds for line #k+1
  (by considering all possible ways line #k+1 could arise)

Let's remain Righteous!

- Recall: a line *i* of a derivation is **righteous** just in case $\Gamma_i \vDash P_i$, i.e. just in case **the set of assumptions/premises accessible from** ***i*** semantically entail the wff on that line.
- Call a derivation *righteous* if every line in it is righteous
- Our goal is to prove that every derivation in QND is righteous!
- We will extend our induction for SND to cover our four new rules!

From righteousness to soundness:

- Let Γ be any set of QL sentences (possibly infinite)
- If $\Gamma \vdash_{QND} \mathcal{P}$, then by definition there is a derivation from finitely-many premises $\Delta \subseteq \Gamma$, such that $\mathcal{P}$ occurs on the final line and lies in the scope of Δ (i.e. $\Delta \vdash_{QND} \mathcal{P}$)
- Then by righteousness, $\Delta \vDash \mathcal{P}$

  - i.e. any model $\mathfrak{M}$ that makes Δ true must make $\mathcal{P}$ true
- So there is no QL-model that makes all the sentences in Γ true while making $\mathcal{P}$ false, so $\Gamma \vDash \mathcal{P}$ as well
- So we will have shown **Soundness**: If $\Gamma \vdash_{QND} \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$

Base Case

- **Base case**: for any QND derivation, show that $\Gamma_1 \vDash \mathcal{P}_1$.
- Proof: Γ_1 is the set of premises accessible at line #1, which comprises exactly the QL-sentence $\mathcal{P}_1$
- (recall that every premise of a derivation lies in its own scope)
- Clearly, $\mathcal{P}_1 \vDash \mathcal{P}_1$, so $\{\mathcal{P}_1\} \vDash \mathcal{P}_1$
- So line #1 is righteous (i.e. $\Gamma_1 \vDash \mathcal{P}_1$)

Stating the Induction Step

- **Induction Hypothesis**: Assume that every line i for $1 < i \leq k$ is righteous (i.e. that $\Gamma_i \vDash \mathcal{P}_i$)
- Induction step: Consider line #k+1; show that $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- We have 16 cases to consider! We have essentially already considered 12 of these from our soundness proof for SND
- We have four new cases: our intro. and elimin. rules for $\forall$ and $\exists$

Cases 1–12: modifying our soundness proof for SND

- For each of these 12 cases, we simply replace “truth-value assignments” with “QL-models” (or interpretations), along with replacing truth-functional semantic notions with ones defined for quantifier logic
- e.g. quantificational entailment, quantificational consistency/satisfiability
- e.g. “$\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$” now means “sentence $\mathcal{P}_{k+1}$ is true in all models that make-true the premise set Γ_{k+1}”.
- Equivalently: $\Gamma_{k+1} \cup \{{\sim}\mathcal{P}_{k+1}\}$ is unsatisfiable in QL

Case 13 (gasp): For-all Elimination

- **Case 13**: $\mathcal{P}_{k+1}$ is derived by Universal Elimination (:$\forall E$)
  Show that $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- $\mathcal{P}_{k+1}$ must have the form $\mathcal{Q}[c/\chi]$ (read "c for χ")
- By the IH, line #h is righteous, so $\Gamma_h \vDash (\forall\chi)\mathcal{Q}$
- Since every assumption that is accessible at line #h is also accessible at line #k+1, we have $\Gamma_h \subseteq \Gamma_{k+1}$
- Hence, $\Gamma_{k+1} \vDash (\forall\chi)\mathcal{Q}$
- **Lemma**: a universally quantified sentence entails each of its substitution instances.
  $\Rightarrow$ any model that makes-true $(\forall\chi)\mathcal{Q}$ also makes-true $\mathcal{Q}[c/\chi]$
- Hence, $\Gamma_{k+1} \vDash \mathcal{Q}[c/\chi]$

Lemma for Case 13

- **Lemma**: a universally quantified sentence entails each of its substitution instances: $(\forall\chi)\mathcal{Q} \vDash \mathcal{Q}[c/\chi]$ for each constant c
- Consider an arbitrary model $\mathfrak{M}$ that makes true $(\forall\chi)\mathcal{Q}$
- Then by defN of true-in-QL, there is some variable assignment d_I that satisfies $(\forall\chi)\mathcal{Q}$ in $\mathfrak{M}$
- By the satisfaction-conditions for univ. quant. sentences, this means that for each $r \in D$, the variant $d_I[r/x]$ satisfies $\mathcal{Q}$
- So for each c, I must assign c an object $r \in D$ s.t. $d_I[I(c)/x]$ satisfies $\mathcal{Q}$.
- Hence, variable assignment d_I satisfies $\mathcal{Q}[c/\chi]$ (Lemma 11.1.1)
- (Intuition: no matter which r in D is assigned to c, $\mathcal{Q}[c/\chi]$ is true)
- So, for each constant c, $\mathfrak{M} \vDash \mathcal{Q}[c/\chi]$, i.e. is true in the model

Case 14: Existential Introduction

- **Case 14**: $\mathcal{P}_{k+1}$ is derived by Existential Introduction ($:\exists I$)
- $\mathcal{P}_{k+1}$ must have the form $(\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$ (read "χ for some c")
- By the IH, line #h is righteous, so $\Gamma_h \vDash \mathcal{Q}$ (where c appears)
- Since every assumption that is accessible at line #h is also accessible at line #k+1, we have $\Gamma_h \subseteq \Gamma_{k+1}$
- Hence, $\Gamma_{k+1} \vDash \mathcal{Q}$
- **Lemma**: a sentence $\mathcal{Q}$ entails any existentially quantified (possibly partial) substitution instance $(\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$.
- Hence, $\Gamma_{k+1} \vDash (\exists \chi)\mathcal{Q}\lceil \chi/c \rceil$

Lemma for Case 14

- **Lemma**: a sentence $\mathcal{Q}$ entails any existentially quantified (possibly partial) substitution instance $(\exists x)\mathcal{Q}\lceil x/c\rceil$
- Consider an arbitrary model $\mathfrak{M}$ that makes true $\mathcal{Q}[c]$
- Then by defN of true-in-QL, there is some variable assignment d_I that satisfies $\mathcal{Q}[c]$ in $\mathfrak{M}$. Let r be the object in D that c stands for.
- Then $d_I[r/x]$ satisfies $\mathcal{Q}\lceil x/c\rceil$ (i.e. the open sentence we get by replacing some c's with x is satisfied by object r)
- Recall: d_I satisfies $(\exists x)\mathcal{Q}\lceil x/c\rceil$ provided there is some object $r \in D$ s.t. $d_I[r/x]$ satisfies $\mathcal{Q}\lceil x/c\rceil$
- Hence, by the defN of true-in-QL, $(\exists x)\mathcal{Q}\lceil x/c\rceil$ is true in $\mathfrak{M}$

Case 16: Existential Elimination

- **Case 16**: $\mathcal{P}_{k+1}$ is derived by Existential Elimination ($:\exists E$)
- We require that c not occur in $(\exists \chi)\mathcal{Q}$, $\mathcal{P}_{k+1}$, or Γ_{k+1}
- By the IH, lines #h and #m are righteous, so $\Gamma_h \vDash (\exists \chi)\mathcal{Q}$ and $\Gamma_m \vDash \mathcal{P}_{k+1}$
- Since every assumption/premise that is accessible at line #h is also accessible at line #k+1, we have $\Gamma_h \subseteq \Gamma_{k+1}$. So $\Gamma_{k+1} \vDash (\exists \chi)\mathcal{Q}$
- Note that every member of Γ_m is accessible at #k+1 except assumption $\mathcal{Q}[c/\chi]$ on line #j. So $\Gamma_m \subseteq \Gamma_{k+1} \cup \{\mathcal{Q}[c/\chi]\}$
- So since $\Gamma_m \vDash \mathcal{P}_{k+1}$, we have $\Gamma_{k+1} \cup \{\mathcal{Q}[c/\chi]\} \vDash \mathcal{P}_{k+1}$
- **Lemma**: if (i) constant c does not occur in $(\exists \chi)\mathcal{Q}$, $\mathcal{P}$, or set Γ, (ii) $\Gamma \vDash (\exists \chi)\mathcal{Q}$ and (iii) $\Gamma \cup \{\mathcal{Q}[c/\chi]\} \vDash \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$
- Hence, $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$, so line #k+1 is righteous!

Locality Lemma (Book's 11.1.7)

- **Locality Lemma**: 'local agreement' on interpretation between models arises iff there is 'agreement' on entailment relations.
- Set-up (for a given sentence $\mathcal{P}$): Consider two QL-models $\mathfrak{M}^1 := (D, I_1)$ and $\mathfrak{M}^2 := (D, I_2)$ with the same domain D, whose interpretation functions I_1 and I_2 give the same interpretations for any constants or predicates appearing in QL-sentence $\mathcal{P}$

  (so any differences between $\mathfrak{M}^1$ and $\mathfrak{M}^2$ arise from how they interpret QL-symbols NOT appearing in $\mathcal{P}$).
- Then $\mathfrak{M}^1 \vDash \mathcal{P}$ if and only if $\mathfrak{M}^2 \vDash \mathcal{P}$
- We will use this lemma for the cases of $\forall I$ and $\exists E$

Lemma for Case 16

- **Lemma**: if (i) constant c does not occur in $(\exists\chi)\mathcal{Q}$, $\mathcal{P}$, or set Γ, (ii) $\Gamma \vDash (\exists\chi)\mathcal{Q}$ and (iii) $\Gamma \cup \{\mathcal{Q}[c/\chi]\} \vDash \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$
- NTS: In a model $\mathfrak{M} := (D, I)$ that makes all members of Γ true (i.e. $\mathfrak{M} \vDash \Gamma$), $\mathcal{P}$ is true (i.e. show that $\mathfrak{M} \vDash \mathcal{P}$)
- Since $\Gamma \vDash (\exists\chi)\mathcal{Q}$, there exists some object $r \in D$ that satisfies $\mathcal{Q}$ (i.e. there exists a d_I s.t. $d_I[r/\chi]$ satisfies $\mathcal{Q}$)
- Since c does not occur in $(\exists\chi)\mathcal{Q}$, $\mathcal{P}$, or set Γ, we can define a new model $\mathfrak{M}'$ that is just like $\mathfrak{M}$ except that $I'(c) = r$.
- Then since $d_I[r/\chi]$ satisfies $\mathcal{Q}$, we have $d_{I'}[r/\chi] = d_{I'}[I'(c)/\chi]$ satisfies $\mathcal{Q}$ as well.
- So by Lemma 11.1.1, $d_{I'}$ satisfies $\mathcal{Q}[c/\chi]$, so $\mathfrak{M}' \vDash \mathcal{Q}[c/\chi]$
- By Locality, since $\mathfrak{M} \vDash \Gamma$, we have $\mathfrak{M}' \vDash \Gamma$ too
- So $\mathfrak{M}' \vDash \Gamma \cup \{\mathcal{Q}[c/\chi]\}$ which by assumption $\vDash \mathcal{P}$. So $\mathfrak{M}' \vDash \mathcal{P}$
- Hence, by Locality, $\mathfrak{M} \vDash \mathcal{P}$, so $\Gamma \vDash \mathcal{P}$

Case 15: For-all Introduction

- **Case 15**: $\mathcal{P}_{k+1}$ is derived by Universal Introduction (:$\forall I$)
- $\mathcal{P}_{k+1}$ must have the form $(\forall \chi)\mathcal{Q}[\chi/c]$, with c not appearing in Γ_h
- By the IH, line #h is righteous, so $\Gamma_h \vDash \mathcal{Q}$ (where c appears)
- Since every assumption/premise that is accessible at line #h is also accessible at line #k+1, we have $\Gamma_h \subseteq \Gamma_{k+1}$
- Hence, $\Gamma_{k+1} \vDash \mathcal{Q}$
- **Lemma**: if c does not appear in any member of set Γ, then if $\Gamma \vDash \mathcal{Q}$, we have $\Gamma \vDash (\forall \chi)\mathcal{Q}[\chi/c]$
- Our rule $\forall I$ requires that c does not appear in Γ_{k+1}, so by the lemma, $\Gamma_{k+1} \vDash (\forall \chi)\mathcal{Q}[\chi/c]$

Lemma for Case 15

- **Lemma**: if c does not appear in any member of set Γ, then if $\Gamma \vDash \mathcal{Q}$, we have $\Gamma \vDash (\forall \chi)\mathcal{Q}[\chi/c]$
- Consider an arbitrary model $\mathfrak{M}$ that makes true all of the sentences in Γ. Then $\mathfrak{M} \vDash \mathcal{Q}$ (i.e. $\mathcal{Q}$ is true in $\mathfrak{M}$)
- So there exists a variable assignment d_I that satisfies $\mathcal{Q}$ in $\mathfrak{M}$
- Goal: show that there exists a variable assignment d'_I that satisfies $(\forall \chi)\mathcal{Q}[\chi/c]$
  - i.e. a d'_I s.t. $d'_I[r/\chi]$ satisfies $\mathcal{Q}[\chi/c]$ for each object $r \in D$
  - We will actually show that the given d_I does the trick!

Lemma for Case 15 continued

- Notice that for $(\forall \mathcal{x})\mathcal{Q}[\mathcal{x}/c]$ to be a wff, $\mathcal{x}$ must not already occur in sentence $\mathcal{Q}$, so $\mathcal{Q}$ can't already have a $\mathcal{x}$–quantifier.
- So $\mathcal{x}$ occurs freely in $\mathcal{Q}[\mathcal{x}/c]$ as the **only** free variable (since $(\forall \mathcal{x})\mathcal{Q}[\mathcal{x}/c]$ is, by assumption, a sentence)
- Hence, a variable assignment d_I of free variables in $\mathcal{Q}[\mathcal{x}/c]$ to objects in D amounts to a choice of object $r \in D$ to assign $\mathcal{x}$
- So d_I must make some choice $r := I(c)$ of object to assign $\mathcal{x}$
- Hence d_I simply equals $d_I[r/\mathcal{x}]$.
- By 11.1.1, d_I satisfies $\mathcal{Q}[\mathcal{x}/c]$ iff $d_I[I(c)/\mathcal{x}]$ satisfies $\mathcal{Q}$
- So d_I and hence $d_I[r/\mathcal{x}]$ satisfies $\mathcal{Q}[\mathcal{x}/c]$

Lemma for Case 15 continued *more*

- For each $r \in D$, we define a new model $\mathfrak{M}_r$ whose interpretation function I_r is just like I except that it assigns the constant c to r
- Now, $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{M}_r$ differs by $\mathfrak{M}$ only in how it interprets a symbol that does not occur in Γ.
- So by Locality, we have $\mathfrak{M}_r \vDash \Gamma$ and hence $\mathfrak{M}_r \vDash \mathcal{Q}$
- So for each object $r \in D$, we have a $d_{I_r}[r/\chi]$ that satisfies $\mathcal{Q}[\chi/c]$.
- This is equivalent to saying that for each $r \in D$, $d_I[r/\chi]$ satisfies $\mathcal{Q}[\chi/c]$, since d_I and each d_{I_r} agree on what to assign every other variable besides possibly χ
- And by defN, this means that d_I satisfies $(\forall\chi)\mathcal{Q}[\chi/c]$, and hence this sentence is true in $\mathfrak{M}$