11. Multiple quantifiers

1. Multiple quantifiers

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11. Multiple quantifiers

a. Two quantifiers

Formulas expressing relations

- A formula $\mathcal{A}x$ with one free variable expresses a **property**
- A formula $\mathcal{B}xy$ with two free variables expresses a **relation**
- $(\forall x)(\forall y)\,\mathcal{B}xy$ is a sentence:
- It's true iff **every pair** of objects α, β stand in the relation expressed by $\mathcal{B}xy$.
- $(\exists x)(\exists y)\,\mathcal{B}xy$ is a sentence:
- It's true iff **at least one pair** of objects α, β stand in the relation expressed by $\mathcal{B}xy$.

Multiple uses of a single quantifier: $\forall$

- Axy: x admires y.
- $(\forall x)(\forall y)\, Axy$: for every pair $\langle \alpha, \beta \rangle$, α admires β.
- In other words: everyone admires everyone.
- Note: "every pair" includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
- $(\forall x)(\forall y)\, Axy$ is true only if all pairs $\langle \alpha, \alpha \rangle$ satisfy Axy.
- That means, everyone admires themselves, in addition to everyone else.
- So: $(\forall x)(\forall y)\, Axy$ does **not** symbolize "everyone admires everyone **else**." (To handle that, we'll need identity!)

Multiple uses of single quantifier: $\exists$

- $(\exists x)(\exists y)\, Axy$: for at least one pair $\langle\alpha, \beta\rangle$, α admires β.
- In other words: at least one person admires at least one person.
- Note: includes pairs $\langle\alpha, \alpha\rangle$, i.e.,
- $(\exists x)(\exists y)\, Axy$ is already true if a single pair $\langle\alpha, \alpha\rangle$ satisfies Axy.
- That means, we could just have one person admiring themselves.
- So: $(\exists x)(\exists y)\, Axy$ does **not** symbolize "someone admires someone **else**." (again, for that, we'll need the identity predicate)

Alternating quantifiers

1. $(\forall x)(\exists y)\, Axy$
   Everyone admires someone
   (possibly themselves)
2. $(\forall y)(\exists x)\, Axy$
   Everyone is admired by someone
   (not necessarily the same person)
3. $(\exists x)(\forall y)\, Axy$
   Someone admires everyone
   (including themselves)
4. $(\exists y)(\forall x)\, Axy$
   Someone is admired by everyone
   (including themselves)

Convergence vs. uniform convergence

- A function f is **point-wise continuous** if

$$(\forall \epsilon)(\forall x)(\forall y)(\exists \delta)(|x - y| < \delta \supset |f(x) - f(y)| < \epsilon)$$

- A function f is **uniformly continuous** if

$$(\forall \epsilon)(\exists \delta)(\forall x)(\forall y)(|x - y| < \delta \supset |f(x) - f(y)| < \epsilon)$$

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b. Using quantifiers to express properties

Our symbolization key

Domain: people alive in 2022 and items of clothing

a: Autumn

g: Greta

Px: $\_\_\_\_\__x$ is a person

Lx: $\_\_\_\_\__x$ is an item of clothing.

Ex: $\_\_\_\_\__x$ is a cape

Rxy: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Hx: $\_\_\_\_\__x$ is a hero

Ix: $\_\_\_\_\__x$ inspires

Yxy: $\_\_\_\_\__x$ is younger than $\_\_\_\_\__y$

Axy: $\_\_\_\_\__x$ admires $\_\_\_\_\__y$

Oxy: $\_\_\_\_\__x$ owns $\_\_\_\_\__y$

Expressing properties, revisited

- One-place predicates express properties, e.g.,

  Hx expresses property “being a hero”

- Combinations of predicates (with connectives, names) can express derived properties, e.g.,

  Axg expresses “x admires Greta”

  $Hx \,\&\, Cx$ expresses “x is a hero who wears a cape”

- Using quantifiers, we can express even more complex properties, e.g.,

  $(\exists y)(Py \,\&\, Axy)$ expresses “x admires someone”

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
  - Greta admires a hero.
    $(\exists y)(Hy \& Agy)$
  - x admires a hero.
    $(\exists y)(Hy \& Axy)$
- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $(\exists y)(Ey \& Rxy)$
  - Greta wears a cape.
    $(\exists y)(Ey \& Rgy)$

Ex: $\_\_\_\_\__x$ is a cape
Rxy: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $(\exists y)(Ey \,\&\, Rxy)$
- x is admired by everyone.
  $(\forall y)(Py \supset Ayx)$
- x admires a hero.
  $(\exists y)(Hy \,\&\, Axy)$
- x admires only heroes.
  $(\forall y)(Axy \supset Hy)$
- x is unclothed (i.e. naked).
  $\sim(\exists y)(Ly \,\&\, Rxy)$
  $(\forall y)(Ly \supset \sim Rxy)$

Px	$\_\_\_\__x$ is a person	Lx	$\_\_\_\__x$ is an item of clothing
Ex	$\_\_\_\__x$ is a cape	Rxy	$\_\_\_\__x$ wears $\_\_\_\__y$

11. Multiple quantifiers

c. Multiple determiners

“Determiner phrases” say what?

- Determiners: quantifiers and indefinite or definite articles (also possessives and demonstratives)
- e.g. many, some, a, the, his, their, this, that
- Determiner phrases: combine a determiner with a (possibily modified) noun:
- ‘all heroes’; ‘a cape’
- ‘some woman’; ‘the donkey’

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?
- Deal with each determiner separately!
- Think of determiner phrase as replaced with name or variable—result has one less determiner.
- When you're down to one determiner, apply known methods for single quantifiers.
- This results in formulas that express properties or relations, but themselves contain quantifiers.

Two separate determiner phrases

- All heroes wear a cape
- All heroes satisfy "x wears a cape"

$$(\forall x)(Hx \supset \text{"}x \text{ wears a cape"})$$

- x wears a cape

$$(\exists y)(Ey \mathbin{\&} Rxy)$$

- Together:

$$(\forall x)(Hx \supset (\exists y)(Ey \mathbin{\&} Rxy))$$

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.
- All things that satisfy "x is a hero who wears a cape" admire Greta.

$$(\forall x)(\text{“x is a hero who wears a cape”} \supset Axg)$$

- x is a hero who wears a cape

$$Hx \,\&\, (\exists y)(Ey \,\&\, Rxy)$$

- Together:

$$(\forall x)((Hx \,\&\, (\exists y)(Ey \,\&\, Rxy)) \supset Axg)$$

Mary Astell, 1666–1731

- British political philosopher
- *Some Reflections upon Marriage* (1700)
- In preface to 3rd ed. 1706 reacts to William Nicholls' claim (in *The Duty of Inferiors towards their Superiors, in Five Practical Discourses* (London 1701), Discourse IV: The Duty of Wives to their Husbands), that women are naturally inferior to men.

Astell TL;DR

- What can Nicholls possibly mean by “women are naturally inferior to men”?
- It can’t be that some woman is inferior to some man, since that’s “no great discovery.”
- After all, surely some men are inferior to some women.
- The obviously intended meaning must be: **all** women are inferior to **all** men.
- But that can’t be right, for then “the greatest Queen ought not to command but to obey her Footman.”
- It can’t even be just: **all** women are inferior to **some** men.
- Since “had they been pleased to remember their Oaths of Allegiance and Supremacy, they might have known that *One* Woman is superior to *All* the Men in these Nations.”

Symbolizing Astell

- Some woman is superior to every man
- Some woman satisfies "x is superior to every man"

$$(\exists x)(Wx \,\&\, \text{“}x \text{ is superior to every man”})$$

- x is superior to every man

$$(\forall y)(My \supset Sxy)$$

- Together:

$$(\exists x)(Wx \,\&\, (\forall y)(My \supset Sxy))$$

Formalizing Astell

- Some woman is superior to some man.

  $(\exists x)(Wx \,\&\, (\exists y)(My \,\&\, Sxy))$

- Every woman is superior to every man.

  $(\forall x)(Wx \supset (\forall y)(My \supset Sxy))$

- Every woman is superior to some man.

  $(\forall x)(Wx \supset (\exists y)(My \,\&\, Sxy))$

- Some woman is superior to every man.

  $(\exists x)(Wx \,\&\, (\forall y)(My \supset Sxy))$

“Any”: sometimes existential

- Any (every) cape is worn by a hero:

$$(\forall x)(Ex \supset (\exists y)(Hy \,\&\, Ryx))$$

- No hero wears any cape:

$$(\forall x)(Hx \supset {\sim}(\exists y)(Ey \,\&\, Rxy))$$

$${\sim}(\exists x)(Hx \,\&\, (\exists y)(Ey \,\&\, Rxy))$$

- No hero wears every cape:

$$(\forall x)(Hx \supset {\sim}(\forall y)(Ey \supset Rxy))$$

$${\sim}(\exists x)(Hx \,\&\, (\forall y)(Ey \supset Rxy))$$

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d. Quantifier scope ambiguity

More scope ambiguity

- "Autumn and Greta admire Isra or Luisa."
- Two logically distinct, natural readings:

1) Autumn admires Isra or Luisa, **and** so does Greta.

$$(Aai \vee Aal)\ \&$$
$$(Agi \vee Agl)$$

2) Autumn and Greta both admire Isra, or they both admire Luisa.

$$(Aai \,\&\, Agi) \vee$$
$$(Aal \,\&\, Agl)$$

Negation and the quantifiers

- “All heroes don’t inspire”
  - Denial of “all heroes inspire”. Ask: “Do all heroes inspire (Answer: No, *it’s not the case that* all heroes inspire ”)

$$\sim(\forall x)(Hx \supset Ix)$$
$$(\exists x)(Hx \,\&\, {\sim}Ix)$$

  - All heroes are not inspiring, i.e., No heroes inspire

$$(\forall x)(Hx \supset {\sim}Ix)$$
$$\sim(\exists x)(Hx \,\&\, Ix)$$

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e., "For every hero, there is a cape they wear"

$$(\forall x)(Hx \supset (\exists y)(Ey \,\&\, Rxy))$$

$$(\forall x)(\exists y)(Hx \supset (Ey \,\&\, Rxy))$$

  - "All heroes" in scope of "a cape", i.e., "There is a cape which every hero wears"

$$(\exists y)(Ey \,\&\, (\forall x)(Hx \supset Rxy))$$

$$(\exists y)(\forall x)(Ey \,\&\, (Hx \supset Rxy))$$

- A (probably bad) joke: ""Every day, a tourist is mugged on the streets of New York. He's going through a lot of wallets."

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e. Donkey sentences

Happy farmers

"Every farmer who owns a donkey is happy"

- Step–by–step symbolization: "All As are Bs"
- x is a farmer who owns a donkey . . .

$$Fx \,\&\, (\exists y)(Dy \,\&\, Oxy)$$

- Every farmer who owns a donkey is happy

$$(\forall x)((Fx \,\&\, (\exists y)(Dy \,\&\, Oxy)) \supset Hx)$$

- Notice how 'a donkey' is bound by an existential here

Unhappy donkeys :(

"Every farmer who owns a donkey beats it"

- Step–by–step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$Fx \,\&\, (\exists y)(Dy \,\&\, Oxy)$$

- Every farmer who owns a donkey beats it:

$$(\forall x)((Fx \,\&\, (\exists y)(Dy \,\&\, Oxy)) \supset Bxy)$$

- PROBLEM: 'y' is unbound! So this is not a QL sentence. Gasp!

Save the donkeys: a failed attempt

- This was our problem: a donkey lay beaten and unbound:

$$(\forall x)((Fx \,\&\, (\exists y)(Dy \,\&\, Oxy)) \supset Bxy)$$

- Can we simply extend the scope of the existential?

$$(\forall x)(\exists y)((Fx \,\&\, (Dy \,\&\, Oxy)) \supset Bxy)$$

- 'y' is now bound, but alas, this sentence is trivially true:
- Provided at least one thing in our UD is not a donkey, that thing makes the antecedent of the conditional false, making the conditional trivially true, for any x.
  In particular, our farmer is not a donkey.
  But he still sounds like kind of a jack@$$!

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\sim(\exists x)(Fx \,\&\, (\exists y)(Dy \,\&\, Oxy \,\&\, \sim Bxy))$$

- For every farmer and every donkey they own: the farmer beats the donkey.

$$(\forall x)(\forall y)((Fx \,\&\, (Dy \,\&\, Oxy)) \supset Bxy)$$

- Every farmer beats every donkey they own.

$$(\forall x)(Fx \supset (\forall y)((Dy \,\&\, Oxy) \supset Bxy))$$

- But what about the case where at least one farmer with a donkey beats only one of his donkeys? #Quitting