Chapter

6

SENTENTIAL LOGIC: METATHEORY

Section 6.1 introduces mathematical induction, a technique that we will use to establish important results about the syntax and semantics of sentential logic. Section 6.2 establishes that the five connectives of *SL* are truth-functionally complete, that is, they can be used to express any truth-function. Section 6.3 establishes that *SD* and *SD+* are sound systems for sentential logic, that is, derivability in these systems establishes truth-functional entailment. Section 6.4 establishes that *SD* and *SD+* are complete for sentential logic, that is, given any truth-functional entailment, there is a corresponding derivation in these systems.

6.1 MATHEMATICAL INDUCTION

In the three previous chapters we concentrated on developing and using techniques of sentential logic, both semantic and syntactic. In this chapter we step back to prove some claims *about* the semantics and syntax of sentential logic. Such results constitute the **metatheory** of sentential logic. Throughout this chapter, unless otherwise noted, when we speak of sets we are speaking of sets of sentences of *SL*, and when we speak of sentences we are speaking of sentences of *SL*. We also adopt the convention of numbering our metatheoretic results to reflect the section in which they occur.

For the language *SL*, the semantic accounts of such logical properties of sentences and sets of sentences of *SL* as validity, consistency, and equivalence given in Chapter 3 are fundamental in the sense that they are the standards by which other accounts of these properties are judged. For instance, although the techniques of Chapter 5 are purely syntactical—all the derivation rules appeal to the structures or forms of sentences, not to their truth-conditions—those techniques are intended to yield results paralleling the results yielded by the semantic techniques of Chapter 3. One of the important metatheoretic results that we shall prove in this chapter is that this parallel does hold. We shall prove this by proving that the natural deduction system *SD* allows us to construct all and only the derivations we want to be able to construct, given the semantics of Chapter 3. Specifically we shall prove that, given any set Γ of sentences of *SL* and any sentence **P** of *SL*, **P** is derivable from Γ in *SD* if and only if **P** is truth–functionally entailed by Γ. It follows from this that all and only the truth-functionally valid arguments of *SL* are valid in *SD*, all and only the truth-functionally true sentences of *SL* are theorems in *SD*, and so on.

We shall use a very powerful method of proof known as **mathematical induction** to establish the foregoing results. We introduce mathematical induction with a simple example. We will use it to prove what appears to be an obvious result: that in every sentence of *SL* the number of left parentheses equals the number of right parentheses. Because there are an infinite number of sentences of *SL*, we cannot establish this result by looking at each sentence and counting the number of left and right parentheses that it contains. Mathematical induction allows us to establish that a claim holds for an infinite number of cases without going through them one at a time. Recall the recursive definition of 'sentence of *SL*' given in Chapter 2:

1. Every sentence letter of *SL* is a sentence of *SL*.
2. If **P** is a sentence of *SL*, then $\sim$ **P** is a sentence of *SL*.
3. If **P** and **Q** are sentences of *SL*, then (**P** & **Q**) is a sentence of *SL*.
4. If **P** and **Q** are sentences of *SL*, then (**P** $\vee$ **Q**) is a sentence of *SL*.
5. If **P** and **Q** are sentences of *SL*, then (**P** $\supset$ **Q**) is a sentence of *SL*.
6. If **P** and **Q** are sentence*s* of *SL*, then (**P** $\equiv$ **Q**) is a sentence of *SL*.
7. Nothing is a sentence of *SL* unless it can be formed by repeated application of clauses 1–6.

It is trivial to show that every atomic sentence of *SL*—that is, every sentence formed in accordance with clause 1—has an equal number of left and right parentheses (namely, zero), because atomic sentences contain no parentheses. All other sentences of *SL* are formed in accordance with clauses 2–6. We note that in each of these cases an equal number of outermost left and right parentheses are added to those already occurring in the sentence's immediate components to form the new sentence (zero of each in clause 2, one of each in clauses 3–6). Therefore, if we can be sure that the immediate components **P** and **Q** of sentences formed in accordance with clauses 2–6 themselves contain an equal number of left and right parentheses, then we may conclude that the sentences produced by these clauses will also contain an equal number of left and right parentheses.

How can we be sure, though, that each of the immediate components of a compound sentence *does* contain an equal number of left and right parentheses? We can start with compound sentences with one occurrence of a connective such as '~ A', '(A ⊃ B)', and '(A & B)'. Every sentence that contains one occurrence of a connective has one of the forms ~ **P**, (**P** & **Q**), (**P** ∨ **Q**), (**P** ⊃ **Q**), or (**P** ≡ **Q**). Moreover, in each case the immediate component or components are atomic. We have already noted that every atomic sentence has an equal number of left and right parentheses (namely, zero), and so, because clauses 2–6 each add an equal number of left and right parentheses to those already occurring in its immediate components, every compound sentence with one occurrence of a connective must also have an equal number of left and right parentheses.

Now consider truth-functionally compound sentences that contain two occurrences of connectives—sentences like '~ ~ A', '~ (A ∨ B)', '(A ∨ ~ B)', '((A ≡ B) ⊃ C)', and '(A ∨ (B & C))'. We may reason as we did in the previous paragraph. That is, every sentence that contains two occurrences of connectives has one of the forms ~ **P**, (**P** & **Q**), (**P** ∨ **Q**), (**P** ⊃ **Q**), or (**P** ≡ **Q**). And in each case the immediate component or components each have fewer than two occurrences of connectives. We have already found that, every sentence that contains fewer than two occurrences of connectives (atomic sentences and sentences containing one occurrence of a connective) has an equal number of left and right parentheses. Therefore, because clauses 2–6 each add an equal number of left and right parentheses to those already occurring in its immediate components, we may conclude that every compound sentence with two occurrences of connectives also has an equal number of left and right parentheses.

The same pattern of reasoning can be used for sentences with *three* occurrences of connectives, such as '~ ~ ~ A', '~ (~ A ∨ B)', '((A ⊃ B) & (A ∨ C))', and '(~ (A ≡ B) ≡ C)'. In every sentence that has three occurrences of connectives, the immediate components each contain fewer than three occurrences of connectives. We have already shown that every sentence of *SL* that contains fewer than three occurrences of connectives has an equal number of left and right parentheses. Therefore, because clauses 2–6 each add an equal number of left and right parentheses, we may conclude that every sentence that contains three occurrences of connectives has an equal number of left and right parentheses. The same reasoning can now be used to show that the claim holds for every sentence with four occurrences of connectives, then for every sentence with five occurrences of connectives, and so on. But of course we cannot consider each case individually—that is, each number of occurrences of connectives a sentence of *SL* might have, because there are an infinite number of such cases. But because there is a commonality of reasoning that can be used for each case we can use *mathematical induction* to generalize, reasoning as follows:

> Every sentence of *SL* with zero occurrences of connectives—that is, every atomic sentence of *SL*—has an equal number of left and right parentheses.

If every sentence of *SL* with **k** or fewer occurrences of connectives has an equal number of left and right parentheses, then every sentence of *SL* with **k** + 1 occurrences of connectives also has an equal number of left and right parentheses.

Therefore every sentence of *SL* has an equal number of left and right parentheses.

(Here we use '**k**' as a variable ranging over the nonnegative integers, that is, the positive integers plus zero.) This argument is logically valid—if the premises are true, then the conclusion is true as well. The first premise is our claim about sentences with no connectives, and the second premise therefore says that it follows that the claim also holds for sentences containing one occurrence of a connective. If the claim holds for all sentences containing zero or one occurrences of connectives, the second premise also assures us that the claim must also hold for sentences containing two occurrences of connectives. If the claim holds for all sentences containing zero, one, or two occurrences of connectives, the second premise also assures us that the claim also holds for sentences containing three occurrences of connectives, and so on for sentences with any number of occurrences of connectives. Because the argument is logically valid, we can establish that its conclusion is true by showing that both premises are true.

We have already shown that the first premise is true. Sentences that contain zero occurrences of connectives are atomic sentences, and atomic sentences are simply sentence letters. The first premise is called the '**basis clause**' of the argument.

The second premise of the argument is called the '**inductive step**', and its antecedent is called the **'inductive hypothesis'**. We shall prove that the inductive step is true by generalizing on the reasoning that we have already used. We shall assume that the inductive hypothesis is true—that is, that every sentence of *SL* containing **k** or fewer occurrences of connectives has an equal number of left and right parentheses—and we will show that it follows that every sentence **P** with **k** + 1 occurrences of connectives also has an equal number of left and right parentheses. Since **k** is nonnegative, **k** + 1 is positive, and hence such a sentence **P** contains at least one occurrence of a connective. So **P** will be a compound sentence that has one of the forms ~ **Q**, (**Q** & **R**), (**Q** ∨ **R**), (**Q** ⊃ **R**), or (**Q** ≡ **R**). We divide these forms into two cases.

Case 1: P has the form ~ **Q**. If ~ **Q** contains **k** + 1 occurrences of connectives, then **Q** contains **k** occurrences of connectives. By the inductive hypothesis (that every sentence containing **k** or fewer connectives has an equal number of left and right parentheses), **Q** has an equal number of left and right parentheses. But ~ **Q** contains all the parentheses occurring in **Q** and no others. So ~ **Q** contains an equal number of left and right parentheses as well.

Case 2: $\mathbf{P}$ has one of the forms $(\mathbf{Q} \& \mathbf{R})$, $(\mathbf{Q} \vee \mathbf{R})$, $(\mathbf{Q} \supset \mathbf{R})$, or $(\mathbf{Q} \equiv \mathbf{R})$. In each instance, if $\mathbf{P}$ contains $\mathbf{k} + 1$ occurrences of connectives, then its immediate components, $\mathbf{Q}$ and $\mathbf{R}$, must each contain $\mathbf{k}$ or fewer occurrences of connectives. By the inductive hypothesis, then:

a. $\mathbf{Q}$ has an equal number of left and right parentheses. Call this number $\mathbf{m}$.
b. $\mathbf{R}$ has an equal number of left and right parentheses. Call this number $\mathbf{n}$.

The number of left parentheses in $\mathbf{P}$ is therefore $\mathbf{m} + \mathbf{n} + 1$ (the 1 is for the outer-left parentheses that is added when $\mathbf{P}$ is formed from $\mathbf{Q}$ and $\mathbf{R}$), and similarly, the number of right parentheses in $\mathbf{P}$ is $\mathbf{m} + \mathbf{n} + 1$. Therefore, the number of left parentheses in $\mathbf{P}$ equals the number of right parentheses in $\mathbf{P}$.

This completes our proof that the second premise, the inductive step, is true. Having established that both premises are true, we may conclude that the conclusion is true as well. Every sentence of *SL* has an equal number of left and right parentheses.

We may now generally characterize arguments by mathematical induction. First, we group the items about which we wish to prove some claim into a series of cases, each associated with a nonnegative integer $\mathbf{k}$. In our example, we arranged the sentences of *SL* into the series: sentences with zero occurrences of connectives, sentences with one occurrence of a connective, sentences with two occurrences of connectives, and so on. Every sentence of *SL* falls into one of these cases. The argument by mathematical induction then takes the following form:[1]

The claim holds for every member of the first group in the series.

If the claim holds for every member of every group associated with an integer less than $\mathbf{k}$, then the claim holds for every member of the group associated with the integer $\mathbf{k}$.

The thesis holds for every member of every group in the series.

All arguments of this form are valid. Of course, only those with true premises are sound. So to establish that the claim holds for every member of every group, we must show that the claim does hold for every member of the first group and that, no matter what subsequent group in the series we may choose, the claim holds for every member of that group if it holds for every member of every prior group. Again, the first premise of an argument by mathematical

[1]Strictly speaking, this is the form of arguments by *strong* mathematical induction. There is another type of mathematical induction, known as *weak* induction. We shall use only the strong variety of mathematical induction in this text. There is no loss here, for every claim that can be proven by weak mathematical induction can also be proven by strong mathematical induction.

induction is the **basis clause,** the second premise is the **inductive step,** and the antecedent of the second premise is the **inductive hypothesis.**

We'll illustrate mathematical induction with another example. Let **P** be a sentence of *SL* that contains only '~', '∨', and '&' as connectives, and let **P′** be the sentence that results from:

a. Replacing each occurrence of '∨' in **P** with '&'
b. Replacing each occurrence of '&' in **P** with '∨'
c. Adding a '~' in front of each atomic component of **P**.

We shall call a sentence that contains only '~', '∨', and '&' as connectives a **TWA** sentence (short for '*t*ilde, *w*edge, and *a*mpersand'), and we shall call the sentence **P′** that results from **P** by (a), (b), and (c) the **dual** of **P**. Here are some examples of duals for TWA sentences:

P	*Dual of* **P**
A	~ A
((A ∨ F) & G)	((~ A & ~ F) ∨ ~ G)
(((B & C) & C) ∨ D)	(((~ B ∨ ~ C) ∨ ~ C) & ~ D)
~ ((A ∨ ~ B) ∨ (~ A & ~ B))	~ ((~ A & ~ ~ B) & (~ ~ A ∨ ~ ~ B))

We shall use mathematical induction to establish the following thesis:

Every TWA sentence **P** is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment (that is, if **P** is true then **P′** is false, and if **P** is false then **P′** is true).

As in the previous example, our series will classify sentences by the number of occurrences of connectives that they contain:

Basis clause: Every TWA sentence **P** with zero occurrences of connectives is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment.

Inductive step: If every TWA sentence **P** with **k** or fewer occurrences of connectives is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment, then every TWA sentence **P** with **k** + 1 occurrences of connectives is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment.

Conclusion: Every TWA sentence **P** of *SL* is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment.

To show that the conclusion of this argument is true, we must show that the first premise, the basis clause, is true and also that the second premise, the inductive step, is true.

Proof of basis clause: A TWA sentence **P** that contains zero occurrences of connectives must be an atomic sentence, and its dual is simply ~ **P**. If **P** is true on a truth-value assignment, then according to the characteristic truth-table for the tilde, ~ **P** must be false. And if **P** is false on a truth-value assignment, then ~ **P** is true. We conclude that **P** and its dual have opposite truth-values on each truth-value assignment.

Proof of inductive step: We assume that the inductive hypothesis is true for all sentences with fewer than **k** + 1 occurrences of connectives—that is, that every TWA sentence **P** with fewer than **k** + 1 occurrences of connectives is such that **P** and its dual **P′** have opposite truth-values on each truth-value assignment. We must show that it follows from this assumption that the claim is also true of all TWA sentences **P** with **k** + 1 occurrences of connectives. A TWA sentence **P** that contains **k** + 1 occurrences of connectives must be compound, and because it is a TWA sentence, it must have one of the three forms ~ **Q**, (**Q** ∨ **R**), or (**Q** & **R**). We will consider each form.

Case 1: **P** has the form ~ **Q**. If ~ **Q** contains **k** + 1 occurrences of connectives, then **Q** contains **k** occurrences of connectives, and **Q** is a TWA sentence (if it were not—if it contained a horseshoe or triple bar—then ~ **Q** would not be a TWA sentence either). Let **Q′** be the dual of **Q**. Then the dual of ~ **Q** is ~ **Q′**, the sentence that results from ~ **Q** by changing **Q** in accordance with (a), (b), and (c) of our definition of dual sentences and leaving the initial tilde of ~ **Q** intact.

Because **Q** is a TWA sentence with fewer than **k** + 1 occurrences of connectives, it follows from the inductive hypothesis that **Q** and its dual **Q′** have opposite truth-values on each truth-value assignment. Therefore, by the characteristic truth-table for negation, ~ **Q** and ~ **Q′** will also have opposite truth-values on each truth-value assignment.

Case 2: **P** has the form (**Q** ∨ **R**). If (**Q** ∨ **R**) contains **k** + 1 occurrences of connectives, then **Q** and **R** each contain **k** or fewer occurrences of connectives. **Q** and **R** must also be TWA sentences. (Again, if either of them were not, then **P** would not be a TWA sentence.) Let **Q′** be the dual of **Q** and **R′** be the dual of **R**. Then the dual of **P** is (**Q′** & **R′**)—the result of making the changes specified by (a), (b), and (c) within **Q** and within **R** and replacing the main connective '∨' of (**Q** ∨ **R**) with '&'.

If (**Q** ∨ **R**) is true on a truth-value assignment, then by the characteristic truth-table for the wedge, either **Q** is true or **R** is true. Because **Q** and **R** each contain **k** or fewer occurrences of connectives, it follows from the inductive hypothesis that either **Q′** is false or **R′** is false. Either way, (**Q′** & **R′**), the dual of (**Q** ∨ **R**), must be false as well. On the other hand, if (**Q** ∨ **R**) is false on a truth-value assignment, then both **Q** and **R** must be false on that assignment. It follows from the inductive hypothesis that both **Q′** and **R′** are true on that assignment, so ($\mathbf{Q}'$ & $\mathbf{R}'$) is true as well. We conclude that ($\mathbf{Q} \vee \mathbf{R}$) and its dual have opposite truth-values on each truth-value assignment.

Case 3: $\mathbf{P}$ has the form ($\mathbf{Q}$ & $\mathbf{R}$). If $\mathbf{P}$ contains $\mathbf{k} + 1$ occurrences of connectives, then $\mathbf{Q}$ and $\mathbf{R}$ each contain $\mathbf{k}$ or fewer occurrences of connectives. And they must also be TWA sentences. Let $\mathbf{Q}'$ be the dual of $\mathbf{Q}$ and $\mathbf{R}'$ the dual of $\mathbf{R}$. Then the dual of $\mathbf{P}$ is ($\mathbf{Q}' \vee \mathbf{R}'$)—the result of making the changes specified by (a), (b), and (c) within $\mathbf{Q}$ and within $\mathbf{R}$ and replacing the main connective of ($\mathbf{Q}$ & $\mathbf{R}$) with '$\vee$'.

If ($\mathbf{Q}$ & $\mathbf{R}$) is true on a truth-value assignment, then, by the characteristic truth-table for the ampersand, both $\mathbf{Q}$ and $\mathbf{R}$ are true on that truth-value assignment. Because $\mathbf{Q}$ and $\mathbf{R}$ each contain $\mathbf{k}$ or fewer occurrences of connectives, it follows from the inductive hypothesis that $\mathbf{Q}'$ and $\mathbf{R}'$ are both false on that assignment, and therefore that the dual of ($\mathbf{Q}$ & $\mathbf{R}$), ($\mathbf{Q}' \vee \mathbf{R}'$), is also false on that assignment. If ($\mathbf{Q}$ & $\mathbf{R}$) is false on a truth-value assignment, then either $\mathbf{Q}$ is false or $\mathbf{R}$ is false on that assignment. If $\mathbf{Q}$ is false, then it follows by the inductive hypothesis that $\mathbf{Q}'$ is true. If $\mathbf{R}$ is false on that assignment, then it follows by the inductive hypothesis that $\mathbf{R}'$ is true. So at least one of $\mathbf{Q}'$ and $\mathbf{R}'$ is true on the assignment in question, and ($\mathbf{Q}' \vee \mathbf{R}'$), the dual of ($\mathbf{Q}$ & $\mathbf{R}$), must also be true on that assignment. We conclude that ($\mathbf{Q}$ & $\mathbf{R}$) and its dual have opposite truth-values on each truth-value assignment.

These three cases establish the inductive step of the mathematical induction, and we may now conclude that its conclusion is true as well. Our argument shows that the thesis about duals is true of every TWA sentence of *SL*.

6.1E EXERCISES

1. Prove the following theses by mathematical induction.

a. No sentence of *SL* that contains only binary connectives, if any, is truth-functionally false (that is, every truth-functionally false sentence of *SL* contains at least one '~').

b. Every sentence of *SL* that contains no binary connectives is truth-functionally indeterminate.

c. If two truth-value assignments $\mathbf{A}'$ and $\mathbf{A}''$ assign the same truth-values to the atomic components of a sentence $\mathbf{P}$, then $\mathbf{P}$ has the same truth-value on $\mathbf{A}'$ and $\mathbf{A}''$.

d. An iterated conjunction (. . . ($\mathbf{P}_1$ & $\mathbf{P}_2$) & . . . & $\mathbf{P}_\mathbf{n}$) of sentences of *SL* is true on a truth-value assignment if and only if $\mathbf{P}_1$, $\mathbf{P}_2$ . . . , $\mathbf{P}_\mathbf{n}$ are all true on that assignment.

e. Where $\mathbf{P}$ is a sentence of *SL* and $\mathbf{Q}$ is a sentential component of $\mathbf{P}$, let [$\mathbf{P}$] ($\mathbf{Q}_1$//$\mathbf{Q}$) be a sentence that is the result of replacing at least one occurrence of $\mathbf{Q}$ in $\mathbf{P}$ with the sentence $\mathbf{Q}_1$. If $\mathbf{Q}$ and $\mathbf{Q}_1$ are truth-functionally equivalent, then $\mathbf{P}$ and [$\mathbf{P}$] ($\mathbf{Q}_1$//$\mathbf{Q}$) are truth-functionally equivalent.

2. Consider this claim:

No sentence of *SL* that contains only binary connectives is truth-functionally true.

Show that this claim is false by producing a sentence that contains only binary connectives and that is truth-functionally true. Explain where an attempt to prove the claim by mathematical induction (in the manner of the answer to Exercise 1.a) would fail.

6.2 TRUTH-FUNCTIONAL COMPLETENESS

In Chapter 2 we defined the truth-functional use of sentential connectives as follows:

> A sentential connective, of a formal or a natural language, is used *truth-functionally* if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determined by the truth-values of those one or more sentences from which the compound is generated, no matter what those truth-values may be.

The connectives of *SL* have only truth-functional uses since their intended interpretations are given wholly by their characteristic truth-tables. Although *SL* contains only five sentential connectives, we found in Chapter 2 that a great variety of English compounds can nevertheless be adequately symbolized using various combinations of these connectives. For instance, an English sentence of the form

Neither **p** nor **q**

can be appropriately symbolized either by a sentence of the form

$\sim (\mathbf{P} \vee \mathbf{Q})$

or by a sentence of the form

$\sim \mathbf{P} \ \& \sim \mathbf{Q}$

An interesting question now arises: Is *SL* capable of representing all possible truth-functionally compound sentences? We want the answer to this question to be 'yes', because we want *SL* to be an adequate vehicle for all of truth-functional logic. If there is some way of truth-functionally compounding sentences that cannot be represented in *SL*, then there may be some truth-functionally valid arguments, for example, that do not have valid symbolizations in *SL* simply because they cannot be adequately symbolized in *SL*.