Chapter

11

PREDICATE LOGIC: METATHEORY

Section 11.1 presents a variety of semantic results that will be used to prove important metatheorems for *PL*, while Section 11.2 does the same for *PLE*. Section 11.3 proves that the derivation systems *PD*, *PD+*, and *PDE* are sound for predicate logic, and Section 11.4 proves that these systems are complete. Section 11.5 proves that the tree method is sound for predicate logic, while Section 11.6 proves that it is complete.

11.1 SEMANTIC PRELIMINARIES FOR *PL*

In this chapter, we shall establish four major results: the soundness and completeness of the natural deduction systems *PD*, *PD+*, and *PDE*, and the soundness and completeness of the truth-tree method developed in Chapter 9. The results we establish are part of the **metatheory** of predicate logic.

In our proofs of the adequacy of the natural deduction systems and the tree method, we shall use some fundamental semantic results that may seem obvious but that nevertheless must be proved. The purpose of this section is to establish these results. The reader may skim over this section on the first reading without working through all the proofs but should keep in mind that later metatheoretic proofs depend on the results presented here.

Given any formula $\mathbf{P}$, variable $\mathbf{x}$, and constant $\mathbf{a}$, let $\mathbf{P}(\mathbf{a}/\mathbf{x})$ be the formula that results from replacing every free occurrence of $\mathbf{x}$ in $\mathbf{P}$ with $\mathbf{a}$. Our first result establishes that every variable assignment $\mathbf{d_I}$ treats $\mathbf{P}(\mathbf{a}/\mathbf{x})$ exactly as

$\mathbf{d_I[I(a)/x]}$ treats $\mathbf{P}$. If $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$, then the variable assignment that is just like $\mathbf{d_I}$ except that it assigns the denotation of $\mathbf{a}$ to $\mathbf{x}$ will satisfy $\mathbf{P}$, and vice versa. This should not be surprising, for if $\mathbf{x}$ is used to refer to exactly the same thing as $\mathbf{a}$, we would expect $\mathbf{P}$ and $\mathbf{P(a/x)}$ to behave the same way.

> **11.1.1:** Let $\mathbf{P}$ be a formula of *PL*, let $\mathbf{P(a/x)}$ be the formula that results from replacing every free occurrence of $\mathbf{x}$ in $\mathbf{P}$ with an individual constant $\mathbf{a}$, let $\mathbf{I}$ be an interpretation, and let $\mathbf{d_I}$ be a variable assignment for $\mathbf{I}$. Then $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$ on $\mathbf{I}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{P}$ on $\mathbf{I}$.

To prove the result, we shall use mathematical induction on the number of occurrences of logical operators—truth-functional connectives and quantifiers—that occur in $\mathbf{P}$.

Basis clause: If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{P}$.

Proof of basis clause: If $\mathbf{P}$ contains zero occurrences of logical operators, then $\mathbf{P}$ is either a sentence letter or a formula of the form $\mathbf{At_1 \ldots t_n}$, where $\mathbf{A}$ is a predicate and $\mathbf{t_1, \ldots, t_n}$ are individual constants or variables. If $\mathbf{P}$ is a sentence letter, then $\mathbf{P(a/x)}$ is simply $\mathbf{P}$—a sentence letter alone does not contain any variables to be replaced. $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$, then, if and only if $\mathbf{I(P) = T}$. And $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{P}$ if and only if $\mathbf{I(P) = T}$. So $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{P}$.

If $\mathbf{P}$ has the form $\mathbf{At_1 \ldots t_n}$, then $\mathbf{P(a/x)}$ is $\mathbf{At_1' \ldots t_n'}$, where $\mathbf{t_i'}$ is $\mathbf{a}$ if $\mathbf{t_i}$ is $\mathbf{x}$ and $\mathbf{t_i'}$ is just $\mathbf{t_i}$ otherwise. By the definition of satisfaction,

- $\mathbf{d_I}$ satisfies $\mathbf{At_1' \ldots t_n'}$ if and only if $<\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_1'}), \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_2'}), \ldots, \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_n'})> \in \mathbf{I(A)}$.

  (Recall from Chapter 8 that if $\mathbf{t_i}$ is a variable, $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_i}) = \mathbf{d_I(t_i)}$, and if $\mathbf{t_i}$ is an individual constant, $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_i}) = \mathbf{I(t_i)}$.)
- $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{At_1 \ldots t_n}$ if and only if $<\mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_1}), \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_2}), \ldots, \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_n})> \in \mathbf{I(A)}$.

But now we note that

- The $\mathbf{n}$-tuples $<\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_1'}), \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_2'}), \ldots, \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_n'})>$ and $<\mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_1}), \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_2}), \ldots, \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_n})>$ are identical.

Consider: If $\mathbf{t_i}$ is a constant, then $\mathbf{t_i'}$ is $\mathbf{t_i}$ and so $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_i'}) = \mathbf{I(t_i)}$ and $\mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_i}) = \mathbf{I(t_i)}$. If $\mathbf{t_i}$ is any variable other than $\mathbf{x}$, then $\mathbf{t_i'}$ is $\mathbf{t_i}$ and so $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_i'}) = \mathbf{d_I(t_i)} = \mathbf{d_I[I(a)/x](t_i)} = \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{t_i})$—the assignment of $\mathbf{I(a)}$ to $\mathbf{x}$ in the variable assignment does not affect the value assigned to $\mathbf{t_i}$ in this case. If $\mathbf{t_i}$ is the variable $\mathbf{x}$, then the variant ensures that the denotations of $\mathbf{x}$ and $\mathbf{a}$ coincide: $\mathbf{t_i'}$ is $\mathbf{a}$ and $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{a}) = \mathbf{I(a)} = \mathbf{d_I[I(a)/x](x)} = \mathrm{den}_{\mathbf{I,d_I[I(a)/x]}}(\mathbf{x})$.

We conclude that $\mathbf{d_I}$ satisfies $\mathbf{At_1' \ldots t_n'}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{At_1 \ldots t_n}$.

The basis clause—in particular, the case where an atomic formula has the form $\mathbf{At_1 \ldots t_n}$—is the crux of our proof. It will be straightforward to show that the addition of connectives and quantifiers to build larger formulas does not change matters. The inductive step in the proof of 11.1.1 is

Inductive step: If every formula $\mathbf{P}$ with $\mathbf{k}$ or fewer occurrences of logical operators is such that $\mathbf{d_I}$ satisfies $\mathbf{P(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{P}$, then the same is true of every formula $\mathbf{P}$ with $\mathbf{k} + 1$ occurrences of logical operators.

Proof of inductive step: Letting $\mathbf{k}$ be an arbitrary positive integer, we assume that the inductive hypothesis holds—that our claim is true of every formula with $\mathbf{k}$ or fewer occurrences of logical operators. To show that it follows that the claim is also true of every formula $\mathbf{P}$ with $\mathbf{k} + 1$ occurrences of logical operators, we consider each form that $\mathbf{P}$ may have.

Case 1: $\mathbf{P}$ has the form $\sim \mathbf{Q}$. Then $\mathbf{P(a/x)}$ is $\sim \mathbf{Q(a/x)}$, the negation of $\mathbf{Q(a/x)}$ (that is, any replacements of $\mathbf{x}$ that were made had to be made within $\mathbf{Q}$).

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $\sim \mathbf{Q(a/x)}$ if and only if it does not satisfy $\mathbf{Q(a/x)}$.
- Because $\mathbf{Q(a/x)}$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{d}$ fails to satisfy $\mathbf{Q(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ fails to satisfy $\mathbf{Q}$.
- By the definition of satisfaction, $\mathbf{d_I[I(a)/x]}$ fails to satisfy $\mathbf{Q}$ if and only if $\mathbf{d_I[I(a)/x]}$ does satisfy $\sim \mathbf{Q}$.

Therefore, $\mathbf{d_I}$ satisfies $\sim \mathbf{Q(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\sim \mathbf{Q}$.

Case 2: $\mathbf{P}$ has the form $\mathbf{Q} \,\&\, \mathbf{R}$. Then $\mathbf{P(a/x)}$ is $\mathbf{Q(a/x)} \,\&\, \mathbf{R(a/x)}$,

that is, all replacements of $\mathbf{x}$ occurred within $\mathbf{Q}$ and $\mathbf{R}$.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $\mathbf{Q(a/x)} \,\&\, \mathbf{R(a/x)}$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{Q(a/x)}$ and $\mathbf{d_I}$ satisfies $\mathbf{R(a/x)}$.
- Both conjuncts contain fewer than $\mathbf{k} + 1$ occurrences of logical operators, so, by the inductive hypothesis, $\mathbf{d_I}$ satisfies $\mathbf{Q(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{Q}$, and $\mathbf{d_I}$ satisfies $\mathbf{R(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{R}$.
- By the definition of satisfaction, $\mathbf{d_I[I(a)/x]}$ satisfies both $\mathbf{Q}$ and $\mathbf{R}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{Q} \,\&\, \mathbf{R}$.

Therefore, $\mathbf{d_I}$ satisfies $\mathbf{Q(a/x)}$ & $\mathbf{R(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $\mathbf{Q}$ & $\mathbf{R}$.

Cases 3–5: The proofs for the cases in which $\mathbf{P}$ has one of the forms $\mathbf{Q} \vee \mathbf{R}$, $\mathbf{Q} \supset \mathbf{R}$, and $\mathbf{Q} \equiv \mathbf{R}$ are similar to that of Case 2 and are left as exercises.

Case 6: $\mathbf{P}$ has the form $(\forall \mathbf{y})\mathbf{Q}$. We must consider two possibilities. If $\mathbf{y}$ is distinct from the variable $\mathbf{x}$ that $\mathbf{a}$ is replacing in $(\forall \mathbf{y})\mathbf{Q}$, then $\mathbf{P(a/x)}$ is $(\forall \mathbf{y})\mathbf{Q(a/x)}$—all replacements of $\mathbf{x}$ are made within $\mathbf{Q}$.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $(\forall \mathbf{y})\mathbf{Q(a/x)}$ if and only if for every member $\mathbf{u}$ of the UD, $\mathbf{d_I[u/y]}$ satisfies $\mathbf{Q(a/x)}$.
- Because $\mathbf{Q}$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that for every member $\mathbf{u}$ of the UD, $\mathbf{d_I[u/y]}$ satisfies $\mathbf{Q(a/x)}$ if and only if $\mathbf{d_I[u/y, I(a)/x]}$ satisfies $\mathbf{Q}$.
- Each variant $\mathbf{d_I[u/y, I(a)/x]}$ is identical to $\mathbf{d_I[I(a)/x, u/y]}$ because $\mathbf{x}$ and $\mathbf{y}$ are distinct variables, and therefore neither of the assignments within the brackets can override the other.
- So every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I[u/y, I(a)/x]}$ satisfies $\mathbf{Q}$ if and only if $\mathbf{d_I[I(a)/x, u/y]}$ satisfies $\mathbf{Q}$.
- By the definition of satisfaction, every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I[I(a)/x, u/y]}$satisfies $\mathbf{Q}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $(\forall \mathbf{y})\mathbf{Q}$.

Therefore, in the case where $\mathbf{y}$ is distinct from the variable $\mathbf{x}$ that $\mathbf{a}$ is replacing, $\mathbf{d_I}$ satisfies $(\forall \mathbf{y})\mathbf{Q(a/x)}$ if and only if $\mathbf{d_I[I(a)/x]}$ satisfies $(\forall \mathbf{y})\mathbf{Q}$.

If $\mathbf{P}$ is $(\forall \mathbf{x})\mathbf{Q}$, where $\mathbf{x}$ is the variable that $\mathbf{a}$ is replacing, then $\mathbf{P(a/x)}$ is also $(\forall \mathbf{x})\mathbf{Q}$. Because $\mathbf{a}$ replaces only *free* occurrences of $\mathbf{x}$ in $\mathbf{P}$ and $\mathbf{x}$ does not occur free in $\mathbf{P}$, no replacements are made within $\mathbf{Q}$.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ (which is our $\mathbf{P(a/x)}$) if and only if every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I[u/x]}$ satisfies $\mathbf{Q}$.
- $\mathbf{d_I[I(a)/x]}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ (which is our $\mathbf{P}$) if and only every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I[I(a)/x, u/x]}$ satisfies $\mathbf{Q}$.
- What is $\mathbf{d_I[I(a)/x, u/x]}$? This variable assignment is just $\mathbf{d[u/x]}$—the first assignment made to $\mathbf{x}$ within the brackets is overridden by the second.
- So every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I[I(a)/x, u/x]}$ satisfies $\mathbf{Q}$ if and only if every member $\mathbf{u}$ of the UD is such that $\mathbf{d[u/x]}$ satisfies $\mathbf{Q}$.

Therefore, $\mathbf{d_I[I(a)/x]}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$.

Case 7: $\mathbf{P}$ has the form $(\exists \mathbf{y})\mathbf{Q}$. Again we consider two possibilities. If $\mathbf{y}$ is distinct from the variable $\mathbf{x}$ that $\mathbf{a}$ is replacing, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if at least one member $\mathbf{u}$ of the UD is such that $\mathbf{d_I}[\mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$.
- Because $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{d_I}[\mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d_I}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- Because $\mathbf{y}$ and $\mathbf{x}$ are different variables, $\mathbf{d_I}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ is the same variable assignment as $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$.
- So $\mathbf{d_I}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}$.
- By the definition of satisfaction, $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}$ if and only if $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{y})\mathbf{Q}$.

It follows that in the case where $\mathbf{y}$ and $\mathbf{x}$ are different variables, $\mathbf{d_I}$ satisfies $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{y})\mathbf{Q}$.

If $\mathbf{P}$ is $(\exists \mathbf{x})\mathbf{Q}$, where $\mathbf{x}$ is the variable that $\mathbf{a}$ is replacing, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $(\exists \mathbf{x})\mathbf{Q}$—no replacements are made within $\mathbf{Q}$ because $\mathbf{x}$ is not free in $(\exists \mathbf{x})\mathbf{Q}$. So we must show in this case that $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if $\mathbf{d_I}$ satisfies $(\exists \mathbf{x})\mathbf{Q}$.

- By the definition of satisfaction, $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if at least one member $\mathbf{u}$ of the UD is such that $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ is just $\mathbf{d}[\mathbf{u}/\mathbf{x}]$—the second assignment to $\mathbf{x}$ overrides the first.
- So $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$ if and only if $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- By the definition of satisfaction, $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$ if and only if $\mathbf{d_I}$ satisfies $(\exists \mathbf{x})\mathbf{Q}$.

Therefore, $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if $\mathbf{d_I}$ satisfies $(\exists \mathbf{x})\mathbf{Q}$.

That establishes the inductive step, so result 11.1.1 is also established—every formula $\mathbf{P}$ is such that $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}$ if and only if $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$ on $\mathbf{I}$.

The next result will enable us to prove a claim that was made in Chapter 8: that for any interpretation and any sentence of *PL*, either all variable assignments satisfy the sentence or none do. We used this claim in defining truth and falsehood for sentences: A sentence is true on an interpretation if it is satisfied by all variable assignments and false if it is satisfied by none. The reason this claim turns out to be true is that there are no free variables in sentences. Result 11.1.2 assures us that only the values that a variable assignment assigns to the variables that are free in a formula play a role in determining whether the formula is satisfied:

11.1.2: Let $\mathbf{I}$ be an interpretation, $\mathbf{d_I}$ a variable assignment for $\mathbf{I}$, and $\mathbf{P}$ a formula of *PL*. Then $\mathbf{d_I}$ satisfies $\mathbf{P}$ on $\mathbf{I}$ if and only if every variable assignment that assigns the same values to the free variables in $\mathbf{P}$ as $\mathbf{d_I}$ satisfies $\mathbf{P}$.

Proof: Let $\mathbf{I}$ be an interpretation, $\mathbf{d_I}$ a variable assignment for $\mathbf{I}$, and $\mathbf{P}$ a formula of *PL*. We shall prove 11.1.2 by mathematical induction on the number of occurrences of logical operators in $\mathbf{P}$.

Basis clause: If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{d_I}$ satisfies $\mathbf{P}$ if and only if every variable assignment that assigns the same values to the free variables in $\mathbf{P}$ as $\mathbf{d_I}$ satisfies $\mathbf{P}$.

Proof of basis clause: If $\mathbf{P}$ contains zero occurrences of logical operators, then $\mathbf{P}$ is either a sentence letter or a formula of the form $\mathbf{At}_1 \ldots \mathbf{t_n}$. If $\mathbf{P}$ is a sentence letter, then any variable assignment satisfies $\mathbf{P}$ on $\mathbf{I}$ if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. Therefore $\mathbf{d_I}$ satisfies $\mathbf{P}$ if and only if every variable assignment that assigns the same values to the free variables in $\mathbf{P}$ as $\mathbf{d_I}$ (that is, every variable assignment) satisfies $\mathbf{P}$.

If $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t_n}$, then by the definition of satisfaction,

- $\mathbf{d_I}$ satisfies $\mathbf{P}$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n})> \in \mathbf{I}(\mathbf{A})$.

And where $\mathbf{d'_I}$ is a variable assignment that assigns the same values to the free variables in $\mathbf{P}$ as $\mathbf{d_I}$,

- $\mathbf{d'_I}$ satisfies $\mathbf{P}$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t_n})> \in \mathbf{I}(\mathbf{A})$.

But now we note that

- the $\mathbf{n}$-tuples $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n})>$ and $<\text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t_n})>$ are identical.

For if $\mathbf{t_i}$ is a constant, then $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i}) = \text{den}_{\mathbf{I},\mathbf{d'_I}}(\mathbf{t_i}) = \mathbf{I}(\mathbf{t_i})$. And if $\mathbf{t_i}$ is a variable, then $\mathbf{t_i}$ is free in $\mathbf{At}_1 \ldots \mathbf{t_n}$ and, by our assumption, $\mathbf{d_I}$ and $\mathbf{d'_I}$ assign the same value to $\mathbf{t_i}$. Hence, we conclude that $\mathbf{d_I}$ satisfies $\mathbf{At}_1 \ldots \mathbf{t_n}$ if and only if every variable assignment $\mathbf{d'_I}$ that assigns the same values to the free variables in $\mathbf{At}_1 \ldots \mathbf{t_n}$ as $\mathbf{d_I}$ satisfies $\mathbf{At}_1 \ldots \mathbf{t_n}$.

Inductive step: If every sentence $\mathbf{P}$ that has $\mathbf{k}$ or fewer occurrences of logical operators is such that $\mathbf{d_I}$ satisfies $\mathbf{P}$ on $\mathbf{I}$ if and only if every variable assignment that assigns the same values to the free variables in $\mathbf{P}$ as $\mathbf{d_I}$ satisfies $\mathbf{P}$, then the same is true of every sentence $\mathbf{P}$ that contains $\mathbf{k} + 1$ occurrences of logical operators.

Proof of inductive step: Assume that, for an arbitrary positive integer **k**, the inductive hypothesis is true. We shall show that on this assumption our claim must also be true of every sentence **P** that contains **k** + 1 occurrences of logical operators. Let **I** be an interpretation and **d** a variable assignment for **I**. We consider each form that **P** may have.

Case 1: P has the form ~ **Q**.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies ~ **Q** if and only if $\mathbf{d_I}$ fails to satisfy **Q**.
- Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{d_I}$ fails to satisfy **Q** if and only if every variable assignment that assigns the same values to the free variables in **Q** fails to satisfy **Q**.
- By the definition of satisfaction, every variable assignment that assigns the same values to the free variables in **Q** as **d** fails to satisfy **Q** if and only if every such assignment does satisfy ~ **Q**.
- The variable assignments that assign the same values to the free variables in **Q** as **d** are the variable assignments that assign the same values to the free variables of ~ **Q** as **d**, because **Q** and ~ **Q** contain the same free variables.

Therefore, $\mathbf{d_I}$ satisfies ~ **Q** if and only if every variable assignment that assigns the same values to the free variables in ~ **Q** satisfies ~ **Q**.

Case 2: P has the form **Q** ∨ **R**.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies **Q** ∨ **R** if and only if either $\mathbf{d_I}$ satisfies **Q** or $\mathbf{d_I}$ satisfies **R**.
- Because **Q** and **R** each contain fewer than **k** + 1 occurrences of logical operators, it follows by the inductive hypothesis that $\mathbf{d_I}$ satisfies **Q** if and only if every variable assignment that assigns the same values to the free variables in **Q** satisfies **Q**, and $\mathbf{d_I}$ satisfies **R** if and only if every variable assignment that assigns the same values to the free variables in **R** satisfies **R**.
- Therefore, $\mathbf{d_I}$ satisfies **Q** ∨ **R** if and only if either every variable assignment that assigns the same values to the free variables in **Q** satisfies **Q** or every variable assignment that assigns the same values to the free variables in **R** satisfies **R**.
- We note that every variable that is free in **Q** is also free in **Q** ∨ **R**, so every variable assignment that assigns the same values as $\mathbf{d_I}$ to the free variables in **Q** ∨ **R** is a variable assignment that assigns the same values as $\mathbf{d_I}$ to the free variables in **Q**; and the same is true of **R.** (The converses do not hold.)

We conclude that $\mathbf{d_I}$ satisfies $\mathbf{Q} \vee \mathbf{R}$ if and only if every variable assignment that assigns the same values as $\mathbf{d_I}$ to the free variables in $\mathbf{Q} \vee \mathbf{R}$ satisfies $\mathbf{Q} \vee \mathbf{R}$.

Cases 3–5: $\mathbf{P}$ has one of the forms $\mathbf{Q}$ & $\mathbf{R}$, $\mathbf{Q} \supset \mathbf{R}$, or $\mathbf{Q} \equiv \mathbf{R}$. These cases are left as an exercise.

Case 6: $\mathbf{P}$ has the form $(\forall \mathbf{x})\mathbf{Q}$.

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- Because $\mathbf{Q}$ contains fewer than $\mathbf{k} + 1$ occurrences of connectives, it follows from the inductive hypothesis that every member $\mathbf{u}$ of the UD is such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$ if and only if every variable assignment that assigns the same values to the free variables in $\mathbf{Q}$ as $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- It follows that $\mathbf{d}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if every member $\mathbf{u}$ of the UD is such that every variable assignment that assigns the same values to the free variables in $\mathbf{Q}$ as $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$.
- Because the variables other than $\mathbf{x}$ that are free in $\mathbf{Q}$ are also free in $(\forall \mathbf{x})\mathbf{Q}$, every variable assignment that assigns the same values to the free variables in $\mathbf{Q}$ as $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ is a variant $\mathbf{d_I'}[\mathbf{u}/\mathbf{x}]$ of a variable assignment $\mathbf{d_I'}$ that assigns the same values to the free variables in $(\forall \mathbf{x})\mathbf{Q}$ as $\mathbf{d_I}$, and vice versa.
- So $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if every member $\mathbf{u}$ of the UD is such that every variant $\mathbf{d_I'}[\mathbf{u}/\mathbf{x}]$ of any variable assignment $\mathbf{d_I'}$ that assigns the same values to the free variables in $(\forall \mathbf{x})\mathbf{Q}$ as $\mathbf{d_I}$ satisfies $\mathbf{Q}$.

It follows by the definition of satisfaction that $\mathbf{d_I}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if every variable assignment that assigns the same values to the free variables in $(\forall \mathbf{x})\mathbf{Q}$ satisfies $(\forall \mathbf{x})\mathbf{Q}$.

Case 7: $\mathbf{P}$ has the form $(\exists \mathbf{x})\mathbf{Q}$. This case is left as an exercise.

It follows immediately from 11.1.2 that

11.1.3: For any interpretation $\mathbf{I}$ and sentence $\mathbf{P}$ of *PL*, either every variable assignment for $\mathbf{I}$ satisfies $\mathbf{P}$ or no variable assignment for $\mathbf{I}$ satisfies $\mathbf{P}$.

Proof: Let $\mathbf{d_I}$ be any variable assignment. Because $\mathbf{P}$ is a sentence and hence contains no free variables, every variable assignment for $\mathbf{I}$ assigns the same values to the free variables in $\mathbf{P}$ as does $\mathbf{d_I}$. By result 11.1.2, then, $\mathbf{d_I}$ satisfies $\mathbf{P}$ if and only if every variable assignment satisfies $\mathbf{P}$. Therefore either every variable assignment satisfies $\mathbf{P}$ or none does.

Each of the following results, which can be established using results 11.1.1–11.1.3, states something that we would hope to be true of quantified sentences of *PL*.

11.1.4: For any universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ of *PL*, $\{(\forall \mathbf{x})\mathbf{P}\}$ quantificationally entails every substitution instance of $(\forall \mathbf{x})\mathbf{P}$.

Proof: Let $(\forall \mathbf{x})\mathbf{P}$ be any universally quantified sentence, let $\mathbf{P}(\mathbf{a}/\mathbf{x})$ be a substitution instance of $(\forall \mathbf{x})\mathbf{P}$, and let $\mathbf{I}$ be an interpretation on which $(\forall \mathbf{x})\mathbf{P}$ is true. Then, by 11.1.3, every variable assignment satisfies $(\forall \mathbf{x})\mathbf{P}$, and so, for every variable assignment $\mathbf{d_I}$ and every member $\mathbf{u}$ of the UD, $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{P}$. In particular, for every variable assignment $\mathbf{d_I}$ the variant $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ must satisfy $\mathbf{P}$. By 11.1.1, then, every variable assignment $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$, so $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is also true on $\mathbf{I}$.

11.1.5: Every substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$ is such that $\{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \vDash (\exists \mathbf{x})\mathbf{P}$.

Proof: See Exercise 3.

11.1.4 and 11.1.5 are results that were used to motivate informally two of the quantifier rules in Chapter 10, Universal Elimination and Existential Introduction, and they will play a role in our proof of the soundness of *PD*. We also want to ensure that the motivations for Universal Introduction and Existential Elimination were correct. We'll first establish two further results that we shall need:

11.1.6: Let $\mathbf{I}$ and $\mathbf{I}'$ be interpretations that have the same UD and that agree on the assignments made to each individual constant, predicate, and sentence letter in a formula $\mathbf{P}$ (that is, $\mathbf{I}$ and $\mathbf{I}'$ assign the same values to those symbols). Then each variable assignment $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}'$.

In stating result 11.1.6, we have made use of the fact that if two interpretations have the same UD, then every variable assignment for one interpretation is a variable assignment for the other. The result should sound obvious: If two interpretations with identical universes of discourse treat the nonlogical symbols of $\mathbf{P}$ in the same way, and if the free variables are interpreted the same way on the two interpretations, then $\mathbf{P}$ says the same thing on both interpretations, and the values that $\mathbf{I}$ and $\mathbf{I}'$ assign to other symbols of *PL* have no bearing on what $\mathbf{P}$ says.

Proof of 11.1.6: Let $\mathbf{P}$ be a formula of *PL* and let $\mathbf{I}$ and $\mathbf{I}'$ be interpretations that have the same UD and that agree on the values assigned to each nonlogical symbol in **P**. We shall prove, by mathematical induction on the number of occurrences of logical operators in **P**, that a variable assignment satisfies **P** on interpretation **I** if and only if it satisfies **P** on interpretation **I**′.

Basis clause: If **P** contains zero occurrences of logical operators, then a variable assignment $\mathbf{d_I}$ satisfies **P** on **I** if and only if it satisfies **P** on **I**′.

Proof of basis clause: If **P** is a sentence letter, then $\mathbf{d_I}$ satisfies **P** on **I** if and only if $\mathbf{I(P)} = \mathbf{T}$, and it satisfies **P** on **I**′ if and only if $\mathbf{I'(P)} = \mathbf{T}$. By our assumption, $\mathbf{I(P)} = \mathbf{I'(P)}$, so $\mathbf{d_I}$ satisfies **P** on **I** if and only if $\mathbf{d_I}$ satisfies **P** on **I**′.

If **P** is an atomic formula $\mathbf{At}_1 \ldots \mathbf{t_n}$,

- By the definition of satisfaction, $\mathbf{d_I}$ satisfies **P** on **I** if and only if $<\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_n})> \in \mathbf{I(A)}$, and $\mathbf{d_I}$ satisfies **P** on **I**′ if and only if $<\mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_2)> \ldots <\mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t_n})> \in \mathbf{I\_(A)}$.
- We note that $<\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_n})>$ and $<\mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t_n})>$ are identical. This is because if $\mathbf{t_i}$ is a constant, then $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_1) = \mathbf{I(t_i)}$, $\mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_1) = \mathbf{I'(t_i)}$, and $\mathbf{I'(t_i)} = \mathbf{I(t_i)}$ since by assumption, **I** and **I**′ assign the same values to the nonlogical symbols in **P**; and if $\mathbf{t_i}$ is a variable, then $\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_i) = \mathbf{d_I(t_i)} = \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_i)$.
- Moreover, $\mathbf{I(A)} = \mathbf{I'(A)}$, by stipulation.
- It follows that $<\mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I,d_I}}(\mathbf{t_n})> \in \mathbf{I(A)}$ if and only if $<\mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I',d_I}}(\mathbf{t_n})> \in \mathbf{I'(A)}$.

Therefore, $\mathbf{d_I}$ satisfies $\mathbf{At}_1 \ldots \mathbf{t_n}$ on **I** if and only if it does so on **I**′.

Inductive step: If every formula **P** that has **k** or fewer occurrences of logical operators is such that a variable assignment satisfies **P** on **I** if and only if it satisfies **P** on **I**′, then the same is true of every formula **P** that has **k** + 1 occurrences of logical operators.

Proof of inductive step: We consider the forms that **P** may have.

Case 1: P has the form ~ **Q**.

- By the definition of satisfaction, a variable assignment $\mathbf{d_I}$ satisfies ~ **Q** on **I** if and only if it fails to satisfy **Q** on **I**.
- Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that a variable assignment fails to satisfy **Q** on **I** if and only if it fails to satisfy **Q** on **I**′.
- A variable assignment fails to satisfy **Q** on **I**′ if and only if it does satisfy ~ **Q** on **I**′.

Therefore, a variable assignment satisfies ~ **Q** on **I** if and only if it satisfies ~ **Q** on **I**′.

Case 2: **P** has the form **Q** & **R**.

- By the definition of satisfaction, a variable assignment satisfies **Q** & **R** on **I** if and only if it satisfies both **Q** and **R** on **I**.
- **Q** and **R** each contain **k** or fewer occurrences of logical operators, and so by the inductive hypothesis, a variable assignment satisfies both **Q** and **R** on **I** if and only if it satisfies both **Q** and **R** on **I**′.
- A variable assignment satisfies both **Q** and **R** on **I**′ if and only if it satisfies **Q** & **R** on **I**′.

Therefore, a variable assignment satisfies **Q** & **R** on **I** if and only if it satisfies **Q** & **R** on **I**′.

Cases 3–5: **P** has the form **Q** ∨ **R**, **Q** ⊃ **R**, or **Q** ≡ **R**. We omit proofs for these cases as they are strictly analogous to Case 2.

Case 6: **P** has the form (∀**x**)**Q**.

- By the definition of satisfaction, a variable assignment $\mathbf{d_I}$ satisfies (∀**x**)**Q** on **I** if and only if every member **u** of **I**'s UD is such that $\mathbf{d_I}$[**u**/**x**] satisfies **Q** on **I**.
- $\mathbf{d_I}$ satisfies (∀**x**)**Q** on **I**′ if and only if every member **u** of **I**′'s UD (which is the same as **I**'s UD) is such that $\mathbf{d_I}$[**u**/**x**] satisfies **Q** on **I**′.
- Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that every member **u** of the common UD is such that $\mathbf{d_I}$[**u**/**x**] satisfies **Q** on **I** if and only if $\mathbf{d_I}$[**u**/**x**] satisfies **Q** on **I**′.

Therefore, a variable assignment satisfies (∀**x**)**Q** on **I** if and only if it satisfies (∀**x**)**Q** on **I**′.

Case 7: **P** has the form (∃**x**)**Q**. This case is similar to Case 6.

That completes the proof of the inductive step, and we may now conclude that 11.1.6 is true.

Result 11.1.7 follows as an immediate consequence of 11.1.6:

11.1.7: Let **I** and **I**′ be interpretations that have the same UD and that agree on the assignments made to each individual constant, predicate, and sentence letter in a sentence **P**. Then **P** is true on **I** if and only if **P** is true on **I**′.

Proof: Let $\mathbf{I}$ and $\mathbf{I}'$ be as specified for a sentence $\mathbf{P}$. If $\mathbf{P}$ is true on $\mathbf{I}$, then, by 11.1.2, $\mathbf{P}$ is satisfied by every variable assignment on $\mathbf{I}$. By 11.1.6, this is the case if and only if $\mathbf{P}$ is satisfied by every variable assignment on $\mathbf{I}'$, that is, if and only if $\mathbf{P}$ is true on $\mathbf{I}'$.

With results 11.1.6 and 11.1.7 at hand, we may now show that our motivations for the rules Universal Introduction and Existential Elimination are correct.

11.1.8: Let $\mathbf{a}$ be a constant that does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of the set Γ. Then if $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, $\Gamma \vDash (\forall \mathbf{x})\mathbf{P}$.

11.1.9: Let $\mathbf{a}$ be a constant that does not occur in the sentences $(\exists \mathbf{x})\mathbf{P}$ and $\mathbf{Q}$ and that does not occur in any member of the set Γ. If $\Gamma \vDash (\exists \mathbf{x})\mathbf{P}$ and $\Gamma \cup \{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \vDash \mathbf{Q}$, then $\Gamma \vDash \mathbf{Q}$ as well.

We shall prove 11.1.8 here; 11.1.9 is left as an exercise.

Proof of 11.1.8: Assume that $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, where $\mathbf{a}$ does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of Γ. We shall assume, contrary to what we want to show, that Γ does not quantificationally entail $(\forall \mathbf{x})\mathbf{P}$—that there is at least one interpretation, call it $\mathbf{I}$, on which every member of Γ is true and $(\forall \mathbf{x})\mathbf{P}$ is false. We shall use $\mathbf{I}$ as the basis for constructing an interpretation $\mathbf{I}'$ on which every member of Γ is true and the substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is false, contradicting our original assumption. Having done so, we may conclude that if Γ does quantificationally entail $\mathbf{P}(\mathbf{a}/\mathbf{x})$, it must also quantificationally entail $(\forall \mathbf{x})\mathbf{P}$.

So assume that $\mathbf{I}$ is an interpretation on which every member of Γ is true and on which $(\forall \mathbf{x})\mathbf{P}$ is false. Because $(\forall \mathbf{x})\mathbf{P}$ is false, there is no variable assignment for $\mathbf{I}$ that satisfies $(\forall \mathbf{x})\mathbf{P}$. That is, for every variable assignment $\mathbf{d_I}$, there is at least one member $\mathbf{u}$ of the UD such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ does not satisfy $\mathbf{P}$. Choose one of these members, calling it $\mathbf{u}$, and let $\mathbf{I}'$ be the interpretation that is just like $\mathbf{I}$ except that it assigns $\mathbf{u}$ to $\mathbf{a}$ (all other assignments made by $\mathbf{I}$ remain the same). It is now straightforward to show that every member of Γ is true on $\mathbf{I}'$ and $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is false. That every member of Γ is true on $\mathbf{I}'$ follows from 11.1.7 because $\mathbf{I}$ and $\mathbf{I}'$ assign the same values to all the nonlogical symbols of *PL* other than $\mathbf{a}$, and, by stipulation, $\mathbf{a}$ does not occur in any member of Γ.

On our assumption that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}$, it follows from 11.1.6 that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}'$. By the way we have constructed $\mathbf{I}'$, $\mathbf{u}$ is $\mathbf{I}'(\mathbf{a})$ and so $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ is $\mathbf{d_I}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$. Result 11.1.1 tells us that $\mathbf{d_I}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$ on $\mathbf{I}'$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}'$. So, because $\mathbf{d_I}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}'$, $\mathbf{d_I}$ does not satisfy $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}'$. By 11.1.3, then, no variable assignment satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}'$, and it is therefore false on this interpretation. But this contradicts our first assumption, that $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, and so we conclude that if $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, then $\Gamma \vDash (\forall \mathbf{x})\mathbf{P}$ as well.

Result 11.1.8 tells us that the rule Universal Introduction is indeed truth-preserving.

We shall state four more semantic results that will be needed in the sections that follow and that the reader should now be able to prove. The proofs are left as exercises. The first result relies on 11.1.6 and 11.1.7, much as the proofs of 11.1.8 and 11.1.9 do.

11.1.10: If $\mathbf{a}$ does not occur in any member of the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ and if the set is quantificationally consistent, then the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is also quantificationally consistent.

Results 11.1.11 and 11.1.12 concern interpretations of a special sort: interpretations on which every member of the UD has a name.

11.1.11: Let $\mathbf{I}$ be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\forall \mathbf{x})\mathbf{P}$ is true on $\mathbf{I}$, so is $(\forall \mathbf{x})\mathbf{P}$.

11.1.12: Let $\mathbf{I}$ be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is false on $\mathbf{I}$, so is $(\exists \mathbf{x})\mathbf{P}$.

Result 11.1.13 says that, if we rename the individual designated by some individual constant in a sentence $\mathbf{P}$ with a constant that does not already occur in $\mathbf{P}$, then, for any interpretation on which $\mathbf{P}$ is true, there is a closely related interpretation (one that reflects the renaming) on which the new sentence is true:

11.1.13: Let $\mathbf{P}$ be a sentence of *PL*, let $\mathbf{b}$ be an individual constant that does not occur in $\mathbf{P}$, and let $\mathbf{P}(\mathbf{b}/\mathbf{a})$ be the sentence that results from replacing every occurrence of the individual constant $\mathbf{a}$ in $\mathbf{P}$ with $\mathbf{b}$. Then if $\mathbf{P}$ is true on an interpretation $\mathbf{I}$, $\mathbf{P}(\mathbf{b}/\mathbf{a})$ is true on the interpretation $\mathbf{I}'$ that is just like $\mathbf{I}$ except that it assigns $\mathbf{I}(\mathbf{a})$ to $\mathbf{b}$ ($\mathbf{I}'(\mathbf{b}) = \mathbf{I}(\mathbf{a})$).

11.1E EXERCISES

***1.** Prove Cases 3–5 in the proof of result 11.1.1.

***2.** Prove Cases 3–5 and 7 in the proof of result 11.1.2.

***3.** Prove result 11.1.5.

***4.** Prove result 11.1.9.

5. Prove result 11.1.10.

6. Prove result 11.1.11.

***7.** Prove result 11.1.12.

***8.** Prove result 11.1.13.

When we turn to the metatheory for *PLE*, we shall need versions of Section 11.1's semantic results that apply to sentences containing the identity operator and complex terms. In this section we discuss the changes that must be made in the statement of the semantic results and in their proofs.

Starting with 11.1.1, we note that we must generalize the result to read:

> Let $\mathbf{P}$ be a formula of *PLE*, let $\mathbf{P}(\mathbf{t}/\mathbf{x})$ be the formula that results from replacing every free occurrence of $\mathbf{x}$ in $\mathbf{P}$ with a closed term $\mathbf{t}$, let $\mathbf{I}$ be an interpretation, let $\mathbf{d_I}$ be a variable assignment for $\mathbf{I}$, and let $\mathbf{d_I}' = \mathbf{d_I}[\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t})/\mathbf{x}]$ (that is, $\mathbf{d_I}'$ is just like $\mathbf{d_I}$ except that it assigns to $\mathbf{x}$ whatever $\mathbf{d_I}$ and $\mathbf{I}$ assign to $\mathbf{t}$). Then $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{t}/\mathbf{x})$ on $\mathbf{I}$ if and only if $\mathbf{d_I}'$ satisfies $\mathbf{P}$ on $\mathbf{I}$.

To modify the proof of 11.1.1, we first establish a result concerning complex terms:

> **11.2.1:** Let $\mathbf{t}$ be a complex term of *PLE*, let $\mathbf{t}(\mathbf{c}/\mathbf{x})$ be the term that results from replacing every occurrence of the variable $\mathbf{x}$ in $\mathbf{t}$ with a closed term $\mathbf{c}$, let $\mathbf{I}$ be an interpretation, let $\mathbf{d_I}$ be a variable assignment for $\mathbf{I}$, and let $\mathbf{d_I}' = \mathbf{d_I}[\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{c})/\mathbf{x}]$. Then $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}(\mathbf{c}/\mathbf{x}))$.

The result states that, if the sole difference between two complex terms $\mathbf{t}_1$ and $\mathbf{t}_2$ is that one contains the closed term $\mathbf{c}$ where the other contains the variable $\mathbf{x}$, then $\mathbf{t}_1$ and $\mathbf{t}_2$ denote the same individual if $\mathbf{x}$ and $\mathbf{c}$ do. We shall prove 11.2.1 by mathematical induction on the number of occurrences of functors in the term.

> *Basis clause:* If a complex term $\mathbf{t}$ contains one functor, then $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}(\mathbf{c}/\mathbf{x}))$.
>
> **Proof of basis clause:** If a complex term $\mathbf{t}$ contains one functor, then $\mathbf{t}$ is $f(\mathbf{t}_1, \ldots, \mathbf{t_n})$ where f is a functor, each $\mathbf{t_i}$ is either a variable or a constant, and $\mathbf{t}(\mathbf{c}/\mathbf{x})$ is $f(\mathbf{t}_1', \ldots, \mathbf{t_n}')$ where $\mathbf{t_i}'$ is $\mathbf{t_i}$ if $\mathbf{t_i}$ is not $\mathbf{x}$, and $\mathbf{t_i}'$ is $\mathbf{c}$ if $\mathbf{t_i}$ is $\mathbf{x}$.
>
> As in the proof of the basis clause of 11.1.1, we note that if $\mathbf{t_i}$ is a constant or variable other than $\mathbf{x}$, $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_i}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i})$—since $\mathbf{d_I}'$ does not differ from $\mathbf{d_I}$ in a way that affects the denotation of these terms. If $\mathbf{t_i}$ is $\mathbf{x}$, then $\mathbf{t_i}'$ is $\mathbf{c}$, and by the definition of how $\mathbf{d_I}'$ was constructed, $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{x}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{c})$. So we know that $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n})>$ and $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n}')>$ are identical. Therefore the $\mathbf{n}$+1-tuple $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n}), \mathbf{u}> \in \mathbf{I}(f)$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n}'), \mathbf{u}> \in \mathbf{I}(f)$, so $\text{den}_{\mathbf{I},\mathbf{d_I}'}(f(\mathbf{t}_1, \ldots, \mathbf{t_n})) = \text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t_n}'))$.

Inductive step: If every complex term $\mathbf{t}$ that contains $\mathbf{k}$ or fewer functors is such that $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}) = \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}(\mathbf{c}/\mathbf{x}))$, then the same is true of every complex term that contains $\mathbf{k} + 1$ functors.

Proof of inductive step: We assume that the inductive hypothesis holds—that the claim is true of every complex term that contains $\mathbf{k}$ or fewer functors, for some arbitrary positive integer $\mathbf{k}$. We must show that the claim is also true of every complex term that contains $\mathbf{k} + 1$ functors. If $\mathbf{t}$ contains $\mathbf{k} + 1$ functors, then $\mathbf{t}$ is $f(\mathbf{t}_1, \ldots, \mathbf{t_n})$, where each $\mathbf{t_i}$ has $\mathbf{k}$ or fewer functors, and $\mathbf{t}(\mathbf{c}/\mathbf{x})$ is $f(\mathbf{t}_1', \ldots, \mathbf{t_n}')$, where each $\mathbf{t_i}'$ is $\mathbf{t_i}(\mathbf{c}/\mathbf{x})$. So each $\mathbf{t_i}$ falls under the inductive hypothesis; that is, $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_i}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i}(\mathbf{t_c}/\mathbf{x}))$, and so $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n})>$ and $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n}')>$ are identical. Therefore, $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n}), \mathbf{u}> \in \mathbf{I}(f)$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n}'), \mathbf{u}> \in \mathbf{I}(f)$, so $\text{den}_{\mathbf{I},\mathbf{d_I}'}(f(\mathbf{t}_1, \ldots, \mathbf{t_n})) = \text{den}_{\mathbf{I},\mathbf{d_I}}(f(\mathbf{t}_1', \ldots, \mathbf{t_n}'))$.

With result 11.2.1 in hand, we can modify the basis clause of result 11.1.1 as follows:

Basis clause: If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{d_I}$ satisfies $\mathbf{P}(\mathbf{t}/\mathbf{x})$ if and only if $\mathbf{d_I}'$ satisfies $\mathbf{P}$.

Proof of basis clause: The proof is identical to the proof of 11.1.1, except that we must change the second part of the basis clause so that it covers complex terms and we must also consider atomic identity formulas.

If $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t_n}$, then $\mathbf{P}(\mathbf{t}/\mathbf{x})$ is $\mathbf{At}_1' \ldots \mathbf{t_n}'$, where $\mathbf{t_i}'$ is $\mathbf{t}$ if $\mathbf{t_i}$ is $\mathbf{x}$ and $\mathbf{t_i}'$ is just $\mathbf{t_i}$ otherwise. By the definition of satisfaction,

- $\mathbf{d_I}$ satisfies $\mathbf{At}_1' \ldots \mathbf{t_n}'$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n}')> \in \mathbf{I}(\mathbf{A})$.
- $\mathbf{d_I}'$ satisfies $\mathbf{At}_1 \ldots \mathbf{t_n}$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n})> \in \mathbf{I}(\mathbf{A})$.

But now we note that

- $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1'), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_n}')>$ and $<\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_n})>$ are identical.

Consider: If $\mathbf{t_i}$ is a constant, then $\mathbf{t_i}'$ is $\mathbf{t_i}$ and so $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i}') = \mathbf{I}(\mathbf{t_i}) = \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_i})$. If $\mathbf{t_i}$ is a variable other than $\mathbf{x}$, then $\mathbf{t_i}'$ is $\mathbf{t_i}$ and so $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i}') = \mathbf{d_I}(\mathbf{t_i}) = \mathbf{d_I}'(\mathbf{t_i}) = \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_i})$—the variation in the variable assignment does not affect the value assigned to $\mathbf{t_i}$ in this case. If $\mathbf{t_i}$ is the variable $\mathbf{x}$, then $\mathbf{t_i}'$ is $\mathbf{t}$ and $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}) = \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{x})$. (The variant $\mathbf{d_I}'$ was defined in a way that ensures that the denotations of $\mathbf{x}$ and of $\mathbf{t}$ coincide.) And if $\mathbf{t_i}$ is a complex term, it follows from 11.2.1 that $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_i}') = \text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t_i})$.

Because the $\mathbf{n}$-tuples are the same, we conclude that $\mathbf{d_I}$ satisfies $\mathbf{At}_1' \ldots \mathbf{t_n}'$ if and only if $\mathbf{d_I}'$ satisfies $\mathbf{At}_1 \ldots \mathbf{t_n}$.

We must also add a new case to the proof of the basis clause, to cover formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. We leave this as an exercise. The rest of the proof of 11.1.1 remains the same, except that we replace $\mathbf{d_I}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ with $\mathbf{d_I'}$ (which is shorthand for $\mathbf{d_I}[\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t_c})/\mathbf{x}]$) throughout.

The proof of 11.1.2 is modified in a similar way. First, we need to prove

> **11.2.2:** Let $\mathbf{I}$ be an interpretation, $\mathbf{d_I}$ a variable assignment for $\mathbf{I}$, and $\mathbf{t}$ a complex term of *PLE*. Then, for any variable assignment $\mathbf{d_I}'$ that assigns the same values to the variables in $\mathbf{t}$ as $\mathbf{d_I}$, $\text{den}_{\mathbf{I},\mathbf{d_I}'}(\mathbf{t}) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t})$.
>
> **Proof:** See Exercise 2.

With this result at hand, the basis clause in the proof of 11.1.2 can now be modified to include atomic formulas containing complex terms and also to include formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. Both modifications are left as exercises.

The proof of result 11.1.6 can be similarly modified, once we have established

> **11.2.3:** Let $\mathbf{t}$ be a complex term of *PLE* and let $\mathbf{I}$ and $\mathbf{I}'$ be interpretations that have the same UD and that agree on the values assigned to each individual constant and functor in $\mathbf{t}$. Then, for any variable assignment $\mathbf{d_I}$, $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}) = \text{den}_{\mathbf{I}',\mathbf{d_I}}(\mathbf{t})$.
>
> **Proof:** See Exercise 11.2.3.

Result 11.1.6 must itself be changed to say:

> Let $\mathbf{I}$ and $\mathbf{I}'$ be interpretations that have the same UD and that agree on the assignments made to each individual constant, functor, predicate, and sentence letter in a formula $\mathbf{P}$. Then each variable assignment $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}'$.

The basis clause must be modified to cover formulas containing complex terms, as well as formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. This is left as an exercise.

The proofs of results 11.1.3–11.1.5 and 11.1.7–11.1.13 for *PLE* are the same as for *PL*, except for the following changes:

1. The proofs must use the modified versions of 11.1.1, 11.1.2, and 11.1.6 in order to apply to *PLE*.
2. Where '$\mathbf{a}$' and '$\mathbf{P}(\mathbf{a}/\mathbf{x})$' are used in results 11.1.4 and 11.1.5 to refer to substitution instances of $\mathbf{P}$, we need to use '$\mathbf{t}$' and '$\mathbf{P}(\mathbf{t}/\mathbf{x})$' instead to allow for instantiation with arbitrary closed terms.
3. In 11.1.7, $\mathbf{I}$ and $\mathbf{I}'$ must also agree on the assignments made to each functor.

4. Results 1.1.11 and 1.1.12 are true for *PLE* in two senses: We can change 'every substitution instance' to 'every substitution instance in which the instantiating individual term is a constant', or we can leave the phrase as it is, to include substitution instances with instantiation by all closed terms, complex ones as well as constants.

Finally we shall need two additional semantic results for *PLE*:

11.2.4: For any closed terms $\mathbf{t}_1$ and $\mathbf{t}_2$, if $\mathbf{P}$ is a sentence that contains $\mathbf{t}_1$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vDash \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$, and if $\mathbf{P}$ is a sentence that contains $\mathbf{t}_2$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vDash \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$.

Proof: See Exercise 11.2.4.

11.2.5: If a quantificationally consistent set Γ contains a sentence with a complex term $f(\mathbf{a}_1, \ldots, \mathbf{a_n})$, where $\mathbf{a}_1, \ldots, \mathbf{a_n}$ are constants, and the constant $\mathbf{b}$ does not occur in Γ, then the set $\Gamma \cup \{\mathbf{b} = f(\mathbf{a}_1, \ldots, \mathbf{a_n})\}$ is also quantificationally consistent.

Proof: See Exercise 11.2.5.

11.2E EXERCISES

***1.** Show the changes that must be made in the basis clauses of the proofs of the following results so that they cover formulas containing complex terms and formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$:

a. Result 11.1.1
b. Result 11.1.2
c. Result 11.1.6

***2.** Prove result 11.2.2.

***3.** Prove result 11.2.3.

4. Prove result 11.2.4.

***5.** Prove result 11.2.5.

11.3 THE SOUNDNESS OF *PD*, *PD+*, AND *PDE*

We shall now establish the soundness of our natural deduction systems. A natural deduction system is said to be **sound** for predicate logic if every rule in that system is truth-preserving. The *Soundness Metatheorem* for *PD* is

Metatheorem 11.3.1: For any set Γ of sentences of *PL* and any sentence $\mathbf{P}$ of *PL*, if $\Gamma \vdash \mathbf{P}$ in *PD*, then $\Gamma \vDash$ P.

(As in Chapter 6, we shall drop 'in *PD*' when we use the single turnstile in this chapter, and we shall use the double turnstile to signify *quantificational* entailment.) We shall prove metatheorem 11.3.1 by mathematical induction and in outline, the proof will be like the proof that we presented in Chapter 6 establishing the soundness of *SD* for sentential logic. In fact, much of the proof in Chapter 6 can be used here—for in Chapter 6 we showed that the rules for the truth-functional connectives are all sound for sentential logic, and with a change from talk of truth-value assignments to talk of interpretations, those rules are established to be sound for predicate logic in the same way. The bulk of the proof will therefore concentrate on the rules for quantifier introduction and elimination.

In our proof we shall use several semantic results that were presented in Section 11.1 along with the following result:

11.3.2: If $\Gamma \vDash \mathbf{P}$ and Γ^* is a superset of Γ, then $\Gamma^* \vDash \mathbf{P}$.

Proof: If every member of Γ^* is true on an interpretation $\mathbf{I}$, then every member of its subset Γ is true on $\mathbf{I}$, and if $\Gamma \vDash \mathbf{P}$, then $\mathbf{P}$ is also true on $\mathbf{I}$. Hence $\Gamma^* \vDash \mathbf{P}$.

Letting $\mathbf{P_i}$ be the sentence at position $\mathbf{i}$ in a derivation and letting $\Gamma_{\mathbf{i}}$ be the set of assumptions that are open at position $\mathbf{i}$ (and hence within whose scope $\mathbf{P_i}$ lies), the proof of Metatheorem 11.3.1 by mathematical induction is

Basis clause: $\Gamma_1 \vDash \mathbf{P}_1$.

Inductive step: If $\Gamma_{\mathbf{i}} \vDash \mathbf{P_i}$ for every position $\mathbf{i}$ in a derivation such that $\mathbf{i} \leq \mathbf{k}$, then $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Conclusion: Every sentence in a derivation is quantificationally entailed by the set of open assumptions in whose scope it lies.

Proof of basis clause: The first sentence in any derivation in *PD* is an assumption, and it lies in its own scope. Γ_1 is just $\{\mathbf{P}_1\}$, and it is trivial that $\{\mathbf{P}_1\} \vDash \mathbf{P}_1$.

Proof of inductive step: We assume the inductive hypothesis for an arbitrary position $\mathbf{k}$, that is, for every position $\mathbf{i}$ such that $\mathbf{i} \leq \mathbf{k}$, $\Gamma_{\mathbf{i}} \vDash \mathbf{P_i}$. We must show that the same holds for position $\mathbf{k} + 1$. We shall show this by considering the justifications that might be used for the sentence at position $\mathbf{k} + 1$.

Cases 1–12: $\mathbf{P}_{\mathbf{k}+1}$ is justified by one of the rules of *SD*. For each of these cases, use the corresponding case from the proof of the soundness of *SD* in Section 6.3, changing talk of truth-value assignments to talk of interpretations, and talk of truth-functional concepts (inconsistency and so on) to talk of quantificational concepts.

Case 13: $\mathbf{P_{k+1}}$ is justified by Universal Elimination. Then $\mathbf{P_{k+1}}$ is a sentence $\mathbf{Q(a/x)}$ derived as follows:

$$\begin{array}{r|ll} \mathbf{h} & (\forall \mathbf{x})\mathbf{Q} & \\ & & \\ \mathbf{k} + 1 & \mathbf{Q}(\mathbf{a}/\mathbf{x}) & \mathbf{h}\ \forall\mathrm{E} \end{array}$$

where every assumption that is open at position $\mathbf{h}$ is also open at position $\mathbf{k} + 1$, so $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash (\forall \mathbf{x})\mathbf{Q}$. It follows, by 11.3.2, that the superset $\Gamma_{\mathbf{k}+1} \vDash (\forall \mathbf{x})\mathbf{Q}$. By 11.1.4, which says that a universally quantified sentence quantificationally entails every one of its substitution instances, $\{(\forall \mathbf{x})\mathbf{Q}\} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$. So $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$ as well.

Case 14: $\mathbf{P_{k+1}}$ is justified by Existential Introduction. Then $\mathbf{P_{k+1}}$ is a sentence $(\exists \mathbf{x})\mathbf{Q}$ derived as follows:

$$\begin{array}{r|ll} \mathbf{h} & \mathbf{Q}(\mathbf{a}/\mathbf{x}) & \\ & & \\ \mathbf{k} + 1 & (\exists \mathbf{x})\mathbf{Q} & \mathbf{h}\ \exists\mathrm{I} \end{array}$$

where every assumption that is open at position $\mathbf{h}$ is also open at position $\mathbf{k} + 1$. So $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$ and so, by 11.3.2, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$. By 11.1.5, $\{\mathbf{Q}(\mathbf{a}/\mathbf{x})\} \vDash (\exists \mathbf{x})\mathbf{Q}$, so $\Gamma_{\mathbf{k}+1} \vDash (\exists \mathbf{x})\mathbf{Q}$ as well.

Case 15: $\mathbf{P_{k+1}}$ is justified by Universal Introduction. Then $\mathbf{P_{k+1}}$ is a sentence $(\forall \mathbf{x})\mathbf{Q}$ derived as follows:

$$\begin{array}{r|ll} \mathbf{h} & \mathbf{Q}(\mathbf{a}/\mathbf{x}) & \\ & & \\ \mathbf{k} + 1 & (\forall \mathbf{x})\mathbf{Q} & \mathbf{h}\ \forall\mathrm{I} \end{array}$$

where every assumption that is open at position $\mathbf{h}$ is also open at position $\mathbf{k} + 1$—so $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$—and in addition, $\mathbf{a}$ does not occur in $(\forall \mathbf{x})\mathbf{Q}$ or in any member of $\Gamma_{\mathbf{k}+1}$ because the rule $\forall$I stipulates this. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$. Because $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}(\mathbf{a}/\mathbf{x})$. And because $\mathbf{a}$ does not occur in $(\forall \mathbf{x})\mathbf{Q}$ or in any member of $\Gamma_{\mathbf{k}+1}$, it follows from 11.1.8, which we repeat here, that $\Gamma_{\mathbf{k}+1} \vDash (\forall \mathbf{x})\mathbf{Q}$ as well:

11.1.8: Let $\mathbf{a}$ be a constant that does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of the set Γ. Then if $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, $\Gamma \vDash (\forall \mathbf{x})\mathbf{P}$.

Case 16: $\mathbf{P_{k+1}}$ is justified by Existential Elimination. Then $\mathbf{P_{k+1}}$ is derived as follows:

$$
\begin{array}{r|l}
\mathbf{h} & (\exists \mathbf{x})\mathbf{Q} \\
& \\
\mathbf{j} & \quad \begin{array}{|l} \mathbf{Q}(\mathbf{a}/\mathbf{x}) \\ \hline \end{array} \\
\mathbf{m} & \quad \begin{array}{|l} \mathbf{P_{k+1}} \end{array} \\
& \\
\mathbf{k} + 1 & \mathbf{P_{k+1}} \qquad\qquad\qquad \mathbf{h}, \mathbf{j}\text{–}\mathbf{m}\ \exists\mathrm{E}
\end{array}
$$

where every member of $\Gamma_{\mathbf{h}}$ is a member of $\Gamma_{\mathbf{k}+1}$ and every member of $\Gamma_{\mathbf{m}}$ except $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is a member of $\Gamma_{\mathbf{k}+1}$ (if any other assumptions in $\Gamma_{\mathbf{m}}$ were closed prior to position $\mathbf{k} + 1$, then the subderivation **j–m** would not be accessible at position $\mathbf{k} + 1$). Because every member of $\Gamma_{\mathbf{m}}$ except $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is a member of $\Gamma_{\mathbf{k}+1}$, $\Gamma_{\mathbf{m}}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}(\mathbf{a}/\mathbf{x})\}$. Moreover, $\mathbf{a}$ does not occur in $(\exists \mathbf{x})\mathbf{Q}$, $\mathbf{P_{k+1}}$, or any member of $\Gamma_{\mathbf{k}+1}$ because the rule $\exists$E stipulates this. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash (\exists \mathbf{x})\mathbf{Q}$, and so because $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k}+1} \vDash (\exists \mathbf{x})\mathbf{Q}$. Also by the inductive hypothesis, $\Gamma_{\mathbf{m}} \vDash \mathbf{P_{k+1}}$, and so, because $\Gamma_{\mathbf{m}}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}(\mathbf{a}/\mathbf{x})\}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}(\mathbf{a}/\mathbf{x})\} \vDash \mathbf{P_{k+1}}$. Because $\mathbf{a}$ does not occur in $(\exists \mathbf{x})\mathbf{Q}$, $\mathbf{P_{k+1}}$, or any member of $\Gamma_{\mathbf{k}+1}$, it follows that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P_{k+1}}$, by 11.1.9, which we repeat here:

11.1.9: Let $\mathbf{a}$ be a constant that does not occur in the sentences $(\exists \mathbf{x})\mathbf{P}$ and $\mathbf{Q}$ and that does not occur in any member of the set Γ. If $\Gamma \vDash (\exists \mathbf{x})\mathbf{P}$ and $\Gamma \cup \{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \vDash \mathbf{Q}$, then $\Gamma \vDash \mathbf{Q}$ as well.

This completes the proof of the inductive step; all of the derivation rules of *PD* are truth-preserving. Note that, in establishing that the two quantifier rules $\forall$I and $\exists$E are truth-preserving, we made essential use of the restrictions that those rules place on the instantiating constant **a**—the restrictions were included in those rules to ensure that they would be truth-preserving. Having established that the inductive step is true, we may conclude that every sentence in a derivation of *PD* is quantificationally entailed by the set of open assumptions in whose scope it lies. Therefore, we have established Metatheorem 11.3.1: if $\Gamma \vdash \mathbf{P}$ in *PD*, then $\Gamma \vDash \mathbf{P}$.

The proof that *PD+* is sound for predicate logic involves the additional steps of showing that the rules of replacement of *SD+*, the three derived rules of *SD+*, and the rule Quantifier Negation are all truth-preserving. The steps in the soundness proof for *SD+* can be converted into steps showing that the rules are truth-preserving for quantificational logic, by talking of interpretations and variable assignments rather than truth-value assignments. We leave the proof that Quantifier Negation is truth-preserving as an exercise.

Finally, we can prove that *PDE* is sound for predicate logic with identity and functions by extending the inductive step of the proof for *PD* to cover Identity Introduction and Identity Elimination and by making one change in the basis clause of the soundness proof. We note that, since we have shown in Section 11.2 that all of the semantic results in Section 11.1 can be extended to predicate logic with identity and functions, a soundness proof for *PDE* can refer to all of those results. In particular, even though the rules $\forall$E and $\exists$I have been changed for *PDE*, the proof for Cases 13 and 14 in the inductive step of the soundness proof for *PD* will remain the same except that in place of the substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ we now have a substitution instance $\mathbf{Q}(\mathbf{t}/\mathbf{x})$, where $\mathbf{t}$ is any closed term.

In the basis clause for *PD*, we said that the first sentence in a derivation is an assumption. This is not always the case in *PDE*; the first sentence *can* be an assumption, but it can also be a sentence of the form $(\forall\mathbf{x})\mathbf{x} = \mathbf{x}$, introduced by Identity Introduction. So the proof of the basis clause will look like this:

> The first sentence in a derivation in *PDE* is either an assumption or a sentence introduced by Identity Introduction. If the first sentence is an assumption, then it lies in its own scope. In this case Γ_1 is just $\{\mathbf{P}_1\}$, and it is trivial that $\{\mathbf{P}_1\} \vDash \mathbf{P}_1$.
>
> If the first sentence is introduced by Identity Introduction, then Γ_1 is empty—there are no assumptions, and hence no open assumptions, at that point. And $\varnothing$ truth-functionally entails every sentence of the form $(\forall\mathbf{x})\mathbf{x} = \mathbf{x}$, because every such sentence is quantificationally true. This was proved in Exercise 8.7.10a.

We add the following two cases to the proof of the inductive step for *PD*:

Case 17: $\mathbf{P}_{\mathbf{k}+1}$ is introduced by Identity Introduction. Then $\mathbf{P}_{\mathbf{k}+1}$ is a sentence of the form $(\forall\mathbf{x})\mathbf{x} = \mathbf{x}$ derived as follows:

$$\begin{array}{r|ll} \mathbf{k}+1 & (\forall\mathbf{x})\mathbf{x} = \mathbf{x} & =\mathrm{I} \end{array}$$

Because $\varnothing$ quantificationally entails every sentence of the form $(\forall\mathbf{x})\mathbf{x} = \mathbf{x}$ and $\varnothing$ is a subset of $\Gamma_{\mathbf{k}+1}$, it follows by 11.3.2 that $\Gamma_{\mathbf{k}+1} \vDash (\forall\mathbf{x})\mathbf{x} = \mathbf{x}$.

Case 18: $\mathbf{P}_{\mathbf{k}+1}$ is introduced by Identity Elimination. Then $\mathbf{P}_{\mathbf{k}+1}$ is derived as follows:

$$\begin{array}{r|ll} \mathbf{h} & \mathbf{t}_1 = \mathbf{t}_2 & \\ \mathbf{j} & \mathbf{P} & \\ \mathbf{k}+1 & \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2) & \mathbf{h}, \mathbf{j}\ =\mathrm{E} \end{array} \qquad \text{or} \qquad \begin{array}{r|ll} \mathbf{h} & \mathbf{t}_1 = \mathbf{t}_2 & \\ \mathbf{j} & \mathbf{P} & \\ \mathbf{k}+1 & \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1) & \mathbf{h}, \mathbf{j}\ =\mathrm{E} \end{array}$$

where both $\Gamma_\mathbf{h}$ and $\Gamma_\mathbf{j}$ are subsets of $\Gamma_{\mathbf{k}+1}$ (because the sentences at positions $\mathbf{h}$ and $\mathbf{j}$ are accessible at position $\mathbf{k} + 1$). By the inductive hypothesis, $\Gamma_\mathbf{h} \vDash \mathbf{t}_1 = \mathbf{t}_2$ and $\Gamma_\mathbf{j} \vDash \mathbf{P}$. Because these are both subsets of $\Gamma_{\mathbf{k}+1}$, it follows by 11.3.2 that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{t}_1 = \mathbf{t}_2$ and $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}$. It follows from 11.2.4, which we repeat here:

11.2.4: For any closed terms $\mathbf{t}_1$ and $\mathbf{t}_2$, if $\mathbf{P}$ is a sentence that contains $\mathbf{t}_1$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vDash \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$, and if $\mathbf{P}$ is a sentence that contains $\mathbf{t}_2$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vDash \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$.

that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$ as well.

These changes establish that *PDE* is sound for predicate logic with identity and functions.

11.3E EXERCISES

1. Using Metatheorem 11.3.1, prove the following:
a. Every argument of *PL* that is valid in *PD* is quantificationally valid.
b. Every sentence of *PL* that is a theorem in *PD* is quantificationally true.
*c. Every pair of sentences $\mathbf{P}$ and $\mathbf{Q}$ of *PL* that are equivalent in *PD* are quantificationally equivalent.

2. Prove the following (to be used in Exercise 3) by mathematical induction:

11.3.4. Let $\mathbf{P}$ be a formula of *PL* and $\mathbf{Q}$ a subformula of $\mathbf{P}$. Let $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ be a sentence that is the result of replacing one or more occurrences of $\mathbf{Q}$ in $\mathbf{P}$ with a formula $\mathbf{Q}_1$. If $\mathbf{Q}$ and $\mathbf{Q}_1$ contain the same nonlogical symbols and variables, and if $\mathbf{Q}$ and $\mathbf{Q}_1$ are satisfied by exactly the same variable assignments on any interpretation, then the same is true of $\mathbf{P}$ and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$.

3. Using 11.3.4, show how we can establish, as a step in an inductive proof of the soundness of *PD+*, that Quantifier Negation is truth-preserving for predicate logic.

***4.**a. Suppose that we changed the rule $\forall$I by eliminating the restriction that the instantiating constant $\mathbf{a}$ in the sentence $\mathbf{P}(\mathbf{a}/\mathbf{x})$ to which $\forall$I applies must not occur in any open assumption. Explain why *PD* would *not* be sound for predicate logic in this case.
b. Suppose that we changed the rule $\exists$E by eliminating the restriction that the instantiating constant $\mathbf{a}$ in the assumption $\mathbf{P}(\mathbf{a}/\mathbf{x})$ must not occur in the sentence $\mathbf{Q}$ that is derived. Explain why *PD* would *not* be sound for predicate logic in this case.

11.4 THE COMPLETENESS OF *PD*, *PD+*, AND *PDE*

In this section we shall prove that our natural deduction systems are **complete** for predicate logic. A natural deduction system is complete for predicate logic if, whenever a sentence is quantificationally entailed by a set of sentences, there is at least one derivation of the sentence from members of that set in the natural deduction system. Metatheorem 11.4.1 is the *Completeness Metatheorem* for *PD*:

Metatheorem 11.4.1: For any set Γ of sentences of *PL* and any sentence $\mathbf{P}$ of *PL*, if $\Gamma \vDash \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in *PD*.

Our proof will be analogous to the proof of the completeness of *SD* for sentential logic in Chapter 6. As in Chapter 6, the Completeness Metatheorem for predicate logic follows almost immediately from the following result:

> **11.4.2:** For any set of sentences of *PL*, if Γ is consistent in *PD* then Γ is quantificationally consistent.

To see how Metatheorem 11.4.1 follows, assume that, for some set Γ and sentence $\mathbf{P}$, $\Gamma \vDash \mathbf{P}$ (this is the antecedent of the metatheorem). Then the set $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent (see Exercise 11.4.1). It follows, from Lemma 11.4.2, that $\Gamma \cup \{\sim \mathbf{P}\}$ is also inconsistent in *PD*. And from this it follows that $\Gamma \vDash \mathbf{P}$ in *PD* (see Exercise 11.4.2).

So the bulk of this section is devoted to proving result 11.4.2.

The major steps in the proof of the analogous result in Chapter 6 involved showing that every set of sentences that is consistent in *SD* is a subset of a set of sentences that is maximally consistent in *SD* and showing that every set of sentences that is maximally consistent in *SD* is also truth-functionally consistent. In addition to **maximal consistency in *PD***, which is defined as

> A set Γ of sentences of *PL* is *maximally consistent in PD* if and only if Γ is consistent in *PD* and for every sentence $\mathbf{P}$ of *PD* that is not a member of Γ, $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *PD*,

we shall rely on an additional property that sets of sentences of *PL* can have, the property of **∃-completeness** (read as *existential completeness*):

> A set Γ of sentences of *PL* is *∃-complete* if and only if, for each sentence in Γ that has the form $(\exists \mathbf{x})\mathbf{P}$, at least one substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is also a member of Γ.

We will be showing that every set of sentences of *PL* that is both maximally consistent in *PD* and ∃-complete is also quantificationally consistent. However, we will not show that every set of sentences that is consistent in *PD* is a subset of a set that is both maximally consistent in *PD* and ∃-complete, for there is a complication. To build a set that has these properties, we need to add the substitution instances required by the property of ∃-completeness, and for this we need to be sure that infinitely many individual constants that do not already occur in the set we are working with are available for the substitution instances. We shall ensure that infinitely many such constants are available by proving 11.4.2 through the following steps:

> *Step 1 in proof of 11.4.2:* We shall prove in result 11.4.3 that, for any set Γ that is consistent in *PD*, if we double the subscript of every individual constant in Γ (so that every resulting subscript will be even), then the resulting set $\Gamma_{\mathbf{e}}$ is also consistent in *PD*. We call such a set an '**evenly subscripted set**'.

Step 2 in proof of 11.4.2: We shall then show that, because there are infinitely many individual constants (namely, all the oddly subscripted constants) that do not occur in the sentences of any evenly subscripted set, every evenly subscripted set Γ that is consistent in *PD* is a subset of a set that is *maximally consistent in PD* and that is ∃-*complete*. This will be established as result 11.4.4.

Step 3 in proof of 11.4.2: We shall next show that there is a straightforward way to construct a model for every set that is maximally consistent in *PD* and that is ∃-complete, from which it follows that every such set is quantificationally consistent. This will be established as result 11.4.7. It follows from this result that the evenly subscripted set from which we built the maximally consistent set in Step 2 must be quantificationally consistent.

Step 4 in proof of 11.4.2: Finally we shall show, in result 11.4.8, that the set Γ that we began with must be quantificationally consistent as well.

We begin with 11.4.3, which establishes *Step 1*:

11.4.3: Let Γ be a set of sentences of *PL* and let $\Gamma_{\mathbf{e}}$ be the set that results from doubling the subscript of every individual constant that occurs in any member of Γ. Then if Γ is consistent in *PD*, $\Gamma_{\mathbf{e}}$ is also consistent in *PD*.

Proof: Assume that Γ is consistent in *PD* and that, contrary to what we wish to prove, $\Gamma_{\mathbf{e}}$ is *in*consistent in *PD*. Then there is a derivation of the sort

$$
\begin{array}{r|l}
1 & \mathbf{P}_1 \\
2 & \mathbf{P}_2 \\
. & . \\
\mathbf{n} & \mathbf{P_n} \\
\cline{2-2}
. & . \\
\mathbf{k} & \mathbf{Q} \\
. & . \\
\mathbf{p} & \sim \mathbf{Q}
\end{array}
$$

where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P_n}$ are members of $\Gamma_{\mathbf{e}}$. We shall convert this derivation into a derivation that shows that Γ is inconsistent in *PD*, contradicting our first assumption. Our strategy, not surprisingly, will be to halve the subscript of every evenly subscripted individual constant occurring in the derivation, thus converting each of $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P_n}$ back to a member of the original set Γ. There is a complication, though—in so doing we may end up with a sequence in which either an ∃E restriction or an a ∀I restriction is violated. For example, let us suppose that Γ is $\{(\exists x)Px, (\forall x)(Px \supset Qa_1), \sim Qa_1\}$. This set is clearly inconsistent. Γ_e is the set $\{(\exists x)Px, (\forall x)(Px \supset Qa_2), \sim Qa_2\}$. Now suppose that we show that Γ_e is inconsistent in *PD* by producing the following derivation:

1	$(\exists x)Px$	Assumption
2	$(\forall x)(Px \supset Qa_2)$	Assumption
3	$\sim Qa_2$	Assumption
4	$\quad Pa_1$	A / $\exists$E
5	$\quad Pa_1 \supset Qa_2$	2 $\forall$E
6	$\quad Qa_2$	4, 5 $\supset$E
7	Qa_2	1, 4–6 $\exists$E
8	$\sim Qa_2$	3 R

Halving each of the even subscripts in this derivation produces the following, which violates two of the $\exists$E restrictions because 'a_1' now occurs in the primary (thus undischarged) assumption on line 3 and also in the sentence that is justified by E on line 7:

1	$(\exists x)Px$	Assumption	
2	$(\forall x)(Px \supset Qa_1)$	Assumption	
3	$\sim Qa_1$	Assumption	
4	$\quad Pa_1$	A / $\exists$E	
5	$\quad Pa_1 \supset Qa_1$	2 $\forall$E	
6	$\quad Qa_1$	4, 5 $\supset$E	
7	Qa_1	1, 4–6 $\exists$E	**MISTAKE!**
8	$\sim Qa_1$	3 R	

We therefore first take a precaution to ensure that this will not happen.

Let $\mathbf{a}_1, \ldots, \mathbf{a_m}$ be the distinct constants that are used as instantiating constants for $\exists$E and $\forall$I in the derivation of $\mathbf{Q}$ and $\sim \mathbf{Q}$ from $\Gamma_e = \{P_1, \ldots, P_{\mathbf{n}}\}$, and let $\mathbf{b}_1, \ldots, \mathbf{b_m}$ be distinct constants that have odd subscripts that are larger than the subscript of any constant occurring in the derivation. (Because every derivation is a finite sequence, we know that among the constants occurring in our derivation, there is one that has the largest subscript—and, whatever this largest subscript may be, there are infinitely many odd numbers that are larger.) We replace each sentence $\mathbf{R}$ in the original derivation with a sentence $\mathbf{R}^*$ that is the result of first replacing each occurrence of $\mathbf{a_i}$ in $\mathbf{R}$, $1 \leq \mathbf{i} \leq \mathbf{m}$, with $\mathbf{b_i}$, and *then* halving every even subscript in a constant in the resulting sentence.

To continue with our example, we first replace the instantiating constant 'a_1' in the previous (good) derivation with 'a_3':

$$\begin{array}{l|l l}
1 & (\exists x)Px & \text{Assumption}\\
2 & (\forall x)(Px \supset Qa_2) & \text{Assumption}\\
3 & \sim Qa_2 & \text{Assumption}\\
\hline
4 & \quad | \; Pa_3 & \text{A / }\exists\text{E}\\
5 & \quad | \; Pa_3 \supset Qa_2 & 2\ \forall\text{E}\\
7 & \quad | \; Qa_2 & 4,\ 5 \supset\text{E}\\
8 & Qa_2 & 1,\ 4\text{–}5\ \exists\text{E}\\
9 & \sim Qa_2 & 3\ \text{R}
\end{array}$$

Then we halve the even subscripts:

$$\begin{array}{l|l l}
1 & (\exists x)Px & \text{Assumption}\\
2 & (\forall x)(Px \supset Qa_1) & \text{Assumption}\\
3 & \sim Qa_1 & \text{Assumption}\\
\hline
4 & \quad | \; Pa_3 & \text{A / }\exists\text{E}\\
5 & \quad | \; Pa_3 \supset Qa_1 & 2\ \forall\text{E}\\
7 & \quad | \; Qa_1 & 4,\ 5 \supset\text{E}\\
8 & Qa_1 & 1,\ 4\text{–}5\ \exists\text{E}\\
9 & \sim Qa_1 & 3\ \text{R}
\end{array}$$

Here we have eliminated the violation of the ∃E restriction.

We claim that the resulting sequence will always be a legitimate derivation in *PD* of **Q*** and ~ **Q*** from members of the set Γ. First note that every new primary assumption $\mathbf{P_i}$* is identical to the primary assumption $\mathbf{P_i}$ in the dervation showing that Γ_e is inconsistent in *PD*. This is because none of $\mathbf{a}_1, \ldots, \mathbf{a_m}$ can occur in a primary assumption of that derivation (lest an instantiating constant restriction be violated). So **P*** is just $\mathbf{P_i}$ with all its individual constant subscripts halved—that is, a member of the original set Γ from which $\Gamma_\mathbf{e}$ was constructed. It remains to be shown that the resulting sequence counts as a derivation in *PD*—that every sentence following the primary assumptions in that sequence can be justified. This is left as an exercise.

Step 2 in our proof of Lemma 11.4.2 is to establish the result

11.4.4: If Γ is an evenly subscripted set of sentences that is consistent in *PD*, then Γ is a subset of at least one set of sentences that is both maximally consistent in *PD* and ∃-complete.

We shall establish 11.4.4 by showing how, beginning with Γ, to construct a superset that has the two properties. We assume that the sentences of *PL* have been enumerated, that is, that they have been placed in a one-to-one correspondence with the positive integers so that there is a first sentence, a second sentence, a third sentence, and so on. The enumeration can be done analogously to the enumeration of the sentences of *SL* in Section 6.4; we leave proof of this as an exercise (Exercise 11.4.4). We shall now build a sequence of sets $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ where Γ_1 is an evenly subscripted set Γ that is consistent in *PD*, by considering each sentence in the enumeration, adding the sentence if it can consistently be added, and, if the added sentence is existentially quantified, adding one of its substitution instances as well. The sequence is constructed as follows:

1. Γ_1 is Γ.
2. $\Gamma_{\mathbf{i}+1}$ is
   (i) $\Gamma_\mathbf{i} \cup \{\mathbf{P_i}\}$, if $\Gamma_\mathbf{i} \cup \{\mathbf{P_i}\}$ is consistent in *PD* and $\mathbf{P_i}$ does not have the form $(\exists \mathbf{x})\mathbf{P}$, or
   (ii) $\Gamma_\mathbf{i} \cup \{\mathbf{P_i}, \mathbf{P_i}^*\}$, if $\Gamma \cup \{\mathbf{P_i}\}$ is consistent in *PD* and $\mathbf{P_i}$ has the form $(\exists \mathbf{x})\mathbf{Q}$, where $\mathbf{P_i}^*$ is a substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x})$ $\mathbf{Q}$ and $\mathbf{a}$ is the alphabetically earliest constant[1] that does not occur in $\mathbf{P_i}$ or in any sentence in $\Gamma_\mathbf{i}$, or
   (iii) $\Gamma_\mathbf{i}$, if $\Gamma_\mathbf{i} \cup \{\mathbf{P_i}\}$ is inconsistent in *PD*.

As an example of (ii), if $\Gamma_\mathbf{i}$ is the set $\{(\forall x)(Fxa \supset Gx), \sim Hc \vee (\exists y)Jyy\}$ and $\mathbf{P_i}$ is '$(\exists z)(Kz \mathbin{\&} (\forall y)Fzy)$', then $\Gamma_\mathbf{i} \cup \{\mathbf{P_i}\}$ is quantificationally consistent, and so $\mathbf{P_i}$ will be added to the set—but we must add a substitution instance of $\mathbf{P_i}$ as well. The alphabetically earliest constant that does not occur in $\mathbf{P_i}$ or in any member of $\Gamma_\mathbf{i}$ is 'b', and so this will be the instantiating constant. $\Gamma_{\mathbf{i}+1}$ is therefore

$$\{(\forall x)(Fxa \supset Gx), \sim Hc \vee (\exists y)Jyy, (\exists z)(Kz \mathbin{\&} (\forall y)Fzy), Kb \mathbin{\&} (\forall y)Fby\}$$

The reason for using an instantiating constant that does not already occur in $\Gamma_\mathbf{i}$ will become clear shortly when we prove that each set in the sequence is consistent in *PD*. Here it is important to note that we can always meet the requirement in condition (ii)—that **a** be a *new* constant—because for any set in the sequence, there is always at least one individual constant that does not already occur in that set. This is because the set that we started with is evenly subscripted, and so we know that infinitely many oddly subscripted individual constants do not occur in Γ.

Because the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is infinitely long, there is no last member in the set. We want a set that contains all the sentences in these sets, so we let Γ^* be the set that contains every sentence that occurs in some set in the infinite sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ We shall show that Γ^* is both maximally consistent in *PD* and $\exists$-complete. To show that Γ^* is maximally consistent in

[1]This is an extended sense of 'alphabetical order' in which the nonsubscripted constants occur first, in ordinary alphabetical order, followed by the constants subscripted with '1'—in ordinary alphabetical order—and so on.

PD, we first prove that each set $\Gamma_{\mathbf{i}}$ in the sequence is consistent in *PD*, using mathematical induction.

Basis clause: Γ_1 is consistent in *PD*.

Proof of basis clause: By definition, Γ_1 is Γ, a set that is consistent in *PD*.

Inductive step: If for every $\mathbf{i} \leq \mathbf{k}$, $\Gamma_{\mathbf{i}}$ is consistent in *PD*, then $\Gamma_{\mathbf{k}+1}$ is consistent in *PD*.

Proof of inductive step: If $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (i), then $\Gamma_{\mathbf{k}+1}$ is obviously consistent in *PD*. If $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (ii), then we need to show that it follows that $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}, \mathbf{P_i}^*\}$, which is what $\Gamma_{\mathbf{k}+1}$ was defined to be in this case, is also consistent in *PD*. Because the instantiating constant in $\mathbf{P_i}^*$ does not occur in any member of $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$, the consistency of $\Gamma_{\mathbf{k}+1}$ follows immediately from result 11.1.10, which we repeat here:

11.1.10: If $\mathbf{a}$ does not occur in any member of the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ and if the set is quantificationally consistent, then the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is also quantificationally consistent.

Finally, if $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (iii), then $\Gamma_{\mathbf{k}+1}$ is $\Gamma_{\mathbf{k}}$, and by the inductive hypothesis, $\Gamma_{\mathbf{k}}$ is consistent in *PD*. So, no matter which condition was applied in its construction, $\Gamma_{\mathbf{k}+1}$ is consistent in *PD*.

We conclude that every set in the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is consistent in *PD*.

We now need to show that the set Γ^*, which contains all the sentences that occur in any set in the sequence, is itself consistent in *PD*. We shall show this by assuming that it is not consistent in *PD* and deriving a contradiction. So assume that Γ^* is not consistent in *PD*. Then there is a finite nonempty subset Γ' of Γ^* that is inconsistent in *PD* (the proof is analogous to that in the proof of 6.4.6). Because Γ' is finite, some sentence in Γ', say, $\mathbf{P_j}$, occurs later in our enumeration of the sentences of *PL* than any other sentence in Γ'. Every member of Γ' is thus a member of $\Gamma_{\mathbf{j}+1}$, by the way we constructed the sets in the sequence. (If the **i**th sentence is added to one of the sets, it is added by the time that $\Gamma_{\mathbf{i}+1}$ is constructed.) It follows that $\Gamma_{\mathbf{j}+1}$ is also inconsistent in *PD* (the proof is analogous to the proof of 6.4.7). But we have just proved that every set in the sequence is consistent in *PD*, so we conclude that, contrary to our assumption, Γ^* is also consistent in *PD*.

That Γ^* is *maximally* consistent in *PD* is proved in exactly the manner that the parallel result in Section 6.4 was proved—for any sentence $\mathbf{P_k}$, if $\Gamma^* \cup \{\mathbf{P_k}\}$ is consistent in *PD*, then the subset $\Gamma_{\mathbf{k}}$ of Γ^* is such that $\Gamma_{\mathbf{k}} \cup \{\mathbf{P_k}\}$ is consistent in *PD*, and so by the construction of the sequence of sets, $\mathbf{P_k}$ is a member of $\Gamma_{\mathbf{k}+1}$ and hence of Γ^*.

Finally, the proof that Γ^* is $\exists$-complete is left as an exercise. This completes the proof of result 11.4.4—every evenly subscripted set Γ of sentences of *PL* that is consistent in *PD* is a subset of at least one set of sentences that is both maximally consistent in *PD* and $\exists$-complete.

We now turn to *Step 3* in our proof of result 11.4.2. We must prove that every set Γ that is both maximally consistent in *PD* and $\exists$-complete is consistent in *PD*. To do this we shall use the following preliminary results:

11.4.5: If $\Gamma \vdash \mathbf{P}$ and Γ is a subset of a set Γ^* that is maximally consistent in *PD*, then $\mathbf{P} \in \Gamma^*$.

Proof: See Exercise 11.4.9.

11.4.6: Every set Γ^* of sentences that is both maximally consistent in *PD* and $\exists$-complete has the following properties:

a. $\mathbf{P} \in \Gamma^*$ if and only if $\sim \mathbf{P} \notin \Gamma^*$.
b. $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$ if and only if $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
c. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
d. $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
e. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$ or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.
f. $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$ if and only if, for every individual constant $\mathbf{a}$, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$.
g. $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$ if and only if, for at least one individual constant $\mathbf{a}$, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$.

Proof: The proofs that (a)–(e) hold for sets of sentences that are maximally consistent in *PD* and $\exists$-complete parallel exactly the corresponding proofs in Section 6.4, using result 11.4.5 instead of 6.4.5.

Proof of (f): Assume that $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$. For any substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\forall \mathbf{x})\mathbf{P}$, $\{(\forall \mathbf{x})\mathbf{P}\} \vdash \mathbf{P}(\mathbf{a}/\mathbf{x})$ (by $\forall$E); so, by 11.4.5, every substitution instance is a member of Γ^* as well. Now assume that $(\forall \mathbf{x})\mathbf{P} \notin \Gamma^*$. Then $\sim (\forall \mathbf{x})\mathbf{P} \in \Gamma^*$, by (a). The following derivation shows that $\{\sim (\forall \mathbf{x})\mathbf{P}\} \vdash (\exists \mathbf{x}) \sim \mathbf{P}$:

1	$\sim (\forall \mathbf{x})\mathbf{P}$	Assumption
2	$\sim (\exists \mathbf{x}) \sim \mathbf{P}$	A / ~ E
3	$\sim \mathbf{P}(\mathbf{a}/\mathbf{x})$	A / ~ E
4	$(\exists \mathbf{x}) \sim \mathbf{P}$	3 $\exists$I
5	$\sim (\exists \mathbf{x}) \sim \mathbf{P}$	2 R
6	$\mathbf{P}(\mathbf{a}/\mathbf{x})$	3–5 ~ E
7	$(\forall \mathbf{x})\mathbf{P}$	6 $\forall$I
8	$\sim (\forall \mathbf{x})\mathbf{P}$	1 R
9	$(\exists \mathbf{x}) \sim \mathbf{P}$	2–8 ~ E

(We assume that the constant **a** does not occur in **P**.) Therefore, by 11.4.5, $(\exists \mathbf{x}) \sim \mathbf{P}$ is also a member of Γ^*. Because Γ^* is $\exists$-complete, some substitution instance $\sim \mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x}) \sim \mathbf{P}$ is a member of Γ^* as well, and it therefore follows from (a) that $\mathbf{P}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$. So, if $(\forall \mathbf{x})\mathbf{P} \notin \Gamma^*$, then there is at least one substitution instance of $(\forall \mathbf{x})\mathbf{P}$ that is not a member of Γ^*.

Proof of (g): Assume that $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$. Then, because Γ^* is $\exists$-complete, at least one substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is also a member of Γ^*. Now assume that $(\exists \mathbf{x})\mathbf{P} \notin \Gamma^*$. If some substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x})\mathbf{P}$ is a member of Γ^*, then because $\{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \vdash (\exists \mathbf{x})\mathbf{P}$ (by $\exists$I), it follows from 11.4.5 that, contrary to our assumption, $(\exists \mathbf{x})\mathbf{P}$ is also a member of Γ^*. So, if $(\exists \mathbf{x})\mathbf{P} \notin \Gamma^*$, then none of its substitution instances is a member of Γ^*.

We can now complete the proof of *Step 3* by establishing the result

11.4.7: Every set of sentences of *PL* that is both maximally consistent in *PD* and $\exists$-complete is quantificationally consistent.

We shall prove this by showing how to construct a model for any set Γ^* that is both maximally consistent in *PD* and $\exists$-complete, that is, an interpretation $\mathbf{I}^*$ on which every member of Γ^* is true. We begin by associating with each individual constant a distinct positive integer—the positive integer **i** will be associated with the alphabetically **i**th constant. The number 1 will be associated with 'a', 2 with 'b', . . . , 22 with 'v', 23 with 'a_1', and so on. $\mathbf{I}^*$ is then defined as follows:

1. The UD is the set of positive integers.
2. For each sentence letter **P**, $\mathbf{I}^*(\mathbf{P}) = \mathbf{T}$ if and only if $\mathbf{P} \in \Gamma^*$.
3. For each individual constant **a**, $\mathbf{I}^*(\mathbf{a})$ is the positive integer associated with **a**.
4. For each **n**-place predicate **A**, $\mathbf{I}^*(\mathbf{A})$ includes all and only those **n**-tuples $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n}) \rangle$ such that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a_n} \in \Gamma^*$.

The major feature of this interpretation is that, for each atomic sentence **P** of *PL*, **P** will be true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$. That is why we defined condition 4 (as well as condition 2) as we did. And to be sure that condition 4 can be met, we must have condition 3, which ensures that each individual constant designates a *different* member of the UD. This is necessary because, for example, if 'Fa' is a member of Γ^* and 'Fb' is not a member, then if 'a' and 'b' designated the same integer—say, 1—condition 4 would require that the 1-tuple $\langle 1 \rangle$ both be and not be a member of $\mathbf{I}^*(\text{F})$. (In addition, condition 3 ensures that every member of the UD is named by a constant, which we shall shortly see is also important when we look at the truth-values that quantified sentences receive on $\mathbf{I}^*$.)

We complete the proof of 11.4.7 by establishing, by mathematical induction on the number of occurrences of logical operators in sentences of *PL*, that each sentence **P** of *PL* is true on **I*** if and only if **P** $\in$ Γ*.

Basis clause: Each sentence **P** that contains zero occurrences of logical operators is true on **I*** if and only if **P** $\in$ Γ*.

Proof of basis clause: Either **P** is a sentence letter or **P** has the form $\mathbf{Aa}_1 \ldots \mathbf{a_n}$. If **P** is a sentence letter, then, by part 2 of the definition of **I***, it follows that **P** is true on **I*** if and only if **P** $\in$ Γ*.

If **P** has the form $\mathbf{Aa}_1 \ldots \mathbf{a_n}$, then **P** is true on **I*** if and only if $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n})\rangle \in \mathbf{I}^*(\mathbf{A})$. Part 4 of the definition of **I*** stipulates that $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n})\rangle \in \mathbf{I}^*(\mathbf{A})$ if and only if $\mathbf{Aa}_1 \ldots \mathbf{a_n} \in \Gamma^*$. So in this case as well, **P** is true on **I*** if and only if **P** $\in$ Γ*.

Inductive step: If each sentence **P** with **k** or fewer occurrences of logical operators is true on **I*** if and only if **P** $\in$ Γ*, then the same is true of each sentence **P** with **k** + 1 occurrences of logical operators.

Proof of inductive step: We assume that, for an arbitrary positive integer **k**, the inductive hypothesis is true. We must show that on this assumption it follows that any sentence **P** that has **k** + 1 occurrences of logical operators is such that **P** is true on **I*** if and only if **P** $\in$ Γ*. We consider the forms that the sentence **P** may have.

Cases 1–5: P has one of the forms ~ **Q**, **Q** & **R**, **Q** $\vee$ **R**, **Q** $\supset$ **R**, or **Q** $\equiv$ **R**. The proofs for these five cases are analogous to the proofs for the parallel cases for *SL* in Section 6.4, so we omit them here.

Case 6: P has the form $(\forall \mathbf{x})\mathbf{Q}$. Assume that $(\forall \mathbf{x})\mathbf{Q}$ is true on **I***. Then every substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\forall \mathbf{x})\mathbf{Q}$ is true on **I*** because, by 11.1.4, $\{(\forall \mathbf{x})\mathbf{Q}\}$ quantificationally entails every one of its substitution instances. Each substitution instance contains fewer than **k** + 1 occurrences of connectives, and so, by the inductive hypothesis, each substitution instance is a member of Γ* since it is true on **I***. It follows from part (f) of 11.4.6 that $(\forall \mathbf{x})\mathbf{Q}$ is also a member of Γ*.

Now assume that $(\forall \mathbf{x})\mathbf{Q}$ is false on **I***. In this case we shall make use of result 11.1.11, which we repeat here:

11.1.11: Let **I** be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\forall \mathbf{x})\mathbf{P}$ is true on **I**, so is $(\forall \mathbf{x})\mathbf{P}$.

I* is an interpretation of the type specified in 11.1.11: Every positive integer in the UD is designated by the individual constant with which we have associated that integer. It follows, then, that if $(\forall \mathbf{x})\mathbf{Q}$ is false on **I***, at least one of its substitution instances $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ must also be false on **I***. Because $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$. And so, by part (f) of 11.4.6, $(\forall \mathbf{x})\mathbf{Q} \notin \Gamma^*$.

Case 7: **P** has the form $(\exists \mathbf{x})\mathbf{Q}$. Assume that $(\exists \mathbf{x})\mathbf{Q}$ is true on **I***. Then it follows from 11.1.12, which we repeat here, that at least one substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x})\mathbf{Q}$ is true on **I***:

11.1.12: Let **I** be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is false on **I**, so is $(\exists \mathbf{x})\mathbf{P}$.

Because the substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$. So, by part (g) of 11.4.6, $(\exists \mathbf{x})\mathbf{Q} \in \Gamma^*$.

Now assume that $(\exists \mathbf{x})\mathbf{Q}$ is false on **I***. Because each substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is such that $\{\mathbf{Q}(\mathbf{a}/\mathbf{x})\} \vDash (\exists \mathbf{x})\mathbf{Q}$ (this is result 11.1.5), it follows that every substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is also false on **I***. Each of these substitution instances contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, and so it follows from the inductive hypothesis that no substitution instance of $(\exists \mathbf{x})\mathbf{Q}$ is a member of Γ^*. Finally, by part (g) of 11.4.6, it follows that $(\exists \mathbf{x})\mathbf{Q} \notin \Gamma^*$.

That completes the proof of the inductive step, and we may now conclude that each sentence **P** of *PL* is true on **I*** if and only if $\mathbf{P} \in \Gamma^*$. So **I*** is a model of Γ^*, and Γ^* is quantificationally consistent. Result 11.4.7 is therefore true: Every set that is both maximally consistent in *PD* and ∃-complete is quantificationally consistent. Results 11.4.4 and 11.4.7 together establish that every evenly subscripted set of sentences of *PL* that is consistent in *PD* is also quantificationally consistent.

Step 4 of the proof of result 11.4.2 is established by result 11.4.8:

11.4.8: Let Γ be a set of sentences of *PL* and let $\Gamma_{\mathbf{e}}$ be the set that results from doubling the subscript of every individual constant that occurs in any member of Γ. Then, if $\Gamma_{\mathbf{e}}$ is quantificationally consistent, Γ is quantificationally consistent as well.

Proof: See Exercise 11.4.8.

We have now completed the four steps in the proof of result 11.4.2, so we may conclude that if a set Γ of sentences of *PL* is quantificationally inconsistent, then Γ is also inconsistent in *PD*. And this establishes the completeness of *PD* for predicate logic. If $\Gamma \vDash \mathbf{P}$, then $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent. By 11.4.2, $\Gamma \cup \{\sim \mathbf{P}\}$ is also inconsistent in *PD*, and hence $\Gamma \vdash \mathbf{P}$ in *PD*.

Because *PD* is complete for predicate logic, so is *PD*+. Every rule of *PD* is a rule of *PD*+, and so every derivation in *PD* is a derivation in *PD*+. So, if $\Gamma \vDash \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in *PD*+ because we know, by Metatheorem 11.4.1, that $\Gamma \vdash \mathbf{P}$ in *PD*.

We also want to be sure that *PDE* is complete for predicate logic with identity and functions. The completeness proof for *PDE* is similar to the completeness proof for *PD*, but there are some important changes. Results 11.4.3 and 11.4.8 must now take into account sentences containing the identity predicate and complex terms; the necessary changes are left as an exercise. Maximal consistency is defined for *PDE* as it was for *PD*, while the definition of ∃-completeness must be modified slightly:

> A set Γ of sentences of *PLE* is ∃-complete if and only if, for each sentence in Γ that has the form $(\exists \mathbf{x})\mathbf{P}$, at least one substitution instance of $(\exists \mathbf{x})\mathbf{P}$ in which the instantiating individual term is a constant is also a member of Γ.

The proof of result 11.4.4 for *PDE*—that every evenly subscripted set of sentences that is consistent in *PDE* is a subset of a set of sentences that is both maximally consistent in *PDE* and ∃-complete—is just like the proof for *PD* except that it speaks of *PLE* and *PDE*, rather than *PL* and *PD*. However, the proof of result 11.4.7 is different because the model $\mathbf{I}^*$ that is constructed for a maximally consistent and ∃-complete set of sentences must be defined differently.

The interpretation $\mathbf{I}^*$ of the maximally consistent and ∃-complete set Γ* that we constructed in the proof of 11.4.7 stipulated that a distinct positive integer be associated with each individual constant and that

> 3. For each individual constant $\mathbf{a}$, $\mathbf{I}^*(\mathbf{a})$ is the positive integer associated with $\mathbf{a}$.

This will not do in the case of *PDE*. For suppose that Γ, and consequently its superset Γ*, contains a sentence $\mathbf{a} = \mathbf{b}$ in which $\mathbf{a}$ and $\mathbf{b}$ are different constants. If we interpret the constants of *PLE* in accordance with condition 3, $\mathbf{a}$ and $\mathbf{b}$ will denote *different* members of the UD, and hence $\mathbf{a} = \mathbf{b}$ will be false. But the interpretation is supposed to make all members of Γ*, including $\mathbf{a} = \mathbf{b}$, true. So we shall have to change condition 3 to take care of the case where a sentence like $\mathbf{a} = \mathbf{b}$ is a member of the set Γ*. We shall also have to interpret the functors in the language, and to do so in a way that makes sentences containing complex terms true if and only if those sentences are members of Γ*.

Before turning to the construction of an interpretation for Γ*, however, we first establish some facts about sets of sentences that are maximally consistent in *PDE* and ∃-complete. As the reader may easily verify, the properties listed in result 11.4.6 remain true for maximally consistent, ∃-complete sets of sentences of *PDE*. We must add three additional properties to the list in result 11.4.6:

> h. For every closed term $\mathbf{t}$, $\mathbf{t} = \mathbf{t} \in \Gamma^*$.
>
> **Proof:** Let $\mathbf{t}$ be any closed term. $\varnothing \vdash \mathbf{t} = \mathbf{t}$, by =I and ∀E. Because the empty set is a subset of Γ*, it follows from 11.4.5 that $\mathbf{t} = \mathbf{t} \in \Gamma^*$.

i. If $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms and $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$, then

a. If $\mathbf{Q}$ is a sentence in which $\mathbf{t}_1$ occurs, $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ (every sentence obtained by replacing one or more occurrences of $\mathbf{t}_1$ in $\mathbf{Q}$ with $\mathbf{t}_2$) is a member of Γ^*.

b. If $\mathbf{Q}$ is a sentence in which $\mathbf{t}_2$ occurs, $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_1//\mathbf{t}_2)$ is a member of Γ^*.

Proof: Let $\mathbf{t}_1 = \mathbf{t}_2$ be a sentence that is a member of Γ^* and let $\mathbf{Q}$ be a sentence in which $\mathbf{t}_1$ occurs. Assume that $\mathbf{Q} \in \Gamma^*$. Every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is derivable from the set $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{Q}\}$ by =E. Therefore, by 11.4.5, every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is a member of Γ^*. Now assume that $\mathbf{Q} \notin \Gamma^*$. Every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is such that $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)\} \vdash \mathbf{Q}$, by =E—use $\mathbf{t}_1$ to replace every occurrence of $\mathbf{t}_2$ that replaced $\mathbf{t}_1$ in $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$, and the result is $\mathbf{Q}$ once again. So, if any sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1) \in \Gamma^*$, then, by 11.4.5, $\mathbf{Q} \in \Gamma^*$ as well. Therefore, if $\mathbf{Q} \notin \Gamma^*$, then no sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is a member of Γ^*.

Similar reasoning shows that if $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ and $\mathbf{Q}$ is a sentence in which $\mathbf{t}_2$ occurs, then $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_1//\mathbf{t}_2)$ is a member of Γ^*.

j. For each $\mathbf{n}$-place functor f and $\mathbf{n}$ terms $\mathbf{t}_1, \ldots, \mathbf{t_n}$, there is at least one constant $\mathbf{b}$ such that $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{b} \in \Gamma^*$.

Proof: By property (h), the formula $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = f(\mathbf{t}_1, \ldots, \mathbf{t_n}) \in \Gamma^*$. Since $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = f(\mathbf{t}_1, \ldots, \mathbf{t_n}) \vdash (\exists \mathbf{x})f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{x}$, the sentence $(\exists \mathbf{x})f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{x}$ must also be a member of Γ^*, by 11.4.5. And because Γ^* is $\exists$-complete, it follows from our revised definition of $\exists$-completeness that there is at least one constant $\mathbf{b}$ such that the formula $f(\mathbf{t}_1, \ldots, \mathbf{t_n}) = \mathbf{b}$ is also a member of Γ^*.

We now turn to the proof of result 11.4.7 for *PDE*—that every set of sentences of *PLE* that is both maximally consistent in *PDE* and $\exists$-complete is also quantificationally consistent. Let Γ^* be a set of sentences that is both maximally consistent in *PLE* and $\exists$-complete. In preparation for constructing a model for this set, we associate positive integers with the individual constants of *PLE* as follows:

First associate the positive integer $\mathbf{i}$ with the alphabetically $\mathbf{i}$th individual constant of *PLE*. Let $\mathbf{p}(\mathbf{a})$ stand for the integer that has been associated with the constant $\mathbf{a}$. Thus $\mathbf{p}$('a') is 1, $\mathbf{p}$('b') is 2, and so on.

Now we define a second association $\mathbf{q}$: For each constant $\mathbf{a}$, $\mathbf{q}(\mathbf{a}) = \mathbf{p}(\mathbf{a}')$, where $\mathbf{a}'$ is the alphabetically earliest constant such that $\mathbf{a} = \mathbf{a}'$ is a member of Γ^*.

Note that for each constant $\mathbf{a}$, property (h) of maximally consistent, $\exists$-complete sets assures us that $\mathbf{a} = \mathbf{a} \in \Gamma^*$, and so we can be certain that $\mathbf{q}$ assigns a value to $\mathbf{a}$. According to the definition of $\mathbf{q}$, $\mathbf{q}$('a') is always 1 since property (h) assures us that 'a = a' is a member of Γ^*, and 'a' is the alphabetically earliest constant of *PLE*. But for every other constant, the value that **q** associates with it depends on the identity sentences that the particular set Γ^* contains. Suppose that 'b = a', 'b = b', 'b = e', and 'b = m_{22}' are the only identity sentences in Γ^* that contain 'b' to the left of the identity predicate. In this case there is an alphabetically earlier constant to the right, namely, 'a', and this is the alphabetically earliest constant so occurring. So **q**('b') = **p**('a') = 1. If 'c = c', 'c = f', and 'c = g_3' are the only identity sentences in Γ^* that contain 'c' to the left of the identity predicate, then 'c' is the alphabetically earliest constant occurring to the right, and so **q**('c') = **p**('c') = 3. The definition of **q** will play a role in ensuring that identity sentences come out true on the interpretation that we shall construct if and only if they are members of Γ^*, as a consequence of

> **11.4.9:** For any constants **a** and **b**, **q**(**a**) = **q**(**b**) if and only if **a** = **b** $\in \Gamma^*$.

> **Proof:** As preliminaries, let **a**′ be the alphabetically earliest constant such that **a** = **a**′ $\in \Gamma^*$ (remember that property (h) guarantees that there is at least one such constant), and let **b**′ be the alphabetically earliest constant such that **b** = **b**′ $\in \Gamma^*$. Then

- **q**(**a**) = **p**(**a**′) by the way **q** is defined, and
- **q**(**b**) = **q**(**b**′).
- It follows that **q**(**a**) = **q**(**b**) if and only if **q**(**a**′) = **p**(**b**′)
- Because **p** associates different values with different constants, **p**(**a**′) = **p**(**b**′) if and only if **a**′ and **b**′ are the same constant.

We conclude that **q**(**a**) = **q**(**b**) if and only if **a**′ and **b**′ are the same constant.

Now assume that **q**(**a**) = **q**(**b**). It follows that **a**′ and **b**′ are the same constant. Therefore, because **b** = **b**′ $\in \Gamma^*$, it follows trivially that **b** = **a**′, which is the same sentence, is a member of Γ^*. And because **a** = **a**′ $\in \Gamma^*$, it follows from property (i) of maximally consistent, $\exists$-complete sets that **a** = **b** $\in \Gamma^*$ (because **a** = **b** is a sentence **a** = **a**′ (**b**//**a**′)).

Now assume that **a** = **b** $\in \Gamma^*$.

- Because **a** = **a**′ $\in \Gamma^*$, it follows from property (i) that **b** = **a**′ $\in \Gamma^*$.
- Because **b** = **b**′ $\in \Gamma^*$, it follows from property (i) that **a** = **b**′ $\in \Gamma^*$.
- **b**′ was defined to be the alphabetically earliest constant that appears to the right of the identity predicate in an identity sentence containing **b**, and so from the fact that **b** = **a**′ $\in \Gamma^*$, we conclude that **a**′ is not alphabetically earlier than **b**′.
- **a**′ was defined to be the alphabetically earliest constant that appears to the right of the identity predicate in an identity

sentence containing $\mathbf{a}$, and so from the fact that $\mathbf{a} = \mathbf{b}' \in \Gamma^*$, we conclude that $\mathbf{b}'$ is not alphabetically earlier than $\mathbf{a}'$.

- Therefore, $\mathbf{a}'$ and $\mathbf{b}'$ must be the same constant.

So, from (e), we may conclude that $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$.

Result 11.4.9 guarantees that if there is an identity sentence in Γ^* that contains the individual constants $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$, and if there is no identity sentence in Γ^* that contains $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{q}(\mathbf{a}) \neq \mathbf{q}(\mathbf{b})$. And this fact will be crucial in our construction of an interpretation on which every member of a set that is both maximally consistent in *PDE* and ∃-complete is true. We turn now to the construction.

Let Γ^* be a set that is both maximally consistent in *PDE* and ∃-complete, and define the interpretation $\mathbf{I}^*$ as follows:

1. The UD is the set of positive integers that $\mathbf{q}$ associates with at least one individual constant of *PLE*.
2. For each sentence letter $\mathbf{P}$, $\mathbf{I}^*(\mathbf{P}) = \mathbf{T}$ if and only if $\mathbf{P} \in \Gamma^*$.
3. For each individual constant $\mathbf{a}$, $\mathbf{I}^*(\mathbf{a}) = \mathbf{q}(\mathbf{a})$.
4. For each $\mathbf{n}$-place functor f, $\mathbf{I}^*(f)$ is the set that includes all and only those $\mathbf{n} + 1$-tuples $<\mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n}), \mathbf{I}^*(\mathbf{b})>$, where $\mathbf{a}_1, \ldots, \mathbf{a_n}$ and $\mathbf{b}$ are individual constants, such that $f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{b} \in \Gamma^*$.
5. For each $\mathbf{n}$-place predicate $\mathbf{A}$ other than the identity predicate, $\mathbf{I}^*(\mathbf{A})$ is the set that includes all and only those $\mathbf{n}$-tuples $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n}) \rangle$ such that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a_n} \in \Gamma^*$.

We must ensure that conditions 4 and 5 can be met.

For condition 4 we must ensure that the interpretation of f is indeed a function on the UD: that for each $\mathbf{n}$ members $\mathbf{u}_1, \ldots, \mathbf{u_n}$ of the UD there is exactly one member $\mathbf{u_{n+1}}$ of the UD such that $<\mathbf{u}_1, \ldots, \mathbf{u_n}, \mathbf{u_{n+1}}> \in \mathbf{I}^*(f)$. That there is *at least one* such member of the UD follows from the fact that every member of the UD is denoted by at least one individual constant (this is guaranteed by condition 1 of our definition of $\mathbf{I}^*$), and property (j) of sets that are maximally consistent in *PDE* and ∃-complete, which we repeat here:

j. For each $\mathbf{n}$-place functor f and $\mathbf{n}$ constants $\mathbf{a}_1, \ldots, \mathbf{a_n}$, there is at least one constant $\mathbf{b}$ such that the formula $f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{b}$ is a member of Γ^*.

Given these two facts, condition 4 ensures that for each $\mathbf{n}$ members $\mathbf{u}_1, \ldots, \mathbf{u_n}$ of the UD there is at least one member $\mathbf{u_{n+1}}$ of the UD such that $<\mathbf{u}_1, \ldots, \mathbf{u_n}, \mathbf{u_{n+1}}> \in \mathbf{I}^*(f)$, for any functor f. To show that there is at most one such member $\mathbf{u_{n+1}}$, let us assume, to the contrary, that there is also a member of the UD $\mathbf{u'_{n+1}}$, where $\mathbf{u'_{n+1}} \neq \mathbf{u_{n+1}}$, such that $<\mathbf{u}_1, \ldots, \mathbf{u_n}, \mathbf{u'_{n+1}}> \in \mathbf{I}^*(f)$.

- Then in addition to the sentence $f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{b}$, Γ includes a sentence $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{c}$, such that $\mathbf{I}^*(\mathbf{c}) = \mathbf{u}'_{\mathbf{n}+1} \neq \mathbf{I}^*(\mathbf{b})$.
- By virtue of clause 3 of the definition of $\mathbf{I}^*$, it follows that $\mathbf{q}(\mathbf{a}) \neq \mathbf{q}(\mathbf{b})$.
- But this is impossible, since $\{f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{b}, f(\mathbf{a}_1, \ldots, \mathbf{a_n}) = \mathbf{c}\} \vdash \mathbf{b} = \mathbf{c}$ by $=$E.
- So $\mathbf{b} = \mathbf{c} \in \Gamma^*$, by 11.4.5, and therefore $\mathbf{q}(\mathbf{c}) = \mathbf{q}(\mathbf{b})$, by 11.4.9,

It follows that $\mathbf{I}^*(\mathbf{c}) = \mathbf{I}^*(\mathbf{b})$, and so there is at most one member $\mathbf{u}_{\mathbf{n}+1}$ of the UD such that $<\mathbf{u}_1, \ldots, \mathbf{u_n}, \mathbf{u}_{\mathbf{n}+1}> \in \mathbf{I}^*(f)$.

We must also ensure that condition 5 can be met, that is, that there are not two atomic sentences $\mathbf{Aa}_1 \ldots \mathbf{a_n}$ and $\mathbf{Aa}'_1 \ldots \mathbf{a}'_\mathbf{n}$ such that one is a member of Γ^* and the other is not, yet $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n})\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_\mathbf{n})\rangle$. In the case of *PD*, it was simple to show this, for distinct constants were interpreted to designate distinct individuals. However, $\mathbf{q}$ may assign the same positive integer to more than one constant, and as a consequence condition 3 may interpret several constants to designate the same value. Here our previous results will be useful. Suppose that the constants $\mathbf{a}_1, \ldots, \mathbf{a_n}$ and $\mathbf{a}'_1, \ldots, \mathbf{a}'_\mathbf{n}$ are such that $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n})\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_\mathbf{n})\rangle$.

- Then, by clause 3 of the definition of $\mathbf{I}^*$, $\mathbf{q}(\mathbf{a_i}) = \mathbf{q}(\mathbf{a}'_\mathbf{i})$ for each $\mathbf{i}$.
- It follows from 11.4.9 that for each $\mathbf{i}$, $\mathbf{a_i} = \mathbf{a_i}' \in \Gamma^*$.
- Therefore, because $\mathbf{a}_1 = \mathbf{a}'_1$ is a member of Γ^*, property (i) assures us that $\mathbf{Aa}_1 \ldots \mathbf{a_n} \in \Gamma^*$ if and only if $\mathbf{Aa}'_1 \ldots \mathbf{a_n} \in \Gamma^*$, and because $\mathbf{a}_2 = \mathbf{a}'_2$ is a member of Γ^*, property (i) assures us that $\mathbf{Aa}_1\mathbf{a}_2 \ldots \mathbf{a_n} \in \Gamma^*$ if and only if $\mathbf{Aa}'_1\ \mathbf{a}'_2 \ldots \mathbf{a_n} \in \Gamma^*$, and so on until we note that because $\mathbf{a_n} = \mathbf{a}'_\mathbf{n}$ is a member of Γ^*, property (i) assures us that $\mathbf{Aa}_1 \ldots \mathbf{a_n} \in \Gamma^*$ if and only if $\mathbf{Aa}'_1 \ldots \mathbf{a}'_\mathbf{n} \in \Gamma^*$.

We conclude that if $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a_n})\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_\mathbf{n})\rangle$, then $\mathbf{Aa}_1 \ldots \mathbf{a_n} \in \Gamma^*$ if and only if $\mathbf{Aa}'_1 \ldots \mathbf{a}'_\mathbf{n} \in \Gamma^*$. So condition 5 can indeed be met.

To establish result 11.4.7 for *PDE*—that every set Γ^* that is both maximally consistent in *PDE* and $\exists$-complete is also quantificationally consistent—we can prove by mathematical induction that a sentence $\mathbf{P}$ of *PDE* is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$. The proof is similar to that for *PD*, except that we must change the basis clause to consider closed complex terms as well as constants, and we must also consider formulas containing the identity operator. We shall use the following result:

11.4.10: For any closed complex term $\mathbf{t}$ and variable assignment $\mathbf{d_I}$, $\mathrm{den}_{\mathbf{I}^*,\mathbf{d_I}}(\mathbf{t}) = \mathbf{I}^*(\mathbf{a})$, where $\mathbf{a}$ is the alphabetically earliest individual constant such that $\mathbf{t} = \mathbf{a} \in \Gamma^*$. (Property (j) of sets that are maximally consistent in *PDE* and $\exists$-complete guarantees that there *is* such a constant $\mathbf{a}$.)

Proof: See Exercise 16.

Here is the revised proof.

Proof of basis clause: Either $\mathbf{P}$ is a sentence letter or $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t_n}$ or $\mathbf{t}_1 = \mathbf{t}_2$. If $\mathbf{P}$ is a sentence letter, then, by clause 2 of the definition of $\mathbf{I}^*$, it follows that $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

If $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t_n}$ then $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if, for every variable assignment $\mathbf{d_{I^*}}$, $<\text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t}_1), \ldots, \text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t_n})> \in \mathbf{I}^*(\mathbf{A})$.

- Property (j) guarantees, for each complex term $\mathbf{t_i}$, that there is an alphabetically earliest constant $\mathbf{a_i}$ such that $\mathbf{t_i} = \mathbf{a_i}$ is a member of Γ^*.
- Moreover, by virtue of the rule =E, $\mathbf{At}_1' \ldots \mathbf{t_n}'$, where $\mathbf{t_i}'$ is $\mathbf{t_i}$ if $\mathbf{t_i}$ is a constant and $\mathbf{t_i}'$ is $\mathbf{a_i}$ otherwise, is derivable from the set consisting of $\mathbf{At}_1 \ldots \mathbf{t_n}$ and each of these identity sentences.
- So, by 11.4.5, $\mathbf{P} \in \Gamma^*$ if and only if the sentence $\mathbf{At}_1' \ldots \mathbf{t_n}' \in \Gamma^*$.
- In addition, $\text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t_i}) = \mathbf{I}^*(\mathbf{t_i}')$, trivially if $\mathbf{t_i}$ is a constant and by 11.4.10 if $\mathbf{t_i}$ is a complex term.
- So $<\text{den}_{\mathbf{I^*,d_I^*}}(\mathrm{t1}), \ldots, \text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t_n})> \in \mathbf{I}^*(\mathbf{A})$ if and only if $<\mathbf{I}^*(\mathbf{t}_1'), \ldots, \mathbf{I}^*(\mathbf{t_n}')> \in \mathbf{I}^*(\mathbf{A})$,
- and clause 5 in the definition of $\mathbf{I}^*$ guarantees that $\mathbf{At}_1' \ldots \mathbf{t_n}' \in \Gamma^*$ if and only if $<\mathbf{I}^*(\mathbf{t}_1'), \ldots, \mathbf{I}^*(\mathbf{t_n}')> \in \mathbf{I}^*(\mathbf{A})$.

We conclude that $\mathbf{At}_1 \ldots \mathbf{t_n} \in \Gamma^*$ if and only if $\mathbf{At}_1 \ldots \mathbf{t_n}$ is true on $\mathbf{I}^*$.

If $\mathbf{P}$ has the form $\mathbf{t}_1 = \mathbf{t}_2$, then

- $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if, for each variable assignment $\mathbf{d_{I^*}}$, $\text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t}_1) = \text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t}_2)$.
- Again, for each complex term $\mathbf{t_i}$, property (j) guarantees that there is an alphabetically earliest constant $\mathbf{a_i}$ such that $\mathbf{t_i} = \mathbf{a_i} \in \Gamma^*$.
- So, by virtue of =E and result 11.4.5, $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ if and only if $\mathbf{t}_1' = \mathbf{t}_2' \in \Gamma^*$, where $\mathbf{t_i}'$ is $\mathbf{t_i}$ if $\mathbf{t_i}$ is a constant and $\mathbf{t_i}'$ is $\mathbf{a_i}$ otherwise.
- Moreover, $\text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t_i}) = \mathbf{I}^*(\mathbf{t_i}')$, trivially if $\mathbf{t_i}$ is a constant and by result 11.4.10 if $\mathbf{t_i}$ is a complex term.
- So $\text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t}_1) = \text{den}_{\mathbf{I^*,d_I^*}}(\mathbf{t}_2)$ if and only if $\mathbf{I}^*(\mathbf{t}_1') = \mathbf{I}^*(\mathbf{t}_2')$.
- By the way in which $\mathbf{I}^*$ was constructed, $\mathbf{I}^*(\mathbf{t}_1') = \mathbf{I}^*(\mathbf{t}_2')$ if and only if $\mathbf{q}(\mathbf{t}_1') = \mathbf{q}(\mathbf{t}_2')$. By result 11.4.9, $\mathbf{q}(\mathbf{t}_1') = \mathbf{q}(\mathbf{t}_2')$ if and only if $\mathbf{t}_1' = \mathbf{t}_2' \in \Gamma^*$.

We may conclude that $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ if and only if $\mathbf{t}_1 = \mathbf{t}_2$ is true on $\mathbf{I}^*$.

Because every member of Γ^* is true on $\mathbf{I}^*$, Γ^* is quantificationally consistent. And, with results 11.4.4 and 11.4.7 established for *PDE*, along with the necessary modifications of 11.4.8 (see Exercise 11.4.15), we know that result 11.4.2 is also true for *PDE*. It follows that *PDE* is complete for predicate logic with identity and functions.

11.4**E** EXERCISES

***1.** Prove that if $\Gamma \vDash \mathbf{P}$ then $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent.

2. Prove that if $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *PD* then $\Gamma \vdash \mathbf{P}$.

3. Using Metatheorem 11.4.1, prove the following:

a. Every argument of *PL* that is quantificationally valid is valid in *PD*.

b. Every sentence of *PL* that is quantificationally true is a theorem in *PD*.

*c. Every pair of sentences **P** and **Q** of *PL* that are quantificationally equivalent are equivalent in *PD*.

4. Prove that the sentences of *PL* can be enumerated. (*Hint:* See Section 6.4.)

5. Prove the following:

If $\Gamma \vdash \mathbf{P}$ and Γ is a subset of Γ', then $\Gamma' \vdash \mathbf{P}$.

6. Prove that any set Γ^* constructed as in our proof of Lemma 11.4.4 is $\exists$-complete.

7. Prove that the sequence of sentences constructed in the proof of 11.4.3 is a derivation in *PD* by showing (by mathematical induction) that each sentence in the new sequence can be justified with the same rule as the corresponding sentence in the original derivation.

***8.** Prove 11.4.8, using result 11.1.13.

***9.** Prove 11.4.5.

10. Explain why, in results 11.4.4 and 11.4.7, we constructed a set that was both $\exists$-complete and maximally consistent in *PD*, rather than a set that was just maximally consistent in *PD*.

11. Let system *PD** be just like *PD* except that the rule $\forall$E is replaced by the following rule:

<u>*Universal Elimination** ($\forall$E*)</u>

$$\begin{array}{|l} (\forall \mathbf{x})\mathbf{P} \\ \\ \sim (\exists \mathbf{x}) \sim \mathbf{P} \end{array}$$

Prove that the system *PD** is complete for predicate logic.

***12.** Let system *PD** be just like *PD* except that the rules ∃E and ∃I are replaced by the following two rules:

<u>*Existential Elimination** (∃E*)</u>

$$\begin{array}{|l} (\exists \mathbf{x})\mathbf{P} \\ \\ \sim (\forall \mathbf{x}) \sim \mathbf{P} \end{array}$$

<u>*Existential Introduction** (∃I*)</u>

$$\begin{array}{|l} \sim (\forall \mathbf{x}) \sim \mathbf{P} \\ \\ (\exists \mathbf{x})\mathbf{P} \end{array}$$

Prove that system *PD** is complete for predicate logic.

13. Using the results in the proof of Metatheorem 11.4.1, prove the following theorem (known as the *Löwenheim Theorem*):

If a sentence **P** of *PL* is not quantificationally false, then there is an interpretation with the set of positive integers as the UD on which **P** is true.

***14.** Prove the following metatheorem (known as the *Löwenheim-Skolem Theorem*):

If a set Γ of sentences of *PL* is quantificationally consistent, then there is an interpretation with the set of positive integers as the UD on which every member of Γ is true.

***15.** Show the changes that must be made in the proofs of 11.4.3 and 11.4.8 so that these results will hold for *PLE* and *PDE*. (*Hint:* Exercise 8 suggested that you use result 11.1.13 in proving 11.4.8; so you must check whether 11.1.13 needs to be changed.)

16. Prove result 11.4.10.

17. Show that the Löwenheim Theorem (and consequently the more general Löwenheim-Skolem Theorem) does *not* hold for *PLE*.

11.5 THE SOUNDNESS OF THE TREE METHOD

We have presented the tree method as a means of testing for semantic properties of sentences and sets of sentences in both sentential logic and predicate logic. In this section and the next we shall prove that the tree method in Chapter 9 fulfills a claim we have made: A finite set of sentences of *PL* is quantificationally inconsistent if and only if every systematic tree for that set closes. In this section we shall prove that the tree method is **sound** for predicate logic—that if a systematic tree for a set of sentences of *PL* closes, then the set is quantificationally inconsistent. We shall prove the same for predicate logic with identity and functions. In both cases we can then be assured that, if we pronounce a