

7. Midterm Review!

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1.1 Recursive Definitions

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1.4 Trees

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a. Recursive Definitions

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- ▶ some trivial cases: “I”, “a” (we’re thinkin ‘base case material!’)
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- ▶ Note that there must be a smaller string δ such that $\Delta = \text{blah} * \delta * \text{blah}'$, where the 'blah's arise from the recursion clause. (all the ways of making a longer Δ from a shorter δ)

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- ▶ Remember to note that since δ has length $< k$, it falls within the induction hypothesis and hence has the property of interest.

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Induction Step: consider an arbitrary *tieeit*-palindrome Δ of length $k > 6$. Then there must exist a shorter *tieeit*-palindrome δ such that Δ equals either $tie * \delta * eit$, $tee * \delta * tee$, or $tte * \delta * ett$. Since δ has length $< k$, the string length of δ is divisible by 6 by the induction hypothesis. In each case, we add 6 letters to δ to form Δ . Hence, the string length of Δ is also divisible by 6.

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c. Induction on SL Sentences

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Example: Induction on SL, using disjunctive syllogism

Prove the following by induction on wffs of SL. Don't forget to explicitly state the **base case** and the **induction step**!

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 - Use induction to show that every wff is baller.

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 - Use induction to show that every wff is baller.
3. Final step (disjunctive syllogism): if (i) every wff is baller and (ii) a given wff doesn't contain any binary connectives, then (iii) it must be baller in virtue of being contingent.

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- ▶ Note that it is contingent: there exists a TVA where it is true, and a distinct TVA where it is false.
- ▶ Hence, every member in the base case is baller

Baller Induction: Induction Step

Induction Step (using our ‘lazy SL’ schema): consider two arbitrary well-formed formula Φ and Ψ that have the property, i.e. are baller. Show that any way of forming a more complex wff from these two also has the property.

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- ii.-v. Then, consider the four cases coming from connecting Φ and Ψ with a binary connective ($\&$, \vee , \supset , or \equiv). Clearly, each of these wffs has a binary connective and so is baller.

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 - ii.-v. Then, consider the four cases coming from connecting Φ and Ψ with a binary connective ($\&$, \vee , \supset , or \equiv). Clearly, each of these wffs has a binary connective and so is baller.
 - Hence, every wff is baller.
- Hence, if a well-formed formula does not contain any binary connectives, since it is still baller, it must be contingent (disjunctive syllogism)

Non-lazy Induction Schema for SL

Induction Hypothesis: assume that every SL wff of string-length n , where $1 \leq n < k$ has the property, i.e. is baller. Show that an arbitrary wff of length k is baller.

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- Proceed to show that in each case, Δ has the property, i.e. is baller.

Key Fact about Conditionals

- Consider a conditional $P \supset Q$. If a truth value assignment satisfies the consequent Q , then it satisfies the conditional (regardless of the truth value of P)

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d. Trees

Tree-contradictions and Tree-tautologies

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- ▶ (again: this definition requires the existence of single such tree)

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 - 2.) You have **a complete open branch**, in which case the argument is **tree-invalid**

Using trees to check for Validity

Since most homework problems follow this pattern, let's make it really explicit!

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Remember that a branch closes whenever a sentence and its negation appear in its nodes (these need not be atomic sentences)

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 - You have **a complete open branch**, in which case the sentence is NOT a tautology
(semantic aside: it is possible to satisfy the sentence's negation, so it's possible to make the sentence in question false)

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- ▶ Don't apply any semantic equivalencies! e.g. to resolve $\sim(\sim D \vee E)$ you write $\sim\sim D$ stacked on $\sim E$. You cannot immediately write ' D '. To get D , you would have to apply the double negation rule to $\sim\sim D$

7. Midterm Review!

**e. Trees Metalogic: Testing
Alternative Rules**

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- ▶ **Equivalently**: ANY tree with root $\Gamma \cup \{\sim\Theta\}$ possesses **at least one complete open branch**
- ▶ (Aside: this is NOT the same as saying that the argument is **tree-invalid**, since that only requires the existence of a single tree with a complete open branch)

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- ▶ You probably want to have this information right in front of you during the exam, rather than to be scrambling looking for it

Checking Soundness: “top-down” reasoning

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- ▶ If our system *were* sound, then whenever a tree closes, it *would* correspond to a semantically valid argument.

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 - Hence, if we construct a **complete open tree** with an **unsatisfiable root**, this is a reductio of completeness

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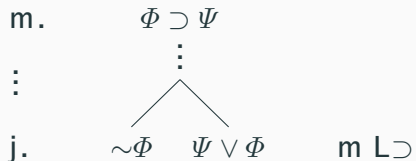
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- ▶ When in doubt, just complete the whole tree

Liberal Conditional and Conservative Biconditional

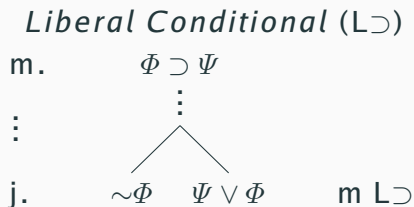
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Liberal Conditional ($L\supset$)

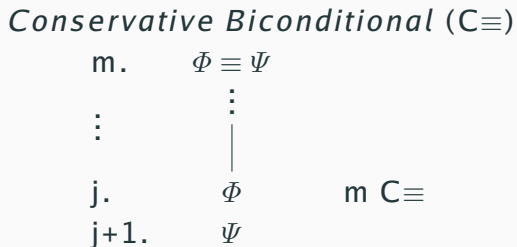


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- ▶ Hence, **to the counterexample!**
(note that the problem REQUIRES this! can't stop won't stop!)

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- But $(P \equiv P) \not\models P$ because $\{(P \equiv P), \sim P\}$ is consistent!
Assign 'P' false! N.B.: your counterexample MUST use actual sentences of SL; not meta-variables!

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 - Hence, $\Phi \equiv \Psi$ is true on \mathcal{I} , which is what we needed to show.

7. Midterm Review!

f. Translation/Symbolization in SL

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- ▶ $\sim\sim V \vee (D \& \sim A)$

Provided that; lonely if; given: these flip the order!

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- ▶ ‘Q if P’ is ‘If P, then Q’
- ▶ Likewise for ‘Q given P’: ‘If P, then Q’

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- ▶ A contrapositive is logically equivalent to its conditional: $P \supset Q$

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- ▶ $(M \& A) \& H$

7. Midterm Review!

g. Truth Tables: (In)Validity and (In)Equivalence

Proving an argument is Valid

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- ▶ Equivalently: every TVA that makes the premises true makes the conclusion true
- ▶ (note that this 'every' claim is vacuously true if your conclusion is a tautology, since a tautology is not false on any TVA)

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- ▶ You must show that the two sentences receive the same truth value for every possible truth value assignment (i.e. every row of the truth table)

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7. Midterm Review!

h. Natural Deduction

Some rules to Definitely Understand

- ▶ Disjunction Elimination

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- ▶ Understand when and how to start a subproof
- ▶ For rules that cite a subproof, remember to use a HYPHEN between line numbers in the justification (you are citing the ENTIRE subproof, even in those cases where the sub proof is itself only 2 lines long)

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- ▶ Derive the consequent within the subproof
- ▶ Exit the subproof by writing the conditional and justifying with $\supset I$

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- ▶ Often, if you are using a sentence that has already occurred as one of the Ψ or $\sim\Psi$, you will need to reiterate it on a line within the subproof
- ▶ *Carnap* seems to be a bit fussy: the Ψ and $\sim\Psi$ cannot themselves be separated by a subproof within your subproof