10. Proofs in QL

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a. Some Equivalences in QL

Motivation: Proving Equivalences

- We'd like to have a method for proving that two sentences are equivalent, or that an argument is valid
- Models only provide counterexamples to equivalence or validity
- By extending our natural deduction system to cover QL, we'll be able to prove two sentences are equivalent!
  - Derive one from the other and vice versa
  - Basically: show that the biconditional of the two sentences is a tautology

Some Equivalences: Quantifiers commuting over connectives

1. $Fa \,\&\, (\exists x)Gx$ is equivalent to $(\exists x)(Fa \,\&\, Gx)$
2. $Fa \,\&\, (\forall x)Gx$ is equivalent to $(\forall x)(Fa \,\&\, Gx)$
3. $Fa \vee (\exists x)Gx$ is equivalent to $(\exists x)(Fa \vee Gx)$
4. $Fa \vee (\forall x)Gx$ is equivalent to $(\forall x)(Fa \vee Gx)$
5. $Fa \supset (\exists x)Gx$ is equivalent to $(\exists x)(Fa \supset Gx)$
6. $Fa \supset (\forall x)Gx$ is equivalent to $(\forall x)(Fa \supset Gx)$
   - But note that '$\supset$' is not symmetric, so we have to examine the converse: e.g. $(\exists x)Gx \supset Fa$!
   - And there are no analogous rules for biconditionals

Quantifiers NOT commuting over some conditionals

'$\supset$' is not symmetric, so we have to examine the converse

bad: $(\exists x)Gx \supset Fa$ is NOT equivalent to $(\exists x)(Gx \supset Fa)$

Instead, the existential converts to a universal when we scope over the consequent:

7. $(\exists x)Gx \supset Fa$ is equivalent to $(\forall x)(Gx \supset Fa)$

bad: $(\forall x)Gx \supset Fa$ is NOT equivalent to $(\forall x)(Gx \supset Fa)$

Instead, the universal converts to an existential when we scope over the consequent:

8. $(\forall x)Gx \supset Fa$ is equivalent to $(\exists x)(Gx \supset Fa)$

Informal Argument: If anyone G's, then c F's

7. $(\exists x)Gx \supset Fc$ is equivalent to $(\forall x)(Gx \supset Fc)$

- Recall: "Any" in antecedents but **without** pronouns referring back to them are **existential**:

  If **anyone** is a hero, Greta is.

  Roughly: if there are heroes (at all), Greta is a hero.

  $(\exists x)\ Hx \supset Hg$

- Intuitively: for everybody, if they are a hero, then Greta is a hero

  $(\forall x)(Hx \supset Hg)$

- We would like to show that if we can derive one of these sentences, then we can derive the other.

  Our proof system will show this *naturally*!

Informal Argument: If everyone G's, then c F's

8. $(\forall x)Gx \supset Fc$ is equivalent to $(\exists x)(Gx \supset Fc)$

- By truth conditions for '$\supset$', first sentence is equivalent to:

  $\sim(\forall x)Gx \vee Fc$

  Apply de Morgan's: $(\exists x)\sim Gx \vee Fc$

- Now apply our rule 3: existential commutes over disjunction (provided the disjunct doesn't contain the bound variable x)

  $(\exists x)(\sim Gx \vee Fc)$, and apply '$\supset$' truth conditions again:

- Yields $(\exists x)(Gx \supset Fc)$

10. Proofs in QL

b. Rules for ∀

Rules for formal proofs

- Need rules for $\forall$ and $\exists$ for formal proofs
- Formal proofs now more important, because no alternative (truth-table method)
- Intro and Elim rules should be
  - simple
  - elegant (not involve other connectives or quantifiers)
  - yield only valid arguments

Candidates for rules

- Only simple sentence close to $(\forall x)\,\mathcal{A}x$ is $\mathcal{A}c$
- Gives simple, elegant $\forall$E rule:

$$\begin{array}{r|ll} k & (\forall x)\,\mathcal{A}x & \\ & & \\ & \mathcal{A}c & :k\ \forall\mathrm{E} \end{array}$$

- This is a good rule: $(\forall x)\,\mathcal{A}x \models \mathcal{A}c$.

Candidates for rules

- Problem: corresponding "intro rule" isn't valid:

$$
\begin{array}{r|l}
k & \mathcal{A}c \\
 & \\
 & (\forall x)\,\mathcal{A}x \qquad :k \textbf{ (doesn't follow)}
\end{array}
$$

- Diagnosis: the c in $\mathcal{A}c$ is a name for a **specific object**.
- We need a name for an **arbitrary, unspecified object**.
- If $\mathcal{A}c$ is true for whatever c **could** name, then $\mathcal{A}x$ is satisfied by **every** object.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.

  Only people who wear capes admire Greta.

  ∴ All heroes wear capes.

  Proof: Let Carl be any hero.
  Since all heroes admire Greta, Carl admires Greta.
  Since only people who wear capes admire Greta, Carl wears a cape. But "Carl" stands for **any** hero.
  So all heroes wear capes.

Universal generalization

$$\begin{array}{l|ll} k & \mathcal{A}c & \\ & (\forall \chi)\,\mathcal{A}\chi & :k\ \forall\text{I} \end{array}$$

- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathcal{A}\chi$ is obtained from $\mathcal{A}c$ by replacing **all** occurrences of c by χ.
- In other words, c must also not occur in $\forall\chi\,\mathcal{A}\chi$.

General conditional proof

Proving "All As are Bs"

k	$\quad Ac$	:AS for $\supset$I
l	$\quad Bc$	
$l+1$	$Ac \supset Bc$	:k-l $\supset$I
$l+2$	$(\forall x)(Ax \supset Bx)$	:$l+1$ $\forall$I

Example

All heroes admire Greta.

Only people who wear capes admire Greta.

$\therefore$ All heroes wear capes.

$(\forall x)(Hx \supset Axg)$

$(\forall x)(Axg \supset Cx)$

$\therefore$ $(\forall x)(Hx \supset Cx)$

Let's do it on Carnap (PP10.2)!

Example

1	$(\forall x)(Hx \supset Axg)$	:PR
2	$(\forall x)(Axg \supset Cx)$	:PR
3	$\quad Hc$	:AS for $\supset$I
4	$\quad Hc \supset Acg$	:1 $\forall$E
5	$\quad Acg$	:4, 3 $\supset$E
6	$\quad Acg \supset Cc$	:2 $\forall$E
7	$\quad Cc$	:6, 5 $\supset$E
8	$Hc \supset Cc$	:3-7 $\supset$I
9	$(\forall x)(Hx \supset Cx)$	:8 $\forall$I

Example

1	$(\forall x)\, Ax \lor (\forall x)\, Bx$	:PR
2	$\quad (\forall x)\, Ax$	:AS for $\lor$E
3	$\quad Ac$	:2 $\forall$E
4	$\quad Ac \lor Bc$	:3 $\lor$I
5	$\quad (\forall x)\, Bx$	:AS for $\lor$E
6	$\quad Bc$	:5 $\forall$E
7	$\quad Ac \lor Bc$	:6 $\lor$I
8	$Ac \lor Bc$	:1, 2–4, 5–7 $\lor$E
9	$(\forall x)(Ax \lor Bx)$	:8 $\forall$I

10. Proofs in QL

c. Rules for ∃

Rules for ∃

- If we know of a specific object that it satisfies $\mathcal{A}x$, we know that at least one object satisfies $\mathcal{A}x$.

- So this rule is valid:

$$
\begin{array}{r|l}
k & \mathcal{A}c \\
\hline
 & (\exists \chi)\,\mathcal{A}\chi \qquad :k\ \exists\mathrm{I}
\end{array}
$$

Arbitrary objects again

- Problem: corresponding "elimination rule" isn't valid:

$$\begin{array}{r|ll} k & (\exists x)\,\mathcal{A}x & \\ & \mathcal{A}c & :k \textbf{ doesn't follow from} \end{array}$$

- If we know that $(\exists x)\,\mathcal{A}x$ is true, we know that **at least one** object satisfies $\mathcal{A}x$, but not which one(s).
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathcal{A}(x)$.

Reasoning from existential information

- To use $(\exists x)\mathcal{A}x$, pretend the x has a name c, and reason from $\mathcal{A}(c)$.
- This is what we do to reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
  $\therefore$ Some heroes admire Greta.

  Proof: We know there are heroes who wear capes.
  Let Cate be an arbitrary one of them.
  So Cate wears a cape. Since anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

Existential elimination (mind the restriction!)

- If
  - we know that some object satisfies $\mathcal{A}\chi$,
  - we hypothetically assume that c is one of them (i.e., assume $\mathcal{A}c$),
  - and we can prove that $\mathcal{B}$ follows from this assumption,

  then $\mathcal{B}$ follows already from $(\exists\chi)\,\mathcal{A}\chi$.
- Rule for existential elimination:

k	$(\exists\chi)\,\mathcal{A}\chi$	
m	$\mathcal{A}c$	:AS for ∃E
n	$\mathcal{B}$	
	$\mathcal{B}$	:k, m-n ∃E

- c is special: c **must NOT appear outside subproof**

Example

There are heroes who wear capes.

Anyone who wears a cape admires Greta.

$\therefore$ Some heroes admire Greta.

$(\exists x)(Hx \mathbin{\&} Cx)$

$(\forall x)(Cx \supset Axg)$

$\therefore (\exists x)(Hx \mathbin{\&} Axg)$

Example (PP10.5)

1	$(\exists x)(Hx \,\&\, Cx)$	:PR
2	$(\forall x)(Cx \supset Axg$	:PR
3	$\quad Hc \,\&\, Cc$	:AS for $\exists$E
4	$\quad Cc$	:3 & E
5	$\quad Cc \supset Acg$	:2 $\forall$E
6	$\quad Acg$	:4, 5 $\supset$E
7	$\quad Hc$	:3 & E
8	$\quad Hc \,\&\, Acg$	:4, 7 & I
9	$\quad (\exists x)(Hx \,\&\, Axg)$	:8 $\exists$I
10	$(\exists x)(Hx \,\&\, Axg)$	:1, 3-9 $\exists$I

10. Proofs in QL

d. The Rules, collected!

Rules for the Universal Quantifier

Universal Elimination (∀E)

$$
\begin{array}{r|ll}
m & (\forall \chi)\Phi(\ldots \chi \ldots \chi \ldots) & \\
\vdots & \quad\vdots & \\
s & \Phi(\ldots c \ldots c \ldots) & :m\ \forall\mathrm{E}
\end{array}
$$

- Note that you replace EVERY instance of χ with c

- Notation: $\Phi[c/\chi]$

- read "c for χ"

Universal Introduction (∀I)

$$
\begin{array}{r|ll}
m & \Phi(\ldots c \ldots c \ldots) & \\
\vdots & \quad\vdots & \\
s & (\forall \chi)\Phi(\ldots \chi \ldots \chi \ldots) & :m\ \forall\mathrm{I}
\end{array}
$$

Provided that both
(i) c does not occur in any undischarged assumptions that Φ is in the scope of.
(ii) χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.

Rules for the Existential Quantifier

Existential Introduction (∃I)

m	$\Phi(\ldots c \ldots c \ldots)$	
$\vdots$	$\vdots$	
s	$(\exists \chi)\Phi(\ldots \chi \ldots \chi \ldots)$	:m ∃I

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.
- Note that χ may replace some or all occurrences of c.

Existential Elimination (∃E)

m	$(\exists \chi)\Phi$	
	$\vdots$	
n	$\quad$ $\Phi(\ldots c \ldots c \ldots)$	:AS for ∃E
	$\quad$ $\vdots$	
s	$\quad$ Ψ	
$s+1$	Ψ	:m, n–s ∃E

provided that c doesn't occur **anywhere else outside** the subproof

10. Proofs in QL

e. Substitution Instances

(Full) Substitution Instances

- $\Phi[c/\chi]$ is the sentence you get from $(\forall\chi)\Phi$ by dropping the $(\forall\chi)$ quantifier and putting c in place of **every** χ in Φ.
- The other variables are untouched!
- $\Phi[c/\chi]$ can also arise from $(\exists\chi)\Phi$ by dropping the $(\exists\chi)$ and putting c in place of **every** χ in Φ.
- Equivalent notation: $\Phi\boxed{\chi \Rightarrow c}$

Some Examples of Substitution Instances

- Instances of $(\forall y)Hy$:
  - Ha, Hb, Hm_{11}
- Instances of $(\exists z)Haz$:
  - Haa, Hab, Haj_3
- Instances of $(\exists z)(Hz \& Fzz)$:
  - Remember to replace **EVERY** occurance of z with the chosen constant:
  - $(Ha \& Faa)$, $(Hc \& Fcc)$
  - The following are **NOT** substitution instances:
  - $(Ha \& Faz)$, $(Hy \& Faa)$, $(Ha \& Fab)$

Some Things to WATCH OUT for!

- Is the wff '$(\forall x)Gx \vee (\forall x)Fx$' universally quantified?
- NO! It is a disjunction, since '$\vee$' is its main connective.
- What are instances of a sentence like $(\forall x)Gx \vee (\forall x)Fx$?
- Trick question! It has no instances, since it is a disjunction, not a quantified sentence!

Finding Formulas that have a given instance

- Goal: figure out some wffs that a given wff is an instance of:
- Fa is an instance of...
  - $(\forall x)Fx$, $(\exists x)Fx$, $(\forall z)Fz$, etc.
- Fab is an instance of...
  - $(\forall x)Fax$, $(\forall x)Fxb$, $(\exists x)Fax$, $(\exists x)Fxb$
- Faa is an instance of...
  - $(\forall x)Fxx$, $(\forall x)Fax$, $(\forall x)Fxa$, $(\exists x)Fxx$, $(\exists x)Fxa$

Partial Substitution Instances

- For Existential Introduction, we can use a partial substitution instance of the wff Φ:
- '$\Phi\lceil \chi / c \rceil$' indicates that the variable χ does not need to replace all occurrences of the constant c in Φ.
- You can decide which occurrences of c to replace and which to leave in place

Examples of Partial Substitution Instances!

- '$\Phi\lceil\chi/c\rceil$' indicates that the variable χ does not need to replace all occurrences of the constant c in Φ

1	Rdd	
2	$(\exists x)Rxx$	:1 $\exists$I
3	$(\exists x)Rxd$	:1 $\exists$I
4	$(\exists z)Rdz$	:1 $\exists$I
5	$(\exists y)(\exists z)Ryz$	:4 $\exists$I

Existential Introduction ($\exists$I)

m	$\Phi(c)$	
$\vdots$	$\vdots$	
s	$(\exists\chi)\Phi\lceil\chi/c\rceil$	:m $\exists$I

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.

Putting this Notation to Work!

- Using the notation of (partial) substitution instances, we can rewrite our rule schemata!
- Payoff: we can drop one of the notes we had about existential introduction
- Is this worth it???? Only one way to find out!

Rules for the Universal Quantifier

Universal Elimination (∀E)

m	$(\forall \chi)\Phi$
$\vdots$	$\vdots$
s	$\Phi[c/\chi]$ $: m\ \forall\mathrm{E}$

- Note that you replace **EVERY** instance of χ with c
- Notation: $\Phi[c/\chi]$
- read "c for χ"

Universal Introduction (∀I)

m	$\Phi(c)$
$\vdots$	$\vdots$
s	$(\forall \chi)\Phi[\chi/c]$ $: m\ \forall\mathrm{I}$

Provided that both
(i) c does not occur in any undischarged assumptions that Φ is in the scope of.
(ii) χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.

Rules for the Existential Quantifier

Existential Introduction (∃I)

$$
\begin{array}{l|ll}
m & \Phi(c) & \\
\vdots & \quad\vdots & \\
s & (\exists\chi)\Phi\lceil\chi/c\rceil & :m\ \exists\text{I}
\end{array}
$$

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.
- As indicated by $\lceil\chi/c\rceil$, χ **may** replace **just some** occurrences of c

Existential Elimination (∃E)

$$
\begin{array}{l|lll}
m & (\exists\chi)\Phi & & \\
 & \quad\vdots & & \\
n & & \Phi[c/\chi] & :\text{AS for } \exists\text{E} \\
 & & \quad\vdots & \\
s & & \Psi & \\
s+1 & \Psi & & :m,\ n\text{-}s\ \exists\text{E}
\end{array}
$$

provided that c doesn't occur **anywhere else outside** the subproof

10. Proofs in QL

f. Tips for an all-natural look!

Working Backwards

- Work backwards (at least in making a plan):
- If the final sentence is...
  - existentially quantified, you will probably introduce it with $\exists I$ somewhere
  - universally quantified: will probably use $\forall I$
  - a conditional: conditional introduction
  - a disjunction: disjunction intro

Working with your Premises

- If a premise is...
  - universally quantified: you will probably eliminate that universal at least once to arrive at an instance
    (you may do this multiple times for different instances)
  - existentially quantified: you will probably eliminate this existential, and if so you probably want to start this subproof ASAP (mind the tricky syntax!)
  - a conjunction: you can get conjuncts using &E
  - a disjunction: think about using disjunction elimination
    (mind the tricky syntax!)

Advice for deriving $(\exists \chi)\,\mathcal{A}\chi$, e.g. $(\exists y)\,Py$

- Ask whether you can derive an instance $\mathcal{A}$[c for χ], e.g. Pc
- If so, you could introduce $(\exists \chi)\,\mathcal{A}\chi$ through $\exists I$
  - Remember: you might have to do this first as the last line of a SUBproof started for $\exists E$, before repeating what you want below the subproof, justified by $\exists$ ELIMINATION
- If not: consider using negation−elimination or disjunction elimination (if a disjunction is available)

Advice for using Existential Elimination

- Remember to begin a sub-proof with a relevant instance of the existential
  - hot tip: use a constant that hasn't appeared yet!
  - in *Carnap*, tab in to a new proof-level by pressing 'tab'

- **PRO TIP**: Usually a good idea to carry out *as much of your proof as possible* **within the scope** of the subproof: this makes obeying the restrictions easier
- Remember that you'll write the same sentence TWICE in a row, once within the subproof and once outside (the second instance being the one justified by $\exists E$)
- Remember that the constant c must NOT appear in the conclusion, existential being eliminated, or an undischarged assumption