- 1. Proofs in SL
- 1.1 Who ordered that???
- 1.2 Conjunction Intro and Elimination (Rules for &)
- 1.3 Conditional Intro and Elim. (Rules for \supset)
- 1.4 Use of subproofs
- 1.5 Disjunction Intro and Elim. (Rules for \lor)
- 1.6 Negation Intro and Elimination
- 1.7 Biconditional Intro and Elimination (<->)
- 1.8 Strategies and examples
- 1.9 The Rules, Reiterated

a. Who ordered that???

The Power of a Proof System

- ▶ We have already seen how powerful a proof system can be compared to truth tables:
- Consider our running example argument:

$$C \vee E$$
, $A \vee M$, $A \supset \sim C$, $\sim S \& \sim M : E$

requires 32 lines and 512 truth values

- ▶ With trees, we showed validity in 10 lines (13 sentences)
- ► Why not just tree always and everywhere?

For the Love of Trees

- ► The advantages of trees create limitations:
 - Mechanical ⇒ not 'normal' pattern of inference
 - Mirror partial truth-tables ⇒ implicitly referencing truth-values
 - Always checking satisfiability of the root ⇒ we don't really 'derive' anything at the end of the proof, unlike in normal inference
 - (this is why so many of us keep forgetting that it is the NEGATION of the conclusion that goes in the root)
- ▶ We would like to form a better model of human inference patterns
- ▶ e.g., common rules such as Modus Ponens, disjunctive syllogism

Idiosyncrasies of Table or Tree Reasoning

- ► Tables:
 - Construct a truth table
 - verify there is no TVA where premises are true but conclusion is false
- ► This is NOT how we typically reason through an argument
- Trees: ask whether a set of sentences is satisfiable:
 - Put premises and NEGATION of conclusion in root
 - If tree closes, then unsatisfiable root (valid argument)
 - If tree remains open, then satisfiable root (invalid argument)
- ► Again, this is NOT how we typically reason through an argument

The Very Idea of 'Natural Deduction'

- We commonly reason according to certain inference rules
- And we often make assumptions in the *middle* of our reasoning, derive an intermediate conclusion, and 'discharge' the assumption (e.g. in proof by contradiction)
- ▶ We would like to see if we can *vindicate* these patterns:
 - Show that these rules never get us into trouble: soundness
 - Show that we have enough rules to handle any valid argument (including additional rules we might want to add): completeness
- Perhaps our natural deduction system explains the success of our ordinary inference patterns

Ordinary Reasoning by "Proofs"

- ► Idea: work our way from premises to conclusion using steps we know are entailed by the premises.
- ► For instance:
 - From "Neither Sarah nor Amir enjoys hiking" we can conclude "Amir doesn't enjoy hiking."
 - From "Either Amir lives in Chicago or he enjoys hiking" and "Amir doesn't enjoy hiking" we can conclude "Amir lives in Chicago" (Disjunctive syllogism DS).
 - etc.
- ► If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

- 1. Sarah lives in Chicago or Erie.
- 2. Amir lives in Chicago unless he enjoys hiking.
- 3. If Amir lives in Chicago, Sarah doesn't.
- 4. Neither Sarah nor Amir enjoy hiking.
- : Sarah lives in Erie.
- 5. Amir doesn't enjoy hiking (from 4).
- 6. Amir lives in Chicago (from 2 and 5).
- 7. Sarah doesn't live in Chicago (from 3 and 6).
- 8. Sarah lives in Erie (from 1 and 7).

A more formal proof

Our argument

- 1. *C* ∨ *E*
- 2. $A \vee M$
- 3. $A \supset \sim C$
- 4. \sim 5 & \sim *M*
- ∴ E
- 5. $\sim M$ (from 4, since $P \& Q \models Q$)
- 6. A (from 2 and 5, since $\{P \lor Q, \sim Q\} \models P$)
- 7. $\sim C$ (from 3 and 6, since $\{P \supset Q, P\} \models Q$, i.e. via 'modus ponens')
- 8. *E* (from 1 and 7, since $\{P \lor Q, \sim P\} \models Q$)

Some aspects of our Formal Deductions

- Numbered lines contain sentences of SL
- ► A line may be a **premise** (:PR).
- ► A line may be an **assumption** (:AS)
- ► If neither a premise nor assumption, it must be justified
- Justification requires:
 - a rule (e.g. '&E'), and
 - prior line(s) invoked by the rule—referenced by line number(s)
 - starting with a colon: e.g. ': 2 &E'
- ▶ But: what are the rules? (very different from 'what IS a rule'?)

Aspects of our Rules for Natural Deduction

- ► Our Rules will (mostly) be ...
 - Simple: cite just a few lines as justification
 - Obvious: new line should clearly be entailed by justifications
 - Schematic: can be described just by forms of sentences involved
 - Few in number: want to make do with just a handful
- We'll have two rules per connective: an introduction and an elimination rule
- ► They'll be used to either:
 - justify (say) P & Q (i.e. to 'introduce' &), or
 - justify something using P & Q (i.e. to 'eliminate' &).

b. Conjunction Intro and Elimination (Rules for &)

Eliminating &

- ▶ What can we justify using P & Q?
- ► A conjunction entails each conjunct:

$$\mathcal{P} \& \mathcal{Q} \models \mathcal{P}$$
 $\mathcal{P} \& \mathcal{Q} \models \mathcal{Q}$

- ▶ Already used this above to get $\sim M$ from $\sim S \& \sim M$, i.e., from "Neither Sarah nor Amir enjoys hiking" we concluded "Amir doesn't enjoy hiking".
- ▶ (Role of \mathcal{P} played by $\sim S$ and that of \mathcal{Q} played by $\sim M$)

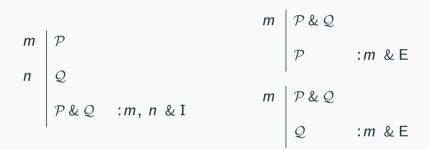
Introducing &

- ▶ What do we **need to justify** P & Q?
- \blacktriangleright We need both \mathcal{P} and \mathcal{Q} :

$$\{\mathcal{P},\mathcal{Q}\} \vDash \mathcal{P} \& \mathcal{Q}$$

- ► For instance, if we have "Sarah doesn't enjoy hiking" and also "Amir doesn't enjoy hiking", we can conclude "Neither Sarah nor Amir enjoys hiking"
- ▶ (Role of \mathcal{P} played by $\sim S$ and \mathcal{Q} by $\sim M$: $\{ \sim S, \sim M \} \vDash \sim S \& \sim M \}$

Rules for &



We'll illustrate using exercises in our Week 6 Practice Problems on Carnap.

1	A & (B & C)	:PR
2	A	:1 & E
3	B & C	:1 & E
4	A B & C C A & C	:3 & E
5	A & C	:2,4 & I

c. Conditional Intro and Elim.

(Rules for ⊃)

Eliminating \supset

- ▶ What can we justify using $P \supset Q$?
- ► We used the conditional "If Amir lives in Chicago, Sarah doesn't" to justify "Sarah doesn't live in Chicago".
- ▶ What is the general rule? What can we justify using $\mathcal{P} \supset \mathcal{Q}$? What do we need in addition to $\mathcal{P} \supset \mathcal{Q}$?
- ► The principle is **modus ponens** (affirming the antecedent):

$$\{\mathcal{P}\supset\mathcal{Q},\mathcal{P}\}\vDash\mathcal{Q}$$

▶ (When inferring from $A \supset \sim C$ and A to $\sim C$, the role of \mathcal{P} is played by A and role of \mathcal{Q} by $\sim C$.)

Elimination rule for \supset

Let's illustrate this rule using an exercise in Carnap: we show that $\{A \& B, A \supset C, B \supset D\} \models C \& D$.

A & B :PR $A\supset C$:PR $B\supset D$:PR :1 & E 5 :2,4⊃E 6 :1 & E В 7 :3, 6 ⊃E D

:5,7 & I

C & D

$\textbf{Introducing}\supset$

- ▶ How do we justify a conditional? What should we require for a proof of $\mathcal{P} \supset \mathcal{Q}$ (say, from some premise \mathcal{R})?
- ▶ We need a proof that shows that $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.
- ▶ Idea: show instead that \mathcal{R} , $\mathcal{P} \models \mathcal{Q}$.
- ightharpoonup The conditional \supset no longer appears, so this seems easier.
- ▶ It's a good move, because if \mathcal{R} , $\mathcal{P} \models \mathcal{Q}$ then $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.

$\textbf{Justifying}\supset \textbf{I}$

Fact

If \mathcal{R} , $\mathcal{P} \models \mathcal{Q}$ then $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.

- ▶ If \mathcal{R} , $\mathcal{P} \models \mathcal{Q}$ then every TVA makes \mathcal{R} or \mathcal{P} false or it makes \mathcal{Q} true
- ▶ Let's show that no valuation is a counterexample to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$:
 - 1. A valuation that makes \mathcal{R} and \mathcal{P} true, but \mathcal{Q} false, is impossible if $\mathcal{R}, \mathcal{P} \models \mathcal{Q}$.
 - 2. So any valuation must make $\mathcal R$ false, $\mathcal P$ false, or $\mathcal Q$ true.
 - 3. If it makes \mathcal{R} false, it's not a counterexample to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.
 - 4. If it makes \mathcal{P} false, it makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.
 - 5. If it makes Q true, it also makes $P \supset Q$ true, so it's not a counterexample.
- ▶ So, there are no counterexamples to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.
- ▶ If $\mathcal{R} = \emptyset$, then from $\mathcal{P} \models \mathcal{Q}$ we can infer $\models \mathcal{P} \supset \mathcal{Q}$

Subproofs (CRUCIAL CONCEPT)

- ▶ We want to justify $\mathcal{P} \supset \mathcal{Q}$ by giving a proof of \mathcal{Q} from assumption \mathcal{P} (and possibly other premises Γ , e.g. \mathcal{R})
- ► How to do this in a proof? We can add something as a premise and *discharge* it later!
- ightharpoonup Solution: add $\mathcal P$ as an assumption (:AS), and keep track of what depends on that assumption (say, by indenting and a vertical line)
- ightharpoonup Once we're done (have proved Q), close this "subproof".
- ▶ Justification of $P \supset Q$ is the **entire** subproof (use a HYPHEN)
- ► Important: nothing inside a subproof is available outside as a justification (since inner lines might depend on the assumption)

Introduction rule for ⊃

$$m \mid \frac{\mathcal{P}}{\mathcal{P}}$$
 :AS for \supset I

i.

 \mathcal{Q}
 $\mathcal{P} \supset \mathcal{Q}$: $m-n \supset$ I

NOTE THE HYPHEN IN THE JUSTIFICATION LINE!!!

We'll illustrate using more exercises from Week 6 Practice Problems

- ▶ Show: $\{A \supset B, B \supset C\} \models A \supset C$.
- ► Show: $A \supset (B \supset C) \models (A \& B) \supset (A \& C)$

1
$$A \supset B$$
 :PR
2 $B \supset C$:PR
3 A :AS for $\supset I$
4 B :1, 3 $\supset E$
5 C :2, 4 $\supset E$
6 $A \supset C$:3-5 $\supset I$

1

$$A \supset (B \supset C)$$
 :PR

 2
 $A \& B$
 :AS for \supset I

 3
 A
 :2 & E

 4
 $B \supset C$
 :1, 3 \supset E

 5
 B
 :2 & E

 6
 C
 :4, 5 \supset E

 7
 $A \& C$
 :3, 6 & I

 8
 $(A \& B) \supset (A \& C)$
 :2-7 \supset I

d. Use of subproofs

Reiteration (for the 11th hour!)

 $P \models P$, so "Reiteration" R is a good rule:

$$m \mid \mathcal{P}$$
 $k \mid \mathcal{P} : m \mid F$

Uses of reiteration (to the Carnap!):

- ▶ Proof of $A \models A$.
- ▶ Proof that $A \supset (B \supset A)$ is a tautology.

$$\begin{array}{c|cc}
1 & A & :AS \text{ for } \supset I \\
2 & A & :1 R \\
3 & A \supset A & :1-2 \supset I
\end{array}$$

Again, note the **HYPHEN!** Even though our subproof is only two lines, we still write ':1-2' and NOT ':1, 2'.

1 |
$$A$$
 : AS for \supset I
2 | B : AS for \supset I
3 | A : 1 R
4 | $B \supset A$: 2-3 \supset I
5 | $A \supset (B \supset A)$: 1-4 \supset I

Rules for justifications and subproofs

- ▶ When a rule calls for a subproof, we cite it as ": m-n", hyphenating the first and last line numbers of the subproof.
- ► Sentences on the Assumption line and last line MUST match rule
- After a subproof is done, you can only cite the whole thing, NOT any line in it (you are 'outside the scope' of these lines)
- ► Subproofs (subproofs can be nested) can be nested
- You also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is completed (since again you'd be 'outside the scope' of those lines)

Reiteration (do's a don'ts) DOs and DON'Ts!

Which are correct applications of R?

2		A	:AS
3		A	:1 √ R
4	/	4	:1 √ R
5	/	4	:2 X R
6	A		:2 X R
_			.1 v D

1 | A :AS

6.d.5

(Rules for ∨)

e. Disjunction Intro and Elim.

Introduction rule for \lor

We have $P \vDash P \lor Q$. So:

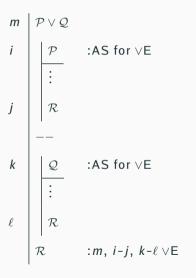
- ► Note that the introduced disjunct can be ANYTHING!
- ► And you can introduce on the left OR right side!
- ► Let's do practice problem 6.10 on Carnap!

$$\begin{array}{c|ccc}
1 & A & :AS & for \supset I \\
2 & B \lor A & :1 \lor I \\
3 & A \supset (B \lor A) & :1-2 \supset I
\end{array}$$

Eliminating \lor (Proof by Cases)

- ▶ What can we justify with disjunction $P \lor Q$?
- ▶ Not \mathcal{P} and also not \mathcal{Q} : neither is entailed by $\mathcal{P} \vee \mathcal{Q}$.
- ▶ But: if both \mathcal{P} and \mathcal{Q} separately entail some third sentence \mathcal{R} , then we know that \mathcal{R} follows from the disjunction!
- ▶ To show this, we need **two** subproofs that show \mathcal{R} , but in each proof we are allowed to use only one of \mathcal{P} , \mathcal{Q} .

Elimination rule for \lor (Proof by Cases)



- ightharpoonup From $\mathcal P$ we derive $\mathcal R$
- Start a subproof for each disjunct
- The subproofs need not be adjacent, but if they are, separate with --
- ightharpoonup From $\mathcal Q$ we derive $\mathcal R$
- You can swap the order of the subproofs
- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction
- Remember to pop out of subproof level at the end!

1
$$A \lor B$$
 :PR
2 $A \lor A$:AS for $\lor E$
3 $B \lor A$:2 $\lor I$
4 $B \lor A$:AS for $\lor E$
5 $B \lor A$:4 $\lor I$
6 $B \lor A$:1, 2-3, 4-5 $\lor E$

- ► In Carnap: Need -- between the subproofs
- ► Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself
- ► Proceed to Carnap PP6.15!

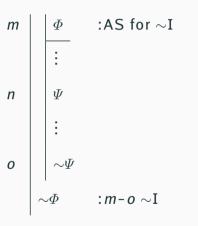
1
$$A \lor B$$
 :PR
2 $A \supset B$:PR
3 $A : AS \text{ for } \lor E$
4 $B : 2, 3 \supset E$
5 $B : AS \text{ for } \lor E$
6 $B : 5 \text{ R}$
7 $B : 1, 3-4, 5-6 \lor E$

f. Negation Intro and Elimination

Introducing \sim

- ► An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.
- ► For instance:
 - $Q \models P$ iff Q and $\sim P$ are jointly unsatisfiable.
 - $Q \vDash \sim P$ iff Q and P are jointly unsatisfiable.
- ▶ This last one gives us idea for \sim I rule: To justify \sim \mathcal{P} , show that \mathcal{P} (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction follows!

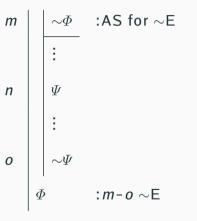
Negation Introduction (\sim I)



- Assume the non-negated wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- ▶ Pop out of the subproof and introduce that negativity!
- ► Remember to cite the WHOLE subproof (hyphen!)
- ► Let's try exercise PP6.21:

1
$$A \supset B$$
:AS for \supset I2 $\sim B$:AS for \supset I3 A :AS for \sim I4 B :1, 3 \supset E5 $\sim B$:2 R6 $\sim A$:3-5 \sim I7 $\sim B \supset \sim A$:2-6 \supset I8 $(A \supset B) \supset (\sim B \supset \sim A)$:1-7 \supset I

Negation Elimination (\sim E)



- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- ▶ Pop out of the subproof and eliminate that negativity!
- ▶ Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)
- ► Let's try exercise PP6.22:

$$\begin{array}{c|cccc}
1 & \sim A \supset \sim B & :AS & \text{for } \supset I \\
2 & B & :AS & \text{for } \supset I \\
3 & \sim A & :AS & \text{for } \sim E \\
4 & \sim B & :1, 3 \supset E \\
5 & B & :2 & R \\
6 & A & :3-5 \sim E \\
7 & B \supset A & :2-6 \supset I \\
8 & (\sim A \supset \sim B) \supset (B \supset A) & :1-7 \supset I
\end{array}$$

g. Biconditional Intro and Elimination (<->)

Biconditional Introduction (\equiv I) (Type <-> !!!)

	:	
j		
k	B	:AS for $\equiv I$
	:	
l	B	
	A - B	:i_i

 $i \mid \mathcal{A}$:AS for $\equiv I$

twice, from both directions
 You can swap the order of the subproofs
 The subproofs need not be

► Like doing conditional intro

adjacent, but if they are,
 separate with - Remember to cite BOTH
 subproofs (hyphens!)

6.q.1

Remember to cite BOTTI subproofs (hyphens!)Remember to pop out of subproof line!

Biconditional Elimination (\equiv E) (Type <-> !!!)

$$m \mid A \equiv B$$
 $n \mid A$
 $B : m, n \equiv E$
 $m \mid A \equiv B$
 $n \mid B$
 $A : m, n \equiv E$

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)
- ► There can be lines between lines m and n
- Remember to cite the lines of both (i) the biconditional and (ii) the side you have already
- ► Carnap issue: must type <->E

Issue with Typing \equiv in Carnap

- ► For Carnap to recognize \equiv I or \equiv E in the justification column, you sadly must type <-> I or <-> E
- ► This is a bummer; I hope we can have it fixed (eventually)
- ► It is still fine to type <> for the biconditional symbol in the sentences
- You can also copy/paste the ≡symbol from elsewhere on the page!

h. Strategies and examples

Working forward and backward

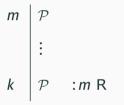
- Working backward from a conclusion (goal) means:
 - Find main connective of goal sentence
 - Match with conclusion of corresponding Intro rule
 - Write out (above the goal!) what you'd need to apply that rule
- Working forward from a premise, assumption, or already justified sentence means:
 - Find main connective of premise, assumption, or sentence
 - Match with top premise of corresponding E rule
 - Write out what else you need to apply the E rule (new goals)
 - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)
- ► Write conclusion at bottom
- Work backward & forward from goals and premises/assumptions in this order:
 - Work backward using & I, \supset I, \equiv I, \sim I/E, or forward using \vee E
 - Work forward using & E
 - Work forward using $\supset E$, $\equiv E$
 - Work backward from ∨I
 - Try Negation Intro or Elimination, working toward a contradiction
- Repeat for each new goal from top

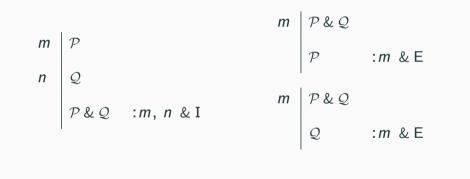
i. The Rules, Reiterated

The rules, one more time: Reiteration



- Remember that you must be in the scope of the line you're reiterating
- e.g. if you're outside a subproof, you can't reiterate anything wholly within the subproof

The rules: Conjunction Intro (& I) and Elimination (& E)



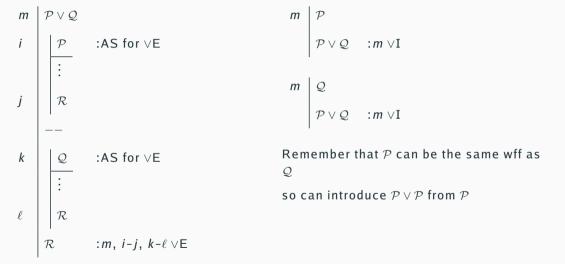
The rules: Conditional Intro (\supset I) and Elim (\supset E)

$$m \mid \frac{\mathcal{P}}{\vdots}$$
 :AS for \supset I

 $m \mid \mathcal{P} \supset \mathcal{Q}$
 $n \mid \mathcal{P}$
 \mathcal{Q}
 $\mathcal{P} \supset \mathcal{Q}$: $m-n \supset$ I

 $m \mid \mathcal{P} \supset \mathcal{Q}$
 $m \mid \mathcal{P} \supset \mathcal{Q}$
 $m \mid \mathcal{P} \supset \mathcal{Q}$

The rules: Disjunction Intro (\lor I) and Elimination (\lor E)



The rules: Negation Intro and Elimination

Negation Intro (\sim I)

m	Φ :AS for \sim I	$m \mid \ \sim \Phi$:AS for \sim E
	:	<u>:</u>
n	Ψ	n Ψ \vdots
	:	
o	$\sim \!\! \Psi$	o $\hspace{-0.5cm} \hspace{-0.5cm} \hspace{-0.5cm} \hspace{-0.5cm} \sim \hspace{-0.5cm} \varPsi$
	$\sim \Phi$: m - o \sim I	Φ :m-o ~E
·		'

Neg. Elimination (\sim E)

6.i.5

Biconditional Elimination ($\equiv E$)

 $egin{array}{c|c} m & \mathcal{A} \equiv \mathcal{B} \\ n & \mathcal{A} \\ \mathcal{B} & :m, \ n \equiv \mathsf{E} \end{array}$

The rules: Biconditional Intro and Elimination (<->)

6.i.6

 $egin{array}{c|c} m & \mathcal{A} \equiv \mathcal{B} \\ n & \mathcal{B} \\ & \mathcal{A} & :m, \ n \equiv \mathsf{E} \end{array}$