Logical Consequence

LOGIC I
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From Last Time…

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \to \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Characteristic Truth Tables: As drawn in the textbook…

Complete Truth Tables

Setup: Write the sentence on the top right, add the constituent sentence letters on the left, and use the characteristic truth tables.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ:

- $[\varphi] = \{\varphi\}$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- For any wfss φ and ψ of $\mathcal{L}^{\text{PL}}$, and $\star \in \{\wedge, \vee, \to, \leftrightarrow\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$;
  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$;

Rows: Add 2^n rows for n constituent sentence letters.

Examples: $[A \wedge (B \vee A)] \to A$, $C \leftrightarrow \neg C$, D.

Tautology: Only 1s under its main connective in its complete truth table.

Contradiction: Only 0s under its main connective in its complete truth table.

Logically Contingent: A 1 and a 0 under its main connective in its complete truth table.

Logical Entailment: On any row of a complete truth table, the consequent has a 1 under its main connective whenever the antecedent does.

Logical equivalence: Identical columns under the main connectives for the sentences.

Satisfiable: There is a row where all wfss have a 1 under all main connectives.

Logical Consequence: The conclusion has a 1 under its main connective in every row in which every premise has a 1 under its main connectives.

Decidability

Effective Procedure: A finitely describable and (in principle) usable procedure that always finishes and produces a correct answer to the question asked, requiring only that the instructions be followed accurately.

Question: How to define the main operators and distribute truth-values?

- Recursively, like the formation rules for the wfs of $\mathcal{L}^{\text{PL}}$.

Question: Is it always possible to construct a complete truth table for a wfs?

- Sentences have a finite number of constituent sentence letters.

Decidable: If there is an effective procedure for determining the answer to a question, that question is *decidable*.

- It is decidable whether a wfs of $\mathcal{L}^{\text{PL}}$ is a tautology, etc.

Question: What about a complete truth table for a set of sentences?

- Could require infinitely many sentence letters.
- We might be able to define an infinite table, but we can't use it.

Question: If one procedure is not effective, couldn't there be another one?

- It turns out that there is no effective procedure…
- There is always an effective procedure for finite sets of sentences.

Validity: So the validity of finite arguments is decidable.

Partial Truth Tables

Worry 1: It is not *that* effective… in practice it is daunting for $n > 4$.

Partial Truth Tables: Sometimes only one or two lines are needed.

- $A \rightarrow \neg(A \vee B)$: not a tautology or contradiction, so contingent.
- $B \leftrightarrow \neg(A \vee B)$ is a contradiction, so we need a complete table.
- $C \vee (A \rightarrow A)$ is a tautology, so we need a complete table.

Complete: To affirm equivalence, entailment, and logical consequence.

Partial: To affirm that a set is satisfiable.

Worry 2: Still daunting sometimes.

Worry 3: Definitions all refer to complete truth tables.

- Definition of a complete truth table has some minor ambiguities.
- These could be fixed, but the result is cumbersome.

Heuristic: The truth table definitions are best taken to be a heuristic guide for grasping the abstract definitions we may now provide.