Topoi

Logical Subtraction as Contextual Equivalence

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Abstract:	The proposition I raise my arm (R) entails my arm goes up (U). But what would be left of R if we subtracted U from R? We can call this leftover content the logical remainder R - U. The straightforward analysis of R - U is something like: I will my arm to go up. One might reject the view that there is such a willing, but the idea of subtracting some content B from another content A has generated a growing literature on (what is now called) logical subtraction. Traditional approaches to logical subtraction identify logical subtraction with logical remainder. I argue that the traditional approach goes wrong because it leads us to build too much indeterminacy into logical remainder. I propose a different approach to defining subtraction. On my approach, we distinguish between logical remainder and logical subtraction. Logical remainder is the formula A - B. Logical subtraction is the idea that A - B and A are equivalent in a context. I take this equivalence as fundamental and use it to make sense of phenomena that broadly fall under the umbrella of logical subtraction.

Keywords:	truthmakers; logical subtraction; logic; aboutness

Logical Subtraction as Contextual Equivalence

Introduction

Wittgenstein (2009, 169) asks, "what is left over if I subtract the fact that my arm goes up from the fact that I raise my arm?" The proposition *I raise my arm* (R) entails *my arm goes up* (U). But what would be left of R if we subtracted U from R? We can call this leftover content, if it is defined, the *logical remainder R - U*. The straightforward answer to Wittgenstein's question is that *R - U* is something like: *I will my arm to go up*. Raising one's arm could be divided into two parts: psychological (willing) and nonpsychological (going up). Wittgenstein rejects the view that there is such a willing, but the idea of subtracting some content B from another content A has generated a growing literature.

Logical subtraction is the logical analogue of arithmetical subtraction; alternatively, it is the inverse of conjunction (Jaeger 1973; Hudson 1975; Hornsby 1980; Humberstone 2000, 2011; Yablo 2014; Fine 2017a, 2017b; Hoek 2018). The notion of logical subtraction has been extended beyond the philosophy of action to phenomena like knowledge, confirmation, mathematics, and loose talk. Much of the literature on the foundations of logical subtraction has centered the following question: which principles govern logical subtraction? What makes a theory of logical subtraction a theory, rather than a summary of intuitions, is that (a) the content of $A - B$ can be systematically generated and (b) the content of $A - B$ can be put into natural logical relations with other propositions. In the first case, we want a systematic way to determine the content of, say, $(A \wedge B) - (A \leftrightarrow B)$ (if it is defined). In the second case, we want to know, say, whether A implies $A - B$.

As it stands, there is no consensus on the principles of subtraction. Even worse: it is unclear what a theory of subtraction should account for. Logical subtraction has various applications—metaphysical, logical, and linguistic. For example, Wittgenstein (2009) appears mainly interested in the existence and nature of particular remainders. In contrast, Jaeger (1973), Hudson (1975), and Humberstone (2011) are interested in the logic of $A - B$. In recent applications of subtraction, Yablo (2014) and Hoek (2018) use the notion to make sense of presuppositions and loose talk. What determines the principles of subtraction will depend on the intended applications of the notion, but there is no existing theory of subtraction that excels at all applications.

To make progress in the theory of logical subtraction, I suggest that we take a radically different approach to defining subtraction. On the traditional approach, one attempts to define A - B in a way that validates certain principles. On my approach, we distinguish between logical remainder and logical subtraction. Logical remainder is the formula A - B. Logical subtraction, however, is a kind of contextual equivalence relation between A - B and A. The animating idea behind most cases of logical subtraction is that A - B and A are equivalent in a context. I take this equivalence as fundamental and use it to make sense of phenomena that broadly fall under the umbrellas of logical subtraction and logical remainder. Here is a preview. Let s be the subject matter of a context c. Then A - B and A are equivalent in a context c just in case the part of A - B that is true about s is identical to the part of A that is true about s. I then understand the content of A - B using truthmakers (or verifying states); a verifier of A - B is a state that, when fused with an arbitrary verifier of B, gives us a verifier of A. The account of subject matter, partial truth, and logical remainder is given within the framework of truthmaker semantics (Fine 2017a, 2017b; Jago 2017, 2020).

I proceed as follows. I start by showing that historical and contemporary theories of logical subtraction have problems satisfying desired applications of logical subtraction (§2-3). Afterward, I develop my novel theory of subtraction. The first component of my theory is that the core of subtraction is the idea of A - B being equivalent to A in a context (§4). The second component of my theory consists of an account of the content of A - B; the remainder $A - B$ corresponds to the relevant conditional $B \rightarrow A$. (§5).

Theories of Subtraction: First Generation

Here are a few more examples of logical subtraction.

- *Conjunction. Sparky is a brown dog - Sparky is brown* has the same content as *Sparky is a dog*.
- *Laws.* A law-like statement is a law except not true. So *X is a law-like statement = X is a law - X is not true*. (Goodman 1983)
- *Kinds.* A fox is a vixen except it need not be female. So *fox = vixen - female*. (Humberstone 2011, 678)
- *Nominalism. The number of planets is greater than zero* entails (A) *abstract numbers exist* (B). A nominalistically acceptable paraphrase of A would be A - B, which intuitively has the same content as *the world is just like it would be if there were more than zero planets except there are no abstract numbers*. (Yablo 2014)

Additional applications of logical subtraction concern loose talk, knowledge, and more. Some of these examples are more complex, and more contentious, than others. I will mainly focus on the conjunction example and Wittgenstein's arm example.

For any two propositions A and B, we want to uniquely identify a proposition $A - B$. One way to do this is by describing principles that govern $A - B$. But before we begin considering principles of subtraction, a few preliminary remarks are in order. First, I will be treating "$-$" as a binary connective. So $A - B$ is identical to some proposition R that is a function of A and B. Second, we will focus on cases of $A - B$ where A entails B. Those are the paradigm cases, as subtracting B requires B to be part of (in some sense) A in the first place. We may extend the theory of subtraction to cover cases in which A doesn't entail B, but we will consider such cases as *value-added*; *viz.,* they would be nice to account for, if we can, but our principal interest is making sense of the paradigm cases. Third, I will start by largely assuming classical logic. So I take the material conditional $\supset$ and semantic entailment relation $\vDash$ to behave classically. To avoid ambiguity, I will consistently refer to material implication as "implication" and semantic entailment as "entailment."

Enough preliminaries. Recall that the proposition *I raise my arm* (R) implies *my arm goes up* (U). What is $R - U$? Intuitively, it is *I will my arm to go up*. To calculate this proposition, Jaeger (1973) initially described five principles constraining logical remainder.[1]

Here is the first principle.

RETURN

It is always true that $[(A - B) \wedge B] \supset A$.

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[1] Strictly speaking, these principles are to be understood as validities. Each principle has the form $\vDash \phi$ or $\nvDash \phi$.

So $[(\textit{Sparky is a brown dog} - \textit{Sparky is brown}) \wedge \textit{Sparky is brown})] \supset \textit{Sparky is a dog}$. And adding *I will my arm to go up* to *my arm goes up* will imply *I raise my arm*. This seems right. Adding B to $A - B$ should give us A. This principle makes sense of the analogy with arithmetical subtraction. Combining $A - B$ and B will get you A in the same way that adding $a - b$ and b will get you a. We *return* to A.[2]

The second principle tells us that, if $A - B$ consists of subtracting B from A, then A should imply the remainder.

PARTS

It is always true that $A \supset (A - B)$.

Sparky is a brown dog implies *Sparky is a brown dog - Sparky is brown* (which implies *Sparky is a dog*). And *I raise my arm* should imply *I will my arm to go up* because the latter is intuitively *part of* the former. PARTS tell us that the whole (A) should imply its parts (including $A - B$).

The next two principles articulate the idea that the subtrahend B will be disjoint from the remainder $A - B$. The third principle says that subtracting A from B should not leave us something that implies B. B should be subtracted from A, already.

RIGHT SUBTRAHEND

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[2] Terminology due to Yablo (2014, 151), though his version of this principle is a biconditional. Yablo compares RETURN to another principle, RECOVERY: $[(A \wedge B) - B] \leftrightarrow A$. The problem with RECOVERY is that, under natural assumptions, it entails that $A - B$ has the same content as A.

It is not always true that $(A - B) \supset B$.

Sparky is a brown dog - Sparky is brown does not imply *Sparky is brown*. And *I raise my arm - my arm goes up* does not imply *my arm goes up*. In general, the subtrahend B cannot be part of the remainder $A - B$.[3]

The fourth principle tells us that B should not necessitate $A - B$; alternatively, $A - B$ is not part of B.

LEFT SUBTRAHEND

It is not always true that $B \supset (A - B)$.

Sparky is brown should not imply *Sparky is a brown dog - Sparky is brown* (which is intuitively identical to *Sparky is a dog*). And *my arm goes up* should not necessitate *I raise my arm - my arm goes up*, given that we take the latter to be identical to *I will my arm to go up*. My arm could certainly go up without my willing it to go up. In general, the remainder cannot be part of the subtrahend.

The fifth and final principle is a higher-order principle.

UNIQUENESS

$A - B$ is the unique formula that satisfies RETURN, PARTS, LEFT SUBTRAHEND, and RIGHT SUBTRAHEND.

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[3] We could institute a principle stronger than RIGHT SUBTRAHEND, one ruling out all cases where $A - B$ implies B. However, it is unclear whether this would give the right verdict for remainders like $A - A$. A similar moral goes for LEFT SUBTRAHEND.

The obvious candidate for $A - B$ is simply the converse of the material conditional: $B \supset A$. The resulting analysis is as follows.

HORSESHOE

$A - B$ is identical to $B \supset A$.

HORSESHOE is an analysis of logical remainder, one that conveniently accounts for the intuitive principles on remainder we have described. This theory has been the starting point for the theory of logical subtraction.

One initial problem with HORSESHOE is that $B \supset A$ does not uniquely satisfy these principles. Just consider $(B \vee C) \supset A$. This leads Jaegar to give up on the idea of logical subtraction. However, Hudson (1975) argues that the converse implication analysis can be preserved. We just say that $A - B$ is the *weakest* statement that, when conjoined with B, implies A.

A more serious problem with HORSESHOE concerns the fact that $\neg B$ entails $B \supset A$ and therefore entails $A - B$. This result has counterintuitive consequences. Here is an example by Humberstone (2011, 690). Intuitively, *Alpha is a vixen - Alpha is female* is equivalent to *Alpha is a fox*. Given HORSESHOE, it follows that *it is not the case that Alpha is female* implies *Alpha is a fox*. But this is intuitively wrong. The failure to be a female does not make you a fox. Yablo (2014, 147) gives another example. Suppose Falstaff is called to give his total testimony *F*. And suppose Falstaff tells the truth about everything except some matters concerning Green; call this false part of his testimony *G*. Intuitively, *F* - *G* is the true part of Falstaff's testimony. However, HORSESHOE implies that: if Falstaff lies about Green, then he must be telling the truth about everything else. This is the wrong result.

These examples show that we want logical subtraction to satisfy another condition.

RELEVANCE

It is not always true that $\neg B \supset (A - B)$.

Given HORSESHOE, the falsity of B implies the truth of $B \supset A$. Relevance logicians find this unacceptable, as they do not think the falsity of an antecedent to guarantees a conditional's truth. We can say something similar about the horseshoe, insofar as it is understood as defining logical remainder. The falsity of B is not necessarily relevant to making A - B true. Even Jaeger (1973, 324) comes to think that RELEVANCE is a condition should be met for any adequate theory of logical remainder. Humberstone (2011, 690) is sympathetic to this condition as well.[4]

There are additional potential problems with HORSESHOE, but the consensus is that it flounders when it comes to RELEVANCE. Next, we turn to modern theories of logical subtraction. Such theories more clearly extend the application of logical subtraction to semantics and metaphysics.

[4] Despite the consensus in favor of RELEVANCE, Hudson (1975) rejects this condition. He says: A and $A - B$ are disjoint, or fail to have common parts, in the sense they only have $A \vee (A - B)$ in common, where this reduces to $A \vee \neg A$ on the horseshoe analysis. However, you may reject the view that $A \vee (A - B)$ is part of, or contained by, A (Gemes 1994, 1997; Angell 2002; Yablo 2014; Fine 2016, 2017a; Brast-McKie 2021).

Theories of Subtraction: Second Generation

The contemporary revival of logical subtraction is chiefly due to Yablo (2014), who charts a new way beyond the failures of the simple horseshoe theory.[5]

The problem with the horseshoe theory is that truth-conditions are inexact. We should distinguish between truth-*conditions* and truth*makers*. The proposition *the coffee is hot* is true in possible worlds in which people serve hot coffee. However, the proposition is exactly *made true* by something—call it **hot coffee**—that is independent of most of what happens in the hot-coffee-worlds. A *truthmaker* for *p* is what exactly verifies, or makes, *p* true. Similarly, a *falsemaker*, or falsifier, is what exactly falsifies *p*; so *the boat is on fire* is falsified by the fact that **the boat is in an oxygen-less container**.

What are truthmakers and falsemakers, exactly? For Yablo, the truthmakers and falsemakers relevant to logical subtraction will be minimal guarantors of truth and falsity.[6] Formally speaking, a truthmaker (falsemaker) for a proposition P is a minimal model (countermodel), and a minimal model (countermodel) is a partial function that is defined just enough to render P true (false). For example, a function v that assigns truth to P and Q is not a minimal model of P because it also makes Q true. But v would be a minimal model of $P \wedge Q$ because it is defined just enough to make $P \wedge Q$ true; making v any less defined

[5] Though I should acknowledge the role that Humberstone (2011) had in reviving logical subtraction. I classify Humberstone as first generation, as opposed to second generation, for thematic reasons. His logical approach has much in common with earlier ways of thinking about logical subtraction.

[6] Or something like it. See Yablo (forthcoming) (forthcoming) for the view that strict minimality is too much to ask for relevance, but minimality relative to a subject matter, on the other hand, could work.

would result in a valuation that does not make $P \wedge P$ true; and making v more defined would result in it unnecessarily making additional formulas true. Let $\{\neg p, q\}$ designate the partial function that makes P false and Q true. It follows that $\{\neg p, q\}$ is a truthmaker—in fact, the only one—for $\neg P \wedge Q$.

If we define truthmakers as minimal models, we are now prepared to understand Yablo's theory of remainder. The idea is to define remainders, not in terms of the truth-conditions of the horseshoe ($B \supset A$), but in terms of *ways of making* $B \supset A$ true. We take a special interest in B-compatible truthmakers for $B \supset A$, for these are ways in which the conditional is true but not because of B's falsity. This condition rules out the possibility that B implies $A - B$, as desired. The basic account of remainder is below.

YABLOVIAN REMAINDER

$A - B$ is

1. true at w just in case $B \supset A$ has a B-compatible truthmaker but $B \supset \neg A$ does not
2. false at w just in case $B \supset \neg A$ has a B-compatible truthmaker but $B \supset A$ does not
3. undefined at w otherwise

The intuition is that $A - B$, when true, adds ways for A to be true without ruling out B; furthermore, $A - B$ does not add ways for A to be false that do not already depend on B's being false. For Yablo, $A - B$ is a principled way to project A beyond the B-worlds.

Now let us consider some applications. For Yablo, the paradigm case concerns numbers. Suppose *the number of planets is greater than zero - abstract numbers exist* is true. Then there must be a B-compatible truthmaker for *abstract numbers exist* ⊃ *the number of planets is greater than zero*. Intuitively, this truthmaker represents the concrete facts about planets. In such a world, there is no corresponding B-compatible truthmaker for *abstract numbers exist* ⊃ *the number of planets is not greater than zero*; given that we have the concrete materials needed for multiple planets, the only way to make the conditional true would be for there to not be any abstract objects. So $A - B$ is true in the worlds that are just like it would be if there were non-zero planets except abstract numbers did not exist.

It will also be useful to consider a case in which $A - B$ is undefined. Consider *the King of France is bald - France has a king*. *France has a king* ⊃ *the King of France is bald* does not have a truthmaker compatible with *France has a king* and neither does *France has a king* ⊃ *it is not the case that the King of France is bald*. All roads to $B \supset A$ and $B \supset A$ require the falsity of B. Given YABLOVIAN REMAINDER, this means $A - B$ is undefined; $A - B$ is neither true nor false. Not every case of remainder in which a presupposition B is subtracted from A will be undefined. For example, Yablo argues that *the King of France is in my garage - France has a king* can be false because the state **my garage is empty** is a B-compatible truthmaker for *France has a king* ⊃ *the King of France is in my garage* but not *France has a king* ⊃ *it is not the case that the King of France is in my garage*; an empty garage is compatible with France's having a king, and the only other way to make it false that the King of France is in my garage is for there to be no king of France.

Yablo's theory of remainder is suited for linguistic applications. He argues that, in many cases, we utter "A," which literally means A, but we nonetheless convey the content A - B. Number-talk is the paradigm case. You say "The number of planets is greater than zero" and literally mean *the number of planets is greater than zero*, but you ultimately want to convey *the number of planets is greater than zero - abstract numbers exist*. The existence of abstract numbers will be irrelevant to the topic of conversation unless the topic is metaphysics. Both the numbers and presupposition cases also illustrate the linguistic phenomenon of *loose talk*—scenarios in which we communicate a weaker message ($A - B$) by using a sentence ("A") that expresses a strictly stronger one (A); we have some literal content A that we loosen by subtracting B from it.

The main problem with Yablo's account of logical subtraction is that YABLOVIAN REMAINDER appears to mandate unwanted indeterminacy. While the paradigm case of remainder is perfectly determinate, it is unclear that how much indeterminacy will crop up in everyday cases of remainder. On Yablo's view, *the King of France is bald - France has a king* is undefined, but this does not strike me as an obviously good result. Even if you think, for linguistic reasons, "The King of France is bald" is neither true nor false, you might not think the theory of logical subtraction should help secure this result. If *the King of France is bald - France has a king* is undefined in this way, then there will be many remainders that are indeterminate.[7] To be clear: the problem is not simply indeterminacy, but the fact that the theory of remainder generates indeterminacy by default and for no intuitive reason.

[7] Jandrić (2014) argues that Yablo's theory does not have the result that Yablo takes it to have. He argues that *The King of France is bald* will be false, so long as we identify the appropriate truthmakers and falsemakers. Even if Yablo is wrong about this particular

A related problem with Yablovian remainder is that it is *non-recursive* (or non-compositional) in the sense that the remainder $A - B$ is not calculated on the basis of the atomic truthmakers of A and B. On a recursive model, an atomic sentence A will be assigned an atomic truthmaker a. So *the emerald is grue* will be assigned the particular atomic truthmaker **the emerald is grue**. Against this view, Yablo (2014, 71) writes: "If you are asked how it comes about that an emerald is grue, you will not answer (as you should, on the recursive model) that it is grue by being grue. An emerald is grue either by being examined and green, or unexamined and blue." Intuitively, simple sentences can be true for complex reasons. So *the emerald is grue* will be verified by $\{\{g\}, \{b\}\}$, where g verifies *green and observed before t* and b verifies *blue and observed after t*. Yablo calls his approach to truthmaking *reductive* as opposed to recursive. The problem with reductive truthmaking, and its consequent non-recursion, is that we know less about how logical remainder works, formally speaking.[8] The advantage of HORSESHOE is that we understand how the horseshoe conditional works on the basis of familiar logical tools. The reductive approach to truthmaking takes these tools away from us.

To be fair, Yablo recognizes both of these potential disadvantages of his theory, but he sees them as necessary tradeoffs. Yablovian remainders may be indeterminate in many cases, but he thinks this is simply what we should expect when we engage in concrete applications. This relates to his defense of non-recursion. He claims that the reductive

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sentence, however, my broad point remains that we should not start with the assumption that remainders will be indeterminate in some cases.

[8] See Hoek (2018, 152) for a similar complaint.

approach is better for modeling the complexities of natural language (Yablo 2014, 71). I am skeptical. Hoek (2018) has recently offered a theory of logical subtraction that seems better suited for linguistic applications than YABLOVIAN REMAINDER. Like Yablo, Hoek thinks that there are linguistic phenomena that are helpfully understood in terms of expressing a logical remainder. Unlike Yablo, however, Hoek mobilizes a battery of tools in natural language semantics to make sense of how the content of remainders are determined.

Here are the basics of the theory. You say "The number of planets is greater than zero" and thereby literally express *the number of planets is greater than zero* (A). This proposition implies *abstract numbers exist* (B). Going further, you might say that A presupposes B. This is how Yablo sets thing up. But for Hoek, there are two differences. The first is that the relevant presuppositions are *contextual* presuppositions; they are attached to an utterance in a context, not the sentence itself (Simons 2005). Second, we cannot compute the remainder if we only have A and B; we must also know the *question under discussion* (QUD) or *subject matter* of the conversation.[9] In the metaphysics room, our topic might be abstract objects; we want to know: do abstract numbers have such-and-such properties? In ordinary life, however, we may use number-talk because we ultimately care about what the concrete world is like; we want to know: did such-and-such concrete event happen? A Hoekean theory of remainder holds that the remainder is a function of the literal content,

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[9] QUDs and subject matters are both ways to treat the broad idea of the topic of conversation, but they are traditionally addressed separately by linguists and philosophers. For discussions of QUDs among linguists, see Büring (2003; Beaver and Clark 2009; Roberts 2012). For discussions of subject matter among philosophers, see Lewis (1988), Yablo (2014, 23–44), Fine (2017b), and Brast-McKie (2021). Hoek (2018) is unique in bringing these discussions together.

its presupposition, and the subject matter of the conversation. Hoek calls this entire linguistic process *conversational exculpature*.

It will be useful to get a sense of the formal machinery involved. Hoek takes a proposition to be an ordered pair of sets of possible worlds, $\langle t, f \rangle$, where t (f) represents the worlds in which the proposition is true (false). A *full* proposition is defined at every possible world; so the union of t and f corresponds to the set of all possible worlds. A (properly) *partial* proposition is a proposition that is not full; it is not defined for all possible worlds. Suppose we have a proposition A which presupposes B. We can construct a partial proposition that restricts A to B; the restricted proposition $A \sqcap B$ will be true at all of the $A \wedge B$ worlds and false at all of the $\neg A \wedge B$ worlds. Think of it this way: $A \sqcap B$ gets rid of the irrelevant bits of A. However, $A \sqcap B$ is too indeterminate to serve as a proper logical remainder. For this reason, we define a function s that takes $A \sqcap B$ and fills in the gaps in a principled way; $(A \sqcap B)^s$ corresponds to $A - B$.[10]

In summary, the analysis is as follows.

HOEKEAN REMAINDER

In a context where B is a presupposition of A and s is the subject matter of conversation, $A - B$ is identical to $(A \sqcap B)^s$.

At first glance, Hoek's theory has the benefit of not putting as much indeterminacy into the logical remainder as Yablo's theory. The subject matter of the conversation serves as a

[10] I have skipped details about how s works, for the sake of brevity. See Hoek (2018, 167–74) for the whole story.

principled way to fill in the gaps of the partial proposition. The proposition may not be fully defined, but we have a reason why: $A - B = (A \sqcap B)^S$ will be defined to the extent that B is irrelevant to the subject matter S. Another benefit of Hoek's account is that it uses familiar notions developed by natural language semanticists—contextual presuppositions, questions under discussion, conversational context, etc.—to do the heavy lifting. Hoek's theory of conversational exculpature is a convincing semantic implementation of some of Yablo's ideas about how logical remainders are used in conversation.

Nonetheless, there is a sense in which Hoek's theory also requires us to be committed to indeterminacy with respect to logical subtraction. Notice that, when interpreted as a theory of logical remainder, there is no such thing as $A - B$ *simpliciter*. The content of $A - B$ is partly defined by the question under discussion, but those can be different depending on the context. This makes sense, from a linguistic perspective, but it leaves us with indeterminacy with respect to logical remainder, itself. I have no complaint with indeterminacy, in general. The problem is that it seems to crop up in logical subtraction even when we are not entertaining standard cases of indeterminacy—like vagueness.

I end my discussion of second generation theories of subtraction with a theory given by Kit Fine. Fine gives at least three accounts of logical remainder, but I will discuss what appears to be his most recent view.[11] Fine (2020, 158–60) gives an account of logical

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[11] Fine (2017b, 684–85) gives an account of what he calls *conjunctive remainder*, where we subtract a proposition and get a weaker one. He also has a theory of *disjunctive remainder*, where we subtract a proposition and get a stronger one Fine (2017b, 695–96). I have omitted discussion of disjunctive remainder as it seems that most people have conjunctive remainder in mind. I will mention his early account of conjunctive remainder in a later section.

remainder that is inspired by the intuitionistic conditional. The intuitionist conditional $B \rightarrow_I A$ is a function that tells us how to construct A from B. In Fine's truthmaker semantics, formulas are verified and falsified by states, so we have to make sense of a state that encodes a function from verifiers of B into verifiers of A. The meaning of logical remainder is expressed informally as follows.[12]

FINEAN REMAINDER

- s verifies $A - B$ iff s encodes a function that takes the verifiers of B into verifiers of A.
- s falsifies $A - B$ iff s encodes a function that takes verifiers of B into falsifiers of A.

So *Sparky is a brown dog* - *Sparky is brown* will be verified by a state that encodes a function that take verifiers of *Sparky is brown* into verifiers of *Sparky is a brown dog*. Intuitively, the same state will also verify *Sparky is a dog*; after all, that is the state that gets us from being brown to being a brown dog.

One immediate problem with this account is that it allows for $A - B$ to be both true and false (Fine 2020, 160). I am sympathetic to truth-value gluts, but I see no reason why we should build their possibility into the theory of logical remainder. Fine (2020) (Ibid.) notes that there is a way to revise the account so that you get truth-value gaps instead of gluts, but this leads us back to the same indeterminacy problem found with Yablo's account.

[12] See Fine (2014) for the truthmaker semantics for intuitionistic logic.

Another problem is that Finean Remainder fails the Relevance condition; this is because the intuitionistic conditional is not a relevance conditional.

A pattern is emerging. Second generation theories of subtraction are plagued by indeterminacy. Indeterminacy is not intrinsically undesirable, even in logic. Strong Kleene logic (or K_3) has no logical truths due to the presence of a third truth-value (representing indeterminacy) and how its connectives are defined. But the lack of logical truths in K_3 is excusable because it is, in some sense, intended to be the logic of indeterminacy. In contrast, the indeterminacy of logical remainder appears, even when we weren't looking for it.

Contextual Equivalence

Here is my diagnosis of the problem. First generation theories of logical subtraction mainly sought to treat logical subtraction as a kind of logical operation. We have Jaeger (1973) to thank for the initial discussion, with Hudson (1975) defending the horseshoe analysis; Humberstone (2011) then gives the most extensive logical treatment of subtraction to date. In contrast, second generation theories mainly focus on linguistic and metaphysical applications of logical subtraction. Yablo (2014) is interested in a battery of metaphysical and linguistic cases of subtraction, while Hoek (2018) treats subtraction more-or-less a linguistic phenomenon. The work of the second generation has been fruitful, but the idea that logical subtraction is a logical operator remains a hangover—a remainder, if you will—from first generation theories. I propose that we think of logical subtraction, not as a logical operator, but as a kind of contextual equivalence.

What made us think logical subtraction was a logical operation, in the first place? The analogy to arithmetic is helpful. We know that $(a - b) + b = a$. We start with the informal principle.

EQUIVALENCE

$A - B$ combined with B is equivalent to A.

We then consider ways to represent equivalence logically. There are two such ways. For the first way, we simply strengthen RETURN by adding its converse. The result is the principle:

RETURN+

It is always true that $[(A - B) \wedge B] \leftrightarrow A$.

For the second way, we do not take the equivalence to hold in the object language (using $\leftrightarrow$); rather, we take the equivalence to be in the metalanguage, using some notion of logical entailment.

RETURN++

$[(A - B) \wedge B]$ entails A, and A entails $[(A - B) \wedge B]$.

However you represent these principles in logic, you then have to give, well, the logic that underwrites them. Given the textbook conception of a mathematical logic, a logic comes equipped with a well-defined syntax, recursive semantic theory, and proof theory.

My suggestion is that the basic idea of logical subtraction is EQUIVALENCE, but we need not cash out equivalence in terms of logical principles like the series of RETURN principles. It is a natural step to make, of course. The most obvious way to make EQUIVALENCE rigorous is to interpret it using logical materials. But as we have seen, there are significant challenges in trying to pin down logical subtraction as a logical operator.

If we do not understand EQUIVALENCE in terms of some kind of logical equivalence, how do we understand it? In intuitive cases of logical subtraction, we do not care about equivalence *simpliciter*. Rather, we are interested in equivalence in a context, or equivalence *modulo* a subject matter. Consider nominalism. Yablo points out the fact that the proposition *the number of planets is greater than zero - abstract number exist* is just as good as the proposition *the number of planets is greater than zero* in many contexts. What contexts? The contexts in which what we care about are the concrete facts. Yablo (2014, 32) discusses the idea that a proposition may be "just as good as true" given a subject matter.

Alternatively, consider an example of loose talk presented by Hoek. To describe Ellen's outfit, we say *Ellen wore the same type of hat as Sherlock Holmes* (A), which implies *Sherlock Holmes exists* (B). However, Sherlock Holmes does not exist. What we really mean is *Ellen wore a deerstalker cap* (A - B). A - B is just as good as A, in this case, because we care about what Ellen is wearing, not whether fictional characters exist. Hoek's theory explicitly builds in the context-sensitivity of subtraction.

We can go back even further. Lewis (1988) alerted us to a perfectly respectable, post-positivist, notion of two statements being observationally equivalent. The empiricist wants to say that two statements are true about what is observed, even if they are not true simpliciter. Lewis develops a notion of subject matter that allows him to articulate what it means for two statements to be equivalent modulo a subject matter.

Among the fans and fellow travelers of logical subtraction, there is agreement that it makes sense to think of two propositions as equivalent relative to a subject matter. My distinct present suggestion is to take this idea as primary to understanding logical subtraction. Instead of starting with the idea that $A - B$ combined with B is equivalent to A, we can take this kind of equivalence to be a special case of a more general form of equivalence: contextual equivalence.

CONTEXTUAL EQUIVALENCE

$A - B$ is equivalent to A in a context c.

In paradigm cases, equivalence depends on adding B to $A - B$, but not every case is a paradigm case. Most notably, you may want $A - B$ to be defined even when A does not entail B. I will argue that there are intuitive cases in which you are not simply adding B.

To make the idea of contextual equivalence rigorous, we employ other notions in the vicinity of logical subtraction: namely, aboutness and subject matter. There is an intuitive sense in which *the number of planets is greater than zero - abstract number exist* and *the number of planets is greater than zero* say the same things about the concrete world. The concrete world is the subject matter of our discussion. It is important to hold a subject matter fixed, because without a subject matter, we could not say A and $A - B$ are equivalent. They are obviously not equivalent in content, and they would not be considered equivalent if we were discussing the existence of abstract numbers.

The ideas of aboutness and subject matter have been given various treatments, but one of the most natural treatments can be found in the framework of truthmaker semantics. I appeal to the conception of truthmakers as worldly states (Fine 2017a, 2017b; Jago 2017, 2020). On this view, truthmakers are worldly entities—fact, object, property, etc. I will refer to truthmakers, in general, as states, and I will refer to particular states using bold typeface. So the state of affairs **Sparky is barking** verifies the proposition. *the dog is barking*. We can say something similar about falsemaking. **Nothing is in the room** falsifies *Jack is in the room*. The *truthmaker content* of a proposition, then, will be an ordered pair of sets of possible states (not possible worlds). The sets will represent the truthmakers and falsemakers for propositions, respectively.

States stand in mereological relations to one another. The state **the house is on fire** is part of **the house is completely aflame**, which is itself part of **a house in the neighborhood is on fire**. Let $s \sqsubseteq u$ mean: s is a part of u. As standard, we take this relation to be reflexive, antisymmetric, and transitive. We also assume that any two arbitrary states can be combined. For any two states s_1, s_2, there is a fusion of such states $s_1 \sqcup s_2$ that has s_1 and s_2 as parts (as well as all parts of s_1 and s_2). So we can fuse **Sparky is barking** with **Chester is meowing** to get **Sparky is barking and Chester is meowing**. It will also be useful to define the greatest common part of two states. So **Obama is a Christian man** and **Obama is a black man** has some state in common, namely **Obama is a man**. We represent the greatest common part of s_1 and s_2 as $s_1 \sqcap s_2$.

The subject matter of a proposition P, given truthmaker semantics, is naturally understood as an ordered pair of (a) the fusion of the possible truthmakers of P and (b) the fusion of the possible falsemakers of P (Fine 2020, 132). You can call components of these ordered pairs the *positive* and *negative* subject matters of the proposition, respectively. Call such propositions, with both positive and negative components, bilateral propositions. I will let σ be a function that takes us from propositions (bilateral or otherwise) to their subject matters. While we can define the subject matter of a proposition, it will also be useful to speak of subject matters more generally. In general, I will speak of a subject matter s as some state.

The idea that propositions P and Q to say the same things about a subject matter s can be analyzed as: (*positive*) the part of P that is true about s is identical to the part of Q that is true about s, and (*negative*) the part of P that is false about s is identical to the part of Q that is false about s. To analyze this, we need a conception of restricted truth, which I will borrow from Fine (2020, 155).

For the sake of simplicity, I start solely with the positive part of a proposition. Given a verifier p of P and subject matter s, we call $s \sqcap p$ the restriction of p to s, or p^s. Think of p^s as a verifier of P when stripped of anything not pertaining to s. We then say that P is true about s at a world w just in case there is some p^s that is part of w. For example, *the number of planets is greater than zero* could be false in a nominalist world w, but it could nonetheless be true about **concreta** in w. Why? Because restricting a possible verifier of P to what it says about **concreta** will give us a state that is part of w. We can define the restriction of (the positive component of) a proposition P to a subject matter s, P^s, as follows: P^s is verified by every p^s such that p verifies P. To this Finean machinery, I add a few simple definitions. We can take the part of P that is true about s to be $\sigma(P^s)$, which (under natural assumptions) is itself identical to $s \sqcap \sigma(P)$. We say P is (positively) equivalent to Q modulo a subject matter s, or $P \equiv_s^+ Q$, just in case $\sigma(P^s) = \sigma(Q^s)$. P and Q may not have the same content, but they say the same true things about the world so far as s is concerned.

To make this account fully general, we need to specify the negative part of the analysis. There are two ways to do this. The first approach is to independently define a negative restricted proposition. Following Fine, we can define a negative restriction P_s that is verified by every falsifier of P that is part of s. The restriction of a bilateral proposition $P = \langle P^s, P_s \rangle$. So $P = \langle P^s, P_s \rangle$ is equivalent to $Q = \langle Q^s, Q_s \rangle$ modulo s, or $P \equiv_s Q$, just in case $\sigma(P^s) = \sigma(Q^s)$ and $\sigma(P_s) = \sigma(Q_s)$. Call this the bilateral definition of equivalence. An alternative approach is to simply say that $P \equiv_s Q$ is true just in case $P \equiv_s^+ Q$ and it is false otherwise. Call this the unilateral definition of equivalence. The second approach is simpler but less fine-grained. It cannot distinguish between cases in which P and Q say the same positive things about a subject matter but say different negative things about it. Suppose P and Q have the same truthmakers but different falsemakers. Then they may be positively contextually equivalent given a subject matter but not negatively contextually equivalent.

While the bilateral definition is the most fine-grained, I will be working with the unilateral definition in what follows. In most cases of logical subtraction, we care principally about whether two propositions are true about the same things. We say that two claims are just as good as true. We do not often say that two claims are just as good as false, where we are explicitly in interested in fine-grained ways of making false.

Now let us connect the idea of logical subtraction to contextual equivalence. Intuitively, a context gives us a subject matter. It tells us that some things are important and on topic. It tells us there is a specific question that is under discussion. And so on. This is plausible linguistically, so I will take it aboard as an assumption. With this assumption in mind, we can see how logical remainder satisfies our desire for contextual equivalence. We are looking for $(A - B) \equiv_s A$, for a subject matter s given by the context. If we want to know whether $(A - B) \equiv_s A$ is true at a possible world w, we simply check to see whether $(A - B) \equiv_s A$ is true and, if so, whether the part of A true about s is part of w.

Now it becomes necessary to distinguish between logical subtraction and logical remainder. I take logical subtraction to describe the general contextual equivalence between A - B and A, and I take logical remainder to identify $A - B$. The outstanding issue, of course, is how to define $A - B$.

Logical Remainder

Since the beginning of the discussion of logical subtraction, there has been an attempt to fit subtraction into the mold of a converse conditional. There is something right about this strategy. Yablo (2014, 178) notes that we can think of $A - B$ as a hidden premise that takes us from B to A. The most obvious choice for $A - B$ would be $B \supset A$, but other conditionals might do. I propose we think of $A - B$ as a kind of relevant conditional.

Jago (2020) produces the following truthmaking clause for the relevant conditional: s verifies $A \rightarrow B$ iff for every verifier b of B, $s \sqcup b$ verifies A. If we identify the converse of this conditional with remainder, we have a truthmaking clause for remainder.

POSITIVE REMAINDER

s verifies $A - B$ iff for every verifier b of B, $s \sqcup b$ verifies A.

I will talk about the falsemaking clause for remainder later. For now, let me explain the choice of this positive clause. The intuitive idea of the logical remainder $A - B$ is that it is what we add to B to get A. I have simply translated this idea into truthmaker form. A verifier of $A - B$ is a state that, when added to (fused with) a verifier of B, gives us a verifier of A. Here is the definition at work in the simplest case, conjunction. Consider *Sparky is a brown dog - Sparky is brown*. The proposition *Sparky is brown* is verified by a state of Sparky being brown, **brown**. *Sparky is brown* is verified by a state of Sparky being a dog, **dog**. What can we add to **brown** to get a verifier of *Sparky is a brown dog*. Easy answer: **dog**. And as it turns out **dog** verifies *Sparky is a dog*, so we get the intuitive conclusion that we would expect.

On to more complex cases. Consider the nominalism case. *The number of planets is greater than zero* is intuitively verified by a fusion of concrete and abstract facts—**concrete** ⊔ **abstract**, let's say. If *abstract numbers exist* is verified by **abstract**, then *the number of planets is greater than zero - abstract numbers exist* is verified by **concrete**. And **concrete** intuitively verifies the proposition *the world is just like it would be if there were more than zero planets except there are no abstract numbers*. To be clear: this analysis requires us to assume (a) that propositions about the number of planets are exactly verified by concrete and abstract states, and (b) that concrete and abstract states can be fused together. But these assumptions are no more controversial than the thesis of nominalism itself. A similar moral goes for the original Wittgenstein case. What is *I raise my arm - my arm goes up*? It is what we add to a verifier of *my arm goes up* to get a verifier of *I raise my arm*. The intuition is that the relevant addition is a state *willing*, which combines with **arm going up** to yield **raise my arm**, which verifies *I raise my arm*. And **willing** intuitively verifies *I will my arm to go up*. You may dispute the existence of the remainder by arguing that *I raise my arm* does not have **willing** ⊔ **my arm goes up** as part of its truthmaker content. Alternatively, you might argue that the remainder exists but insist that the state **willing** is less metaphysically fundamental than surrounding states.

We have (at least part of) a semantic clause of for remainder. It is tempting to go on to define logical subtraction using remainder alone. However, this definition of remainder will not validate many of the intuitive principles of remainder. For instance, recall RETURN+: $[(A - B) \wedge B] \leftrightarrow A$. Suppose we take conjunction to be defined as follows: s verifies $A \wedge B$ just in case s is a fusion of verifiers of A and B. Then $(A - B) \wedge B$ will imply A. One way to see this is to note that, if $A - B$ is a converse conditional, then the relevant inference is modus ponens. However, A will not generally imply $(A - B) \wedge B$. As a logical matter, we cannot conclude that A will imply B without more information about what A contains.

To capture the idea of logical subtraction, we need more than remainder; we additionally need a notion of subject matter and contextual equivalence. In each case, $A - B$ is equivalent to A modulo a subject matter. Start with nominalism. *The number of planets is greater than zero - abstract numbers exist* is equivalent to *the number of planets is greater than zero* modulo the concrete facts, **concreta**. What is the part of *the number of planets is greater than zero* that is true about **concreta**? To identify this state, just take the verifiers of *the number of planets is greater than zero*, strip them of their abstract parts, and fuse the remaining concrete parts together. The only thing left will be concrete facts about planets. We do the same procedure for the remainder. The part of the remainder that is true about **concreta** will consist of the concrete facts that we need to add to the mathematical facts to give us facts about the number of planets. These parts will be identical.

Now consider whether $A - B$ and A are equivalent modulo the abstract facts, **abstracta**? They are not. The part of A that is true about **abstract** will not overlap with the part of $A - B$ that is true about **abstracta**; there is no part of $A - B$ that is true about **abstracta**! So the account gives the right verdict about non-equivalence, as well.

Here are the structural features of the case that lead, like magic, to the consequence that the part about A is identical to the part about A - B. First, we are assuming that A contains A - B, in the following sense: every verifier of A - B is part of a verifier of A and every verifier of A has a verifier of $A - B$ as a part. Second, we choose our subject matter **s** in a way that ensures that the subject matter of B does not overlap with it. With these two constraints in place, we find that A and A - B say the same (positive) thing — in a sense — when restricted to **s**.

What do we say, then, about cases in which A does not contain or entail B? One possibility is that we simply let logical subtraction be undefined, no matter what the context. After all, one might think: logical subtraction presupposes there is something to be subtracted; we regard cases in which there is nothing to subtracted as akin to dividing by zero.

Though there is more to said for this line of thought, Yablo gives reasons to think that there is a way to understand cases of $A - B$ in which A does not entail B. His idea is to think of logical remainder as a missing premise, an interpolant, between B and A. Consider the planets case.

$$\frac{\begin{array}{c} B\text{: } Abstract\ numbers\ exist \\ R \end{array}}{A\text{: } The\ number\ of\ planets\ is\ greater\ than\ zero}$$

The remainder R is what allows us to complete the gap from B to A. Yablo suggests we can apply this same model to cases in which A does not entail B.

Here is the extended example he discusses (Yablo 2014, 181).

$$\frac{\begin{array}{c} B\text{: } All\ and\ only\ firefighters\ are\ goalkeepers \\ R \end{array}}{A\text{: } No\ firefighters\ are\ horticulturalists}$$

In this case, A clearly does not entail B, but there are some things we can substitute for R to bridge the gap between the two propositions. For example, R might be: *no goalkeepers are horticulturalists*. This makes the inference valid. Yablo then considers constraints on R that will ensure that we get the right kind of interpolant. The two most obvious conditions are sufficiency and necessity: sufficiency tells us that R, when combined with A, should suffice for B; necessity tell us that B implies that A is false when R is false.

I think Yablo is right to make a connection between interpolation and remainder, though I believe there is an important unacknowledged difference between cases like the firefighter case and the planets case. To see this difference, let us try to fill in for the value of R in the firefighters case. My suggestion is simple: we let R be $A - B$. Roughly speaking, we ask: what should be added to *no firefighters are horticulturalists* to get *all and only firefighters are goalkeepers*? Whatever verifies *no goalkeepers are horticulturalists*. We add facts about goalkeepers not being horticulturalists to the facts about firefighters not being horticulturalists. The exact details will require us to make sense of quantificational truthmaking, which is complicated, but the general idea is intuitive enough.

Even if we set aside quantification, we still need a full account of $A - B$ in the propositional setting. So far, I have only presented the truthmaking clause for $A - B$ and a possible truthmaking clause for conjunction. We need a falsemaking clause for $A - B$ as well as truthmaking and falsemaking clauses for the standard stock of logical connectives. Luckily, Jago (2020) has done this work for us, proving that A - B, when understood as the relevant conditional $B \rightarrow A$ (in the logic **R**), has a truthmaker semantics. Jago's full account is too complex to dive into, here.[13] But I should emphasize that $A - B$ will logically behave like a relevant conditional. So it follows that $A - B$, when conjoined with B, implies A. In doing so, my account satisfies Yablo's sufficiency constraint. It also satisfies the necessity

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[13] Jago requires three additional constructions that I cannot fully explain here. First, you need an ordering $\preccurlyeq$ on states that expresses the idea that one state refines or determines another. Second, you need a set of states that are (treated as) absolutely determinate called the *prime states*; we write $u \preccurlyeq_p s$ when s is prime. Third, you need a function m that pairs every state with one that is maximally compatible with it. Jago uses this to deliver the following clauses.

- *Atomic*: s verifies (falsifies) A iff s is a truthmaker (falsemaker) for A.
- *Negation*: s verifies (falsifies) $\neg A$ iff s is a falsemaker (truthmaker) for A.
- *Conjunction*: s verifies $A \wedge B$ iff $s = a \sqcup b$, where a verifies A and b verifies B; s falsifies $A \wedge B$ iff s falsifies A or s falsifies B.
- *Disjunction.* s verifies $A \vee B$ iff for every u such that $u \preccurlyeq s$, either u verifies P or u verifies Q; s falsifies $P \vee Q$ iff s verifies P and s verifies Q
- *Implication (Falsity).* **s** falsifies $A \rightarrow B$ iff for every t such that $t \preccurlyeq_p s$, there exists a verifier a of A that, when fused with $m(s)$, does not verify B.

constraint. In both cases, this is because I take $A - B$ to be the converse relevant conditional $B \rightarrow A$.

In fact, $A - B$ will always be defined, regardless of whether A contains or entails B; this simply follows from the fact that the relevant conditional will always be defined in **R**. Given the fact that $A - B$ will always be defined, we can assess whether $A - B$ satisfied the conditions on remainder described earlier. When we substitute the relevant conditional for the horseshoe conditional, we will find that RETURN, RIGHT SUBTRAHEND, LEFT SUBTRAHEND, and RELEVANCE all hold. PARTS, and RELEVANCE do not hold because they are paradoxes of material implication that relevant conditionals are designed to avoid. Finally, RETURN+ and RETURN++ do not hold because, intuitively, *no* conditional should make these principles hold.

Remainder, I believe, can be given a fully logical treatment. Theorists are subtraction are right that there is something that can be given a proper logic — syntax, semantics, and proof theory — in the vicinity of logical subtraction. However, logical remainder is not the same as logical subtraction, and the fact that remainder is always defined does not mean that you will always have an interesting case of logical subtraction. Consider the fact that Yablo's chief example is: *no firefighters are horticulturalists - all and only firefighters are goalkeepers*. While we can make sense of the content of this proposition, there is no clear subject matter that we want to use to define a contextual equivalence.

No firefighters are horticulturalists - all and only firefighters are goalkeepers (A - B) is equivalent to *No firefighters are horticulturalists* (A) modulo X. We have to solve for X. We cannot say that X consists of facts about horticulturalists, because A and A - B do not even appear to say the same things about horticulturalists; A - B presumably has to say something about goalkeepers not being horticulturalists while A does not. The subject matter cannot be about firefighters or goalkeepers for the same reason. We might say that A and A - B are equivalent modulo a subject matter that is independent of them both, like **cupcakes**; alternatively, we could say that A and A - B are equivalent modulo the empty subject matter—the subject matter that is part of every state. However, these are both uninteresting cases of contextual equivalence.

Put it another way: A - B makes logical sense in every case, but it may not always be what we have in mind when we are interested in logical subtraction. We are interested in logical subtraction because we care about contextual equivalences. This contextual equivalence presupposes some kind of logical operation, but the operation should not be outright identified with logical subtraction itself.

If we think of things this way, we come to resolve, in Hegelian fashion, the struggle between different theorists of logical subtraction. First generation theorists of subtraction mainly focused on defining remainder (e.g., Jaeger (1973; Hudson 1975; Humberstone 2011)), while second generation theorists either (a) focused on defining contextual equivalence (e.g., Hoek (2018)) or (b) focused on defining remainder and contextual equivalence as the same thing (e.g., Yablo (2014)).[14] The current account takes remainder and contextual equivalence (or subtraction) to be related but distinct. In this way, we

[14] Fine (2017a, 2020) is not easily classified by my generational breakdown. He is first generation by theme, though second generation temporally.

resolve the tensions between the logic of logical subtraction and the practical deployment of the concept.

Conclusion

I have given a theory of logical subtraction that distinguish between logical subtraction and logical remainder. Logical subtraction is contextual equivalence; it is to say that two propositions are equivalent A and $A - B$ are equivalent modulo a subject matter. Logical remainder is the converse of the relevant conditional as interpreted using truthmaker semantics; as such, logical remainder is defined in every case. Think of it this way: logical remainder is the logical core of logical subtraction, while contextual equivalence is the general notion used to make sense of particular applications of remainder—metaphysical, linguistic, philosophical. By enacting a new division of labor in the theory of subtraction, we synthesize the insights of the friends of subtraction and open new avenues for further research.

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