Derivation Rules of Natural Deduction (System SND)

Conjunction Introduction (& I)

$$\begin{array}{l|ll} m & \Phi & \\ n & \Psi & \\ & \Phi \mathbin{\&} \Psi & :m,\ n\ \ \&\,\mathrm{I} \\ & \Psi \mathbin{\&} \Phi & :m,\ n\ \ \&\,\mathrm{I} \end{array}$$

Conditional Introduction (⊃I)

$$\begin{array}{l|ll} m & \quad\begin{array}{|l} \Phi \\ \hline \end{array} & :\text{AS for } \supset\mathrm{I} \\ n & \quad\begin{array}{|l} \Psi \end{array} & \\ & \Phi \supset \Psi & :m\text{–}n\ \supset\mathrm{I} \end{array}$$

Negation Introduction (∼I)

$$\begin{array}{l|ll} m & \quad\begin{array}{|l} \Phi \\ \hline \end{array} & :\text{AS for } {\sim}\mathrm{I} \\ n & \quad\begin{array}{|l} \Psi \end{array} & \\ o & \quad\begin{array}{|l} {\sim}\Psi \end{array} & \\ & {\sim}\Phi & :m\text{–}o\ {\sim}\mathrm{I} \end{array}$$

Disjunction Introduction (∨I)

$$\begin{array}{l|ll} m & \Phi & \\ & \Phi \vee \Psi & :m\ \vee\mathrm{I} \\ & \Psi \vee \Phi & :m\ \vee\mathrm{I} \end{array}$$

Biconditional Introduction (≡I)

$$\begin{array}{l|ll} i & \quad\begin{array}{|l} \Phi \\ \hline \end{array} & :\text{AS for } \equiv\mathrm{I} \\ j & \quad\begin{array}{|l} \Psi \end{array} & \\ & -- & \\ k & \quad\begin{array}{|l} \Psi \\ \hline \end{array} & :\text{AS for } \equiv\mathrm{I} \\ l & \quad\begin{array}{|l} \Phi \end{array} & \\ & \Phi \equiv \Psi & :i\text{–}j,\ k\text{–}l\ \equiv\mathrm{I} \end{array}$$

Conjunction Elimination (& E)

$$\begin{array}{l|ll} m & \Phi \mathbin{\&} \Psi & \\ & \Phi & :m\ \ \&\,\mathrm{E} \\ & \Psi & :m\ \ \&\,\mathrm{E} \end{array}$$

Conditional Elimination (⊃E)

$$\begin{array}{l|ll} m & \Phi \supset \Psi & \\ n & \Phi & \\ & \Psi & :m,\ n\ \supset\mathrm{E} \end{array}$$

Negation Elimination (∼E)

$$\begin{array}{l|ll} m & \quad\begin{array}{|l} {\sim}\Phi \\ \hline \end{array} & :\text{AS for } {\sim}\mathrm{E} \\ n & \quad\begin{array}{|l} \Psi \end{array} & \\ & \quad\begin{array}{|l} \vdots \end{array} & \\ o & \quad\begin{array}{|l} {\sim}\Psi \end{array} & \\ & \Phi & :m\text{–}o\ {\sim}\mathrm{E} \end{array}$$

Disjunction Elimination (∨E)

$$\begin{array}{l|ll} m & \Phi \vee \Psi & \\ i & \quad\begin{array}{|l} \Phi \\ \hline \end{array} & :\text{AS for } \vee\mathrm{E} \\ j & \quad\begin{array}{|l} \Omega \end{array} & \\ & -- & \\ k & \quad\begin{array}{|l} \Psi \\ \hline \end{array} & :\text{AS for } \vee\mathrm{E} \\ l & \quad\begin{array}{|l} \Omega \end{array} & \\ & \Omega & :m,\ i\text{–}j,\ k\text{–}l\ \vee\mathrm{E} \end{array}$$

Biconditional Elimination (≡E)

$$\begin{array}{l|ll} m & \Phi \equiv \Psi & \\ n & \Psi & \\ & \Phi & :m,\ n\ \equiv\mathrm{E} \end{array}$$

Derivation Rules of Natural Deduction System QND

All the rules of SND, plus the following rules for the quantifiers! The rules of SND govern inferences where the main logical operator is one of the connectives from *SL*. Reiteration is also allowed.

Recall: $\Phi[c/\chi]$ is the sentence you get from $(\forall\chi)\,\Phi$ by dropping the $(\forall\chi)$ quantifier and putting c in place of every χ in Φ. The other variables are untouched! Equivalent notation: $\Phi\boxed{\chi \Rightarrow c}$. But we'll write things out long-hand in the schemata below!

Universal Elimination (∀E)

$$
\begin{array}{l|ll}
m & (\forall\chi)\,\Phi(\ldots\chi\ldots\chi\ldots) & \\
\vdots & \quad\vdots & \\
s & \Phi(\ldots c\ldots c\ldots) & :m\ \forall\mathrm{E}
\end{array}
$$

Existential Introduction (∃I)

$$
\begin{array}{l|ll}
m & \Phi(\ldots c\ldots c\ldots) & \\
\vdots & \quad\vdots & \\
s & (\exists\chi)\,\Phi(\ldots\chi\ldots\chi\ldots) & :m\ \exists\mathrm{I}
\end{array}
$$

Provided that χ does not occur already in $\Phi(\ldots c\ldots c\ldots)$. Note that χ may replace some or all occurrences of c.

Universal Introduction (∀I)

$$
\begin{array}{l|ll}
m & \Phi(\ldots c\ldots c\ldots) & \\
\vdots & \quad\vdots & \\
s & (\forall\chi)\,\Phi(\ldots\chi\ldots\chi\ldots) & :m\ \forall\mathrm{I}
\end{array}
$$

Provided that both
(i) c does not occur in any undischarged assumptions that Φ is in the scope of.
(ii) χ does not occur already in $\Phi(\ldots c\ldots c\ldots)$.

Existential Elimination (∃E)

$$
\begin{array}{l|lll}
m & (\exists\chi)\,\Phi & & \\
 & \quad\vdots & & \\
n & & \begin{array}{|l}\Phi(\ldots c\ldots c\ldots)\\ \hline \quad\vdots \end{array} & :\mathrm{AS\ for}\ \exists\mathrm{E} \\
r & & \begin{array}{|l}\Psi\end{array} & \\
 & \quad\vdots & & \\
s & \Psi & & :m,\ n\text{–}r\ \exists\mathrm{E}
\end{array}
$$

Provided that
(i) c does not occur in any undischarged assumptions that Φ is in the scope of
(ii) c does not occur already in $(\exists\chi)\,\Phi$
(iii) c does not occur in Ψ

Simplified restriction (easier to remember): **provided c doesn't occur anywhere else outside the subproof**. (Moral: always use a distinct constant for existential elimination, and you'll satisfy the three restrictions above automatically!)