Natural Deduction in $\text{QL}^=$

LOGIC I
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Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Quantifier Rules

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall\alpha\varphi$ where β is a constant, α is a variable, and β does not occur in $\forall\alpha\varphi$ or in any undischarged assumption.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Identity Rules

(=I) $\vdash \alpha = \alpha$ for any constant α.

Axiom: This rule is better referred to as an axiom schema.

Note: Easy to use, but not always obvious when to use.

Task 1: Derive the following in QD:

- $\forall x(x = x \supset \exists yFyx) \vdash \exists y(Fyy)$.
- Everything is something.
- Something exists.

(=E) $\varphi[\alpha/\gamma], \alpha = \beta \vdash \varphi[\beta/\gamma]$.

Note: Also easy to use, but not always obvious how to use.

Task 2: Derive the following in QD:

- $m = n \vee n = o,\ An \ \vdash Am \vee Ao$
- Every symmetric antisymmetric relation is lonely.
- Every irreflexive antisymmetric relation is asymmetric.

Relations

Task 4: Regiment and derive the following in QD.

1. Every transitive symmetric relation is left and right euclidean.
2. Every nonempty transitive and symmetric relation is reflexive.
3. Only the empty relation is symmetric and asymmetric.
4. Every intransitive relation is irreflexive.
5. Every intransitive relation is asymmetric.

Further Examples

Task 3: Regiment and derive the following in QD.

1. $\forall x(x = m), Rma \vdash \exists xRxx$
2. $\forall x(x{=}n \equiv Mx), \forall x(Ox \lor \neg Mx) \vdash On$
3. $\exists x(Kx \land \forall y(Ky \rightarrow x{=}y) \land Bx), Kd \vdash Bd$
4. $\vdash Pa \supset \forall x(Px \lor x{\neq}a)$