10. Proofs in QL

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a. Some Equivalences in QL

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  - Basically: show that the biconditional of the two sentences is a tautology

Some Equivalences: Quantifiers commuting over connectives

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   - And there are no analogous rules for biconditionals

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Informal Argument: If anyone G's, then c F's

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  Our proof system will show this *naturally*!

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  $(\exists x)(\sim Gx \vee Fc)$, and apply '$\supset$' truth conditions again:

- Yields $(\exists x)(Gx \supset Fc)$

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b. Rules for ∀

Rules for formal proofs

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  - yield only valid arguments

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- This is a good rule: $(\forall x)\,\mathcal{A}x \vDash \mathcal{A}c$.

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- Diagnosis: the c in $\mathcal{A}c$ is a name for a **specific object**.
- We need a name for an **arbitrary, unspecified object**.
- If $\mathcal{A}c$ is true for whatever c **could** name, then $\mathcal{A}x$ is satisfied by **every** object.

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  Only people who wear capes admire Greta.

  ∴ All heroes wear capes.

  Proof: Let Carl be any hero.
  Since all heroes admire Greta, Carl admires Greta.
  Since only people who wear capes admire Greta, Carl wears a cape. But “Carl” stands for **any** hero.
  So all heroes wear capes.

Universal generalization

$$\begin{array}{r|ll} k & \mathcal{A}c & \\ & & \\ & (\forall x)\,\mathcal{A}x & :k\ \forall\mathrm{I} \end{array}$$

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- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathcal{A}x$ is obtained from $\mathcal{A}c$ by replacing **all** occurrences of c by x.
- In other words, c must also not occur in $\forall x\,\mathcal{A}x$.

General conditional proof

Proving “All *A*s are *B*s”

k	$\quad$ Ac	:AS for $\supset$I
l	$\quad$ Bc	
$l+1$	$Ac \supset Bc$	:k-l $\supset$I
$l+2$	$(\forall x)(Ax \supset Bx)$	:$l+1$ $\forall$I

Example

All heroes admire Greta.

Only people who wear capes admire Greta.

∴ All heroes wear capes.

$(\forall x)(Hx \supset Axg)$

$(\forall x)(Axg \supset Cx)$

$\therefore (\forall x)(Hx \supset Cx)$

Let's do it on Carnap (PP10.2)!

Example

1	$(\forall x)(Hx \supset Axg)$	:PR
2	$(\forall x)(Axg \supset Cx)$	:PR
3	$\quad Hc$	:AS for $\supset$I
4	$\quad Hc \supset Acg$	:1 $\forall$E
5	$\quad Acg$	:4, 3 $\supset$E
6	$\quad Acg \supset Cc$	:2 $\forall$E
7	$\quad Cc$	:6, 5 $\supset$E
8	$Hc \supset Cc$	:3-7 $\supset$I
9	$(\forall x)(Hx \supset Cx)$	:8 $\forall$I

Example

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2	$\quad (\forall x)\, Ax$	:AS for $\vee$E
3	$\quad Ac$	:2 $\forall$E
4	$\quad Ac \vee Bc$	:3 $\vee$I
5	$\quad (\forall x)\, Bx$	:AS for $\vee$E
6	$\quad Bc$	:5 $\forall$E
7	$\quad Ac \vee Bc$	:6 $\vee$I
8	$Ac \vee Bc$	:1, 2–4, 5–7 $\vee$E
9	$(\forall x)(Ax \vee Bx)$	:8 $\forall$I

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c. Rules for ∃

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Arbitrary objects again

- Problem: corresponding "elimination rule" isn't valid:

$$\begin{array}{r|ll} k & (\exists x)\,\mathcal{A}x & \\ & \mathcal{A}c & :k\ \textbf{doesn't follow from} \end{array}$$

- If we know that $(\exists x)\,\mathcal{A}x$ is true, we know that **at least one** object satisfies $\mathcal{A}x$, but not which one(s).
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathcal{A}(x)$.

Reasoning from existential information

- To use $(\exists x)\,\mathcal{A}x$, pretend the x has a name c, and reason from $\mathcal{A}(c)$.
- This is what we do to reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
  $\therefore$ Some heroes admire Greta.

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  Proof: We know there are heroes who wear capes.
  Let Cate be an arbitrary one of them.
  So Cate wears a cape. Since anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

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m	$\quad \mathcal{A}c$	:AS for $\exists$E
n	$\quad \mathcal{B}$	
	$\mathcal{B}$	:k, m–n $\exists$E

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	$\mathcal{B}$		:k, m-n $\exists$E

- c is special: c **must NOT appear outside subproof**

Example

There are heroes who wear capes.

Anyone who wears a cape admires Greta.

$\therefore$ Some heroes admire Greta.

$(\exists x)(Hx \,\&\, Cx)$

$(\forall x)(Cx \supset Axg)$

$\therefore$ $(\exists x)(Hx \,\&\, Axg)$

Example (PP10.5)

1	$(\exists x)(Hx \,\&\, Cx)$	:PR
2	$(\forall x)(Cx \supset Axg$	:PR
3	$Hc \,\&\, Cc$	:AS for $\exists$E
4	Cc	:3 & E
5	$Cc \supset Acg$	:2 $\forall$E
6	Acg	:4, 5 $\supset$E
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8	$Hc \,\&\, Acg$	:4, 7 & I
9	$(\exists x)(Hx \,\&\, Axg)$	:8 $\exists$I
10	$(\exists x)(Hx \,\&\, Axg)$	:1, 3-9 $\exists$I

10. Proofs in QL

d. The Rules, collected!

Rules for the Universal Quantifier

Universal Elimination (∀E)

$$
\begin{array}{c|ll}
m & (\forall \chi)\Phi(\ldots \chi \ldots \chi \ldots) & \\
\vdots & \quad \vdots & \\
s & \Phi(\ldots c \ldots c \ldots) & : m\ \forall\mathrm{E}
\end{array}
$$

- Note that you replace EVERY instance of χ with c

- Notation: $\Phi[c/\chi]$

- read "c for χ"

Universal Introduction (∀I)

$$
\begin{array}{c|ll}
m & \Phi(\ldots c \ldots c \ldots) & \\
\vdots & \quad \vdots & \\
s & (\forall \chi)\Phi(\ldots \chi \ldots \chi \ldots) & : m\ \forall\mathrm{I}
\end{array}
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Provided that both
(i) c does not occur in any undischarged assumptions that Φ is in the scope of.
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Rules for the Existential Quantifier

Existential Introduction (∃I)

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\vdots & \quad\vdots & \\
s & (\exists \chi)\Phi(\ldots \chi \ldots \chi \ldots) & :m\ \exists\text{I}
\end{array}
$$

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.

- Note that χ may replace some or all occurrences of c.

Existential Elimination (∃E)

$$
\begin{array}{l|lll}
m & (\exists \chi)\Phi & & \\
 & \quad\vdots & & \\
n & & \Phi(\ldots c \ldots c \ldots) & :\text{AS for } \exists\text{E} \\
 & & \quad\vdots & \\
s & & \Psi & \\
s+1 & \Psi & & :m,\ n\text{–}s\ \exists\text{E}
\end{array}
$$

provided that c doesn't occur **anywhere else outside** the subproof

10. Proofs in QL

e. Substitution Instances

(Full) Substitution Instances

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- The other variables are untouched!
- $\Phi[c/\chi]$ can also arise from $(\exists\chi)\Phi$ by dropping the $(\exists\chi)$ and putting c in place of **every** χ in Φ.
- Equivalent notation: $\Phi\boxed{\chi \Rightarrow c}$

Some Examples of Substitution Instances

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  - The following are **NOT** substitution instances:
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Some Things to WATCH OUT for!

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- Is the wff '$(\forall x)Gx \vee (\forall x)Fx$' universally quantified?
- NO! It is a disjunction, since '$\vee$' is its main connective.
- What are instances of a sentence like $(\forall x)Gx \vee (\forall x)Fx$?
- Trick question! It has no instances, since it is a disjunction, not a quantified sentence!

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- Goal: figure out some wffs that a given wff is an instance of:

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- Fab is an instance of...
  - $(\forall x)Fax$, $(\forall x)Fxb$, $(\exists x)Fax$, $(\exists x)Fxb$
- Faa is an instance of...
  - $(\forall x)Fxx$, $(\forall x)Fax$, $(\forall x)Fxa$, $(\exists x)Fxx$, $(\exists x)Fxa$

Partial Substitution Instances

- For Existential Introduction, we can use a partial substitution instance of the wff Φ:

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- For Existential Introduction, we can use a partial substitution instance of the wff Φ:
- '$\Phi\lceil\chi/c\rceil$' indicates that the variable χ does not need to replace all occurrences of the constant c in Φ.
- You can decide which occurrences of c to replace and which to leave in place

Examples of Partial Substitution Instances!

- '$\Phi\lceil\chi/c\rceil$' indicates that the variable χ does not need to replace all occurrences of the constant c in Φ

1	Rdd	
2	$(\exists x)Rxx$	:1 $\exists$I
3	$(\exists x)Rxd$	:1 $\exists$I
4	$(\exists z)Rdz$	:1 $\exists$I
5	$(\exists y)(\exists z)Ryz$	:4 $\exists$I

Existential Introduction ($\exists$I)

m	$\Phi(c)$	
$\vdots$	$\vdots$	
s	$(\exists\chi)\Phi\lceil\chi/c\rceil$	:m $\exists$I

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.

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- Is this worth it???? Only one way to find out!

Rules for the Universal Quantifier

Universal Elimination (∀E)

m	$(\forall \chi)\Phi$
$\vdots$	$\vdots$
s	$\Phi[c/\chi]$: m ∀E

- Note that you replace **EVERY** instance of χ with c
- Notation: $\Phi[c/\chi]$
- read "c for χ"

Universal Introduction (∀I)

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Rules for the Existential Quantifier

Existential Introduction (∃I)

m	$\Phi(c)$	
$\vdots$	$\vdots$	
s	$(\exists\chi)\Phi\lceil\chi/c\rceil$	$: m\ \exists\mathrm{I}$

- **Provided that** χ does not occur already in $\Phi(\ldots c \ldots c \ldots)$.
- As indicated by $\lceil\chi/c\rceil$, χ **may** replace **just some** occurrences of c

Existential Elimination (∃E)

m	$(\exists\chi)\Phi$	
	$\vdots$	
n	$\quad\Phi[c/\chi]$	:AS for ∃E
	$\quad\vdots$	
s	$\quad\Psi$	
$s+1$	Ψ	$: m,\ n\text{-}s\ \exists\mathrm{E}$

provided that c doesn't occur **anywhere else outside** the subproof

10. Proofs in QL

f. Tips for an all-natural look!

Working Backwards

- Work backwards (at least in making a plan):

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  - a disjunction: disjunction intro

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  - existentially quantified: you will probably eliminate this existential, and if so you probably want to start this subproof ASAP (mind the tricky syntax!)
  - a conjunction: you can get conjuncts using &E
  - a disjunction: think about using disjunction elimination
    (mind the tricky syntax!)

Advice for deriving $(\exists x)\,\mathcal{A}x$, e.g. $(\exists y)\,Py$

- Ask whether you can derive an instance $\mathcal{A}[\mathsf{c}\text{ for }x]$, e.g. Pc

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Advice for deriving $(\exists \chi)\,\mathcal{A}\chi$, e.g. $(\exists y)\,Py$

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  - Remember: you might have to do this first as the last line of a SUBproof started for $\exists E$, before repeating what you want below the subproof, justified by $\exists$ ELIMINATION
- If not: consider using negation–elimination or disjunction elimination (if a disjunction is available)

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- Remember that you'll write the same sentence TWICE in a row, once within the subproof and once outside
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- Remember that the constant c must NOT appear in the conclusion, existential being eliminated, or an undischarged assumption