The Completeness of SL Tree Proofs

LOGIC I
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The Proof

Completeness: Every unsatisfiable root has a closed tree: $\Gamma \vDash \bot \Rightarrow \Gamma \vdash \bot$.

Contrapositive: If there is no closed tree with root Γ, then Γ is satisfiable.

Lemma 6: For any tree X with root Γ, there is a complete tree X' with root Γ.

- Assume there is no closed tree with root Γ.
- Roots are trees, and so Γ has a complete tree X.
- So X is a complete open tree with a complete open branch $\mathcal{B}$.

Note: This result is purely syntactic.

Lemma 7: Every complete open branch in an SL tree is satisfiable.

- So $\mathcal{B}$ is satisfiable, and so Γ is satisfiable.
- By contraposition, if $\Gamma \vDash \bot$, then $\Gamma \vdash \bot$.

Resolution

Let the *resolution* $\texttt{Res}(\varphi)$ provide an upper bound on the number of times that φ and its descendants could be resolved in an SL tree.

1. $\texttt{Res}(\varphi) = 0$ if φ is a literal.
2. For any SL sentences φ and ψ:
   - $\texttt{Res}(\neg\neg\varphi) = \texttt{Res}(\varphi) + 1$.
   - $\texttt{Res}(\varphi \wedge \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \wedge \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \vee \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \vee \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \supset \psi) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \supset \psi)) = \texttt{Res}(\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\varphi \equiv \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.
   - $\texttt{Res}(\neg(\varphi \equiv \psi)) = \texttt{Res}(\varphi) + \texttt{Res}(\neg\psi) + \texttt{Res}(\neg\varphi) + \texttt{Res}(\psi) + 1$.

Resolution Set: Let $[X]$ be the set of SL sentences that are resolvable in a branch of X.

Tree Resolution: Let $\texttt{Res}(X) = \sum_{\varphi \in [X]} \texttt{Res}(\varphi)$ be an upper bound on resolutions in X.

Supporting Lemmas

Lemma 4: Every SL tree X has a finite number of branches.

Lemma 5: For any SL tree X with root Γ and $\varphi \in [X]$, there is an SL tree Y with root Γ where $\texttt{Res}(Y) < \texttt{Res}(X)$.

- Let X be an SL tree with root Γ where $\varphi \in [X]$.
- By *Lemma 4*, φ is resolvable in finitely many branches of X.
- So there is a tree Y with root Γ that resolves φ throughout X.
- So $\varphi \notin [Y]$ but the children of φ could be in $[Y]$.

Case 1: Assume $\varphi = \neg\neg\psi$ where $\psi \in [Y]$ and $\psi \notin [X]$.

- So $\texttt{Res}(\psi) < \texttt{Res}(\varphi)$, and so $\texttt{Res}(Y) < \texttt{Res}(X)$.

Case n: The other cases are similar.

Lemma 6

Proof: For any Γ-tree X, there is a complete Γ-tree X'.

Base: Assume X is a Γ-tree where $\texttt{Res}(X) = 0$.

- So every $[X]$ is empty, so X is complete.

Hypothesis: Every Γ-tree X where $\texttt{Res}(X) \leq n$ has a complete Γ-tree X'.

Induction: Let X be a Γ-tree where $\texttt{Res}(X) = n + 1$.

- Since $\texttt{Res}(X) > 0$, there is some $\varphi \in [X]$.
- By *Lemma 5*, there is some Γ-tree Y where $\texttt{Res}(Y) < \texttt{Res}(X)$.
- By hypothesis, there is a complete Γ-tree Y'.

Conclusion: By strong induction, QED.

Finite Lemma

Proof: Every branch $\mathcal{B}$ in an SL tree contains finitely many sentences.

Base: Assume $\mathcal{B}$ belongs to an SL tree X where $\texttt{Length}(X) = 0$, so finite.

Hypothesis: Assume that every branch $\mathcal{B}$ of an SL tree X of $\texttt{Length}(X) = n$ has a finite number of sentences.

Induction: Assume that $\mathcal{B}'$ belongs to an SL tree X' of $\texttt{Length}(X) = n + 1$.

- Let X be a tree where X' is the result of resolving a sentence in X.
- So $\texttt{Length}(X) = n$.

- By hypothesis, every branch $\mathcal{B}$ of X has a finite number of branches.
- $\mathcal{B}'$ includes at most two more sentences than any branch $\mathcal{B}$ in X.
- Thus $\mathcal{B}'$ has a finite number of sentences.

Lemma 7

Proof: Every complete open branch in an SL tree is satisfiable.

Assume: Let $\mathcal{B}$ be a complete open branch in an SL tree.

- Let $\mathcal{I}(\varphi) = 1$ *iff* φ is a sentence letter in $\mathcal{B}$.
- By the *Finite Lemma*, we may assign sentences in $\mathcal{B}$ a position number where the leaf is 0.

Base: Assume φ has position 0.

- Since $\mathcal{B}$ is complete and open, φ is a literal.

Case 1: If φ is a sentence letter, $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi) = 1$.

Case 2: Assume $\varphi = \neg\psi$ where ψ is a sentence letter.

- Since $\mathcal{B}$ is open, ψ does not occur in $\mathcal{B}$.
- So $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{I}(\psi) = 0$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg\psi) = 1$.

Hypothesis: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every φ with position $k \leq n$ in $\mathcal{B}$.

Induction: Assume φ has position $n + 1$ in $\mathcal{B}$.

Case 1: φ is a literal, so $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ as above.

Case 2: $\varphi = \neg\neg\psi$.

- Since $\mathcal{B}$ is complete, ψ occurs in $\mathcal{B}$ in position $k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\psi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg\neg\psi) = 1$.

Case 3: $\varphi = \psi \wedge \chi$.

Case 4: $\varphi = \neg(\psi \wedge \chi)$.

- Since $\mathcal{B}$ is complete, $\neg\psi, \neg\chi$ occur in $\mathcal{B}$ in positions $j, k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\neg\psi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\neg\chi) = 1$.
- So $\mathcal{V}_{\mathcal{I}}(\psi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\chi) = 0$, and so $\mathcal{V}_{\mathcal{I}}(\psi \wedge \chi) = 0$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg(\psi \wedge \chi)) = 1$.

Case n: $\varphi = \neg(\psi \equiv \chi)$.

- Since $\mathcal{B}$ is complete, ψ and $\neg\chi$ occur in $\mathcal{B}$ in positions $j, k \leq n$, or else $\neg\psi$ and χ occur in $\mathcal{B}$ in positions $j, k \leq n$.
- By hypothesis, $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{I}}(\neg\chi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\neg\psi) = \mathcal{V}_{\mathcal{I}}(\chi) = 1$.
- In either case, $\mathcal{V}_{\mathcal{I}}(\psi) \neq \mathcal{V}_{\mathcal{I}}(\chi)$, and so $\mathcal{V}_{\mathcal{I}}(\psi \equiv \chi) = 0$.
- Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\neg(\psi \equiv \chi)) = 1$.