Problem Set 12

Keep the internet from becoming logically complete! Never share these!

Apologies for typos! Let me know if you catch any! (Gotta catch-em all!)

1.) Case 10 of Soundness proof for *SND* (Negation Elimination). We start by drawing a schematic derivation, labeling the relevant lines with letters so that we can easily refer to them. Let "Γ_{k+1}" denote the set of open assumptions at line k+1. Our goal is to show that line k+1 satisfies the inductive property (i.e. is "righteous", so that $\Gamma_{k+1} \vDash \mathcal{P}$).
(Note: you could start your own diagram at line j with just some vertical dots above it)

1	First premise	:PR
	$\vdots$	
f	Last premise	:PR
	$\vdots$	
j	$\quad \sim\mathcal{P}$	:AS for $\sim$E
	$\quad \vdots$	
ℓ	$\quad \mathcal{R}$	
m	$\quad \sim\mathcal{R}$	
$k+1$	$\mathcal{P}$	:j–m $\sim$E

By assumption, we derive $\mathcal{R}$ in l-steps from Γ_l and $\sim\mathcal{R}$ in m-steps from Γ_m. Hence, by the Induction Hypothesis, we have $\Gamma_l \vDash \mathcal{R}$ and $\Gamma_m \vDash \sim\mathcal{R}$ (in the IH, we assume that each line less than k+1 is "righteous", i.e. if $h < k+1$ and $\Gamma_h \vdash \mathcal{S}$, then $\Gamma_h \vDash \mathcal{S}$).

Next we note the relevant subset relations connecting these assumption-sets and the assumption set Γ_{k+1} for line k+1: $\Gamma_l \subseteq \Gamma_{k+1} \cup \{\sim\mathcal{P}\}$ and $\Gamma_m \subseteq \Gamma_{k+1} \cup \{\sim\mathcal{P}\}$. To see these relations, note that every assumption that is open at line ℓ is open at line k+1 except for $\sim\mathcal{P}$ on line j. Likewise for Γ_m.

Hence, any truth-value assignment that makes every sentence in $\Gamma_{k+1} \cup \{\sim\mathcal{P}\}$ true must also make true every sentence in Γ_l and every sentence in Γ_m. Since these latter sets semantically entail $\mathcal{R}$ and $\sim\mathcal{R}$ respectively, we see that the superset must entail these sentences as well: $\Gamma_{k+1} \cup \{\sim\mathcal{P}\} \vDash \mathcal{R}$ and $\Gamma_{k+1} \cup \{\sim\mathcal{P}\} \vDash \sim\mathcal{R}$ (here, we have justified and applied the book's lemma 6.3.2).

Therefore, any TVA that makes true every sentence in $\Gamma_{k+1} \cup \{\sim\mathcal{P}\}$ must make true both $\mathcal{R}$ and $\sim\mathcal{R}$. But this is impossible: there is no such TVA. Hence, the superset $\Gamma_{k+1} \cup \{\sim\mathcal{P}\}$ must be unsatisfiable (i.e. semantically inconsistent). Hence, any TVA that makes true every sentence in Γ_{k+1} must make true $\mathcal{P}$ (this applies the book's lemma 6.3.5, noting that $\sim\sim\mathcal{P}$ is semantically equivalent to $\mathcal{P}$). Hence, line k+1 has the inductive property of righteousness.

2.) We prove case (c) of the membership lemma (book's 6.4.11), used in the completeness proof for *SND* (and, suitably modified, for *QND* as well): if Γ^* is a maximally syntactically-consistent set of SL sentences, then: $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$.

Remember that we need to prove both the forwards and backwards directions, possibly splitting each of these into further subcases.

($\Rightarrow$-Direction): Assume that $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$. Need to show (NTS): either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$. We split this into two cases, since $\mathcal{P}$ is either in the club or it is not in the club.
Case (i): Note that if $\mathcal{P} \in \Gamma^*$ the relevant either-or claim is true. So move to case (ii), where we assume that $\mathcal{P} \notin \Gamma^*$. Our goal is to show that $\mathcal{Q} \in \Gamma^*$. To show this, it suffices to derive $\mathcal{Q}$ from finitely-many premises assumed to be in Γ^*, since we can then apply The Door lemma to conclude that $\mathcal{Q} \in \Gamma^*$.

At this point, there are two ways to proceed directly. We could either note that by Case (a) of the membership lemma, $\mathcal{P} \notin \Gamma^*$ entails that $\sim\mathcal{P} \in \Gamma^*$. We could then derive $\mathcal{Q}$ in SND from the finite premise set $\{\mathcal{P} \vee \mathcal{Q}, \sim\mathcal{P}\} \subset \Gamma^*$ by constructing a derivation similar to that below. Alternatively, we can recall the definition of a maximally SND-consistent set and note that $\mathcal{P} \notin \Gamma^*$ entails that $\Gamma^* \cup \{\mathcal{P}\}$ is SND-INconsistent. So there exists an SL-sentence $\mathcal{R}$ such that $\Gamma^* \cup \{\mathcal{P}\} \vdash \mathcal{R}$ and $\Gamma^* \cup \{\mathcal{P}\} \vdash \sim\mathcal{R}$. You ought to memorize this criterion, so we'll use it below to reinforce that memory!

Let $\mathcal{A}_1; \mathcal{A}_2; \ldots; \mathcal{A}_n; \mathcal{P}$ be finitely-many premises from $\Gamma^* \cup \{\mathcal{P}\}$ that derive $\mathcal{R}$ and $\sim\mathcal{R}$. We show that there is a derivation of $\mathcal{Q}$ from $\mathcal{A}_1; \mathcal{A}_2; \ldots; \mathcal{A}_n; \mathcal{P} \vee \mathcal{Q}$ (given shamelessly without line numbers below). You should include schematic line numbers in the justifications!:

$\vert$ [Assumptions from Γ^*]	:PRs
$\vert$ $\mathcal{P} \vee \mathcal{Q}$	:PR from Γ^*
$\vert\ \vert$ $\mathcal{Q}$	A/ $\vee E$
$\vert\ \vert$ $\mathcal{Q}$	R
$\vert\ \vert$ $\mathcal{P}$	A/ $\vee E$
$\vert\ \vert\ \vert$ $\sim\mathcal{Q}$	A/ $\sim E$
$\vert\ \vert\ \vert$ $\vdots$	
$\vert\ \vert\ \vert$ $\mathcal{R}$	
$\vert\ \vert\ \vert$ $\sim\mathcal{R}$	
$\vert\ \vert$ $\mathcal{Q}$	$\sim E$
$\vert$ $\mathcal{Q}$	$\vee E$

Since $\{\mathcal{A}_1; \mathcal{A}_2; \ldots; \mathcal{A}_n; \mathcal{P} \vee \mathcal{Q}\} \subseteq \Gamma^*$, this proves that $\Gamma^* \vdash \mathcal{Q}$. So by The Door Lemma (book's 6.4.9), $\mathcal{Q} \in \Gamma^*$. We proceed to the next direction!

($\Leftarrow$-direction): Now, assume that either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$ (where our inclusive-or includes the possibility that both are in Γ^*). In subcase (i), we assume $\mathcal{P} \in \Gamma^*$. Note that we can derive $\mathcal{P} \vee \mathcal{Q}$ from $\mathcal{P}$ with one application of $\vee$ introduction: $\{\mathcal{P}\} \subset \Gamma^* \vdash \mathcal{P} \vee \mathcal{Q}$. So by the Door Lemma, $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$. In subcase (ii), we assume $\mathcal{Q} \in \Gamma^*$. Then similarly, $\{\mathcal{Q}\} \vdash \mathcal{P} \vee \mathcal{Q}$ by $\vee$I. So in either case, $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$, which is what we needed to show.

This completes both directions. Welcome to the club haha!

3.) We aim to leverage the membership lemma case we just proved in problem #2, to complete missing case (3) in our induction over SL. Recall that in this induction, we construct a TVA $\mathcal{I}$ such that a sentence $\mathcal{P}$ is true on $\mathcal{I}$ if and only if $\mathcal{P}$ belongs to the club, i.e. $\mathcal{P} \in \Gamma^*$. We induct over the number of connectives in SL sentences, assuming that this property (they be "clubbin' ") holds for each sentence with less than k+1 connectives. Note that we again have two directions to prove, since this is an "iff" statement.

In case 3, $\mathcal{P}$ has the form $\mathcal{Q} \vee \mathcal{R}$, with k+1-many connectives.

($\Rightarrow$-Direction): Assume that $\mathcal{Q} \vee \mathcal{R}$ is true on $\mathcal{I}$. Then by the truth-table for $\vee$, at least one of $\mathcal{Q}$ or $\mathcal{R}$ is true on $\mathcal{I}$. Since each of these sentences contains less than k+1 connectives, they are clubbin' by the Induction Hypothesis. So at least one of $\mathcal{Q}$ or $\mathcal{R}$ belongs to Γ^*. Hence, by case (c) of the membership lemma, $\mathcal{Q} \vee \mathcal{R} \in \Gamma^*$ as well.

($\Leftarrow$-direction): Assume that $\mathcal{Q} \vee \mathcal{R} \in \Gamma^*$. NTS: $\mathcal{Q} \vee \mathcal{R}$ is true on $\mathcal{I}$. We can immediately apply membership lemma case (3) to note that since $\mathcal{Q} \vee \mathcal{R} \in \Gamma^*$, at least one of $\mathcal{Q}$ or $\mathcal{R}$ belongs to Γ^*. Since $\mathcal{Q}$ and $\mathcal{R}$ each have less than k+1 connectives, by the IH they are true on the TVA $\mathcal{I}$. So by the truth-table for $\vee$, $\mathcal{Q} \vee \mathcal{R}$ is also true on $\mathcal{I}$.

Note that we could streamline our proof by taking advantage of relevant iff-claims at each step, thereby handling both directions in one go (NB: "$\Leftrightarrow$" means bi-directional entailment, NOT the biconditional connective $\equiv$):

$\mathcal{Q} \vee \mathcal{R}$ is true on $\mathcal{I} \Leftrightarrow$ $\mathcal{Q}$ is true on $\mathcal{I}$ or $\mathcal{R}$ is true on $\mathcal{I}$ by the truth table for $\vee$.
$\Leftrightarrow$ $\mathcal{Q} \in \Gamma^*$ or $\mathcal{R} \in \Gamma^*$ by the induction hypothesis $\Leftrightarrow$ by 6.4.11 c) $\mathcal{Q} \vee \mathcal{R} \in \Gamma^*$.
But I don't recommend this shortcut since it is harder to parse the argument in the backwards direction (since we write from left to right).

4.) Recall that a natural deduction is always finite. So no matter how many sentences there might be in a set Γ, a derivation can draw on *only finitely-many* sentences from Γ.
So if $\Gamma \vDash S$, then by the completeness theorem, $\Gamma \vdash_{SND} S$. Hence, there exists a finite subset $\Delta \subset \Gamma$ such that $\Delta \vdash_{SND} S$. By the soundness theorem, we can convert this single turnstile to a dolla dolla double: $\Delta \vDash S$ (i.e. the finite set Δ semantically entails S).

5.) Assume for *reductio* that you could derive a contradictory sentence pair $\mathcal{R}$ and $\sim\mathcal{R}$ from the atomic sentence letter B. Then we'd have $B \vdash_{SND} \mathcal{R}$ and $B \vdash_{SND} \sim\mathcal{R}$.

Applying the soundness theorem, we could convert these singles to doubles, so we'd have $B \vDash \mathcal{R}$ and $B \vDash \sim\mathcal{R}$. Yet, recall that a sentence (or set of sentences) can entail a contradiction only if it is unsatisfiable! But B is satisfiable: it is true on any TVA that assigns "true" to B. Hence, we have reached a contradiction: It is impossible for any TVA that makes B true to make both $\mathcal{R}$ and $\sim\mathcal{R}$ true. Hence, the set $\{B\}$ is syntactically-consistent in SND.

An alternative, very slick *reductio*: if you could derive a contradictory sentence pair $\mathcal{R}$ and $\sim\mathcal{R}$ from the atomic sentence letter B, then you could derive $\sim B$ from B by negation introduction. So then we'd have $B \vdash_{SND} \sim B$, and by soundness $B \vDash \sim B$. This would mean that on any TVA where B is true, $\sim B$ is also true, which is clearly absurd.

6.) By the completeness theorem for *SND*, $\Gamma \vDash S \Rightarrow \Gamma \vdash_{SND} S$. Hence, we can appeal to the contrapositive of completeness: if $\Gamma \nvdash_{SND} S$, then $\Gamma \nvDash S$.

We are told that for some SL-set Γ, $\Gamma \nvdash_{SND} S$. Hence, by completeness$_{contra}$, $\Gamma \nvDash S$. This is a counterexample to the soundness of modified system SND*, i.e. a case where $\Gamma \vdash_{SND^*} S$ but it's *not the case* that $\Gamma \vDash S$.

Spelled out in more detail: since $\Gamma \nvDash S$, the set $\Gamma \cup \{\sim S\}$ is satisfiable: i.e. there is a TVA that makes true every sentence in Γ while making $\sim S$ true and hence S false.
So since $\Gamma \vdash_{SND^*} S$, our modified system allows a case where we go from true premises to derive a false conclusion, which just means that *SND** is unsound.