PHIL 24.241, Connectives and Truth tables

Fall 2022

1 Statements and argument structure

Statement/proposition: a sentence that is true or false. All statements are declarative/indicative (i.e., questions, commands, etc. are *not* statements).

Argument: a series of premises (i.e. reasons) that purport to support a conclusion. Both the premises and the conclusion must be statements.

Three kinds of arguments:

1. **Deductive**: "risk-free;" premises imply the conclusion with 100% certainty. If the premises are true, then the conclusion is true.
   E.g., all humans are mortal. I am a human. Therefore, I am mortal.

2. **Inductive**: "risky;" premises make the conclusion probable. Nevertheless, the premises could be true while the conclusion is false.
   E.g., every raven I've ever seen has been black. Therefore, the next raven I see will be black.

3. **Abductive** (inference to the best explanation): among a set of competing explanations, we select the one that is best according to some set of methodological criteria.
   E.g., reasoning method of Sherlock Holmes

Arguments can have two good truth-related properties: validity and soundness.

Valid argument: whenever its premises are true, the conclusion is true. Valid arguments have ideal logical structure: they preserve truth. Given true premises, they spit out true conclusions.

Sound argument: a valid argument whose premises are true (so the conclusion is also true). Sound arguments not only have ideal logical structure, they also track the truth. They reflect the way the world is.

2 Logical connectives and truth tables

A **logical connective** (propositional operator) connects or modifies one or more propositions. A string of propositions joined by one or more operators is also a proposition. We will use the following connectives:

- **Negation**: $\sim P$. $\sim P$ is true if and only if P is false.
- **Conjunction**: $(P \& Q)$. True if and only if P and Q are both true.
- **Disjunction**: $(P \vee Q)$. True if and only if *at least one* of P or Q is true (including both)
- **If...then (conditional)**: $(P \supset Q)$. True if and only if either P is false or Q is true (false only when P is true and Q is false).
- **Biconditional**: $(P \equiv Q)$. True if and only if both P and Q have the same truth value (either both true or both false).

A set of truth-functional connectives is *expressively adequate* provided that given any truth-functional compound, we can express it using just the connectives in this set. Negation and conjunction are expressively adequate, but adding disjunction and conditionals makes it easier to analyze arguments.

Truth Table: A diagram showing all the possible truth values that a statement can have.

- It shows the ways in which the truth-values of complex statements depend on the truth-values of the parts, for each connective. (Truth functionality)

However, not all statements are truth-functional. For example, "I got the flu because I did not get a flu vaccine" is not a a function of its constituents' truth-values. The "because" operator is not a truth function.

In contrast, the truth value of compound statements that involve only the truth-functional operators negation, conjunction, disjunction, conditional, and biconditional depend only on the truth-values of the constituent statements. This is because these operators are truth-functional (as shown by the truth-tables below).

2.1 Negation

How to read "∼P": affix *it is not the case that* before the proposition P you are considering. The negation of a statement is true provided that the negated statement is false; the negation is false provided that the original statement is true.

For instance, *a mast cell is a type of white blood cell* vs. *it is not the case that a mast cell is a type of white blood cell.*

P	**∼ P**
T	F
F	T

Double negation elimination: *it is not the case that it is not the case that P* is truth-functionally equivalent to *it is the case that P.* I.e., ∼∼*P* is truth-functionally equivalent to *P*.

2.2 Conjunction: P & Q

- A conjunction (e.g. P & Q) is true provided that both of its conjuncts (P, Q) are true.

E.g. *mast cells and basophils are types of white blood cells* is the same as *mast cells are a type of white blood cell & basophils are a type of white blood cell.* This statement is true if and only if both of the conjuncts are true.

P	**Q**	**P & Q**
T	T	T
T	F	F
F	T	F
F	F	F

2.3 Disjunction: P ∨ Q

- A disjunction (e.g. P ∨ Q) is true provided that *at least one* of the disjuncts (P, Q) are true.

P	**Q**	**P ∨ Q**
T	T	T
T	F	T
F	T	T
F	F	F

Note that we are using “inclusive-or” rather than “exclusive-or:”

Inclusive-or: either or both of the disjuncts can be true.

Exclusive-or: true if and only if exactly one of the disjuncts is true. We represent this formally as an inclusive-or conjoined with a negation of the conjunction: $(P \vee Q)\,\&\,{\sim}(P\,\&\,Q)$.

2.4 Conditional: P ⊃ Q

Read “P ⊃ Q” as *if P then Q*. The statement in the P-position is the *antecedent*. The statement in the Q-position is the *consequent*.

Classical logic uses the *material conditional*, which is true provided that either the antecedent is false or the consequent is true. If neither of these conditions is met, then the conditional is false (i.e. it is only false when the antecedent is true and the consequent is false).

P	**Q**	**P ⊃ Q**
T	T	T
T	F	F
F	T	T
F	F	T

In other words, $P \supset Q$ is truth-functionally equivalent to ${\sim}(P\,\&\,{\sim}Q)$, and hence also to ${\sim}P \vee Q$.

E.g., *if I have the flu, then I have a fever.* This conditional is false if I have the flu but do not have a fever. However (and strangely), this conditional is true provided I have a fever, even if I don’t have the flu. It is also true if I don’t have the flu, independently of whether or not I have a fever. Yet, we’re generally not interested in such conditionals if the antecedent is false. Think about a conditional as a promise: it is a promise that if the antecedent obtains, then the consequent obtains as well. Keep in mind that the material conditional is NOT the same as “implies” since the antecedent and consequent don’t have to be connected to each other in any special way.

Another example: *if you get a vaccine, then you won’t get the flu.* Again, think about this as a promise. If you didn’t get a vaccine and still didn’t get the flu, you wouldn’t have any reason to criticize this conditional. You wouldn’t have any reason to think that the flu vaccine didn’t keep its promise.

Contrapositive: given a conditional statement *if P then Q*, the contrapositive is the conditional *if not Q then not P* (${\sim}Q \supset {\sim}P$). The contrapositive is truth-functionally equivalent to its conditional.

One is true if and only if the other is true. One is false if and only if the other is false. Consider *if I don't have the fever, then I don't have the flu.*

Often, when constructing a proof, it is much easier to demonstrate a conditional statement by demonstrating its contrapositive. (N.B. this trick doesn't apply to using truth tables).

Language constructions that are equivalent to the conditional (formed by putting the antecedent second): "Q if P," "Q provided that P," and "Q in case P."

Additionally, "P only if Q" is equivalent to "if P then Q." To check this, note that "P only if Q" says that if it is not the case that Q, then it is not the case that P. This is truth-functionally equivalent to the contrapositive $\sim Q \supset \sim P$, which is truth-functionally equivalent to the conditional $P \supset Q$.

2.5 Biconditional

- A biconditional (e.g. P ≡ Q) is true only if both propositions have the same truth value. We read this as "P if and only if Q," often abbreviated to "P iff Q."
- To see why this is so, you can think of the biconditional as: (P ⊃ Q) & (Q ⊃ P).

P	Q	P ≡ Q
T	T	T
T	F	F
F	T	F
F	F	T

2.6 De Morgan's Laws

- $\sim(P \& Q)$ is logically equivalent to $\sim P \vee \sim Q$
- To show a conjunction is false, it suffices to show one of the conjuncts is false
- $\sim(P \vee Q)$ is logically equivalent to $\sim P \& \sim Q$
- To show a disjunction is false, one must show both disjuncts are false

2.7 Some rules of inference

Modus ponens (affirming the antecedent): a very common rule of inference. Given a conditional statement $P \supset Q$ and the antecedent P, we can infer the consequent Q of the conditional. For instance, if you get a vaccine, then you probably don't get the flu. Jason got a vaccine. Hence, (by modus ponens), Jason probably won't get the flu.

Modus tollens (denying the consequent): Given a conditional statement $P \supset Q$, and a false consequent, i.e. $\sim Q$, we infer $\sim P$.

2.8 Constructing Truth Tables

1. Determine how many columns you need.

   (a) 1 column for each sentence letter and 1 column for each premise and conclusion.[1]

   Ex: There are two sentences in Modus Tollens (MT), namely P and Q. We need two columns, one for each sentence. We also need a column for premise 1 ($P \supset Q$), premise 2 ($\sim$Q), and the conclusion ($\sim$P)

P	Q	$P \supset Q$	$\sim Q$	$\sim P$

2. Determine how many rows you need.

   (a) Remember the formula, # of rows = 2^n, where n = number of sentences

   (b) Thus, for instance, if there are 2 sentences (as above), there are four rows. If there are three sentences, you need 8 rows. Four sentences, 16 rows. And and so on.

   (c) Recall also that # of rows = # of possible truth-value combinations

3. Fill out the single sentence columns (e.g. P, Q, R, etc.). Notice the patterns below.

   Ex: 2 sentences

P	Q	$P \supset Q$	$\sim Q$	$\sim P$
T	T			
T	F			
F	T			
F	F			

   Ex: 3 sentences

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

4. Fill out the rest of the columns (the ones with the connectives, e.g. $\sim$P, $P \supset Q$, $P \vee Q$, etc.) Reference sec. 2 if you have any questions about how to do this.

[1] If one of your premises is an atomic sentence (e.g. if one of your premises is just 'P'), you do not need another column for that premise (e.g. you do not need to include two columns with 'P').

P	Q	P ⊃ Q	∼Q	∼P
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

3 Proving Validity

First, recall the definition of **validity**: If the premises are true, then the conclusion, *must* be true.

1. Identify the premises in the truth table (look at the columns)

P	Q	**P ⊃ Q**	**∼Q**	∼P
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

2. Identify the row(s) in which both premises are *true*

P	Q	**P ⊃ Q**	**∼Q**	∼P
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	**T**	**T**	T

3. Look at the truth-value of the conclusion in that row (or, those rows, if there are multiple)

P	Q	**P ⊃ Q**	**∼Q**	∼P
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	**T**	**T**	**T**

   (a) If the conclusion is also true (on each row where all premises are true), then the argument is **valid**.

   (b) If the conclusion is false on one of these rows where all premises are true, then the argument is **invalid**.[2]

   (c) *Validity on the cheap*: If there is no row where all premises are true, then the premises are **inconsistent**. In this case, the argument is (vacuously) **valid**. This is because in all cases where the premises are true (which is none), the conclusion is not false. *Inconsistent arguments are always valid.* Impress your friends.

[2]Remember, you only need one row in the truth table where the premises are all true and the conclusion false to prove that the argument is invalid!