Natural Deduction in SL: Part II

LOGIC I
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Negation

Elimination Rule: $\neg\neg A \vdash A$. (*Double Negation Elimination*)

1. $A \vee \neg A$. (*Law of Excluded Middle*)
2. $A, \neg A \vdash B$. (*Ex Falso Quodlibet*)

Introduction Rule: $\neg(A \wedge \neg A)$. (*Law of Non-Contradiction*)

3. $A \vdash \neg\neg A$. (*Double Negation Introduction*)

Proof

Proof: A natural deduction PROOF (or DERIVATION) of a conclusion φ from a set of premises Γ in SD is any finite sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) A premise in Γ;

(2) A discharged assumption; or

(3) Follows from previous lines by the rules for SD.

Provable: An SL sentence φ is PROVABLE (or DERIVABLE) from Γ in SD *iff* there is a natural deduction proof (derivation) of φ from Γ in SD, i.e., $\Gamma \vdash \varphi$.

Theorems: An SL sentence φ is a THEOREM of SD *iff* $\vdash \varphi$.

Equivalent: Sentences φ and ψ are PROVABLY EQUIVALENT (or INTERDERIVABLE) if and only if both $\varphi \vdash \psi$ and $\psi \vdash \varphi$, i.e., $\varphi \dashv\vdash \psi$.

Inconsistent: A set of sentences Γ is PROVABLY INCONSISTENT *iff* $\Gamma \vdash \bot$ where $\bot$ is the arbitrarily contradiction we chose, i.e., $A \wedge \neg A$.

Soundness and Completeness

Assume: $\Gamma \vdash \varphi$ *iff* $\Gamma \vDash \varphi$.

Tautologies: Coextensive with the theorems.

Validity: The valid SL arguments are derivable in SD, and *vice versa*.

Task 1: Can we ever use SD to determine that an argument is invalid?

Uncertainty: If we haven't found a proof, that doesn't mean one doesn't exist.

Logical Analysis

Task 2: How can we tell if an argument is valid?

- Use a semantic argument: true premises and false conclusion.
- Construct a tree proof.

Pro: Both methods provide a countermodel if there is one.

Con: Neither method derives the conclusion from the premises if valid.

Task 3: How can we tell if a theorem is valid?

Tautology?	If YES, prove $\vdash \varphi$.	If NO, provide a countermodel.
Contradiction?	If YES, prove $\vdash \neg\varphi$.	If NO, provide a model.
Contingent?	If YES, provide a models.	If NO, prove $\vdash \varphi$ or $\vdash \neg\varphi$.
Equivalent?	If YES, prove $\varphi \dashv\vdash \psi$.	If NO, provide a countermodel.

Schemata

Observe: Compare rules of inference in SD to SL proofs in SD.

- Whereas the rules are general, SL proofs are particular.
- But nothing in our SL proofs depend on the particulars.

Task 3: How might we generalise our proofs beyond any instance?

Rule Schemata: Replace sentence letters in SL proofs with metavariables.

- Premises are replaced with the lines cited by that rule.
- New rules require new names if we are to use them.

Task 4: Can we also generalise proofs of theorems?

Axiom Schemata: Amount to lines that can be added without citing lines.

Goal: We want to derive intuitive rule schemata.

Derivable Schemata

Double Negation: $\neg\neg\varphi \dashv\vdash \varphi$.

Ex Falso Quodlibet: $\varphi, \neg\varphi \vdash \psi$.

Law of Excluded Middle: $\vdash \varphi \vee \neg\varphi$.

Law of Non-Contradiction: $\vdash \neg(\varphi \wedge \neg\varphi)$.

Hypothetical Syllogism: $\varphi \supset \psi, \psi \supset \chi \vdash \varphi \supset \chi$.

Modus Tollens: $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi$.
Contraposition: $\varphi \supset \psi \vdash \neg\psi \supset \neg\varphi$.
Dilemma: $\varphi \vee \psi,\ \varphi \supset \chi,\ \psi \supset \chi \vdash \chi$.
Disjunctive Syllogism: $\varphi \vee \psi,\ \neg\varphi \vdash \psi$.
$\vee$-Commutativity: $\varphi \vee \psi \vdash \psi \vee \varphi$.
$\wedge$-Commutativity: $\varphi \wedge \psi \vdash \psi \wedge \varphi$.
Biconditional MP: $\varphi \equiv \psi,\ \neg\varphi \vdash \neg\psi$.
$\equiv$-Commutativity: $\varphi \equiv \psi \vdash \psi \equiv \varphi$.
$\wedge$-De Morgan's: $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$.
$\vee$-De Morgan's: $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$.
$\vee\wedge$-Distribution: $\varphi \vee (\psi \wedge \chi) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$.
$\wedge\vee$-Distribution: $\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.
$\vee\wedge$-Absorption: $\varphi \vee (\varphi \wedge \psi) \dashv\vdash \varphi$.
$\wedge\vee$-Absorption: $\varphi \wedge (\varphi \vee \psi) \dashv\vdash \varphi$.
$\wedge$-Associativity: $\varphi \wedge (\psi \wedge \chi) \dashv\vdash (\varphi \wedge \psi) \wedge \chi$.
$\vee$-Associativity: $\varphi \vee (\psi \vee \chi) \dashv\vdash (\varphi \vee \psi) \vee \chi$.

Axiom System for SL

Axiom System: Consider the axiom and rule schemata, writing '/' for deduction.

- $\varphi \supset (\psi \supset \varphi)$.
- $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$.
- $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$.
- $\varphi \supset \psi,\ \varphi\ /\ \psi$.

PL-Proof: $\Gamma \vdash_{PL} \varphi$ *iff* there is a finite sequence of SL sentences where every sentence in the sequence is either: (1) a member of Γ; (2) an axiom schemata; or (3) follows from previous sentences in the sequence by the single rule schemata given above.

Equivalence: Amazingly, it is possible to show that $\Gamma \vdash_{PL} \varphi$ *iff* $\Gamma \vdash_{SD} \varphi$.

Definitions: Given that the axioms and rule schemata only include $\neg$ and $\supset$, we may take these to be the *only* primitive logical connectives, defining all other connectives in their terms.

- This makes for a very compact description of the same logic.
- This logic is much less natural to use, requiring that a lot of derived rules be added to system.
- We don't have this problem, though our system is more complex.