Problem Set 13 (12th graded PS; 24.241 Symbolic Logic)

Due Saturday **Dec. 10th** by 1pm Eastern

Please scan and upload to Canvas as a pdf

Choose FOUR questions total to answer, but not *both* #2 and #5
(at least, you're not allowed to prove #2 by appealing to #5! Das ist verboten!)

If you worked with up to two classmates, please list their names!

1. Problem #12 in *The Logic Book* §11.4E (p. 584). Have fun!

2. Prove that there can be no formula B of quantifier logic that is both
(i) true in all finite models and (ii) false in all countably infinite models
(where the cardinality of a model is the cardinality of its domain of discourse).
Hint: use the compactness theorem for QL. But I recommend #5 or #6 instead!

3. Let Γ be a set of QL-sentences that is unsatisfiable (i.e. there is no QL-model that makes true all of the sentences in Γ). Show that there must be sentences $\mathcal{P}_1$, $\mathcal{P}_2$, ..., $\mathcal{P}_n \in \Gamma$, such that the disjunction of the negations of these $\mathcal{P}_k$'s is a tautology (i.e. true in every QL-model):

$$\mathcal{T} := {\sim}\mathcal{P}_1 \vee {\sim}\mathcal{P}_2 \vee \ldots \vee {\sim}\mathcal{P}_n$$

(For readability, I have left out the otherwise requisite parentheses in $\mathcal{T}$).

You may take for granted that ${\sim}\mathcal{T}$ is logically equivalent to the following:

$$\mathcal{T}' := {\sim}{\sim}\mathcal{P}_1 \,\&\, {\sim}{\sim}\mathcal{P}_2 \,\&\, \ldots \,\&\, {\sim}{\sim}\mathcal{P}_n$$

Hint: use the compactness theorem for QL. And you don't need to use $\mathcal{T}'$.

4. Provide at least one reason why our proof of the soundness of system STD (trees) from Week 5 requires the premise set Γ to be *finite*, whereas our soundness proofs for SND and QND allow Γ to be infinite.
Hint: compare the relevant inductive properties and base cases in the soundness proofs.
(*completely optional & ungraded follow-up*: is it possible to modify either a definition(s) or our proof to show tree-soundness for a countably infinite premise-set Γ?)

5. Consider a set of QL-sentences Γ that has arbitrarily large finite models (i.e. for arbitrarily-many $n \in \mathbb{N}$, there exists a model $\mathfrak{M}$ whose domain $\mathcal{D}_n$ has cardinality n, s.t. $\mathfrak{M}$ satisfies Γ). Using compactness, prove that Γ has an infinite model.
(A corollary: no set of QL-sentences can be true in all and only finite models).
Hints: Let L_n be a QL-sentence that says that there are at least n-things.
Define the set $\mathcal{K}$ to be the set of natural numbers k such that Γ has a model of size k.

6. Is it possible to say in quantifier logic with identity that "there are finitely many things", without being more specific about how many things there are in the domain of discourse? If 'yes', explain how. If 'no', explain why not.
Hints: as above, let L_n be a QL-sentence that says that there are at least n-things.
Let F be a QL-sentence that says that there are finitely many things.
Consider the set $X := \{F, L_1, L_2, \ldots\}$, containing F and each L_k for $k \in \mathbb{N}$.

7. Prove that if every finite subgraph of a graph Γ can be n-colored, then so can Γ.

   Relevant background and definitions:
   A *graph* Γ is a finite or countably infinite set $X := \{x_1, x_2, \dots\}$ of *nodes* together with an irreflexive and symmetric relation called *adjacency*.
   An n-coloring of a graph is a function c that assigns a number in $\{1, 2, \dots, n\}$ to each node x_k—called the *color* of x_k—such that adjacent nodes never receive the same color.

   Hints: (i) Recall that our atomic wffs have at most a single subscript, e.g. P_3. For convenience, introduce "$P_{k,n}$" as a nickname for the atomic wff $P_{2^k 3^n}$, for any $k, n \in \mathbb{N}$.
   (ii) Construct a set S of SL-sentences representing the claim "Γ can be n-colored".
   (iii) Use the compactness theorem for SL

8. Use the compactness theorem for a first-order language (e.g. QL with functions) to construct a non-standard model of arithmetic. Here, we take arithmetic to be a model $\mathfrak{N}$ whose domain of discourse is the natural numbers $\mathbb{N}$, equipped with the ordinary less-than ordering relation and the usual arithmetic functions of successor, addition, multiplication, and exponentiation. A non-standard model of arithmetic $\mathfrak{U}$ has a domain with more element(s) than $\mathfrak{N}$, but such that the same set of sentences $\mathcal{E}$ is true, i.e. such that both $\mathfrak{N} \models \mathcal{E}$ and $\mathfrak{U} \models \mathcal{E}$.
   (*Optional reflection question*: why is the existence of non-standard models of arithmetic kind of a bummer?)

Compactness Theorems and Definitions

Compactness of SL: for any set Γ of SL-sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. there is a truth-value assignment that makes all sentences in Δ true).

Compactness of QL: for any set Γ of QL-sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable (i.e. there is a QL-model $\mathfrak{M}_\Delta$ that makes true every sentence in Δ).

Compactness of a first-order language $\mathcal{L}$: for any set Γ of $\mathcal{L}$-sentences (possibly infinite), Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is satisfiable.

Note: for all problems, it suffices to consider sets Γ that are countably infinite. By using the axiom of choice, we could modify our completeness proof to show completeness (and thus compactness) for first-order languages that allow *uncountably* infinite sets.

"Saying in a model" (relevant for problem #6): for a given claim C about how many things there are (e.g. the claim that there are finitely-many things), a set of sentences Γ *say that claim* C just in case those sentences are true in *all and only* models in which the domain D is as claim C describes.

First-order language $\mathcal{L}$: a set of well-formed formulae specified by a recursion clause like the one we gave for QL, where the symbols of $\mathcal{L}$ include variables, our five connectives, our two quantifiers, left and right parentheses, a set of names, a non-empty set of predicates, and a set of function symbols (interpreted as mapping terms to terms). Different first-order languages differ in their names, predicates, and functions.