Natural Deduction in SL

LOGIC I
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Motivation

Proof Trees: Proof trees provide an efficient way to evaluate validity.

- If an argument is valid, the tree will close.
- If an argument is invalid, the tree will give us an interpretation.

Unnatural: But proof trees do not provide a natural line of reasoning.

- Proof trees go by *reductio* which are not explanatory.
- Rules for proof trees are not entirely unnatural.
- But trees do not resemble natural reasoning.

Natural Deduction: How would we describe the patterns of natural deduction?

- Identify a range of intuitively compelling basic inferences in SL.
- Such inferences hold in virtue of the meanings of the connectives.
- Define a proof to be any composition of basic inferences.

Rules: Our system will include introduction and elimination rules.

- These rules will describe how to reason with the connectives.

Conditional

Elimination: $A,\ A \supset B,\ B \supset C \vdash C$.

- Premises justified by ':PR'.
- Easy to derive C.
- What if A was excluded from the premises?

Introduction: $A \supset B,\ B \supset C \vdash A \supset C$.

- Need something to work with.
- Want to conclude with a conditional claim.
- Assumption of A justified by ':AS'.

Subproofs: Lines in a closed subproof are dead and all else are live.

- $\supset$E can only cite to live lines.
- $\supset$I can only cite an appropriate subproof.

Reiteration

Example: $A \vdash D \supset [C \supset (B \supset A)]$.

Conjunction

Elimination: $A \supset (B \wedge C),\ B \supset D \vdash A \supset D$.

Introduction: $A \wedge B,\ B \supset C \vdash A \wedge C$.

Disjunction

Introduction: $A \vdash B \vee ((A \vee C) \vee D)$.

Elimination: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$.

Biconditional

Elimination: $A \equiv (B \supset [(A \wedge C) \equiv D]) \vdash (A \wedge B) \supset (D \supset C)$.

Introduction: $A \supset (B \wedge C),\ C \supset (B \wedge A) \vdash A \equiv C$.

Negation

Elimination: $\neg\neg A \vdash A$.

Introduction: $A \supset (B \wedge C),\ C \supset (B \wedge A) \vdash A \equiv C$.

Proof

Proof: A natural deduction PROOF (or DERIVATION) of a conclusion φ from a set of premises Γ in SD is any sequence of lines ending with φ on a live line where every line in the sequence is either:

(1) a premise in Γ;
(2) a discharged assumption; or
(3) follows from previous lines by the rules for SD.

Provable: An SL sentence φ is PROVABLE (or DERIVABLE) from Γ in SD *iff* there is a natural deduction proof (derivation) of φ from Γ in SD, i.e., $\Gamma \vdash \varphi$.

Equivalent: Two sentences φ and ψ are PROVABLY EQUIVALENT (or INTERDERIVABLE) if and only if both $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Inconsistent: A set of sentences Γ is PROVABLY INCONSISTENT if and only if $\Gamma \vdash \bot$ where $\bot$ is our arbitrarily chosen contradiction, e.g., $A \wedge \neg A$.