7. Using the results of Exercises 1.a and 1.b in Section 6.1E, prove that the following sets of connectives are not truth-functionally complete: {'$\sim$'}, {'&', '$\vee$', '$\supset$', '$\equiv$'}.

8. Prove that the set {'$\sim$', '$\equiv$'} is not truth-functionally complete. *Hint:* Show that the truth-table for any sentence **P** that contains only these two connectives and just two atomic components will have an even number of **T**s and an even number of **F**s in the column under the main connective.

9. Prove that if a truth-functionally complete set of connectives consists of exactly one binary connective, then that connective has either the characteristic truth-table for '$\downarrow$' or the characteristic truth-table for '$|$'. (That is, show that the connective must be either '$\downarrow$' or '$|$', though possibly under a different name.) (*Hint:* In the proofs for Exercises 7 and 8 above, it became apparent that characteristic truth-tables for truth-functionally complete sets of connectives must have certain properties. Show that only two characteristic truth-tables with just four rows have these properties.)

6.3 THE SOUNDNESS OF *SD* AND *SD+*

We now turn to the results announced at the beginning of this chapter. In this section we shall prove that, if a sentence **P** of *SL* is derivable in *SD* from a set Γ of sentences of *SL*, then Γ truth-functionally entails **P**. A natural deduction system for which this result holds is said to be **sound** for sentential logic. In the next section we shall prove the converse—that if a set Γ of sentences of *SL* truth-functionally entails a sentence **P** of *SL*, then **P** is derivable in *SD* from Γ. A natural deduction system for which this second result holds is said to be **complete** for sentential logic. Soundness and completeness are important properties for natural deduction systems. A natural deduction system that is not sound will sometimes lead us from true sentences to false ones, and a natural deduction system that is not complete will not allow us to construct all the derivations that we want to construct. In either case the natural deduction system would not be adequate for the purposes of sentential logic.

Metatheorem 6.3.1 is the *Soundness Metatheorem* for *SD*.

Metatheorem 6.3.1: For any set Γ of sentences of *SL* and any sentence **P** of *SL*, if $\Gamma \vdash \mathbf{P}$ in *SD* then $\Gamma \vDash \mathbf{P}$[3].

Recall that $\Gamma \vDash \mathbf{P}$ if and only if there is no truth-value assignment on which all the members of Γ are true and **P** is false. Metatheorem 6.3.1 therefore says that the derivation rules of *SD* are *truth-preserving*.

Our proof will use mathematical induction to establish that each sentence in a derivation is true if all the open assumptions in whose scope the sentence lies are true. The basis clause will show that this claim is true of the

[3]In what follows we shall abbreviate '$\Gamma \vdash \mathbf{P}$ in *SD*' as '$\Gamma \vdash \mathbf{P}$'.

first sentence in a derivation. The inductive step will show that, if the claim is true for the first **k** sentences in a derivation, then the claim is also true for the ($\mathbf{k}$ + 1)th sentence—that is, each application of another derivation rule in the derivation is truth-preserving. We will then be able to conclude that the last sentence in any derivation, no matter how long the derivation is, is true if all the open assumptions in whose scope the sentence lies are true, which is what Metatheorem 6.3.1 says.

In the course of the proof, we shall use some set-theoretic terminology that we will explain here: Let Γ and Γ' be sets. If every member of Γ is also a member of Γ', then Γ is said to be a **subset** of Γ'. Note that every set is a subset of itself, and the empty set is trivially a subset of every set (because the empty set has no members, it has no members that are *not* members of every set). As an example, the set of sentences

{A, B, C}

has eight subsets: {A, B, C}, {A, B}, {B, C}, {A, C}, {A}, {B}, {C}, and $\varnothing$. If a set Γ is a subset of a set Γ', then Γ' is said to be a **superset** of Γ. Thus {A, B, C} is a superset of each of its eight subsets.

We will also make use of several semantic results that we prove here. First, if **P** is truth-functionally entailed by a set of sentences Γ, then **P** is truth-functionally entailed by every superset of Γ:

6.3.2: If $\Gamma \vDash \mathbf{P}$, then for every superset Γ' of Γ, $\Gamma' \vDash \mathbf{P}$.

Proof: Assume that $\Gamma \vDash \mathbf{P}$ and let Γ' be any superset of Γ. If every member of Γ' is true on some truth-value assignment, then every member of its subset Γ is true on that assignment, and so, because $\Gamma \vDash \mathbf{P}$, **P** is also true on the assignment. Therefore $\Gamma' \vDash \mathbf{P}$.

Second, we have two results that were proved in the exercises for Chapter 3:

6.3.3: If $\Gamma \cup \{\mathbf{Q}\} \vDash \mathbf{R}$, then $\Gamma \vDash \mathbf{Q} \supset \mathbf{R}$ (see Exercise 2.b in Section 3.6E).

6.3.4: If $\Gamma \vDash \mathbf{Q}$ and $\Gamma \vDash {\sim}\,\mathbf{Q}$ for some sentence **Q**, then Γ is truth-functionally inconsistent (see Exercise 3.b in Section 3.6E).

Finally, if a set of sentences is truth-functionally inconsistent, then every sentence **Q** in the set is such that the set consisting of all the *other* sentences in the set truth-functionally entails ~ **Q**:

6.3.5: If $\Gamma \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent, then $\Gamma \vDash {\sim}\,\mathbf{Q}$.

Proof: Assume that $\Gamma \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent. Then there is no truth-value assignment on which every member of $\Gamma \cup \{\mathbf{Q}\}$ is true. Therefore, if every member of Γ is true on some truth-value assignment, **Q** must be false on that assignment, and so ~ **Q** will be true. So $\Gamma \vDash {\sim}\,\mathbf{Q}$.

In our proof that each sentence in a derivation is truth-functionally entailed by the set of the open assumptions in whose scope the sentence lies, we use the following notation: For any derivation, let $\mathbf{P_k}$ be the **k**th sentence in the derivation, and let $\Gamma_{\mathbf{k}}$ be the set of open assumptions in whose scope $\mathbf{P_k}$ lies. Here is our argument by mathematical induction on the position **k** in a derivation:

Basis clause: $\Gamma_1 \vDash \mathbf{P}_1$.

Inductive step: If $\Gamma_{\mathbf{i}} \vDash \mathbf{P_i}$ for every positive integer $\mathbf{i} \leq \mathbf{k}$, then $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Conclusion: For every positive integer **k**, $\Gamma_{\mathbf{k}} \vDash \mathbf{P_k}$.

Proof of basis clause: $\mathbf{P}_1$ is the first sentence in a derivation. Moreover, because every derivation in *SD* begins with one or more assumptions, $\mathbf{P}_1$ is an open assumption that lies in its own scope. (We remind the reader that, by definition, every assumption of a derivation lies within its own scope.) Therefore, the set Γ_1 of open assumptions in whose scope $\mathbf{P}_1$ lies is $\{\mathbf{P}_1\}$. Because $\{\mathbf{P}_1\} \vDash \mathbf{P}_1$, we conclude that the basis clause is true.

Proof of inductive step: Let **k** be an arbitrary positive integer and assume the inductive hypothesis: for every positive integer $\mathbf{i} \leq \mathbf{k}$, $\Gamma_{\mathbf{i}} \vDash \mathbf{P_i}$. We must show that it follows that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$. We shall consider each way in which $\mathbf{P}_{\mathbf{k}+1}$ might be justified and show that our thesis holds no matter what justification is used. We now turn to cases.

Case 1: $\mathbf{P}_{\mathbf{k}+1}$ is an Assumption. Then $\mathbf{P}_{\mathbf{k}+1}$ is a member of $\Gamma_{\mathbf{k}+1}$, the set of open assumptions in whose scope $\mathbf{P}_{\mathbf{k}+1}$ lies. Therefore, if every member of $\Gamma_{\mathbf{k}+1}$ is true, $\mathbf{P}_{\mathbf{k}+1}$, being a member of the set, is true as well. So $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Case 2: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Reiteration. Then $\mathbf{P}_{\mathbf{k}+1}$ occurs earlier in the derivation as sentence $\mathbf{P_i}$ at some position **i**. Moreover every assumption that is open at position **i** must remain open at position **k** + 1—for if even one assumption in whose scope $\mathbf{P_i}$ lies were closed before position **k** + 1, $\mathbf{P_i}$ would not be accessible at position **k** + 1. Therefore, $\Gamma_{\mathbf{i}}$ is a subset of $\Gamma_{\mathbf{k}+1}$. By our inductive hypothesis, $\Gamma_{\mathbf{i}} \vDash \mathbf{P_i}$. Because $\Gamma_{\mathbf{i}}$ is a subset of $\Gamma_{\mathbf{k}+1}$, it follows, by 6.3.2, that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P_i}$. $\mathbf{P}_{\mathbf{k}+1}$ is the same sentence as $\mathbf{P_i}$, so $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Case 3: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Conjunction Introduction. The conjuncts of $\mathbf{P}_{\mathbf{k}+1}$ occur earlier in the derivation, say at positions **h** and **j**, both of which are accessible from position **k** + 1:

h	**Q**	
j	**R**	
k + 1	**Q** & **R** (= $\mathbf{P}_{\mathbf{k}+1}$)	**h**, **j** &I

(There may be open assumptions between positions **h** and **j** and between positions **j** and **k** + 1. Moreover it may be that **R** occurs earlier in the derivation than **Q** does—the order is immaterial.) By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q}$ and $\Gamma_{\mathbf{j}} \vDash \mathbf{R}$. Moreover, because both **Q** and **R** are accessible at position **k** + 1, $\Gamma_{\mathbf{h}}$ and $\Gamma_{\mathbf{j}}$ are both subsets of $\Gamma_{\mathbf{k}+1}$ and so, by 6.3.2, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}$ and $\Gamma_{\mathbf{k}+1} \vDash \mathbf{R}$. But whenever both **Q** and **R** are true, $\mathbf{P}_{\mathbf{k}+1}$, which is **Q** & **R**, is also true. So $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$ as well.

Case 4: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Conjunction Elimination:

h	**Q** & $\mathbf{P}_{\mathbf{k}+1}$			**h**	$\mathbf{P}_{\mathbf{k}+1}$ & **Q**	
			or			
k + 1	$\mathbf{P}_{\mathbf{k}+1}$	**h** &E		**k** + 1	$\mathbf{P}_{\mathbf{k}+1}$	**h** &E

By the inductive hypothesis, $\Gamma_{\mathbf{h}}$ truth-functionally entails the conjunction at position **h**. And whenever the conjunction is true, both conjuncts must be true. So $\Gamma_{\mathbf{h}} \vDash \mathbf{P}_{\mathbf{k}+1}$. Moreover, $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$, because the conjunction at position **h** is accessible at position **k** + 1. It follows, by 6.3.2, that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Case 5: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Disjunction Introduction:

h	**Q**			**h**	**R**	
			or			
k + 1	$\mathbf{Q} \vee \mathbf{R}$ 3(= $\mathbf{P}_{\mathbf{k}+1}$)	**h** $\vee$I		**k** + 1	$\mathbf{Q} \vee \mathbf{R}$ (= $\mathbf{P}_{\mathbf{k}+1}$)	**h** $\vee$I

By the inductive hypothesis, $\Gamma_{\mathbf{h}}$ truth-functionally entails the sentence at position **h**. That sentence is one of the disjuncts of $\mathbf{Q} \vee \mathbf{R}$, so whenever it is true, so is $\mathbf{Q} \vee \mathbf{R}$. Thus $\Gamma_{\mathbf{h}} \vDash \mathbf{P}_{\mathbf{k}+1}$. $\Gamma_{\mathbf{h}}$ must be a subset of $\Gamma_{\mathbf{k}+1}$ if the sentence at position **h** is accessible at position **k** + 1, and so, by 6.3.2, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Case 6: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Conditional Elimination:

h	**Q**	
j	$\mathbf{Q} \supset \mathbf{P}_{\mathbf{k}+1}$	
k + 1	$\mathbf{P}_{\mathbf{k}+1}$	**h**, **j** $\supset$E

By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q}$ and $\Gamma_{\mathbf{j}} \vDash \mathbf{Q} \supset \mathbf{P}_{\mathbf{k}+1}$. Both $\Gamma_{\mathbf{h}}$ and $\Gamma_{\mathbf{j}}$ must be subsets of $\Gamma_{\mathbf{k}+1}$ because the sentences at positions **h** and **j** are accessible at position **k** + 1. By 6.3.2, then, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}$ and $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q} \supset \mathbf{P}_{\mathbf{k}+1}$. Because $\mathbf{P}_{\mathbf{k}+1}$ must be true whenever both **Q** and $\mathbf{Q} \supset \mathbf{P}_{\mathbf{k}+1}$ are true, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$ as well.

Case 7: $\mathbf{P_{k+1}}$ is justified by Biconditional Elimination:

$$\begin{array}{r|ll} \mathbf{h} & \mathbf{Q} & \\ \mathbf{j} & \mathbf{Q} \equiv \mathbf{P_{k+1}} & \\ & & \\ \mathbf{k}+1 & \mathbf{P_{k+1}} & \mathbf{h}, \mathbf{j} \ {\equiv}\mathrm{E} \end{array} \qquad \text{or} \qquad \begin{array}{r|ll} \mathbf{h} & \mathbf{Q} & \\ \mathbf{j} & \mathbf{P_{k+1}} \equiv \mathbf{Q} & \\ & & \\ \mathbf{k}+1 & \mathbf{P_{k+1}} & \mathbf{h}, \mathbf{j} \ {\equiv}\mathrm{E} \end{array}$$

By the inductive hypothesis, $\Gamma_\mathbf{h} \vDash \mathbf{Q}$ and $\Gamma_\mathbf{j}$ truth-functionally entails the biconditional at position **j**. $\Gamma_\mathbf{h}$ and $\Gamma_\mathbf{j}$ must be subsets of $\Gamma_{\mathbf{k}+1}$ because the sentences at positions **h** and **j** are accessible at position **k** + 1. By 6.3.2, then, $\Gamma_{\mathbf{k}+1}$ truth-functionally entails both **Q** and the biconditional at position **j**. Because the sentence $\mathbf{P_{k+1}}$ must be true whenever both **Q** and the biconditional at position **j** are true, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P_{k+1}}$ as well.

Case 8: $\mathbf{P_{k+1}}$ is justified by Conditional Introduction:

$$\begin{array}{r|l|ll} \mathbf{h} & & \mathbf{Q} & \\ \cline{3-3} & & & \\ \mathbf{j} & & \mathbf{R} & \\ & & & \\ \mathbf{k}+1 & \multicolumn{2}{l}{\mathbf{Q} \supset \mathbf{R} \ (= \mathbf{P_{k+1}})} & \mathbf{h}\text{–}\mathbf{j} \ {\supset}\mathrm{I} \end{array}$$

By the inductive hypothesis, $\Gamma_\mathbf{j} \vDash \mathbf{R}$. Because the subderivation in which **R** is derived from **Q** is accessible at position **k** + 1, every assumption that is open at position **j** is open at position **k** + 1, except for the assumption **Q** that begins the subderivation. So the set of open assumptions $\Gamma_\mathbf{j}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$. Because $\Gamma_\mathbf{j} \vDash \mathbf{R}$, it follows, by 6.3.2, that $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{R}$. And from this it follows, by 6.3.3, that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q} \supset \mathbf{R}$.

Case 9: $\mathbf{P_{k+1}}$ is justified by Negation Introduction:

$$\begin{array}{r|l|ll} \mathbf{h} & & \mathbf{Q} & \\ \cline{3-3} & & & \\ \mathbf{j} & & \mathbf{R} & \\ \mathbf{m} & & {\sim}\,\mathbf{R} & \\ & & & \\ \mathbf{k}+1 & \multicolumn{2}{l}{{\sim}\,\mathbf{Q} \ (= \mathbf{P_{k+1}})} & \mathbf{h}\text{–}\mathbf{m} \ {\sim}\,\mathrm{I} \end{array}$$

By the inductive hypothesis, $\Gamma_\mathbf{j} \vDash \mathbf{R}$ and $\Gamma_\mathbf{m} \vDash {\sim}\,\mathbf{R}$. Because the subderivation that derives **R** from **Q** is accessible at position **k** + 1, every assumption that is open at position **j** is open at position **k** + 1 except for the assumption **Q** that begins the subderivation. That is, the set of open assumptions $\Gamma_\mathbf{j}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$. By similar reasoning $\Gamma_\mathbf{m}$ must be a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$. Therefore, by 6.3.2,

$\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{R}$ and $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \sim \mathbf{R}$. From this it follows, by 6.3.4, that $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent and then, by 6.3.5, that $\Gamma_{\mathbf{k}+1} \vDash \sim \mathbf{Q}$.

Case 10: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Negation Elimination. See Exercise 3.

Case 11: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Disjunction Elimination:

$$
\begin{array}{r|l l}
\mathbf{h} & \mathbf{Q} \vee \mathbf{R} & \\
\mathbf{j} & \quad \mathbf{Q} & \\
 & \quad \vdots & \\
\mathbf{m} & \quad \mathbf{P}_{\mathbf{k}+1} & \\
\mathbf{n} & \quad \mathbf{R} & \\
 & \quad \vdots & \\
\mathbf{p} & \quad \mathbf{P}_{\mathbf{k}+1} & \\
\mathbf{k}+1 & \mathbf{P}_{\mathbf{k}+1} & \mathbf{h}, \mathbf{j}\text{–}\mathbf{m}, \mathbf{n}\text{–}\mathbf{p} \ \vee\mathrm{E}
\end{array}
$$

By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q} \vee \mathbf{R}$, $\Gamma_{\mathbf{m}} \vDash \mathbf{P}_{\mathbf{k}+1}$, and $\Gamma_{\mathbf{p}} \vDash \mathbf{P}_{\mathbf{k}+1}$. Because the two subderivations are accessible at position $\mathbf{k} + 1$, the open assumptions $\Gamma_{\mathbf{m}}$ form a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$ and the open assumptions $\Gamma_{\mathbf{p}}$ form a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\}$. By 6.3.2, then, $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{P}_{\mathbf{k}+1}$ and $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\} \vDash \mathbf{P}_{\mathbf{k}+1}$. Moreover, because $\mathbf{Q} \vee \mathbf{R}$ at position $\mathbf{h}$ is accessible at position $\mathbf{k} + 1$, $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k}+1}$. So, because $\Gamma_{\mathbf{h}} \vDash \mathbf{Q} \vee \mathbf{R}$, it follows, by 6.3.2, that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q} \vee \mathbf{R}$. Now consider any truth-value assignment on which every member of $\Gamma_{\mathbf{k}+1}$ is true. Because $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q} \vee \mathbf{R}$, $\mathbf{Q} \vee \mathbf{R}$ is also true on this assignment. So either $\mathbf{Q}$ or $\mathbf{R}$ is true. If $\mathbf{Q}$ is true, then every member of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$ is true and hence $\mathbf{P}_{\mathbf{k}+1}$ is true as well because $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{P}_{\mathbf{k}+1}$. Similarly, if $\mathbf{R}$ is true, then every member of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\}$ is true, and hence $\mathbf{P}_{\mathbf{k}+1}$ is true as well because $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\} \vDash \mathbf{P}_{\mathbf{k}+1}$. Either way, it follows that $\mathbf{P}_{\mathbf{k}+1}$ must be true on any truth-value assignment on which every member of $\Gamma_{\mathbf{k}+1}$ is true. So $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

Case 12: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Biconditional Introduction:

$$
\begin{array}{r|l l}
\mathbf{h} & \quad \mathbf{Q} & \\
 & \quad \vdots & \\
\mathbf{j} & \quad \mathbf{R} & \\
\mathbf{m} & \quad \mathbf{R} & \\
 & \quad \vdots & \\
\mathbf{n} & \quad \mathbf{Q} & \\
\mathbf{k}+1 & \mathbf{Q} \equiv \mathbf{R} \ (= \mathbf{P}_{\mathbf{k}+1}) & \mathbf{h}\text{–}\mathbf{j}, \mathbf{m}\text{–}\mathbf{n} \ \equiv\mathrm{I}
\end{array}
$$

By the inductive hypothesis, $\Gamma_{\mathbf{j}} \vDash \mathbf{R}$ and $\Gamma_{\mathbf{n}} \vDash \mathbf{Q}$. Because the two subderivations are accessible at position $\mathbf{k} + 1$, $\Gamma_{\mathbf{j}}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\}$ and $\Gamma_{\mathbf{n}}$ is a subset of $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\}$. By 6.3.2, then, $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{R}$ and $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\} \vDash \mathbf{Q}$. Now consider any truth-value assignment on which every member of

$\Gamma_{\mathbf{k}+1}$ is true. If **R** is also true on that assignment, then so is **Q** because $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{R}\} \vDash \mathbf{Q}$. If **R** is false on that assignment, then **Q** must also be false because if **Q** were true, **R** would have to be true as well since $\Gamma_{\mathbf{k}+1} \cup \{\mathbf{Q}\} \vDash \mathbf{R}$. Either way, **Q** and **R** have the same truth-value, and so $\mathbf{Q} \equiv \mathbf{R}$ is true on every truth-value assignment on which every member of $\Gamma_{\mathbf{k}+1}$ is true. So $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$.

This completes the proof of the inductive step; we have considered every way in which the sentence at position $\mathbf{k} + 1$ of a derivation might be justified and have shown that in each case $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}_{\mathbf{k}+1}$ if the same is true of all earlier positions in the derivation. We have therefore established the conclusion of the mathematical induction. The sentence at any position in a derivation is truth-functionally entailed by the set of open assumptions in whose scope it lies. And this establishes the soundness metatheorem for *SD*: If $\Gamma \vdash \mathbf{P}$ in *SD*, then $\Gamma \vDash \mathbf{P}$. It follows from Metatheorem 6.3.1 that every argument of *SL* that is valid in *SD* is truth-functionally valid, every sentence of *SL* that is a theorem in *SD* is truth-functionally true, every pair of sentences of *SL* that are equivalent in *SD* are truth-functionally equivalent, and every set of sentences of *SL* that is inconsistent in *SD* is truth-functionally inconsistent. (see Exercise 14 in Exercise set 5.3E).

6.3**E** EXERCISES

1. List all the subsets of each of the following sets:

a. $\{A \supset B, C \supset D\}$

b. $\{C \vee {\sim} D, {\sim} D \vee C, C \vee C\}$

c. $\{(B \& A) \equiv K\}$

d. $\varnothing$

2. Of which of the following sets is $\{A \supset B, C \& D, D \supset A\}$ a superset?

a. $\{A \supset B\}$

b. $\{D \supset A, A \supset B\}$

c. $\{A \supset D, C \& D\}$

d. $\varnothing$

e. $\{C \& D, D \supset A, A \supset B\}$

***3.** Prove Case 10 of the inductive step in the proof of Metatheorem 6.3.1.

4.

a. Suppose that system *SD** is just like *SD* except that it also contains a new rule of inference:

<u>*Negated Biconditional Introduction* (~≡I)</u>

| **P**
|
| ~ **Q**
|
| ~ (**P** ≡ **Q**)

Prove that system *SD** is a sound system for sentential logic; that is, prove that if $\Gamma \vdash \mathbf{P}$ in *SD** then $\Gamma \vDash \mathbf{P}$. (You may use Metatheorem 6.3.1.)

*b. Suppose that system *SD** is just like *SD* except that it also contains a new rule of inference:

Backward Conditional Introduction (B⊃I)

$$\begin{array}{l|l} & \begin{array}{|l} \sim \mathbf{Q} \\ \hline \\ \sim \mathbf{P} \end{array} \\ & \mathbf{P} \supset \mathbf{Q} \end{array}$$

Prove that system *SD** is sound for sentential logic.

c. Suppose that system *SD** is just like *SD* except that it also contains a new rule of inference:

Crazy Disjunction Elimination (C∨E)

$$\begin{array}{l|l} & \mathbf{P} \vee \mathbf{Q} \\ \\ \triangleright & \mathbf{P} \end{array} \qquad \text{or} \qquad \begin{array}{l|l} & \mathbf{P} \vee \mathbf{Q} \\ \\ \triangleright & \mathbf{Q} \end{array}$$

Prove that *SD** is not a sound system for sentential logic.

*d. Suppose that system *SD** is just like *SD* except that it also contains a new rule of inference:

Crazy Conditional Introduction (C⊃I)

$$\begin{array}{l|l} & \begin{array}{|l} \sim \mathbf{P} \\ \hline \\ \mathbf{Q} \end{array} \\ & \mathbf{P} \supset \mathbf{Q} \end{array}$$

Prove that *SD** is not a sound system for sentential logic.

e. Suppose that the rules of a system *SD** form a subset of the rules of *SD*. Is *SD** a sound system for sentential logic? Explain.

5. Suppose that we changed the characteristic truth-table for '&' to

P	**Q**	**P & Q**
T	**T**	**T**
T	**F**	**T**
F	**T**	**F**
F	**F**	**F**

while the characteristic truth-tables for the other sentential connectives remained the same. Would *SD* still be a sound system for sentential logic? Explain.

6. Using Metatheorem 6.3.1 and Exercise 1.e in Section 6.1E, prove that *SD+* is sound for sentential logic.

We proved in the last section that *SD* is sound, and so every derivation in *SD* is semantically acceptable. This fact alone does not establish that *SD* is an adequate natural deduction system for sentential logic. To establish that *SD* is such a system we must also show that if a set of sentences of *SL* truth-functionally entails a sentence **P** of *SL*, then there is a derivation in *SD* of **P** from that set of sentences. If there is even one case of truth-functional entailment for which a derivation cannot be constructed in *SD*, then *SD* is not adequate to sentential logic. Our final metatheorem assures us that we can derive all that we want to derive in *SD*; it is called the *Completeness Metatheorem:*

Metatheorem 6.4.1: For every sentence **P** of *SL* and every set Γ of sentences of *SL*, if $\Gamma \vDash \mathbf{P}$ then $\Gamma \vdash \mathbf{P}$ in *SD*.

That is, if a set Γ truth-functionally entails a sentence **P**, then **P** can be derived from Γ in *SD*. It follows from this metatheorem that every argument of *SL* that is truth-functionally valid is valid in *SD*, that every sentence of *SL* that is truth-functionally true is a theorem in *SD* (see Exercise 20 in Section 5.4E), that every pair of sentences of *SL* that are truth-functionally equivalent are equivalent in *SD*, and that every set of sentences of *SL* that is truth-functionally inconsistent is inconsistent in *SD*. A system for which Metatheorem 6.4.1 holds is said to be *complete* for sentential logic.

There are several well-known ways to prove completeness theorems. It may seem that the obvious approach is to show, given that $\Gamma \vDash \mathbf{P}$, how to construct a derivation of **P** from Γ. There are such proofs and they are termed **constructive** proofs precisely because they not only prove that there is a derivation of **P** from Γ but also provide instructions for constructing such a derivation. We will pursue a different course. The proof we will offer will establish that if $\Gamma \vDash \mathbf{P}$ then there is a derivation of **P** from Γ but *it will not show us how to construct that derivation.*[4]

The bulk of the work in our proof of 6.4.1 will be in establishing what may at first seem to be an unrelated result:

6.4.2: For any set Γ of sentences of *SL*, if Γ is consistent in *SD* then Γ is truth-functionally consistent.

We begin by proving that Metatheorem 6.4.1 does follow from 6.4.2. To do this, we will use the following result:

6.4.3: If $\Gamma \vDash \mathbf{P}$ then $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent.

Proof: Assume $\Gamma \vDash \mathbf{P}$. Then there is no truth-value assignment on which every member of Γ is true and **P** is false. Therefore, there is no truth-

[4]The proof we will present, while complex, is actually simpler than are constructive proofs. Moreover, our proof of the completeness of *SD* will serve as a model for the completeness proof for *PD* in Chapter 11, for which a constructive proof is not possible. The method that we use to prove completeness is due to Leon Henkin, "The Completeness of the First-Order Functional Calculus," *Journal of Symbolic Logic,* 14 (1949), pp. 159–166.

value assignment on which every member of Γ is true and $\sim$ **P** is also true. So $\Gamma \cup \{\sim$ **P**$\}$ is truth-functionally inconsistent.

Here's how metatheorem 6.4.1 follows from 6.4.2:

- Assume that $\Gamma \vDash$ **P**.
- Then, by 6.4.3, if follows that $\Gamma \cup \{\sim$ **P**$\}$ is truth-functionally inconsistent.
- Since 6.4.2 (which we have not yet proven) tells us that a set of sentences of *SL* that is consistent in *SD* is also truth-functionally consistent, it follows that $\Gamma \cup \{\sim$ **P**$\}$ is *not* consistent in *SD*.
- And if $\Gamma \cup \{\sim$ **P**$\}$ is *not* consistent in *SD*, it follows from the following result that $\Gamma \vdash$ **P**:

  6.4.4: If $\Gamma \cup \{\sim$ **P**$\}$ is inconsistent in *SD* then $\Gamma \vdash$ **P**.

  Proof: Assume that $\Gamma \cup \{\sim$ **P**$\}$ is inconsistent in *SD*. Then by definition, there is a derivation **D** of some sentence **Q** and its negation $\sim$ **Q** from $\Gamma \cup \{\sim$ **P**$\}$. But then we can show that there is also a derivation **D′** of **P** from Γ.

  - The primary assumptions of derivation **D′** are the same as the primary assumptions of derivation **D**, *except that* they do not include $\sim$ **P**.
  - Rather, $\sim$ **P** is assumed *as an auxiliary assumption* in **D′** immediately after the primary assumptions.
  - Once we add this auxiliarly assumption, all of the primary assumptions in **D** are open and accessible assumptions in **D′**.
  - **Q** and $\sim$ **Q** are then derived in **D′** in the same way they were derived in **D.**
  - Then Negation Elimination is used to close the auxiliary assumption $\sim$ **P** and derive **P**.
  - Since **P** falls only within the scope of the primary assumptions in **D′**, which are the members of Γ, this establishes that $\Gamma \vdash$ **P**.

All that remains to be done to prove Metatheorem 6.4.1, then, is proving result 6.4.2.

As noted earlier, our proof of 6.4.2 is quite complex. We start by outlining the structure of that proof, proving the simpler parts of it as we go but leaving proof of the more complex section until we have completed the outline. In proving metatheorem 6.4.2 our overall strategy will be to show that if a set Γ of sentences of *SL* is consistent in *SD* then we can construct a truth-value assignment on which every member of Γ is true, thereby showing that Γ is truth-functionally consistent.

We shall construct the truth-value assignment in two steps. First, we shall form a superset of Γ (a set that includes all the members of Γ and possibly other sentences) that is *maximally consistent in SD*. A maximally consistent set is, intuitively, a consistent set that contains as many sentences as it can without being inconsistent in *SD*:

> A set Γ of sentences of *SL* is **maximally consistent in *SD*** if and only if Γ is consistent in *SD* and, for every sentence **P** of *SL* that is not a member of Γ, $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*.

If a set is maximally consistent in *SD*, then if we add to the set any sentence that is not already a member, it will be possible to derive some sentence and its negation from the augmented set.

Having constructed a maximally consistent superset of Γ, we then construct a model for the maximally consistent superset, that is, a truth-value assignment on which every member of the maximally consistent superset is true. We construct the model for a superset of Γ that is maximally consistent in *SD*, rather than simply for the original set Γ, because there is a straightforward way to construct models for maximally consistent sets. Of course, because every member of Γ will be in the maximally consistent superset, it will follow that every member of Γ is true on the model that we have constructed and therefore that Γ is truth-functionally consistent.

We now need to fill in the details of our proof. We first need to establish that if Γ is consistent in *SD* then it is a subset of a set Γ that is maximally consistent in *SD*:

> **6.4.5** (The *Maximal Consistency Lemma*): If Γ is a set of sentences of *SL* that is consistent in *SD*, then Γ is a subset of at least one set of sentences that is maximally consistent in *SD*.

In proving the Maximal Consistency Lemma (6.4.5), we shall make use of the fact that the sentences of *SL* can be *enumerated,* that is, placed in a definite order in one-to-one correspondence with the positive integers so that each sentence of *SL* is associated with exactly one positive integer. Here is one method of enumerating the sentences of *SL*. First, we associate with each symbol of *SL* the two-digit numeral occurring to its right:

Symbol	*Numeral*	*Symbol*	*Numeral*
~	10	A	30
$\vee$	11	B	31
&	12	C	32
$\supset$	13	D	33
$\equiv$	14	E	34
(	15	F	35
)	16	G	36
0	20	H	37
1	21	I	38
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	29	Z	55

(The ellipses mean that the next two-digit numeral is assigned to the next digit or letter of the alphabet.) Next we associate with each sentence of *SL*, atomic or compound, the integer designated by the numeral that consists of the numerals associated with the symbols in the sentence, in the order in which those symbols occur. For example, the integers associated with the sentences

$$(A \vee C_2) \qquad {\sim}{\sim}(A \supset (B \ \& \ {\sim} C))$$

are, respectively,

153011322216   10101530131531121032161 6

It is obvious that each sentence of *SL* will thus have a distinct integer associated with it. Finally we enumerate all the sentences of *SL* in the order of their associated integers: The first sentence in the enumeration is the sentence with the smallest associated integer, the second sentence is the one with the next smallest associated integer, and so on. In effect, we have imposed an alphabetical order on the sentences of *SL* so that we may freely talk of the first sentence of *SL* (which turns out to be 'A'—because only atomic sentences will have two-digit associated numbers, and the number for 'A' is the smallest of these), the second sentence of *SL* (which turns out to be 'B'), and so on.

Starting with a set Γ of sentences that is consistent in *SD* (as provided for in the antecedent of the Maximal Consistency Lemma), we use our enumeration to construct a superset of Γ that is maximally consistent in *SD*. The construction considers in sequence each sentence in the enumeration we have just described and adds the sentence to the set if and only if the resulting set is consistent in *SD*. In the end the construction will have added as many sentences as can be added to the original set without producing a set that is inconsistent in *SD*. More formally, as the construction goes through the sentences of *SL*, deciding whether to add each sentence, it produces an infinite sequence Γ_1, Γ_2, Γ_3, . . . of sets of sentences of *SL*:

1. Γ_1 is the original set Γ.
2. If $\mathbf{P_i}$ is the **i**th sentence in the enumeration, then $\Gamma_{\mathbf{i}+1}$ is $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$ if $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$ is consistent in *SD*; otherwise $\Gamma_{\mathbf{i}+1}$ is $\Gamma_{\mathbf{i}}$.

As an example, if $\Gamma_{\mathbf{i}}$ is $\{{\sim} B, {\sim} C \vee {\sim} B\}$ and $\mathbf{P_i}$ is 'A', then $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$, which is $\{{\sim} B, {\sim} C \vee {\sim} B, A\}$, is consistent in *SD*. In this case $\Gamma_{\mathbf{i}+1}$ will be the expanded set $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$. If $\Gamma_{\mathbf{i}}$ is $\{A, {\sim} B, {\sim} C \vee {\sim} B\}$ and $\mathbf{P_i}$ is 'B', then $\Gamma_{\mathbf{i}} \cup \{\mathbf{P_i}\}$, which is $\{A, {\sim} B, {\sim} C \vee {\sim} B, B\}$, is inconsistent in *SD* (this is readily verified). In this case $\mathbf{P_i}$ is not added and the set $\Gamma_{\mathbf{i}+1}$ is defined to be Γ_{i} , that is, the set $\{A, {\sim} B, {\sim} C \vee {\sim} B\}$.

Because we have an infinite sequence of sets, we cannot take the last member of the series as the maximally consistent set desired—because there is no last member! Instead, we form a set Γ^* that is the union of all the sets in the series: Γ^* is defined to contain every sentence that is a member of at least one set in the series and no other sentences. Γ^* is a superset of Γ because it follows from the definition of Γ^* that every sentence in Γ_1 (as well as Γ_2, Γ_3, . . .) is a member of Γ^*, and Γ_1 is the original set Γ.

Having formed the set Γ^*, it remains to be proved that Γ^* is consistent in *SD* and that it is *maximally* consistent in *SD*. To prove the first claim, we first prove that every set in the sequence Γ_1, Γ_2, Γ_3, . . . is consistent in *SD*. This is easily established by mathematical induction:

> *Basis clause:* The first member of the sequence, Γ_1, is consistent in *SD*.
>
> **Proof:** Γ_1 is defined to be the original set Γ, which is consistent in *SD*.
>
> *Inductive step:* If every set in the sequence prior to $\Gamma_{\mathbf{k}+1}$ is consistent in *SD*, then $\Gamma_{\mathbf{k}+1}$ is consistent in *SD*.
>
> **Proof:** $\Gamma_{\mathbf{k}+1}$ was defined to be $\Gamma_{\mathbf{k}} \cup \{\mathbf{P_k}\}$ if the latter set is consistent in *SD* and to be $\Gamma_{\mathbf{k}}$ otherwise. In the first case $\Gamma_{\mathbf{k}+1}$ is obviously consistent in *SD*. In the second case $\Gamma_{\mathbf{k}+1}$ is consistent because, by the inductive hypothesis, $\Gamma_{\mathbf{k}}$ is consistent in *SD* and $\Gamma_{\mathbf{k}+1}$ just is $\Gamma_{\mathbf{k}}$.
>
> ---
>
> *Conclusion:* Every member of the series Γ_1, Γ_2, Γ_3, . . . is consistent in *SD*.

We must next show that Γ^* is also consistent in *SD*. We will do so by using the following easily established result:

> **6.4.6:** If a set Γ of sentences of *SL* is inconsistent in *SD*, then some finite subset of Γ is also inconsistent in *SD* (see Exercise 6.4.2).

Assume that Γ^* is inconsistent in *SD*. It then follows from 6.4.6 that there is a finite subset of Γ^*, call it Γ', that is inconsistent in *SD*. Γ' must be nonempty, for the empty set is consistent in *SD* (see Exercise 6.4.3). Moreover, because Γ' is finite, there is a sentence in Γ' that comes after all the other members of Γ' in our enumeration—call this sentence $\mathbf{P_j}$. (That is, any other member of Γ' is $\mathbf{P_h}$ for some $\mathbf{h} < \mathbf{j}$.) Then every member of Γ' is a member of $\Gamma_{\mathbf{j}+1}$, by the way we constructed the series Γ_1, Γ_2, Γ_3, ... (We have constructed the sets in such a way that if a sentence that is the **i**th sentence in our enumeration is a member of *any* set in the sequence—and hence of Γ^*—it must be in the set $\Gamma_{\mathbf{i}+1}$ and every set thereafter.) But if Γ' is inconsistent in *SD*, and every member of Γ' is a member of $\Gamma_{\mathbf{j}+1}$, then $\Gamma_{\mathbf{j}+1}$ is inconsistent in *SD* as well, by 6.4.7:

> **6.4.7:** If Γ is inconsistent in *SD*, then every superset of Γ is inconsistent in *SD*.
>
> **Proof:** Assume that Γ is inconsistent in *SD*. Then for some sentence **P** there is a derivation of **P** in which all the primary assumptions are members of Γ, and also a derivation of ~ **P** in which all the primary assumptions are members of Γ. The primary assumptions of both derivations are members of every superset of Γ, so **P** and ~ **P** are both derivable from every superset of Γ. Therefore every superset of Γ is inconsistent in *SD*.

But we have already proved by mathematical induction that every set in the infinite sequence is consistent in *SD*. So $\Gamma_{\mathbf{j}+1}$ *cannot* be inconsistent in *SD*, and our supposition that led to this conclusion is wrong—we may conclude that Γ^* is consistent in *SD*.

Having established that Γ^* is consistent in *SD*, it remains to be shown that it is *maximally* consistent in *SD*. Suppose that Γ^* is not maximally consistent in *SD*. Then there is at least one sentence $\mathbf{P_k}$ of *SL* that is not a member of Γ^* and is such that $\Gamma^* \cup \{\mathbf{P_k}\}$ is consistent in *SD*. We showed, in 6.4.7, that every superset of a set that is inconsistent in *SD* is itself inconsistent, so every subset of a set that is *consistent* in *SD* must itself be consistent in *SD*. In particular, the subset $\Gamma_{\mathbf{k}} \cup \{\mathbf{P_k}\}$ of $\Gamma^* \cup \{\mathbf{P_k}\}$ must be consistent in *SD*. But then, by step 2 of the construction of the sequence of sets, $\Gamma_{\mathbf{k+1}}$ is defined to be $\Gamma_{\mathbf{k}} \cup \{\mathbf{P_k}\}$—$\mathbf{P_k}$ is a member of $\Gamma_{\mathbf{k+1}}$. $\mathbf{P_k}$ is therefore a member of Γ^*, contradicting our supposition that it is not a member of Γ^*. Therefore Γ^* must be maximally consistent in *SD*—every sentence that can be consistently added to Γ^* is already a member of Γ^*. This and the result of the previous paragraph establish the Maximal Consistency Lemma (6.4.5); we have shown that, given any set of sentences that is consistent in *SD*, we can construct a superset that is maximally consistent in *SD*.

Finally, we will show that we can construct a truth-value assignment for every set that is maximally consistent in *SD* such that every member of that set is true on that truth-value assignment. From this we will have the following:

> **6.4.8** (the *Consistency Lemma*): Every set of sentences of *SL* that is maximally consistent in *SD* is truth-functionally consistent.

In establishing the Consistency Lemma, we shall appeal to the following important facts about sets that are maximally consistent in *SD*:

> **6.4.9:** If $\Gamma \vdash \mathbf{P}$ and Γ^* is a maximally consistent superset of Γ, then $\mathbf{P}$ is a member of Γ^*.
>
> **Proof:** Assume that $\Gamma \vdash \mathbf{P}$ and let Γ^* be a maximally consistent superset of Γ. By the definition of derivability in *SD*, $\Gamma^* \vdash \mathbf{P}$ as well. Now suppose, contrary to what we wish to prove, that $\mathbf{P}$ is *not* a member of Γ^*. Then, by the definition of maximal consistency, $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in *SD*. Therefore by 6.4.4 (if $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*, then $\Gamma \vdash \sim \mathbf{P}$), it follows that $\Gamma^* \vdash \sim \mathbf{P}$. But then, because both $\mathbf{P}$ and $\sim \mathbf{P}$ are derivable in *SD* from Γ^*, it follows that Γ^* is inconsistent in *SD*. But this is impossible if Γ^* is maximally consistent in *SD*. We conclude that our supposition about $\mathbf{P}$, that it is not a member of Γ^*, is wrong—that is, if $\Gamma^* \vdash \mathbf{P}$, then $\mathbf{P}$ *is* a member of Γ^*.

In what follows, we will use the standard notation

$$\mathbf{P} \in \Gamma$$

to mean

> $\mathbf{P}$ is a member of Γ

and the standard notation

$$\mathbf{P} \notin \Gamma$$

to mean

$\mathbf{P}$ is not a member of Γ.

The next result concerns the composition of the membership of any set that is maximally consistent in *SD*:

6.4.11: If Γ^* is maximally consistent in *SD* and $\mathbf{P}$ and $\mathbf{Q}$ are sentences of *SL*, then:

a. $\sim \mathbf{P} \in \Gamma^*$ if and only if $\mathbf{P} \notin \Gamma^*$.
b. $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$ if and only if both $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
c. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
d. $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
e. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.

Proof of (a): Assume that $\sim \mathbf{P} \in \Gamma^*$. Then $\mathbf{P} \notin \Gamma^*$ for, if it were a member, then Γ^* would have a finite subset that is inconsistent in *SD*, namely, $\{\mathbf{P}, \sim \mathbf{P}\}$, and according to 6.4.7 this is impossible if Γ^* is consistent in *SD*. Now assume that $\mathbf{P} \notin \Gamma^*$. Then, by the definition of maximal consistency in *SD*, $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in *SD*. So, by reasoning similar to that used in proving 6.4.9, some finite subset Γ' of Γ^* is such that $\Gamma' \cup \{\mathbf{P}\}$ is inconsistent in *SD*, and therefore such that $\Gamma' \cup \{\sim \sim \mathbf{P}\}$ is inconsistent in *SD*. So $\Gamma' \vdash \sim \mathbf{P}$, by 6.4.4. It follows, by 6.4.9, that $\sim \mathbf{P} \in \Gamma^*$.

Proof of (b): Assume that $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$. Then $\{\mathbf{P} \,\&\, \mathbf{Q}\}$ is a subset of Γ^*. Because $\{\mathbf{P} \,\&\, \mathbf{Q}\} \vdash \mathbf{P}$ and $\{\mathbf{P} \,\&\, \mathbf{Q}\} \vdash \mathbf{Q}$ (both by Conjunction Elimination), it follows, by 6.4.9, that $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$. Now suppose that $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$. Then $\{\mathbf{P}, \mathbf{Q}\}$ is a subset of Γ^* and, because $\{\mathbf{P}, \mathbf{Q}\} \vdash \mathbf{P} \,\&\, \mathbf{Q}$ (by Conjunction Introduction), it follows, by 6.4.9, that $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$.

Proof of (c): See Exercise 6.4.5.

Proof of (d): Assume that $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$. If $\mathbf{P} \notin \Gamma^*$, then it follows trivially that either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. If $\mathbf{P} \in \Gamma^*$, then $\{\mathbf{P}, \mathbf{P} \supset \mathbf{Q}\}$ is a subset of Γ^*. Because $\{\mathbf{P}, \mathbf{P} \supset \mathbf{Q}\} \vdash \mathbf{Q}$ (by Conditional Elimination), it follows, by 6.4.9, that $\mathbf{Q} \in \Gamma^*$. So, if $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$, then either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. Now assume that either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. In the former case, by (a), $\sim \mathbf{P} \in \Gamma^*$. So either $\{\sim \mathbf{P}\}$ is a subset of Γ^* or $\{\mathbf{Q}\}$ is a subset of Γ^*. $\mathbf{P} \supset \mathbf{Q}$ is derivable from both subsets:

1	$\sim \mathbf{P}$	Assumption
2	$\mathbf{P}$	A / $\sim$E
3	$\sim \mathbf{Q}$	A / $\sim$E
4	$\mathbf{P}$	2 R
5	$\sim \mathbf{P}$	1 R
6	$\mathbf{Q}$	3–5 $\sim$E
7	$\mathbf{P} \supset \mathbf{Q}$	2–6 $\supset$I

1	$\mathbf{Q}$	Assumption
2	$\mathbf{P}$	A / $\supset$I
3	$\mathbf{Q}$	1 R
4	$\mathbf{P} \supset \mathbf{Q}$	2–3 $\supset$I

Either way, there is a finite subset of Γ^* from which $\mathbf{P} \supset \mathbf{Q}$ is derivable; so, by 6.4.9, it follows that $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$.

Proof of (e): See Exercise 6.4.5.

Turning now to the Consistency Lemma (6.4.8), let Γ be a set of sentences that is maximally consistent in *SD*. We said earlier that it is easy to construct a truth-value assignment on which every member of a maximally consistent set is true, and it is; we need only consider the atomic sentences in the set. Let $\mathbf{A}^*$ be the truth-value assignment that assigns the truth-value **T** to every atomic sentence of *SL* that is a member of Γ^* and assigns the truth-value **F** to every other atomic sentence of *SL*. We shall prove by mathematical induction that each sentence of *SL* is true on the truth-value assignment $\mathbf{A}^*$ if and only if it is a member of Γ^*—from which it follows that every member of Γ^* is true on $\mathbf{A}^*$, thus establishing truth-functional consistency. The induction will be based on the number of occurrences of connectives in the sentences of *SL*:

Basis clause: Each atomic sentence of *SL* is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

Inductive step: If every sentence of *SL* with **k** or fewer occurrences of connectives is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*, then every sentence of *SL* with $\mathbf{k} + 1$ occurrences of connectives is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

Conclusion: Every sentence of *SL* is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

The basis clause is obviously true; we defined $\mathbf{A}^*$ to be an assignment that assigns **T** to all and only the atomic sentences of *SL* that are members of Γ^*. To prove the inductive step, we will assume that the inductive hypothesis holds for an arbitrary integer **k**: that each sentence containing **k** or fewer occurrences of connectives is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*. We must now show that the same holds true for every sentence **P** containing $\mathbf{k} + 1$ occurrences of connectives. We consider five cases, reflecting the five forms that a compound sentence of *SL* might have.

Case 1: **P** has the form $\sim \mathbf{Q}$. If $\sim \mathbf{Q}$ is true on $\mathbf{A}^*$, then **Q** is false on $\mathbf{A}^*$. Because **Q** contains fewer than $\mathbf{k} + 1$ occurrences of connectives, it follows by the inductive hypothesis that $\mathbf{Q} \notin \Gamma^*$. Therefore, by 6.4.11(a), $\sim \mathbf{Q} \in \Gamma^*$. If $\sim \mathbf{Q}$ is false on $\mathbf{A}^*$, then **Q** is true on $\mathbf{A}^*$. It follows by the inductive hypothesis that $\mathbf{Q} \in \Gamma^*$. Therefore, by 6.4.11(a), $\sim \mathbf{Q} \notin \Gamma^*$.

Case 2: **P** has the form $\mathbf{Q}\ \&\ \mathbf{R}$. If $\mathbf{Q}\ \&\ \mathbf{R}$ is true on $\mathbf{A}^*$, then both **Q** and **R** are true on $\mathbf{A}^*$. Because **Q** and **R** each contain fewer than $\mathbf{k} + 1$ occurrences of connectives, it follows by the inductive hypothesis that $\mathbf{Q} \in \Gamma^*$ and $\mathbf{R} \in \Gamma^*$. Therefore, by 6.4.11(b), $\mathbf{Q}\ \&\ \mathbf{R} \in \Gamma^*$. If $\mathbf{Q}\ \&\ \mathbf{R}$ is false on $\mathbf{A}^*$, then either **Q** is false on $\mathbf{A}^*$ or **R** is false on **A***. Therefore, by the inductive hypothesis, either $\mathbf{Q} \notin \Gamma^*$ or $\mathbf{R} \notin \Gamma^*$ and so, by 6.4.11(b), $\mathbf{Q} \;\&\; \mathbf{R} \notin \Gamma^*$.

Case 3: **P** has the form $\mathbf{Q} \vee \mathbf{R}$. See Exercise 6.4.6.

Case 4: **P** has the form $\mathbf{Q} \supset \mathbf{R}$. If $\mathbf{Q} \supset \mathbf{R}$ is true on **A***, then either **Q** is false on **A*** or **R** is true on **A***. Because **Q** and **R** each contain fewer than $\mathbf{k} + 1$ occurrences of connectives, it follows from the inductive hypothesis that either $\mathbf{Q} \notin \Gamma^*$ or $\mathbf{R} \in \Gamma^*$. By 6.4.11(d), then, $\mathbf{Q} \supset \mathbf{R} \in \Gamma^*$. If $\mathbf{Q} \supset \mathbf{R}$ is false on **A***, then **Q** is true on **A*** and **R** is false on **A***. By the inductive hypothesis, then, $\mathbf{Q} \in \Gamma^*$ and $\mathbf{R} \notin \Gamma^*$. And by 6.4.11(d), it follows that $\mathbf{Q} \supset \mathbf{R} \notin \Gamma^*$.

Case 5: See Exercise 6.4.6.

This completes the proof of the inductive step. Hence we may conclude that each sentence of *SL* is a member of Γ^* if and only if it is true on **A***. So every member of a set Γ^* that is maximally consistent in *SD* is true on **A***, and the set Γ^* is therefore truth-functionally consistent. This establishes the Consistency Lemma (6.4.8).

We now know that result 6.4.2, which we repeat here, is true:

6.4.2: For any set Γ of sentences of *SL*, if Γ is consistent in *SD* then Γ is truth-functionally consistent.

Because every set of sentences Γ that is consistent in *SD* is a subset of a set of sentences that is maximally consistent in *SD* (the Maximal Consistency Lemma (6.4.5)), and because every set of sentences that is maximally consistent in *SD* is truth-functionally consistent (the Consistency Lemma (6.4.8)), it follows that every set of sentences that is consistent in *SD* is a subset of a truth-functionally consistent set and is therefore itself truth-functionally consistent.

And Metatheorem 6.4.1 follows from 6.4.2:

If $\Gamma \vDash \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$.

For if $\Gamma \vDash \mathbf{P}$, then, by 6.4.3, $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent. Then, by result 6.4.2, $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*. And if $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*, then, by 6.4.4, $\Gamma \vdash \mathbf{P}$ in *SD*. So *SD* is complete for sentential logic—for every truth-functional entailment, at least one corresponding derivation can be constructed in *SD*. This, together with the proof of the Soundness Metatheorem in Section 6.3, shows that *SD* is an adequate system for sentential logic.

We conclude by noting that another important result, the *Compactness Theorem* for sentential logic, follows from Result 6.4.2 and Metatheorem 6.3.1:

Metatheorem 6.4.12: A set Γ of sentences of *SL* is truth-functionally consistent if and only if every finite subset of Γ is truth-functionally consistent.

And, as a consequence, a set of sentences of *SL* is truth-functionally inconsistent if and only if at least one finite subset of Γ is inconsistent.

6.4E EXERCISES

1. Prove 6.4.4 and 6.4.10.

2. Prove 6.4.6.

***3.** Prove that the empty set is consistent in *SD*.

4. Using Metatheorem 6.4.1, prove that *SD*+ is complete for sentential logic.

***5.** Prove that every set that is maximally consistent in *SD* has the following properties:

a. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.

b. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$. (These are clauses c and e of 6.4.11.)

***6.** Establish Cases 3 and 5 of the inductive step in the proof of the Consistency Lemma 6.4.8.

7.a. Suppose that *SD** is like *SD* except that it lacks Reiteration. Show that *SD** is complete for sentential logic.

*b. Suppose that *SD** is like *SD* except that it lacks Negation Introduction. Show that *SD** is complete for sentential logic.

8. Suppose that *SD** is like *SD* except that it lacks Conjunction Elimination. Show where our completeness proof for *SD* will fail as a completeness proof for *SD**.

9. Using Result 6.4.2 and Metatheorem 6.3.1, prove Metatheorem 6.4.12.