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a. Checking Soundness for Alt. Rules

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  then figure out what a sound rule would give you.

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  - If yes, then the new rule preserves soundness (proceed to formally extend the proof!)

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- Strategy: first do a heuristic: do the earlier accessible sentences semantically entail the final sentence?
  - If yes, then the new rule preserves soundness (proceed to formally extend the proof!)
  - If no, then you should be able to construct a concrete counterexample to soundness (i.e. case where $\Gamma \vdash_{SND^*} P$ but $\Gamma \nvDash P$ for a concrete set of SL sentences Γ

Notation for Soundness Cases

- Γ_i stands for the set of assumptions that are open at the i–th line, i.e. these are the accessible premises/assumptions at line i. They are every premise/assumption (sentence sitting on a horizontal line) such that its scope line (vertical line) travels all the way down to line i, and line i is to the right of this vertical line.
- $\mathbf{P_i}$ stands for the sentence that is on the i–th line.
- $\Delta \subseteq \Gamma$ means that the set Δ is a subset of Γ.
- $\Gamma \cup \{\mathbf{Q}\}$ means that we have added the sentence $\mathbf{Q}$ to the set of sentences Γ (we have taken their union).

Induction Hypothesis and Key Fact

- *Induction hypothesis* for Soundness: assume that the soundness/righteousness property holds for all lines i less than the $k+1$–st line, i.e. if $i \leq k$ and if $\Gamma_i \vdash \mathbf{P_i}$, then $\Gamma_i \vDash \mathbf{P_i}$.
- In words: we are assuming that if we can derive a sentence $\mathbf{P_i}$ from a set of assumptions Γ_i, then those assumptions semantically entail that sentence.
- Lemma 6.3.2 (a.k.a. Useful Fact 1): if $\Gamma \vDash \mathbf{P}$ and Γ is a subset of a larger set Γ', then the larger set semantically entails the sentence $\mathbf{P}$ as well, i.e. $\Gamma' \vDash \mathbf{P}$.

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- Reason about relevant semantic entailment claims by using the truthtables for the connectives

Schematic Solution Steps (if you're totally lost)

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   - If the sentence P_j at line j is an additional open assumption that is not open at line $k+1$, then you need to tack this on, using the union operation: $\Gamma_j \subseteq (\Gamma_{k+1} \cup P_j)$.

Schematic Solution Steps continued

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6. Next, consider an arbitrary truth value assignment that makes all of the sentences in Γ_{k+1} true. Use whatever double turnstiles are at your disposal to infer that some other sentence(s) is true.

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8. Pat yourself on the back (soundly)!

For additional guidance on Soundness, see...

- Section 6.3 of *The Logic Book* (reading for Week 12)
- pages 246–250 contain most of the cases for our system SND
- PS12 #1 handles negation elimination (case 10)
- §6.3 Exercises on page 250–251, problem #4 parts a thru d
- Think of your own cases by throwing in negation symbols, thinking about de Morgan's or other semantically equivalent sentences, etc.!

14. Final Review!

b. Applications of soundness & completeness

Applying soundness and/or completeness theorems

- PS12, problems #4–7 illustrate simple applications of the soundness and/or completeness theorems

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- The final might contain problems of a similar flavor

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- Answer: $\Gamma \vDash \mathcal{P}$ (and vice-versa)
- If $\Gamma \vDash \sim\mathcal{Q}$, what else can we say?
- Answer: $\Gamma \cup \{\mathcal{Q}\}$ is unsatisfiable (and vice-versa)
- See p. 245 if you don't believe this; but should be able to give valid arguments for these claims verbally!

Practice w/ Applying Soundness & Completeness

To avoid ambiguity, let the sentences and sets of sentences be from QL, and let '$\vdash$' denote $\vdash_{QND}$

1. Prove or provide a counterexample to the following statement: If $\Gamma \vDash \mathcal{P}$ and $\Delta \vdash \mathcal{Q}$, then $\Gamma \cup \Delta \vdash \mathcal{P} \,\&\, \mathcal{Q}$

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3. If $\Gamma \cup \{\sim\mathbf{P}\}$ is unsatisfiable and $\Delta \vdash \mathbf{R}$, prove or provide a counterexample to $(\Gamma \cup \Delta) \vdash (\sim\mathbf{P} \equiv \mathbf{R})$.

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2. If $\Gamma \vdash (S \vee R)$ and $\Gamma \vdash {\sim}(S \vee R)$, prove or provide a counterexample that $\Gamma \vDash S$
3. If $\Gamma \cup \{{\sim}\mathbf{P}\}$ is unsatisfiable and $\Delta \vdash \mathbf{R}$, prove or provide a counterexample to $(\Gamma \cup \Delta) \vdash ({\sim}\mathbf{P} \equiv \mathbf{R})$.
4. Prove or give a counterexample to the following statement: If Γ is satisfiable, then $\{{\sim}S \mid S \in \Gamma\}$ is satisfiable.

Concept Review (if totally lost)

- Soundness theorem for SND: if you have a single turnstile (in SND), then you have a double turnstile. In words: if a set of assumptions gives you a derivation (in SND) for a sentence S, then those assumptions semantically entail that sentence S. In symbols: if $\Gamma \vdash_{SND} \mathbf{S}$, then $\Gamma \vDash \mathbf{S}$.
- Completeness theorem for SND: if you have a double turnstile, then you have a single turnstile (in SND). In words: if a set of assumptions semantically entails a sentence S, then those assumptions gives you a derivation (in SND) for that sentence S. In symbols: if $\Gamma \vDash \mathbf{S}$, then $\Gamma \vdash_{SND} \mathbf{S}$.
- Likewise for QL and QND

Solution Tips for Logically Complete Students

1. Use the soundness theorem to convert any single turnstiles you have (from system SND) into double turnstiles.
2. Convert claims about unsatisfiability into double turnstile relations
3. Use the completeness theorem to convert any double turnstiles you have into single turnstiles.
4. If you get stuck, write out the definitions of any key terms involved. These will guide you on your path to victory.
5. If you have to provide a counterexample, think about the simplest counterexample that gets the job done. Your counterexample must involve ACTUAL sentences; not metavariables
6. Pray for a stroke of insight! (Jk! Try reasoning backwards to figure out what you need!)

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c. Alt. Cases of Membership Lemma

Alternative Cases of Membership Lemma

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- We can prove variants of these cases, e.g. the following: modified version of case (e): $\sim\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either i) both $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$ or ii) both $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$.

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- Alternately, one can be given an alternative SND rule (replacing one of our 11 sanctioned rules) from which to reprove a given case of the membership lemma (using the Door lemma)

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- Note that all of these cases have TWO directions, and you need to prove BOTH directions to complete the problem.

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- Sometimes a case involves subcases, each of which can require its own non-trivial SND deduction (e.g. cases (c) and (d) for disjunction and conditional)
- Finally, remember that the membership lemma is purely syntactic! No mention of truth-value assignments here!

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- **Membership Lemma** for club Γ^*: if $\mathcal{P}$ and $\mathcal{Q}$ are SL wffs, then:

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  a.) $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$

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  d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$

  e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$

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  - c.) $\mathcal{P} \vee \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \in \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - d.) $\mathcal{P} \supset \mathcal{Q} \in \Gamma^*$ if and only if either $\mathcal{P} \notin \Gamma^*$ or $\mathcal{Q} \in \Gamma^*$
  - e.) $\mathcal{P} \equiv \mathcal{Q} \in \Gamma^*$ iff either (i) $\mathcal{P} \in \Gamma^*$ and $\mathcal{Q} \in \Gamma^*$ or (ii) $\mathcal{P} \notin \Gamma^*$ and $\mathcal{Q} \notin \Gamma^*$
- Notice how these syntactic constraints mirror truth-conditions!

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- (modified version of case b):
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- Additional practice problem (modified version of case c):
  prove that $\mathbf{P} \vee \sim\mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$.

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- Study case (d) for the conditional (bottom of p. 258)!

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  prove that $\mathbf{P} \vee \sim\mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \notin \Gamma^*$.
- Study case (d) for the conditional (bottom of p. 258)!
- Note that the 7-line derivation for case d) has a serious typo on line 2: the justification should be ":A / $\supset$ I", i.e. :AS for conditional intro.

Maximally Consistent-in-SND

- Γ^* is maximally-SND-consistent provided that both (i) Γ^* is consistent in SND (i.e. can't derive any contradictions) and (ii) if $\mathbf{P}$ is not in Γ^*, then $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in SND.
- In other words: you can't derive a contradiction from assumptions in Γ^*. And if $\mathbf{P} \notin \Gamma^*$, then $\Gamma^* \cup \{\mathbf{P}\}$ lets you derive a contradictory pair (i.e. you can derive both $\mathbf{R}$ and $\sim\mathbf{R}$).
- Assuming that you are not asked to prove a variant of case (a), you can help yourself to this result. Hence, if a sentence $\mathbf{P} \notin \Gamma^*$, then case (a) lets you conclude that $\sim\mathbf{P} \in \Gamma^*$, and vice versa: if $\sim\mathbf{P} \in \Gamma^*$, then you can conclude that $\mathbf{P} \notin \Gamma^*$.

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  - Hence the strategy: if you are trying to show that $S \in \Gamma^*$, figure out how to derive S in SND from sentences you have assumed are in Γ^*. Then, apply The Door.

14. Final Review!

d. Translations in QL, with Identity

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- Q if P; Q provided that P; Q given that P; if P, then Q: $P \supset Q$

Some Simple Examples not involving identity

Domain: all people;

Predicates: **Dx**: x went to Disneyland; **Kxy**: x knows y;

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- So plural possessives are NOT definite descriptions.

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- Watch out for conditionals, which are trivially satisfied if the antecedent is false

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- $\mathcal{A} \supset \mathcal{B}$ is true iff $\mathcal{A}$ is false or $\mathcal{B}$ is true

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14. Final Review!

f. Derivations in QND

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- Another practice problem!:
$(\exists x)Gx \supset Fa \ \vdash_{QND} (\forall x)(Gx \supset Fa)$
- The other direction is MUCH trickier (but can be done in 10 lines)!
$(\forall x)(Gx \supset Fa) \vdash_{QND} (\exists x)Gx \supset Fa$

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- In general, each rule applies only to the WHOLE sentence, not a part. So you CANNOT apply a rule to just part of a sentence.

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- If you get stuck, try to work from the bottom up. Think about what you would first have to derive to build your ultimate goal.
- If you get stuck on a subgoal, assume the opposite of your subgoal to try using either negation introduction or negation elimination to keep going.