Completeness of $\text{FOL}^=$

LOGIC I
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Basic Lemmas

L9.1 If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

L11.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

L11.6 If $\mathcal{M}$ and $\mathcal{M}'$ have the same domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in a wff φ, then $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any v.a. $\hat{a}$ defined over $\mathbb{D}$.

L12.1 If α is a constant and X is an $\text{FOL}^=$ derivation in which the constant β does not occur, then $X[\beta/\alpha]$ is also an $\text{FOL}^=$ derivation.

L12.3 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

L12.4 If $\Lambda \vdash \varphi$ and $\Pi \cup \{\varphi\} \vdash \psi$, then $\Lambda \cup \Pi \vdash \psi$.

L12.6 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

L12.9 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

L12.11 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Completeness

T12.1 Every consistent set of $\mathcal{L}^=$ wfss Γ is satisfiable.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- Assuming $\Gamma \vDash \varphi$, we know $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable (check).
- So $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **T12.1**.
- So $\Gamma \vdash \neg\neg\varphi$ by **L12.3**, and so $\Gamma \vdash \varphi$ by DN and **L12.4**.

Assume: Let Γ be a set of $\mathcal{L}^=$ wfss that is consistent in $\text{FOL}^=$.

Saturation

Extension: Let $\mathcal{L}^=_{\mathbb{N}}$ include the extra constants $\mathbb{N}$ where $\mathbb{C}$ is the set of all constants.

Free: Let $\varphi(\alpha)$ be a wff of $\mathcal{L}^=_{\mathbb{N}}$ with at most one free variable α.

Saturated: A set of wfss Σ is SATURATED in $\mathcal{L}^=_{\mathbb{N}}$ just in case for each wff $\varphi(\alpha)$ of $\mathcal{L}^=_{\mathbb{N}}$, there is a constant β where $(\exists\alpha\varphi \rightarrow \varphi[\beta/\alpha]) \in \Sigma$.

L12.2 Γ is consistent in $\text{FOL}^=_{\mathbb{N}}$. (We will drop 'in $\text{FOL}^=_{\mathbb{N}}$')

Free Enumeration: Let $\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\alpha_3), \ldots$ enumerate all wffs of $\mathcal{L}^=_{\mathbb{N}}$ with at most one free variable.

Witnesses: $\theta_1 = (\exists\alpha_1\varphi_1 \rightarrow \varphi_1[n_1/\alpha_1])$ where $n_1 \in \mathbb{N}$ is the first constant not in φ_1.
$\theta_{k+1} = (\exists\alpha_{k+1}\varphi_{k+1} \rightarrow \varphi_{k+1}[n_{k+1}/\alpha_{k+1}])$ where $n_{k+1} \in \mathbb{N}$ is the first constant not in φ_{k+1} or θ_j for any $j \leqslant k$.

Saturation: $\Sigma_0 = \Gamma$,
$\Sigma_{n+1} = \Sigma_n \cup \{\theta_{n+1}\}$, and
$\Sigma_\Gamma = \bigcup_{i\in\mathbb{N}} \Sigma_n$.

L12.5 Σ_Γ is consistent and saturated in $\mathcal{L}^{=}_{\mathbb{N}}$.

Base: $\Sigma_0 = \Gamma$ is consistent.

Induction: Assume Σ_m is consistent.

- Assume $\Sigma_{m+1} = \Sigma_m \cup \{\theta_{m+1}\}$ is inconsistent for contradiction.
- So $\Sigma_m \vdash \neg\theta_{m+1}$ by **L12.3**, and so $\Sigma_m \vdash \neg(\exists\alpha_{m+1}\varphi_{m+1} \rightarrow \varphi_{m+1}[n_{m+1}/\alpha_{m+1}])$.
- So $\Sigma_m \vdash \exists\alpha_{m+1}\varphi_{m+1}$ and $\Sigma_m \vdash \neg\varphi_{m+1}[n_{m+1}/\alpha_{m+1}]$ by derived PL rules.
- So $\Sigma_m \vdash \forall\alpha_{m+1}\neg\varphi_{m+1}$ by $\forall$I since n_{m+1} is not in $\forall\alpha_{m+1}\neg\varphi_{m+1}$ or Σ_m.
- So $\Sigma_m \vdash \neg\exists\alpha_{m+1}\varphi_{m+1}$ by $\forall\neg$, and so Σ_m is inconstant by **L12.9**.
- It follows by *reductio* that Σ_{m+1} is consistent.
- By weak induction, we know that Σ_k is consistent for all $k \in \mathbb{N}$.

Limit: If Σ_Γ is inconsistent, then X derives $\bot$ from Σ_Γ in $\text{FOL}^{=}_{\mathbb{N}}$.

- Since X is finite, $\Sigma_m \vdash \bot$ for some $m \in \mathbb{N}$ including all premises in X.
- So Σ_m is inconsistent, contradicting the above.
- By *reductio*, Σ_Γ is consistent.

Maximization

Maximal: A set of wfss Δ is MAXIMAL in $\mathcal{L}^{=}_{\mathbb{N}}$ just in case either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every wfs ψ in $\mathcal{L}^{=}_{\mathbb{N}}$.

Full Enumeration: Let $\psi_0, \psi_1, \psi_2, \ldots$ enumerate all wfss in $\mathcal{L}^{=}_{\mathbb{N}}$.

Maximization: $\Delta_0 = \Sigma$,

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases},$$

$\Delta_\Sigma = \bigcup_{i\in\mathbb{N}} \Delta_n$.

L12.7 $\Delta = \Delta_{\Sigma_\Gamma}$ is maximal in $\mathcal{L}^{=}_{\mathbb{N}}$ and consistent.

Case 1: $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$.

- If $\Delta_n \cup \{\neg\psi_n\}$ is inconsistent, then Δ_n is inconsistent by **L12.6**.
- So Δ_{n+1} is consistent in both cases.

- If Δ_Σ is inconsistent, then $\Delta_m \vdash \bot$ for some $m \in \mathbb{N}$.
- Maximality is immediate.

L12.8 $\Gamma \subseteq \Sigma_\Gamma \subseteq \Delta$ where Δ is saturated.

L12.10 $\varphi \in \Delta$ whenever $\Delta \vdash \varphi$.

- Assuming $\Delta \vdash \varphi$, we know $\Delta \nvdash \neg\varphi$ by **L12.9**.
- So $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$.
- Thus $\varphi \in \Delta$ by maximality.

Henkin Model

Element: $[\alpha]_\Delta = \{\beta \in \mathbb{C} : \alpha = \beta \in \Delta\}$.

Domain: $\mathbb{D}_\Delta = \{[\alpha]_\Delta \subseteq \mathbb{C} : \alpha \in \mathbb{C}\}$.

L12.12 $\alpha \in [\alpha]_\Delta$ for any constant $\alpha \in \mathbb{C}$.

- Still need to check that elements in $\mathbb{D}_\Delta$ are well defined.

L12.13 If $\alpha = \beta \in \Delta$, then $[\alpha]_\Delta = [\beta]_\Delta$.

- Assuming $\alpha = \beta \in \Delta$ where $\gamma \in [\alpha]_\Delta$, we know $\alpha = \gamma \in \Delta$.
- So $\alpha = \beta, \alpha = \gamma \vdash \beta = \gamma$ by =E, and so $\Delta \vdash \beta = \gamma$ by **L12.11**.
- Thus $\beta = \gamma \in \Delta$ by **L12.10**, and so $\gamma \in [\beta]_\Delta$, hence $[\alpha]_\Delta \subseteq [\beta]_\Delta$.

Constants: $\mathcal{I}_\Delta(\alpha) = [\alpha]_\Delta$ for all constants $\alpha \in \mathbb{C}$.

Predicates: $\mathcal{I}_\Delta(\mathcal{F}^n) = \{\langle [\alpha_1]_\Delta, \ldots, [\alpha_n]_\Delta \rangle \in \mathbb{D}_\Delta^n : \mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta\}$.

L12.14 If $\alpha_i = \beta_i \in \Delta$, then $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$ *iff* $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$.

- Assume $\alpha_i = \beta_i \in \Delta$ where $\mathcal{F}^n\alpha_1, \ldots, \alpha_n \in \Delta$.
- $\Delta \vdash \mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i]$ by =E, so $\mathcal{F}^n\alpha_1, \ldots, \alpha_n[\beta_i/\alpha_i] \in \Delta$ by **L12.10**.
- Parity of reasoning completes the proof.

Henkin Lemmas

L12.15 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some constant $\beta \in \mathbb{C}$.

- Letting $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$, we know $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.
- So $\hat{c}(\alpha) = [\beta]_\Delta$ for some $\beta \in \mathbb{C}$, so $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$ since $\mathcal{I}_\Delta(\beta) = [\beta]_\Delta$.
- Thus $\mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L11.5**.
- So $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ by **L9.1**.
- Assume instead that $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$.
- Let $\hat{c}$ be the α-variant of $\hat{a}$ where $\hat{c}(\alpha) = \mathcal{I}_\Delta(\beta)$, so $\mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\alpha) = \mathfrak{v}^{\hat{c}}_{\mathcal{I}}(\beta)$.

- Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha])$ by **L11.5**, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\varphi) = 1$.

L12.16 $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\forall\alpha\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi[\beta/\alpha]) = 1$ for all constants $\beta \in \mathbb{C}$.

- Similar to **L12.15**.

L12.17 $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$. (Let $\hat{a}$ be arbitrary.)

Base: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\alpha_1 = \alpha_2) = 1$ *iff* $\mathcal{I}_\Delta(\alpha_1) = \mathcal{I}_\Delta(\alpha_2)$ *iff* $[\alpha_1]_\Delta = [\alpha_2]_\Delta$ *iff* $\alpha_1 = \alpha_2 \in \Delta$.

- If $[\alpha_1]_\Delta = [\alpha_2]_\Delta$, then $\alpha_2 \in [\alpha_2]_\Delta$ by **L12.12**, and so $\alpha_2 \in [\alpha_1]_\Delta$.
- Thus $\alpha_1 = \alpha_2 \in \Delta$ by definition, and the converse holds by **L12.13**.

Induction: Assume $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$ whenever $\texttt{Comp}(\varphi) \leqslant n$.

- Let φ be a wfs of $\mathcal{L}^{=}_{\mathbb{N}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\neg\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) \neq 1$ *iff* $\psi \notin \Delta$ *iff* $\neg\psi \in \Delta$.

Case 2: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\chi) = 1$ *iff* $\psi, \chi \in \Delta$ *iff* $\psi \wedge \chi \in \Delta$.

Case 6: $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\exists\alpha\psi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\psi[\beta/\alpha]) = 1$ for some $\beta \in \mathbb{C}$ by **L12.15**.

... *iff* $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$ by hypothesis.

($\exists$) ... *iff* $\exists\alpha\psi \in \Delta$ by $\exists$I and **L12.10** given saturation.

- If $\psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$, then $\Delta \vdash \exists\alpha\psi$ by $\exists$I, so $\exists\alpha\psi \in \Delta$ by **L12.10**.
- If $\exists\alpha\psi \in \Delta$ instead, then $\psi = \varphi_i(\alpha_i)$ for some $i \in \mathbb{N}$ where $\alpha_i = \alpha$.
- Thus $\exists\alpha_i\varphi_i \to \varphi_i[n_i/\alpha_i] \in \Delta$ by the saturation assumed of Δ.
- Since $n_i \in \mathbb{C}$, it follows that $\exists\alpha\psi \to \psi[\beta/\alpha] \in \Delta$ for some $\beta \in \mathbb{C}$, and so $\Delta \vdash \psi[\beta/\alpha]$ by conditional elimination $\to$E.
- Thus **L12.10** that $\psi[\beta/\alpha] \in \Delta$, thereby establishing ($\exists$).

Conclusion: So $\mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, from which the lemma follows.

Restriction

Restriction: $\mathcal{I}'_\Delta(\alpha) = [\alpha]_\Delta$ for every constant α in $\mathcal{L}^{=}_{\mathbb{N}}$.

$\mathcal{I}'_\Delta(\mathcal{F}^n) = \mathcal{I}_\Delta(\mathcal{F}^n)$ for all n-place predicates $\mathcal{F}^n$.

L12.18 For any wfs φ of $\mathcal{L}^{=}_{\mathbb{N}}$, $\mathcal{M}'_\Delta$ satisfies φ just in case $\mathcal{M}_\Delta$ satisfies φ.

- $\mathcal{V}^{\hat{a}}_{\mathcal{I}'_\Delta}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}_\Delta}(\varphi)$ for any variable assignment $\hat{a}$ by **L11.6**.

T12.1 Every consistent set Γ of wfss in $\mathcal{L}^{=}_{\mathbb{N}}$ is satisfiable.

- Let Γ be a consistent set of $\mathcal{L}^{=}$ sentences in FOL$^{=}$.
- By **L12.2**, Γ is a set of $\mathcal{L}^{=}_{\mathbb{N}}$ sentences that is consistent in FOL$^{=}_{\mathbb{N}}$.
- So Σ_Γ is consistent and saturated in $\mathcal{L}^{=}_{\mathbb{N}}$ by **L12.5**.
- Given **L12.7** and **L12.8**, Δ_{Σ_Γ} is a saturated maximal consistent set of sentences in $\mathcal{L}^{=}_{\mathbb{N}}$ where $\Gamma \subseteq \Delta_{\Sigma_\Gamma}$.

- Letting $\Delta = \Delta_{\Sigma_\Gamma}$, **L12.7** shows that the Henkin model $\mathcal{M}_\Delta$ satisfies φ just in case $\varphi \in \Delta$, and so $\mathcal{M}_\Delta$ satisfies Δ.
- Having shown that $\Gamma \subseteq \Delta$, we know that $\mathcal{M}_\Delta$ satisfies Γ.
- Since Γ is a set of $\mathcal{L}^=$ sentences, it follows by **L12.18** that there is a model $\mathcal{M}'_\Delta$ of $\mathcal{L}^=$ that satisfies Γ.
- Thus Γ is satisfiable.

Compactness

C12.2 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

C12.3 Γ is satisfiable if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable.

Final Exam Review

Regimentation: (a) No two individuals are at least as tall as each other. Sanna is at least as tall as the finalist, and the finalist is at least as tall as Sanna. Thus, Sanna is the finalist.

Models: (a) $Qab, Qba \nvDash a = b$.

(b) $\forall x \forall y(Px \supset (Py \supset x \neq y)) \nvDash \exists x \exists y\, x \neq y$.

Equivalence: $\exists x(\forall y(Py \supset x = y) \land Px) \dashv\vDash \exists x \forall y(Py \equiv x = y)$.

Relations: (a) R is symmetric and antisymmetric. Therefore R is reflexive.

(b) R is asymmetric. Therefore R is antisymmetric.