Soundness: Part II

LOGIC I
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Soundness of QD

Assume: $\Gamma \vdash_{\text{QD}} \varphi$, so there is a QD proof X of φ from Γ.

Lines: Let φ_i be the i^{th} line of X.

Dependencies: Let Γ_i be the undischarged assumptions at line i.

Proof: The proof goes by induction on length of X:

BASE: $\Gamma_1 \vDash \varphi_i$.

HYPOTHESIS: Assume $\Gamma_k \vDash \varphi_k$ for all $k \leq n$.

INDUCTION: If φ_{n+1} follows by the proof rules for QD from sentences in Γ_{n+1}, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Finite: Since X is finite, there is some m where $\Gamma_m = \Gamma$ and $\varphi_m = \varphi$, so $\Gamma \vDash \varphi$.

SD Lemmas

L12.1 If $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vDash \varphi$.

L12.2 For any QD proof X, if φ_k is live at line n where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

L12.3 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

L12.4 If $\Gamma \cup \{\varphi\}$ is unsatisfiable, then $\Gamma \vDash \neg\varphi$.

L12.5 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$ if $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ.

L12.6 $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ for every v.a. $\hat{a}$ over $\mathbb{D}$.

L12.7 If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \supset \psi$.

SD Rules

(R) $\varphi_k = \varphi_{n+1}$ for live $k \leq n$. Thus $\Gamma_k \vDash \varphi_k$ by hypothesis and $\Gamma_k \subseteq \Gamma_{n+1}$ by **L12.2**. Thus $\Gamma_{n+1} \vDash \varphi_k$ by **L12.1**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\neg$I)
- There is a proof of ψ at line h and $\neg\psi$ at line j from φ on line i.
- By hypothesis $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$, where $\Gamma_h, \Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$.
- By **L12.1**, $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$.
- So $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable by **L12.3**, so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by **L12.4**.

($\wedge$E) • $\varphi_{n+1} \wedge \psi$ is live on line $i \leq n$.

- By hypothesis, $\Gamma_i \vDash \varphi_{n+1} \wedge \psi$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**
- Thus $\Gamma_{n+1} \vDash \varphi_{n+1} \wedge \psi$ by **L12.1**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by semantics.

($\supset$I) • There is a proof of ψ at line j from φ on line i.

- By hypothesis $\Gamma_j \vDash \psi$, where $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi\}$.
- So $\Gamma_{n+1} \cup \{\varphi\} \vDash \psi$, and so $\Gamma_{n+1} \vDash \varphi \supset \psi$ by **L12.7**.

QD Lemmas

L12.8 $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ if $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ and β is free for α in φ.

Base: Assume φ is $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ or $\alpha_1 = \alpha_2$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

- Let $\gamma_i = \beta$ if $\alpha_i = \alpha$ and otherwise $\gamma_i = \alpha_i$.
- $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n)$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_1) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\gamma_2)$.

Induction: If $\texttt{Comp}(\varphi) \leq n$, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha])$ whenever $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

Case 2: Assume $\varphi = \psi \wedge \chi$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ for all $\hat{a}$.

- So $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi \wedge \chi) = 1$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\chi) = 1$ *iff* ...

Case 6: Assume $\varphi = \forall\gamma\psi$ where $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$.

- If $\gamma = \alpha$, then $\varphi = \varphi[\beta/\alpha]$.
- If $\gamma \neq \alpha$, $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\gamma\psi) = 1$ *iff* $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff*...
- Let $\hat{e}$ be an arbitrary γ-variant of $\hat{a}$.
- Since $\gamma \neq \alpha$, $\hat{e}(\alpha) = \hat{a}(\alpha)$ if α is a variable, so $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha)$.
- Thus $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$ follows from the assumption.
- Since β is free for α in $\forall\gamma\psi$, we know that $\gamma \neq \beta$.
- If β is a variable, then $\hat{e}(\beta) = \hat{a}(\beta)$ since $\hat{e}$ is a γ-variant of $\hat{a}$.
- Thus $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\beta) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\beta)$, and so $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\alpha) = \mathcal{V}^{\hat{e}}_{\mathcal{I}}(\beta)$.
- By hypothesis, $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi[\beta/\alpha])$, where $\hat{e}$ was arbitrary.
- ...*iff* $\mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi[\beta/\alpha]) = 1$ for all γ-variants $\hat{e}$ of $\hat{a}$ *iff* $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi[\beta/\alpha]) = 1$.

L12.9 If $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}' \rangle$ where $\mathcal{I}$ and $\mathcal{I}'$ agree about every constant α and n-place predicate $\mathcal{F}^n$ that occurs in φ, it follows that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$.

Base: $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$ *iff* $\langle \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_1), \ldots, \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_n) \rangle \in \mathcal{I}'(\mathcal{F}^n)$.

- $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ is immediate from the assumption.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \mathcal{I}(\alpha_i) = \mathcal{I}'(\alpha_i) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a constant.
- $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_i) = \hat{a}(\alpha_i) = \mathcal{V}^{\hat{a}}_{\mathcal{I}'}(\alpha_i)$ if α_i is a variable.

L12.10 For any constant β that does not occur in $\forall\alpha\varphi$ or in any sentence $\psi \in \Gamma$, if $\Gamma \vDash \varphi[\beta/\alpha]$, then $\Gamma \vDash \forall\alpha\varphi$.

1. Assume $\Gamma \vDash \varphi[\beta/\alpha]$ for constant β not in $\forall\alpha\varphi$ or Γ.
2. Assume $\Gamma \nvDash \forall\alpha\varphi$, and so $\mathcal{M}$ satisfies Γ but $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) \neq 1$.
3. So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) \neq 1$ for some α-variant $\hat{c}$ of $\hat{a}$.
4. Let $\mathcal{M}'$ by like $\mathcal{M}$ but for $\mathcal{I}'(\beta) = \hat{c}(\alpha)$.
5. By **L12.9**, $\mathcal{M}'$ satisfies Γ since β does not occur in Γ.
6. So $\mathcal{M}'$ satisfies $\varphi[\beta/\alpha]$ since $\Gamma \vDash \varphi[\beta/\alpha]$.
7. By **L12.6**, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$ for all $\hat{c}$, and so for $\hat{c}$ in particular.
8. Since β is not in $\forall\alpha\varphi$, we know β is not in φ.
9. So $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) \neq 1$ by **L.12.9** given (3) above.
10. By (4) above, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\alpha) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\beta)$ where β is free for α.
11. By **L12.8**, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha])$.
12. Thus $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) \neq 1$, contradicting the above.

L12.11 $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$ where α is a variable and $\varphi[\beta/\alpha]$ is a sentence.

- Let $\mathcal{M}$ satisfy $\forall\alpha\varphi$, so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ for some $\hat{a}$.
- So $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ where $\hat{c}(\alpha) = \mathcal{I}(\beta)$ for an α-variant $\hat{c}$ of $\hat{a}$.
- By **L12.8**, $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi[\beta/\alpha])$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$.

L12.12 If $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then $\Gamma \cup \Sigma \vDash \psi$.

L12.13 $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ where α is a variable and $\varphi[\beta/\alpha]$ is a sentence.

L12.14 For any constant β that does not occur in $\exists\alpha\varphi$, ψ, or in any sentence $\chi \in \Gamma$, if $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$, then $\Gamma \vDash \psi$.

L12.15 If α and β are constants, then $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$.

QD Rules

($\forall$I)

- $\varphi_i = \varphi[\beta/\alpha]$ for $i \leq n$ live at $n+1$ where β is not in φ_{n+1} or Γ_{n+1}.
- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L12.1**, so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$.
- So $\Gamma_{n+1} \vDash \forall\alpha\varphi$ by **L12.10** since β not in $\forall\alpha\varphi$ or Γ_{n+1}.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.

($\forall$E)

- $\varphi_i = \forall\alpha\varphi$ for $i \leq n$ live at $n+1$ where $\varphi_{n+1} = \varphi[\beta/\alpha]$.
- So $\Gamma_i \vDash \varphi_i$ by hypothesis, and $\Gamma_i \subseteq \Gamma_{n+1}$ by **L12.2**.
- Thus $\Gamma_{n+1} \vDash \varphi_i$ by **L12.1**, so $\Gamma_{n+1} \vDash \forall\alpha\varphi$.
- By **L12.11** $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$, and so $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$ by **L12.12**.
- Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$.