Mathematical Induction

LOGIC I
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Continuing from Last Time...

Task 1: Let $\mathcal{I}^+(\alpha) = 1$ for every sentence letter α in SL. Show that $\mathcal{V}_{\mathcal{I}^+}(\varphi) = 1$ for every SL sentence φ that does not include negation.

Task 2: Every tree has a finite number of branches.

Task 3: Show that for every SL sentence φ, if $\texttt{Simple}(\varphi)$, then there are SL interpretations $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{J}}(\varphi) = 0$.

Task 4: For any SL sentences φ, ψ, χ and SL sentence letter α, if $\vDash \varphi \equiv \psi$, then $\vDash \chi_{[\varphi/\alpha]} \equiv \chi_{[\psi/\alpha]}$.

Task 5: Every tree can be completed in a finite number of steps.

Recursive Definitions

Length: For any SL tree X, we define $\texttt{Length}(X)$ to be the number of resolution rules that have been applied.

- $\texttt{Length}(X) = 0$ for any root X.
- For any SL tree X, if $\texttt{Length}(X) = n$ and X' is the result of resolving a sentence in exactly one branch in X, then $\texttt{Length}(X') = n + 1$.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ.

- If $\texttt{Comp}(\varphi) = 0$, then $[\varphi] = \{\varphi\}$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$; and

  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$.

Simplicity: We define $\texttt{Simple}(\varphi)$ to hold just in case the SL sentence φ has at most one occurrence of each sentence letter in SL.

- If $\texttt{Comp}(\varphi) = 0$, then $\texttt{Simple}(\varphi)$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$; and

  ($\star$) $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$.

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If $\texttt{Comp}(\varphi) = 0$, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

($\neg$) $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$; and
($\star$) $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$.

Resolution: We define $\texttt{Res}(\varphi)$ to be the maximum number of times that we may resolve φ and any of its descendants.

- $\texttt{Res}(\varphi) = 0$ if φ is a literal.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

($\neg$) $\texttt{Res}(\neg\neg\varphi) = \texttt{Res}(\varphi) + 1$; and
($\star$) $\texttt{Res}(\varphi \star \psi) = \texttt{Res}(\varphi) + \texttt{Res}(\psi) + 1$.
($\star$) $\texttt{Res}(\neg(\varphi \star \psi)) = \texttt{Res}(\neg\varphi) + \texttt{Res}(\neg\psi) + 1$.

Set Binary: We extend the definition of Res to sets of sentences as follows:

$$\texttt{Res}(\Gamma) = \sum_{\varphi \in \Gamma} \texttt{Res}(\varphi).$$

Unresolved: We define $[X]$ to be the set of unresolved sentences in the SL tree X.

- $\varphi \in [X]$ if X is a root and φ occurs in X.
- For any SL tree X' which results from resolving $\varphi \in [X]$ on every open branch in X in which φ occurs:

($\neg$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi\}$ if $\varphi = \neg\neg\psi$;
($+$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \chi\}$ if $\varphi \in \{\psi \wedge \chi, \psi \vee \chi\}$;
($-$) $[X'] = ([X]/\{\varphi\}) \cup \{\neg\psi, \neg\chi\}$ if $\varphi \in \{\neg(\psi \wedge \chi), \neg(\psi \vee \chi)\}$;
($\supset$) $[X'] = ([X]/\{\varphi\}) \cup \{\neg\psi, \chi\}$ if $\varphi = \psi \supset \chi$;
($\not\supset$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \neg\chi\}$ if $\varphi = \neg(\psi \supset \chi)$;
($\equiv$) $[X'] = ([X]/\{\varphi\}) \cup \{\psi, \chi, \neg\psi, \neg\chi\}$ if $\varphi \in \{\psi \equiv \chi, \neg(\psi \equiv \chi)\}$.

Resolvable: Letting $\mathbb{L}$ be the set of SL literals, we define $X_U = [X]/\mathbb{L}$ to be the set of SL sentences in X that can be resolved.

Task 5

Proof: Given any SL tree X, let X_U be the set of resolvable sentences in X. The proof goes by induction on $\texttt{Res}(X_U)$ for any SL tree X.

Base Case: Let X be an SL tree where $\texttt{Res}(X_U) = 0$. By definition, every branch is complete, and so the tree is complete. Accordingly, X can be completed in a finite number of steps, namely 0.

Hypothesis: Assume that for every SL tree X, if $\texttt{Res}(X_U) \leq n$, then X can be completed in a finite number of steps.

Induction: Let X be an SL tree where $\texttt{Res}(X_U) = n + 1$. Thus there is some $\varphi \in X_U$. Letting X' be the SL tree that results from resolving φ on every open branch in X, we may observe that $\varphi \notin X'_U$. Consider the following cases where $\star \in \{\wedge, \vee, \supset, \equiv\}$ is a binary connective:

Case 1: If $\varphi = \neg\neg\psi$ and $\texttt{Res}(\psi) \neq 0$, then $X'_U = (X_U/\{\varphi\}) \cup \{\psi\}$. If instead $\texttt{Res}(\psi) = 0$, then $X'_U = X_U/\{\varphi\}$. Since $\texttt{Res}(\psi) = \texttt{Res}(\varphi) - 1$, it follows either way that $\texttt{Res}(X'_U) \leq \texttt{Res}(X_U) - 1 = n$.

Case 2: If $\varphi = \psi \star \chi$ and $\texttt{Res}(\psi) \neq 0$ and $\texttt{Res}(\chi) \neq 0$, then we know that $X'_U = (X_U/\{\varphi\}) \cup \{\psi, \chi\}$. Since $\texttt{Res}(\psi) + \texttt{Res}(\chi) = \texttt{Res}(\varphi) - 1$, it follows that $\texttt{Res}(X'_U) = \texttt{Res}(X_U) - 1 = n$. If instead $\texttt{Res}(\psi) = 0$ or $\texttt{Res}(\chi) = 0$, then $\texttt{Res}(X'_U)$ will be even smaller, and so $\texttt{Res}(X'_U) \leq n$.

Case 3: If $\varphi = \neg(\psi \star \chi)$ and $\texttt{Res}(\neg\psi) \neq 0$ and $\texttt{Res}(\neg\chi) \neq 0$, then $X'_U = (X_U/\{\varphi\}) \cup \{\neg\psi, \neg\chi\}$. Since $\texttt{Res}(\neg\psi) + \texttt{Res}(\neg\chi) = \texttt{Res}(\varphi) - 1$, it follows that $\texttt{Res}(X'_U) = \texttt{Res}(X_U) - 1 = n$. If instead $\texttt{Res}(\neg\psi) = 0$ or $\texttt{Res}(\neg\chi) = 0$, then $\texttt{Res}(X'_U)$ will be even smaller, and so $\texttt{Res}(X'_U) \leq n$.

Since in all cases $\texttt{Res}(X'_U) \leq n$, it follows by hypothesis that X' can be completed in a finite number of steps. We know by **Task 2** that X' is the result of resolving φ in at most a finite number of branches in X. Since the sum of finite numbers is finite, we may conclude that X may be completed in a finite number of steps. Thus it follows by induction that every tree X can be completed in a finite number of steps.