6. Proofs in SL

- 1. Proofs in SL
- 1.1 Who ordered that???
- 1.2 Conjunction Intro and Elimination (Rules for &)
- 1.3 Conditional Intro and Elim. (Rules for \supset)
- 1.4 Use of subproofs
- 1.5 Disjunction Intro and Elim. (Rules for \lor)
- 1.6 Negation Intro and Elimination
- 1.7 Biconditional Intro and Elimination (<->)
- 1.8 Strategies and examples
- 1.9 The Rules, Reiterated

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- ▶ e.g., common rules such as Modus Ponens, disjunctive syllogism

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- Perhaps our natural deduction system explains the success of our ordinary inference patterns

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 - etc.
- ► If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

- 1. Sarah lives in Chicago or Erie.
- 2. Amir lives in Chicago unless he enjoys hiking.
- 3. If Amir lives in Chicago, Sarah doesn't.
- 4. Neither Sarah nor Amir enjoy hiking.
- : Sarah lives in Erie.
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- 7. Sarah doesn't live in Chicago (from 3 and 6).
- 8. Sarah lives in Erie (from 1 and 7).

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 - justify something using P & Q (i.e. to 'eliminate' &).

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b. Conjunction Intro and Elimination (Rules for &)

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Rules for &

We'll illustrate using exercises in our Week 6 Practice Problems on Carnap.

1	A & (B & C)	:PR
2	A	:1 & E
3	B & C	:1 & E
4	A B & C C A & C	:3 & E
5	A & C	:2,4 & I

c. Conditional Intro and Elim.

(Rules for ⊃)

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$\textbf{Eliminating}\supset$

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▶ (When inferring from $A \supset \sim C$ and A to $\sim C$, the role of \mathcal{P} is played by A and role of \mathcal{Q} by $\sim C$.)

Elimination rule for >

$$m \mid \mathcal{P} \supset \mathcal{Q}$$
 $n \mid \mathcal{P}$
 $\mathcal{Q} : m, n \supset \mathsf{E}$

Let's illustrate this rule using an exercise in Carnap: we show that $\{A \& B, A \supset C, B \supset D\} \models C \& D$.

A & B :PR $A\supset C$:PR $B\supset D$:PR :1 & E 5 :2,4⊃E 6 :1 & E В 7 :3, 6 ⊃E D

:5,7 & I

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- ▶ It's a good move, because if $\mathcal{R}, \mathcal{P} \models \mathcal{Q}$ then $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.

Fact

If \mathcal{R} , $\mathcal{P} \models \mathcal{Q}$ then $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.

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 - 2. So any valuation must make \mathcal{R} false, \mathcal{P} false, or \mathcal{Q} true.
 - 3. If it makes \mathcal{R} false, it's not a counterexample to $\mathcal{R} \vDash \mathcal{P} \supset \mathcal{Q}$.

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 - 4. If it makes \mathcal{P} false, it makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.

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 - 5. If it makes \mathcal{Q} true, it also makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.

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 - 2. So any valuation must make $\mathcal R$ false, $\mathcal P$ false, or $\mathcal Q$ true.
 - 3. If it makes \mathcal{R} false, it's not a counterexample to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.
 - 4. If it makes \mathcal{P} false, it makes $\mathcal{P} \supset \mathcal{Q}$ true, so it's not a counterexample.
 - 5. If it makes $\mathcal Q$ true, it also makes $\mathcal P\supset \mathcal Q$ true, so it's not a counterexample.
- ▶ So, there are no counterexamples to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.

Fact

- ▶ If $\mathcal{R}, \mathcal{P} \models \mathcal{Q}$ then every TVA makes \mathcal{R} or \mathcal{P} false or it makes \mathcal{Q} true
- ▶ Let's show that no valuation is a counterexample to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$:
 - 1. A valuation that makes $\mathcal R$ and $\mathcal P$ true, but $\mathcal Q$ false, is impossible if $\mathcal R, \mathcal P \vDash \mathcal Q$.
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 - 5. If it makes $\mathcal Q$ true, it also makes $\mathcal P\supset\mathcal Q$ true, so it's not a counterexample.
- ▶ So, there are no counterexamples to $\mathcal{R} \models \mathcal{P} \supset \mathcal{Q}$.
- ▶ If $\mathcal{R} = \emptyset$, then from $\mathcal{P} \models \mathcal{Q}$ we can infer $\models \mathcal{P} \supset \mathcal{Q}$

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- ▶ Justification of $P \supset Q$ is the **entire** subproof (use a HYPHEN)
- ► Important: nothing inside a subproof is available outside as a justification (since inner lines might depend on the assumption)

Introduction rule for ⊃

$$m \mid \frac{\mathcal{P}}{\mathcal{P}}$$
 :AS for $\supset I$

$$\vdots$$

$$p \mid \mathcal{Q}$$

$$\mathcal{P} \supset \mathcal{Q} : m-n \supset I$$

NOTE THE HYPHEN IN THE JUSTIFICATION LINE!!!

We'll illustrate using more exercises from Week 6 Practice Problems

- ▶ Show: $\{A \supset B, B \supset C\} \models A \supset C$.
- ► Show: $A \supset (B \supset C) \models (A \& B) \supset (A \& C)$

1
$$A \supset B$$
 :PR
2 $B \supset C$:PR
3 $A : AS \text{ for } \supset I$
4 $B : 1, 3 \supset E$
5 $C : 2, 4 \supset E$
6 $A \supset C : 3-5 \supset I$

1

$$A \supset (B \supset C)$$
 :PR

 2
 $A \& B$
 :AS for \supset I

 3
 A
 :2 & E

 4
 $B \supset C$
 :1, 3 \supset E

 5
 B
 :2 & E

 6
 C
 :4, 5 \supset E

 7
 $A \& C$
 :3, 6 & I

 8
 $(A \& B) \supset (A \& C)$
 :2-7 \supset I

6. Proofs in SL

d. Use of subproofs

Reiteration (for the 11th hour!)

 $P \models P$, so "Reiteration" R is a good rule:

$$\begin{array}{c|c}
m & \mathcal{P} \\
k & \mathcal{P} & :m & \mathbf{R}
\end{array}$$

Uses of reiteration (to the Carnap!):

▶ Proof of $A \models A$.

Reiteration (for the 11th hour!)

$$P \models P$$
, so "Reiteration" R is a good rule:

$$k \mid \mathcal{P} : m \mid R$$

Uses of reiteration (to the Carnap!):

- ▶ Proof of $A \models A$.
- ▶ Proof that $A \supset (B \supset A)$ is a tautology.

$$\begin{array}{c|cc}
1 & A & :AS \text{ for } \supset I \\
2 & A & :1 R \\
3 & A \supset A & :1-2 \supset I
\end{array}$$

Again, note the **HYPHEN!** Even though our subproof is only two lines, we still write ':1-2' and NOT ':1, 2'.

1 |
$$A$$
 : AS for \supset I
2 | B : AS for \supset I
3 | A : 1 R
4 | $B \supset A$: 2-3 \supset I
5 | $A \supset (B \supset A)$: 1-4 \supset I

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- ► Subproofs (subproofs can be nested) can be nested
- You also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is completed (since again you'd be 'outside the scope' of those lines)

Which are correct applications of R?

Τ		A		:AS		
2			A	:AS	5	
3			A	:1	R	
4		A	A	:1	R	
5		1	A	:2	R	
6	Α			:2	R	
7	Α			:1	R	

1 | 4 .40

Which are correct applications of R?

1		A		:AS	5
2			Α	:AS	5
3			Α	:1、	/ R
4		A	١	:1	R
5		A	١	:2	R
6	A			:2	R
7	Α			:1	R

Which are correct applications of R?

1		<i>A</i>	:AS	
2		Α	:AS	
3		Α	:1 √ R	
4		A	:1 √ R	
5		A	:2 R	
6	A		:2 R	
7	Α		:1 R	

Which are correct applications of R?

1		A	\	:AS	
2			Α	:AS	
3			Α	:1 √ R	
4		A	١	:1 √ R	
5		A	١	:2 X R	
6	A			:2 R	
7	Α			:1 R	

Which are correct applications of R?

1		A		:AS
2			Α	:AS
3			Α	:1 √ R
4		A	١	:1 √ R
5		A	١	:2 X R
6	A			:2 X R
7	Α			:1 R

Which are correct applications of R?

1		4	:AS
2		Α	:AS
3		Α	:1 √ R
4		4	:1 √ R
5		4	:2 X R
6	A		:2 X R
7	Α		:1 x R

6. Proofs in SL

(Rules for ∨)

e. Disjunction Intro and Elim.

Introduction rule for \lor

We have $P \models P \lor Q$. So:

► Note that the introduced disjunct can be ANYTHING!

Introduction rule for \lor

We have $P \models P \lor Q$. So:

- ► Note that the introduced disjunct can be ANYTHING!
- ► And you can introduce on the left OR right side!

Introduction rule for \(\neq \)

We have $P \models P \lor Q$. So:

- ► Note that the introduced disjunct can be ANYTHING!
- ► And you can introduce on the left OR right side!
- ► Let's do practice problem 6.10 on Carnap!

$$\begin{array}{c|ccc}
1 & A & :AS & for \supset I \\
2 & B \lor A & :1 \lor I \\
3 & A \supset (B \lor A) & :1-2 \supset I
\end{array}$$

Eliminating \vee (Proof by Cases)

 \blacktriangleright What can we justify with disjunction $\mathcal{P}\vee\mathcal{Q}?$

Eliminating \(\text{(Proof by Cases)} \)

▶ What can we justify with disjunction $P \lor Q$?

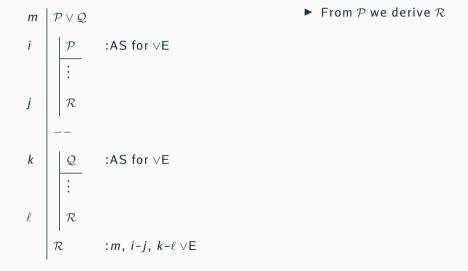
▶ Not $\mathcal P$ and also not $\mathcal Q$: neither is entailed by $\mathcal P \vee \mathcal Q$.

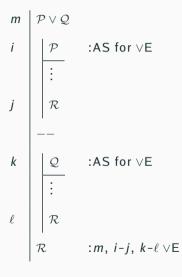
Eliminating \vee (**Proof by Cases**)

- ▶ What can we justify with disjunction $P \lor Q$?
- ▶ Not \mathcal{P} and also not \mathcal{Q} : neither is entailed by $\mathcal{P} \vee \mathcal{Q}$.
- ▶ But: if both \mathcal{P} and \mathcal{Q} separately entail some third sentence \mathcal{R} , then we know that \mathcal{R} follows from the disjunction!

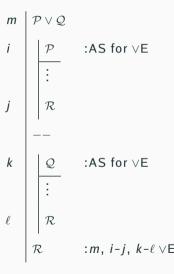
Eliminating \(\text{(Proof by Cases)} \)

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- ▶ But: if both \mathcal{P} and \mathcal{Q} separately entail some third sentence \mathcal{R} , then we know that \mathcal{R} follows from the disjunction!
- ▶ To show this, we need **two** subproofs that show \mathcal{R} , but in each proof we are allowed to use only one of \mathcal{P} , \mathcal{Q} .

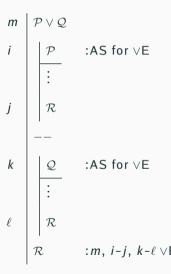




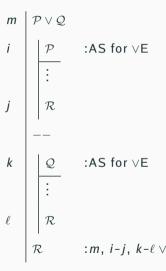
- ightharpoonup From $\mathcal P$ we derive $\mathcal R$
- ► Start a subproof for each disjunct



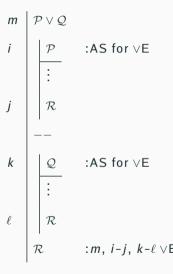
- ightharpoonup From $\mathcal P$ we derive $\mathcal R$
- Start a subproof for each disjunct
- ► The subproofs need not be adjacent, but if they are, separate with --



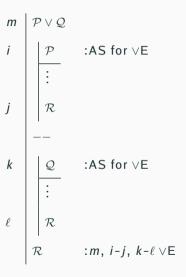
- ightharpoonup From $\mathcal P$ we derive $\mathcal R$
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- ightharpoonup From $\mathcal Q$ we derive $\mathcal R$



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- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction



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- Remember to cite BOTH subproofs (hyphens!), AND the line with the disjunction
- Remember to pop out of subproof level at the end!

$$\begin{array}{c|cccc}
1 & A \lor B & :PR \\
\hline
2 & A & :AS for \lor E \\
\hline
3 & B \lor A & :2 \lor I \\
\hline
4 & B & :AS for \lor E \\
\hline
5 & B \lor A & :4 \lor I \\
\hline
6 & B \lor A & :1, 2-3, 4-5 \lor E
\end{array}$$

► In Carnap: Need -- between the subproofs

$$\begin{array}{c|cccc}
1 & A \lor B & :PR \\
2 & A & :AS \text{ for } \lor E \\
3 & B \lor A & :2 \lor I \\
4 & B & :AS \text{ for } \lor E \\
5 & B \lor A & :4 \lor I \\
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\end{array}$$

- ► In Carnap: Need -- between the subproofs
- ▶ Note: need the **SAME sentence** as the last line of each subproof

1
$$A \lor B$$
 :PR
2 A :AS for \lor E
3 $B \lor A$:2 \lor I
4 B :AS for \lor E
5 $B \lor A$:4 \lor I
6 $B \lor A$:1, 2-3, 4-5 \lor E

- ► In Carnap: Need -- between the subproofs
- ▶ Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself

1
$$A \lor B$$
 :PR
2 $A \lor A$:AS for $\lor E$
3 $B \lor A$:2 $\lor I$
4 $B \lor A$:AS for $\lor E$
5 $B \lor A$:4 $\lor I$
6 $B \lor A$:1, 2-3, 4-5 $\lor E$

- ► In Carnap: Need -- between the subproofs
- ► Note: need the **SAME sentence** as the last line of each subproof
- Note the complex justification structure: (a) line with disjunction, (b) first subproof, (c) second subproof, (d) the rule itself
- ► Proceed to Carnap PP6.15!

1
$$A \lor B$$
 :PR
2 $A \supset B$:PR
3 $A : AS \text{ for } \lor E$
4 $B : 2, 3 \supset E$
5 $B : AS \text{ for } \lor E$
6 $B : 5 \text{ R}$
7 $B : 1, 3-4, 5-6 \lor E$

6. Proofs in SL

f. Negation Intro and Elimination

Introducing \sim

► An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.

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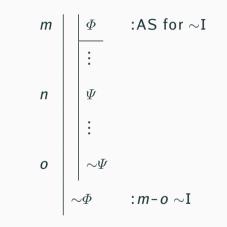
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- ► For instance:
 - $Q \models \mathcal{P}$ iff Q and $\sim \mathcal{P}$ are jointly unsatisfiable.

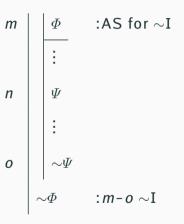
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- ► For instance:
 - $Q \models P$ iff Q and $\sim P$ are jointly unsatisfiable.
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- ▶ This last one gives us idea for \sim I rule: To justify \sim \mathcal{P} , show that \mathcal{P} (together with all other premises) is unsatisfiable.

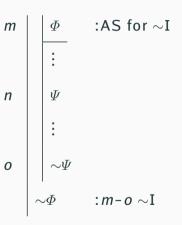
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 - $Q \models P$ iff Q and $\sim P$ are jointly unsatisfiable.
 - $Q \vDash \sim P$ iff Q and P are jointly unsatisfiable.
- ▶ This last one gives us idea for \sim I rule: To justify \sim \mathcal{P} , show that \mathcal{P} (together with all other premises) is unsatisfiable.
- ► Unsatisfiable means: a contradiction follows!



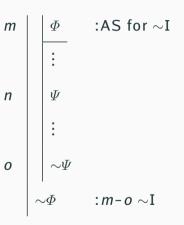
► Assume the **non**-negated wff!



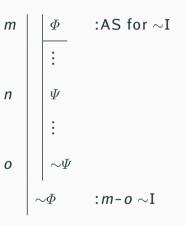
- Assume the non-negated wff!
- ▶ Derive a sentence and its negation (could be Φ !)



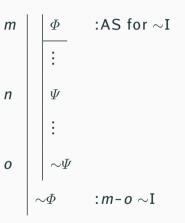
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- Pop out of the subproof and introduce that negativity!

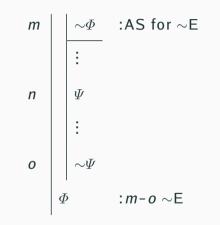


- Assume the non-negated wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
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- Remember to cite the WHOLE subproof (hyphen!)

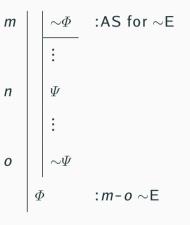


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- ► Let's try exercise PP6.21:

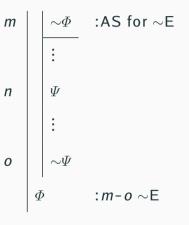
1
$$A \supset B$$
:AS for \supset I2 $\sim B$:AS for \supset I3 A :AS for \sim I4 B :1, 3 \supset E5 $\sim B$:2 R6 $\sim A$:3-5 \sim I7 $\sim B \supset \sim A$:2-6 \supset I8 $(A \supset B) \supset (\sim B \supset \sim A)$:1-7 \supset I



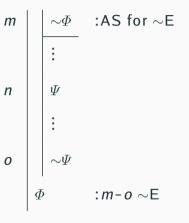
► Assume the **negated** wff!



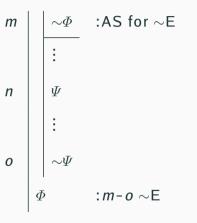
- ► Assume the **negated** wff!
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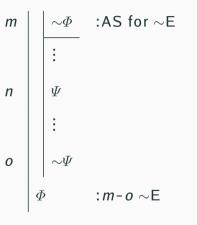
- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)



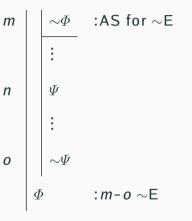
- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and eliminate that negativity!



- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- ▶ Pop out of the subproof and eliminate that negativity!
- ▶ Put a smile on!



- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- Pop out of the subproof and eliminate that negativity!
- ▶ Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)



- ► Assume the **negated** wff!
- ▶ Derive a sentence and its negation (could be Φ !)
- (Ψ and $\sim\Psi$ can appear in opposite order)
- ▶ Pop out of the subproof and eliminate that negativity!
- ► Put a smile on!
- Remember to cite the WHOLE subproof (hyphen!)
- ► Let's try exercise PP6.22:

$$\begin{array}{c|cccc}
1 & \sim A \supset \sim B & :AS \text{ for } \supset I \\
2 & B & :AS \text{ for } \supset I \\
3 & \sim A & :AS \text{ for } \sim E \\
4 & \sim B & :1, 3 \supset E \\
5 & B & :2 R \\
6 & A & :3-5 \sim E \\
7 & B \supset A & :2-6 \supset I \\
8 & (\sim A \supset \sim B) \supset (B \supset A) & :1-7 \supset I
\end{array}$$

6. Proofs in SL

g. Biconditional Intro and Elimination (<->)

Biconditional Introduction (\equiv I) (Type <-> !!!)

Like doing conditional intro twice, from both directions

Biconditional Introduction (\equiv I) (Type <->!!!)

$$i$$
 A
 A
 A
 A
 A
 $A \equiv B$
 A
 A
 $A = I$
 A
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- Like doing conditional intro twice, from both directions
- You can swap the order of the subproofs

Biconditional Introduction (\equiv I) (Type <-> !!!)

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- ► Remember to pop out of subproof line!

6.q.1

$$egin{array}{c|c} m & \mathcal{A} \equiv \mathcal{B} \\ n & \mathcal{A} \\ \mathcal{B} & :m, \ n \equiv \mathbb{E} \\ \hline m & \mathcal{A} \equiv \mathcal{B} \\ n & \mathcal{B} \end{array}$$

Just like conditional elimination!

$$m \mid \mathcal{A} \equiv \mathcal{B}$$
 $n \mid \mathcal{A}$
 $\mathcal{B} : m, n \equiv \mathbf{E}$
 $m \mid \mathcal{A} \equiv \mathcal{B}$

- Just like conditional elimination!
- Only now you can eliminate from either side! (power!)

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:*m*, *n* ≡E

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- ► There can be lines between lines m and n

$$m \mid A \equiv B$$
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 $B : m, n \equiv E$
 $m \mid A \equiv B$

:*m*, *n* ≡E

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▶ For Carnap to recognize \equiv I or \equiv E in the justification column, you sadly must type <-> I or <-> E

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- You can also copy/paste the ≡symbol from elsewhere on the page!

6. Proofs in SL

h. Strategies and examples

Working forward and backward

► Working backward from a conclusion (goal) means:

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 - Find main connective of premise, assumption, or sentence
 - Match with top premise of corresponding E rule

- Working backward from a conclusion (goal) means:
 - Find main connective of goal sentence
 - Match with conclusion of corresponding Intro rule
 - Write out (above the goal!) what you'd need to apply that rule
- Working forward from a premise, assumption, or already justified sentence means:
 - Find main connective of premise, assumption, or sentence
 - Match with top premise of corresponding E rule
 - Write out what else you need to apply the E rule (new goals)

- Working backward from a conclusion (goal) means:
 - Find main connective of goal sentence
 - Match with conclusion of corresponding Intro rule
 - Write out (above the goal!) what you'd need to apply that rule
- Working forward from a premise, assumption, or already justified sentence means:
 - Find main connective of premise, assumption, or sentence
 - Match with top premise of corresponding E rule
 - Write out what else you need to apply the E rule (new goals)
 - If necessary, write out conclusion of the rule

Write out premises at the top (if there are any)

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 - Work forward using & E

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 - Work backward using & I, \supset I, \equiv I, \sim I/E, or forward using \vee E
 - Work forward using & E
 - Work forward using $\supset E$, $\equiv E$

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- Work backward & forward from goals and premises/assumptions in this order:
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 - Work forward using & E
 - Work forward using $\supset E$, $\equiv E$
 - Work backward from ∨I

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- Work backward & forward from goals and premises/assumptions in this order:
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 - Try Negation Intro or Elimination, working toward a contradiction
- Repeat for each new goal from top

6. Proofs in SL

i. The Rules, Reiterated

The rules, one more time: Reiteration

```
m \mid \mathcal{P}
\vdots
k \mid \mathcal{P} : m \mid R
```

- Remember that you must be in the scope of the line you're reiterating
- e.g. if you're outside a subproof, you can't reiterate anything wholly within the subproof

The rules: Conjunction Intro (& I) and Elimination (& E)

The rules: Conditional Intro $(\supset I)$ and Elim $(\supset E)$

$$m \mid \frac{\mathcal{P}}{\mathcal{P}} : AS \text{ for } \supset I$$

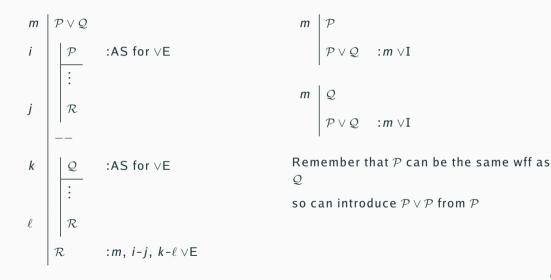
$$\vdots \qquad \qquad m \mid \mathcal{P} \supset \mathcal{Q}$$

$$n \mid \mathcal{Q} \qquad \qquad n \mid \mathcal{P} \supset \mathcal{Q}$$

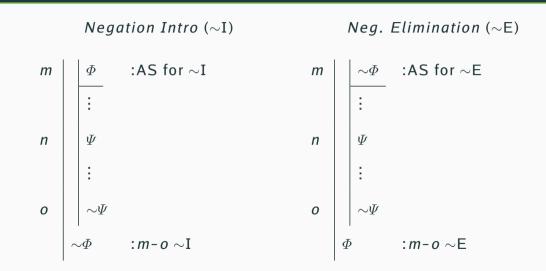
$$\mathcal{P} \supset \mathcal{Q} : m-n \supset I$$

$$m \mid \mathcal{P} \supset \mathcal{Q} \qquad m, n \supset E$$

The rules: Disjunction Intro (\lor I) and Elimination (\lor E)



The rules: Negation Intro and Elimination



Note that you can swap the order of Ψ and $\sim \Psi$ in the subproofs!

6.i.5

The rules: Biconditional Intro and Elimination (<->)