Semantic Proofs

LOGIC I
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From Before...

Semantics: For any interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, the VALUATION function $\mathcal{V}_{\mathcal{I}}$ from the wfs of $\mathcal{L}^{\text{PL}}$ to truth-values is defined:

- $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ if φ is a sentence letter of $\mathcal{L}^{\text{PL}}$.
- $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).
- $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.
- $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).
- $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Formal Definitions

Interpretation: $\mathcal{I}$ is an *interpretation* of $\mathcal{L}^{\text{PL}}$ *iff* $\mathcal{I}(\varphi) \in \{1, 0\}$ for every sentence letter φ of $\mathcal{L}^{\text{PL}}$.

Tautology: φ is a *tautology iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\mathcal{I}$.

Contradiction: φ is a *contradiction iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ for all $\mathcal{I}$.

Logically Contingent: φ is *contingent iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \neq \mathcal{V}_{\mathcal{J}}(\varphi)$ for some $\mathcal{I}$ and $\mathcal{J}$.

Logical Entailment: φ *entails* ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) \leq \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Logical Equivalence: φ is *equivalent* to ψ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$ for all $\mathcal{I}$.

Satisfiable: Γ is *satisfiable iff* $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$ for some $\mathcal{I}$.

Logical Consequence: $\Gamma \vDash \varphi$ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ whenever $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Satisfiability

Which sets of sentences are satisfiable?

Taller

(1) Liza is taller than Sue.

(2) Sue is taller than Paul.

(3) Paul is taller than Liza.

Lost

(4) Kim is either in Somerville or Cambridge.

(5) If Kim is in Somerville, then she is not far from home.

(6) If Kim is not far from home, then she is in Cambridge.

(7) Kim is not in Cambridge.

Validity

Arguments: Sequences of wfss of $\mathcal{L}^{\text{PL}}$, not sets.

Valid: For any argument, it is valid *iff* its conclusion is a logical consequence of its set of premises.

- Many arguments may have the same set of premises.
- An argument is valid *iff* its conclusion is true in every interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ to satisfy the set of premises.

Tautology: A wfs φ of $\mathcal{L}^{\text{PL}}$ is a *tautology* just in case $\vDash \varphi$.

- Every $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ satisfies the empty set.
- Each premise constrains the set of interpretations the conclusion must be true in where the empty set has no constraints.

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

- Each wfs of $\mathcal{L}^{\text{PL}}$ corresponds to a set of all interpretations which make that sentence true: $|\varphi| := \{\mathcal{I} : \mathcal{V}_{\mathcal{I}}(\varphi) = 1\}$.
- Is the interpretation set for the conclusion a subset of the intersection of the premise interpretation sets?

Examples

1. Show that $\neg R \to \neg Q,\ P \wedge Q \vDash P \wedge R$.
2. Show that $A \vee B,\ B \to C,\ A \leftrightarrow C \vDash C$.
3. Show that $P,\ P \to Q,\ \neg Q \vDash A$.
4. Show that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology.
5. Show that $A \leftrightarrow \neg A$ is a contradiction.
6. Show that $\{P,\ P \to Q,\ Q \to \neg P\}$ is unsatisfiable.
7. Show that $\{P \to Q,\ \neg P \vee \neg Q,\ Q \to P\}$ is satisfiable.

Observe: There seem to be patterns.

Question: How could we systematize these proofs?

Methods

Truth Tables:	Mechanical but tedious.
•	Bad if there are lots of sentence letters.
•	Good for counterexamples. $A \leftrightarrow (B \rightarrow C),\ A \wedge \neg B,\ D \vee \neg A \vDash C$.
Semantic Arguments:	Good if there are lots of sentence letters. $(A \vee B) \rightarrow (C \wedge D),\ \neg C \wedge \neg E \vDash \neg A$.

The Material Conditional

Roses

A1. Sugar is sweet.

A2. The roses are only red if sugar is sweet.

Observe: First paradox of the material conditional.

Vacation

B1. Casey is not on vacation.

B2. If Casey is on vacation, then he is in Paris.

Observe: Second paradox of the material conditional.

Crimson

C1. Mary doesn't like the ball unless it is crimson.

C2. Mary likes the ball.

C3. If the ball is blue, then Mary likes it.

The Biconditional

Rectangle

D1. The room is a square.

D2. The room is a rectangle.

D3. The room is a square if and only if it is a rectangle.

Work

E1. Kin isn't a professor.

E2. Sue isn't a chef.

E3. Kin is a professor just in case Sue is a chef.