The Connectives

LOGIC I
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Definitions

Previously: We considered the sentences that could be constructed from the sentence letters with the connectives. We will now seek to specify this construction precisely.

Object Language: We will be concerned to define the sentences of SL, where the language of SL $= \langle \mathbb{L}, \neg, \wedge, \vee, \supset, \equiv, (,) \rangle$ will be referred to as the OBJECT LANGUAGE.

Strings: An EXPRESSION of SL is any finite string of symbols from the language of SL.

Quotation: To talk about strings we will need to name them, where a quoted string is the CANONICAL NAME for the string quoted.

Use/Mention: We mention expressions by putting them in quotes, where otherwise they are used.

Example 1: 'Sue' is a three letter name but Sue is not.

Example 2: The complex sentence '$A \supset B$' includes the sentence letters 'A' and 'B'.

Metalanguage: We talk about the expressions of SL with the resources of our METALANGAUGE mathematical English.

Metalinguistic Variables: Letting $\varphi, \psi, \chi, \ldots$ be variables whose values are expressions, we may quantify over the expressions of SL in order to define the wffs of SL.

The Sentences of SL

1. Every atomic sentence in $\mathbb{L}$ is a wff of SL.
2. If φ and ψ are wffs of SL, then:
   (a) $\neg\varphi$ is a wff of SL;
   (b) $(\varphi \wedge \psi)$ is a wff of SL;
   (c) $(\varphi \vee \psi)$ is a wff of SL;
   (d) $(\varphi \supset \psi)$ is a wff of SL; and
   (e) $(\varphi \equiv \psi)$ is a wff of SL.
3. Nothing else is a wff of SL.

Observations and Conventions

Corner Quotes: Strictly speaking, this definition is non-sense and we need to use corner quotes to fix it.

Well-formed Formulas: The wffs are the grammatical expressions of SL of type t, and so candidates for interpretation.

Sentences: Since all wffs in SL are *good* candidates for interpretation (it makes sense to assign them truth-values), we may identify the wffs with the sentences of SL. By contrast, not all the wffs of QL are sentences of QL.

Sentential Variables: Restrict $\varphi, \psi, \chi, \ldots$ to sentences of SL.

Task 1: Build increasingly complex sentences from just A.

Conventions: We will often drop quotes and parentheses for ease: $A \vee B \vee C$ vs $A \vee B \wedge C$.

Therefore: $\therefore$ is not part of SL.

Truth Functionality

Sentential Operators: The connectives are SENTENTIAL OPERATORS which map sentences to sentences.

Interpretations: Last time we said that an INTERPRETATION $\mathcal{I}$ assigns truth-values to sentence letters.

Valuation: We may then define a VALUATION function $\mathcal{V}_{\mathcal{I}}$ which assigns truth-values to every sentence of SL by way of the following semantic clauses:

(A) $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{I}(\varphi)$ iff φ is a sentence letter of SL.

($\neg$) $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ (i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$).

($\wedge$) $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$.

($\vee$) $\mathcal{V}_{\mathcal{I}}(\varphi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).

($\supset$) $\mathcal{V}_{\mathcal{I}}(\varphi \supset \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ (or both).

($\equiv$) $\mathcal{V}_{\mathcal{I}}(\varphi \equiv \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$.

Homophonic Semantics: The semantics for $\neg$, $\wedge$, and $\vee$ use analogous operators in the metalanguage, but not so for $\supset$ and $\equiv$.

Quantified Logic: Later in the course, we will provide semantic clauses for the quantifiers 'for all' and 'there is' which will not have homophonic semantic clauses.

Negation

Uninitiated

(1) If Sam attended the gathering, then he has been initiated.

(2) Sam is uninitiated.

∴ Sam did not attend the gathering.

Observe: Being uninitiated is the same as not being initiated.

Uninvited

(1) Arden is not invited.

∴ Arden is uninvited.

Observe: Arden can fail to be invited without being uninvited.
Question: What about the converse?

The Material Conditional

Square

(1) The rug is square.

∴ If the rug is triangular, then it is square.

Regiment: Use S and T to regiment this argument.
Observe: It is natural to provide a modal reading for the conclusion.
Conclude: The material conditional does not track possibilities.

1 Valid Arguements

1.1 Definitions

V1: An argument is **valid if and only if** the premises are true and you can't imagine a scenario where the conclusion is false.

- Problem: requires the premises to be true. But we don't necessarily know if the premises are true. And we still want to be able to draw logical inferences from the premises.

V2: An argument is **valid if and only if** the premises logically entail the conclusion.

- Problem: this is better, but 'logically entails' is just a synonym for validity. So this is not that informative. Since we'd need to know what logical entailment is.

(Good) V3: An argument is **valid if and only if** in *every interpretation/conceptual-scenario/"possibility"* in which the premises are true, then the conclusion must also be true.

- Correct: it is important that the "possibility" is something conceptual/interpretational not something about physics or the way the world must be as a matter of its physical nature.

Argument 1: P1 Gyre is a mome rath.

$\therefore$ Gyre or Jake is a mome rath.

(Unsound because premises are false)

P1 Ben is sitting.

P2 If Ben is sitting, then Jake is standing.

$\therefore$ Ben or Jake is sitting.

P1 Ben is sitting.

P2 If Ben is sitting, then Jake is standing.

$\therefore$ Jake is standing.

P1 Jake is not standing.

P2 If Ben is sitting, then Jake is standing.

$\therefore$ Ben is not sitting.

Try 1: to invalidate by making conclusion false and premises true.

Since conclusion is false: then Ben is sitting.

But if P2 is true: then P1 is false

So: no way to make premises both true if conclusion is false.

Try 2: assume premises are true; check to see if conclusion is true.

So: **assume P1** and P2 are true.

Assume that the conclusion is false (to see if we get into trouble).

If the conclusion is false, then Ben is sitting.

But by P2, it follows that P1 must then be false, contrary to our assumption.

So we were wrong to think it is possible for the conclusion to be false given that the premises are true.

(Unsound because premises are false)

P1 Ben is sitting.
P2 If Ben is sitting, then Jake is standing
P3 Pigs can fly.
∴ Ben or Jake is sitting.

2 Invalid Arguements

Argument 1: P1 The speed of light is c.