The Soundness of SL Tree Proofs

LOGIC I
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Informal Proof

Motive: We want to know which arguments are valid.

Equivalence: $\Sigma \vDash \varphi$ *iff* $\Sigma, \neg\varphi \vDash \bot$.

Soundness: Letting $\Gamma = \Sigma \cup \{\neg\varphi\}$, we want to show that $\Gamma \vDash \bot$ if $\Gamma \vdash \bot$.

Informally: We want to show that every closed tree has an unsatisfiable root.

Question 1: Why can't we use our tree method (or similar) to prove soundness?

Definitions

Root: An SL tree whose root contains the sentences in Γ is a tree *with* root Γ.

Branch Satisfaction: An SL interpretation $\mathcal{I}$ *satisfies* a branch $\mathcal{B}$ in an SL tree X just in case $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for every φ which occurs in $\mathcal{B}$.

Setting up the Proof

Contrapositive: Every SL tree with a satisfiable root is closed.

Lemma 3: Every SL tree with a satisfiable root has a satisfiable branch.

Question 2: How can we derive soundness from this stronger claim?

Question 3: How can we prove *Lemma 4*?

Supporting Lemmas

Lemma 1: Every satisfiable branch $\mathcal{B}$ in an SL tree X is open.

Lemma 2: If X is an SL tree with a satisfiable branch $\mathcal{B}$, then any tree X' which is the result of resolving a sentence in $\mathcal{B}$ has a satisfiable branch $\mathcal{B}'$.

- Assume X has a satisfiable branch $\mathcal{B}$.
- So there is some $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all φ in $\mathcal{B}$.
- By *Lemma 1*, $\mathcal{B}$ is open.
- If $\mathcal{B}$ is complete, then the consequent holds vacuously.
- If $\mathcal{B}$ is not complete, then $\mathcal{B}$ has a resolvable sentence φ.
- There are nine cases to check given our nine resolution rules.

Lemma 3

Proof: Every SL tree with a satisfiable root has a satisfiable branch.

Antecedent: Assume $\Gamma \nvDash \perp$.

Base: Let X be an tree with root Γ where $\texttt{Length}(X) = 0$.

Hypothesis: Every tree X with root Γ of $\texttt{Length}(X) = n$ has a satisfiable branch $\mathcal{B}$.

Induction: Assume X' is a tree with root Γ of $\texttt{Length}(X') = n + 1$.

1. Let X be any tree with root Γ where X' is the result of resolving a sentence φ in a branch $\mathcal{B}$ of X.
2. So X is a tree with root Γ of $\texttt{Length}(X) = n$.
3. By hypothesis, X has a satisfiable branch $\mathcal{B}^\star$.
4. So either $\mathcal{B}^\star = \mathcal{B}$ or not.

Case 1: Assume $\mathcal{B}^\star = \mathcal{B}$.

(a) We know that X' is the result of resolving φ in $\mathcal{B}$.
(b) By the case assumption $\mathcal{B} = \mathcal{B}^\star$.
(c) Since $\mathcal{B}^\star$ is satisfiable, X' has a satisfiable branch $\mathcal{B}'$ by *Lemma 2*.

Case 2: Assume $\mathcal{B}^\star \neq \mathcal{B}$.

(a) We know $\mathcal{B}^\star$ is a satisfiable branch of X.
(b) X' is the result of resolving φ in $\mathcal{B} \neq \mathcal{B}^\star$ of X.
(c) So $\mathcal{B}^\star$ is also a branch of X'.
(d) Since $\mathcal{B}^\star$ is satisfiable, X' has a satisfiable branch.

5. Thus X' has a satisfiable branch whether $\mathcal{B}^\star = \mathcal{B}$ or not.
6. Every tree X' with root Γ of $\texttt{Length}(X') = n + 1$ has a satisfiable branch $\mathcal{B}$.

Conclusion: By weak induction, QED.

Proving Soundness

Proof: If there is a closed SL tree with root Γ, then Γ is unsatisfiable.

1. Assume Γ is satisfiable.
2. Let X be an SL tree with root Γ.
3. So X has a satisfiable branch $\mathcal{B}$ by *Lemma 3*.
4. So $\mathcal{B}$ is open by *Lemma 1*.
5. So X is not closed.
6. More generally, there is no closed SL tree with root Γ.
7. By contraposition, QED.