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a. A Meta-refresher

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- But our derivations are justified only if system SND is *sound*
- And guaranteed to have a derivation for every valid argument only if system SND is *complete*

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- Single Turnstile Natural $\vdash_{SND}$: **derivability** in SND

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- We'll use this last fact A LOT in our proof that SND is complete!

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  - (Means: we wrote down *enough* rules!)

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- Not in practice: no valid SL argument ever requires infinitely-many premises to entail its conclusion (PS 12 #4)

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- If there is no such TVA, then $\Gamma \vDash \Theta$

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- In this case, we write $\Gamma \vdash_{SND} \Theta$ (also: $\Delta \vdash_{SND} \Theta$)
- If no such derivation exists, then we say that Θ is NOT SND-derivable from Γ, and we write $\Gamma \nvdash_{SND} \Theta$

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b. Soundness of System SND

Soundness: Proof Idea and notation

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- (like with soundness for trees, we reason "from the top down")

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- Induction step: show the property holds for line #k+1 (by considering all possible ways line #k+1 could arise)

Let's get Righteous!

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- Our goal is to prove that every derivation in SND is righteous!

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- So there is no truth-value assignment that makes all the sentences in Γ true while making $\mathcal{P}$ false, so $\Gamma \vDash \mathcal{P}$ as well
- So we will have shown **Soundness**: If $\Gamma \vdash_{SND} \mathcal{P}$, then $\Gamma \vDash \mathcal{P}$

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- So line #1 is righteous (i.e. $\Gamma_1 \vDash \mathcal{P}_1$)

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- What is the 12th case?? (We could say 13, but that is BAD LUCK)

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- So $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$; so this case be righteous!

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Case 2: Reiteration

- **Case 2**: $\mathcal{P}_{k+1}$ arises from an application of rule R, reiteration
- Then wff $\mathcal{P}_{k+1}$ appears on an earlier line #i as the wff $\mathcal{P}_i$
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- Since $\Gamma_i \vDash \mathcal{P}_{k+1}$ and $\Gamma_i \subseteq \Gamma_{k+1}$, we have $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$
- Draw a schematic derivation to better understand $\Gamma_i \subseteq \Gamma_{k+1}$!

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- So $\Gamma_{k+1} \vDash \mathcal{P}_{k+1}$ and line #k+1 is righteous

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- Otherwise, Γ_{k+1} satisfies $\mathcal{Q}$. Since $\Gamma_j \subseteq \Gamma_{k+1} \cup \{\mathcal{Q}\}$, this means that Γ_j is satisfied in this case. Then since line #j is righteous, we have $\Gamma_{k+1} \cup \{\mathcal{Q}\} \vDash \mathcal{R}$. So in this case, Γ_{k+1} satisfies $\mathcal{Q} \supset \mathcal{R}$ as well.

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  $\Rightarrow \Gamma_{k+1} \cup \{\mathcal{Q}\}$ is unsatisfiable. Hence, $\Gamma_{k+1} \vDash \sim\mathcal{Q}$

12. Metalogic for SL

c. Completeness of System SND

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- We introduce a new **syntactic** notion of consistency relative to our SND derivation system:

  - a set of SL wffs is **SND-consistent** provided that you can't derive contradictory sentences from it in SND
- Core proof idea: we'll show that if a set of sentences is **consistent-in-SND**, then it is also semantically consistent (i.e. **satisfiable**). So by the contrapositive: if a set is **un**satisfiable, then it is **in**consistent-in-SND.

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- Recall: a set of SL sentences is **satisfiable** provided there is a TVA that makes all of them true
- This is a *semantic* notion of consistency
- i.e. **truth-functionally consistent**
- Contrast this with the syntactic notion of **consistency in SND**:

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- Just remember: this syntactic notion has nothing to do with truth value assignments!

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- Goal: prove the completeness of SL: for every SL wff $\mathcal{P}$ and every set Γ of SL sentences, if $\Gamma \vDash \mathcal{P}$ then $\Gamma \vdash \mathcal{P}$

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  no TVA satisfies the premises and negated conclusion
- We now appeal to a **Consistency lemma** that is the heart of the enterprise: any SND-consistent set of SL sentences is satisfiable (i.e. semantically consistent)

Proof Sketch: Using the consistency lemma

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- **Contrapositive** of CL: any set of SL sentences that is Unsatisfiable is SND-Inconsistent
- From $\Gamma \vDash \mathcal{P}$ we know that $\Gamma \cup \{\sim\mathcal{P}\}$ is unsatisfiable
- So by the contrapositive of CL, we see that $\Gamma \cup \{\sim\mathcal{P}\}$ is SND-inconsistent
- This means that we can derive a pair of contradictory sentences R and $\sim R$ from $\Gamma \cup \{\sim\mathcal{P}\}$! So using the power of negation elimination, we can derive $\mathcal{P}$ from Γ, i.e. $\Gamma \vdash \mathcal{P}$. So we are 'done'!

Negation Elimination Refresher (book's claim 6.4.4)

- Claim: if $\Gamma \cup \{\sim\mathcal{P}\}$ is **SND-inconsistent**, then $\Gamma \vdash \mathcal{P}$

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- The third stage comprises a tedious lemma and induction! PS12 problems 2 and 3 provide practice with this tedium!

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  - So we piggyback on an appropriate Γ^* to show that any SND-consistent set Γ is also satisfiable

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- More precisely, $\Gamma^* := \bigcup_{k=1}^{\infty} \Gamma_k$

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- So with our ordering, 'A' is the first wff; 'B' the second ... up to Z, and then we hit $\sim A$ ($\mapsto$ 1030), then $\sim B$ ($\mapsto$ 1031), etc.

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- We prove these in turn

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- (note: the book's proof, p. 256, is way more complicated than necessary...)

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- Then by the recursive structure of SL wffs, $\mathcal{I}(\mathcal{Q}) = \mathit{True}$ iff $\mathcal{Q} \in \Gamma^*$

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- Moral: We all want to belong, but sometimes our enemies get in the way!

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- Induction step: show that an arbitrary SL wff with k+1–many connectives is clubbin'

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- We proceed to do induction using our SL induction schema: an arbitrary sentence $\mathcal{P}$ with k+1–many connectives has one of five forms, coming from our five connectives.

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So by the truth conditions for negation, $\mathcal{I}(\mathcal{P}) = 1$

Induction on SL: Cases 2–5

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- Case 5: $\mathcal{P}$ has the form $\mathcal{Q} \equiv \mathcal{R}$ (we'll do this case if and only if we accomplish all other goals in our lives)

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- (Schematize this sentence in quantifier logic)