Mathematical Induction

LOGIC I
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Review from Last Time

1. Show that $A \vee B, B \supset C, A \equiv C \vDash C$.
2. Show that $\{P, P \supset Q, Q \supset \neg P\}$ is unsatisfiable.
3. Show that $\{P \supset Q, \neg P \vee \neg Q, Q \supset P\}$ is satisfiable.
4. Evaluate $P, P \supset Q, \neg Q \vDash A$.
5. Evaluate $(A \wedge B) \supset C, C \equiv (D \wedge E), \neg D \wedge B \vdash \neg A$.

Soundness and Completeness

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Induction: These proofs will require mathematical induction.

Definitions: We will also need a few more recursive definitions.

Recursive Definitions

Complexity: We define $\texttt{Comp}(\varphi)$ to be the number of connectives in φ.

- If φ is a sentence letter of SL, then $\texttt{Comp}(\varphi) = 0$.
- For any SL sentences φ and ψ:

  ($\neg$) $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$;

  ($\wedge$) $\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;

  $\vdots$

Note: Could avoid redundancy by taking $\star$ to be any binary connective.

Constituents: We define $[\varphi]$ to be the set of sentence letters that occur in φ.

- If $\texttt{Comp}(\varphi) = 0$, then $[\varphi] = \{\varphi\}$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $[\neg\varphi] = [\varphi]$;

  ($\star$) $[\varphi \star \psi] = [\varphi] \cup [\psi]$;

Simplicity: We define $\texttt{Simple}(\varphi)$ to hold just in case the SL sentence φ has at most one occurrence of each sentence letter in SL.

- If $\texttt{Comp}(\varphi) = 0$, then $\texttt{Simple}(\varphi)$.
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $\texttt{Simple}(\neg\varphi)$ if $\texttt{Simple}(\varphi)$;

  ($\star$) $\texttt{Simple}(\varphi \star \psi)$ if $\texttt{Simple}(\varphi)$, $\texttt{Simple}(\psi)$, and $[\varphi] \cap [\psi] = \varnothing$;

Substitution: We define $\varphi_{[\chi/\alpha]}$ to be the result of replacing every occurrence of the sentence letter α in φ with χ.

- If $\texttt{Comp}(\varphi) = 0$, then $\varphi_{[\chi/\alpha]} = \begin{cases} \chi & \text{if } \varphi = \alpha, \\ \varphi & \text{otherwise.} \end{cases}$
- For any SL sentences φ and ψ, and binary connective $\star \in \{\wedge, \vee, \supset, \equiv\}$:

  ($\neg$) $(\neg\varphi)_{[\chi/\alpha]} = \neg(\varphi_{[\chi/\alpha]})$;

  ($\star$) $(\varphi \star \psi)_{[\chi/\alpha]} = \varphi_{[\chi/\alpha]} \star \psi_{[\chi/\alpha]}$;

Strong Induction

Step 1: Identify the set of elements and the property in question.

Step 2: Partition the set into a sequence of stages to run induction on.

Step 3: Establish that every element in the base stage has the property.

Step 4: Assume every element in stage n and below have the property.

Step 5: Show that every element in stage $n + 1$ have the property.

Examples

Task 1: Show that every SL sentence has an even number of parentheses.

Task 2: Show that for every SL sentence φ, if $\texttt{Simple}(\varphi)$, then there are SL interpretations $\mathcal{I}$ and $\mathcal{J}$ where $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{J}}(\varphi) = 0$.

Task 3: For any SL sentences φ, ψ, χ and SL sentence letter α, if $\vDash \varphi \equiv \psi$, then $\vDash \chi_{[\varphi/\alpha]} \equiv \chi_{[\psi/\alpha]}$.

Task 4: Let $\mathcal{I}^+(\alpha) = 1$ for every sentence letter α in SL. Show that $\mathcal{V}_{\mathcal{I}^+}(\varphi) = 1$ for every SL sentence φ that does not include negation.