Logical Entailment

LOGIC I
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Satisfaction: An interpretation $\mathcal{I}$ of SL *satisfies* a set of SL sentences Γ *iff* $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Gamma$. Derivatively, an interpretation $\mathcal{I}$ of SL *satisfies* a sentence φ of SL *iff* $\mathcal{I}$ satisfies $\{\varphi\}$.

Logical Entailment: $\Gamma \vDash \varphi$ *iff* every SL interpretation $\mathcal{I}$ that satisfies Γ also satisfies φ.

Validity: An argument in SL is *valid* just in case its conclusion is true in any interpretation in which its premises are true.

Question: How are we to describe the space of all valid arguments?

Answer: In terms of entailment.

Task 1: Show that validity and entailment are distinct:

- $\Gamma \vDash \varphi$ does not determine a unique argument.
- Entailment does not order the premises.
- Entailment admits of infinitely many premises.
- Entailment admits of no premises.

Tautology: An SL sentence φ is a *tautology* just in case $\vDash \varphi$.

Weakening: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

Unsatisfiable

Absurdity: A contradiction entails everything: $A \wedge \neg A \vDash B$.

Bottom: Let '$\bot$' abbreviate any contradiction.

Unsatisfiable: A sentence is *unsatisfiable* just in case $\Gamma \vDash \bot$.

Task 2: Show that a set of SL sentences is unsatisfiable just in case no SL interpretation satisfies it.

Consistency: Recall: a set of SL sentences is *consistent* just in case there is a line on the complete truth table for those sentences which makes them all true, and *inconsistent* otherwise.

Task 3: Show that consistency and satisfiability are co-extensional.

Examples

Which sets of sentences are consistent? (e.g., is $\{(1),(2)\}$ consistent?)

Taller

(1) Liza is taller than Sue.

(2) Sue is taller than Paul.

(3) Paul is taller than Liza.

Lost

(4) Kim is either in Somerville or Cambridge.

(5) If Kim is in Somerville, then she is not far from home.

(6) If Kim is not far from home, then she is in Cambridge.

(7) Kim is not in Cambridge.

Methods

Truth Tables: Mechanical but tedious.

- Bad if there are lots of sentence letters.
- Good for counterexamples.
  $A \equiv (B \supset C),\ A \wedge \neg B,\ D \vee \neg A \therefore C.$

Semantic Arguments: Good if there are lots of sentence letters.
$(A \vee B) \supset (C \wedge D),\ \neg C \wedge \neg E \therefore \neg A.$

Task 4: Provide a semantic argument.

Inference Rules: Suppose we were to schematize inferences.

- $\varphi \wedge \psi \vdash \varphi.$
- $\neg\varphi \vdash \neg(\varphi \wedge \psi).$
- $\varphi \supset \psi,\ \neg\psi \vdash \neg\varphi.$
- $\neg(\varphi \vee \psi) \vdash \neg\varphi.$

Observe: Rules are valid.

Task 5: Use rules to derive above.

Proof Theory: How many rules are there, and how should we describe the space of all of them?