Natural Deduction in $\text{QL}^=$

LOGIC I
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Motivation

Entailment: We have defined entailment in $\text{QL}^=$.

Completeness: We want a complete natural deduction system for $\text{QL}^=$.

Question 1: What rules do we need to derive the following?

-	All humans are mortal.	-	$\forall x(Hx \supset Mx)$
-	Socrates is human.	-	Hs
-	Socrates is mortal.	-	Ms
$\therefore$	Someone is mortal.	$\therefore$	$\exists x Mx$

Substitution

Free For: β is FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β.

Constants: If β is a constant, then β is free for any α and φ.

Substitution: If β is free for α in φ, then the SUBSTITUTION $\varphi[\beta/\alpha]$ is the result of replacing all free occurrences of α in φ with β.

Instance: $\varphi[\beta/\alpha]$ is a substitution instance of $\forall\alpha\varphi$ and $\exists\alpha\varphi$ if β is a constant.

Universal Elimination and Existential Introduction

($\forall$E) $\forall\alpha\varphi \vdash \varphi[\beta/\alpha]$ where β is a constant and α is a variable.

($\exists$I) $\varphi[\beta/\alpha] \vdash \exists\alpha\varphi$ where β is a constant and α is a variable.

Task 1: Derive the argument above.

Universal: Everyone is either great or unfortunate $\forall x(Gx \vee Ux)$.

Existential: Tom is either great or unfortunate $(Gt \vee Ut)$.

$\therefore$	$\exists x(Gx \vee Ux)$.	$\therefore$	$\exists y \exists x(Gy \vee Uy)$.
$\therefore$	$\exists x(Gx \vee Ut)$.	$\therefore$	$\exists y \exists x(Gx \vee Uy)$.
$\therefore$	$\exists x(Gt \vee Ut)$.	#	$\exists x \exists x(Gx \vee Ux)$.

Universal Introduction

Generalising: It would seem that we cannot universally generalise from instances.

Invalid: The following argument is invalid and should not be derivable.

- Socrates is mortal. (Ms)

\# Everything is mortal. $(\forall x Mx)$

Valid: Compare the following valid argument which should be derivable:

- $\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$.
- $\forall x \neg Rxx$.

$\therefore$ $\forall x \forall y(Rxy \supset \neg Ryx)$.

Task 2: Use the rules we have to derive as much as we can.

1. $\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$
2. $\underline{\forall x \neg Rxx}$
3. $\forall y \forall z((Ray \wedge Ryz) \supset Raz)$:$\forall$E
4. $\forall z((Rab \wedge Rbz) \supset Raz)$:$\forall$E
5. $(Rab \wedge Rba) \supset Raa$:$\forall$E
6. $\neg Raa$:$\forall$E
7. $\mid Rab$:AS for $\supset$I
8. $\mid \mid Rba$:AS for $\neg$I
9. $\mid \mid Rab \wedge Rba$:$\wedge$I
10. $\mid \mid Raa$:$\supset$E
11. $\mid \neg Rba$:$\neg$I
12. $Rab \supset \neg Rba$:$\supset$I
13. $\forall y(Ray \supset \neg Rya)$:$\forall$I
14. $\forall x \forall y(Rxy \supset \neg Ryx)$:$\forall$I

Question 2: How are we going to introduce universal quantifiers without making the invalid argument above derivable?

($\forall$I) $\varphi[\beta/\alpha] \vdash \forall \alpha \varphi$ where β is a constant, α is a variable, and β does not occur in $\forall \alpha \varphi$ or in any undischarged assumption.

Arbitrary: The constraints on ($\forall$E) require β to be arbitrary.

Review: Bad inference above is blocked.

In Premise: Anu loves every dog.
$\forall x(Dx \supset Lax) \vdash Da \supset Laa \nvdash \forall x(Dx \supset Lxx)$.

In Conclusion: All dogs love themselves.
$\forall x(Dx \supset Lxx) \vdash Da \supset Laa \nvdash \forall x(Dx \supset Lax)$.

Existential Elimination

Task 3: Compare the following invalid inference.

- Someone is mortal.

\# Zeus is mortal.

Question 3: How are we going to eliminate existential quantifiers without making the argument above derivable?

Example: Consider the following argument:

- Everyone who applied found a position $\forall x(Ax \supset \exists yFxy)$.
- Someone applied $\exists xAx$.

$\therefore$ Someone found a position $\exists x\exists yFxy$.

($\exists$E) If $\exists\alpha\varphi, \varphi[\beta/\alpha] \vdash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any undischarged assumption, then $\exists\alpha\varphi \vdash \psi$.

Derivation: We can derive the example without deriving the invalid inference.

Quantifier Exchange Rules

($\neg\exists$) $\neg\exists\alpha\varphi \vdash \forall\alpha\neg\varphi$.

($\neg\forall$) $\neg\forall\alpha\varphi \vdash \exists\alpha\neg\varphi$.

($\forall\neg$) $\forall\alpha\neg\varphi \vdash \neg\exists\alpha\varphi$.

($\exists\neg$) $\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$.

Task 4: $\forall\alpha\neg\varphi \vdash \neg\exists\alpha\varphi$.

1. $\underline{\forall\alpha\neg\varphi}$
2. $\mid \underline{\exists\alpha\varphi}$
3. $\mid \mid \underline{\varphi[\beta/\alpha]}$
4. $\mid \mid \mid \underline{\exists\alpha\varphi}$
5. $\mid \mid \mid \varphi[\beta/\alpha]$
6. $\mid \mid \mid \neg\varphi[\beta/\alpha]$
7. $\mid \mid \neg\exists\alpha\varphi$
8. $\mid \neg\exists\alpha\varphi$
9. $\neg\exists\alpha\varphi$

Task 5: $\exists\alpha\neg\varphi \vdash \neg\forall\alpha\varphi$.

10. $\underline{\exists\alpha\neg\varphi}$
11. $\mid \underline{\forall\alpha\varphi}$
12. $\mid \mid \underline{\neg\varphi[\beta/\alpha]}$
13. $\mid \mid \mid \underline{\forall\alpha\varphi}$
14. $\mid \mid \mid \neg\varphi[\beta/\alpha]$
15. $\mid \mid \mid \varphi[\beta/\alpha]$
16. $\mid \mid \neg\forall\alpha\varphi$
17. $\mid \neg\forall\alpha\varphi$
18. $\neg\forall\alpha\varphi$

Task 6: Prove the rules below:

(MCP) If $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

($\forall$DN) $\forall\alpha\neg\neg\varphi \vdash \forall\alpha\varphi$.

($\exists$DN) $\exists\alpha\neg\neg\varphi \vdash \exists\alpha\varphi$.

Task 7: Use the rules above to derive ($\neg\exists$) and ($\neg\forall$).