13. Completeness of QND

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- Completeness guarantees that for any valid QL-argument, there is at least one corresponding deduction in QND.
- So we need not reason about arbitrary models to determine if a QL-argument is valid; reasoning in QND suffices! WOW COOL

"$\vDash$": our Semantic Double Turnstile

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- We’ll use this last fact A LOT in our proof that QND is complete!

13. Completeness of QND

a. Semantic vs. Syntactic Consistency

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- We introduce a new **syntactic** notion of consistency relative to QND:

  - a set of QL sentences is **QND–consistent** provided that you can't derive contradictory sentences from it in QND
- Core proof idea: we'll show that if a set of sentences is **QND–consistent**, then it is also semantically consistent (i.e. **satisfiable**). So by the contrapositive: if a set is **un**satisfiable, then it is **in**consistent–in–QND.

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- This is a *semantic* notion of consistency (aka "quantificational consistency")
- Contrast this with the syntactic notion of **consistency in QND**:

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- Just remember: this syntactic notion has nothing to do with models or interpretations!

13. Completeness of QND

b. Proof Sketch

Proof Sketch: Just like what we did for SL!

- Goal: prove the completeness of QL: for every QL sentence $\mathcal{P}$ and every set Γ of QL sentences, if $\Gamma \vDash \mathcal{P}$ then $\Gamma \vdash_{QND} \mathcal{P}$

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- This means that $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**:
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- This means that $\Gamma \cup \{\sim\mathcal{P}\}$ is **unsatisfiable**:
  no QL-model satisfies the premises and negated conclusion (i.e. $\Gamma \cup \{\sim\mathcal{P}\}$ is *semantically* inconsistent)
- We now appeal to a **Consistency lemma** that remains the heart of the enterprise: any QND-consistent set of QL sentences is satisfiable (i.e. semantically consistent)

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- From $\Gamma \vDash \mathcal{P}$ we know that $\Gamma \cup \{{\sim}\mathcal{P}\}$ is unsatisfiable
- So by the contrapositive of CL, we see that $\Gamma \cup \{{\sim}\mathcal{P}\}$ is QND–inconsistent
- This means that we can derive a pair of contradictory sentences R and ${\sim}R$ from $\Gamma \cup \{{\sim}\mathcal{P}\}$! So using the power of negation elimination, we can derive $\mathcal{P}$ from Γ, i.e. $\Gamma \vdash_{QND} \mathcal{P}$. So we are 'done'!

Negation Elimination Refresher (book's Exercise 11.4.2)

- Claim: if $\Gamma \cup \{\sim\mathcal{P}\}$ is **QND-inconsistent**, then $\Gamma \vdash_{QND} \mathcal{P}$
- Proof: starting with (finitely-many) premises Δ from Γ, introduce $\sim\mathcal{P}$ as a subproof assumption for negation elimination
- Since $\Gamma \cup \{\sim\mathcal{P}\}$ is QND-inconsistent, we can derive a contradictory pair R and $\sim R$ within the scope of sentences in $\Delta \cup \{\sim\mathcal{P}\}$
- Then discharge this assumption $\sim\mathcal{P}$ by negation elimination, writing $\mathcal{P}$, now in the scope of Δ. So $\Delta \vdash_{QND} \mathcal{P}$
- Since $\Delta \subseteq \Gamma$, we have $\Gamma \vdash_{QND} \mathcal{P}$

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- From our QL$'$–model, we generate a QL–model that satisfies Γ

13. Completeness of QND

c. Stage 0: $\exists$-Completeness and QL′

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- Recall that our purely syntactic membership lemma is motivated by the truth-conditions for QL sentences: sentences belong to Γ^* iff the relevant "truth-condition pieces" belong to Γ^* as well
- To extend our membership lemma to quantified sentences, we require that every existential sentence in Γ^* has a substitution instance also in Γ^*. So we introduce a new property:

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- **$\exists$-completeness**: a set Γ of QL or QL$'$ sentences is *existentially-complete* just in case for every sentence in Γ of the form $(\exists x)\mathcal{P}$, at least one substitution instance $\mathcal{P}[c/x]$ is in Γ

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- *Problem*: our starting Γ might be infinite and so already use infinitely-many constants from QL. What are we to do?

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- Moral: reach *for the stars*, **not** drugs

13. Completeness of QND

d. Stage 1: Constructing Γ^*

Stage 1(i): first enumerate the sentences of QL′!

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- Assign each symbol an **index** between '10' and '84' (skip 17-19)
- Then each QL′ sentence corresponds to a unique positive integer, constructed by replacing each symbol in the sentence with its index, from left to right.

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- To construct Γ^*, we first **enumerate** the QL′ sentences, so that every QL′ sentence is associated with a unique positive integer $\{1, 2, 3, \ldots\}$
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- $\sim, \vee, \&, \supset, \equiv, (,), 0, 1, \ldots, 9, A, B, \ldots, Z, a, \ldots, v, w, x, y, z, \forall, \exists, '$
- Assign each symbol an **index** between '10' and '84' (skip 17-19)
- Then each QL′ sentence corresponds to a unique positive integer, constructed by replacing each symbol in the sentence with its index, from left to right.
- So with our ordering, 'A' is the first sentence; 'B' the second ...up to Z, and then we hit $\sim A$ ($\mapsto$ 1030), then $\sim B$ ($\mapsto$ 1031), etc.

Recall what we did in SL to form Γ^* Max.–SND–Consist.

- We considered the first sentence 'A' in our enumeration. If A could be added to Γ without the resulting set being SND–inconsistent, then we let $\Gamma_1 := \Gamma \cup \{A\}$.

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- Γ^* was the result of 'doing' this procedure for every SL sentence
- Now we need to complicate matters a bit, to handle sentences of the form $(\exists \chi)\mathcal{P}$ and ensure we add a suitable substitution instance whenever we can add $(\exists \chi)\mathcal{P}$ while preserving QND–consistency

Building up Γ^*

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  - So the countable–infinity of primed subscripted constants from QL$'$ are available at each stage if needed.
- Then $\Gamma^* := \bigcup_{k=0}^{\infty} \Gamma_k$

13. Completeness of QND

e. Stage 2: Γ^* is M-QND-C & $\exists$-complete

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- We prove these in turn

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    - It remains to show that $\Gamma_k \cup \{(\exists \mathcal{X})\mathcal{Q}, \mathcal{Q}[c/\mathcal{X}]\}$ is QND-consistent

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  - **Lemma**: if c does not occur in a QND-C set $\Gamma_k \cup \{(\exists \mathcal{x})\mathcal{Q}\}$, then $\Gamma_k \cup \{(\exists \mathcal{x})\mathcal{Q}, \mathcal{Q}[c/\mathcal{x}]\}$ is QND-consistent

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- Hence, Γ^* must be QND-consistent, on pain of *reductio*

Stage 2 (ii): Γ^* is maximally QND-consistent

- Assume for *reductio* that Γ^* weren't maximally QND-consistent, despite being QND-consistent
- i.e. assume *it is not the case that* for all other sentences, adding it to Γ^* would result in a QND-inconsistent set
  $\Rightarrow$ there exists a sentence $\mathcal{Q}$ that we could add to Γ^* while preserving QND-consistency (i.e. there is some sentence we neglected that could make Γ^* a 'bigger' QND-consistent set)
- Yet, $\mathcal{Q}$ would appear in our enumeration as some sentence P_k, 'considered' at the k-th stage of our construction of Γ^*.
- So if $\mathcal{Q}$ isn't in Γ^*, then this is because adding it 'would have' made $\Gamma_k \subset \Gamma^*$ QND-inconsistent.
  So $\{\mathcal{Q}\} \cup \Gamma^*$ must be QND-inconsistent (*reductio*!)
- So we can't add any $\mathcal{Q}$ to Γ^* while preserving QND-consistency

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- We simply need to show that for each sentence of the form $(\exists \chi)\mathcal{Q} \in \Gamma^*$, a substitution instance $\mathcal{Q}[c/\chi]$ also belongs to Γ^*

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- This is so even if $(\exists \mathcal{X})\mathcal{Q}$ is already in $\Gamma_0 := \Gamma$, since by condition (iii) $\Gamma_{k+1} := \Gamma_k \cup \{(\exists \mathcal{X})\mathcal{Q}, \mathcal{Q}[c/\mathcal{X}]\}$ which in this case would equal $\Gamma_k \cup \{\mathcal{Q}[c/\mathcal{X}]\}$ (since in this case, $(\exists \mathcal{X})\mathcal{Q} \in \Gamma_k$)

13. Completeness of QND

f. Stage 3: Model Construction

Stage 3: The Maximal Consistency Lemma ($\approx$ book's 11.4.7)

- **$\exists$-C Maximal Consistency Lemma**: every QL$'$ set that is maximally-QND-consistent and $\exists$-complete is satisfiable

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- **∃-C Maximal Consistency Lemma**: every QL′ set that is maximally-QND-consistent and ∃-complete is satisfiable
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  We construct this model, calling it "$\mathfrak{M}^*$" ($\approx$book's "$\mathbf{I}^*$")

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  This holds in particular for each QL′–atomic sentence
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- Construct a QL′–model $\mathfrak{M}^*$ such that for each atomic QL′–sentence $\mathcal{A}$, $\mathfrak{M}^* \vDash \mathcal{A}$ iff $\mathcal{A} \in \Gamma^*$
- Then by the recursive structure of QL′ sentences, $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$

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- Basically, Γ^* is *THE* club with the BADDEST MOTHA F$***$in' bouncer you've eva seen, who enforces maximal consistency. Before this ma$$–hole lets a sentence into Γ^*, he checks who else is GOOD. You hear?
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  - Hence, if $\Gamma \vdash P$ and $\Gamma \subseteq \Gamma^*$, then P must belong to Γ^*

Membership Lemma: Cases (a)–(e)

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Membership Lemma: Case (a)

- **Case (a)**: $\sim\mathcal{P} \in \Gamma^*$ if and only if $\mathcal{P} \notin \Gamma^*$
- Two directions to prove:

  $\Rightarrow$: Assume $\sim\mathcal{P} \in \Gamma^*$. Then if $\mathcal{P}$ were in Γ^*, we could derive contradictory sentences.

  So since Γ^* is QND-consistent, we must have $\mathcal{P} \notin \Gamma^*$

  $\Leftarrow$: Assume $\mathcal{P} \notin \Gamma^*$. Then adding $\mathcal{P}$ to Γ^* results in an QND-inconsistent set. Hence, there is some finite subset $\Delta \subset \Gamma^*$ s.t. $\Delta \cup \{\mathcal{P}\}$ is QND-inconsistent (i.e. derives contradictory sentence pair).
- So by negation introduction, $\Delta \vdash \sim\mathcal{P}$
- So by **The Stairway**, $\sim\mathcal{P} \in \Gamma^*$

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  - Then for any substitution instance $\mathcal{P}[c/x]$, we note that $(\forall x)\mathcal{P} \vdash_{QND} \mathcal{P}[c/x]$ by $\forall E$. So by the Stairway, $\mathcal{P}[c/x] \in \Gamma^*$

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  - Then $\sim(\forall x)\mathcal{P} \in \Gamma^*$ by membership clause (a)

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  - Then since Γ^* is $\exists$-complete, there is at least one substitution instance $\sim\mathcal{P}[b/\chi] \in \Gamma^*$. So by (a), $\mathcal{P}[b/\chi] \notin \Gamma^*$, which is what we needed to show.

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  So by the Stairway, $(\exists \mathcal{X})\mathcal{P} \in \Gamma^*$
- This completes the Membership Lemma, so we proceed to construct a model that satisfies Γ^* (in virtue of being maximally–QND–consistent and $\exists$–complete)!

Stage 3 (ii): Model construction (smart choices=lazy choices)

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  2. For the 0-th place predicates, i.e. the sentence letters B, $I^*(B) = \mathit{true}$ iff $B \in \Gamma^*$
  3. For each QL′-constant c, define $I^*(c) = c$ (each names itself)
  4. For each k-place predicate P, $I^*(P) := \mathit{Ext}(P)$ includes all and only those k-tuples $\langle c_1, \ldots, c_k \rangle$ such that $P c_1 \ldots c_k \in \Gamma^*$

Some important properties of our Model $\mathfrak{M}^*$

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- By condition 3, every member of the domain is named by a constant, namely itself
- We will occasionally rely on these properties in our induction

Stage 3 (iii): Induction on QL$'$ (i.e. we still be clubbin')

- Goal: construct a QL$'$-model $\mathfrak{M}^*$ that satisfies the $\exists$-C M-QND-C set Γ^*, i.e. that makes true everything in Γ^* ($\mathfrak{M}^* \models \Gamma^*$)

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Stage 3 (iii): Induction on QL′ (i.e. we still be clubbin')

- Goal: construct a QL′–model $\mathfrak{M}^*$ that satisfies the ∃–C M–QND–C set Γ^*, i.e. that makes true everything in Γ^* ($\mathfrak{M}^* \vDash \Gamma^*$)
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  Say that a sentence is "**clubbin'** " whenever it meets this property

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  Say that a sentence is "**clubbin'** " whenever it meets this property
- We induct on the number of logical operators in a QL$'$ sentence: i.e. the five connectives and two quantifiers ("conquans")
- **Base case**: show that each QL$'$–sentence with zero logical operators is clubbin' (i.e. the QL$'$–atomics be clubbin')

Stage 3 (iii): Induction on QL$'$ (i.e. we still be clubbin')

- Goal: construct a QL$'$–model $\mathfrak{M}^*$ that satisfies the $\exists$–C M–QND–C set Γ^*, i.e. that makes true everything in Γ^* ($\mathfrak{M}^* \vDash \Gamma^*$)
  - Suffices to construct $\mathfrak{M}^*$ s.t. $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$
  Say that a sentence is "**clubbin'** " whenever it meets this property
- We induct on the number of logical operators in a QL$'$ sentence: i.e. the five connectives and two quantifiers ("conquans")
- **Base case**: show that each QL$'$–sentence with zero logical operators is clubbin' (i.e. the QL$'$–atomics be clubbin')
- (Strong) **Induction hypothesis**: assume every QL$'$ sentence with 1 to k–many operators is clubbin'

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- Induction step: show that an arbitrary QL$'$ sentence with k+1–many operators is clubbin'

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- If $\mathcal{A}$ is of form $P c_1 \ldots c_n$, then by definition $\mathfrak{M}^* \vDash P c_1 \ldots c_n$ iff $\langle c_1^D, \ldots, c_n^D \rangle \in Ext(P)$.

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- If $\mathcal{A}$ is of form $Pc_1 \dots c_n$, then by definition $\mathfrak{M}^* \models Pc_1 \dots c_n$ iff $\langle c_1^D, \dots, c_n^D \rangle \in Ext(P)$.
  By part 4, $\langle c_1^D, \dots, c_n^D \rangle = \langle c_1, \dots, c_n \rangle \in Ext(P)$ iff $Pc_1 \dots c_n \in \Gamma^*$

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  By part 4, $\langle c_1^D, \ldots, c_n^D \rangle = \langle c_1, \ldots, c_n \rangle \in Ext(P)$ iff $P c_1 \ldots c_n \in \Gamma^*$
- We proceed to do induction using our QL′ induction schema: an arbitrary sentence $\mathcal{P}$ with k+1–many connectives has one of seven forms, coming from our seven operators

Induction on QL′: Cases 1–5

- Cases 1-5 are just like what did to prove the completeness of SND
- See the next slide for a refresher (*mutatis mutandis*)!
- Need to show: $\mathcal{P}$ be clubbin', i.e. $\mathcal{P}$ is true on $\mathfrak{M}^*$ iff $\mathcal{P} \in \Gamma^*$, where $\mathcal{P}$ is arbitrary QL′ sentence with k+1–many operators
- Induction hypothesis: assume every QL sentence with 1 to k–many operators is clubbin'
- Case 1: $\mathcal{P}$ has the form $\sim\mathcal{Q}$
- Case 2: $\mathcal{P}$ has the form $\mathcal{Q} \,\&\, \mathcal{R}$
- Case 3: $\mathcal{P}$ has the form $\mathcal{Q} \vee \mathcal{R}$
- Case 4: $\mathcal{P}$ has the form $\mathcal{Q} \supset \mathcal{R}$
- Case 5: $\mathcal{P}$ has the form $\mathcal{Q} \equiv \mathcal{R}$

Induction on QL′: Case 1

- **Case 1**: $\mathcal{P}$ has the form $\sim\mathcal{Q}$, where since $\mathcal{Q}$ has k−operators, it is clubbin by the IH (i.e. $\mathfrak{M}^* \vDash \mathcal{Q}$ if and only if $\mathcal{Q} \in \Gamma^*$)

Induction on QL$'$: Case 1

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- NTS: (i) (the $\Rightarrow$direction) if $\mathfrak{M}^* \vDash \mathcal{P}$ then $\mathcal{P} \in \Gamma^*$ and (ii) (the $\Leftarrow$direction) if $\mathcal{P} \in \Gamma^*$, then $\mathfrak{M}^* \vDash \mathcal{P}$

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By Membership lemma (a), $\sim\mathcal{Q} \in \Gamma^*$, so $\mathcal{P} \in \Gamma^*$

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By Membership lemma (a), $\sim\mathcal{Q} \in \Gamma^*$, so $\mathcal{P} \in \Gamma^*$

$\Leftarrow$ if $\mathcal{P} \in \Gamma^*$, then $\sim\mathcal{Q} \in \Gamma^*$. So by Membership lemma (a), $\mathcal{Q} \notin \Gamma^*$.

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Since $\mathcal{Q}$ is clubbin', we have $\mathfrak{M}^* \nvDash \mathcal{Q}$. (i.e. $\mathcal{Q}$ is false in $\mathfrak{M}^*$)

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Since $\mathcal{Q}$ is clubbin', we have $\mathfrak{M}^* \nvDash \mathcal{Q}$. (i.e. $\mathcal{Q}$ is false in $\mathfrak{M}^*$)

So by the truth conditions for negation, $\mathcal{P}$ is true in $\mathfrak{M}^*$, i.e. $\mathfrak{M}^* \vDash \mathcal{P}$

Induction on QL′: Case 7 (existential quantifier)

- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$
  (warning: “$\mathcal{Q}$” is not a sentence, so sadly it can’t be clubbin’)

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- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$
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- We will use Membership Lemma **Case (g)**:
  $(\exists \chi)\mathcal{Q} \in \Gamma^*$ iff for at least one constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

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$\Rightarrow$ Assume $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$. (need to show that $(\exists \chi)\mathcal{Q} \in \Gamma^*$)

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$\Rightarrow$ Assume $\mathfrak{M}^* \models (\exists \chi)\mathcal{Q}$. (need to show that $(\exists \chi)\mathcal{Q} \in \Gamma^*$)
- Then by the truth-conditions for existential, there is some object $r \in D$ that satisfies $\mathcal{Q}$.

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- This substitution instance has less than $k+1$-operators, so it is clubbin'. Hence, by the IH, $\mathcal{Q}[r/\chi] \in \Gamma^*$ (since $\mathfrak{M}^* \vDash \mathcal{Q}[r/\chi]$)

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- So by membership case (g), $(\exists \chi)\mathcal{Q} \in \Gamma^*$

Induction on QL′: Case 7 backwards direction

- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$

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Induction on QL′: Case 7 backwards direction

- **Case 7**: $\mathcal{P}$ has the form $(\exists \mathcal{x})\mathcal{Q}$
- Use Membership Lemma **Case (g)**:
  $(\exists \mathcal{x})\mathcal{Q} \in \Gamma^*$ iff for at least one constant c, $\mathcal{Q}[c/\mathcal{x}] \in \Gamma^*$

$\Leftarrow$ Assume $(\exists \mathcal{x})\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\exists \mathcal{x})\mathcal{Q}$

Induction on QL′: Case 7 backwards direction

- **Case 7**: $\mathcal{P}$ has the form $(\exists \chi)\mathcal{Q}$
- Use Membership Lemma **Case (g)**:
  $(\exists \chi)\mathcal{Q} \in \Gamma^*$ iff for at least one constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

$\Leftarrow$ Assume $(\exists \chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$
- Then by membership case (g), there is at least one substitution instance $\mathcal{Q}[c/\chi] \in \Gamma^*$, for some constant c

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$\Leftarrow$ Assume $(\exists \chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$

- Then by membership case (g), there is at least one substitution instance $\mathcal{Q}[c/\chi] \in \Gamma^*$, for some constant c

- Since $\mathcal{Q}[c/\chi]$ has fewer than $k+1$–operators, it is clubbin'.

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$\Leftarrow$ Assume $(\exists \chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\exists \chi)\mathcal{Q}$
- Then by membership case (g), there is at least one substitution instance $\mathcal{Q}[c/\chi] \in \Gamma^*$, for some constant c
- Since $\mathcal{Q}[c/\chi]$ has fewer than $k+1$–operators, it is clubbin'.
- So by the Induction Hypothesis, $\mathfrak{M}^* \vDash \mathcal{Q}[c/\chi]$.

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- Since 'c' names object c, we see that c satisfies $\mathcal{Q}$ in $\mathfrak{M}^*$

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- Then by membership case (g), there is at least one substitution instance $\mathcal{Q}[c/\chi] \in \Gamma^*$, for some constant c
- Since $\mathcal{Q}[c/\chi]$ has fewer than $k+1$–operators, it is clubbin'.
- So by the Induction Hypothesis, $\mathfrak{M}^* \vDash \mathcal{Q}[c/\chi]$.
- Since 'c' names object c, we see that c satisfies $\mathcal{Q}$ in $\mathfrak{M}^*$
- So by the truth–conditions for existentials, $(\exists\chi)\mathcal{Q}$ is true in $\mathfrak{M}^*$

Induction on QL′: Case 6 (universal quantifier)

- **Case 6**: $\mathcal{P}$ has the form $(\forall \chi)\mathcal{Q}$
  (warning: "$\mathcal{Q}$" is not a sentence, so it can't be clubbin')

Induction on QL$'$: Case 6 (universal quantifier)

- **Case 6**: $\mathcal{P}$ has the form $(\forall \chi)\mathcal{Q}$
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- We will use Membership Lemma **Case (f)**:
  $(\forall \chi)\mathcal{Q} \in \Gamma^*$ iff for each constant c, $\mathcal{Q}[c/\chi] \in \Gamma^*$

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$\Rightarrow$ Assume $\mathfrak{M}^* \vDash (\forall \chi)\mathcal{Q}$. Show that $(\forall \chi)\mathcal{Q} \in \Gamma^*$

- Then every object satisfies $\mathcal{Q}$, so every substitution instance for every constant is true in $\mathfrak{M}^*$ (since each object is named by itself)

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- These $\mathcal{Q}[c/\chi]$ are clubbin' by the IH, so they all belong to Γ^*. So then by Membership Lemma case (f), $(\forall \chi)\mathcal{Q} \in \Gamma^*$

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$\Leftarrow$ Assume $(\forall \chi)\mathcal{Q} \in \Gamma^*$. Show that $\mathfrak{M}^* \vDash (\forall \chi)\mathcal{Q}$

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- Then every object satisfies $\mathcal{Q}$, so every substitution instance for every constant is true in $\mathfrak{M}^*$ (since each object is named by itself)
- These $\mathcal{Q}[c/\chi]$ are clubbin' by the IH, so they all belong to Γ^*. So then by Membership Lemma case (f), $(\forall \chi)\mathcal{Q} \in \Gamma^*$

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- Practice this yourself!

Upshots of our Induction

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- Having handled every case (in spirit), we conclude that every sentence of QL′ is clubbin’:
- For all QL′–sentences $\mathcal{P}$, $\mathfrak{M}^* \vDash \mathcal{P}$ iff $\mathcal{P} \in \Gamma^*$

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- Hence, we have proven the **∃–C Maximal Consistency Lemma**: every QL$'$ set that is maximally–QND–consistent and ∃–complete is satisfiable in QL$'$
- It remains to prove the **Consistency Lemma**, i.e. that any QND–consistent QL–set (like our O.G. Γ) is satisfiable **in QL**!

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- $\square$ Q.E.D. motha f$***$ers!!! (i.e. *quod erat demonstrandum*)