Logic Notes

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Propositional Logic: Syntax and Semantics

Canonical Name: A quoted symbol is the *canonical name* for the symbol quoted.

Language: The propositional language $\mathcal{L}$ includes *symbols* for: *sentence letters* 'p_1','p_2', …, *sentential operators* '$\neg$', '$\rightarrow$', and *punctuation* '(' and ')'.

Sentence Letters: Let $\mathbb{L}$ be the set of all sentence letters of $\mathcal{L}$.

Strings: The concatenation of a finite number of symbols in $\mathcal{L}$ is a *string* of $\mathcal{L}$.

Schematic Variables: Let 'φ', 'ψ', 'χ',… be *schematic variables* for strings of $\mathcal{L}$.

Corner Quotes: Let $\ulcorner\cdot\urcorner$ map concatenations of strings of $\mathcal{L}$ to names for those strings.

Well-Formed Sentences: The set of *well-formed sentences* $\texttt{wfs}(\mathcal{L})$ is the smallest set to satisfy:

- $\varphi \in \texttt{wfs}(\mathcal{L})$ *if* φ is a sentence letter of $\mathcal{L}$.
- $\ulcorner\neg\varphi\urcorner \in \texttt{wfs}(\mathcal{L})$ *if* φ is a wfs of $\mathcal{L}$.
- $\ulcorner(\varphi \rightarrow \psi)\urcorner \in \texttt{wfs}(\mathcal{L})$ *if* φ and ψ are wfss of $\mathcal{L}$.

Abbreviations: Letting $\ulcorner\varphi \coloneqq \psi\urcorner$ signify that φ abbreviates ψ, assume the following:

- $(\varphi \vee \psi) \coloneqq (\neg\varphi \rightarrow \psi)$.
- $(\varphi \wedge \psi) \coloneqq \neg(\varphi \rightarrow \neg\psi)$.
- $(\varphi \leftrightarrow \psi) \coloneqq [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$.

Models: Let $\mathcal{M}$ be a *model* of $\mathcal{L}$ *iff* for every sentence letter φ of $\mathcal{L}$, either $\mathcal{M}(\varphi) = 0$ or $\mathcal{M}(\varphi) = 1$, but not both.

Semantics: We may extend a model $\mathcal{M}$ to interpret all wfss of $\mathcal{L}$ by taking $\vDash_{\text{PL}}$ to be the smallest relation to satisfy the following:

$\mathcal{M} \vDash$ 'p_i' *iff* $\mathcal{M}(p_i) = 1$.
$\mathcal{M} \vDash \ulcorner\neg\varphi\urcorner$ *iff* it is not the case that $\mathcal{M} \vDash \varphi$ (i.e., $\mathcal{M} \nvDash \varphi$).
$\mathcal{M} \vDash \ulcorner(\varphi \rightarrow \psi)\urcorner$ *iff* $\mathcal{M} \nvDash \varphi$ or $\mathcal{M} \vDash \psi$.

Quotes, corner quotes, and the outermost parentheses of a wfs will often be omitted, relying on the reader to know where they belong.

Logical Consequence: $\Gamma \vDash_{\text{PL}} \varphi$ *iff* for all models $\mathcal{M}$, if $\mathcal{M} \vDash_{\text{PL}} \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash_{\text{PL}} \varphi$.

Logical Equivalence: $\varphi \dashv\vDash_{\text{PL}} \psi$ *iff* $\varphi \vDash_{\text{PL}} \psi$ and $\psi \vDash_{\text{PL}} \varphi$.

Logical Truth: A wfs φ of $\mathcal{L}$ is *logical truth* (or *valid*) *iff* $\varnothing \vDash_{\text{PL}} \varphi$ (written $\vDash_{\text{PL}} \varphi$).

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}^+$ include the symbols in $\mathcal{L}$ together with the sentential operators '$\vee$', '$\wedge$', and '$\leftrightarrow$' which are to be read 'or', 'and', and 'if and only if', respectively.

1. Provide a recursive definition of the set $\texttt{wfs}(\mathcal{L}^+)$ of wfss of $\mathcal{L}^+$.
2. Provide a semantics for $\mathcal{L}^+$ by defining the models of $\mathcal{L}^+$ and $\vDash^+$.
3. Prove that $\varphi \vee \psi$, $\varphi \vee (\varphi \wedge \psi)$, and $\varphi \leftrightarrow \psi$ are logically equivalent to wfss of $\mathcal{L}$.
4. For each new operator in $\mathcal{L}^+$, provide two logical truths including that operator.

PROPOSITIONAL LOGIC: FITCH SYSTEM

Reiteration (R)

$$\begin{array}{r|ll} m & \varphi & \\ & \varphi & :m \text{ R} \end{array}$$

Assumption (AS)

$$\begin{array}{r|l|l} m & \underline{\varphi} & :\text{AS} \end{array}$$

Conjunction Introduction ($\wedge$I)

$$\begin{array}{r|ll} m & \varphi & \\ n & \psi & \\ & \varphi \wedge \psi & :m, n \wedge\text{I} \\ & \psi \wedge \varphi & :m, n \wedge\text{I} \end{array}$$

Conjunction Elimination ($\wedge$E)

$$\begin{array}{r|ll} m & \varphi \wedge \psi & \\ & \varphi & :m \wedge\text{E} \\ & \psi & :m \wedge\text{E} \end{array}$$

Conditional Introduction ($\rightarrow$I)

$$\begin{array}{r|l|l} m & \underline{\varphi} & :\text{AS for} \rightarrow\text{I} \\ n & \psi & \\ \hline & \varphi \rightarrow \psi & :m\text{–}n \rightarrow\text{I} \end{array}$$

Conditional Elimination ($\rightarrow$E)

$$\begin{array}{r|ll} m & \varphi \rightarrow \psi & \\ n & \varphi & \\ & \psi & :m, n \rightarrow\text{E} \end{array}$$

Negation Introduction ($\neg$I)

$$\begin{array}{r|l|l} m & \underline{\varphi} & :\text{AS for } \neg\text{I} \\ n & \psi & \\ o & \neg\psi & \\ \hline & \neg\varphi & :m\text{–}o\ \neg\text{I} \end{array}$$

Negation Elimination ($\neg$E)

$$\begin{array}{r|l|l} m & \underline{\neg\varphi} & :\text{AS for } \neg\text{E} \\ & \vdots & \\ n & \psi & \\ o & \neg\psi & \\ \hline & \varphi & :m\text{–}o\ \neg\text{E} \end{array}$$

Disjunction Introduction ($\vee$I)

$$\begin{array}{r|ll} m & \varphi & \\ & \varphi \vee \psi & :m \vee\text{I} \\ & \psi \vee \varphi & :m \vee\text{I} \end{array}$$

Disjunction Elimination ($\vee$E)

$$\begin{array}{r|l|l} m & \varphi \vee \psi & \\ i & \underline{\varphi} & :\text{AS for } \vee\text{E} \\ j & \chi & \\ k & \underline{\psi} & :\text{AS for } \vee\text{E} \\ l & \chi & \\ \hline & \chi & :m, i\text{–}j, k\text{–}l \vee\text{E} \end{array}$$

Biconditional Introduction ($\leftrightarrow$I)

$$\begin{array}{r|l|l} i & \underline{\varphi} & :\text{AS for} \leftrightarrow\text{I} \\ j & \psi & \\ k & \underline{\psi} & :\text{AS for} \leftrightarrow\text{I} \\ l & \varphi & \\ \hline & \varphi \leftrightarrow \psi & :i\text{–}j, k\text{–}l \leftrightarrow\text{I} \end{array}$$

Biconditional Elimination ($\leftrightarrow$E)

$$\begin{array}{r|ll} m & \varphi \leftrightarrow \psi & \\ n & \psi/\varphi & \\ & \varphi/\psi & :m, n \leftrightarrow\text{E} \end{array}$$

Proof Lines: A *proof line* may be represented by a tuple $\langle n, i_n, \varphi_n, J_n \rangle$ consisting of a *position number* n naming the line, *indentation number* i_n, wfs φ_n of $\mathcal{L}^+$, and *justification* J_n.

Live: A proof line n is *live* from all lines $m > n$ if $i_n = 0$, and otherwise n is *live* from all lines $m > n$ in the same subproof as n (including sub-subproofs). A subproof k-l is *live* from all lines $m > l$ if $i_l = 1$, and otherwise k-l is *live* from all lines $m > l$ in the same subproof as the first line $n > l$ where $i_n = i_l - 1$ (including sub-subproofs).

Fitch Proof: A *Fitch proof* of ψ from Γ is a finite sequence of proof lines X with consecutive position numbers starting from 1 that ends at ψ where every line $\langle n, i_n, \varphi_n, J_n \rangle$ in X is: (1) a *premise* where $i_n = 0$, $\varphi_n \in \Gamma$, and $J_n = \text{PR}$; (2) a *discharged assumption* where $i_n > 0$ and $J_n = \text{AS}$; or (3) follows by a *Fitch rule* where J_n cites live lines/subproofs.

Derivable: A wfs ψ of $\mathcal{L}^+$ is *derivable* (or *provable*) from Γ by the Fitch proof system given above— i.e., $\Gamma \vdash_F \psi$— just in case there is a Fitch proof X of ψ from Γ.

Problem Set: Regimentation and Deduction[1]

Regimentation: Resolve the following ambiguities (if any) by regimenting each in $\mathcal{L}^+$:

1. Figaro exulted, and Basilio fretted, or the Court had a plan.
2. Fred danced and sang or Ginger went home.
3. If we are not in Paris then today is not Tuesday.
4. The senator will not testify unless he is granted immunity.
5. The senator will testify only if he is granted immunity.
6. If Figaro does not expose the Count and force him to reform, then the Countess will discharge Susanna and resign to loneliness.
7. The trade deficit will diminish and agriculture or industry will lead a recovery provided that both the dollar drops and neither Japan nor the EU raise their tariffs.

Arguments: Regiment the following arguments in the propositional language $\mathcal{L}^+$:

1. Basilio fretted. Thus, if Figaro exulted, then Basilio fretted.
2. Fred danced if Ginger went home. Fred didn't dance. And so Ginger didn't go home.
3. If Figaro exulted, then the Court had a plan if Basilio fretted. Thus if Basilio fretted, then the Court had a plan if Figaro exulted.
4. Fred danced or else Ginger sang and danced. It follows that either Fred danced or Ginger sang.
5. If Lucy and Mary beat the record, then Paul will have to go. If Ian wins the race, then Paul can stay. Mary beat the record and Ian won the race. Therefore Lucy did not beat the record.
6. If we are in Paris, then we are in Paris.
7. It is not the case that we both are, and are not in Paris.
8. Either Ginger or Fred danced. But Fred did not dance. Thus Ginger must have been the one who danced.
9. Basilio fretted or Gigaro exulted. If Basilio fretted, the Court had a plan. But Gigaro did not exult, if David did not save the day. And so either the Court had a plan, or David saved the day.

[1] I have adapted the following problems from Goldfarb (2003) and Laboreo (2005).

10. Kant is out for a walk just in case it is half noon. So either Kant is out for a walk and it is half noon, or Kant is not out for a walk and it is not half noon.
11. It is not the case that Fred either sang or danced. It follows that Fred did not sing, nor did he dance.
12. It is not the case that Fred sang and danced. It follows that Fred did not sing, or else did he did not dance.
13. If we are in Paris, then we are in France. We are not in France. So we are not in Paris.
14. If we are in Paris, then we are in France. If we are in France, we are in Europe. It follows that if we are in Paris, we are in Europe.

Deduction: Prove that the conclusion of each of the regimented arguments above is derivable from its premises by constructing a proof.

Propositional Logic: Hilbert System

Hilbert System: The *Hilbert proof system* includes the following *axiom schemata* **A1** – **A3** and *rule schema* **MP**, using the abbreviations above when convenient:

A1 $\varphi \to (\psi \to \varphi)$.

A2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

A3 $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$.

MP $\{\varphi, \varphi \to \psi\} \vdash \psi$.

Set notation will typically be omitted, writing '$\Gamma, \varphi, \psi \vdash \chi$' in place of '$\Gamma \cup \{\varphi, \psi\} \vdash \chi$', and writing '$\vdash \varphi$' in place of '$\varnothing \vdash \varphi$'.

Instances: The *axioms* of the Hilbert proof system are the instances of **A1** – **A3** and the *rules* are the instances of **MP**.

Hilbert Proof: A *Hilbert proof* of ψ from Γ is a finite sequence X of wfss of $\mathcal{L}$ ending in ψ where every wfs φ in X is either: (1) a *premise* in Γ; (2) an *axiom*; or (3) follows from previous wfss χ and $\chi \to \varphi$ in X by a *rule*.

Derivable: A wfs ψ of $\mathcal{L}$ is *derivable* (*provable*) from Γ by the Hilbert proof system—i.e., $\Gamma \vdash_{\text{PL}} \psi$— just in case there is a Hilbert proof X of ψ from Γ.

Metalogic: The Hilbert proof system is *sound* insofar as $\Gamma \vDash_{\text{PL}} \psi$ whenever $\Gamma \vdash_{\text{PL}} \psi$, and *complete* insofar as $\Gamma \vdash_{\text{PL}} \psi$ whenever $\Gamma \vDash_{\text{PL}} \psi$.

Problem Set: Derived Metarules

1. *Weakening* (**WK**): if $\Gamma \vdash_{\text{PL}} \varphi$ and $\Gamma \subseteq \Sigma$, then $\Sigma \vdash_{\text{PL}} \varphi$.
2. *Cut elimination* (**CUT**): if $\Gamma \vdash_{\text{PL}} \varphi$ and $\Sigma, \varphi \vdash_{\text{PL}} \psi$, then $\Sigma, \Gamma \vdash_{\text{PL}} \psi$.
3. *Principle of detachment* (**PD**): if $\Gamma \vdash_{\text{PL}} \varphi \to \psi$ and $\Sigma \vdash_{\text{PL}} \varphi$, then $\Gamma, \Sigma \vdash_{\text{PL}} \psi$.
4. *Deduction theorem* (**DT**): if $\Gamma, \varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \varphi \to \psi$.
5. *Reverse deduction* (**RD**): if $\Gamma \vdash_{\text{PL}} \varphi \to \psi$, then $\Gamma, \varphi \vdash_{\text{PL}} \psi$.

Problem Set: Axiomatic Proofs

1. *Hypothetical syllogism* (**HS**): $\varphi \to \psi, \psi \to \chi \vdash_{\text{PL}} \varphi \to \chi$.
2. *Hypothetical exchange* (**HE**): $\varphi \to (\psi \to \chi) \vdash_{\text{PL}} \psi \to (\varphi \to \chi)$.
3. *Reductio ad absurdum* (**RAA**): $\vdash_{\text{PL}} \varphi \to (\neg\varphi \to \psi)$.
4. *Ex falso quidlobet* (**EFQ**): $\vdash_{\text{PL}} \neg\varphi \to (\varphi \to \psi)$.
5. *Reverse contraposition* (**RCP**): $\neg\varphi \to \neg\psi \vdash_{\text{PL}} \psi \to \varphi$.
6. *Double negation elimination* (**DNE**): $\vdash_{\text{PL}} \neg\neg\varphi \to \varphi$.
7. *Double negation introduction* (**DNI**): $\vdash_{\text{PL}} \varphi \to \neg\neg\varphi$.
8. *Contraposition* (**CP**): $\varphi \to \psi \vdash_{\text{PL}} \neg\psi \to \neg\varphi$.
9. *Negation elimination* (**NE**): if $\Gamma, \neg\varphi \vdash_{\text{PL}} \neg\psi$ and $\Gamma, \neg\varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \varphi$.
10. *Negation introduction* (**NI**): if $\Gamma, \varphi \vdash_{\text{PL}} \neg\psi$ and $\Gamma, \varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \neg\varphi$.
11. *Ex contradictione quidlobet* (**ECQ**): $\varphi, \neg\varphi \vdash_{\text{PL}} \psi$.
12. *Left disjunction introduction* (**LDI**): $\varphi \vdash_{\text{PL}} \varphi \vee \psi$.
13. *Right disjunction introduction* (**RDI**): $\psi \vdash_{\text{PL}} \varphi \vee \psi$.
14. *Conjunction introduction* (**CI**): $\varphi, \psi \vdash_{\text{PL}} \varphi \wedge \psi$.
15. *Left conjunction elimination* (**LCE**): $\varphi \wedge \psi \vdash_{\text{PL}} \varphi$.
16. *Right conjunction elimination* (**RCE**): $\varphi \wedge \psi \vdash_{\text{PL}} \psi$.
17. *Disjunction elimination* (**DE**): if $\Gamma, \varphi \vdash_{\text{PL}} \chi$ and $\Gamma, \psi \vdash_{\text{PL}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\text{PL}} \chi$.
18. *Biconditional introduction* (**BI**): if $\Gamma, \varphi \vdash_{\text{PL}} \psi$ and $\Gamma, \psi \vdash_{\text{PL}} \varphi$, then $\Gamma \vdash_{\text{PL}} \varphi \leftrightarrow \psi$.
19. *Left biconditional elimination* (**LBE**): $\varphi \leftrightarrow \psi, \varphi \vdash_{\text{PL}} \psi$.
20. *Right biconditional elimination* (**RBE**): $\varphi \leftrightarrow \psi, \psi \vdash_{\text{PL}} \varphi$.

Propositional Modal Logic: Motivation

Paradox: The following schemata may be derived in the Hilbert proof system:

1. $\varphi \to (\psi \to \varphi)$.
2. $\neg\varphi \to (\varphi \to \psi)$.

These are paradoxical insofar as a true proposition is not implied by any proposition, nor does a false proposition imply any proposition.

(X) If sugar is sweet, then if roses are red, sugar is sweet.

(Y) If snow is not green, then if snow is green, roses are red.

Desiderata: Lewis (1912) and Lewis and Langford (1932) developed modal logic in attempt to better capture what they call the "usual sense" of 'implies'.

Propositional Modal Logic: Syntax

Language $\mathcal{L}^{\Box}$: The propositional language $\mathcal{L}^{\Box}$ extends $\mathcal{L}$ to include the unary sentential operator '$\Box$' where $\ulcorner\Box\varphi\urcorner$ reads $\ulcorner$It is necessary that $\varphi\urcorner$.

Well-Formed Sentences: The set of *well-formed sentences* $\texttt{wfs}(\mathcal{L}^{\Box})$ is the smallest set to satisfy:

- $\varphi \in \texttt{wfs}(\mathcal{L})$ *if* φ is a sentence letter of $\mathcal{L}$.
- $\ulcorner\neg\varphi\urcorner \in \texttt{wfs}(\mathcal{L})$ *if* φ is a wfs of $\mathcal{L}$.
- $\ulcorner(\varphi \to \psi)\urcorner \in \texttt{wfs}(\mathcal{L})$ *if* φ and ψ are wfss of $\mathcal{L}$.
- $\ulcorner\Box\varphi\urcorner \in \texttt{wfs}(\mathcal{L})$ *if* φ is a wfs of $\mathcal{L}$.

Abbreviations: We maintain the previous abbreviations along with:

- $\Diamond\varphi \coloneqq \neg\Box\neg\varphi$.
- $\varphi \mathrel{-\!\!3} \psi \coloneqq \Box(\varphi \to \psi)$.

Lewis and Langford (1932) took the *strict conditional* '$\mathrel{-\!\!3}$' to capture the "usual sense" of 'implies', improving on the material conditional '$\to$'.

Problem Set: Syntax

Regimentation: Regiment the following in $\mathcal{L}^\Box$ using metalinguistic abbreviations.

1. If sugar is sweet, then if roses are red, sugar is sweet.
2. If snow is not green, then if snow is green, roses are red.
3. Either it could rain or not.
4. It's necessarily possible that it either rains or doesn't.
5. If rain is possibly necessary, then it's necessarily possible.
6. It cannot both necessarily rain and necessarily not rain.
7. If rain is necessary, then it cannot necessarily not rain.
8. If rain and snow are jointly possible, then each is possible individually.
9. If rain could imply snow, then it could snow if it necessarily rains.
10. If rain or snow are necessary, then either rain is necessary or snow is possible.

Propositional Modal Logic: Axiomatic Systems

Axiom Schemata: Consider the following axiom schemata:

K	$\Box(\varphi \to B) \to (\Box\varphi \to \Box B)$.	*None.*
D	$\Box\varphi \to \Diamond\varphi$.	$\exists w' R(w, w')$.
T	$\Box\varphi \to \varphi$.	$R(w, w)$.
B	$\varphi \to \Box\Diamond\varphi$.	$R(w, w') \to R(w', w)$.
4	$\Box\varphi \to \Box\Box\varphi$.	$[R(w, w') \wedge R(w', w'')] \to R(w, w'')$.
5	$\Diamond\varphi \to \Box\Diamond\varphi$.	$[R(w, w') \wedge R(w, w'')] \to R(w', w'')$.

Necessitation Rule: Consider the following metarule:

N If $\vdash \varphi$, then $\vdash \Box\varphi$.

Axioms: The *axioms* for the following systems include all $\mathcal{L}^\Box$ instances of the Hilbert axioms together with the following additions respectively:

$\mathcal{K}$	All **K** axioms.	$\mathcal{B}$	All **K**, **B**, and **T** axioms.
$\mathcal{T}$	All **K** and **T** axioms.	$\mathcal{S}4$	All **K**, **T**, and **4** axioms.
$\mathcal{D}$	All **K** and **D** axioms.	$\mathcal{S}5$	All **K**, **T**, and **5** axioms.

Modal Proofs: Letting $\mathcal{Q}$ be the set of axioms for one of the systems above, a *$\mathcal{Q}$-proof* of ψ from Γ is defined recursively as follows:

BASE: Any finite sequence X of wfss of $\mathcal{L}^{\Box}$ ending in ψ is a *$\mathcal{Q}$-proof* of ψ from Γ if every wfs φ in X is either: (1) a *premise* in Γ; (2) an *axiom* in $\mathcal{Q}$; or (3) follows from previous wfss χ and $\chi \to \varphi$ in X by a *rule*.

RECURSIVE: Given a $\mathcal{Q}$-proof Y of φ from $\varnothing$ and a $\mathcal{Q}$-proof Z with premises Γ which includes every wfs in Y, the result of appending $\Box\varphi$ to the end of Z is a *$\mathcal{Q}$-proof* of $\Box\varphi$ from Γ.

Derivable: A wfs ψ of $\mathcal{L}^{\Box}$ is *derivable* (*provable*) from Γ by the proof system $\mathcal{Q}$, i.e., $\Gamma \vdash_{\mathcal{Q}} \psi$, just in case there is a $\mathcal{Q}$-proof X of ψ from Γ.

Modal Systems: Letting the *$\mathcal{Q}$-modal proof system* includes all $\mathcal{Q}$-proofs we may abuse notation by referring to the $\mathcal{Q}$-modal proof system simply as $\mathcal{Q}$.

Problem Set: Axiomatic Proofs

Credence: Evaluate the plausibility of each modal axioms when '$\Box$' and '$\Diamond$' are read:

1. ($\Box$) 'It is necessary that'. ($\Diamond$) 'It is possible that'.
2. ($\Box$) 'It is obligatory that'. ($\Diamond$) 'It is permissible that'.
3. ($\Box$) 'It is always going to be the case that'. ($\Diamond$) 'It is going to be the case that'.
4. ($\Box$) 'It has always been the case that'. ($\Diamond$) 'It has been the case that'.
5. ($\Box$) 'It must be the case that'. ($\Diamond$) 'It might be the case that'.

Proofs: Provide a proof of each of the following:

6. If $\vdash_{\mathcal{K}} \varphi \to \psi$, then $\vdash_{\mathcal{K}} \Box\varphi \to \Box\psi$.
7. If $\varphi \vdash_{\mathcal{K}} \psi$, then $\Box\varphi \vdash_{\mathcal{K}} \Box\psi$.
8. $\vdash_{\mathcal{K}} \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$.
9. $\vdash_{\mathcal{K}} (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.
10. $\vdash_{\mathcal{K}} \Box(\varphi \to \psi) \to \Box(\neg\psi \to \neg\varphi)$.
11. If $\vdash_{\mathcal{K}} \varphi \to \psi$, then $\vdash_{\mathcal{K}} \Diamond\varphi \to \Diamond\psi$.
12. $\vdash_{\mathcal{K}} \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.
13. $\vdash_{\mathcal{T}} \Box\varphi \to \Diamond\varphi$.
14. $\vdash_{\mathcal{T}} \neg\Box(\varphi \wedge \neg\varphi)$.
15. $\vdash_{\mathcal{D}} \Diamond(\varphi \to \varphi)$.
16. $\vdash_{\mathcal{B}} \Box\varphi \to \Diamond\varphi$.
17. $\vdash_{\mathcal{S}4} (\Diamond\varphi \wedge \Box\psi) \to \Diamond(\varphi \wedge \Box\psi)$.
18. $\vdash_{\mathcal{S}4} \Box\varphi \to \Box\Diamond\Box\varphi$.
19. $\vdash_{\mathcal{S}4} \Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$.
20. $\vdash_{\mathcal{S}4} \Diamond\Box\Diamond\varphi \leftrightarrow \Diamond\varphi$.
21. $\vdash_{\mathcal{S}5} \Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$.
22. $\vdash_{\mathcal{S}5} \Diamond\Box\varphi \leftrightarrow \Box\varphi$.
23. $\vdash_{\mathcal{S}5} \Box\varphi \to \Box\Box\varphi$.
24. $\vdash_{\mathcal{S}5} \varphi \to \Box\Diamond\varphi$.
25. $\vdash_{\mathcal{S}5} \Diamond\Box\varphi \to \varphi$.

PROPOSITIONAL MODAL LOGIC: FRAMES

Frame: A *modal frame* is an ordered pair $\mathcal{F} = \langle W, R\rangle$, where W is any nonempty set (called *worlds*) and $R \subseteq W^2$ is any relation (called *accessibility*).

Frame Constraints: Consider the following constraints on an arbitrary frame $\langle W, R \rangle$:

SERIAL (SER): For all $w \in W$ there is some $u \in W$ where $R(w, u)$.

REFLEXIVE (REF): $R(w, w)$ for every $w \in W$.

EMPTY (EMP): $\neg R(w, u)$ for all $w, u \in W$.

SYMMETRIC (SYM): $R(u, w)$ whenever $R(w, u)$.

TRANSITIVE (TRA): $R(w, v)$ whenever both $R(w, u)$ and $R(u, v)$.

LEFT EUCLIDEAN (LEU): $R(u, v)$ whenever both $R(u, w)$ and $R(v, w)$.

RIGHT EUCLIDEAN (REU): $R(u, v)$ whenever both $R(w, u)$ and $R(w, v)$.

TOTAL (TOT): Either $R(w, u)$ or $R(u, w)$ for all $w, u \in W$.

Accessible Worlds: Let $(w)_R := \{u \in W : R(w, u)\}$ be the set of worlds that are *accessible* from $w \in W$ by the relation R.

Partition: Given a set X, a *partition* of X is any set of subsets $Y \subseteq \wp(X)$ where:

Empty Set: The empty set is excluded $\varnothing \notin Y$.

Disjoint: $A \cap B = \varnothing$ for all $A, B \in Y$ where $A \neq B$.

Covering: For all $x \in X$ there is a $A \in Y$ where $x \in A$, i.e., $X \subseteq \bigcup Y$.[2]

Relational Image: Let $\mathtt{img}(R) := \{(w)_R : w \in W\}$ be the *relational image* of R.

Problem Set: Frames

Relations: Evaluate the following, providing a proof or counterexample:

1. Every reflexive frame is serial.
2. Every serial frame is reflexive.
3. How many frames are both symmetric and transitive but not reflexive.
4. Every frame that is left and right Euclidean is symmetric and transitive.
5. Every frame that is symmetric and transitive is left and right Euclidean.
6. Every frame that is transitive and both left and right Euclidean is symmetric.
7. Every frame that is symmetric and left Euclidean is transitive.
8. There is a finite serial transitive frame that is neither reflexive nor symmetric.
9. A symmetric frame is left Euclidean just in case it is right Euclidean.
10. The relational image of a transitive, symmetric, reflexive frame is a partition.
11. Every total frame is a partition.
12. There is a symmetric total frame that is not a partition.

[2]The *union* of a set of sets Y is defined $\bigcup Y := \{x : \exists A \in Y \text{ where } x \in A\}$.

Propositional Modal Logic: Logical Consequence

Interpretation: An *interpretation* of $\mathcal{L}^{\Box}$ over a frame $\mathcal{F}$ is any function $\mathcal{I} : \mathbb{L} \to \wp(W)$, i.e., where $\mathcal{I}(p_i) \subseteq W$ for every sentence letter $p_i \in \mathbb{L}$.

Model: A *C-model* of $\mathcal{L}^{\Box}$ is an ordered triple $\mathcal{M} = \langle W, R, \mathcal{I} \rangle$ where $\langle W, R \rangle$ is a frame satisfying the constraints in C and $\mathcal{I}$ is an interpretation of $\mathcal{L}^{\Box}$.

Semantics: Let $\vDash$ be the smallest relation to satisfy the following conditions where $\mathcal{M} = \langle W, R, \mathcal{I} \rangle$ is any model of $\mathcal{L}^{\Box}$, both $w, u \in W$, $p_i \in \mathbb{L}$, $\varphi, \psi \in \mathtt{wfs}(\mathcal{L}^{\Box})$, and we adopt the abbreviation $R(w, u) \coloneqq \langle w, u \rangle \in R$:

$\mathcal{M}, w \vDash p_i$ *iff* $w \in \mathcal{I}(p_i)$.

$\mathcal{M}, w \vDash \neg\varphi$ *iff* $\mathcal{M}, w \nvDash \varphi$.

$\mathcal{M}, w \vDash \varphi \to \psi$ *iff* $\mathcal{M}, w \nvDash \varphi$ or $\mathcal{M}, w \vDash \psi$.

$\mathcal{M}, w \vDash \Box\varphi$ *iff* $\mathcal{M}, u \vDash \varphi$ for every $u \in W$ such that $R(w, u)$.

Truth-Condition: The *truth-condition* $|\varphi|_{\mathcal{M}} \coloneqq \{w \in W : \mathcal{M}, w \vDash \varphi\}$ for a wfs $\varphi \in \mathtt{wfs}(\mathcal{L}^{\Box})$ on a model $\mathcal{M}$ is the set of worlds at which φ is true.

Restricted: The truth-condition *restricted* to $(u)_R$ for a wfs $\varphi \in \mathtt{wfs}(\mathcal{L}^{\Box})$ on a model $\mathcal{M}$ is the set of worlds $|\varphi|^{w}_{\mathcal{M}} \coloneqq \{u \in W : R(w, u) \text{ and } \mathcal{M}, u \vDash \varphi\}$.

Logical Consequence: $\Gamma \vDash_C \varphi$ just in case for any C-model $\mathcal{M} = \langle W, R, \mathcal{I} \rangle$ of $\mathcal{L}^{\Box}$ and world $w \in W$, if $\mathcal{M}, w \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w \vDash \varphi$.

Logical Equivalence: $\varphi \equiv_C \psi$ just in case $\varphi \vDash_C \psi$ and $\psi \vDash_C \varphi$.

Logical Truth: A wfs φ of $\mathcal{L}^{\Box}$ is a *C-logical truth* (or *C-valid*) just in case $\vDash_C \varphi$.

Constraint Set: Let C be an arbitrary set of frame constraints (not just those above).

Extension: A set of constraints C' *extends* C just in case $C \subseteq C'$.

Strength: $\vDash_{C'}$ is *at least as strong* as $\vDash_C$ just in case C' extends C.

Problem Set: Logical Consequence

Nonempty: Why are frames required to be nonempty?

Countermodels: Evaluate the following, providing a proof or countermodel:

1. $\Box\varphi \vDash_K \Box\Box\varphi$
2. $\Box\varphi \vDash_K \varphi$
3. $\varphi \vDash_K \Box\varphi$
4. $\Box(\varphi \vee \psi) \vDash_K \Box\varphi \vee \Box\psi$
5. $\Box(\varphi \wedge \psi) \vDash_K \Box\varphi \wedge \Box\psi$
6. $\Box\varphi \vee \Box\psi \vDash_K \Box(\varphi \vee \psi)$
7. $\Box\varphi \wedge \Box\psi \vDash_K \Box(\varphi \wedge \psi)$
8. $\Box\varphi \vDash_K \Diamond\varphi$
9. $\Diamond\varphi \vDash_K \Box\varphi$
10. $\Diamond\Box\varphi \vDash_K \Box\Diamond\varphi$
11. $\Box\Diamond\varphi \vDash_K \Diamond\Box\varphi$
12. $\Box\varphi \vDash_K \Box\Diamond\varphi$
13. $\vDash_K \neg\Box(\varphi \wedge \neg\varphi)$
14. $\Diamond\psi \vDash_K \neg\Box(\varphi \wedge \neg\varphi)$
15. $\Box\varphi \vDash_K \Box(\psi \to \varphi)$.
16. $\neg\Box\varphi \vDash_K \Box(\varphi \to \psi)$.

Logical Consequence: Replace K with the weakest set of constraints C to make the above hold.

Truth-Conditions: Draw on the semantic definitions above to establish the following:

1. $|\neg\varphi|_{\mathcal{M}} = |\varphi|^c_{\mathcal{M}}$.[3]
2. $|\varphi \wedge \psi|_{\mathcal{M}} = |\varphi|_{\mathcal{M}} \cap |\psi|_{\mathcal{M}}$.[4]
3. $|\varphi \vee \psi|_{\mathcal{M}} = |\varphi|_{\mathcal{M}} \cup |\psi|_{\mathcal{M}}$.[5]
4. $|\varphi \rightarrow \psi|_{\mathcal{M}} = |\varphi|^c_{\mathcal{M}} \cup |\psi|_{\mathcal{M}}$.
5. $|\varphi \leftrightarrow \psi|_{\mathcal{M}} = (|\varphi|_{\mathcal{M}} \cap |\psi|_{\mathcal{M}}) \cup (|\varphi|^c_{\mathcal{M}} \cap |\psi|^c_{\mathcal{M}})$.
6. $|\Box\varphi|_{\mathcal{M}} = \{w \in W : (w)_R \subseteq |\varphi|_{\mathcal{M}}\}$.

Relative: Establish the following by appealing to the definitions:

7. $\mathcal{M}, w \vDash \varphi \wedge \psi$ *iff* $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$.
8. $\mathcal{M}, w \vDash \Diamond\varphi$ *iff* $\mathcal{M}, u \vDash \varphi$ for some $u \in W$ such that $R(w, u)$.
9. $|\varphi|^w_{\mathcal{M}} = (w)_R \cap |\varphi|_{\mathcal{M}}$.
10. $\mathcal{M}, w \vDash \Box\varphi$ *iff* $(w)_R \subseteq |\varphi|_{\mathcal{M}}$.
11. $\mathcal{M}, w \vDash \Box(\varphi \rightarrow \psi)$ *iff* $|\varphi|^w_{\mathcal{M}} \subseteq |\psi|_{\mathcal{M}}$.
12. $\mathcal{M}, w \vDash \Box(\varphi \leftrightarrow \psi)$ *iff* $|\varphi|^w_{\mathcal{M}} = |\psi|^w_{\mathcal{M}}$.
13. $\mathcal{M}, w \vDash \Diamond\varphi$ *iff* $(w)_R \cap |\varphi|_{\mathcal{M}} \neq \varnothing$.
14. $\mathcal{M}, w \vDash \Diamond\Box\varphi$ *iff* there exists $u \in (w)_R$ where $(u)_R \subseteq |\varphi|_{\mathcal{M}}$.

Propositional Modal Logic: Metalogic

Sound: The system $\mathcal{Q}$ is *sound* with respect to $\vDash_C$ just in case $\Gamma \vDash_C \psi$ whenever $\Gamma \vdash_{\mathcal{Q}} \psi$.

Complete: The system $\mathcal{Q}$ is *complete* with respect to $\vDash_C$ just in case $\Gamma \vdash_{\mathcal{Q}} \psi$ whenever $\Gamma \vDash_C \psi$.

Characterization: The frame constraints C *characterize* the modal proof system $\mathcal{Q}$ just in case $\mathcal{Q}$ is both sound and complete with respect to $\vDash_C$.

Modal Logics: The modal proof systems are characterized the following sets of frame constraints:

- $K = \varnothing$ characterizes $\mathcal{K}$.
- $D = \{\text{SER}\}$ characterizes $\mathcal{D}$.
- $T = \{\text{REF}\}$ characterizes $\mathcal{T}$.
- $B = \{\text{REF}, \text{SYM}\}$ characterizes $\mathcal{B}$.
- $S4 = \{\text{REF}, \text{TRA}\}$ characterizes $\mathcal{S}4$.
- $S5 = \{\text{REF}, \text{REU}\}$ characterizes $\mathcal{S}5$.

Problem Set: Metalogic

Semantic Proofs: Provide semantic proofs of the following:

1. $\vDash_K \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
2. $\vDash_K \Box\varphi \rightarrow \Box\varphi$
3. $\vDash_D \Box\varphi \rightarrow \Diamond\varphi$
4. $\vDash_T \Box\varphi \rightarrow \varphi$
5. $\vDash_T \Box\Box\varphi \rightarrow \Box\varphi$
6. $\vDash_B \varphi \rightarrow \Box\Diamond\varphi$
7. $\vDash_4 \Box\varphi \rightarrow \Box\Box\varphi$
8. $\vDash_5 \Diamond\varphi \rightarrow \Box\Diamond\varphi$

Equivalences: Provide semantic proofs of the following equivalences:

[3] The *complement* X^c in W is the set of elements in W that are not in X, i.e., $X^c := \{z \in W : z \notin X\}$.
[4] The *intersection* $X \cap Y$ is the set of elements in both X and Y, i.e., $X \cap Y := \{z \in W : z \in X \text{ and } z \in Y\}$.
[5] The *union* $X \cup Y$ is the set of elements in either X or Y, i.e., $X \cup Y := \{z \in W : z \in X \text{ or } z \in Y\}$.

(9) $\neg\Box\varphi \equiv_K \Diamond\neg\varphi$.

(10) $\neg\Diamond\varphi \equiv_K \Box\neg\varphi$.

(11) $\Diamond\Box\varphi \equiv_B \Diamond\Box\Diamond\Box\varphi$

(12) $\Box\Box\varphi \equiv_4 \Box\varphi$.

(13) $\Box\Diamond\varphi \equiv_4 \Diamond\varphi$.

(14) $\Diamond\Box\varphi \equiv_5 \Box\varphi$.

Propositional Tense Logic: Syntax

Language $\mathcal{L}^{\mathrm{T}}$: The propositional language $\mathcal{L}^{\mathrm{T}}$ extends $\mathcal{L}$ to include:

- '$\boxed{\mathrm{P}}$' where $\ulcorner\boxed{\mathrm{P}}\varphi\urcorner$ reads $\ulcorner$It has always been that $\varphi\urcorner$.
- '$\boxed{\mathrm{F}}$' where $\ulcorner\boxed{\mathrm{F}}\varphi\urcorner$ reads $\ulcorner$It is always going to be that $\varphi\urcorner$.
- A zero-place operator $\bot$ called the *falsum*.

Well-Formed Sentences: Letting $p_i \in \mathcal{L}$, the *well-formed sentences* $\mathcal{L}^{\mathrm{T}}$ may be defined succinctly:

$$\varphi ::= p_i \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid \boxed{\mathrm{P}}\varphi \mid \boxed{\mathrm{F}}\varphi$$

This is a compact notation for a standard recursive definition. We may then let $\mathtt{wfs}(\mathcal{L}^{\mathrm{T}})$ be the set of all well-formed sentences of $\mathcal{L}^{\mathrm{T}}$.

Abbreviations: We maintain the abbreviations given for $\mathcal{L}$ along with:

- $\Diamond_{\mathrm{P}}\varphi \coloneqq \neg\boxed{\mathrm{P}}\neg\varphi$.
- $\Diamond_{\mathrm{F}}\varphi \coloneqq \neg\boxed{\mathrm{F}}\neg\varphi$.
- $\top \coloneqq \neg\bot$.
- $\triangle\varphi \coloneqq \boxed{\mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi$.
- $\triangledown\varphi \coloneqq \Diamond_{\mathrm{P}}\varphi \vee \varphi \vee \Diamond_{\mathrm{F}}\varphi$.
- $\mathcal{Q}\Gamma \coloneqq \{\mathcal{Q}\varphi \mid \varphi \in \Gamma\}$

Problem Set: Syntax

Negation: Provide a definition of negation using the other operators in $\mathcal{L}^{\mathrm{T}}$.

Regimentation: Regiment the following in $\mathcal{L}^{\mathrm{T}}$ disambiguating as needed.

1. If it is raining, it will stop.
2. If it wasn't that cold before, it might still be that cold at some point.
3. Either it has rained or it will snow.
4. If it will rain, then it has always been so.
5. If it will always rain, then it must have rained before.
6. It has always been true that it either will rain or it won't.
7. If it has always rained, then it will always been that it rained before.
8. If it has always rained, then it has always been that it has rained.
9. If it will always rain, then it cannot have always not rained.
10. If has rained and snowed, then it could tomorrow.
11. If rain has always implied clouds, it will always be cloudy if it always rains.
12. If rain lead to snow before, then snow might lead to rain.

Propositional Tense Logic: Frames

Frame: A *temporal frame* is an ordered pair $\mathcal{T} = \langle T, < \rangle$, where T is a nonempty set (called *times*) and $<$ is a binary relation on T (called *earlier-than*).

Abbreviations: We maintain the usual infix notation for convenience:

- $x < y \coloneqq \langle x, y\rangle \in <$
- $x \nless y \coloneqq \neg(x < y)$
- $x \leqslant y \coloneqq (x < y) \vee (x = y)$
- $y > x \coloneqq x < y$
- $y \ngtr x \coloneqq x \nless y$
- $x \geqslant y \coloneqq y \leqslant x$.

Frame Constraints: Consider the following constraints on an arbitrary frame $\langle T, <\rangle$:

INFINITE FUTURE (INF): For all $x \in T$ there is some $y \in T$ where $x < y$.

INFINITE PAST (INP): For all $x \in T$ there is some $y \in T$ where $y < x$.

IRREFLEXIVE (IRR): $x \nless x$ for every $x \in T$.

ASYMMETRIC (ASM): $y \nless x$ whenever $x < y$.

TRANSITIVE (TRA): $x < z$ whenever both $x < y$ and $y < z$.

LEFT LINEAR (LLN): Either $y < z$, $y = z$, or $z < y$ whenever both $y < x$ and $z < x$.

RIGHT LINEAR (RLN): Either $y < z$, $y = z$, or $z < y$ whenever both $x < y$ and $x < z$.

TOTAL (TOT): Either $x < y$, $x = y$, or $y < x$ for all $x, y \in T$.

DENSITY (DEN): If $x < z$, there is some $y \in T$ where $x < y$ and $y < z$.

LEFT DISCRETE (LDI): If $x < y$, there is some $z < y$ where $u \leqslant z$ for all $x \leqslant u < y$.

RIGHT DISCRETE (RDI): If $x < y$, there is some $z > x$ where $u \geqslant z$ for all $x < u \leqslant y$.

Problem Set: Frames

Relations: Evaluate the following, providing a proof or counterexample:

1. Every asymmetric frame is irreflexive.
2. Every irreflexive transitive frame is asymmetric.
3. Every frame that is not irreflexive has neither beginning nor end.
4. Every left and right linear frame is total.
5. Every total frame is left and right linear.
6. Every frame that is left and right linear is transitive.
7. Every frame that is not right linear is right discrete.
8. Every frame that is dense is both left and right linear.
9. Every frame that is asymmetric and left linear is transitive.
10. There is a dense frame with both a beginning and end.
11. The relational image of a frame with a beginning and end is finite.
12. The relational image of an asymmetric frame is not a partition.

PROPOSITIONAL TENSE LOGIC: LOGICAL CONSEQUENCE

Interpretation: An *interpretation* of $\mathcal{L}^{\mathrm{T}}$ over a frame $\mathcal{T}$ is any function $\mathcal{I} : \mathbb{L} \to \wp(T)$, i.e., where $\mathcal{I}(p_i) \subseteq T$ for every sentence letter $p_i \in \mathbb{L}$.

Model: A *C-model* of $\mathcal{L}^{\mathrm{T}}$ is an ordered triple $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ where $\langle T, <\rangle$ is a frame satisfying the constraints in C and $\mathcal{I}$ is an interpretation of $\mathcal{L}^{\mathrm{T}}$.

Semantics: Let $\vDash$ be the smallest relation to satisfy the following conditions where $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ is a model of $\mathcal{L}^{\mathrm{T}}$, and $x, y \in T$, $p_i \in \mathbb{L}$, $\varphi, \psi \in \mathtt{wfs}(\mathcal{L}^{\mathrm{T}})$:

$\mathcal{M}, x \nvDash \bot$
$\mathcal{M}, x \vDash p_i$ *iff* $x \in \mathcal{I}(p_i)$.
$\mathcal{M}, x \vDash \neg\varphi$ *iff* $\mathcal{M}, x \nvDash \varphi$.
$\mathcal{M}, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, x \nvDash \varphi$ or $\mathcal{M}, x \vDash \psi$.
$\mathcal{M}, x \vDash \boxed{\text{P}}\varphi$ *iff* $\mathcal{M}, y \vDash \varphi$ for every $y \in T$ such that $y < x$.
$\mathcal{M}, x \vDash \boxed{\text{F}}\varphi$ *iff* $\mathcal{M}, y \vDash \varphi$ for every $y \in T$ such that $x < y$.

Truth-Condition: The *truth-condition* $|\varphi|_{\mathcal{M}} \coloneqq \{x \in T : \mathcal{M}, x \vDash \varphi\}$ for a wfs $\varphi \in \texttt{wfs}(\mathcal{L}^{\text{T}})$ on a model $\mathcal{M}$ is the set of times when φ is true.

Constraint Set: Let C be an arbitrary set of frame constraints (not just those above).

Logical Consequence: $\Gamma \vDash_C \varphi$ just in case for any C-model $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ of $\mathcal{L}^{\text{T}}$ and time $x \in T$, if $\mathcal{M}, x \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, x \vDash \varphi$.

Logical Equivalence: $\varphi \equiv_C \psi$ just in case $\varphi \vDash_C \psi$ and $\psi \vDash_C \varphi$.

Logical Truth: A wfs φ of $\mathcal{L}^{\text{T}}$ is a *C-logical truth* (or *C-valid*) just in case $\vDash_C \varphi$.

Tense Systems: Consider the following sets of temporal frame constraints:

$$\begin{array}{rlrl} \textit{Strict Partial:} & P = \{\text{IRR}, \text{TRA}\}. & \textit{Dense Infinite:} & Q = E \cup \{\text{INF}, \text{INP}\}. \\ \textit{Open Future:} & O = P \cup \{\text{LLN}\}. & \textit{Discrete Linear:} & I = L \cup \{\text{LDI}, \text{RDI}\}. \\ \textit{Linear:} & L = O \cup \{\text{RLN}\}. & \textit{Discrete Infinite:} & Z = I \cup \{\text{INF}, \text{INP}\}. \\ \textit{Dense Linear:} & E = L \cup \{\text{DEN}\}. & \textit{Discrete Finite:} & F = I \cup \{\neg\text{INF}, \neg\text{INP}\}. \end{array}$$

Propositional Tense Logic: Axiomatic System

Axiom Schemata: Consider the following temporal axiom schemata:

TK $\boxed{\text{F}}(\varphi \to \psi) \to (\boxed{\text{F}}\varphi \to \boxed{\text{F}}\psi)$ (future distribution).

TL $\triangle\varphi \to \boxed{\text{F}}\boxed{\text{P}}\varphi$ (linearity).

T4 $\boxed{\text{F}}\varphi \to \boxed{\text{F}}\boxed{\text{F}}\varphi$ (temporal transitivity).

TA $\varphi \to \boxed{\text{F}}\Diamond\varphi$ (temporal accessibility).

TD If $\vdash \varphi$, then $\vdash \varphi[\boxed{\text{P}}/\boxed{\text{F}}]$ (temporal duality).

where $\triangle\varphi \coloneqq \boxed{\text{P}}\varphi \wedge \varphi \wedge \boxed{\text{F}}\varphi$, $\Diamond\varphi \coloneqq \neg\boxed{\text{P}}\neg\varphi$, and $\varphi[\boxed{\text{P}}/\boxed{\text{F}}]$ denotes the result of simultaneously swapping all occurrences of $\boxed{\text{P}}$ with $\boxed{\text{F}}$ and vice versa.

Future Necessitation Rule: Consider the following metarule:

FN If $\Gamma \vdash \varphi$, then $\boxed{\text{F}}\Gamma \vdash \boxed{\text{F}}\varphi$ (future necessitation).

where $\boxed{\text{F}}\Gamma = \{\boxed{\text{F}}\chi \mid \chi \in \Gamma\}$.

TL System: The *TL proof system* for linear tense logic includes all $\mathcal{L}^{\text{T}}$ instances of the Hilbert axioms together with axioms **TK**, **TL**, **T4**, and **TA**, along with the **TD** duality rule and **FN** future necessitation rule.

Tense Proofs: A -*proof* of ψ from Γ is a finite sequence $\varphi_1, \ldots, \varphi_n$ where $\varphi_n = \psi$ and each φ_i is either: (1) a *premise* in Γ; (2) a propositional *axiom*; (3) a **TL** *axiom* instance; or (4) follows from previous formulas by modus ponens, the **FN** rule, or the **TD** duality rule.

Derivable: A wfs ψ of $\mathcal{L}^{\mathrm{T}}$ is *derivable* from Γ by the proof system , written $\Gamma\psi$, just in case there is a -proof of ψ from Γ.

Problem Set: Tense Proofs

Derivations: Provide formal -proofs for the following theorems:

1. $\boxed{\mathrm{F}}(\varphi \to \psi) \to (\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\psi)$
2. $\boxed{\mathrm{P}}(\varphi \to \psi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\psi)$
3. $\boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$
4. $\boxed{\mathrm{P}}\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\varphi$
5. $\varphi \to \boxed{\mathrm{F}}\Diamond_{\mathrm{P}}\varphi$
6. $\varphi \to \boxed{\mathrm{P}}\Diamond_{\mathrm{F}}\varphi$

Duality Applications: Use the **TD** duality rule to derive past axioms from future axioms:

1. Show that if $\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi$ (T4), then $\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\boxed{\mathrm{P}}\varphi$ by TD.
2. Show that if $\varphi \to \boxed{\mathrm{F}}\Diamond_{\mathrm{P}}\varphi$ (TA), then $\varphi \to \boxed{\mathrm{P}}\Diamond_{\mathrm{F}}\varphi$ by TD.

Linearity: Prove that the **TL** axiom $\triangle\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{P}}\varphi$ characterizes linear time by:

1. Providing a countermodel in a branching time frame where TL fails.
2. Proving TL holds in all linear frames.

Problem Set: Characterization

Countermodels: Evaluate the following, providing a proof or countermodel:

1. $\vDash \boxed{\mathrm{P}}(\varphi \to \psi) \to (\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\psi)$.
2. $\vDash \Diamond_{\mathrm{P}}\top$.
3. $\vDash \varphi \to \boxed{\mathrm{F}}\Diamond_{\mathrm{P}}\varphi$.
4. $\vDash \boxed{\mathrm{P}}\boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\varphi$.
5. $\vDash \boxed{\mathrm{P}}\bot \vee \Diamond_{\mathrm{P}}\boxed{\mathrm{P}}\bot$.
6. $\vDash \boxed{\mathrm{P}}\boxed{\mathrm{F}}\varphi \to \triangle\varphi$.
7. $\vDash \boxed{\mathrm{P}}\varphi \to \boxed{\mathrm{P}}\boxed{\mathrm{P}}\varphi$.
8. $\vDash (\Diamond_{\mathrm{P}}\top \wedge \varphi \wedge \boxed{\mathrm{F}}\varphi) \to \Diamond_{\mathrm{P}}\boxed{\mathrm{F}}\varphi$.
9. $\vDash \boxed{\mathrm{F}}(\varphi \to \psi) \to (\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\psi)$.
10. $\vDash \Diamond_{\mathrm{F}}\top$.
11. $\vDash \varphi \to \boxed{\mathrm{P}}\Diamond_{\mathrm{F}}\varphi$.
12. $\vDash \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\varphi$.
13. $\vDash \boxed{\mathrm{F}}\bot \vee \Diamond_{\mathrm{F}}\boxed{\mathrm{F}}\bot$.
14. $\vDash \boxed{\mathrm{F}}\boxed{\mathrm{P}}\varphi \to \triangle\varphi$.
15. $\vDash \boxed{\mathrm{F}}\varphi \to \boxed{\mathrm{F}}\boxed{\mathrm{F}}\varphi$.
16. $\vDash (\Diamond_{\mathrm{F}}\top \wedge \varphi \wedge \boxed{\mathrm{P}}\varphi) \to \Diamond_{\mathrm{F}}\boxed{\mathrm{P}}\varphi$.

Logical Consequence: For each of the claims above, strengthen $\vDash$ by imposing the weakest set of constraints C which make that claim valid.

Bimodal Logic: Tense and Modality

Natural language frequently expresses relationships between temporal and modal concepts. Consider these intuitions:

Perpetuity: "If something is necessary, then it has always been and will always be true." Formally: if $\Box\varphi$, then φ holds at all times (past, present, and future).

Possibility: "If something is sometimes true, then it is possibly true." Formally: if φ holds at some time, then $\Diamond\varphi$.

Interaction: "Necessary truths persist through time." If $\Box\varphi$ at time t, then $\Box\varphi$ at all future times.

These principles cannot be captured by modal logic alone (which lacks temporal operators) nor by tense logic alone (which lacks modal operators). We require a *bimodal logic* combining both dimensions. The **TM system** (Logic of Tense and Modality) provides such a framework, integrating S5 modal logic with linear tense logic through specific interaction axioms.

Bimodal Logic: Syntax

Language: The *bimodal language* $\mathcal{L}^{\mathrm{B}}$ combines the modal operator $\Box$ (defined in modal logic) with temporal operators $\boxed{\scriptstyle \mathrm{P}}$ and $\boxed{\scriptstyle \mathrm{F}}$ (defined in tense logic):

$$\mathcal{L}^{\mathrm{B}} = \langle \mathbb{L}, \bot, \rightarrow, \Box, \mathsf{Past}, \mathsf{Future} \rangle$$

where $\mathbb{L} = \{p_i \mid i \in \mathbb{N}\}$ is the set of sentence letters.

Well-Formed Sentences: The set of well-formed sentences $\mathtt{wfs}(\mathcal{L}^{\mathrm{B}})$ is defined inductively:

$$\varphi ::= p_i \mid \bot \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \boxed{\scriptstyle \mathrm{P}}\varphi \mid \boxed{\scriptstyle \mathrm{F}}\varphi$$

Abbreviations: Derived operators $\Diamond$, $\Diamond_{\mathrm{P}}$, $\Diamond_{\mathrm{F}}$, $\triangle$, and $\triangledown$ are defined as in modal and tense logic. Scope conventions follow modal logic: operators bind more tightly than connectives.

Problem Set: Syntax

Well-Formedness: Determine which of the following are well-formed sentences of $\mathcal{L}^{\mathrm{B}}$:

1. $\Box p_1 \rightarrow \boxed{\scriptstyle \mathrm{F}} p_2$
2. $\boxed{\scriptstyle \mathrm{P}}\Box p_1$
3. $\triangle\triangledown p_1$
4. $\Box\boxed{\scriptstyle \mathrm{F}}\Diamond p_1$
5. $p_1 \Box p_2$
6. $\boxed{\scriptstyle \mathrm{P}} \rightarrow p_1$
7. $\triangle(\Box p_1 \leftrightarrow \boxed{\scriptstyle \mathrm{F}} p_2)$
8. $\Diamond\Diamond_{\mathrm{P}}\boxed{\scriptstyle \mathrm{F}} p_1$

Abbreviations: Expand the following abbreviated formulas into primitive $\mathcal{L}^{\mathrm{B}}$ notation (using only $\bot$, $\rightarrow$, $\Box$, $\boxed{\scriptstyle \mathrm{P}}$, $\boxed{\scriptstyle \mathrm{F}}$):

1. $\triangle p_1$
2. $\Diamond\triangledown p_1$
3. $\Diamond_{\mathrm{P}}\varphi \vee \Diamond_{\mathrm{F}}\psi$
4. $\Box\varphi \wedge \triangle\psi$

Scope Ambiguity: Provide fully parenthesized versions of the following to clarify intended scope:

1. $\Box\varphi \rightarrow \boxed{\scriptstyle \mathrm{F}}\psi \vee \chi$
2. $\triangle\varphi \wedge \Box\psi \rightarrow \Diamond_{\mathrm{P}}\chi$

BIMODAL LOGIC: SEMANTICS

Task Frame: A **task frame** is a triple $\mathcal{F}_{\mathrm{T}} = \langle \mathcal{W}, \mathcal{T}, \Rightarrow \rangle$ where:

- $\mathcal{W}$ is a non-empty set of **world states**
- $\mathcal{T} = \langle T, +, \leqslant \rangle$ is a **totally ordered abelian group**
- $\Rightarrow \subseteq \mathcal{W} \times T \times \mathcal{W}$ is a parameterized **task relation**

We write $w \Rightarrow_x v$ to mean $\langle w, x, v \rangle \in \Rightarrow$.

Nullity Constraint: For all $w \in \mathcal{W}$, we have $w \Rightarrow_0 w$.

Compositionality Constraint: For all $w, u, v \in \mathcal{W}$ and all $x, y \in T$, if $w \Rightarrow_x u$ and $u \Rightarrow_y v$, then $w \Rightarrow_{x+y} v$.

World History: A **world history** over a frame $\mathcal{F}_{\mathrm{T}} = \langle \mathcal{W}, \mathcal{T}, \Rightarrow \rangle$ is a function $\tau : X \to \mathcal{W}$ where $X \subseteq T$ is a convex subset of times satisfying:

- **Task Coherence:** For all times $x, y \in T$ where both $x, x + y \in X$, we have $\tau(x) \Rightarrow_y \tau(x + y)$.

We write $\mathcal{H}$ for the set of all world histories over a given frame.

Task Model: A **task model** is a tuple $\mathcal{M}_{\mathrm{T}} = \langle \mathcal{W}, \mathcal{T}, \Rightarrow, \ulcorner\cdot\urcorner \rangle$ where $\langle \mathcal{W}, \mathcal{T}, \Rightarrow \rangle$ is a task frame and $\ulcorner p_i \urcorner \subseteq \mathcal{W}$ is the **interpretation** of sentence letter p_i.

Truth Conditions: Given a task model $\mathcal{M}_{\mathrm{T}}$, world history τ, and time x, we define $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$ inductively:

(a) $\mathcal{M}_{\mathrm{T}}, \tau, x \models p_i$ iff $x \in dom(\tau)$ and $\tau(x) \in \ulcorner p_i \urcorner$

(b) $\mathcal{M}_{\mathrm{T}}, \tau, x \not\models \bot$

(c) $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi \to \psi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, x \not\models \varphi$ or $\mathcal{M}_{\mathrm{T}}, \tau, x \models \psi$

(d) $\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi$ for all $\sigma \in \mathcal{H}$

(e) $\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle P}\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, y \models \varphi$ for all $y \in dom(\tau)$ with $y < x$

(f) $\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle F}\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, y \models \varphi$ for all $y \in dom(\tau)$ with $y > x$

Validity: A formula φ is **valid over task frames** (written $\models_{\mathrm{TM}} \varphi$) iff for all task models $\mathcal{M}_{\mathrm{T}}$, all world histories τ, and all times $x \in dom(\tau)$, we have $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$.

Problem Set: Semantics

Countermodels: Evaluate the following, providing a proof or countermodel:

1. $\models \Box\varphi \to \triangle\varphi$
2. $\models \triangledown\varphi \to \Diamond\varphi$
3. $\models \Box\varphi \to \Box\triangle\varphi$
4. $\models \Diamond\triangledown\varphi \to \Diamond\varphi$
5. $\models \Box\varphi \to \Box\boxed{\scriptstyle F}\varphi$
6. $\models \Box\varphi \to \boxed{\scriptstyle F}\Box\varphi$

Duality: Show that the following pairs are duals by providing semantic argument or counterexample:

1. $\Box\varphi$ and $\Diamond\varphi$
2. $\boxed{\scriptstyle \mathrm{P}}\varphi$ and $\langle\mathrm{P}\rangle\varphi$
3. $\triangle\varphi$ and $\triangledown\varphi$

Bimodal Logic: Axiomatic System

TM System: The *TM proof system* combines the S5 modal system with the TL temporal system, adding two bimodal interaction axioms. We write $\Gamma \vdash_{\mathrm{TM}} \varphi$ for derivability in TM.

System Components: TM includes:

(a) All S5 axioms and rules (MT, M4, MB, MK, MP)

(b) All TL axioms and rules (TK, TL, T4, TA, TD, FN)

(c) **MF:** $\Box\varphi \rightarrow \Box\boxed{\scriptstyle \mathrm{F}}\varphi$ (modal-future interaction)

(d) **TF:** $\Box\varphi \rightarrow \boxed{\scriptstyle \mathrm{F}}\Box\varphi$ (temporal-future interaction)

Perpetuity Principles: The bimodal interaction axioms yield the following TM-derivable theorems:

P1: $\Box\varphi \rightarrow \triangle\varphi$

P2: $\triangledown\varphi \rightarrow \Diamond\varphi$

P3: $\Box\varphi \rightarrow \Box\triangle\varphi$

P4: $\Diamond\triangledown\varphi \rightarrow \Diamond\varphi$

P5: $\Diamond\triangledown\varphi \rightarrow \triangle\Diamond\varphi$

P6: $\triangledown\Box\varphi \rightarrow \Box\triangle\varphi$

Problem Set: Derivations

Perpetuity Proofs: Provide formal TM-proofs for the perpetuity principles:

1. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \triangle\varphi$ (P1)
2. $\vdash_{\mathrm{TM}} \triangledown\varphi \rightarrow \Diamond\varphi$ (P2)
3. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \Box\triangle\varphi$ (P3)
4. $\vdash_{\mathrm{TM}} \Diamond\triangledown\varphi \rightarrow \Diamond\varphi$ (P4)
5. $\vdash_{\mathrm{TM}} \Diamond\triangledown\varphi \rightarrow \triangle\Diamond\varphi$ (P5)
6. $\vdash_{\mathrm{TM}} \triangledown\Box\varphi \rightarrow \Box\triangle\varphi$ (P6)

Interaction Axioms: Show that the bimodal interaction axioms characterize the relationship between modality and time by proving:

1. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \Box\boxed{\scriptstyle \mathrm{F}}\varphi$ using MF
2. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \boxed{\scriptstyle \mathrm{F}}\Box\varphi$ using TF
3. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \Box\boxed{\scriptstyle \mathrm{P}}\varphi$ using MF and TD
4. $\vdash_{\mathrm{TM}} \Box\varphi \rightarrow \boxed{\scriptstyle \mathrm{P}}\Box\varphi$ using TF and TD

Bimodal Logic: Metatheory

We have now defined both the proof-theoretic (axiomatic) and semantic (task frames) aspects of TM logic. The axiomatic system showed us *how* to derive theorems syntactically, while the task semantics showed us *what* formulas mean and *why* perpetuity principles hold. The central metalogical questions are: (1) **Soundness** – does every derivable formula hold in all task models? and (2) **Completeness** – does every semantically valid formula have a proof? These questions establish the correspondence between syntactic derivability and semantic validity.

Logical Consequence: A formula φ is a **logical consequence** of a set Γ of formulas (written $\Gamma \models_{\mathrm{TM}} \varphi$) iff for all task models $\mathcal{M}_{\mathrm{T}}$, all world histories τ, and all times $x \in dom(\tau)$, if $\mathcal{M}_{\mathrm{T}}, \tau, x \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$.

Soundness Theorem: *If* $\Gamma \vdash_{\mathrm{TM}} \varphi$*, then* $\Gamma \models_{\mathrm{TM}} \varphi$*.*

That is, every TM-derivable formula is semantically valid over all task frames. This means the proof system is **reliable**: if we can prove something syntactically, it is guaranteed to be true semantically.

Soundness Proof Sketch: The proof proceeds by induction on the length of TM-proofs. We show: (1) all axiom schemata are valid over task frames, and (2) all inference rules preserve validity. We demonstrate representative cases:

Case MP: If φ and $\varphi \to \psi$ are both valid, then ψ is valid.

Proof: Suppose $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$ and $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi \to \psi$ for arbitrary model, history, and time. By truth condition for $\to$, either $\mathcal{M}_{\mathrm{T}}, \tau, x \not\models \varphi$ or $\mathcal{M}_{\mathrm{T}}, \tau, x \models \psi$. Since we have $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$, it must be that $\mathcal{M}_{\mathrm{T}}, \tau, x \models \psi$. Thus ψ is valid. $\square$

Case MK: The axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is valid.

Proof: Suppose $\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box(\varphi \to \psi)$. This means for all histories σ with $x \in dom(\sigma)$, we have $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi \to \psi$. Now suppose $\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box\varphi$, meaning for all such histories σ, we have $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi$. Take any history σ with $x \in dom(\sigma)$. We have both $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi$ and $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi \to \psi$, so by truth condition for $\to$, we get $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \psi$. Since σ was arbitrary, $\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box\psi$. $\square$

Case MF: The bimodal axiom $\Box\boxed{\scriptstyle F}\varphi \to \boxed{\scriptstyle F}\Box\varphi$ is valid.

Proof: Suppose $\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box\boxed{\scriptstyle F}\varphi$. This means for all histories σ with $x \in dom(\sigma)$, we have $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \boxed{\scriptstyle F}\varphi$, i.e., for all $y > x$ in $dom(\sigma)$, we have $\mathcal{M}_{\mathrm{T}}, \sigma, y \models \varphi$. To show $\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle F}\Box\varphi$, take any $y > x$ in $dom(\tau)$. We need to show $\mathcal{M}_{\mathrm{T}}, \tau, y \models \Box\varphi$. Take any history ρ with $y \in dom(\rho)$. By assumption applied to ρ, we have $\mathcal{M}_{\mathrm{T}}, \rho, y \models \varphi$. Since ρ was arbitrary, $\mathcal{M}_{\mathrm{T}}, \tau, y \models \Box\varphi$. Since y was arbitrary, $\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle F}\Box\varphi$. $\square$

The remaining axioms (MT, M4, MB, TD, TL, T4, TA, TF) and rules are shown valid by similar arguments. The key insight is that the bimodal axioms MF and TF ensure that modal and temporal operators interact correctly across world histories and times.

Completeness Theorem: *If* $\Gamma \models_{\mathrm{TM}} \varphi$*, then* $\Gamma \vdash_{\mathrm{TM}} \varphi$*.*

That is, every semantically valid formula is TM-derivable. This means the proof system is **adequate**: if something is true in all task models, we can find a proof of it.

Completeness Proof Strategy: The proof uses the **canonical model construction** method standard in modal logic. The idea is to build a task model from maximal consistent sets of formulas, where:

- World states are maximal consistent sets (sets of formulas closed under consequence and not containing contradictions)
- The task relation $w \Rightarrow_x v$ is defined by formula consistency: if $\boxed{\text{F}}^x \varphi \in w$, then $\varphi \in v$
- Histories are constructed as maximal linear extensions of consistent temporal sequences
- The valuation assigns truth to sentence letters at worlds where they appear

The key lemma is the **Truth Lemma**: for any formula φ and maximal consistent set w representing a world state in the canonical model, $\varphi \in w$ iff the canonical model validates φ at w. This is proved by induction on formula complexity, with the modal and temporal cases following from the construction of the task relation and histories.

Given the Truth Lemma, completeness follows: if $\Gamma \models_{\text{TM}} \varphi$ but $\Gamma \nvdash_{\text{TM}} \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent (by negation completeness). This set can be extended to a maximal consistent set, which forms a world in the canonical model where Γ holds but φ fails – contradicting semantic validity.

Metalogical Consequences: The soundness and completeness theorems have several important consequences:

- **Consistency:** TM is consistent (no formula is both provable and refutable) since the task semantics provides models
- **Decidability:** TM is decidable (there exists an algorithm to determine provability) inherited from decidability of S5 modal logic and tense logic
- **Compactness:** If every finite subset of Γ is satisfiable, then Γ is satisfiable (standard consequence of completeness)
- **Expressiveness:** TM can express perpetuity principles (like P1-P6) that are not derivable in either pure modal logic or pure tense logic alone

Problem Set: Metatheory

Axiom Validity: Show that the following TM axioms are valid over all task frames by providing explicit semantic proofs:

1. MT: $\Box\varphi \to \varphi$ (*Hint:* Use reflexivity of task relation via nullity)
2. T4: $\boxed{\text{P}}\varphi \to \boxed{\text{P}}\boxed{\text{P}}\varphi$ (*Hint:* Transitivity of temporal order)

Consistency: For each of the following sets of formulas, determine whether it is consistent (satisfiable in some task model) or inconsistent. If consistent, describe a task model that satisfies it. If inconsistent, derive a contradiction using TM axioms.

1. $\{\Box p, \Diamond\neg p\}$
2. $\{\triangle p, \triangledown \neg p\}$
3. $\{\Box p, \neg\triangle p\}$
4. $\{\triangledown p, \triangle\neg p\}$

Countermodels: For each invalid formula, use the completeness theorem to argue that a countermodel must exist, then construct an explicit task model refuting it:

1. $\triangle\varphi \to \Box\varphi$ (temporal necessity doesn't imply modal necessity)
2. $\Diamond\varphi \to \triangledown\varphi$ (modal possibility doesn't imply temporal possibility)

Bimodal Logic: Extensions

The TM logic we have developed makes minimal assumptions about the temporal order structure – it is dense, unbounded, and linear. We can extend TM by adding operators that are useful for reasoning about specific temporal and modal relationships. The extensions presented here build on the core TM system while preserving its soundness and completeness properties.

Extended Language: Letting $p_i \in \mathbb{L}$, the *well-formed sentences* of the extended bimodal language $\mathcal{L}^{\mathrm{B}}_{+}$ are defined:

$$\varphi ::= p_i \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid \boxed{\mathrm{P}}\varphi \mid \boxed{\mathrm{F}}\varphi \mid \Box\varphi \mid \boxed{\cdot}\varphi \mid \boxed{\triangleleft}\varphi \mid \boxed{\triangleright}\varphi.$$

Define $\texttt{wfs}(\mathcal{L}^{\mathrm{B}}_{+})$ to be the set of all well-formed sentences of $\mathcal{L}^{\mathrm{B}}_{+}$.

Abbreviations: We maintain the abbreviations from $\mathcal{L}^{\mathrm{B}}$ along with the conventions:

Will Always: $\boxed{!}\varphi \coloneqq \boxed{\cdot}\boxed{\mathrm{F}}\varphi$. *Could Always:* $\boxed{?}\varphi \coloneqq \Diamond_{\cdot}\boxed{\mathrm{F}}\varphi$.

Will Eventually: $\Diamond_{!}\varphi \coloneqq \boxed{\cdot}\Diamond_{\mathrm{F}}\varphi$. *Could Eventually:* $\Diamond_{?}\varphi \coloneqq \Diamond_{\cdot}\Diamond_{\mathrm{F}}\varphi$.

Intersection: $\langle\tau\rangle_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(x) = \tau(x)\}$.

Open Futures: $|\tau\rangle_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \leqslant x\}$.

Open Pasts: $\langle\tau|_x \coloneqq \{\sigma \in H_{\mathcal{F}} \mid \sigma(y) = \tau(y) \text{ for all } y \geqslant x\}$.

Model: A *model* of $\mathcal{L}^{\mathrm{B}}_{+}$ is an $\mathcal{M} = \langle W, \to, \mathbb{Z}, <, \mathcal{I}\rangle$ where $\langle W, \to, \mathbb{Z}, <\rangle$ is a bimodal frame and $\mathcal{I}$ is a bimodal interpretation of $\mathcal{L}^{\mathrm{B}}$.

Semantics: We define $\vDash$ recursively for a model $\mathcal{M}$ of $\mathcal{L}^{\mathrm{B}}_{+}$, worlds $\tau, \sigma \in H_{\mathbb{Z}}$, times $x, y \in T$, sentence letter $p_i \in \mathbb{L}$, and $\varphi, \psi \in \texttt{wfs}(\mathcal{L}^{\mathrm{B}}_{+})$:

$\mathcal{M}, \tau, x \nvDash \bot$.

$\mathcal{M}, \tau, x \vDash p_i$ *iff* $\tau(x) \in \mathcal{I}(p_i)$.

$\mathcal{M}, \tau, x \vDash \neg\varphi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$.

$\mathcal{M}, \tau, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, \tau, x \nvDash \varphi$ or $\mathcal{M}, \tau, x \vDash \psi$.

$\mathcal{M}, \tau, x \vDash \boxed{\mathrm{P}}\varphi$ *iff* $\mathcal{M}, \tau, y \vDash \varphi$ for every $y \in T$ such that $y < x$.

$\mathcal{M}, \tau, x \models \boxed{\mathrm{F}}\varphi$ *iff* $\mathcal{M}, \tau, y \models \varphi$ for every $y \in T$ such that $x < y$.

$\mathcal{M}, \tau, x \models \Box\varphi$ *iff* $\mathcal{M}, \sigma, x \models \varphi$ for every $\sigma \in H_{\mathbb{Z}}$.

$\mathcal{M}, \tau, x \models \boxdot\varphi$ *iff* $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in \langle\tau\rangle_x$.

$\mathcal{M}, \tau, x \models \boxed{\triangleright}\varphi$ *iff* $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in |\tau\rangle_x$.

$\mathcal{M}, \tau, x \models \boxed{\triangleleft}\varphi$ *iff* $\mathcal{M}, \sigma, x \models \varphi$ for all $\sigma \in \langle\tau|_x$.

Logical Consequence: $\Gamma \models \varphi$ just in case for any model $\mathcal{M} = \langle W, \rightarrow, \mathbb{Z}, <, \mathcal{I}\rangle$ of $\mathcal{L}^{\mathrm{B}}_{+}$, world $\tau \in H_{\mathbb{Z}}$, and time $x \in T$, if $\mathcal{M}, \tau, x \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, \tau, x \models \varphi$.

Problem Set: Extensions

Regimentation: Regiment the following in $\mathcal{L}^{\mathrm{B}}$ and $\mathcal{L}^{\mathrm{B}}_{+}$ disambiguating as needed.

1. If it could rain, it will eventually stop.
2. Either it will always eventually rain or it could eventually always rain.
3. If it has always been raining, then it will eventually stop.
4. Either it could always rain, or it has been raining and will rain again.
5. If it could eventually always rain, then it eventually it could always rain.
6. Either it has been raining or it will eventually rain.
7. If it could eventually stop raining, then it has been raining.
8. If it will be raining, then it could have always been raining.
9. If it has always been raining, then it could eventually stop.
10. If it could eventually always rain, then it will always rain.

Countermodels: Evaluate the following, providing a proof or countermodel:

1. $\boxdot\varphi \models \boxed{\triangleleft}\varphi$.
2. $\boxed{\triangleleft}\varphi \wedge \boxed{\triangleright}\varphi \models \varphi$.
3. $\boxed{!}\varphi \models \boxed{\mathrm{P}}\boxed{!}\varphi$
4. $\boxdot\varphi \models \boxed{\triangleright}\varphi$.
5. $\varphi \models \boxed{\triangleleft}\varphi \wedge \boxed{\triangleright}\varphi$.
6. $\lozenge\!\!\!\!?\;\varphi \models \boxed{\mathrm{P}}\lozenge\!\!\!\!?\;\varphi$

Additional Materials

The following sections present alternative semantic approaches to bimodal logic that complement the main task semantics presentation.

Propositional Tense Logic: Indeterminacy

Well-Formed Sentences: Letting $p_i \in \mathbb{L}$, the *well-formed sentences* $\mathcal{L}^{\text{T}}_{\Box}$ are defined:

$$\varphi ::= p_i \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid \boxed{\text{P}}\varphi \mid \boxed{\text{F}}\varphi \mid \boxdot\varphi$$

Define $\texttt{wfs}(\mathcal{L}^{\text{T}}_{\Box})$ to be the set of all well-formed sentences of $\mathcal{L}^{\text{T}}_{\Box}$.

Abbreviation: In addition to the abbreviations for $\mathcal{L}^{\text{T}}$, we may define $\Diamond\!\!\!\cdot\,\varphi := \neg\boxdot\neg\varphi$.

Index: Let $\mathcal{T}_i = \langle T_i, <_i \rangle$ share the same index.

Subframe: $\mathcal{T}_i$ is a *subframe* of $\mathcal{T}_j$ *iff* $T_i \subseteq T_j$ and $x <_j y$ whenever $x <_i y$.

History: $\mathcal{T}_i$ is a *history* of the frame $\mathcal{T}$ *iff:* (1) $\mathcal{T}_i$ is a subframe of $\mathcal{T}$ that satisfies TOT; and (2) for every subframe $\mathcal{T}_j$ of $\mathcal{T}$ that satisfies TOT, if $\mathcal{T}_i$ is a subframe of $\mathcal{T}_j$, then $\mathcal{T}_i = \mathcal{T}_j$ (i.e., $\mathcal{T}_i$ is a *maximal total subframe* of $\mathcal{T}$).

Possible Histories: Let $H_{\mathcal{T}}$ be the set of all histories of $\mathcal{T}$.

Inevitability Set: Let $H^x_{\mathcal{T}} := \{\mathcal{T}_i \in H_{\mathcal{T}} \mid x \in T_i\}$ be the histories of $\mathcal{T}$ in which x occurs.

Language $\mathcal{L}^{\text{T}}_{\Box}$: Let $\mathcal{L}^{\text{T}}_{\Box}$ extend $\mathcal{L}^{\text{T}}$ to include '$\boxdot$' where $\ulcorner\boxdot\varphi\urcorner$ reads $\ulcorner$Inevitably $\varphi\urcorner$.

Inevitability: We define $\vDash$ recursively for a model $\mathcal{M} = \langle T, <, \mathcal{I} \rangle$ of $\mathcal{L}^{\text{T}}$, history $\mathcal{T}_i = \langle T_i, <_i \rangle$, times $x, y \in T$, sentence letter $p_i \in \mathbb{L}$, and $\varphi, \psi \in \texttt{wfs}(\mathcal{L}^{\text{T}})$:

$\mathcal{M}, \mathcal{T}_i, x \nvDash \bot$.
$\mathcal{M}, \mathcal{T}_i, x \vDash p_i$ *iff* $x \in \mathcal{I}(p_i)$.
$\mathcal{M}, \mathcal{T}_i, x \vDash \neg\varphi$ *iff* $\mathcal{M}, \mathcal{T}_i, x \nvDash \varphi$.
$\mathcal{M}, \mathcal{T}_i, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, \mathcal{T}_i, x \nvDash \varphi$ or $\mathcal{M}, \mathcal{T}_i, x \vDash \psi$.
$\mathcal{M}, \mathcal{T}_i, x \vDash \boxed{\text{P}}\varphi$ *iff* $\mathcal{M}, \mathcal{T}_i, y \vDash \varphi$ for every $y \in T_i$ such that $y <_i x$.
$\mathcal{M}, \mathcal{T}_i, x \vDash \boxed{\text{F}}\varphi$ *iff* $\mathcal{M}, \mathcal{T}_i, y \vDash \varphi$ for every $y \in T_i$ such that $x <_i y$.
$\mathcal{M}, \mathcal{T}_i, x \vDash \boxdot\varphi$ *iff* $\mathcal{M}, \mathcal{T}_j, x \vDash \varphi$ for every $\mathcal{T}_j \in H^x_{\mathcal{T}}$.

Logical Consequence: $\Gamma \vDash \varphi$ just in case for any model $\mathcal{M} = \langle T, <, \mathcal{I} \rangle$ of $\mathcal{L}^{\text{T}}_{\Box}$, history $\mathcal{T}_i \in H_{\mathcal{T}}$, and time $x \in T$, if $\mathcal{M}, \mathcal{T}_i, x \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, \mathcal{T}_i, x \vDash \varphi$.

Problem Set: Indeterminacy

Logical Consequence: Without imposing any restriction on the models of $\mathcal{L}^{\text{T}}_{\Box}$, evaluate the following where $p_i \in \mathbb{L}$, providing a proof or countermodel:

1. $\vDash p_i \to \Diamond\!\!\!\cdot\, p_i$.
2. $\vDash \varphi \to \Diamond\!\!\!\cdot\,\varphi$.
3. $\vDash \Diamond_{\text{P}}\varphi \vee \Diamond_{\text{P}}\neg\varphi$.
4. $\vDash \varphi \to \boxed{\text{F}}\Diamond_{\text{P}}\varphi$.
5. $\vDash \boxed{\text{P}}\boxed{\text{F}}\varphi \to \triangle\varphi$.
6. $\vDash p_i \to \boxdot p_i$.
7. $\vDash \varphi \to \boxdot\varphi$.
8. $\vDash \Diamond_{\text{F}}\varphi \vee \Diamond_{\text{F}}\neg\varphi$.
9. $\vDash \varphi \to \boxed{\text{P}}\Diamond_{\text{F}}\varphi$.
10. $\vDash \boxed{\text{F}}\boxed{\text{P}}\varphi \to \triangle\varphi$.

Propositional Bimodal Logic: Cartesian Semantics

Well-Formed Sentences: Letting $p_i \in \mathbb{L}$, the *well-formed sentences* $\mathcal{L}^{\text{B}}_{\times}$ are defined:

$$\varphi ::= p_i \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid \boxed{\scriptstyle P}\varphi \mid \boxed{\scriptstyle F}\varphi \mid \boxtimes\varphi \mid \Box\varphi$$

Define $\texttt{wfs}(\mathcal{L}^{\text{B}}_{\times})$ to be the set of all well-formed sentences of $\mathcal{L}^{\text{B}}_{\times}$.

Abbreviation: We maintain the abbreviations from $\mathcal{L}$, $\mathcal{L}^{\Box}$, and $\mathcal{L}^{\text{T}}$ along with the convention $\mathcal{Q}\Gamma := \{\mathcal{Q}\gamma : \gamma \in \Gamma\}$ for any $\mathcal{Q} \in \{\boxed{\scriptstyle P}, \boxed{\scriptstyle F}, \boxtimes, \Box\}$.

Frame: A *Cartesian frame* is an ordered quadruple $\mathcal{C} = \langle W, R, T, < \rangle$ where $\langle W, R \rangle$ is a modal frame and $\langle T, < \rangle$ is a temporal frame.

Interpretation: A *Cartesian interpretation* of $\mathcal{L}^{\text{B}}_{\times}$ over $\mathcal{C}$ is a function $\mathcal{I} : \mathbb{L} \to \wp(W \times T)$, i.e., where $\mathcal{I}(p_i) \subseteq W \times T$ for every sentence letter $p_i \in \mathbb{L}$.

Model: A Cartesian model of $\mathcal{L}^{\text{B}}_{\times}$ is an $\mathcal{M} = \langle W, R, T, <, \mathcal{I} \rangle$ where $\langle W, R, T, < \rangle$ is a Cartesian frame and $\mathcal{I}$ is a Cartesian interpretation of $\mathcal{L}^{\text{B}}_{\times}$.

Semantics: We define $\vDash$ recursively for model $\mathcal{M} = \langle W, R, T, <, \mathcal{I} \rangle$ of $\mathcal{L}^{\text{B}}_{\times}$, worlds $w, u \in W$, times $x, y \in T$, sentence letter $p_i \in \mathbb{L}$, and $\varphi, \psi \in \texttt{wfs}(\mathcal{L}^{\text{B}}_{\times})$:

$\mathcal{M}, w, x \nvDash \bot$.
$\mathcal{M}, w, x \vDash p_i$ *iff* $x \in \mathcal{I}(p_i)$.
$\mathcal{M}, w, x \vDash \neg\varphi$ *iff* $\mathcal{M}, w, x \nvDash \varphi$.
$\mathcal{M}, w, x \vDash \varphi \to \psi$ *iff* $\mathcal{M}, w, x \nvDash \varphi$ or $\mathcal{M}, w, x \vDash \psi$.
$\mathcal{M}, w, x \vDash \boxed{\scriptstyle P}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for every $y \in T$ such that $y < x$.
$\mathcal{M}, w, x \vDash \boxed{\scriptstyle F}\varphi$ *iff* $\mathcal{M}, w, y \vDash \varphi$ for every $y \in T$ such that $x < y$.
$\mathcal{M}, w, x \vDash \boxtimes\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for every $u \in W$ and every $x \in T$.
$\mathcal{M}, w, x \vDash \Box\varphi$ *iff* $\mathcal{M}, u, x \vDash \varphi$ for every $u \in W$.

Logical Consequence: $\Gamma \vDash \varphi$ just in case for any model $\mathcal{M} = \langle W, R, T, <, \mathcal{I} \rangle$ of $\mathcal{L}^{\text{B}}_{\times}$, world $w \in W$, and time $x \in T$, if $\mathcal{M}, w, x \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w, x \vDash \varphi$.

Problem Set: Cartesian Semantics

Countermodels: Evaluate the following, providing a proof or countermodel:

1. *If* $\Gamma \vDash \varphi$, *then* $\Box\Gamma \vDash \Box\varphi$.
2. $\vDash \Box\varphi \to \varphi$.
3. $\vDash \Box\varphi \to \Box\Box\varphi$.
4. $\vDash \varphi \to \Box\Diamond\varphi$.
5. $\vDash \Box\varphi \to \triangle\varphi$.
6. $\vDash \boxtimes\varphi \leftrightarrow \Box\triangle\varphi$.
7. $\vDash \boxtimes\varphi \to \varphi$.
8. $\vDash \boxtimes\varphi \to \boxtimes\boxtimes\varphi$.
9. $\vDash \varphi \to \boxtimes\diamondtimes\varphi$.
10. $\vDash \boxtimes\varphi \to \triangle\varphi$.

First-Order Logic: Syntax

Language $\mathcal{L}^1$: The first-order language $\mathcal{L}^1$ includes: constants 'c_1', 'c_2', ..., variables 'x_1', 'x_2', ..., n-place predicates 'p^n_1', 'p^n_2', ..., for each natural number $n \geqslant 0$, sentential operators '$\neg$','$\to$', '$\forall$', and parentheses '(' and ')'.

Terms: A symbol is a *term* just in case that symbol is a constant or variable.

Well Formed Formulas: Let 't_1',..., 't_n' be terms of $\mathcal{L}^1$, 'x' be a variable of $\mathcal{L}^1$, 'H^n' be an n-place predicate of $\mathcal{L}^1$, and 'A' and 'B' name arbitrary sentences of $\mathcal{L}^1$. We may then let $\mathcal{G}_1$ be the set of wff of $\mathcal{L}^1$, defined recursively as follows:

- The 0-place predicates 'p_1^0','p_2^0',... are all wff of $\mathcal{L}^1$.
- If H^n is an n-place predicate of $\mathcal{L}^1$, and $t_1, \ldots, t_n$ are terms of $\mathcal{L}^1$, then the atomic sentence $\ulcorner H^n(t_1, \ldots, t_n)\urcorner$ is a wff of $\mathcal{L}^1$.
- If A is a wff of $\mathcal{L}^1$, then $\ulcorner\neg A\urcorner$ is a wff of $\mathcal{L}^1$.
- If A and B are wffs of $\mathcal{L}^1$, then $\ulcorner(A \vee B)\urcorner$ is a wff of $\mathcal{L}^1$.
- If A is a wff of $\mathcal{L}^1$, then $\ulcorner\forall x A\urcorner$ is a wff of $\mathcal{L}^1$.

Abbreviations: (i) $\ulcorner(A \wedge B)\urcorner$ abbreviates $\ulcorner\neg(\neg A \vee \neg B)\urcorner$;
(ii) $\ulcorner(A \rightarrow B)\urcorner$ abbreviates $\ulcorner(\neg A \vee B)\urcorner$;
(iii) $\ulcorner(A \leftrightarrow B)\urcorner$ abbreviates $\ulcorner[(A \rightarrow B) \wedge (B \rightarrow A)]\urcorner$;
(iv) $\ulcorner\exists x A\urcorner$ abbreviates $\ulcorner\neg\forall x\neg A\urcorner$.

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}^1$ include the symbols in $\mathcal{L}^1$ together with the sentential operators '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', and '$\exists x_i$' which are to be read 'and', '(materially) implies that', 'just in case', and 'every x_i is such that', respectively. Provide a definition $\mathcal{G}_1^+$ of the wfss of $\mathcal{L}^1$.

FIRST-ORDER LOGIC: PROOF THEORY

Free Variable: Every variable which occurs in an atomic sentence of $\mathcal{L}^1$ is *free*. If x is free in the wff A, then x is *bound* in the wff $\exists x A$. The wfss of $\mathcal{L}^1$ are those wff of $\mathcal{L}^1$ with no free variables.

Substitution: For any wfs A and terms t and k, let $\ulcorner A(t/k)\urcorner$ be the wfs which result from replacing every occurrence of k in the wfs A with t.

Available: A term t is *available* (written $t^\star$) for substitution in A iff t does not occur in A or in any premise or undischarged assumption used to prove A.

Rules of Inference: Let $\mathcal{R}_1^+$ extend $\mathcal{R}^+$ to also include the following rules of inference:

FIRST-ORDER LOGIC: SEMANTICS

Domain: Let the *domain* $\mathcal{D}$ be a set of objects.

Cartesian Domain: Let $\mathcal{D}^n$ be the set of all ordered tuples $\langle d_1, \ldots, d_n\rangle$ where each d_i is an object in the domain $\mathcal{D}$, i.e., $\mathcal{D}^n = \{\langle d_1, \ldots, d_n\rangle : d_i \in \mathcal{D} \text{ for } 1 \leqslant i \leqslant n\}$.

Interpretation: Let $\vDash_1$ be an *interpretation* of $\mathcal{L}^1$ over $\mathcal{D}$ just in case: (i) $\vDash_1 (p_i^n) \subseteq \mathcal{D}^n$ for every $i \geqslant 1$ and $n \geqslant 0$; and (ii) $\vDash_1 (c_i) \in \mathcal{D}$ for every $i \geqslant 1$.

Assignment: An *assignment* $\underline{a}$ is a function from the variables in $\mathcal{L}^1$ to the members of $\mathcal{D}$ such that $\underline{a}(x_i)$ is a member of the domain $\mathcal{D}$ for every $i \geqslant 1$.

Denotation: Let $I(t) = \begin{cases} \vDash_1 (t) & \text{if } t = c_i \text{ for any } i \geqslant 1 \\ \underline{a}(t) & \text{if } t = x_i \text{ for any } i \geqslant 1 \end{cases}$

Variant: The function $\underline{a}[d/x]$ is an *x-variant* of the assignment $\underline{a}$ just in case $\underline{a}[d/x]$ differs from $\underline{a}$ at most by setting $\underline{a}[d/x](x) = d$.

Model: A *model* of $\mathcal{L}^1$ is any ordered pair $\mathcal{M} = \langle \mathcal{D}, \vDash_1 \rangle$, where $\mathcal{D}$ is a domain of individuals, and $\vDash_1$ an interpretation over $\mathcal{D}$.

Semantics: Given a model $\mathcal{M}$ of $\mathcal{L}^1$, and assignment $\underline{a}$, we may recursively define $\mathcal{M}, \underline{a} \vDash A$ for all wfss A of $\mathcal{L}^1$ as follows:

(p_i) $\mathcal{M}, \underline{a} \vDash p_i^n(t_1, \ldots, t_n)$ *iff* $\langle I(t_1), \ldots, I(t_n) \rangle \in \vDash_1 (p_i^n)$.
($\exists$) $\mathcal{M}, \underline{a} \vDash \exists x_i A$ *iff* $\mathcal{M}, \underline{a}[d/x_i] \vDash A$, for some $d \in \mathcal{D}$.
($\neg$) $\mathcal{M}, \underline{a} \vDash \neg A$ *iff* $\mathcal{M}, \underline{a} \nvDash A$.
($\vee$) $\mathcal{M}, \underline{a} \vDash A \vee B$ *iff* $\mathcal{M}, \underline{a} \vDash A$ or $\mathcal{M}, \underline{a} \vDash B$.

It is important that in the case where $n = 0$, we adopt the convention that $\vDash_1 (p_i^0) = \{\varnothing\}$ indicates truth, and $\vDash_1 (p_i^0) = \varnothing$ indicates falsity.

First-Order Logic: Metalogic

Truth on a Model: $\mathcal{M} \vDash_1 A$ *iff* $\mathcal{M}, \underline{a} \vDash A$ for all variable assignments $\underline{a}$.

Logical Consequence: $\Gamma \vDash_1 A$ *iff* for all models $\mathcal{M}$, if $\mathcal{M} \vDash G$ for all $G \in \Gamma$, then $\mathcal{M} \vDash A$.

Logical Equivalence: $A \equiv_1 B$ *iff* $A \vDash_1 B$ and $B \vDash_1 A$.

Logical Truth: A wfs A of $\mathcal{L}^1$ is *valid* (or a logical truth) just in case $\vDash_1 A$.

First-Order Logic: The first-order formal system of natural deduction $\mathcal{F}_1^+ = \langle \mathcal{L}^1, \mathcal{G}_1^+, \mathcal{A}_1^+, \mathcal{R}_1^+ \rangle$ is sound and complete, where $\mathcal{A}_1^+ = \varnothing$.

Problem Set: First-Order Logic[6]

Semantics: Provide a semantics for the wfss of $\mathcal{L}^1$.

Regimentation: Regiment the following arguments into $\mathcal{L}^1$.

(1) Everything that is beautiful is beautiful.
(2) Every philosopher is happy. So if everything is a philosopher, everything is happy.
(3) Everything is a philosopher and everything is happy. It follows that everything is a happy philosopher.
(4) Something is such that it is happy if Ella is a philosopher. So if Ella is a philosopher, then something is happy.
(5) There is a beautiful country. And so something is beautiful and something is a country.
(6) Nothing is ugly, and so everything is not ugly.
(7) Something is not right. It follows that not everything is right.
(8) Not everything is free. And so something is not free.
(9) Everything is not free. It follows that nothing is free.
(10) Every philosopher is wise, and everything wise is happy. Thus, every philosopher is happy.

[6] I have adapted some of the following problems from Carr (2013). See also Halbach (2010).

(11) Every philosopher is happy. There is a wise philosopher. And something is wise and happy.

(12) Everything loves everything. Thus, everything loves itself.

(13) Something loves itself. And so something loves something.

(14) Nothing loves something which returns its loves.

Deduction: Use the natural deduction rules $\mathcal{R}_1^+$ to prove that the conclusion of each of the regimented arguments above follows from its premises.

Metalogic: Prove that every theorem of $\mathcal{F}^+$ is also a theorem of $\mathcal{F}_1^+$.

Bonus: Regiment the following into $\mathcal{L}^1$:

(1) Everybody loves somebody.

(2) Everybody everybody loves loves somebody.

(3) Everybody everybody everybody loves loves loves somebody.

(4) You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

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