Updated: February 2, 2025

Logic Notes

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Propositional Logic: Syntax and Semantics

Canonical Name: A quoted symbol is the *canonical name* for the symbol quoted.

Language: The propositional language $\mathcal{L}$ includes *symbols* for: *sentence letters* 'p_1','p_2', ..., *sentential operators* '$\vee$','$\neg$', and *punctuation* '(' and ')'.

Strings: The concatenation of a finite number of symbols in $\mathcal{L}$ is a *string* of $\mathcal{L}$.

Schematic Variables: Let 'φ', 'ψ', 'χ',... be *schematic variables* for strings of $\mathcal{L}$.

Corner Quotes: Let $\ulcorner\cdot\urcorner$ map strings of $\mathcal{L}$ to the canonical names for those strings.

Well-Formed Sentences: The set of *well-formed sentences* $\mathtt{wfs}(\mathcal{L})$ is the smallest set to satisfy:

- $\varphi \in \mathtt{wfs}(\mathcal{L})$ *if* φ is a sentence letter of $\mathcal{L}$.
- $\ulcorner\neg\varphi\urcorner \in \mathtt{wfs}(\mathcal{L})$ *if* φ is a wfs of $\mathcal{L}$.
- $\ulcorner(\varphi \to \psi)\urcorner \in \mathtt{wfs}(\mathcal{L})$ *if* φ and ψ are wfss of $\mathcal{L}$.

Abbreviations: Letting $\ulcorner\varphi \coloneqq \psi\urcorner$ signify that φ abbreviates ψ, assume the following:

- $(\varphi \vee \psi) \coloneqq (\neg\varphi \to \psi)$.
- $(\varphi \wedge \psi) \coloneqq \neg(\varphi \to \neg\psi)$.
- $(\varphi \leftrightarrow \psi) \coloneqq [(\varphi \to \psi) \wedge (\psi \to \varphi)]$.

Models: Let $\mathcal{M}$ be a *model* of $\mathcal{L}$ *iff* for every sentence letter φ of $\mathcal{L}$, either $\mathcal{M}(\varphi) = 0$ or $\mathcal{M}(\varphi) = 1$, but not both.

Semantics: We may extend a model $\mathcal{M}$ to interpret all wfss of $\mathcal{L}$ by taking $\vDash_{\text{PL}}$ to be the smallest relation to satisfy the following:

$\mathcal{M} \vDash$ 'p_i' *iff* $\mathcal{M}(p_i) = 1$.
$\mathcal{M} \vDash \ulcorner\neg\varphi\urcorner$ *iff* it is not the case that $\mathcal{M} \vDash \varphi$ (i.e., $\mathcal{M} \nvDash \varphi$).
$\mathcal{M} \vDash \ulcorner(\varphi \to \psi)\urcorner$ *iff* $\mathcal{M} \nvDash \varphi$ or $\mathcal{M} \vDash \psi$.

We rely on our grasp of the English expressions 'it is not the case that' and 'or' to interpret all wfss of $\mathcal{L}$ given the model $\mathcal{M}$.

Logical Consequence: $\Gamma \vDash_{\text{PL}} \varphi$ *iff* for all models $\mathcal{M}$, if $\mathcal{M} \vDash_{\text{PL}} \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash_{\text{PL}} \varphi$.

Logical Equivalence: $\varphi \dashv\vDash_{\text{PL}} \psi$ *iff* $\varphi \vDash_{\text{PL}} \psi$ and $\psi \vDash_{\text{PL}} \varphi$.

Logical Truth: A wfs φ of $\mathcal{L}$ is *logical truth* (or *valid*) *iff* $\varnothing \vDash_{\text{PL}} \varphi$ (written $\vDash_{\text{PL}} \varphi$).

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}^+$ include the symbols in $\mathcal{L}$ together with the sentential operators '$\vee$', '$\wedge$', and '$\leftrightarrow$' which are to be read 'or', 'and', and 'if and only if', respectively.

(1) Provide a natural definition of the set $\mathtt{wfs}(\mathcal{L}^+)$ of wfss of $\mathcal{L}^+$.

(2) Provide a semantics for $\mathcal{L}^+$, defining the models of $\mathcal{L}^+$ and logical consequence $\vDash^+$.

(3) Prove $(\varphi \vee \psi)$, $(\varphi \vee (\varphi \wedge \psi))$, and $(\varphi \leftrightarrow \psi)$ from $\mathcal{L}^+$ are logically equivalent to wfss of $\mathcal{L}$.

(4) For each operator in $\mathcal{L}^+$, provide two examples of logical truths including that operator.

PROPOSITIONAL LOGIC: PROOF THEORY

Rules of Inference: Consider the following *Fitch rules of inference* for $\mathcal{L}^+$:

Reiteration (R)

$$\begin{array}{r|ll} m & \varphi & \\ & \varphi & :m\ \mathrm{R} \end{array}$$

Conjunction Introduction ($\wedge$I)

$$\begin{array}{r|ll} m & \varphi & \\ n & \psi & \\ & \varphi \wedge \psi & :m, n\ \wedge\mathrm{I} \\ & \psi \wedge \varphi & :m, n\ \wedge\mathrm{I} \end{array}$$

Conditional Introduction ($\rightarrow$I)

$$\begin{array}{r|ll} m & \big|\ \underline{\varphi} & :\mathrm{AS\ for} \rightarrow\mathrm{I} \\ n & \big|\ \psi & \\ & \varphi \rightarrow \psi & :m\text{–}n \rightarrow\mathrm{I} \end{array}$$

Negation Introduction ($\neg$I)

$$\begin{array}{r|ll} m & \big|\ \underline{\varphi} & :\mathrm{AS\ for}\ \neg\mathrm{I} \\ n & \big|\ \psi & \\ o & \big|\ \neg\psi & \\ & \neg\varphi & :m\text{–}o\ \neg\mathrm{I} \end{array}$$

Disjunction Introduction ($\vee$I)

$$\begin{array}{r|ll} m & \varphi & \\ & \varphi \vee \psi & :m\ \vee\mathrm{I} \\ & \psi \vee \varphi & :m\ \vee\mathrm{I} \end{array}$$

Biconditional Introduction ($\leftrightarrow$I)

$$\begin{array}{r|ll} i & \big|\ \underline{\varphi} & :\mathrm{AS\ for} \leftrightarrow\mathrm{I} \\ j & \big|\ \psi & \\ k & \big|\ \underline{\psi} & :\mathrm{AS\ for} \leftrightarrow\mathrm{I} \\ l & \big|\ \varphi & \\ & \varphi \leftrightarrow \psi & :i\text{–}j, k\text{–}l \leftrightarrow\mathrm{I} \end{array}$$

Assumption (AS)

$$\begin{array}{r|ll} m & \big|\ \underline{\varphi} & :\mathrm{AS} \end{array}$$

Conjunction Elimination ($\wedge$E)

$$\begin{array}{r|ll} m & \varphi \wedge \psi & \\ & \varphi & :m\ \wedge\mathrm{E} \\ & \psi & :m\ \wedge\mathrm{E} \end{array}$$

Conditional Elimination ($\rightarrow$E)

$$\begin{array}{r|ll} m & \varphi \rightarrow \psi & \\ n & \varphi & \\ & \psi & :m, n \rightarrow\mathrm{E} \end{array}$$

Negation Elimination ($\neg$E)

$$\begin{array}{r|ll} m & \big|\ \underline{\neg\varphi} & :\mathrm{AS\ for}\ \neg\mathrm{E} \\ & \big|\ \vdots & \\ n & \big|\ \psi & \\ o & \big|\ \neg\psi & \\ & \varphi & :m\text{–}o\ \neg\mathrm{E} \end{array}$$

Disjunction Elimination ($\vee$E)

$$\begin{array}{r|ll} m & \varphi \vee \psi & \\ i & \big|\ \underline{\varphi} & :\mathrm{AS\ for} \vee\mathrm{E} \\ j & \big|\ \chi & \\ k & \big|\ \underline{\psi} & :\mathrm{AS\ for} \vee\mathrm{E} \\ l & \big|\ \chi & \\ & \chi & :m, i\text{–}j, k\text{–}l\ \vee\mathrm{E} \end{array}$$

Biconditional Elimination ($\leftrightarrow$E)

$$\begin{array}{r|ll} m & \varphi \leftrightarrow \psi & \\ n & \psi/\varphi & \\ & \varphi/\psi & :m, n \leftrightarrow\mathrm{E} \end{array}$$

Proof Lines: A *proof line* may be represented by a tuple $\langle n, i, \varphi, J\rangle$ consisting of a *position number* n, *indentation number* i, wfs φ of $\mathcal{L}^+$, and *justification* J.

Fitch Proof: A *Fitch proof* of ψ from Γ is a finite sequence X of proof lines with consecutive position numbers starting from 1 that ends in $\langle m, 0, \psi, K\rangle$ where every line $\langle n, i, \varphi, J\rangle$ in X is either: (1) a *premise* where $i = 0$, $\psi \in \Gamma$, and $J = \mathrm{PR}$; (2) a *discharged assumption* where $i > 0$ and $J = \mathrm{AS}$; or (3) follows by a *Fitch rule* where J cites previous lines.

Derivable: A wfs ψ of $\mathcal{L}^+$ is *derivable* (or *provable*) from Γ by the *Fitch proof system* given above, i.e., $\Gamma \vdash_{\mathrm{F}} \psi$, just in case there is a Fitch proof X of ψ from Γ.

Problem Set: Translation and Deduction[1]

Translation: Resolve the following ambiguities (if any) by regimenting each in $\mathcal{L}^+$:

(1) Figaro exulted, and Basilio fretted, or the Court had a plan.

(2) Fred danced and sang or Ginger went home.

(3) If we are not in Paris then today is not Tuesday.

(4) The senator will not testify unless he is granted immunity.

(5) The senator will testify only if he is granted immunity.

(6) If Figaro does not expose the Count and force him to reform, then the Countess will discharge Susanna and resign to loneliness.

(7) The trade deficit will diminish and agriculture or industry will lead a recovery provided that both the dollar drops and neither Japan nor the EU raise their tariffs.

Arguments: Regiment the following arguments in the propositional language $\mathcal{L}^+$:

(1) Basilio fretted. Thus, if Figaro exulted, then Basilio fretted.

(2) Fred danced if Ginger went home. Fred didn't dance. And so Ginger didn't go home.

(3) If Figaro exulted, then the Court had a plan if Basilio fretted. Thus if Basilio fretted, then the Court had a plan if Figaro exulted.

(4) Fred danced or else Ginger sang and danced. It follows that either Fred danced or Ginger sang.

(5) If Lucy and Mary beat the record, then Paul will have to go. If Ian wins the race, then Paul can stay. Mary beat the record and Ian won the race. Therefore Lucy did not beat the record.

(6) If we are in Paris, then we are in Paris.

(7) It is not the case that we both are, and are not in Paris.

(8) Either Ginger or Fred danced. But Fred did not dance. Thus Ginger must have been the one who danced.

(9) Basilio fretted or Gigaro exulted. If Basilio fretted, the Court had a plan. But Gigaro did not exult, if David did not save the day. And so either the Court had a plan, or David saved the day.

(10) Kant is out for a walk just in case it is half noon. So either Kant is out for a walk and it is half noon, or Kant is not out for a walk and it is not half noon.

(11) It is not the case that Fred either sang or danced. It follows that Fred did not sing, nor did he dance.

(12) It is not the case that Fred sang and danced. It follows that Fred did not sing, or else did he did not dance.

(13) If we are in Paris, then we are in France. We are not in France. So we are not in Paris.

(14) If we are in Paris, then we are in France. If we are in France, we are in Europe. It follows that if we are in Paris, we are in Europe.

Deduction: Prove that the conclusion of each of the regimented arguments above is derivable from its premises by constructing a proof.

[1] I have adapted the following problems from Goldfarb (2003) and Laboreo (2005).

Propositional Logic: Naturalness and Metalogic

Fitch System: The Fitch proof system has the following metalogical properties:

Sound: $\Gamma \vDash^{+}_{\text{PL}} \psi$ whenever $\Gamma \vdash_{\text{F}} \psi$.

Complete: $\Gamma \vdash_{\text{F}} \psi$ whenever $\Gamma \vDash^{+}_{\text{PL}} \psi$.

Equivalent: Two proof systems A and B for the same language are *extensionally equivalent* just in case $\Gamma \vdash_{A} \psi$ if and only if $\Gamma \vdash_{B} \psi$.

Hilbert System: The *Hilbert proof system* consists of the following *axiom schemata* **A1 – A3** and *rule schema* **MP**:

A1 $\varphi \to (\psi \to \varphi)$.

A2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

A3 $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

MP $\{\varphi, \varphi \to \psi\} \vdash \psi$.

Set notation will be typically be omitted, writing '$\Gamma, \varphi, \psi \vdash \chi$' in place of '$\Gamma \cup \{\varphi, \psi\} \vdash \chi$', and writing '$\vdash \varphi$' in place of '$\varnothing \vdash \varphi$'.

Instances: The *axioms* of the Hilbert proof system are the instances of **A1 – A3** and the *rules* are the instances of **MP**.

Hilbert Proof: A *Hilbert proof* of ψ from Γ is a finite sequence X of wfss of $\mathcal{L}$ ending in ψ where every wfs in X is either: (1) a *premise* in Γ; (2) an axiom; or (3) follows from previous wfss φ and $\varphi \to \psi$ in X by a rule.

Derivable: A wfs ψ of $\mathcal{L}$ is *derivable* (*provable*) from Γ by the Hilbert proof system, i.e., $\Gamma \vdash_{\text{PL}} \psi$, just in case there is a Hilbert proof X of ψ from Γ.

Metalogic: The Hilbert proof system has the following metalogical properties:

Sound: $\Gamma \vDash_{\text{PL}} \psi$ whenever $\Gamma \vdash_{\text{PL}} \psi$.

Complete: $\Gamma \vdash_{\text{PL}} \psi$ whenever $\Gamma \vDash_{\text{PL}} \psi$.

Naturalness: Whereas the Fitch system idealizes natural forms of reasoning, the Hilbert system is easier to state and study but harder to use.

Problem Set: Naturalness and Metalogic

(1)

TO BE CONTINUED...

FIRST-ORDER LOGIC: SYNTAX

Language $\mathcal{L}^1$: The first-order language $\mathcal{L}^1$ includes: constants 'c_1', 'c_2', ..., variables 'x_1', 'x_2', ..., n-place predicates 'p_1^n', 'p_2^n', ..., for each natural number $n \geqslant 0$, sentential operators '$\neg$','$\rightarrow$', '$\forall$', and parentheses '(' and ')'.

Terms: A symbol is a *term* just in case that symbol is a constant or variable.

Well Formed Formulas: Let 't_1',..., 't_n' be terms of $\mathcal{L}^1$, 'x' be a variable of $\mathcal{L}^1$, 'H^n' be an n-place predicate of $\mathcal{L}^1$, and 'A' and 'B' name arbitrary sentences of $\mathcal{L}^1$. We may then let $\mathcal{G}_1$ be the set of wff of $\mathcal{L}^1$, defined recursively as follows:

- The 0-place predicates 'p_1^0','p_2^0',... are all wff of $\mathcal{L}^1$.
- If H^n is an n-place predicate of $\mathcal{L}^1$, and $t_1, \ldots, t_n$ are terms of $\mathcal{L}^1$, then the atomic sentence $\ulcorner H^n(t_1, \ldots, t_n)\urcorner$ is a wff of $\mathcal{L}^1$.
- If A is a wff of $\mathcal{L}^1$, then $\ulcorner \neg A\urcorner$ is a wff of $\mathcal{L}^1$.
- If A and B are wffs of $\mathcal{L}^1$, then $\ulcorner(A \vee B)\urcorner$ is a wff of $\mathcal{L}^1$.
- If A is a wff of $\mathcal{L}^1$, then $\ulcorner\forall x A\urcorner$ is a wff of $\mathcal{L}^1$.

Abbreviations: (i) $\ulcorner(A \wedge B)\urcorner$ abbreviates $\ulcorner\neg(\neg A \vee \neg B)\urcorner$;
(ii) $\ulcorner(A \rightarrow B)\urcorner$ abbreviates $\ulcorner(\neg A \vee B)\urcorner$;
(iii) $\ulcorner(A \leftrightarrow B)\urcorner$ abbreviates $\ulcorner[(A \rightarrow B) \wedge (B \rightarrow A)]\urcorner$;
(iv) $\ulcorner\exists x A\urcorner$ abbreviates $\ulcorner\neg\forall x \neg A\urcorner$.

Problem Set: Metalinguistic Abbreviation

Let $\mathcal{L}^1$ include the symbols in $\mathcal{L}^1$ together with the sentential operators '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', and '$\exists x_i$' which are to be read 'and', '(materially) implies that', 'just in case', and 'every x_i is such that', respectively. Provide a definition $\mathcal{G}_1^+$ of the wfss of $\mathcal{L}^1$.

FIRST-ORDER LOGIC: PROOF THEORY

Free Variable: Every variable which occurs in an atomic sentence of $\mathcal{L}^1$ is *free*. If x is free in the wff A, then x is *bound* in the wff $\exists x A$. The wfss of $\mathcal{L}^1$ are those wff of $\mathcal{L}^1$ with no free variables.

Substitution: For any wfs A and terms t and k, let $\ulcorner A(t/k)\urcorner$ be the wfs which result from replacing every occurrence of k in the wfs A with t.

Available: A term t is *available* (written $t^\star$) for substitution in A iff t does not occur in A or in any premise or undischarged assumption used to prove A.

Rules of Inference: Let $\mathcal{R}_1^+$ extend $\mathcal{R}^+$ to also include the following rules of inference:

FIRST-ORDER LOGIC: SEMANTICS

Domain: Let the *domain* $\mathcal{D}$ be a set of objects.

Cartesian Domain: Let $\mathcal{D}^n$ be the set of all ordered tuples $\langle d_1, \ldots, d_n\rangle$ where each d_i is an object in the domain $\mathcal{D}$, i.e., $\mathcal{D}^n = \{\langle d_1, \ldots, d_n\rangle : d_i \in \mathcal{D}$ for $1 \leqslant i \leqslant n\}$.

Interpretation: Let $\vDash_1$ be an *interpretation* of $\mathcal{L}^1$ over $\mathcal{D}$ just in case: (i) $\vDash_1 (p_i^n) \subseteq \mathcal{D}^n$ for every $i \geqslant 1$ and $n \geqslant 0$; and (ii) $\vDash_1 (c_i) \in \mathcal{D}$ for every $i \geqslant 1$.

Assignment: An *assignment* $\underline{a}$ is a function from the variables in $\mathcal{L}^1$ to the members of $\mathcal{D}$ such that $\underline{a}(x_i)$ is a member of the domain $\mathcal{D}$ for every $i \geqslant 1$.

Denotation: Let $I(t) = \begin{cases} \vDash_1 (t) & \text{if } t = c_i \text{ for any } i \geqslant 1 \\ \underline{a}(t) & \text{if } t = x_i \text{ for any } i \geqslant 1 \end{cases}$

Variant: The function $\underline{a}[d/x]$ is an *x-variant* of the assignment $\underline{a}$ just in case $\underline{a}[d/x]$ differs from $\underline{a}$ at most by setting $\underline{a}[d/x](x) = d$.

Model: A *model* of $\mathcal{L}^1$ is any ordered pair $\mathcal{M} = \langle \mathcal{D}, \vDash_1 \rangle$, where $\mathcal{D}$ is a domain of individuals, and $\vDash_1$ an interpretation over $\mathcal{D}$.

Semantics: Given a model $\mathcal{M}$ of $\mathcal{L}^1$, and assignment $\underline{a}$, we may recursively define $\mathcal{M}, \underline{a} \vDash A$ for all wfss A of $\mathcal{L}^1$ as follows:

(p_i) $\mathcal{M}, \underline{a} \vDash p_i^n(t_1, \ldots, t_n)$ *iff* $\langle I(t_1), \ldots, I(t_n) \rangle \in \vDash_1 (p_i^n)$.
($\exists$) $\mathcal{M}, \underline{a} \vDash \exists x_i A$ *iff* $\mathcal{M}, \underline{a}[d/x_i] \vDash A$, for some $d \in \mathcal{D}$.
($\neg$) $\mathcal{M}, \underline{a} \vDash \neg A$ *iff* $\mathcal{M}, \underline{a} \nvDash A$.
($\vee$) $\mathcal{M}, \underline{a} \vDash A \vee B$ *iff* $\mathcal{M}, \underline{a} \vDash A$ or $\mathcal{M}, \underline{a} \vDash B$.

It is important that in the case where $n = 0$, we adopt the convention that $\vDash_1 (p_i^0) = \{\varnothing\}$ indicates truth, and $\vDash_1 (p_i^0) = \varnothing$ indicates falsity.

First-Order Logic: Metalogic

Truth on a Model: $\mathcal{M} \vDash_1 A$ *iff* $\mathcal{M}, \underline{a} \vDash A$ for all variable assignments $\underline{a}$.

Logical Consequence: $\Gamma \vDash_1 A$ *iff* for all models $\mathcal{M}$, if $\mathcal{M} \vDash G$ for all $G \in \Gamma$, then $\mathcal{M} \vDash A$.

Logical Equivalence: $A \equiv_1 B$ *iff* $A \vDash_1 B$ and $B \vDash_1 A$.

Logical Truth: A wfs A of $\mathcal{L}^1$ is *valid* (or a logical truth) just in case $\vDash_1 A$.

First-Order Logic: The first-order formal system of natural deduction $\mathcal{F}_1^+ = \langle \mathcal{L}^1, \mathcal{G}_1^+, \mathcal{A}_1^+, \mathcal{R}_1^+ \rangle$ is sound and complete, where $\mathcal{A}_1^+ = \varnothing$.

Problem Set: First-Order Logic[2]

Semantics: Provide a semantics for the wfss of $\mathcal{L}^1$.

Translation: Translate the following arguments into $\mathcal{L}^1$.

(1) Everything that is beautiful is beautiful.
(2) Every philosopher is happy. So if everything is a philosopher, everything is happy.
(3) Everything is a philosopher and everything is happy. It follows that everything is a happy philosopher.
(4) Something is such that it is happy if Ella is a philosopher. So if Ella is a philosopher, then something is happy.

[2] I have adapted some of the following problems from Carr (2013). See also Halbach (2010).

(5) There is a beautiful country. And so something is beautiful and something is a country.

(6) Nothing is ugly, and so everything is not ugly.

(7) Something is not right. It follows that not everything is right.

(8) Not everything is free. And so something is not free.

(9) Everything is not free. It follows that nothing is free.

(10) Every philosopher is wise, and everything wise is happy. Thus, every philosopher is happy.

(11) Every philosopher is happy. There is a wise philosopher. And something is wise and happy.

(12) Everything loves everything. Thus, everything loves itself.

(13) Something loves itself. And so something loves something.

(14) Nothing loves something which returns its loves.

Deduction: Use the natural deduction rules $\mathcal{R}_1^+$ to prove that the conclusion of each of the regimented arguments above follows from its premises.

Metalogic: Prove that every theorem of $\mathcal{F}^+$ is also a theorem of $\mathcal{F}_1^+$.

Bonus: Translate the following into $\mathcal{L}^1$:

(1) Everybody loves somebody.

(2) Everybody everybody loves loves somebody.

(3) Everybody everybody everybody loves loves loves somebody.

(4) You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

Propositional Modal Logic: Motivation

Paradox: Substitution instances of the following schemata are theorems of $\mathcal{F}^+$:

(1) $A \rightarrow (B \rightarrow A)$. (2) $\neg A \rightarrow (A \rightarrow B)$.

But intuitively, a true proposition is not implied by any proposition whatsoever, nor does a false proposition imply any proposition.

Examples:
- If sugar is sweet, then if roses are red, sugar is sweet.
- If snow is not green, then if snow is green, roses are red.

Problem: The material conditional '$\rightarrow$' fails to adequately capture a strong enough sense of 'implies', sometimes represented in natural language by means of conditional constructions such as 'if. . . , then. . . '.

Desiderata: Lewis (1912) and Lewis and Langford (1932) developed modal logic in attempt to better capture the "usual sense" of 'implies'.

Propositional Modal Logic: Syntax

Language $\mathcal{L}_\Box$: The propositional language $\mathcal{L}_\Box$ includes: sentence letters 'p_1','p_2', ..., the sentential operators '$\vee$','$\neg$', '$\Box$', and parentheses '(' and ')'.

Well Formed Sentences: Let 'A' and 'B' name arbitrary sentences of $\mathcal{L}$. We may then let $\mathcal{G}_\Box$ be the set of wfs of $\mathcal{L}_\Box$, defined recursively as follows:

- The sentence letters 'p_1','p_2',... are all wfs of $\mathcal{L}_\Box$.
- If A is a wfs of $\mathcal{L}_\Box$, then $\ulcorner\Box A\urcorner$ is a wfs of $\mathcal{L}_\Box$.
- If A is a wfs of $\mathcal{L}_\Box$, then $\ulcorner\neg A\urcorner$ is a wfs of $\mathcal{L}_\Box$.
- If A and B are wfss of $\mathcal{L}_\Box$, then $\ulcorner(A \vee B)\urcorner$ is a wfs of $\mathcal{L}_\Box$.

Abbreviations: (i) $\ulcorner(A \wedge B)\urcorner$ abbreviates $\ulcorner\neg(\neg A \vee \neg B)\urcorner$;
(ii) $\ulcorner(A \rightarrow B)\urcorner$ abbreviates $\ulcorner(\neg A \vee B)\urcorner$;
(iii) $\ulcorner(A \leftrightarrow B)\urcorner$ abbreviates $\ulcorner[(A \rightarrow B) \wedge (B \rightarrow A)]\urcorner$;
(iv) $\ulcorner\Diamond A\urcorner$ abbreviates $\ulcorner\neg\Box\neg A\urcorner$.

Strict Conditional: Lewis and Langford (1932) took the *strict conditional* '$⥽$' to better approximate the "usual sense" of 'implies', where '$A ⥽ B$' abbreviates '$\Box(A \rightarrow B)$'. It is typical to maintain the latter as standard notation.

Problem Set: Motivation and Translation

Motivation: Prove that the paradoxes of the material conditional (1) and (2) given above are theorems of $\mathcal{F}^+$.

Abbreviation: Let $\mathcal{L}^\Box$ include the symbols in $\mathcal{L}_\Box$ as well as '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', and '$\Diamond$' which are read 'and', '(materially) implies that', 'just in case', and 'possibly', respectively. Provide a definition of $\mathcal{G}_\Box^+$ which includes all and only the wfss of $\mathcal{L}^\Box$ where $\mathcal{G}_\Box \subseteq \mathcal{G}_\Box^+$.

Translation: Translate the following into $\mathcal{L}^\Box$ as well as $\mathcal{L}_\Box$.

(1) It could rain or it could not rain.
(2) If it is necessary that it rains, then it is necessary that it could rain.
(3) It is necessary that it could either rain or not.

Propositional Modal Logic: Semantics

Frame: A *Kripke frame* $\mathcal{K}$ is an ordered pair $\langle W, R\rangle$, where W is a set of points called *possible worlds*, R is an *accessibility* relation between worlds.

Interpretation: $\vDash_\Box$ is an *interpretation* of $\mathcal{L}_\Box$ over W just in case for each $w \in W$ and $i \geqslant 1$, either $\vDash_\Box (p_i)(w) = 1$ or $\vDash_\Box (p_i)(w) = 0$, but not both.

Model: A *model* of $\mathcal{L}_\Box$ is any ordered triple $\mathcal{M}_\Box = \langle W, R, \vDash_\Box\rangle$ where $\langle W, R\rangle$ is a Kripke frame and $\vDash_\Box$ is an interpretation of $\mathcal{L}_\Box$.

Semantics: Given a model $\mathcal{M}_\Box$ of $\mathcal{L}_\Box$, and a world $w \in W$, we may recursively define $\mathcal{M}_\Box, w \vDash A$ for all wfss A of $\mathcal{L}_\Box$ as follows:

(p_i) $\quad \mathcal{M}_\Box, w \vDash p_i$ *iff* $\vDash_\Box (p_i)(w) = 1$.

($\Box$) $\mathcal{M}_\Box, w \vDash \Box A$ *iff* $\mathcal{M}_\Box, w' \vDash A$ for every $w' \in W$ such that $R(w, w')$.

($\neg$) $\mathcal{M}_\Box, w \vDash \neg A$ *iff* $\mathcal{M}, \underline{a} \nvDash A$.

($\vee$) $\mathcal{M}_\Box, w \vDash A \vee B$ *iff* $\mathcal{M}, \underline{a} \vDash A$ or $\mathcal{M}, \underline{a} \vDash B$.

Proposition: The proposition $[\![A]\!]_{\mathcal{M}_\Box}$ that a wfs A of $\mathcal{L}_\Box$ expresses on a model $\mathcal{M}_\Box$ is the set of worlds $\{w \in W : \mathcal{M}_\Box, w \vDash A\}$ at which A is true. Every model $\mathcal{M}_\Box$ of $\mathcal{L}_\Box$ may then be though of as assigning each wfs of $\mathcal{L}_\Box$ to a proposition, conceived of as a subset of W.

Propositional Modal Logic: Axiomatic Systems

Axioms: Consider the following axiom schemata and frame constraints:

(K)	$\Box(A \to B) \to (\Box A \to \Box B)$.	*None.*
(T)	$\Box A \to A$.	$R(w, w)$.
(B)	$A \to \Box\Diamond A$.	$R(w, w') \to R(w', w)$.
(4)	$\Box A \to \Box\Box A$.	$[R(w, w') \wedge R(w', w'')] \to R(w, w'')$.

Rules of Inference: Let $\mathcal{R}^+_\Box$ include the following rules of inference:

Systems:

(K) The formal system $K = \langle \mathcal{L}^\Box, \mathcal{G}^+_\Box, \mathcal{A}^+_K, \mathcal{R}^+_\Box \rangle$, where $\mathcal{A}^+_K$ includes the theorems of $\mathcal{F}^+$ together with all instances of K.

(T) The formal system $T = \langle \mathcal{L}^\Box, \mathcal{G}^+_\Box, \mathcal{A}^+_T, \mathcal{R}^+_\Box \rangle$, where $\mathcal{A}^+_T$ includes the theorems of $\mathcal{F}^+$ together with all instances of K and T.

($S4$) The formal system $S4 = \langle \mathcal{L}^\Box, \mathcal{G}^+_\Box, \mathcal{A}^+_4, \mathcal{R}^+_\Box \rangle$, where $\mathcal{A}^+_4$ includes the theorems of $\mathcal{F}^+$ together with all instances of K, T, and 4.

($S5$) The formal system $S5 = \langle \mathcal{L}^\Box, \mathcal{G}^+_\Box, \mathcal{A}^+_5, \mathcal{R}^+_\Box \rangle$, where $\mathcal{A}^+_5$ includes the theorems of $\mathcal{F}^+$ together with all instances of K, T, B, and 4.

Problem Set: Axiomatic Proofs[3]

Credence: Evaluate the plausibility of each of the modal axioms when '$\Box$' and '$\Diamond$' are read as metaphysical necessity and possibility, respectively.

Translation: Translate the axioms belonging to $\mathcal{A}^+_5$ into natural language.

Proofs: Provide a proof of each of the following:

(1) $\vdash_K \Box(P \to Q) \to \Box(\neg Q \to \neg P)$.

(2) $\vdash_K (\Box P \wedge \Box Q) \to \Box(P \to Q)$.

(3) $\vdash_T \Box P \to \Diamond P$.

(4) $\vdash_T \neg\Box(P \wedge \neg P)$.

(5) $\vdash_{S4} \Box P \to \Box\Diamond\Box P$.

(6) $\vdash_{S4} \Diamond\Diamond\Diamond P \to \Diamond P$.

(7) $\vdash_{S5} \Diamond(P \wedge \Diamond Q) \leftrightarrow (\Diamond P \wedge \Diamond Q)$.

[3] I have adapted some of the following exercises from Studd (2016) and Sider (2010).

Propositional Modal Logic: Metalogic

Truth on a Model: $\mathcal{M} \models A$ *iff* $\mathcal{M}, w \models A$ for all $w \in W$.

Logical Consequence: $\Gamma \models_{\mathcal{C}} A$ *iff* for all $\mathcal{M} \in \mathcal{C}$, if $\mathcal{M} \models G$ for all $G \in \Gamma$, then $\mathcal{M} \models A$.

Logical Equivalence: $A \equiv_{\mathcal{C}} B$ *iff* $A \models_{\mathcal{C}} B$ and $B \models_{\mathcal{C}} A$.

Logical Truth: A wfs A of $\mathcal{L}_\Box$ is *valid* on a class of models $\mathcal{C}$ just in case $\models_{\mathcal{C}} A$.

Reflexive: A frame $\mathcal{K} = \langle W, R\rangle$ is *reflexive* just in case $R(w, w)$ for every $w \in W$. A model $\mathcal{M}_\Box = \langle W, R, \models_\Box\rangle$ is *reflexive* just in case $\langle W, R\rangle$ is a reflexive frame. Let $\mathcal{C}_r$ be the class of all reflexive models of $\mathcal{L}_\Box$.

Symmetric: A frame $\mathcal{K} = \langle W, R\rangle$ is *symmetric* just in case $R(w', w)$ whenever $R(w, w')$. A model $\mathcal{M}_\Box = \langle W, R, \models_\Box\rangle$ is *symmetric* just in case $\langle W, R\rangle$ is symmetric. Let $\mathcal{C}_s$ be the class of all symmetric models of $\mathcal{L}_\Box$.

Transitive: A frame $\mathcal{K} = \langle W, R\rangle$ is *transitive* just in case $R(w, w'')$ whenever $R(w, w')$ and $R(w', w'')$. A model $\mathcal{M}_\Box = \langle W, R, \models_\Box\rangle$ is *transitive* just in case $\langle W, R\rangle$ is transitive. Let $\mathcal{C}_t$ be the class of transitive models of $\mathcal{L}_\Box$.

Modal Logics:

(K) The modal system K is sound and complete over the class of all models $\mathcal{C}_K$, i.e., $\vdash_K A$ if and only if $\models_{\mathcal{C}_K} A$.

(T) The modal system T is sound and complete over the class of all reflexive models $\mathcal{C}_T = \mathcal{C}_r$, i.e., $\vdash_T A$ if and only if $\models_{\mathcal{C}_T} A$.

($S4$) The system $S4$ is sound and complete over the reflexive and transitive models $\mathcal{C}_{S4} = \mathcal{C}_r \bigcap \mathcal{C}_t$, i.e., $\vdash_{S4} A$ if and only if $\models_{\mathcal{C}_{S4}} A$.[4]

($S5$) The modal system $S5$ is sound and complete over the class of all reflexive, symmetric, and transitive models $\mathcal{C}_{S5} = \mathcal{C}_r \bigcap \mathcal{C}_s \bigcap \mathcal{C}_t$, i.e., $\vdash_{S5} A$ if and only if $\models_{\mathcal{C}_{S5}} A$.[5]

Counter Model: A *counter model* for a wfs A of $\mathcal{L}_\Box$ is a model of $\mathcal{L}_\Box$ in which A is false.

Invalidity: To demonstrate that a wfs A of $\mathcal{L}_\Box$ is *invalid* on the class of models $\mathcal{C}$ (i.e., $\not\models_{\mathcal{C}} A$), it is sufficient to specify a single counter model to A in $\mathcal{C}$.

Problem Set: Further Exercises

Semantic Proofs: Give semantic arguments to demonstrate each of the following:

(1) $\models_{\mathcal{C}_T} \Box A \to A$.

(2) $\models_{\mathcal{C}_T} \Box A \to \Box A$.

(3) $\models_{\mathcal{C}_{S4}} \Box\Box A \to \Box A$.

(4) $\models_{\mathcal{C}_{S4}} \Box A \to \Box\Box A$.

(5) $\models_{\mathcal{C}_{S5}} A \to \Box\Box A$.

(6) $\models_{\mathcal{C}_{S5}} \Box A \to \Box\Box A$.

Equivalences: Provide semantic proofs of the following equivalences:

(7) $\neg\Box A \equiv_{\mathcal{C}_K} \Box\neg A$.

(8) $\neg\Box A \equiv_{\mathcal{C}_K} \Box\neg A$.

(9) $\neg\Box\neg \equiv_{\mathcal{C}_K} \Box A$.

(10) $\neg\Box\neg \equiv_{\mathcal{C}_K} \Box A$.

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[4] The intersection $X \bigcap Y$ is the set of elements in both X and Y, i.e., $X \bigcap Y = \{z : z \in X \text{ and } z \in Y\}$.

[5] See Hughes and Cresswell (1996) for proofs of soundness and completeness for $K, T, S4$, and $S5$.

Counter Models: Provide counter models to demonstrate the following:

(11) $\nvDash_{\mathcal{C}_K} \Box A \rightarrow A$.

(12) $\nvDash_{\mathcal{C}_K} \Box A \rightarrow \Box A$.

(13) $\nvDash_{\mathcal{C}_{S4}} A \rightarrow \Box\Box A$.

(14) $\nvDash_{\mathcal{C}_{S4}} \Box A \rightarrow \Box\Box A$.

(15) $\nvDash_{\mathcal{C}_T} \Box A \rightarrow \Box\Box A$.

(16) $\nvDash_{\mathcal{C}_T} \Box A \rightarrow \Box\Box A$.

Propositions: Draw on the semantic definitions above to establish the following:

(17) $\mathcal{M}_\Box, w \vDash A$ *iff* $w \in [\![A]\!]_{\mathcal{M}_\Box}$.

(18) $\mathcal{M}_\Box, w \vDash A \rightarrow B$ *iff* $w \in [\![A]\!]^c_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.[6]

(19) $\mathcal{M}_\Box, w \vDash \Box(A \rightarrow B)$ *iff* $[\![A]\!]_{\mathcal{M}_\Box} \subseteq [\![B]\!]_{\mathcal{M}_\Box}$.

(20) $\mathcal{M}_\Box, w \vDash \Box(A \leftrightarrow B)$ *iff* $[\![A]\!]_{\mathcal{M}_\Box} = [\![B]\!]_{\mathcal{M}_\Box}$.

(21) $[\![\neg A]\!]_{\mathcal{M}_\Box} = [\![A]\!]^c_{\mathcal{M}_\Box}$.[7]

(22) $[\![A \wedge B]\!]_{\mathcal{M}_\Box} = [\![A]\!]_{\mathcal{M}_\Box} \bigcap [\![B]\!]_{\mathcal{M}_\Box}$.

(23) $[\![A \vee B]\!]_{\mathcal{M}_\Box} = [\![A]\!]_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.

(24) $[\![A \rightarrow B]\!]_{\mathcal{M}_\Box} = [\![A]\!]^c_{\mathcal{M}_\Box} \bigcup [\![B]\!]_{\mathcal{M}_\Box}$.

Paradoxes: Prove the following analogues of the paradoxes given above:

(25) $\Box A \rightarrow \Box(B \rightarrow A)$.

(26) $\neg\Box A \rightarrow \Box(A \rightarrow B)$.

Irrelevance: Prove the following for an arbitrary A, B and $\mathcal{M}_\Box$ of $\mathcal{L}_\Box$:

(27) If $\Box A$, then $[\![B]\!]_{\mathcal{M}_\Box} = [\![B \wedge A]\!]_{\mathcal{M}_\Box}$.

(28) If $\neg\Box A$, then $[\![B]\!]_{\mathcal{M}_\Box} = [\![B \vee A]\!]_{\mathcal{M}_\Box}$.

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[6] The union $X \bigcup Y$ is the set of elements in both X and Y, i.e., $X \bigcup Y = \{z : z \in X \text{ or } z \in Y\}$.

[7] The complement X^c is the set of elements in W that are not in X, i.e., $X^c = \{z \in W : z \notin X\}$.