Propositional Bimodal Logic: Task Semantics

Language: The *bimodal language* $\mathcal{L}^{\mathrm{B}}$ extends propositional logic with modal and temporal operators:

$$\mathcal{L}^{\mathrm{B}} = \langle \mathbb{L}, \bot, \to, \Box, \boxed{\scriptstyle \mathrm{P}}, \boxed{\scriptstyle \mathrm{F}} \rangle$$

where $\mathbb{L} = \{p_i \mid i \in \mathbb{N}\}$ is the set of sentence letters.

Well-Formed Sentences: The set of well-formed sentences $\texttt{wfs}(\mathcal{L}^{\mathrm{B}})$ is defined inductively:

$$\varphi ::= p_i \mid \bot \mid \varphi \to \varphi \mid \Box\varphi \mid \boxed{\scriptstyle \mathrm{P}}\varphi \mid \boxed{\scriptstyle \mathrm{F}}\varphi$$

where $p_i \in \mathbb{L}$.

Abbreviations: We define the following derived operators:

(a) $\Diamond\varphi := \neg\Box\neg\varphi$ (possibility)

(b) $\Diamond_{\mathrm{P}}\varphi := \neg\boxed{\scriptstyle \mathrm{P}}\neg\varphi$ (past possibility)

(c) $\Diamond_{\mathrm{F}}\varphi := \neg\boxed{\scriptstyle \mathrm{F}}\neg\varphi$ (future possibility)

(d) $\triangle\varphi := \boxed{\scriptstyle \mathrm{P}}\varphi \wedge \varphi \wedge \boxed{\scriptstyle \mathrm{F}}\varphi$ (always)

(e) $\triangledown\varphi := \Diamond_{\mathrm{P}}\varphi \vee \varphi \vee \Diamond_{\mathrm{F}}\varphi$ (sometimes)

(f) $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \leftrightarrow \psi$ as usual.

Scope: As in modal logic, $\Box$, $\boxed{\scriptstyle \mathrm{P}}$, and $\boxed{\scriptstyle \mathrm{F}}$ bind more tightly than connectives. Thus:

(a) $\Box\varphi \to \psi$ means $(\Box\varphi) \to \psi$

(b) $\boxed{\scriptstyle \mathrm{F}}\varphi \vee \psi$ means $(\boxed{\scriptstyle \mathrm{F}}\varphi) \vee \psi$

(c) $\triangle\varphi \wedge \psi$ means $(\triangle\varphi) \wedge \psi$

Task Frame: A **task frame** is a triple $\mathcal{F}_{\mathrm{T}} = \langle \mathcal{W}, \mathcal{T}, \Rightarrow \rangle$ where:

- $\mathcal{W}$ is a non-empty set of **world states**
- $\mathcal{T} = \langle T, +, \leqslant \rangle$ is a totally ordered abelian group of times
- $\Rightarrow \subseteq \mathcal{W} \times T \times \mathcal{W}$ is a parameterized task relation

satisfying **nullity** ($w \Rightarrow_0 w$ for all $w \in \mathcal{W}$) and **compositionality** (if $w \Rightarrow_x u$ and $u \Rightarrow_y v$, then $w \Rightarrow_{x+y} v$).

World History: A **world history** over frame $\mathcal{F}_{\mathrm{T}}$ is a function $\tau : X \to \mathcal{W}$ where $X \subseteq T$ is convex (if $x_1, x_2 \in X$ and $x_1 \leqslant z \leqslant x_2$, then $z \in X$) satisfying **task coherence**: for all $x, y \in T$ where $x, x + y \in X$, we have $\tau(x) \Rightarrow_y \tau(x + y)$. We write $\mathcal{H}$ for the set of all world histories over a given frame.

Task Model: A **task model** is a tuple $\mathcal{M}_{\mathrm{T}} = \langle \mathcal{W}, \mathcal{T}, \Rightarrow, \ulcorner \cdot \urcorner \rangle$ where $\langle \mathcal{W}, \mathcal{T}, \Rightarrow \rangle$ is a task frame and $\ulcorner p_i \urcorner \subseteq \mathcal{W}$ is the interpretation of sentence letter p_i.

Semantics: Given task model $\mathcal{M}_{\mathrm{T}}$, world history τ, and time x, we define truth at $\mathcal{M}_{\mathrm{T}}, \tau, x$ (written $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$) inductively:

$\mathcal{M}_{\mathrm{T}}, \tau, x \models p_i$ iff $x \in \mathit{dom}(\tau)$ and $\tau(x) \in \ulcorner p_i \urcorner$

$\mathcal{M}_{\mathrm{T}}, \tau, x \not\models \bot$

$\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi \rightarrow \psi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, x \not\models \varphi$ or $\mathcal{M}_{\mathrm{T}}, \tau, x \models \psi$

$\mathcal{M}_{\mathrm{T}}, \tau, x \models \Box\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \sigma, x \models \varphi$ for all $\sigma \in \mathcal{H}$

$\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle P}\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, y \models \varphi$ for all $y \in \mathit{dom}(\tau)$ with $y < x$

$\mathcal{M}_{\mathrm{T}}, \tau, x \models \boxed{\scriptstyle F}\varphi$ iff $\mathcal{M}_{\mathrm{T}}, \tau, y \models \varphi$ for all $y \in \mathit{dom}(\tau)$ with $y > x$

Logical Consequence: $\Gamma \models \varphi$ just in case for all task models $\mathcal{M}_{\mathrm{T}}$, all world histories τ, and all times $x \in \mathit{dom}(\tau)$, if $\mathcal{M}_{\mathrm{T}}, \tau, x \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}_{\mathrm{T}}, \tau, x \models \varphi$.