The Modern History of Modal Logic

PROBLEM SET 03: DUE MARCH 3RD

Updated: May 13, 2025

1 Truth-Conditions

1. Why are frames required to be nonempty?
2. $\mathcal{M}, w \vDash \varphi \vee \psi$ *iff* $\mathcal{M}, w \vDash \varphi$ or $\mathcal{M}, w \vDash \psi$.
3. $\mathcal{M}, w \vDash \Diamond\varphi$ *iff* $\mathcal{M}, u \vDash \varphi$ for some $u \in W$ such that $R(w, u)$.
4. $\mathcal{M}, w \vDash \Box\varphi$ *iff* $(w)_R \subseteq |\varphi|_{\mathcal{M}}$.
5. $\mathcal{M}, w \vDash \Box(\varphi \to \psi)$ *iff* $|\varphi|^w_{\mathcal{M}} \subseteq |\psi|_{\mathcal{M}}$.
6. $\mathcal{M}, w \vDash \Box(\varphi \leftrightarrow \psi)$ *iff* $|\varphi|^w_{\mathcal{M}} = |\psi|^w_{\mathcal{M}}$.
7. $\mathcal{M}, w \vDash \Diamond\varphi$ *iff* $(w)_R \cap |\varphi|_{\mathcal{M}} \neq \varnothing$.

2 Frames

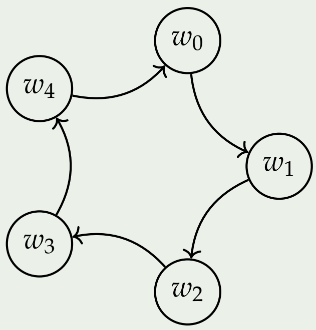
Relations: Evaluate the following, providing a proof or countermodel:

1. Every serial frame is reflexive.

Solution **Ben**

Letting $W = \{w_i : 0 \leqslant i \leqslant 4\}$ where $R = \{\langle w_x, w_y\rangle : y = x + 1 \pmod 5\}$, we may observe that every world sees some world but no world sees itself. Thus $\langle W, R\rangle$ is serial without being reflexive, providing a countermodel to the stated claim.

We may produce infinitely many countermodels by simply varying the cardinality of $|W| > 1$. $\Diamond$



NOTE: This solution is needlessly complex. I use it as an example of what to avoid though it is a perfectly correct countermodel. Instead, try to find the simplest countermodels that you can, adding complexity only as needed.

2. Every serial symmetric frame is reflexive.
3. How many frames are both symmetric and transitive but not reflexive.
4. Every frame that is left and right Euclidean is symmetric and transitive.
5. Every frame that is left and right Euclidean is symmetric and transitive.
6. Every frame that is symmetric and transitive is left and right Euclidean.

7. Every frame that is transitive and both left and right Euclidean is symmetric.
8. Every frame that is symmetric and left Euclidean is transitive.

Solution — Ben

Let $\langle W, R\rangle$ be a symmetric (SYM) and left Euclidean (LEU) frame. Since R is transitive (TRA) if $R = \varnothing$, we may assume otherwise. Letting $R(x, y)$ and $R(y, z)$ for arbitrary $x, y, z \in W$, we aim to show that $R(x, z)$.

Since $R(z, y)$ follows by SYM, we know $R(x, z)$ by LEU. Generalizing on $x, y, z \in W$, we may conclude that $\langle W, R\rangle$ is transitive as desired. □

9. There is a finite serial transitive frame that is neither reflexive nor symmetric.
10. A symmetric frame is left Euclidean just in case it is right Euclidean.
11. The relational image of a transitive, symmetric, reflexive frame is a partition.
12. Every total frame is a partition.
13. There is a symmetric total frame that is not a partition.

Countermodels: Evaluate the following, providing a proof or countermodel. If there is a countermodel, replace K with the weakest set of constraints C to make the claim hold. You do not need to prove that it is the weakest set of constraints.

14. $\Box\varphi \vDash_K \Box\Box\varphi$
15. $\Box\varphi \vDash_K \varphi$
16. $\varphi \vDash_K \Box\varphi$
17. $\Box(\varphi \vee \psi) \vDash_K \Box\varphi \vee \Box\psi$
18. $\Box\varphi \vDash_K \Diamond\varphi$
19. $\Diamond\varphi \vDash_K \Box\varphi$
20. $\Diamond\Box\varphi \vDash_K \Box\Diamond\varphi$
21. $\Box\Diamond\varphi \vDash_K \Diamond\Box\varphi$
22. $\Box\varphi \vDash_K \Box\Diamond\varphi$
23. $\vDash_K \neg\Box(\varphi \wedge \neg\varphi)$
24. $\Diamond\psi \vDash_K \neg\Box(\varphi \wedge \neg\varphi)$
25. $\Box\varphi \vDash_K \Box(\psi \to \varphi)$.
26. $\neg\Box\varphi \vDash_K \Box(\varphi \to \psi)$.

3 Characterization

Semantic Proofs: Provide semantic proofs of the following:

1. If $\vDash_K \varphi$, then $\vDash_K \Box\varphi$
2. $\vDash_K \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\vDash_D \Box\varphi \to \Diamond\varphi$
4. $\vDash_T \Box\varphi \to \varphi$
5. $\vDash_B \varphi \to \Box\Diamond\varphi$
6. $\vDash_4 \Box\varphi \to \Box\Box\varphi$
7. $\vDash_5 \Diamond\varphi \to \Box\Diamond\varphi$

Equivalences: Provide semantic proofs of the following equivalences:

8. $\neg\Box\varphi \equiv_K \Diamond\neg\varphi$.
9. $\neg\Diamond\varphi \equiv_K \Box\neg\varphi$.
10. $\Diamond(\varphi \vee \psi) \equiv_K \Diamond\varphi \vee \Diamond\psi$
11. $\Diamond\Box\varphi \equiv_B \Diamond\Box\Diamond\Box\varphi$
12. $\Box\Box\varphi \equiv_4 \Box\varphi$.
13. $\Box\Diamond\varphi \equiv_5 \Diamond\varphi$.
14. $\Diamond\Box\varphi \equiv_5 \Box\varphi$.