The Modern History of Modal Logic

PROBLEM SET 02: FEBRUARY 10TH

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1 Metalinguistic Abbreviation

Let $\mathcal{L}^+$ include the symbols in $\mathcal{L}$ together with the sentential operators '$\vee$', '$\wedge$', and '$\leftrightarrow$' which are to be read 'or', 'and', and 'if and only if', respectively.

1. Although they are not typically written, why are quotes included in the semantics?
2. Provide a recursive definition of the set $\texttt{wfs}(\mathcal{L}^+)$ of wfss of $\mathcal{L}^+$.
3. Provide a semantics for $\mathcal{L}^+$ by defining the models of $\mathcal{L}^+$ and $\vDash^+$.
4. Prove that $\varphi \vee \psi$, $\varphi \vee (\varphi \wedge \psi)$, and $\varphi \leftrightarrow \psi$ are logically equivalent to wfss of $\mathcal{L}$.
5. For each new operator in $\mathcal{L}^+$, provide two logical truths including that operator.

2 Derived Metarules

Derive the following metarules in the Hilbert proof system:

1. *Weakening* (**WK**): if $\Gamma \vdash_{\text{PL}} \psi$ and $\Gamma \subseteq \Sigma$, then $\Sigma \vdash_{\text{PL}} \psi$.

   Solution — Ben

   Assume $\Gamma \vdash_{\text{PL}} \psi$ and $\Gamma \subseteq \Sigma$ for arbitrary Γ, Σ, and ψ. We aim to show that $\Sigma \vdash_{\text{PL}} \psi$ by appealing to the definitions.

   By the definition of $\vdash_{\text{PL}}$, there is some Hilbert proof X of φ from Γ. By the definition of a Hilbert proof, every wfs φ in X is either: (1) a *premise* in Γ; (2) an *axiom*; or (3) follows from previous wfss χ and $\chi \to \varphi$ in X by a *rule*. Since $\Gamma \subseteq \Sigma$, any premise in X also belongs to Σ. Thus X satisfies the definition of a Hilbert proof of ψ from Σ. $\square$

   NOTE: This proof is rather verbose for such a trivial result, but it is good to err on the side of being *too explicit* so that every step is easy to follow for all.

2. *Cut elimination* (**CUT**): if $\Gamma \vdash_{\text{PL}} \varphi$ and $\Sigma, \varphi \vdash_{\text{PL}} \psi$, then $\Sigma, \Gamma \vdash_{\text{PL}} \psi$.
3. *Principle of detachment* (**PD**): if $\Gamma \vdash_{\text{PL}} \varphi \to \psi$ and $\Sigma \vdash_{\text{PL}} \varphi$, then $\Gamma, \Sigma \vdash_{\text{PL}} \psi$.
4. *Deduction theorem* (**DT**): if $\Gamma, \varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \varphi \to \psi$.
5. *Reverse deduction* (**RD**): if $\Gamma \vdash_{\text{PL}} \varphi \to \psi$, then $\Gamma, \varphi \vdash_{\text{PL}} \psi$.

3 Axiomatic Proofs

You may appeal previous results (proofs before the proof you are working on) to derive the following in the Hilbert proof system, justifying each step where premises are indicated with '**PR**' and any assumed rules of the form $\Gamma \vdash_{\text{PL}} \psi$ with '**AS**':

1. *Hypothetical syllogism* (**HS**): $\varphi \to \psi, \psi \to \chi \vdash_{\text{PL}} \varphi \to \chi$.

Solution — Ben

The proof goes by an easy derivation given the added premise φ to start.

1.	$\varphi \to \psi$	**PR**
2.	$\psi \to \chi$	**PR**
3.	φ	**PR**
4.	ψ	**MP: 1,3**
5.	χ	**MP: 2,4**

The proof above establishes that $\varphi \to \psi, \psi \to \chi, \varphi \vdash_{\text{PL}} \chi$. It follows from **DT** that $\varphi \to \psi, \psi \to \chi \vdash_{\text{PL}} \varphi \to \chi$ as desired. □

2. *Hypothetical exchange* (**HE**): $\varphi \to (\psi \to \chi) \vdash_{\text{PL}} \psi \to (\varphi \to \chi)$.

3. *Reductio ad absurdum* (**RAA**): $\vdash_{\text{PL}} \varphi \to (\neg\varphi \to \psi)$.

4. *Ex falso quidlobet* (**EFQ**): $\vdash_{\text{PL}} \neg\varphi \to (\varphi \to \psi)$.

5. *Reverse contraposition* (**RCP**): $\neg\varphi \to \neg\psi \vdash_{\text{PL}} \psi \to \varphi$.

6. *Double negation elimination* (**DNE**): $\vdash_{\text{PL}} \neg\neg\varphi \to \varphi$.

7. *Double negation introduction* (**DNI**): $\vdash_{\text{PL}} \varphi \to \neg\neg\varphi$.

8. *Contraposition* (**CP**): $\varphi \to \psi \vdash_{\text{PL}} \neg\psi \to \neg\varphi$.

9. *Negation elimination* (**NE**): if $\Gamma, \neg\varphi \vdash_{\text{PL}} \neg\psi$ and $\Gamma, \neg\varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \varphi$.

10. *Negation introduction* (**NI**): if $\Gamma, \varphi \vdash_{\text{PL}} \neg\psi$ and $\Gamma, \varphi \vdash_{\text{PL}} \psi$, then $\Gamma \vdash_{\text{PL}} \neg\varphi$.

11. *Ex contradictione quidlobet* (**ECQ**): $\varphi, \neg\varphi \vdash_{\text{PL}} \psi$.

12. *Left disjunction introduction* (**LDI**): $\varphi \vdash_{\text{PL}} \varphi \vee \psi$.

13. *Right disjunction introduction* (**RDI**): $\psi \vdash_{\text{PL}} \varphi \vee \psi$.

14. *Conjunction introduction* (**CI**): $\varphi, \psi \vdash_{\text{PL}} \varphi \wedge \psi$.

15. *Left conjunction elimination* (**LCE**): $\varphi \wedge \psi \vdash_{\text{PL}} \varphi$.

Solution — Ben

Expressed in primitive notation, we must show that $\neg(\varphi \to \neg\psi) \vdash_{\text{PL}} \varphi$. By drawing on previous results, we use the metarules to reason as follows:

1.	$\vdash_{\text{PL}} \neg\varphi \to (\varphi \to \neg\psi)$	**EFQ: $\neg\psi/\psi$**
2.	$\vdash_{\text{PL}} \neg(\varphi \to \neg\psi) \to \neg\neg\varphi$	**CUT: 1, CP**
3.	$\vdash_{\text{PL}} \neg(\varphi \to \neg\psi) \to \varphi$	**CUT: 2, DNE**
4.	$\neg(\varphi \to \neg\psi) \vdash_{\text{PL}} \varphi$	**RD: 3**

Instead of providing an explicit Hilbert derivation, we prove that there is such a derivation by relying on the previous results as cited above. □

16. *Right conjunction elimination* (**RCE**): $\varphi \wedge \psi \vdash_{\text{PL}} \psi$.

17. *Disjunction elimination* (**DE**): if $\Gamma, \varphi \vdash_{\text{PL}} \chi$ and $\Gamma, \psi \vdash_{\text{PL}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\text{PL}} \chi$.
18. *Biconditional introduction* (**BI**): if $\Gamma, \varphi \vdash_{\text{PL}} \psi$ and $\Gamma, \psi \vdash_{\text{PL}} \varphi$, then $\Gamma \vdash_{\text{PL}} \varphi \leftrightarrow \psi$.
19. *Left biconditional elimination* (**LBE**): $\varphi \leftrightarrow \psi, \varphi \vdash_{\text{PL}} \psi$.
20. *Right biconditional elimination* (**RBE**): $\varphi \leftrightarrow \psi, \psi \vdash_{\text{PL}} \varphi$.

4 Regimentation

Regiment the following in $\mathcal{L}^{\Box}$ using metalinguistic abbreviations, resolving ambiguities.

1. If sugar is sweet, then if roses are red, sugar is sweet.
2. If snow is not green, then if snow is green, roses are red.

Solution — Ben

Taking 'If. . . , then' to express the material conditional, we have:

$$\neg G \to (G \to R)$$

However, if Lewis (1912) is to have his way, we get

$$\neg G \prec\!\!\!\!\;3\; (G \prec\!\!\!\!\;3\; R)$$

This reading follows from taking 'If. . . , then' to express the strict conditional rather than the material conditional. This is equivalent to the following:

$$\Box(\neg G \to \Box(G \to R))$$

That is, in any possibility in which snow is not green it is also the case that in any possibility in which snow is green, roses are red. We need only consider a possibility in which snow is green and roses are not read to make the consequent false. If there is a possibility in which snow is not green (like the actual world) where it is also possible that snow is green and roses are not red, then this regimentation of the sentence is false. Lewis takes it to be an advantage of his system of strict implication that such claims are not theorems, and so are permitted to have false instances.

3. Either it could rain or not.
4. It's necessarily possible that it either rains or doesn't.
5. If rain is possibly necessary, then it's necessarily possible.
6. It cannot both necessarily rain and necessarily not rain.
7. If rain is necessary, then it cannot necessarily not rain.
8. If rain and snow are jointly possible, then each is possible individually.
9. If rain could imply snow, then it could snow if it necessarily rains.
10. If rain or snow are necessary, then either rain is necessary or snow is possible.

4.1 Interpretations

Evaluate the plausibility of each modal axioms when '$\Box$' and '$\Diamond$' are read:

1. ($\Box$) 'It is necessary that'. ($\Diamond$) 'It is possible that'.
2. ($\Box$) 'It is obligatory that'. ($\Diamond$) 'It is permissible that'.
3. ($\Box$) 'It is always going to be the case that'. ($\Diamond$) 'It is going to be the case that'.
4. ($\Box$) 'It has always been the case that'. ($\Diamond$) 'It has been the case that'.
5. ($\Box$) 'It must be the case that'. ($\Diamond$) 'It might be the case that'.

5 Modal Axiomatic Proofs

As above, you may appeal to previous results to derive the following rules in the modal proof system indicated by the turnstile, justifying each step:

1. If $\vdash_{\mathcal{K}} \varphi \to \psi$, then $\vdash_{\mathcal{K}} \Box\varphi \to \Box\psi$.
2. If $\varphi \vdash_{\mathcal{K}} \psi$, then $\Box\varphi \vdash_{\mathcal{K}} \Box\psi$.
3. $\vdash_{\mathcal{K}} \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$.
4. $\vdash_{\mathcal{K}} (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.

Solution Ben

Here it is natural to think about the last step first. If we can show that both $\vdash_{\mathcal{K}} \Box\varphi \to \Box(\varphi \vee \psi)$ and $\vdash_{\mathcal{K}} \Box\psi \to \Box(\varphi \vee \psi)$, then the theorem follows from **DE**. In primitive terms, this amount to showing that $\Box\varphi \vdash_{\mathcal{K}} \Box(\neg\varphi \to \psi)$ and $\Box\psi \vdash_{\mathcal{K}} \Box(\neg\varphi \to \psi)$. We may establish these as follows:

1.	$\vdash_{\mathcal{K}} \varphi \to (\neg\varphi \to \psi)$	**RAA**
2.	$\vdash_{\mathcal{K}} \Box\varphi \to \Box(\neg\varphi \to \psi)$	**#5.1: 1**
3.	$\Box\varphi \vdash_{\mathcal{K}} \Box(\neg\varphi \to \psi)$	**RD: 2**
4.	$\vdash_{\mathcal{K}} \psi \to (\neg\varphi \to \psi)$	**A1:** ψ/φ, $\neg\varphi/\psi$
5.	$\vdash_{\mathcal{K}} \Box\psi \to \Box(\neg\varphi \to \psi)$	**#5.1: 4**
6.	$\Box\psi \vdash_{\mathcal{K}} \Box(\neg\varphi \to \psi)$	**RD: 5**
7.	$\vdash_{\mathcal{K}} (\Box\varphi \vee \Box\psi) \to \Box(\neg\varphi \to \psi)$	**DE: 3, 6**

By metalinguistic abbreviation, **7** is $\vdash_{\mathcal{K}} (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.

5. $\vdash_{\mathcal{K}} \Box(\varphi \to \psi) \to \Box(\neg\psi \to \neg\varphi)$.
6. If $\vdash_{\mathcal{K}} \varphi \to \psi$, then $\vdash_{\mathcal{K}} \Diamond\varphi \to \Diamond\psi$.
7. $\vdash_{\mathcal{K}} \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.
8. $\vdash_{\mathcal{T}} \Box\varphi \to \Diamond\varphi$.
9. $\vdash_{\mathcal{T}} \neg\Box(\varphi \wedge \neg\varphi)$.
10. $\vdash_{\mathcal{D}} \Diamond(\varphi \to \varphi)$.

11. $\vdash_{\mathcal{B}} \Box\varphi \to \Diamond\varphi$.
12. $\vdash_{\mathcal{S}4} (\Diamond\varphi \wedge \Box\psi) \to \Diamond(\varphi \wedge \Box\psi)$.
13. $\vdash_{\mathcal{S}4} \Box\varphi \to \Box\Diamond\Box\varphi$.
14. $\vdash_{\mathcal{S}4} \Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$.
15. $\vdash_{\mathcal{S}4} \Diamond\Box\Diamond\varphi \leftrightarrow \Diamond\varphi$.
16. $\vdash_{\mathcal{S}5} \Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$.
17. $\vdash_{\mathcal{S}5} \Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$.
18. $\vdash_{\mathcal{S}5} \Diamond\Box\varphi \leftrightarrow \Box\varphi$.
19. $\vdash_{\mathcal{S}5} \Box\varphi \to \Box\Box\varphi$.
20. $\vdash_{\mathcal{S}5} \varphi \to \Box\Diamond\varphi$.
21. $\vdash_{\mathcal{S}5} \Diamond\Box\varphi \to \varphi$.