The Modern History of Modal Logic

Problem Set 04: Due March 17rd

Updated: May 13, 2025

1 Regimentation

Negation: Provide a definition of negation using the other operators in $\mathcal{L}^{\mathsf{T}}$.

Regimentation: Regiment the following in $\mathcal{L}^{\mathsf{T}}$ and $\mathcal{L}^{\mathsf{T}}_{\Box}$ disambiguating as needed.

1. If it is raining, it will stop.

   Solution — Ben

   Letting R symbolize 'It is raining', we may regiment the claim by $R \rightarrow \Diamond_{\mathrm{F}}\neg R$ in $\mathcal{L}^{\mathsf{T}}$. This asserts that if it is raining, there is a time in the actual future in which it stops raining.

   Alternatively, we may take 'will' to convey that if it is raining, then for every future there is a time in which the rain stops. We may capture this reading in $\mathcal{L}^{\mathsf{T}}_{\Box}$ with the regimentation $R \rightarrow \boxdot\Diamond_{\mathrm{F}}\neg R$.

2. If it wasn't that cold before, it might still be that cold at some point.
3. Either it has rained or it will snow.
4. If it will rain, then it has always been so.
5. If it will always rain, then it must have rained before.
6. It has always been true that it either will rain or it won't.
7. If it has always rained, then it will always have been that it rained before.
8. If it has always rained, then it has always been that it has rained.
9. If it will always rain, then it cannot have always not rained.
10. If has rained and snowed, then it could tomorrow.
11. If rain has always implied clouds, it will always be cloudy if it always rains.
12. If rain lead to snow before, then snow might lead to rain.

2 Temporal Frame Constraints

Relations: Evaluate the following, providing a proof or counterexample:

1. Every asymmetric frame is irreflexive.
2. Every irreflexive transitive frame is asymmetric.
3. Every frame that is not irreflexive has neither beginning nor end.
4. Every left and right linear frame is total.
5. Every total frame is left and right linear.

Solution Ben

Let $\mathcal{F} = \langle T, < \rangle$ be a total frame where both $y < x$ and $z < x$ for arbitrary $x, y, z \in T$. By TOT, either $y < z$, $y = z$, or $y > z$. Since $x, y, z \in T$ were arbitrary, we may conclude that $\mathcal{F}$ is left linear.

Assuming instead that $y > x$ and $z > x$ for arbitrary $x, y, z \in T$, either $y < z$, $y = z$, or $y > z$ follows by TOT. Generalizing on $x, y, z \in T$, $\mathcal{F}$ is also right linear. Since $\mathcal{F}$ was an arbitrary total frame, we may conclude that every total frame is left and right linear. □

6. Every frame that is left and right linear is transitive.
7. Every frame that is not right linear is right discrete.
8. Every frame that is dense is both left and right linear.
9. Every frame that is asymmetric and left linear is transitive.
10. There is a dense frame with both a beginning and end.

Solution Ben

Consider the frame $\mathcal{F} = \langle [0,1], < \rangle$ where $[0,1] \subseteq \mathbb{Q}$ and $<$ is the standard ordering of rational numbers. Thus for all $i \in (0,1)$, we have:

$$0 \longrightarrow \cdots \; i \; \cdots \longrightarrow 1$$

Since $0 < i < 1$ for all $i \in (0,1)$, it follows that $\mathcal{F}$ has both a beginning and end (it is bounded below and above) and so neither INF or INP hold.

Given any $x, z \in [0,1]$ where $x < z$, we may let $y = x + \frac{z-x}{2}$ where this is the rational number between x and z, and so $x < y < z$. Since $x, z \in [0,1]$ were arbitrary, $\mathcal{F}$ satisfies DEN as desired. ◇

11. The relational image of a frame with a beginning and end is finite.
12. The relational image of an asymmetric frame is not a partition.

3 Characterization

Countermodels: Evaluate the following, providing a proof or $\mathcal{L}^{\mathrm{T}}$ countermodel. If there is a countermodel, strengthen $\vDash$ by imposing the weakest set of constraints C which make that claim valid. (You do not need to prove that it is the weakest set of constraints.)

1. $\vDash \boxed{\scriptstyle P}(\varphi \rightarrow \psi) \rightarrow (\boxed{\scriptstyle P}\varphi \rightarrow \boxed{\scriptstyle P}\psi)$.
2. $\vDash \Diamond_{P}\top$.
3. $\vDash \varphi \rightarrow \boxed{\scriptstyle F}\Diamond_{P}\varphi$.
4. $\vDash \boxed{\scriptstyle P}\boxed{\scriptstyle P}\varphi \rightarrow \boxed{\scriptstyle P}\varphi$.
5. $\vDash \boxed{\scriptstyle P}\bot \vee \Diamond_{P}\boxed{\scriptstyle P}\bot$.
6. $\vDash \boxed{\scriptstyle P}\boxed{\scriptstyle F}\varphi \rightarrow \triangle\varphi$.

7. $\vDash \boxed{\scriptstyle \mathrm{P}}\varphi \to \boxed{\scriptstyle \mathrm{P}}\boxed{\scriptstyle \mathrm{P}}\varphi$.

8. $\vDash (\Diamond_{\mathrm{P}}\top \wedge \varphi \wedge \boxed{\scriptstyle \mathrm{F}}\varphi) \to \Diamond_{\mathrm{P}}\boxed{\scriptstyle \mathrm{F}}\varphi$.

9. $\vDash \boxed{\scriptstyle \mathrm{F}}(\varphi \to \psi) \to (\boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\psi)$.

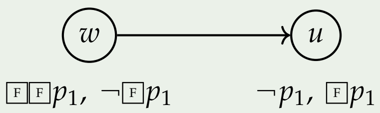
10. $\vDash \Diamond_{\mathrm{F}}\top$.

11. $\vDash \varphi \to \boxed{\scriptstyle \mathrm{P}}\Diamond_{\mathrm{F}}\varphi$.

12. $\vDash \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\varphi$.

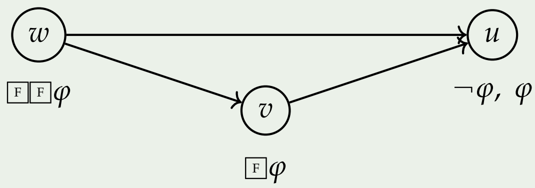
Solution **Ben**

Consider an $\mathcal{L}^{\mathrm{T}}$ model $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ where $T = \{w, u\}$, only $w < u$, and $\mathcal{I}(p_1) = \varnothing$ (the interpretation of all other sentence letters is arbitrary):



Vacuously, every $v \in T$ where $u < v$ is such that $v \in \mathcal{I}(p_1)$, and so $\mathcal{M}, v \vDash p_1$. Thus $\mathcal{M}, u \vDash \boxed{\scriptstyle \mathrm{F}}p_1$ by the semantics for $\boxed{\scriptstyle \mathrm{F}}$, and so $\mathcal{M}, w \vDash \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}p_1$ since u is the only element of T where $w < u$. At the same time, $u \notin \mathcal{I}(p_1)$, and so $\mathcal{M}, u \nvDash p_1$. Since $w < u$, it follows that $\mathcal{M}, w \nvDash \boxed{\scriptstyle \mathrm{F}}p_1$ by the semantics for $\boxed{\scriptstyle \mathrm{F}}$. Thus $\mathcal{M}, w \nvDash \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}p_1 \to \boxed{\scriptstyle \mathrm{F}}p_1$ by the semantics for $\to$.

Nevertheless, we may show that $\nvDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\varphi$ by assuming for contradiction that there is an $\mathcal{L}^{\mathrm{T}}$ model $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ that satisfies DEN where $\mathcal{M}, w \nvDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\varphi$ for some $w \in T$. It follows by the semantics for $\to$ that both: (1) $\mathcal{M}, w \vDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi$; and (2) $\mathcal{M}, w \nvDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\varphi$. It follows from the latter that $\mathcal{M}, u \nvDash_{\mathrm{DEN}} \varphi$ for some $u \in T$ where $w < u$. Since $\mathcal{M}$ satisfies DEN, there is some $v \in T$ where $w < v < u$, and so we have:



Since $w < v$, it follows from (1) that $\mathcal{M}, v \vDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\varphi$, and so $\mathcal{M}, u \vDash_{\mathrm{DEN}} \varphi$, contradicting the above. Thus $\vDash_{\mathrm{DEN}} \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\varphi$ as desired. □

13. $\vDash \boxed{\scriptstyle \mathrm{F}}\bot \vee \Diamond_{\mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\bot$.

14. $\vDash \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{P}}\varphi \to \triangle\varphi$.

15. $\vDash \boxed{\scriptstyle \mathrm{F}}\varphi \to \boxed{\scriptstyle \mathrm{F}}\boxed{\scriptstyle \mathrm{F}}\varphi$.

4 Indeterminacy

Evaluate: Without imposing any restriction on the models of $\mathcal{L}^{\mathrm{T}}_{\Box}$, evaluate the following where $p_i \in \mathbb{L}$, providing a proof or countermodel:

1. $\vDash p_i \to \Diamond p_i$.

2. $\vDash \varphi \to \langle\cdot\rangle\varphi$.

3. $\vDash \langle\mathrm{P}\rangle\varphi \vee \langle\mathrm{P}\rangle\neg\varphi$.

4. $\vDash \varphi \to \boxed{\mathrm{F}}\langle\mathrm{P}\rangle\varphi$.

5. $\vDash \boxed{\mathrm{P}}\boxed{\mathrm{F}}\varphi \to \triangle\varphi$.

6. $\vDash \varphi \to \boxed{\cdot}\varphi$.

7. $\vDash \langle\mathrm{F}\rangle\varphi \vee \langle\mathrm{F}\rangle\neg\varphi$.

Solution **Ben**

Consider a minimal model $\mathcal{M} = \langle T, <, \mathcal{I}\rangle$ for $\mathcal{L}^{\mathrm{T}}_{\Box}$ where $T = x$ has just one time, $x \nless x$, and $\mathcal{I}$ is arbitrary. Letting $\mathcal{T}_i = \langle T, <\rangle$, we may observe that $\mathcal{M}, \mathcal{T}_i, x \nvDash \langle\mathrm{F}\rangle\varphi$ since there is no $y \in T_i$ where $x < y$ and $\mathcal{M}, \mathcal{T}_i, y \nvDash \varphi$, and so $\mathcal{M}, \mathcal{T}_i, y \vDash \neg\varphi$ by the semantics for negation. Moreover, $\mathcal{M}, \mathcal{T}_i, x \nvDash \langle\mathrm{F}\rangle\neg\varphi$ since neither is there a $y \in T_i$ where $x < y$ and $\mathcal{M}, \mathcal{T}_i, y \nvDash \neg\varphi$. It follows that $\mathcal{M}, \mathcal{T}_i, x \nvDash \neg\langle\mathrm{F}\rangle\varphi \to \langle\mathrm{F}\rangle\neg\varphi$ by the semantics for $\to$, and so $\nvDash \neg\langle\mathrm{F}\rangle\varphi \to \langle\mathrm{F}\rangle\neg\varphi$ by the definition of logical consequence. Thus $\nvDash \langle\mathrm{F}\rangle\varphi \vee \langle\mathrm{F}\rangle\neg\varphi$ by abbreviation. $\Diamond$

8. $\vDash \varphi \to \boxed{\mathrm{P}}\langle\mathrm{F}\rangle\varphi$.

9. $\vDash \boxed{\mathrm{F}}\boxed{\mathrm{P}}\varphi \to \triangle\varphi$.