Type Theory

Paradox and Infinity
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Against Ramification

Orders: Ramsey rejects Russell's divisions of the types into orders.

Autological: A predicate is *autological iff* it expresses a property it has.

Heterological: A predicate is *heterological iff* it is not autological.

- Whereas 'short' is autological, 'long' is heterological.
- Is 'heterological' heterological?
- Use/mention distinction.

Solution: In the style of *Principia Mathematica* we have:

- $H(w)$ *iff* there is a property P where w expresses P and $\neg P(w)$.
- The variable 'P' ranges over first-order properties which H is not.
- "This theory of a hierarchy of orders of functions of individuals escapes the contradictions; but it lands us in an almost equally serious difficulty, for it invalidates many important mathematical arguments..."

Simple Type Theory

Logical: Ramsey accepts Russell's solution to the logical paradoxes.

Simple Types: Recall the function application clause from the simple theory of types:

- If $\varphi_1 : t_1, \ldots, \varphi_n : t_n$, then $z(\varphi_1, \ldots, \varphi_n) : ((t_1, \ldots, t_n), t_1, \ldots, t_n)$.
- Observe that z is of *higher type* than its arguments.

Self-Application: A predicate, "cannot significantly take itself as argument..."

- Neither $P(P)$ nor $\neg P(P)$ can be simply typed, and so meaningless.
- This case differs from *heterological* since no semantic terms occur.

Question: Is there any self application in $R \coloneqq \{x : \neg\mathtt{In}(x, x)\}$?

Sets: Russell identifies classes (sets) with functions (properties).

- '$x \in y$' is notational variant of '$y(x)$'.
- Thus '$x \in x$' abbreviates '$x(x)$' which cannot be simply typed.

Question: Why not respond by faulting the identification of sets with properties?

- Blocking this identification further undoes Russell's solution.
- But it does not solve the paradox.
- Ramsey thought we should leave this part of Russell's solution.

Dividing the Paradoxes

Russell: Solves all the paradoxes by accepting RTT + AR.

Ramsey: Rejects AR, and also rejects RTT to preserve mathematics.

- Ramsey divides the paradoxes into the logical and semantic.
- The semantic paradoxes include naming, expressing, meaning, etc.
- The semantic paradoxes concern the interpretation of language.
- Ramsey thought they deserved their own solution.

Tarski: Recall Tarski's levels of languages.

- Could distinguish between true_1, true_2, etc.
- The semantic terms for a language cannot belong to that language.
- The object/metalanguage distinction is widely accepted.

Truth: What of a theory of truth, meaning, reference, etc.?

- The object/metalanguage distinction doesn't provide a theory of truth.
- Nor does it provide a theory of meaning, or reference, etc.
- Lots remains to be done to understand how language works.

Higher-Order Logic

Mathematics: Truth theories remain controversial but mathematics is preserved.

- The logical paradoxes are solved.
- We also get the theory of simple types as a result.

Higher-Order: Since talk of orders has been banished, we may reclaim the term.

- Today, 'order' refers to the type of the variable bound by a quantifier.
- The quantifiers of *first-order logic* bind variables in name position.
- The quantifiers of *higher-order logics* bind variables of higher type.

Quine: Thought higher-order logic was "set theory in sheep's clothing."

- Instead of recasting sets as properties, *first-orderists* go the other way.
- But then how are we to escape Russell's paradox (this is for next week).

Types: Consider the following defense of higher-order logic:

- Properties, relations, etc., are not first-order objects.
- If we recognize type distinctions, why only first-order quantifiers?
- If there are some higher-order things, why not quantify over them?

Controvercy: Quine's shadow remains long in philosophy but not computer science.