Infinite Cardinalities

PARADOX AND INFINITY
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May 14, 2024

Where to Begin...

Assumptions: Theories have to begin somewhere.

- A theory that is neutral on everything is no theory at all.
- But a conclusion cannot come from nothing.

Concepts: Can't define everything.

- Must take some concepts as primitive.
- Intuitions from patterns of use.
- Principles encode theoretical function.

Example: Consider the concept of *number*.

- Numbers are answers to 'How many?'-Questions.
- What principles might we affirm?

Finite Intuitions

Proper Subset Principle: $A \subset B \rightarrow |A| < |B|$.

- There are more mammals than lamas.
- There are more reals than rationals.

Bijection Principle: $A \simeq B \leftrightarrow |A| = |B|$.

- $A \simeq B$ means the As and Bs can be paired one-to-one with no remainders.
- We can define this without recourse to the concept of number.

Ordered Pair: $\langle a, b\rangle \coloneqq \{\{a\}, \{a, b\}\}$.

Relation: $A \times B \coloneqq \{\langle a, b\rangle : a \in A,\ b \in B\}$.

Function: A *(total) function* $f : A \rightarrow B$ is any relation $f \subseteq A \times B$ where for every $a \in A$: (1) there is some $b \in B$ where $\langle a, b\rangle \in f$; and (2) if there are some $b, c \in B$ where both $\langle a, b\rangle, \langle a, c\rangle \in f$, then $b = c$.

- If f is a function, then we may take '$f(a) = b$' to abbreviate '$\langle a, b\rangle \in f$'.

Injective: $f : A \rightarrow B$ is *injective iff* for any $a, b \in A$, if $f(a) = f(b)$, then $a = b$.

Surjective: $f : A \rightarrow B$ is *surjective iff* for all $b \in B$ there is some $a \in A$ where $f(a) = b$.

Bijection: $f : A \rightarrow B$ is *bijective iff* f is an injective and surjective function.

Equinumerous: $A \simeq B$ *iff* there is a bijection $f : A \rightarrow B$.

Infinity and Paradox?

Hilbert's Hotel: Always room for (countably many) more guests.

- Given a domain $\mathbb{D}$, $f : x \mapsto t(x)$ defines the function $\{\langle x, t(x)\rangle : x \in \mathbb{D}\}$ where '$t(x)$' is a term that may include 'x', e.g., '$x+1$'.
- $f_m : n \mapsto n + m$ is a bijection $f_m : \mathbb{N} \to \mathbb{N}_m$ where $\mathbb{N}_m = \{k \in \mathbb{N} : k \geqslant m\}$.
- $g_m : n \mapsto n \times m$ is a bijection $g_m : \mathbb{N} \to \mathbb{N}_{(m)} = \{k \times m : k \in \mathbb{N}\}$.

(?) What about countably many countable groups of new guests?

Galileo's Roots: "Every square has its own root and every root has its own square, while no square has more than one root and no root has more than one square."

Paradox: There are many equinumerous proper subsets of infinite sets.

- By the principles above, both $|\mathbb{N}_2| < |\mathbb{N}|$ and $|\mathbb{N}_2| = |\mathbb{N}|$.
- But $x < y$ *iff* $x \leqslant y$ and $x \neq y$.
- Thus $|\mathbb{N}_2| < |\mathbb{N}|$ entails $|\mathbb{N}_2| \neq |\mathbb{N}|$: contradiction.

(?) Which principle should we give up?

The Abductive Method

Deductively Closed: Good theories include all of their implications.

Consistency: Good theories exclude contradictions.

Simplicity: Good theories are easy to understand, e.g., are finitely axiomatizable in terms of intuitively compelling and conceptually elegant concepts.

Strength: Good theories say more rather than less.

Utility: Good theories serve our aims, e.g., have useful applications.

A Metaphysical Aside

Subjectivity: Is theory choice by abduction a reflection of human psychology?

- The abductive method describes how we typically choose theories.
- Are we right to use the abductive method and why?

Realism: Is the abductive method well suited to the task of describing reality?

- We don't need to decide this before using the abductive method.
- It will help to put the method to work.

Example: Which of the principles above should we give up?

Towards a Theory of Number

Hypothesis: Suppose we were to retain the *Proper Subset Principle* (PSP).

- Then $|\mathbb{N}| > |\mathbb{N}_2| > |\mathbb{N}_3| > \ldots$ and $|\mathbb{N}| > |\mathbb{N}_{(2)}| > |\mathbb{N}_{(4)}| > \ldots$ etc.
- How are we to compare $|\mathbb{N}_{(2)}|$ and $|\mathbb{N}_{(3)}|$?

Linear Ordering: Numbers are linearly ordered by $\leqslant$ and so must satisfy the following.

`Reflexive`: $x \leqslant x$ for any number x.

`Transitive`: If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

`Anti-Symmetric`: If $x \leqslant y$ and $y \leqslant x$, then $x = y$.

`Total`: Either $x \leqslant y$ or $y \leqslant x$ for any numbers x and y.

- Compare giving a theory of identity where symmetry fails.
- We wouldn't really be talking about identity.

Incomplete: Converse of PSP is false and so PSP does not define $<$.

(?) Could PSP be supplemented in some way?

(?) How would we compare the cardinality of two bags of stones?

- By trying to line them up one-to-one.

Injection Principle: $|A| \leqslant |B|$ *iff* $A \simeq C$ for some $C \subseteq B$.

- This assumes *Bijection Principle* (BP) which is in tension with PSP.
- Restricting the *Injection Principle* (IP) and BP to finite sets is *ad hoc*.
- Better to take Hilbert's Hotel to be a counterexample to PSP.

Countable Infinity

Principles: Given IP and BP, we may show that numbers are linearly ordered.

- `Anti-Symmetric` is proven by the Cantor-Schroeder-Bernstein theorem.
- `Total` is equivalent to the Axiom of Choice.

Countable Sets: A set A is *countably infinite iff* $|A| = |\mathbb{N}|$.

Identities: Bijection Principle makes many sets countably infinite.

- $|\mathbb{N}| = |\mathbb{N}_m| = |\mathbb{N}_{(m)}|$.
- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.

(?) What about $|\mathbb{N}| = |\mathbb{R}|$?

Next Time: We will show that there are different sizes of infinity.