Set Theory

PARADOX AND INFINITY
Benjamin Brast-McKie
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Axioms and Theorems

Restriction: We will restrict the quantifiers to sets.

SEPARATION: $\forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \wedge \varphi))$ where 'y' does not occur in 'φ'.

- Given any set z, there is a set of z's members that are φ.

Question: Could there be more than one subset of φs?

EXTENSIONALITY: $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$.

- Extensionality guarantees uniqueness.
- Whereas *Extensionality* is an axiom, *Separation* is an *axiom schema*.

Question: Could there be a universal set U?

- Assume there is a universal set U where $\forall x(x \in U)$.
- $\exists y \forall x(x \in y \leftrightarrow (x \in U \wedge \varphi))$ from *Separation* with U for z.
- $\exists y \forall x(x \in y \leftrightarrow \varphi)$ since $x \in U$ for all x.
- $\exists y \forall x(x \in y \leftrightarrow x \notin x)$ by replacing φ with $x \notin x$.
- $\forall x(x \in r \leftrightarrow x \notin x)$ by existential elimination.
- $r \in r \leftrightarrow r \notin r$ by instantiating x with r.
- Hence $\neg\exists y \forall x(x \in y)$ is a theorem.

Theorems: We don't need an axiom to rule out U.

Question: What other axioms do we need to describe the concept?

Zermelo's Theory of Sets

NULL SET: $\exists y \forall x(x \notin y)$.

- There is a set with no members.

PAIRS: $\forall z \forall w \exists y \forall x(x \in y \leftrightarrow (x = z \vee x = w))$.

- For any sets x and w, there is a set whose only members are x and w.

UNIONS: $\forall z \exists y \forall x(x \in y \leftrightarrow \exists w(x \in w \wedge w \in z))$.

- For any set z, there is a set of all members of members of z.

Subset Definition: $x \subseteq z \coloneqq \forall w(w \in x \rightarrow w \in z)$

- Every member of x is a member of z.

POWER SET: $\forall z \exists y \forall x(x \in y \leftrightarrow x \subseteq z)$.

- For any set z, there is a set y of all subsets of z.

Infinite Sets

Question: Does anything guarantee that there are infinite sets?

- If we want there to be infinite sets, how would we guarantee this?

Contains the Null Set: $\varnothing \in y \coloneqq \exists x(x \in y \wedge \forall z(z \notin x))$.

Successor: $z = x' \coloneqq \forall y(y \in z \leftrightarrow (y \in x \vee y = x))$.[1]

INFINITY: $\exists y[\varnothing \in y \wedge \forall x(x \in y \rightarrow \exists z(z \in y \wedge z = x'))]$.

REGULARITY: $\exists x\varphi \rightarrow \exists x(\varphi \wedge \forall y(y \in x \rightarrow \neg\varphi[y/x]))$ where φ does not contain 'y' and $\varphi[y/x]$ is the result of replacing all occurrences of 'x' in φ with 'y'.

- If some set is such that φ, there is "smallest set" x that is such that φ.

Example: Letting φ be '$\exists z(z \in x)$', there is a set that only contains the empty set.

- $\exists x \exists z(z \in x) \rightarrow \exists x(\exists z(z \in x) \wedge \forall y(y \in x \rightarrow \forall z(z \notin y)))$.

Example: Assume there is a set that belongs to itself, i.e., $\exists x(x \in x)$.

- $\exists x(x \in x \wedge \forall y(y \in x \rightarrow y \notin y))$ by *Regularity*.
- $r \in r$ and $\forall y(y \in r \rightarrow y \notin y)$ by conjunction and existential elimination.
- $r \in r \rightarrow r \notin r$ by universal elimination.
- $r \notin r$ by conditional elimination.
- Hence $\neg\exists x(x \in x)$ is a theorem.

Stage Theory

Motivation: Why believe these axioms and not some others?

Iterative Conception: Because they conform to an iterative conception of set.

Stages: Here are the stage axioms.

- No stage is earlier than itself.
- Earlier than is transitive.
- Earlier than is connected/total.
- There is an earliest stage.
- Every stage has a next stage.

Formation: Here are the formation axioms.

- There is a limit stage which does not have a latest predecessor.
- Every set is formed at a unique stage.
- Every member of a set is formed earlier than that set.
- If the members of a set are formed before a stage the set is formed at that stage.

[1] Better to define the successor function $'$.