24.118: Paradox and Infinity, Spring 2024
Problem Set 1: Infinite Cardinalities

Please include your name and list all of your collaborators on the last page of your problem set, preferably a separate page so that no one sees it till the end. *Failing to list collaborators constitutes a violation of academic integrity.*

How your answers will be graded:

- In Part I there is no need to justify your answers. Submit answers in a quiz that you will access on Canvas.
- In Part II you must justify your answers unless stated otherwise in the problem. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on your justification. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- No answer may comprise more than 180 English words. Longer answers will be penalized. However, showing your work in a calculation or proof does *not* count toward the word limit.
- You may consult published literature and the web, but you must credit all sources where failure to do so constitutes plagiarism and can have serious consequences. For advice about how and when to credit sources see `https://integrity.mit.edu/`. Note that merely citing a source does *not* count as a good justification.

All submissions must be in PDF format. Type-written submissions may be strongly preferred by your TA; handwritten submissions are acceptable only if:

1. Your handwriting is easily legible (as judged by your TA);
2. You produce a clean version of the document (as opposed to the sheet of paper you used to work out the problems); and
3. Your manuscript has been scanned to high enough standards (as judged by your TA). Consider using, e.g. *Scannable* or *Adobe Scan*.

Part I (Quiz on Canvas: 31 points)

1. A **relation** with **domain** A and **range** B is any set of ordered pairs $\langle a, b\rangle$ where $a \in A$ and $b \in B$, i.e., any subset of all such pairs $A \times B = \{\langle a, b\rangle : a \in A, b \in B\}$.

   A relation is a **function** $f : A \to B$ just in case every element in A is assigned to exactly one element of B, i.e., if $f(x) = f(y)$ whenever $x = y$.

   A function $f : A \to B$ is an **injection** just in case no two elements in A are assigned to the same element in B, i.e., if $x = y$ whenever $f(x) = f(y)$.

   A function $f : A \to B$ is a **surjection** just in case for every element $b \in B$ there is some $a \in A$ where $f(a) = b$.

   Determine whether the following functions are injective or surjective.

   (a) (2 points)

   $$\begin{array}{rcl} f(x) & = & x+2 \\ A & = & \mathbb{N} \\ B & = & \mathbb{N} - \{0,1\} \end{array}$$

   (b) (2 points)

   $$\begin{array}{rcl} f(x) & = & 2x+3 \\ A & = & \mathbb{Z} \\ B & = & \mathbb{Z} \end{array}$$

   (c) (2 points)

   $$\begin{array}{rcl} f(x) & = & x^2 \\ A & = & \mathbb{R} \\ B & = & \mathbb{R} \end{array}$$

   (d) (2 points)

   $$\begin{array}{rcl} f(x) & = & x+\sqrt{2} \\ A & = & \mathbb{R} \\ B & = & \mathbb{R} \end{array}$$

2. A function $f : A \to B$ is a **bijection** just in case f is both injective and surjective. For which of the following pairs of sets is there a bijection between them?

   (a) The set of negative integers and the set of non-negative integers excluding finitely-many natural numbers (2 points.)

   (b) The set of prime numbers and the set of real numbers between 0 and 0.0001. (2 points.)

   (c) The rational numbers and the set of rational numbers between 2023 and 2024. (2 points.)

   (d) The irrational numbers and the rational numbers (2 points.)

3. The following principles give conflicting answers to cardinality questions:

   The Proper Subset Principle Suppose A is a proper subset of B. Then A and B are *not* of the same size: B has more members than A.

   The Bijection Principle Set A has the same size as set B if and only if there is a *bijection* from A to B.

   For each of the questions below determine which of the following answers is correct: "yes", "no", or "not determined".

   (a) Are $\{1992, 1993, 2019\}$ and $\{1992, 1993, 2019, 2024\}$ of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

   (b) Are $\{0, 1, 2\}$ and $\{1, 2, 3, 4\}$ of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

   (c) Are the set of prime numbers and the set of natural numbers of the same size, according to the Proper Subset Principle? Are they of the same size according to the Bijection Principle? (3 points.)

   Reminder: Although you'll need to think about the Proper Subset Principle for the purposes of this question, it won't be relevant for the rest of the PSet. At least in our PSets, we follow Cantor—and current mathematical practice—in rejecting the Proper Subset Principle and assessing cardinality questions on the basis of the Bijection Principle. Food for thought: *Is this merely a convention?*

4. Every room in Hilbert's Hotel is occupied. New guests show up.

   (a) You're in charge of room assignments, and initially you believe there'll be one new guest for each rational number. But then just before arrival, each new guest invites their parents! Understandably, each pair of parents would prefer to stay in a room adjacent to their child. Can all of these new guests be accommodated without asking anyone to share a room? (3 points.)

   (b) Initially, you think there'll be one new guest for each real number (and you start feeling a bit overwhelmed). Then you hear what seems like good news! The new guests for the real numbers from 2023 to 2024 have decided they have to bail. Should this change how you feel, i.e., can all of these new guests be accommodated without asking anyone to share a room? (3 points.)

Part II (Submit PDF on Canvas: 69 points)

5. Describe a set that contains no integers but has the same cardinality as the set of integers. (For this one, no need to provide a justification) (3 points.)

6. Construct a bijection between the set of integers {...-2, -1, 0, 1, 2, ...} and the set of squares of integers $\{0, 1, 4, 9, 16, \ldots\}$. (9 points)

7. The Cantor-Schroeder-Bernstein Theorem states that if there is an injection from A to (a subset of) B and an injection from B to (a subset of) A, then there is a bijection from A to B.

   (a) Construct an injection from $\mathbb{Z}$ to the set of prime numbers $\{2, 3, 5, 7, \ldots\}$. (10 points) (You may assume that whenever $p_1, \ldots, p_n$ are primes, there is a smallest prime greater than each of $p_1, \ldots, p_n$.)

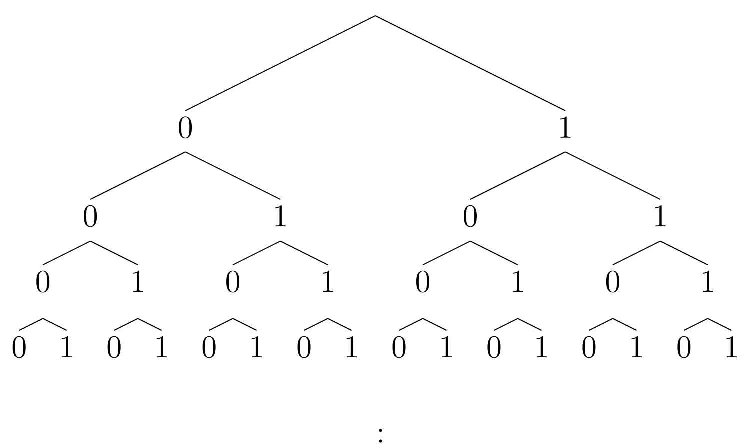
   (b) Use the Cantor-Schroeder-Bernstein Theorem to prove that there is a bijection from $\mathbb{Z}$ to the set of prime numbers. (5 points)

8. Let $S = \{f : f \text{ is a function from the natural numbers to } \{\oplus, \odot, ☾\}\}$ where $\oplus$ is the Earth, $\odot$ is the Sun, and ☾ is the Moon. An example of a member of S is the function $g : \mathbb{N} \to \{\oplus, \odot, ☾\}$ such that:

$$g(n) = \begin{cases} \oplus \text{ if } n \text{ is a power of seven} \\ \odot \text{ otherwise} \end{cases}$$

   Prove that there cannot be a bijection from the set of natural numbers to S. (10 points)

9. Consider the following infinite tree:



   (When fully spelled out, the tree contains one row for each natural number. The zero-th row contains one node, the first row contains two nodes, the second row contains four nodes, and, in general, the nth row contains 2^n nodes.)

(a) Is there a bijection between the *nodes* of this tree and the natural numbers? Don't forget to justify your answer! (10 points)

(b) Is there a bijection between the *paths* of this tree and the natural numbers? A path is an infinite sequence of nodes which starts at the top of the tree and contains a node at every row, with each node connected to its successor by an edge. (Paths can be represented as infinite sequences of zeroes and ones.) (10 points; don't forget to justify your answers!)

10. Let $\mathscr{P}^\star(A)$ be the set of **non-empty** subsets of A.

(a) Show that the following analogue of Cantor's Theorem is false: for any set A, $|A| < |\mathscr{P}^\star(A)|$. (2 points)

(b) Show that the following analogue of Cantor's Theorem is true: for any set A with two or more members, $|A| < |\mathscr{P}^\star(A)|$. (10 points)

Notation: $|A|$ is the cardinality of A and $|A| < |B|$ means that there is an injection from A to B but no bijection from A to B.