Infinite Cardinalities

PARADOX AND INFINITY
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Cardinality Principles

Bijection Principle: $|A| = |B|$ *iff* $A \simeq B$.

`Reflexive:` $A \simeq A$.
`Symmetric:` if $A \simeq B$, then $B \simeq A$.

* What's an inverse of a relation?
* Do functions always have inverses?
* Observe: the inverse of a bijection is a bijection.

`Transitive:` if $A \simeq B$ and $B \simeq C$, then $A \simeq C$.

Observe: We get equivalence classes but no ordering.

Injection Principle: $|A| \leqslant |B|$ *iff* $A \simeq C$ for some $C \subseteq B$.

`Reflexive:` $|A| \leqslant |A|$.
`Transitive:` if $|A| \leqslant |B|$ and $|B| \leqslant |C|$, then $|A| \leqslant |C|$.
`Anti-Symmetric:` if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.

Cantor-Schroeder-Bernstein Theorem: If there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

`Total:` $|A| \leqslant |B|$ or $|B| \leqslant |A|$. (Requires the Axiom of Choice)

Could Define: $|A| = |B|$ *iff* $|A| \leqslant |B|$ and $|B| \leqslant |A|$.
$|A| < |B|$ *iff* $|A| \leqslant |B|$ and $|B| \nleqslant |A|$.

Countably Infinite

Countable: A set A is *countable iff* $|A| \leqslant |\mathbb{N}|$.
Infinite: A set A is *infinite iff* $|\mathbb{N}| \leqslant |A|$.

- $\mathbb{N}_m$ is countably infinite since $f(n) = n + m$ is a bijection.
- $\mathbb{N}_{(m)}$ is countably infinite since $f(n) = n \times m$ is a bijection.
- $\mathbb{Z}$ is countably infinite since there is a bijection $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{otherwise.} \end{cases}$.
- The positive rational numbers $\mathbb{Q}^+$ are countably infinite since:
  - There is an injection from $\mathbb{Q}^+$ to $\mathbb{N}^2$.
  - And $f(\langle n, m \rangle) = 2^n \cdot 3^m$ is an injection from $\mathbb{N}^2$ to $\mathbb{N}$.
  - Hence $\mathbb{Q}^+$ is countable, and so $\mathbb{Q}$ is also countable.
  - Infinite since identity is an injection from $\mathbb{N}$ to $\mathbb{Q}$.

Real Numbers

Real Interval: The real interval $(0,1)$ is uncountably infinite.

1. $|\mathbb{N}_2| \leqslant |(0,1)|$ since $f(x) = 1/x$ is an injection $f : \mathbb{N}_2 \to (0,1)$.
2. $|\mathbb{N}_2| \neq |(0,1)|$ by Cantor's diagonal argument.
3. Thus $|\mathbb{N}_1| < |(0,1)|$.
4. Observe that $g(x) = \pi(x - 1/2)$ is a bijection $g : (0,1) \to (-\pi/2, \pi/2)$.
5. Additionally $\texttt{tan} : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection.
6. By the bijection principle, $|(0,1)| = |(-\pi/2, \pi/2)| = |\mathbb{R}|$.
7. Thus $|\mathbb{N}_2| < |\mathbb{R}|$ where $|\mathbb{N}_2| = |\mathbb{N}|$, so $|\mathbb{N}| < |\mathbb{R}|$.

Cantor's Theorem

Theorem: $|A| < |\wp(A)|$ for any set A where $\wp(A) = \{X : X \subseteq A\}$.

1. $|A| \leqslant |\wp(A)|$ since $f(a) = \{a\}$ is an injection.
2. Assume there is a bijection $f : A \to \wp(A)$.
3. Let $D = \{a \in A : a \notin f(a)\}$.
4. Since $D \subseteq A$, we know that $D \in \wp(A)$.
5. Since f is surjective, $f(d) = D$ for some $d \in A$.
6. But $d \in f(d)$ *iff* $d \in D$ *iff* $d \notin f(d)$.
7. This has the form $P \leftrightarrow \neg P$ which is equivalent to $P \wedge \neg P$.
8. Thus there is no bijection $f : A \to \wp(A)$, and so $|A| \neq |\wp(A)|$.
9. Given the above, $|A| < |\wp(A)|$.

Corollary

Universal Set: There is no set of all sets.

1. Suppose there were a set U of all sets.
2. Since every $X \in \wp(U)$ is a set, $\wp(U) \subseteq U$.
3. So $f(x) = x$ is an injection $f : \wp(U) \to U$.
4. Thus $|\wp(U)| \leqslant |U|$.
5. Moreover, $g(x) = \{x\}$ is an injection $g : U \to \wp(U)$.
6. So $|U| \leqslant |\wp(U)|$.
7. Thus $|U| = |\wp(U)|$.
8. By Cantor's Theorem, $|U| < |\wp(U)|$, so $|U| \neq |\wp(U)|$.
9. Hence there is no set U of all sets, so no set of everything!

Axioms and Intuitions

Continuum Hypothesis: There is no set A where $|\mathbb{N}| < |A| < |\mathbb{R}|$.

Independent: Adding CH or its negation to ZFC is consistent if ZFC is consistent.

- Is it up to us to choose?
- Neither intuition nor mathematical practice seems to decide the issue.

Compare: Gödel showed that ZFC is consistent if ZF is consistent.

Axiom of Choice: Every set of sets X has a function f where $f(Y) \in Y$ for all $Y \in X$.

Well-Ordering Theorem: Every set X can be well-ordered (its subsets all have least elements).

- AC and WOT are equivalent, intuitive, and extremely useful.
- Not so for CH!