Self Reference

Paradox and Infinity
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From Cantor to Russell

Cantor's Theorem: Recall the proof that $|A| \neq |\wp(A)|$.

- Assume there is a bijection $f : A \to \wp(A)$.
- Let $D = \{a \in A : a \notin f(a)\}$.
- Since $D \subseteq A$, we know that $D \in \wp(A)$.
- Since f is surjective, $f(d) = D$ for some $d \in A$.
- But $d \in f(d)$ *iff* $d \in D$ *iff* $d \notin f(d)$.
- This has the form $P \leftrightarrow \neg P$ which is equivalent to $P \wedge \neg P$.
- Thus there is no bijection $f : A \to \wp(A)$, and so $|A| \neq |\wp(A)|$.

Universal Set: There is no set of all sets.

- Suppose there were a set U of all sets.
- Consider the identity map $f : U \to U$.
- Let $R = \{a \in U : a \notin f(a)\}$.
- Since $R \in U$, we may ask whether $R \in R$.
- But $R \in R$ *iff* $R \notin f(R)$ *iff* $R \notin R$.
- Hence there is no set U of all sets.

Burali-Forti Paradox

Ordinals: There is no set of all ordinals.

- Suppose there were a set Ω of all ordinals.
- Ω is set-transitive: if $x \in \Omega$ and $y \in x$, then $y \in \Omega$.
- Ω is well-ordered: if $X \subseteq \Omega$, then some $y <_o x$ for all $x \in X$.
  - If x and y are ordinals, then $x <_o y$ or $y <_o x$.
  - Ordinals contain all of their predecessors.
- So Ω is an ordinal, and hence $\Omega \in \Omega$, and so $\Omega <_o \Omega$.
- But $x \nless_o x$ for any ordinal x.
- Or, observe that $\Omega <_o \Omega'$ where $\Omega' = \Omega \cup \{\Omega\}$.
- Hence Ω does not include all ordinals.

Properties Paradox

Horse: The property *being a horse* is not a horse, i.e., does not instantiate itself.

Property: The property *being a property* is a property, i.e., instantiates itself.

Paradox: Let P be the property of not instantiate itself, i.e., $P(X) \coloneqq \neg X(X)$.

- But then $P(P)$ *iff* $\neg P(P)$.
- $\exists Y[\forall Z(Z = Y \leftrightarrow \forall X[Z(X) \leftrightarrow \neg X(X)]) \wedge Y = P]$.

Universal Liar

Liar: The proposition that *Liar* expresses is false.

- If the *Liar* is true, then by its own lights it is false.
- If the *Liar* is false, then by its own lights it is true.

Analysis: $\exists\varphi(\forall\psi[\texttt{Expresses}(\mathit{Liar}, \psi) \leftrightarrow \varphi = \psi] \wedge \neg\varphi)$.

Nonexistence?

Response: Isn't the most natural response to just deny that there is a set R, or property P, or proposition expressed by *Liar*.

Ad Hoc: Need to explain why there is no such set, property, or proposition.

Proposition: Why doesn't *Liar* express a proposition?

- Can't simply appeal to paradox to explain its nonexistence.

Properties: Why isn't there such a property as P?

- Seems like most properties have this property, e.g., *being a horse*.

Sets: Why isn't there a Russell set R?

- All sets do not belong to themselves, and there is no set of all sets.

Vicious Circle Principle

Diagnosis: "No totality can contain members defined in terms of itself."

- Want something that explains all of the "reflexive paradoxes."

Take Two: "Whatever contains an apparent variable must not be a possible value of that variable."

- $R \coloneqq \{x : x \notin x\}$ i.e., $\exists X(R = Y \wedge \forall Y[Y = X \leftrightarrow \forall z(z \in Y \leftrightarrow z \notin z)])$.

Types: "Whatever contains an apparent variable must be of a different type from the possible values of that variable…"