Type Theory

Paradox and Infinity
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Reals from the Rationals

Construction: Dedekind aimed to construct the real numbers to ground analysis.

Definition: A *real number* is any nonempty proper subset $X \subset \mathbb{Q}$ where:

1. X is downward closed, i.e., $x \in X$ whenever $y \in X$ and $x < y$.
2. X has no greatest element, i.e., for every $x \in X$, there is some $y \in X$ where $x < y$.

Roots: Let $\sqrt{2} = \{x \in \mathbb{Q} : x^2 < 2\}$.

Higher-Order: The real numbers are of higher-order than the rationals.

Subsumption: The rationals are *subsumed* by the reals $\frac{2}{5} = \{x \in \mathbb{Q} : x < \frac{2}{5}\}$.

Greatest Lower Bounds

Ordering: For $x, y \in \mathbb{R}$, $x \leqslant y$ *iff* $x \subseteq y$.

Lower Bound: x is a *lower bound* of $X \subseteq R$ *iff* $x \leqslant y$ for all $y \in X$.

Greatest Lower Bound: x is a *greatest lower bound* of $X \subseteq R$ *iff* x is a lower bound of X and $x \geqslant y$ for any lower bound of y of X.

- The definition of the glb of X quantifies over real numbers in X.
- Mathematics is full of definitions of properties of every higher-order.
- We want to quantify over all properties of real numbers at once.

Properties: '$P(6)$' vs. '$6 \in \pi$' where $\pi = \{x \in \mathbb{N} : P(x)\}$.

Type Restrictions: "Whatever contains an apparent variable must be of a different type from the possible values of that variable..."

Predicative Properties

Typed Expresions: Recall the recursive clauses from last time:

- If $(t_1^{o_1}, \ldots, t_n^{o_n})^a$ is predicative and $\varphi_1 : t_1^{o_1}, \ldots, \varphi_n : t_n^{o_n}$ for expressions $\varphi_1, \ldots, \varphi_n$, then $z(\varphi_1, \ldots, \varphi_n) : ((t_1^{o_1}, \ldots, t_n^{o_n})^a, t_1^{o_1}, \ldots, t_n^{o_n})^{a+1}$.
- If $\varphi : (t_1^{o_1}, \ldots, t_n^{o_n})^a$, then $\forall x : t_i^{o_i} \varphi : (t_1^{o_1}, t_{i-1}^{o_{i-1}}, \ldots, t_{i+1}^{o_{i+1}}, t_n^{o_n})^a$.

Stratification: Quantification is stratified by ramified types.

Predicative Types: t^o is *predicative* if o is as low as it can be, c.f., $(0^0)^1$ and $(0^0)^2$.

Axiom of Reducibility

Axiom of Reducibility: "[E]very propositional function is equivalent, for all its values, to some predicative function." (pp. 242-3)

- Want all properties of real numbers to be on a par.
- Russell assumes there always are equivalent predicative properties.
- What is the property *being the glb of X* equivalent to?
- Neither Russell nor anyone else can say.

Construction: This undermines the spirit of Dedekind's project.

- The aim is to construct the numbers (ultimately from the empty set).
- But Russell was a logicist, not an constructivist/intuitionist.
- Logicists have a much more realist conception of mathematics where logic describes objective universal principles (the laws of thought) and mathematics reduces to logic.
- Nevertheless, one might worry that the AR is not logical, i.e., it is thinkable (consistent) for it to be false.

Mathematics or Metaphysics

Planets Example: Let Hesperus and Phosphorus have the same first-order properties.

- Hesperus is shining *iff* Phosphorus is shining, etc.
- Could they differ in higher-order properties?
- Maybe someone loves Hesperus but no one loves Phosphorus?
- There is a relation someone bears to Hesperus but not to Phosphorus?
- Given that Hesperus is Phosphorus, this might seem unlikely.

Iron Spheres: Consider two iron spheres that have all the same properties.

- Could there be a higher-order property upon which they differ?

Logic: Logic is concerned with what must be the case.

- By 'must' we mean: it would be contradictory for it to not be the case.
- Compare the axioms of set theory which are contingent.
- Must the axiom of reducibility hold?
- Many have thought it contingent (e.g., Ramsey and Wittgenstein).

Logicists: Ensuring AR is a truth of logic is required for logicism.

- Reducing math to logic is an ambitious program.
- Neither logicists and intuitionists can easily accommodate AR.