24.118: Paradox and Infinity, Spring 2024
Problem Set 3: Omega-Sequence Paradoxes

Please include your name and list all of your collaborators on the last page of your problem set, preferably a separate page so that no one sees it till the end. *Failing to list collaborators constitutes a violation of academic integrity.*

How your answers will be graded:

- There is no quiz this week so all answers will be written and submitted as a PDF on Canvas.
- You must justify your answers unless stated otherwise in the problem. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on your justification. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- No answer may comprise more than 180 English words. Longer answers will be penalized. However, showing your work in a calculation or proof does *not* count toward the word limit.
- You may consult published literature and the web, but you must credit all sources where failure to do so constitutes plagiarism and can have serious consequences. For advice about how and when to credit sources see `https://integrity.mit.edu/`. Note that merely citing a source does *not* count as a good justification.

All submissions must be in PDF format. Type-written submissions may be strongly preferred by your TA; handwritten submissions are acceptable only if:

1. Your handwriting is easily legible (as judged by your TA);
2. You produce a clean version of the document (as opposed to the sheet of paper you used to work out the problems); and
3. Your manuscript has been scanned to high enough standards (as judged by your TA). Consider using, e.g. *Scannable* or *Adobe Scan*.

Note: A murky problem can be addressed in more or less thoughtful ways. Good answers will reveal that you've understood the complexity of the underlying terrain and that you've given the issue serious thought.

1. Lazy wants to run from A to B, but he likes to take one-second breaks. He first stops halfway between A and B and takes a one-second break. He then stops halfway between that point and B and takes a one-second break, and so on. (More generally: for each $k \geq 1$, Lazy takes a break at a distance of $(B - A)/2^k$ from B.) Assume that the traveling itself takes Lazy no time at all. Is there a positive integer n such that after n seconds Lazy has reached point B? If so, what is it? (10 points)

2. There are two prisoners in a room. They both close their eyes and each of them is approached by a guard. Each guard flips a fair coin. If the coin lands Heads, she gives her prisoner a red hat; if it lands Tails, she gives her prisoner a blue hat. Once the prisoners have been assigned hats, they are both allowed to open their eyes.

   As soon as they each see the color of the other's hat (but not the color of their own hat), the prisoners are taken into separate rooms and are unable to communicate with one another. At that point, they are each asked to name the color of their hat.

   - if at least one of the prisoners answers correctly, they will both be set free;
   - otherwise, they will both remain in prison. Note that both MUST answer.

   (a) Find a strategy that the prisoners can agree upon ahead of time which guarantees that they are both set free. Make sure your strategy is *deterministic*: it must determine a definite outcome for each prisoner, given the prisoner's situation at the time of his decision. (8 points)

   (b) Given that the prisoners have no access to information about the color of their own hats, and that the colors were chosen using independent coin tosses, what explains the possibility of a deterministic strategy that brings the prisoners' chance of freedom above 50%? (12 points)

3. Imagine an island on which everyone is either a knight or a knave. Knights only assert truths, knaves only assert falsehoods.

   For each of the scenarios below, determine whether one can settle the question of whether S_0 is a knight or a knave. If the answer is "yes", state whether S_0 is a knight or a knave and explain how you know this; if the answer is "no", explain why.

   (a) Ten islanders, $S_0, S_1, \ldots, S_9$, are lined up. S_0 is at the back; in front of her is S_1; in front of him is S_2; and so on. Each islander says: "There is at least one person in front of me, and everyone in front of me is a knave." (10 points)

   (b) An ω-sequence of islanders, $S_0, S_1, S_2, \ldots$, are lined up. S_0 is still at the back of the line; in front of her is S_1; in front of him is S_2, and so on. Each islander says: "There is at least one person in front of me, and everyone in front of me is a knave." *You may go over the word limit here*! (14 points)

   (c) As in the previous case, but this time each islander says: "There is at least one person in front of me, and everyone in front of me is a knight." (14 points)

- Assassin 1 will kill Oscar if and only if Oscar is still alive when any part of his body crosses the threshold of the room by more than 1 mm.
- Assassin 2 will kill Oscar if and only if Oscar is still alive when any part of his body crosses the threshold of the room by more than 1/2 mm.

$$\vdots$$

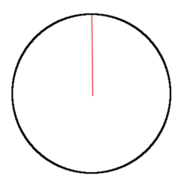
- Assassin n will kill Oscar if and only if Oscar is still alive when any part of his body crosses the threshold of the room by more than $1/n$ mm.

$$\vdots$$

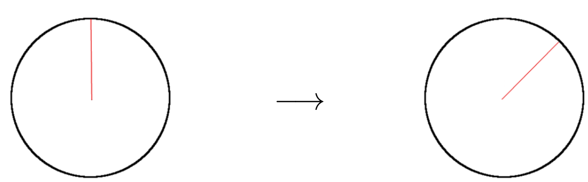
All assassins kill instantly if they kill at all. On the assumption that Oscar can only die if he's killed by someone, is it possible for him to cross the threshold? Possible for him to cross the threshold and remain alive? If not, what stops him? (10 points; don't forget about the word limit.)

4. The following questions are intended to test your understanding of continuity assumptions. At least one of them might get into "murky" territory. If you think there is a *determinate way* the wheel will look or direction a line will point, explain why. If instead you think that the way the wheel will look or where a line will point is *undetermined* by the scenario described, explain why.

   (a) You have a wheel with a radius of 1 unit. You draw a red line going from the center of the wheel to the twelve o'clock position:

   You then rotate the wheel, in steps. At each step, you rotate the wheel one radian clockwise, so that the outermost point of the red line travels 1 unit around the perimeter of the wheel. Here is the first rotation:

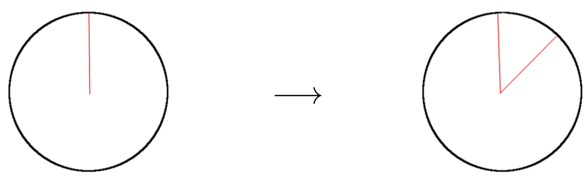
   

   Because the tip of the red line travels 1 unit around the perimeter after each rotation, and because the perimeter of the wheel is 2π units (which is an irrational number of units), there is no positive integer n such that, after n steps, the red line returns to a position it had occupied before.

Suppose you perform this operation infinitely many times, once for each positive integer. At noon the red line is at the twelve o'clock position. At 12:30 you rotate the wheel one radian clockwise. At 12:45 you rotate the wheel another radian clockwise. And so forth: for each $n \geq 1$, you perform the nth rotation $\frac{1}{2^n}$ hours before 1pm.

Question: in which direction will the red line be pointing at 1pm? Don't forget to justify your answer. (10 points)

(b) Consider a variant of the case in part (a). As before, you start by drawing a red line from the center of the wheel to the twelve o'clock position. But this time you draw new lines, in steps. At each step, you draw a red line at a one radian angle from the last line you drew. At noon you draw the initial red line, pointing to the twelve o'clock position. At 12:30 you draw an additional line, a radian away in the clockwise direction. At 12:45 you draw an additional line, a radian away in the clockwise direction. And so forth: for each $n \geq 1$, you draw a line $\frac{1}{2^n}$ hours before 1pm. Here are the first two steps of the process:



At 1pm, you're done drawing lines on the wheel. Note that there's a well-defined fact of the matter about what the wheel is like. Namely: a radial line r will be colored red if and only if, for some number n, r is exactly n radians away from twelve o'clock, going clockwise.

The next step is to rotate the wheel. Consider the result of rotating the wheel one radian in the clockwise direction. It'll look exactly the way the unrotated wheel would have looked if you had skipped drawing the 12pm line. And if you rotate the wheel one radian further, for a total rotation of two radians, it'll look exactly the way the unrotated wheel would have looked if you had skipped drawing the 12pm and 12:30pm lines. And so forth: after a total of n rotations, the wheel will look exactly the way the original unrotated wheel would have looked if you had skipped drawing all lines up to and including the 1pm$-\frac{1}{2^{n-1}}$ hours line.

Suppose that you rotate the wheel infinitely many times, once for each positive integer. At 1pm you rotate the wheel one radian in the clockwise direction. At 1:30 you rotate the wheel another radian in the clockwise direction. Again at 1:45. And so forth. For each $n \geq 0$, you perform the nth rotation $\frac{1}{2^n}$ hours before 2:00pm.

Question: What does the wheel look like at 2pm? Assume that the only way for a line to disappear is for someone to erase it. (12 points)

(c) Suppose you start by drawing lines on the wheel as in part (b). But this time you perform no rotations. Instead, you *erase* red lines using the following procedure. At 1:00pm you erase the red line pointing to the twelve o'clock position. At 1:30 you erase the line pointing one radian in the clockwise direction from the last line erased. And so forth: for each $n \geq 1$, you erase the line one radian away in the clockwise direction from the last line erased.

Question: Is there a difference between the way our wheel looks at 2pm and the way the wheel in the previous exercise looked at 2pm? Assume that the only way for an erased line to reappear is for someone to redraw it. (10 points)