Omega Sequences

PARADOX AND INFINITY
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Ordinals into Obscurity?

Definition: Let $\mathfrak{B}_\alpha = \begin{cases} \mathbb{N} & \text{if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta) & \text{if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} & \text{otherwise.} \end{cases}$

Obscurity: But all of this is only studied within set theory.

- Neither standard mathematics nor the sciences need ordinals beyond ω nor cardinalities beyond the continuum.
- But there are puzzles that arise even for ω sequences.
- Simplest case of the infinite worth exploring.

Motivations: Why care about all of this?

- Because we can.
- Because it's awesome (in the religious sense).
- We are probing the limits of what is thinkable, not useful.

Zeno's Analysis

Dichotomy Paradox: "That which is in locomotion must arrive at the half-way stage before it arrives at the goal." – Aristotle, Physics VI:9, 239b10

Infinite Task: Doing infinitely many tasks, each taking a non-zero amount of time.

- Some infinite tasks are cannot be performed in a finite amount of time.

Question: What about infinite tasks with strictly decreasing times for each task?

- The harmonic series is a counterexample: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- $H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots \geqslant 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \ldots = \frac{1}{2} + H.$

Super Task: An infinite task that is performed in finite time.

- Example: walking across the room since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1.
- $f(1) = \frac{1}{2}, f(2) = \frac{3}{4}, f(3) = \frac{7}{8}, f(4) = \frac{15}{16}, \ldots$, so $f(n) = 1 - \frac{1}{2^n}$.
- For any $\epsilon > 0$, there is some $n \in \mathbb{N}$ where $|1 - f(m)| < \epsilon$ for any $m > n$.

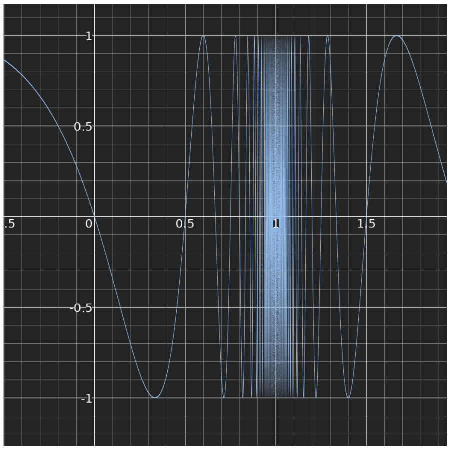
Paradox: So there are super tasks, though this would have surprised Zeno.

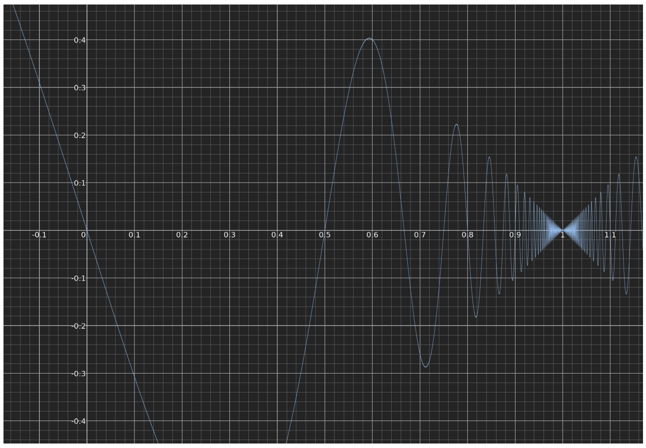
- It took the development of analysis in 17th century to solve.
- Analysis was not put on solid foundations until the 19th century by Bolzano, Cauchy, and Weierstrass.
- Poster child of illuminating paradoxes, but not all are like this.

Thomson's Lamp

Descriptions: 60s (off); 30s (on); 15s (off); ...

- Compare $f(n) = \mathtt{sin}(\frac{\pi}{1-x})$ to $g(n) = \mathtt{sin}(\frac{\pi}{1-x})(1-x)$.





- No continuous function satisfying the description is defined at $n = 1$.
- Subsequences of a convergent sequence converge to the same limit as the original, so neither the above nor $1, -1, 1, -1, \ldots$ converge.

Demon's Game

Setup: "As long as only finitely many of you say *aye*, each of you will receive as many gold coins as there are people who said *aye*."

- Assumes everyone is "optimally rational" and cannot collaborate.
- Is maximizing really "optimally rational" in this scenario?

Individual Version: "If you answer *aye* at most finitely many times, you will receive as many gold coins as the *aye*-answers that you give."

- Assumes no diachronic collaboration between time-slices.
- It is wrong to assume that the will is unable to persist across times.

Video Rental: \$5 to rent, \$2 late fee, but it is always worth it to Daniel to pay the fee.

- This is not a paradox, just a problem that Daniel has.
- Also a problem for a theory of rationality that takes Daniel to be ideally rational given his preferences at each time.

Buridan's Ass: Compare infinite ever larger bales of hay to the duplicate bale case.

- The ass does not starve, but not because of its sins against rationality.
- We can choose between duplicates/an arbitrary cut-off point.