24.118: Paradox and Infinity, Spring 2024
Response Set 3: Set Theory

Please include your name and list all of your collaborators on the last page of your response set, preferably a separate page so that no one sees it till the end. *Failing to list collaborators constitutes a violation of academic integrity.*

How your answers will be graded:

- There is no quiz this week so all answers are to be submitted by PDF.
- Each response should be a maximum of 250 words.
- Responses will be evaluated by the clarity, concision, and the degree to which they reflect careful reading and comprehension of the text. Begin by working out your thoughts separately as you might with a problem set, writing up your response only once you have a good sense of what you want to say and how you want to say it.
- You are encouraged to read critically rather than accepting every word of a text as true. At the same time, it is important to be charitable to what the author intends. When something in the text is less than clear, it is important to do your best to get clarity about what the author intends to express, assuming the best interpretation consistent with the text before delivering any criticisms. For instance, saying that something in the text is not clear is not a good criticism even if it is true.
- You may consult published literature and the web, but you must credit all sources where failure to do so constitutes plagiarism and can have serious consequences. For advice about how and when to credit sources see `https://integrity.mit.edu/`. Note that merely citing a source does *not* count as a good justification.
- Reading old texts is hard! So don't worry if you struggle to understand every point. Instead, do your best to learn what the author is trying to express, using this as a chance to sharpen your close reading skills.

All submissions must be in PDF format. Type-written submissions may be strongly preferred by your TA; handwritten submissions are acceptable only if:

1. Your handwriting is easily legible (as judged by your TA);
2. You produce a clean version of the document (as opposed to the sheet of paper you used to work out the problems); and
3. Your manuscript has been scanned to high enough standards (as judged by your TA). Consider using, e.g. *Scannable* or *Adobe Scan*.

1. Here is a logic-to-mathematical-English dictionary where φ and ψ are to be replaced with sentences (possibly including variables) and $\ulcorner\alpha\urcorner$ names the expression α:[1]

   - '$(\exists x)$__' reads 'there is some x where__'.
   - '(x)__' reads 'every x is such that__'.
   - 'Sx' reads 'x is a set'.
   - '$x \in y$' reads 'x is a member of y'.
   - '$x = y$' reads 'x is identical to y'.
   - $\ulcorner\sim\varphi\urcorner$ reads $\ulcorner$it is not the case that $\psi\urcorner$.
   - $\ulcorner\varphi \,\&\, \psi\urcorner$ reads $\ulcorner\varphi$ and $\psi\urcorner$.
   - $\ulcorner\varphi \vee \psi\urcorner$ reads $\ulcorner\varphi$ or $\psi\urcorner$.
   - $\ulcorner\varphi \rightarrow \psi\urcorner$ reads $\ulcorner$if φ then $\psi\urcorner$.
   - $\ulcorner\varphi \leftrightarrow \psi\urcorner$ reads $\ulcorner\varphi$ if and only if $\psi\urcorner$.

   Given the readings above, provide a translation of the following formal sentence after first replacing φ with the sentence 'x is a natural number':

   (a) $(\exists y)(Sy \,\&\, (x)(x \in y \leftrightarrow \varphi))$.

   Even the naive conception of set restricts the sentences that we may substitute for 'φ' by forbidding the variable 'y' to occur free.[2] What could go wrong if 'y' were permitted to occur free in the sentence φ that we substitute in (a)? Note that φ is permitted to include the variable 'x'. What sort of trouble does this create for the naive theory given that it asserts all instances of (a) where 'y' is not free in φ?

2. In considering whether there is a set of all sets, Boolos (1971) writes:

   > Of course a set can and must include itself (as a subset). But *contain* itself? Whatever tenuous hold on the concepts of *set* and *member* were give none by Cantor's definitions of 'set' and one's ordinary understanding of 'element', 'set', 'collection', etc. is altogether lost if one is to suppose that some sets are members of themselves. (p. 119)

   What does Boolos say in defense of this claim? Why does he also believe that there can't be sets that belong to each other? Do you find his arguments convincing? Which is harder to believe: (a) some sets are members of themselves; or (b) there is no set of all sets? Briefly explain what supports your conclusion.

[1] You do not need to know how $\ulcorner\alpha\urcorner$ differs from 'α' to succeed at this translation exercise but $\ulcorner\cdot\urcorner$ is a function from expressions to names for expressions, and 'α' is a variable for expressions. By contrast, quotes name whatever is inside them, e.g., 'α' names the first letter of the Greek alphabet, not the value of the variable α. You are also encouraged to adapt your translation to be natural sounding in English, rephrasing 'is such that', 'it is not the case that', etc., while preserving the same meaning.

[2] For instance, 'y' is *free* in 'y admires Cantor' but *bound* and so not free in 'every y is such that y admires Cantor'. For this example, it is fine to restrict consideration to simple sentences which do not include quantifiers like 'every y'. Every variable in a sentence without quantifiers is free.

3. Boolos (1971) describes the iterative conception of set in three parts, writing:

   > The first is a rough statement of the idea. It contains such expressions as 'stage', 'is formed at', 'earlier than', 'keep on going', which must be exorcised from any formal theory of sets. From the rough description it sounds as if sets were continually being created, which is not the case (p. 229)

   Setting these convenient ways of speaking to one side, in what sense are some sets "earlier than" others if they are not earlier in time?

4. Boolos (1971) claims towards the end of his paper:

   > The axiom of extensionality enjoys a special epistemological status shared by none of the other axioms of ZF. (p. 229)

   Do you share in his finding. If so, what convinces you that the axiom of extensionality enjoys this special status? If not, explain what gives rise to your doubts.

References

Boolos, George. 1971. "The Iterative Conception of Set." *The Journal of Philosophy* 68:215–231. ISSN 0022-362X. doi: 10.2307/2025204.