The Higher Infinite

PARADOX AND INFINITY
Benjamin Brast-McKie
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The Ordinals

Cardinalities: Observe that $|\omega| = |\omega + 1| = |\omega + 2| = \ldots = |\omega + \omega| = \ldots$

- This is just Hilbert's hotel all over again.

Structure: But $\omega \neq \omega + 1 \neq \omega + 2 \neq \ldots \neq \omega + \omega \neq \ldots$

- Ordinals have more structure than is encoded by their cardinalities.
- 2 represents the class of two things where one is *after* the other.

Order: The ordinals have order structure— they are *well-ordered*.

- α and β have the same order-type *iff* $\langle \alpha, <_o \rangle \cong \langle \beta, <_o \rangle$.
- Even though $\omega \simeq \omega + 1$, we don't get that $\langle \omega, <_o \rangle \cong \langle \omega + 1, <_o \rangle$.
- Moreover, $1 + \omega = \omega$ even though $\omega + 1 \neq \omega$.

Sequences: If we care about numbers, why introduce the ordinals at all?

- Isn't this a bate and switch?
- Why do we need transfinite order structure?

Ordinals and Algorithms

Algorithms: Programs/instructions to add 1 some (ordinal) number of times.

- Roughly: '3' says 'add 1 three times'.
- Note that we use 'three' to say what '3' means.

First Pass: $3 + 2 = 0''' + 0'' = 0'''' + 0' = 0''''' + 0 = 0''''' = 5$.

- $\alpha + \beta' = \alpha' + \beta$.
- Doesn't generalize since $\omega + \omega = \omega' + ?$

Adding Limit Ordinals: How should we think about adding 1 $\omega + \omega$ times?

- What if we could parallel process (add)?
- Then we would get $\omega + \omega = \omega$.
- Need to preserve order and "parallel processing" ignores order.

Simpler Case: $\omega + 1 \neq 1 + \omega$.

- Think of the simplest case you can to exhibit some feature.
- Not always the first case you might think of.

Conclusion: If we ignore order, we are back to cardinality.

Ordinal Arithmetic

Addition: $\alpha + 0 = \alpha$
$\alpha + \beta' = (\alpha + \beta)'$
$\alpha + \lambda = \bigcup\{\alpha + \beta : \beta <_o \lambda\}$ where $\lambda \neq 0$ is a limit ordinal.

Examples: $3 + 2 = 3 + 1' = (3 + 1)' = (3 + 0')' = (3 + 0)'' = 3'' = 5$.
$\omega + \omega = \bigcup\{\omega + 0, \omega + 1, \omega + 2, \ldots\} = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$.
$1 + \omega = \bigcup\{1 + 0, 1 + 1, 1 + 2, \ldots\} = \{0, 1, 2, \ldots\} = \omega$.

Multiplication: $\alpha \times 0 = 0$
$\alpha \times \beta' = (\alpha \times \beta) + \alpha$
$\alpha \times \lambda = \bigcup\{\alpha \times \beta : \beta <_o \lambda\}$ where $\lambda \neq 0$ is a limit ordinal.

Examples: $\omega \times 2 = (\omega \times 1) + \omega = ((\omega \times 0) + \omega) + \omega = (0 + \omega) + \omega = \omega + \omega$.

$$\omega \times \omega = \bigcup\{\omega \times 0, \omega \times 1, \omega \times 2, \ldots\}$$

$$= \begin{cases} 0, 1, \ldots \\ \omega, \omega + 1, \ldots \\ \omega + \omega, (\omega + \omega) + 1, \ldots \\ \quad \vdots \quad , \quad \vdots \quad , \quad \vdots \quad , \end{cases}$$

"Small" Cardinals

Recall: $|\mathbb{N}| < |\wp(\mathbb{N})| < |\wp^2(\mathbb{N})| < \ldots < |U|$.

- Remember that '$<$' means 'injection but no bijection' here.
- We can use ω to define U.

Definition: Let $\mathfrak{B}_\alpha = \begin{cases} \mathbb{N} & \text{if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta) & \text{if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} & \text{otherwise.} \end{cases}$

Example: $\mathfrak{B}_\omega = \bigcup\{\mathbb{N}, \wp(\mathbb{N}), \wp^2(\mathbb{N}), \ldots\} = U$.

Uncountable Ordinals

Countable Ordinals: We have only seen countable ordinals so far.

Beth: Let $\beth_\alpha = \beta$ *iff* $|\beta| = |\mathfrak{B}_\alpha|$ and $\beta <_o \gamma$ for all γ where $|\gamma| = |\mathfrak{B}_\alpha|$.

- $\beth_0 = \omega$ since $|\omega| = |\mathfrak{B}_0| = |\mathbb{N}|$.
- We can then make big ordinals to make even bigger cardinals, etc.

Motivations: Why care about all of this?

- Because math (is awesome).
- We are probing the limits of what is thinkable, not useful.