Set Theory

Paradox and Infinity
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Motivations

Dialectic: Recall Ramsey's simple theory of types.

- Sets are replaced with properties.
- '$x \in x$' is treated as '$x(x)$' which cannot be typed.
- If '$\in$' is intelligible in its own right, the paradox remains.
- Set theory is more intuitive than simple type theory.
- Worth developing in place of or alongside type theory.

Naive Set Theory

Language: $\forall, \exists, \neg, \vee, \wedge, \rightarrow, \leftrightarrow, =, \in, S, x_1, x_2, \ldots, (,)$ are primitive symbols.

- For purposes of illustration we may add other predicates and names.
- But strictly speaking, the language of set theory is very austere.

Comprehension: Every open sentence with one free variable corresponds to a set.

- $\exists y \forall x(x \in y \leftrightarrow \varphi)$ where 'y' does not occur in 'φ'.
- From open sentences to predicates *vs.* definite descriptions.
- Sets are objects not properties.

Question: Why assume uniqueness.

Extensinoality: Sets are defined by their members.

- $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$.
- The set of fish that walk is identical to the set of pigs that fly.
- Properties need not be identified with sets.
- Set theory restricts attention to the extensions of predicates.

Russell's Paradox

Russell Set: The open sentence '$x \notin x$' can be used to define a set.

- $R \coloneqq \{x : x \notin x\}$, i.e., $\exists y \forall x(x \in y \leftrightarrow x \notin x)$.
- $R \in R$ *iff* $R \notin R$.
- Restricting comprehension looks *ad hoc*.

The Iterative Conception of Set

Intuitions: No reason to expect our intuitions to be univocal.

- Lots of reasons to expect otherwise, e.g., *same number as*, etc.
- First impressions often have to be revised.

Extensinoality: Compare the following metaphysical theses.

- *What it is to be* (identical to) a set is to have the members it has.
- *What it is for* a set to exist is for its members to exist.
- Sets *ontologically depend* on their members: part of what it is for a set to exist is for all of its members to exist.
- Put otherwise: for a set to exist it is *necessary for* its members to exist.
- But then sets can't be members of themselves.

Sufficiency: Is the existence of some entities *sufficient for* a set to exist?

- Do you have to "put a lasso around" some entities to make a set?
- Intuitionists of a certain stripe might claim so.
- What about the empty set? Is it a product of our conceptual exertion.
- Platonists reject this, taking sets to exist objectively and necessarily.
- Existence of the members to be sufficient for the existence of a set.

Separation Axiom

Construction: Sets are constructed not in time, but in nature.

- The ingredients precede the product in constitution.
- Given any things, we have a set from which we can build new sets.

Comprehension: $\forall z \exists y \forall x (x \in y \leftrightarrow (y \in z \wedge \varphi))$, i.e., some $k \coloneqq \{x \in z : \varphi\}$ for any set z.

- What of Russell's paradox?
- For any set z, there is a set $R_z \coloneqq \{x \in z : x \notin x\}$.

Indefinite Extensibility: Whence the contradiction?

- Assume $R_z \in z$ for contradiction.
- Then $R_z \in R_z$ *iff* $R_z \in z$ and $R_z \notin R_z$ *iff* $R_z \notin R_z$.
- So $R_z \notin z$.
- Letting $z' \coloneqq z \cup \{R_z\}$, then $R_{z'} \notin z'$, so $z'' \coloneqq z' \cup R_{z'}$, etc.

Solution: Consider the following conclusions.

- There is no universal set.
- No set belongs to itself and so $R_z = z$ for any set z.
- Every set is indefinitely extensible.