Omega Sequences

PARADOX AND INFINITY
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Paradox Grading Rubric

Grading: Let 0 be low and 10 be high for the following qualities:

- Informative/illuminating.
- Intelligible/salient/compelling on a first take.
- Difficult to analyze/make progress.
- Examined/well-studied.
- Controversial/unsolved.
- Dangerous/risk of devouring one's career.
- Fundamental/influential for other areas.

Filthy Liars

Blackboard: The only sentence on the blackboard in room 32-141 is false.

Self-Reference: This sentence is false.

Pairs: (A) B is true. (B) A is false.

Truth Predicate: $\ulcorner T(\ulcorner A \urcorner)\urcorner$ is a sentence for every sentences A.

Truth Schema: $T(\ulcorner A \urcorner) \leftrightarrow A$.

Fixed Point Lemma: If $\varphi(x)$ has at most x free , $PA \vdash \varphi^\star \leftrightarrow \varphi(\varphi^\star)$ for some $\varphi^\star$.

Liar: Letting $\varphi(x) = \neg T(x)$, then $\vdash L \leftrightarrow \neg T(\ulcorner L \urcorner)$ where $\varphi^\star = L$.

- If $T(\ulcorner L \urcorner)$, then L by the *Truth Schema*.
- So $\neg T(\ulcorner L \urcorner)$ by *Liar*, and so $\bot$.
- If $\neg T(\ulcorner L \urcorner)$, then L by *Liar*.
- So $T(L)$ by *Truth Schema*, hence $\bot$.
- So $T(\ulcorner L \urcorner) \vee \neg T(\ulcorner L \urcorner) \vdash \bot$, and so $LEM \vdash \bot$.

Classical Logic

Excluded Middle: $\vdash A \vee \neg A$ (Every sentence is either true or not true).

Ex Falso Quidlebet: $P, \neg P \vdash Q$.

- EFQ follows from *Disjunction Introduction* and *Disjunctive Syllogism*.
- Dialetheists like Priest even give up *Modus Ponens*.

Self Reference

Claim: The problem arises from self-reference.

Insufficient: Some self-reference is OK.

- I hope this letter finds you well.
- This sentence contains five words.

Not Necessary: The sentence $\neg T(L)$ does not refer to itself.

Infinite Liar

Yablo: Even more trouble without self-reference.

- s_k: s_n is false for every $n > k$.
- s_k is true *iff* s_n is false for every $n > k$.

Finite Sequences: No paradox arises since we reach a semantic bottom.

- But if that bottom is always deferred, we are in trouble.

Solution?

Question: What would a solution look like?

- Restrict language to avoid paradoxes, i.e., deriving contradictions.
- On its own, even a successful restriction would be *ad hoc*.
- Also need to explain why that restriction is in place.

Metalanguages

Truth: We can only define truth for $\mathcal{L}_n$ in a metalanguage $\mathcal{L}_{n+1}$ for $\mathcal{L}$.

- Assume $\mathcal{L}_0$ contains no truth-predicates.
- $\mathcal{L}_{n+1}$ may include $A \leftrightarrow T_n(\ulcorner A \urcorner)$ for each sentence A of $\mathcal{L}_n$.
- T_n does not apply to sentences in $\mathcal{L}_n$, and so no paradox.

Question: Is this a good response for natural languages?

- Truth would then be radically polysemous.
- What if we define super-true as true in any sense?