The Higher Infinite

PARADOX AND INFINITY
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The Continuum Hypothesis

Sizes of Infinity: We have seen that $|\mathbb{N}| < |\wp(\mathbb{N})|$ where $|\wp(\mathbb{N})| = |\mathbb{R}|$.

- $B = \{.b_0b_1b_2\ldots : b_i \in \{0,1\}$ for all $i \in \mathbb{N}\}$.
- $|\wp(\mathbb{N})| = |B| = |B/\{.\bar{0}, .\bar{1}\}| = |(0,1)| = |(-\pi/2, \pi/2)| = |\mathbb{R}|$.
- $|S| = |S \cup A|$ whenever $|\mathbb{N}| \leqslant |S|$ and $|A| \leqslant |\mathbb{N}|$.

Continuum Hypothesis: There is no set A where $|\mathbb{N}| < |A| < |\mathbb{R}|$.

Independence: Adding CH or its negation to ZFC is consistent if ZFC is consistent.

- $ZFC + CH$ is consistent if ZFC is consistent (Kurt Gödel 1940).
- $ZFC + \neg CH$ is consistent if ZFC is consistent (Paul Cohen 1963).

Convention: Is it up to us to choose which we include?

- Neither intuition nor mathematical practice seems to decide the issue.
- Platonism, conventionalism, and pragmatism.

The Axiom of Choice

Axiom of Choice: Every set of sets X has a function f where $f(Y) \in Y$ for all $Y \in X$.

- Gödel (1938) showed that ZFC is consistent if ZF is consistent.
- Cohen (1963) showed that ZF¬C is consistent if ZF is consistent.
- How does AC compare to CH?

Well-Ordering Theorem: Every set X can be well-ordered (its subsets all have least elements).

- AC and WOT are equivalent, intuitive, and extremely useful.
- `Totality`: $|A| \leqslant |B|$ or $|B| \leqslant |A|$ for all sets A and B.
- That `Totality` is equivalent to the WOT is good reason to accept AC.

Orderings

Weak Total Ordering: $\langle X, \leqslant \rangle$ reflexive, anti-symmetric, transitive, and total.

Strict Total Ordering: $\langle X, < \rangle$ asymmetric, transitive, and total.

- The irreflexive kernel of WTO is STO; reflexive closure is the inverse.

Total Well-Ordering: A WTO/STO where every subset has a least element.

The Ordinals

Something from Nothing: $\varnothing$, $\{\varnothing\}$, $\{\varnothing,\{\varnothing\}\}$, $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\},\ldots$

Infinite Succession: $0, 0', 0'', 0''', \ldots$ where taking $n' = n + 1$ makes these look familiar.

Successor: $\alpha' = \alpha \cup \{\alpha\}$.

Successor Ordinal: α is a *successor ordinal iff* $\alpha = \beta'$ for some ordinal β.

- Every ordinal has a successor and contains all of its predecessors.
- And its predecessors contain their predecessors, and so on.

Set-Transitive: For any ordinal α, if $\beta \in \alpha$ and $\gamma \in \beta$, then $\gamma \in \alpha$.

Ordering: $\alpha <_o \beta \coloneqq \alpha \in \beta$.

Question: Are the successor ordinals all of the ordinals there are?

Omega: Let $\omega = \{0, 0', 0''', \ldots\}$ be the smallest set to contain 0 that is closed under the successor operation, i.e, $\alpha' \in \omega$ whenever $\alpha \in \omega$.

- ω is not a successor ordinal.

Question: Is ω an ordinal? What's an ordinal?

Ordinal: α is an *ordinal iff* α is set-transitive and well-ordered by $<_o$.

Key Ideas: Ordinals contain their predecessors and always bottom out.

- Not all ordinals have a greatest predecessor, i.e, are successor ordinals.

Limit Ordinal: α is a *limit ordinal iff* α is an ordinal that is not a successor ordinal.

Continuation: $0, 0', 0'', 0''', \ldots, \omega, \omega', \omega'', \omega''', \ldots$ where $'$ is defined as before.

Question: How shall we write the next limit ordinal?

- $\omega + \omega = \omega \times 0''$ but $0'' \times \omega = \omega$ and $\omega + 0'' \neq 0'' + \omega$.
- $|\omega + \omega| = |\omega|$.

Well-Order Types

Cantor Ordinals: Consider c, $\{c\}$, $\{c,\{c\}\}$, $\{c,\{c\},\{c,\{c\}\}\},\ldots$ where 'c' names Cantor.

Question: Couldn't we repeat all the same tricks, substituting 'c' for '$\varnothing$'?

- What it Dedekind gets jealous and wants a hierarchy? Then Hilbert…

Well-Order Type: Every ordinal in any hierarchy is a well-ordered set.

Isomorphism: Let $\alpha \cong \beta$ *iff* there is a bijection $f : \alpha \to \beta$ such that $\gamma <_a \delta$ just in case $f(\gamma) <_b f(\delta)$ for all $\gamma, \delta \in \alpha$ where $<_a$ orders α and $<_b$ orders β.

Ordinals: The ordinals represent their own well-order type.