24.118: Paradox and Infinity, Spring 2024
Problem Set 2: The Higher Infinite

Please include your name and list all of your collaborators on the last page of your problem set, preferably a separate page so that no one sees it till the end. *Failing to list collaborators constitutes a violation of academic integrity.*

How your answers will be graded:

- In Part I there is no need to justify your answers. Submit answers in a quiz that you will access on Canvas.
- In Part II you must justify your answers unless stated otherwise in the problem. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on your justification. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- No answer may comprise more than 180 English words. Longer answers will be penalized. However, showing your work in a calculation or proof does *not* count toward the word limit.
- You may consult published literature and the web, but you must credit all sources where failure to do so constitutes plagiarism and can have serious consequences. For advice about how and when to credit sources see `https://integrity.mit.edu/`. Note that merely citing a source does *not* count as a good justification.

All submissions must be in PDF format. Type-written submissions may be strongly preferred by your TA; handwritten submissions are acceptable only if:

1. Your handwriting is easily legible (as judged by your TA);
2. You produce a clean version of the document (as opposed to the sheet of paper you used to work out the problems); and
3. Your manuscript has been scanned to high enough standards (as judged by your TA). Consider using, e.g. *Scannable* or *Adobe Scan.*

Part I (Quiz on Canvas: 46 points)

Notation: $\emptyset$ is the empty set. If A and B are sets, $\wp(A)$ is the set of A's subsets and $A - B$ is the set whose members are the elements of A that are not also elements of B (so, for instance, $\{1,2\} - \{1\} = \{2\}$).

1. Answer the following questions. (2 points each)

   (a) Is $\wp(\mathbb{Z}) - \wp(\mathbb{N})$ well-ordered by $\subseteq$?

   (b) Is $\wp(\wp(\emptyset))$ well-ordered by $\in$?

   (c) Is $\wp^4(\emptyset) = \wp(\wp(\wp(\wp(\emptyset))))$ well-ordered by $\in$?

   (d) Is $\wp^3(\emptyset) - \{\{\{\emptyset\}\}\} = \wp(\wp(\wp(\emptyset))) - \{\{\{\emptyset\}\}\}$ well-ordered by $\in$?

   (e) Is $\wp^3(\emptyset) - \{\{\{\emptyset\}\}\}$ an ordinal number?

2. Recall the following definitions:

   - $\mathbb{N}$ is the set of natural numbers.
   - $\wp^n(A) = \underbrace{\wp(\wp(\ldots\wp(}_{n \text{ times}} A \underbrace{)\ldots))}_{n \text{ times}}$ $(n \in \mathbb{N})$.
   - $\bigcup A = \{x : x \in B \text{ for } B \in A\}$.

   Determine whether each of the following statements is true or false. (2 points each)

   (a) $\wp^n(\mathbb{Z}) \subseteq \wp^m(\mathbb{Z})$, for $n < m$ and $n, m \in \mathbb{N}$.

   (b) $|\wp^n(\mathbb{Z})| < |\wp^m(\mathbb{Z})|$, for $n < m$ and $n, m \in \mathbb{N}$.

   (c) For arbitrary $n \in \mathbb{N}$, $\wp^n(\mathbb{Z}) \subseteq \{\wp^m(\mathbb{Z}) : m \in \mathbb{N}\}$.

   (d) For arbitrary $n \in \mathbb{N}$, $\wp^n(\mathbb{Z}) \subseteq \bigcup\{\wp^m(\mathbb{Z}) : m \in \mathbb{N}\}$.

   (e) $|\wp^n(\mathbb{Q})| < |\{\wp^m(\mathbb{Q}) : m \in \mathbb{N}\}|$, $n \in \mathbb{N}$

   (f) $|\wp^n(\mathbb{R})| < |\bigcup\{\wp^m(\mathbb{R}) : m \in \mathbb{N}\}|$, $n \in \mathbb{N}$

3. Recall the following definitions:

   - $\alpha <_o \beta \leftrightarrow_{df} \alpha \in \beta$ (α, β ordinals)
   - $|A| < |B| \leftrightarrow_{df}$ there is an injection from A to B but no bijection
   - $\mathfrak{B}_\alpha = \begin{cases} \mathbb{N}, \text{ if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta), \text{ if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} \text{ if } \alpha \text{ is a limit ordinal greater than } 0 \end{cases}$
   - $\beth_\alpha$ is the $<_o$-smallest ordinal of cardinality $|\mathfrak{B}_\alpha|$

   Which of the following are true? (2 points each)

(a) $\omega <_o \omega + \omega$

(b) $|\omega| < |\omega \times \omega|$

(c) $\omega \times \omega <_o \beth_0$

(d) $|\omega \times \omega| < |\beth_0|$

(e) Let '118^o' denote the ordinal named by 0 followed by 118 prime symbols. Claim: $\beth_{118^o} <_o \beth_\omega$

(f) $|\beth_{118^o}| < |\beth_\omega|$

4. Let $\mathscr{U} = \bigcup\{\wp^m(\mathbb{N}) : m \in \mathbb{N}\}$. Answer the following questions (2 points each).

(a) Does $\mathscr{U}$ also contain a set $\underbrace{\{\{\ldots\{\{}_{\infty \text{ times}} 118 \underbrace{\}\}\ldots\}\}}_{\infty \text{ times}}$?

(b) Is the cardinality of the following set greater than the cardinality of $\wp^n(\mathscr{U})$ for each $n \in \mathbb{N}$?

$$\bigcup\{\mathbb{N}, \wp(\mathbb{N}), \wp(\wp(\mathbb{N})), \ldots, \mathscr{U}, \wp(\mathscr{U}), \wp(\wp(\mathscr{U})), \ldots\}$$

5. Determine whether the following are true or false (2 points each)

(a) $(\omega + 0'') + (\omega + 0') = (\omega + \omega) + (\omega + 0')$

(b) $(0' + \omega) \times 0''' = (0' \times 0''') + (\omega \times 0''')$

(c) $(\omega + 0'') \times 0'''' = (\omega \times 0'') + (0'' \times 0'''')$

(d) $(\omega + 0') + 0'' = (0' + \omega) + 0''$

Part II (Submit PDF on Canvas: 54 points)

6. Draw a diagram (or use prose) to give an informal characterization of the well-ordering types represented by each of the following ordinals. (2 points each; no need to justify answer but feel free to show work)

What does it mean to use prose to give an informal characterization of a well-order type? Suppose, for example, that the well-order type in question corresponded to ω. Then you might say something like "A countably infinite sequence of items which is ordered like the natural numbers, with an nth member for each $n \in \mathbb{N}$)—but no last member." Drawing diagrams is probably easier than using your words!

(a) $(\omega + 0'') \times 0'''$

(b) $(\omega \times 0''') \times 0''$

(c) $(0'''''' \times \omega) \times 0''$

(d) $(\omega \times \omega) + \omega$

(e) $(\omega \times \omega) \times \omega$

7. Recall that a relation constitutes a strict total order on a set just in case it is (i) asymmetric (so irreflexive as well and hence 'strict'), (ii) transitive, and (iii) total. Using an example not found in the course material/lecture:

   (a) Specify (i) a set whose members are not numbers and (ii) an ordering on that set that is not a strict total ordering. (5 points)

   (b) Specify (i) a set whose members are not numbers and (ii) a strict total ordering on that set that is not a well-ordering. You do not need to rigorously prove that the relation constitutes a strict total ordering. (5 points)

8. Recall that $\alpha <_o \beta$ is defined as $\alpha \in \beta$, for α and β ordinals. Does $\alpha <_o \beta$ entail $|\alpha| < |\beta|$? If so explain, why. If not, give a counterexample. (5 points)

9. Recall that we think of the ordinals as introduced in stages, in accordance with the following principles:

   Open-Endedness Principle However many stages have occurred, there is always a "next" stage: a first stage after every stage considered so far.

   Construction Principle At each stage, we introduce a new[1] ordinal, namely: the set of all ordinals that have been introduced at previous stages.

   Use these principles to give an informal justification of each of the following propositions. (7 points each; don't forget to justify your answers)

   (a) No ordinal is a member of itself.

   (b) For any ordinal α, either $\alpha = \{\}$ or $\{\} <_o \alpha$.

   (c) If α is an ordinal with infinitely many members, then either $\alpha = \omega$ or $\omega <_o \alpha$.

10. Give an example of a set whose cardinality is "much greater" than $|\mathfrak{B}_{\omega \times \omega^\omega}|$, in the sense that there are infinitely many sizes of infinity between the set you identify and $|\mathfrak{B}_{\omega \times \omega^\omega}|$. (8 points; don't forget to justify your answer.)

[1] A new ordinal is an ordinal that has not been introduced at previous stages.