Paradox and Infinity

— Infinite Cardinalities —
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Where do theories begin?

- A theory that is neutral on everything is no theory at all.
- No assumptions, no conclusions.
- Assumptions are only as meaningful as their terms.

Where do meanings begin?

- Can't define everything.
- Intuitions and patterns of use.
- Principles encode theoretical roles.

Cardinal Numbers

- Provide answers to 'How many?'-questions.
- Which cardinality principles should we adopt?

Cardinality: '$|A|$' reads 'The number of As'.

Proper Subset Principle: $|A| < |B|$ *if* $A \subset B$.

- Subset: $A \subseteq B$ *iff* for every x, if $x \in A$, then $x \in B$.
- Extensionality: $A = B$ *iff* for any x: $x \in A$ just in case $x \in B$.
- Proper Subset: $A \subset B$ *iff* $A \subseteq B$ and $A \neq B$.

Bijection Principle: $|A| = |B|$ *iff* $A \simeq B$.

- Equinumerous: $A \simeq B$ *iff* there is a bijection $f : A \rightarrow B$.
- Bijection: "A one-to-one pairing with no remainder."
- Can be defined without recourse to the concepts of number.

Ordered Pair: $\langle a, b \rangle \coloneqq \{\{a\}, \{a, b\}\}$.

Cartesian Product: $A \times B \coloneqq \{\langle a, b \rangle : a \in A,\ b \in B\}$.

Function: $f \subseteq A \times B$ is a *function* $f : A \to B$ *iff* every $a \in A$:

1. There is some $b \in B$ where $\langle a, b \rangle \in f$; and
2. For any $b, c \in B$, if $\langle a, b \rangle \in f$ and $\langle a, c \rangle \in f$, then $b = c$.

Abbreviation: '$f(a) = b$' abbreviates '$\langle a, b \rangle \in f$' if f is a function.

Injective: $f : A \to B$ is *injective iff* for any $a, b \in A$, if $f(a) = f(b)$, then $a = b$.

Surjective: $f : A \to B$ is *surjective iff* for all $b \in B$ there is some $a \in A$ where $f(a) = b$.

Bijection: $f : A \to B$ is *bijective iff* f is injective and surjective.

Equinumerous: $A \simeq B$ *iff* there is a bijection $f : A \to B$.

Bijection Principle: $|A| = |B|$ *iff* $A \simeq B$.

Paradox: There are equinumerous proper subsets of infinite sets.

- $\mathbb{N}_1 \subset \mathbb{N}$ where $\mathbb{N}_i := \{x \in \mathbb{N} : i \leqslant x\}$.
- The successor function $x' = x + 1$ is a bijection from $\mathbb{N}$ to $\mathbb{N}_1$.

Hilbert's Hotel: Always room for (countably many) more guests.

- $f_m(n) = n + m$ is a bijection from $\mathbb{N}$ to $\mathbb{N}_m$.
- $g_m(n) = n \times m$ is a bijection where $\mathbb{N}_{(m)} = \{k \times m : k \in \mathbb{N}\}$.

Question: Can Hilbert's Hotel accommodate an infinite number of groups of infinitely many new guests?

Contradiction:

1. Both $|\mathbb{N}_1| < |\mathbb{N}|$ and $|\mathbb{N}_1| = |\mathbb{N}|$ by the principles.
2. Totality: $x < y$, $x = y$, or $x > y$ (exclusive).
3. Thus $|\mathbb{N}_1| < |\mathbb{N}|$ entails $|\mathbb{N}_1| \neq |\mathbb{N}|$: contradiction.

Options:

- Accept contradictions?
- Give up totality?
- Give up one of the principles above?

Abductive Method: A good theory Γ ought to be...

1. Deductively Closed: $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$.
2. Consistent: $\Gamma \nvdash \varphi \wedge \neg\varphi$.
3. Simple: finitely axiomatizable in intuitive terms.
4. Strong: says more rather than less.
5. Practical: has useful applications.

Metaphysical Aside: Is theory choice subjective?

Proper Subset Principle: $|A| < |B|$ *if* $A \subset B$.

1. $|\mathbb{N}| > |\mathbb{N}_2| > |\mathbb{N}_3| > \ldots$ and $|\mathbb{N}| > |\mathbb{N}_{(2)}| > |\mathbb{N}_{(4)}| > \ldots$
2. How are we to compare $|\mathbb{N}_{(2)}|$ and $|\mathbb{N}_{(3)}|$ for size?
3. Could PSP be supplemented?
4. Example: $|\text{set of people}| < |\text{set of seats}|$.

Injection Principle: $|A| \leqslant |B|$ *iff* $A \simeq C$ for some $C \subseteq B$.

1. Reflexive: $|A| \leqslant |A|$.
2. Transitive: if $|A| \leqslant |B|$ and $|B| \leqslant |C|$, then $|A| \leqslant |C|$.
3. Anti-Symmetric: if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.
4. Total: $|A| \leqslant |B|$ or $|B| \leqslant |A|$.

Conclusion: The *Proper Subset Principle* must go!

The Infinite: A set A is *countably infinite iff* $|A| = |\mathbb{N}|$.

- $|\mathbb{N}| = |\mathbb{N}_m| = |\mathbb{N}_{(m)}|$.
- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.

Next Time: We will show that there are different sizes of infinity.

- $|\mathbb{N}| \neq |\mathbb{R}|$.
- $|A| \neq |\mathcal{P}(A)|$.