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# Bimodal Reference Manual

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*A Logic for Tense and Modality*

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— January 12, 2026 —

**Primary Reference:**

“*The Construction of Possible Worlds*”, Brast-Mckie, (under review), 2025.

## Abstract

This reference manual provides the formal specification of the Bimodal logic **TM** for tense and modality as implemented in the [ProofChecker](#) project. **TM** is a bimodal logic combining an S5 historical necessity operator with linear temporal operators for the past and future tenses. The soundness theorem has been established along with the deduction theorem. Completeness infrastructure is in place but the core lemmas remain axiomatized, requiring further work to derive these lemmas.

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## 1 Introduction

**TM** is a bimodal logic combining S5 a historical necessity operator ( $\Box$ ) with linear temporal operators for past ( $H$ ) and future ( $G$ ). The logic provides a framework for reasoning about necessary truths across time to train AI systems to reason about past and future contingency.

The semantics is based on *task frames*, which extend Kripke frames with temporal structure. A task frame consists of world states connected by a *task relation* indexed by temporal durations. World histories are temporal slices through a task frame, representing the unfolding of a world over time.

### Project Structure

The Lean 4 implementation is in the `Bimodal/` directory:

- `Syntax/` – Defines the formula language with 6 primitive constructors and derived operators. **Complete**.
- `ProofSystem/` – Axioms (14 schemata) and inference rules (7 rules) forming a Hilbert-style proof system. **Complete**.
- `Semantics/` – Task frames model possible worlds; world histories model time; truth conditions define meaning. **Complete**.
- `Metalogic/` – Soundness theorem (proven: all axioms valid, rules preserve validity), deduction theorem (proven: enables assumption introduction), and completeness infrastructure (Lindenbaum lemma, canonical model axiomatized). **Soundness and deduction complete; completeness pending**.
- `Theorems/` – Perpetuity principles and modal theorems derived from the axiom system. **Partial**.

## 2 Syntax

### 2.1 Formulas

Formulas are defined inductively with six primitive constructors.

**Definition 2.1** (Formula). The type `Formula` is defined by:

$$\varphi, \psi ::= p \mid \perp \mid \varphi \rightarrow \psi \mid \Box\varphi \mid H\varphi \mid G\varphi$$

where  $p$  ranges over propositional atoms (type `String`).

Symbol	Name	Lean	Reading
$p, q, r$	Atom	atom s	propositional atom
$\perp$	Bottom	bot	falsity
$\varphi \rightarrow \psi$	Implication	imp	"if $\varphi$ then $\psi$ "
$\Box\varphi$	Necessity	box	"necessarily $\varphi$ "
$H\varphi$	Always past	all_past	"always in the past, $\varphi$ "
$G\varphi$	Always future	all_future	"always in the future, $\varphi$ "

## 2.2 Derived Operators

The following operators are defined in terms of the primitives.

**Definition 2.2** (Propositional).

$$\begin{aligned}\neg\varphi &:= \varphi \rightarrow \perp \\ \varphi \wedge \psi &:= \neg(\varphi \rightarrow \neg\psi) \\ \varphi \vee \psi &:= \neg\varphi \rightarrow \psi\end{aligned}$$

Symbol	Name	Lean	Reading
$\neg\varphi$	Negation	neg	"not $\varphi$ "
$\varphi \wedge \psi$	Conjunction	and	" $\varphi$ and $\psi$ "
$\varphi \vee \psi$	Disjunction	or	" $\varphi$ or $\psi$ "

**Definition 2.3** (Modal).

$$\Diamond\varphi := \neg\Box\neg\varphi$$

Symbol	Name	Lean	Reading
$\Diamond\varphi$	Possibility	pos	"possibly $\varphi$ "

**Definition 2.4** (Temporal).

$$\begin{aligned}P\varphi &:= \neg H\neg\varphi \\ F\varphi &:= \neg G\neg\varphi \\ \Delta\varphi &:= H\varphi \wedge \varphi \wedge G\varphi \\ \nabla\varphi &:= P\varphi \vee \varphi \vee F\varphi\end{aligned}$$

Symbol	Name	Lean	Reading
$P\varphi$	Sometime past	<code>some_past</code>	"at some past time, $\varphi$ "
$F\varphi$	Sometime future	<code>some_future</code>	"at some future time, $\varphi$ "
$\triangle\varphi$	Always	<code>always</code>	"at all times, $\varphi$ "
$\nabla\varphi$	Sometimes	<code>sometimes</code>	"at some time, $\varphi$ "

## 2.3 Temporal Duality

The `swap_temporal` function exchanges past and future operators.

**Definition 2.5** (Temporal Swap). The function  $\langle S \rangle : \text{Formula} \rightarrow \text{Formula}$  is defined by structural induction on formulas:

$$\begin{aligned}\langle S \rangle p &= p \\ \langle S \rangle \perp &= \perp \\ \langle S \rangle (\varphi \rightarrow \psi) &= \langle S \rangle \varphi \rightarrow \langle S \rangle \psi \\ \langle S \rangle \Box \varphi &= \Box \langle S \rangle \varphi \\ \langle S \rangle H\varphi &= G \langle S \rangle \varphi \\ \langle S \rangle G\varphi &= H \langle S \rangle \varphi\end{aligned}$$

**Theorem 2.6** (Involution).  $\langle S \rangle \langle S \rangle \varphi = \varphi$

# 3 Task Semantics

## 3.1 Task Frames

Task frames are the fundamental semantic structures for **TM**. They abstract from universal laws governing transitions between world states while still retaining the temporal duration for a transition to complete.

The following primitives are required to define a task frame:

Primitive	Type	Description
$W$	Type	World states
$T$	Type	Temporal durations
$w \Rightarrow_x u$	$W \rightarrow T \rightarrow W \rightarrow \text{Prop}$	Task relation

**Definition 3.1** (Task Frame). A **task frame** over temporal type  $T$  is a triple  $\mathcal{F} = (W, T, \Rightarrow)$  satisfying:

1. **Nullity:** For all  $w : W$ , we have  $w \Rightarrow_0 w$ .
2. **Compositionality:** For all  $w, u, v : W$  and  $x, y : T$ , if  $w \Rightarrow_x u$  and  $u \Rightarrow_y v$ , then  $w \Rightarrow_{x+y} v$ .

Nullity ensures that zero-duration tasks leave the world state unchanged. Compositionality ensures that executing tasks sequentially yields results consistent with a single task of combined duration.

### 3.2 World Histories

A world history is a function from times to world states that respects the task relation over a convex temporal domain. World histories represent possible paths through the space of world states.

**Definition 3.2** (Convex Domain). A domain  $\text{dom} : T \rightarrow \text{Prop}$  is **convex** if whenever  $a, c \in \text{dom}$  with  $a \leq c$ , every time  $b$  with  $a \leq b \leq c$  is also in  $\text{dom}$ . More precisely, for all  $a, b, c : T$ , if  $\text{dom}(a)$  and  $\text{dom}(c)$  and  $a \leq b \leq c$ , then  $\text{dom}(b)$ . Convexity ensures the domain has no temporal gaps.

**Definition 3.3** (World History). A **world history** in a task frame  $\mathcal{F}$  is a dependent function  $\tau : (x : T) \rightarrow \text{dom}(x) \rightarrow W$  where  $\text{dom} : T \rightarrow \text{Prop}$  is a convex subset of  $T$  and  $\tau(x) \Rightarrow_{y-x} \tau(y)$  for all times  $x, y : T$  with  $\text{dom}(x)$ ,  $\text{dom}(y)$ , and  $x \leq y$ . We write  $H_{\mathcal{F}}$  for all world histories over frame  $\mathcal{F}$ .

### 3.3 Task Models

A task model extends a task frame with an interpretation function that assigns truth values to sentence letters at world states. **Propositions** are subsets of  $W$  representing instantaneous ways for the system to be. Sentence letters express propositions, which can be realized by zero or more world states. World states themselves are specific configurations of the total system at an instant.

**Definition 3.4** (Task Model). A **task model** defined over a frame  $\mathcal{F}$  is a pair  $\mathcal{M} = (\mathcal{F}, I)$  where the **interpretation function**  $I : W \rightarrow \text{String} \rightarrow \text{Prop}$  assigns to each world state  $w : W$  and sentence letter  $p : \text{String}$  a truth value  $I(w, p) : \text{Prop}$ . We write  $I(w, p)$  to indicate that sentence letter  $p$  is true at  $w$ .

### 3.4 Truth Conditions

Truth is evaluated relative to a model  $\mathcal{M}$  providing the interpretation, a world history  $\tau$  representing a possible path through the space of world

states, and a time  $x : T$ . Whereas the model fixes the interpretation of the language, the contextual parameters  $\tau$  and  $x$  determine the truth value of every sentence of the language.

**Definition 3.5** (Truth). For model  $\mathcal{M}$ , history  $\tau \in H_{\mathcal{F}}$ , and time  $x : T$ :

$$\begin{aligned} \mathcal{M}, \tau, x \models p &\text{ iff } x \in \text{dom}(\tau) \text{ and } I(\tau(x), p) \\ \mathcal{M}, \tau, x \not\models \perp & \\ \mathcal{M}, \tau, x \models \varphi \rightarrow \psi &\text{ iff } \mathcal{M}, \tau, x \not\models \varphi \text{ or } \mathcal{M}, \tau, x \models \psi \\ \mathcal{M}, \tau, x \models \Box \varphi &\text{ iff } \mathcal{M}, \sigma, x \models \varphi \text{ for all } \sigma \in H_{\mathcal{F}} \\ \mathcal{M}, \tau, x \models H\varphi &\text{ iff } \mathcal{M}, \tau, y \models \varphi \text{ for all } y : T \text{ where } y < x \\ \mathcal{M}, \tau, x \models G\varphi &\text{ iff } \mathcal{M}, \tau, y \models \varphi \text{ for all } y : T \text{ where } x < y \end{aligned}$$

The modal operator  $\Box$  quantifies over all world histories  $\sigma : H_{\mathcal{F}}$  at the current time  $x : T$ . The temporal operators  $H$  and  $G$  quantify over all earlier and later times  $y : T$ .

### 3.5 Time-Shift

The time-shift operation is used to establish the **perpetuity principles**:

- P1:  $\Box\varphi \rightarrow \Diamond\varphi$  (what is necessary is always the case)
- P2:  $\nabla\varphi \rightarrow \Diamond\varphi$  (what is sometimes the case is possible)

It is natural to assume that whatever is necessary is always the case, or equivalently, whatever is sometimes the case is possible. Time-shift enables proofs of the bimodal axioms MF ( $\Box\varphi \rightarrow \Box G\varphi$ ) and TF ( $\Box\varphi \rightarrow G\Box\varphi$ ) which together imply the perpetuity principles.

**Definition 3.6** (Time-Shift). For  $\tau, \sigma \in H_{\mathcal{F}}$  and  $x, y : T$ , world histories  $\tau$  and  $\sigma$  are **time-shifted from  $y$  to  $x$** , written  $\tau \approx_y^x \sigma$ , if and only if there exists an order automorphism  $\bar{a} : T \rightarrow T$  where  $y = \bar{a}(x)$ ,  $\text{dom}_{\sigma} = \bar{a}^{-1}(\text{dom}_{\tau})$ , and  $\sigma(z) = \tau(\bar{a}(z))$  for all  $z \in \text{dom}_{\sigma}$ .

Time-shifting preserves the essential structure of histories:

**Theorem 3.7** (Convexity Preservation). *If  $\tau$  has a convex domain and  $\tau \approx_y^x \sigma$ , then  $\sigma$  has a convex domain.*

**Theorem 3.8** (Task Preservation). *If  $\tau$  respects the task relation and  $\tau \approx_y^x \sigma$ , then  $\sigma$  respects the task relation.*

### 3.6 Logical Consequence and Validity

Logical consequence and validity are defined uniformly across all temporal types, frames, models, histories, and times.

**Definition 3.9** (Logical Consequence). A formula  $\varphi$  is a **logical consequence** of  $\Gamma$  (written  $\Gamma \models \varphi$ ) just in case for every temporal type  $T : \text{Type}$ , frame  $\mathcal{F} : \text{TaskFrame}(T)$ , model  $\mathcal{M} : \text{TaskModel}(\mathcal{F})$ , history  $\tau \in H_{\mathcal{F}}$ , and time  $x : T$ , if  $\mathcal{M}, \tau, x \models \psi$  for all  $\psi \in \Gamma$ , then  $\mathcal{M}, \tau, x \models \varphi$ .

**Definition 3.10** (Validity). A formula  $\varphi$  is **valid** (written  $\models \varphi$ ) just in case  $\varphi$  is a logical consequence of the empty set:  $\emptyset \models \varphi$ . Equivalently,  $\varphi$  is true at every model-history-time triple.

**Definition 3.11** (Satisfiability). A context  $\Gamma$  is **satisfiable** in temporal type  $T : \text{Type}$  if there exist a frame  $\mathcal{F} : \text{TaskFrame}(T)$ , model  $\mathcal{M} : \text{TaskModel}(\mathcal{F})$ , history  $\tau \in H_{\mathcal{F}}$ , and time  $x : T$  such that  $\mathcal{M}, \tau, x \models \psi$  for all  $\psi \in \Gamma$ .

**Theorem 3.12** (Monotonicity). If  $\Gamma \subseteq \Delta$  and  $\Gamma \models \varphi$ , then  $\Delta \models \varphi$ .

## 4 Proof Theory

### 4.1 Axiom Schemata

The TM proof system has 14 axiom schemata.

#### 4.1.1 Propositional Axioms

**Axiom 4.1** (K - Distribution).  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

**Axiom 4.2** (S - Weakening).  $\varphi \rightarrow (\psi \rightarrow \varphi)$

**Axiom 4.3** (EFQ - Ex Falso Quodlibet).  $\perp \rightarrow \varphi$

**Axiom 4.4** (Peirce).  $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$

#### 4.1.2 Modal Axioms (S5)

**Axiom 4.5** (MT - Modal T).  $\Box\varphi \rightarrow \varphi$

**Axiom 4.6** (M4 - Modal 4).  $\Box\varphi \rightarrow \Box\Box\varphi$

**Axiom 4.7** (MB - Modal B).  $\varphi \rightarrow \Box\Diamond\varphi$

**Axiom 4.8** (M5 - Modal 5 Collapse).  $\Diamond\Box\varphi \rightarrow \Box\varphi$

**Axiom 4.9** (MK - Modal Distribution).  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

### 4.1.3 Temporal Axioms

**Axiom 4.10** (TK - Temporal Distribution).  $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$

**Axiom 4.11** (T4 - Temporal 4).  $G\varphi \rightarrow GG\varphi$

**Axiom 4.12** (TA - Temporal A).  $\varphi \rightarrow GP\varphi$

**Axiom 4.13** (TL - Temporal L).  $\triangle\varphi \rightarrow GH\varphi$

### 4.1.4 Modal-Temporal Interaction

**Axiom 4.14** (MF - Modal-Future).  $\square\varphi \rightarrow \square G\varphi$

**Axiom 4.15** (TF - Temporal-Future).  $\square\varphi \rightarrow G\square\varphi$

Axiom	Lean Constructor	Pattern
K	Axiom.prop_k	Distribution
S	Axiom.prop_s	Weakening
EFQ	Axiom.ex_falso	Explosion
Peirce	Axiom.peirce	Classical
MT	Axiom.modal_t	Reflexivity
M4	Axiom.modal_4	Transitivity
MB	Axiom.modal_b	Symmetry
M5	Axiom.modal_5Collapse	S5 collapse
MK	Axiom.modal_k_dist	Modal distribution
TK	Axiom.temp_k_dist	Temporal distribution
T4	Axiom.temp_4	Temporal transitivity
TA	Axiom.temp_a	Connectedness
TL	Axiom.temp_l	Introspection
MF	Axiom.modal_future	Modal-future
TF	Axiom.temp_future	Temporal-modal

## 4.2 Inference Rules

The proof system has 7 inference rules.

**Definition 4.16** (Axiom Rule). If  $\varphi$  matches an axiom schema, then  $\Gamma \vdash \varphi$ .

**Definition 4.17** (Assumption Rule). If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .

**Definition 4.18** (Modus Ponens).

$$\frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

**Definition 4.19** (Necessitation).

$$\frac{\vdash \varphi}{\vdash \Box\varphi}$$

Applies only to theorems (empty context).

**Definition 4.20** (Temporal Necessitation).

$$\frac{\vdash \varphi}{\vdash G\varphi}$$

Applies only to theorems (empty context).

**Definition 4.21** (Temporal Duality).

$$\frac{\vdash \varphi}{\vdash \langle S \rangle \varphi}$$

Applies only to theorems (empty context).

**Definition 4.22** (Weakening).

$$\frac{\Gamma \vdash \varphi \quad \Gamma \subseteq \Delta}{\Delta \vdash \varphi}$$

Rule	Lean Constructor	Context Requirement
Axiom	DerivationTree.axiom	Any
Assumption	DerivationTree.assumption	Any
Modus Ponens	DerivationTree.modus_ponens	Any
Necessitation	DerivationTree.necessitation	Empty only
Temp. Necessitation	DerivationTree.temporal_necessitation	Empty only
Temporal Duality	DerivationTree.temporal_duality	Empty only
Weakening	DerivationTree.weakening	Any

### 4.3 Derivation Trees

Derivations are represented as inductive trees.

**Definition 4.23** (Derivation Tree). `DerivationTree`  $\Gamma \vdash \varphi$  (written  $\Gamma \vdash \varphi$ ) is an inductive type representing a derivation of  $\varphi$  from context  $\Gamma$ .

**Definition 4.24** (Height). The height of a derivation tree:

- Base cases (axiom, assumption): height 0
- Unary rules: height of subderivation + 1
- Modus ponens: max of both subderivations + 1

The height measure enables well-founded recursion in metalogical proofs.

#### 4.4 Notation

Notation	Lean
$\Gamma \vdash \varphi$	<code>DerivationTree Γ φ</code>
$\vdash \varphi$	<code>DerivationTree [] φ</code>

### 5 Metalogic

#### 5.1 Soundness

The soundness theorem establishes that derivability implies semantic validity.

**Theorem 5.1** (Soundness). *If  $\Gamma \vdash \varphi$  then  $\Gamma \vDash \varphi$ .*

The proof proceeds by induction on the derivation structure:

- **Axioms:** Each of the 14 axiom schemata is proven valid
- **Assumptions:** Assumed formulas are true by hypothesis
- **Modus ponens:** Validity preserved under implication elimination
- **Necessitation:** Valid formulas become necessarily valid
- **Temporal necessitation:** Valid formulas become always-future valid
- **Temporal duality:** Past-future swap preserves validity
- **Weakening:** Adding premises preserves semantic consequence

Axiom Validity	Lean Theorem	Technique
prop_k_valid	Propositional K	Propositional reasoning
prop_s_valid	Propositional S	Propositional reasoning
ex_falso_valid	EFQ	Vacuous implication
peirce_valid	Peirce	Classical case analysis
modal_t_valid	MT	Reflexivity of accessibility
modal_4_valid	M4	Transitivity of accessibility
modal_b_valid	MB	Symmetry of accessibility
modal_5Collapse_valid	M5	S5 equivalence structure
modal_k_dist_valid	MK	Distribution
temp_k_dist_valid	TK	Temporal distribution
temp_4_valid	T4	Transitivity of time
temp_a_valid	TA	Temporal connectedness
temp_l_valid	TL	Always implies recurrence
modal_future_valid	MF	Time-shift invariance
temp_future_valid	TF	Time-shift invariance

The MF and TF axioms use time-shift invariance (via `WorldHistory.time_shift`) to relate truth at different times.

## 5.2 Deduction Theorem

**Theorem 5.2** (Deduction Theorem). *If  $A :: \Gamma \vdash B$  then  $\Gamma \vdash A \rightarrow B$ .*

The proof uses well-founded induction on derivation height, handling each rule:

- **Axiom:** Use S axiom to weaken
- **Assumption:** Identity if same, S axiom if different
- **Modus ponens:** Use K axiom distribution
- **Weakening:** Case analysis on assumption membership
- **Modal/temporal rules:** Do not apply (require empty context)

## 5.3 Consistency

**Definition 5.3** (Consistent). A context  $\Gamma$  is **consistent** if  $\Gamma \not\vdash \perp$ .

**Definition 5.4** (Maximal Consistent). A context  $\Gamma$  is **maximal consistent** if it is consistent and for all  $\varphi \notin \Gamma$ , the context  $\varphi :: \Gamma$  is inconsistent.

## 5.4 Completeness Infrastructure

The completeness proof uses canonical model construction.

**Lemma 5.5** (Lindenbaum). *Every consistent context can be extended to a maximal consistent context.*

**Definition 5.6** (Canonical Frame). The canonical frame has:

- World states: Maximal consistent sets
- Times: Integers ( $\mathbb{Z}$ )
- Task relation: Defined via modal/temporal formula transfer

**Definition 5.7** (Canonical Valuation). An atom  $p$  is true at world  $\Gamma$  iff  $p \in \Gamma$ .

**Lemma 5.8** (Truth Lemma). *In the canonical model,  $\varphi \in \Gamma$  iff  $\varphi$  is true at  $\Gamma$ .*

**Theorem 5.9** (Weak Completeness (Axiomatized)). *If  $\vdash \varphi$  then  $\vdash \varphi$ .*

**Theorem 5.10** (Strong Completeness (Axiomatized)). *If  $\Gamma \vDash \varphi$  then  $\Gamma \vdash \varphi$ .*

## 5.5 Implementation Status

Component	Status	Lean
Soundness	Fully proven	<code>soundness</code>
Deduction Theorem	Fully proven	<code>deduction_theorem</code>
Lindenbaum Lemma	Axiomatized	<code>lindenbaum</code>
Canonical Frame	Axiomatized	<code>canonical_frame</code>
Truth Lemma	Axiomatized	<code>truth_lemma</code>
Weak Completeness	Axiomatized	<code>weak_completeness</code>
Strong Completeness	Axiomatized	<code>strong_completeness</code>

# 6 Theorems

## 6.1 Perpetuity Principles

The perpetuity principles establish deep connections between modal necessity ( $\Box$ ) and temporal operators ( $\Diamond, \nabla$ ).

**Theorem 6.1** (P1: Necessity Implies Always).  $\vdash \Box\varphi \rightarrow \Diamond\varphi$

**Theorem 6.2** (P2: Sometimes Implies Possible).  $\vdash \nabla\varphi \rightarrow \Diamond\varphi$

**Theorem 6.3** (P3: Necessity of Perpetuity).  $\vdash \Box\varphi \rightarrow \Box\Box\varphi$

**Theorem 6.4** (P4: Possibility of Occurrence).  $\vdash \Diamond\Box\varphi \rightarrow \Diamond\varphi$

**Theorem 6.5** (P5: Persistent Possibility).  $\vdash \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$

**Theorem 6.6** (P6: Occurrent Necessity is Perpetual).  $\vdash \Box\Box\varphi \rightarrow \Box\Box\Box\varphi$

All six perpetuity principles are fully proven in the Lean implementation.

Principle	Lean Theorem	Key Lemmas
P1	perpetuity_1	MF, TF, MT
P2	perpetuity_2	Contraposition of P1
P3	perpetuity_3	P1, box_mono
P4	perpetuity_4	Contraposition
P5	perpetuity_5	modal_5, temporal K
P6	perpetuity_6	P5, bridge lemmas

## 6.2 Modal S5 Theorems

**Theorem 6.7** (T-Box-to-Diamond).  $\vdash \Box\varphi \rightarrow \Diamond\varphi$

**Theorem 6.8** (Box Distributes Over Disjunction).  $\vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

**Theorem 6.9** (Box Preserves Contraposition).  $\vdash \Box(\varphi \rightarrow \psi) \rightarrow \Box(\neg\psi \rightarrow \neg\varphi)$

**Theorem 6.10** (K Distribution for Diamond).  $\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$

**Theorem 6.11** (S5 Collapse).  $\vdash \Diamond\Box\varphi \leftrightarrow \Box\varphi$

**Theorem 6.12** (Box-Conjunction Biconditional).  $\vdash \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

**Theorem 6.13** (Diamond-Disjunction Biconditional).  $\vdash \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$

## 6.3 Modal S4 Properties

The following S4 properties are derived from the TM axiom system.

**Theorem 6.14** (Modal 5).  $\vdash \Diamond\varphi \rightarrow \Box\Diamond\varphi$

**Theorem 6.15** (Diamond 4).  $\vdash \Diamond\Diamond\varphi \rightarrow \Diamond\varphi$

**Theorem 6.16** (Box Monotonicity). *If  $\vdash \varphi \rightarrow \psi$  then  $\vdash \Box\varphi \rightarrow \Box\psi$ .*

**Theorem 6.17** (Diamond Monotonicity). *If  $\vdash \varphi \rightarrow \psi$  then  $\vdash \Diamond\varphi \rightarrow \Diamond\psi$ .*

## 6.4 Propositional Theorems

**Theorem 6.18** (Identity).  $\vdash \varphi \rightarrow \varphi$

**Theorem 6.19** (Double Negation Introduction).  $\vdash \varphi \rightarrow \neg\neg\varphi$

**Theorem 6.20** (Double Negation Elimination).  $\vdash \neg\neg\varphi \rightarrow \varphi$

**Theorem 6.21** (Contraposition). *If  $\vdash \varphi \rightarrow \psi$  then  $\vdash \neg\psi \rightarrow \neg\varphi$ .*

**Theorem 6.22** (De Morgan (Disjunction)).  $\vdash \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

**Theorem 6.23** (De Morgan (Conjunction)).  $\vdash \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

## 6.5 Combinator Infrastructure

The combinator infrastructure provides Hilbert-style proof tools.

**Theorem 6.24** (B Combinator).  $\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

**Theorem 6.25** (Implication Transitivity). *If  $\vdash A \rightarrow B$  and  $\vdash B \rightarrow C$  then  $\vdash A \rightarrow C$ .*

**Theorem 6.26** (Pairing).  $\vdash A \rightarrow (B \rightarrow (A \wedge B))$

**Theorem 6.27** (Classical Merge).  $\vdash (P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q)$

## 6.6 Generalized Necessitation

**Theorem 6.28** (Generalized Modal Necessitation). *If  $\Gamma \vdash \varphi$  then  $\Box\Gamma \vdash \Box\varphi$  where  $\Box\Gamma = [\Box\psi \mid \psi \in \Gamma]$ .*

**Theorem 6.29** (Generalized Temporal Necessitation). *If  $\Gamma \vdash \varphi$  then  $G\Gamma \vdash G\varphi$  where  $G\Gamma = [G\psi \mid \psi \in \Gamma]$ .*

## 6.7 Module Organization

Module	Contents
Perpetuity.lean	P1-P6 principles
ModalS5.lean	S5 characteristic theorems
ModalS4.lean	S4 properties (modal_5, diamond_4)
Propositional.lean	Classical propositional theorems
Combinators.lean	B, I, S combinators, imp_trans
GeneralizedNecessitation.lean	Context-level necessitation

## 7 Notes

### 7.1 Implementation Status

Component	Status	Notes
Syntax	Complete	6 primitives, derived operators
Semantics	Complete	Task frames, world histories, truth
Proof System	Complete	14 axioms, 7 inference rules
Soundness	Fully Proven	All 14 axioms valid, 7 rules sound
Deduction Theorem	Fully Proven	Well-founded recursion on height
Completeness	Infrastructure Only	Axiomatized (Lindenbaum, truth lemma)
Perpetuity Principles	Fully Proven	P1-P6 all proven

### 7.2 Discrepancy Notes

This section documents differences between the paper “The Construction of Possible Worlds” and the Lean implementation.

#### 7.2.1 Terminology

- The paper uses “perpetuity principles” for P1-P6; the Lean code uses the same terminology.
- The paper’s notation  $\triangle$  and  $\nabla$  for “always” and “sometimes” is preserved in the Lean implementation as `always` and `sometimes`.

#### 7.2.2 Axiom Naming

Paper Name	Lean Name	Notes
MT (Modal T)	<code>Axiom.modal_t</code>	$\Box\varphi \rightarrow \varphi$
M4 (Modal 4)	<code>Axiom.modal_4</code>	$\Box\varphi \rightarrow \Box\Box\varphi$
MB (Modal B)	<code>Axiom.modal_b</code>	$\varphi \rightarrow \Box\Diamond\varphi$
MK	<code>Axiom.modal_k_dist</code>	K distribution
TK	<code>Axiom.temp_k_dist</code>	Temporal K distribution
T4	<code>Axiom.temp_4</code>	Temporal transitivity
TA	<code>Axiom.temp_a</code>	Temporal connectedness
TL	<code>Axiom.temp_l</code>	Temporal introspection
MF	<code>Axiom.modal_future</code>	Modal-future interaction
TF	<code>Axiom.temp_future</code>	Temporal-future interaction

### 7.2.3 M5 Collapse Axiom

The implementation includes an explicit M5 collapse axiom (`Axiom.modal_5_collapse`):

$$\Diamond \Box \varphi \rightarrow \Box \varphi$$

This is derivable from the other S5 axioms (MB + M4) but is included as a primitive for proof convenience in the S5 collapse theorem.

### 7.2.4 Temporal Type Generalization

The paper uses a fixed temporal type  $T = \mathbb{Z}$  (integers). The implementation generalizes to any type  $T$  with `LinearOrderedAddCommGroup` instance, allowing for integers, rationals, reals, or custom bounded time structures.

### 7.2.5 Completeness Status

The paper proves completeness via canonical model construction. The Lean implementation has the completeness infrastructure (Lindenbaum lemma statement, canonical frame types, truth lemma statement) but the core lemmas remain axiomatized. Estimated effort to complete: 70-90 hours of focused development.

## 7.3 Source Files

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Directory	Contents
<code>Bimodal/Syntax/</code>	Formula, Context, temporal swap
<code>Bimodal/Semantics/</code>	TaskFrame, WorldHistory, Truth, Validity
<code>Bimodal/ProofSystem/</code>	Axioms, Derivation
<code>Bimodal/Metalogic/</code>	Soundness, DeductionTheorem, Completeness
<code>Bimodal/Theorems/</code>	Perpetuity, ModalS5, Propositional, Combinators

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