SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B1.1

Honour School of Mathematics and Computer Science Part B: Paper B1.1

Honour School of Mathematics and Philosophy Part B: Paper B1.1

Honour School of Mathematics and Statistics Part B: Paper B1.1

Honour School of Computer Science and Philosophy Part B: Paper B1.1

Honour School of Philosophy, Politics, and Economics: Paper B1.1

LOGIC

TRINITY TERM 2020

Friday 12 June

Opening time: 09:30 (BST)

You have 2 hours 45 minutes to complete the paper and upload your answer file

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part B)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - add to the first page, in numerical order, the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit a single PDF document with your answers for this paper.

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. Let $\mathcal{L}_0 = \{\neg, \to\}$ be the language of propositional calculus with connectives $\neg$ and $\to$, and with propositional variables $p_0, p_1, p_2, \ldots$. Let L_0 be the deductive system of propositional calculus with axioms

   A1 $(\alpha \to (\beta \to \alpha))$

   A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$

   A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$

   where α, β, γ may be any $\mathcal{L}_0$-formulas, and with modus ponens (**MP**) as the only rule of inference.

   (a) [8 marks]

   (i) What is a *valuation*? Explain how to extend a valuation v to $\widetilde{v} : \mathrm{Form}(\mathcal{L}_0) \to \{T, F\}$, where $\mathrm{Form}(\mathcal{L}_0)$ is the set of $\mathcal{L}_0$-formulas.

   (ii) For any integer $n > 0$, let V_n be the set of functions $\{p_0, p_1, \ldots, p_{n-1}\} \to \{T, F\}$ ('partial valuations') and let $\mathrm{Form}_n(\mathcal{L}_0)$ be the set of $\mathcal{L}_0$-formulas containing only propositional variables among $p_0, p_1, \ldots, p_{n-1}$. Determine the number of $v \in V_3$ with

   $$\widetilde{v}(((p_0 \to \neg p_1) \to (\neg p_2 \to p_1))) = T.$$

   (iii) Show that, for any $n > 0$ and any $k \in \{0, 1, 2, 3, \ldots, 2^n\}$, there is some $\phi \in \mathrm{Form}_n(\mathcal{L}_0)$ with $\sharp\{v \in V_n : \widetilde{v}(\phi) = T\} = k$.
   [If you use the adequacy of $\mathcal{L}_0$ you should prove it.]

   (b) [7 marks] Let ϕ be an $\mathcal{L}_0$-formula and let Γ be a subset of $\mathrm{Form}(\mathcal{L}_0)$.

   (i) What does it mean to say that
   - ϕ is a *logical consequence* of Γ (denoted by $\Gamma \models \phi$)
   - ϕ is *derivable* in L_0 from Γ (denoted by $\Gamma \vdash \phi$)
   - Γ is *satisfiable*
   - Γ is *consistent*
   - Γ is *maximal consistent*?

   (ii) Show that if Γ is consistent and $\Gamma \vdash \phi$ then $\Gamma \cup \{\phi\}$ is consistent and $\Gamma \cup \{\neg\phi\}$ is inconsistent.

   (c) [10 marks] State and prove the Completeness Theorem **CT** for L_0.
   [You may use **DT** *(the Deduction Theorem) and* **PC** *(Proof by Contradiction) for L_0, where* **PC** *says: If $\Gamma \cup \{\neg\phi\}$ is inconsistent then $\Gamma \vdash \phi$.
   This is for part (c) only.]*

2. Let $\mathcal{L}$ be a first-order language and let $K(\mathcal{L})$ be the deductive system for first-order predicate calculus with axioms

A1 $(\alpha \to (\beta \to \alpha))$
A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$
A4 $(\forall x_i \alpha \to \alpha[t/x_i])$, where t is free for x_i in α
A5 $(\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta))$, provided that $x_i \notin \text{Free}(\alpha)$
A6 $\forall x_i\ x_i \doteq x_i$
A7 $(x_i \doteq x_j \to (\phi \to \phi'))$, where ϕ is *atomic* and ϕ' is obtained from ϕ by replacing some (not necessarily all) occurrences of x_i in ϕ by x_j,

where α, β and γ are arbitrary $\mathcal{L}$-formulas, t is an arbitrary $\mathcal{L}$-term, and with Modus Ponens (**MP**), Generalisation ($\forall$) and the Thinning Rule as rules of inference.

(a) [8 marks] Let ϕ be an $\mathcal{L}$-formula and let Γ be a set of $\mathcal{L}$-formulas. What does it mean to say that ϕ is *derivable* from Γ in $K(\mathcal{L})$ (denoted by $\Gamma \vdash \phi$)? Carefully explain the Generalisation Rule ($\forall$) and explain, by way of an example, what goes wrong when you allow passing from a formula ψ to $\forall x_j \psi$ without any restrictions on x_j.
What is the Thinning Rule and why do we need it?

(b) [9 marks]

(i) Let ϕ and ψ be any $\mathcal{L}$-formulas.
Prove, without using the Deduction Theorem, that $\{(\phi \to \phi)\} \vdash (\psi \to \psi)$.
Is it always true that $\{(\phi \to \phi)\} \vdash (\phi \to \forall x_j \phi)$? Briefly justify your answer.

(ii) Assuming that x_j does not occur in the formula ϕ, show that $\{\forall x_i \phi\} \vdash \forall x_j \phi[x_j/x_i]$.

(c) [8 marks] Let $K'(\mathcal{L})$ be the deductive system with axioms **A1 - A7** and with the same rules of inference as $K(\mathcal{L})$ (so **(MP)**, ($\forall$) and the Thinning Rule), but where we only allow α, β and γ in **A1 - A5** to be arbitrary **$\mathcal{L}$-sentences**. Show that we still have the Soundness Theorem ('$\Gamma \vdash_{K'(\mathcal{L})} \phi$ implies $\Gamma \models \phi$') for $\Gamma \cup \{\phi\}$ any set of $\mathcal{L}$-sentences. What about the Completeness Theorem ('$\Gamma \models \phi$ implies $\Gamma \vdash_{K'(\mathcal{L})} \phi$')?
[You may use the Soundness and Completeness Theorems for $K(\mathcal{L})$.]

3. Let $\mathcal{L} = \{E\}$ be the first-order language with a single binary relation symbol E. Let $K(\mathcal{L})$ be the deductive system for first-order predicate calculus given in Question 2.

(a) [8 marks]

(i) What is an *$\mathcal{L}$-structure*?

(ii) What is an *assignment* in an $\mathcal{L}$-structure $\mathcal{A}$? Given an $\mathcal{L}$-formula ϕ and an assignment v in an $\mathcal{L}$-structure $\mathcal{A}$, what does it mean to say that ϕ *holds in* $\mathcal{A}$ *under* v (denoted by $\mathcal{A} \models \phi[v]$)?

(iii) Define $\mathtt{Th}(\mathcal{A})$, the *$\mathcal{L}$-theory* of an $\mathcal{L}$-structure $\mathcal{A}$. What does it mean for two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ to be *elementarily equivalent* (denoted by $\mathcal{A} \equiv \mathcal{B}$)?

(iv) What does it mean to say that a set Σ of $\mathcal{L}$-sentences
is *maximal consistent*;
has a *model*?

(b) [8 marks]

(i) Show that a set Σ of $\mathcal{L}$-sentences is maximal consistent if and only if Σ has a model and any two models of Σ are elementarily equivalent.
[You may use the Soundness and Completeness Theorem for $K(\mathcal{L})$ if you state them clearly.]

(ii) Define two $\mathcal{L}$-structures $\mathcal{A} = \langle A; E_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B; E_{\mathcal{B}} \rangle$ to be *isomorphic* (denoted by $\mathcal{A} \cong \mathcal{B}$) if there is a bijective map $g : A \to B$ such that, for all $a, a' \in A$, $E_{\mathcal{A}}(a, a')$ holds in $\mathcal{A}$ if and only if $E_{\mathcal{B}}(g(a), g(a'))$ holds in $\mathcal{B}$. Show that $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$.

(c) [9 marks]

(i) Write down $\mathcal{L}$-sentences ρ, σ and τ expressing that E is reflexive, symmetric and transitive respectively.

(ii) Find a set Σ of $\mathcal{L}$-sentences whose models are precisely those $\mathcal{L}$-structures $\mathcal{A} = \langle A; E_{\mathcal{A}} \rangle$ where $E_{\mathcal{A}}$ is an equivalence relation on A having infinitely many equivalence classes, but no finite equivalence classes.

(iii) Can Σ in (ii) be chosen finite? Carefully justify your answer.

(iv) Show that any two countable models of Σ as in (ii) are isomorphic and deduce that Σ is maximal consistent.

[You may use any major theorems if you state them clearly.]