“Feynman’s Derivation of the Schrödinger Equation”
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This is part of a weekly journal club series held by the Society of Physics Students under the University at Buffalo chapter.

For the original paper discussed at this meeting, click: here

1 Introduction

For this week, we began our discussion by reviewing Lagrangian Mechanics, global vs. local concepts of minimization, and a brief historical context for the importance of this work by Feynman.

The aim of this paper and discussion was to show the transition between classical mechanics and quantum mechanics by means of minimizing the action of an object (classically) and how this relates to the phase of a wave-function in the quantum regime, and then further discussion was conducted by Lisong to show how this can be generalized further by means of the path integral.

2 Discussion

A note to the reader: It is strongly advised (and encouraged) that they read through the paper themselves to grasp the full meaning - as well as the historical context - of this work, for not all details will be explicitly covered in this review. It is good practice to understand a derivation independently before reading what others have to say on the matter, for it helps sharpen ones own intuition instead of relying on someone else’s, as well as giving one independent ideas so that they may be better prepared to prompt later discussion.

As is introduced in undergraduate mechanics courses, the Euler-Lagrange equations are a result of minimizing functions within the functional framework and having chose suitable generalized coordinates for the system. It can be seen that this minimization occurs globally, but the question arises: does it hold locally?

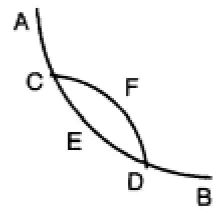

Figure 1: Local vs. Global minimization

As seen in Figure 1, we know ACEDB to be a global minimum, but is a segment of this, CED a minimum as well? One can determine another path, CFD, that is of the same length as CED, but not a minimum, thus the path ACFDB has the same length as ACEDB, but is not at all a minimum - this contradiction means that CED must be minimum if ACEDB is to be a minimum, i.e., local minimums hold as long as they are a segment of a global minimum. Keep this idea of local vs. global minimization in mind for later discussion below.

We return to the action integral, i.e., Hamilton's Principle, from which we obtain our Euler-Lagrange equations. The action is stated as:

$$\int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} \big(K - U\big)\, dt \tag{1}$$

from which the Euler-Lagrange equations for our motion of the system follow, for our selected generalized coordinates.

We continued with a brief review of the form of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U\psi = i\hbar\frac{\partial \psi}{\partial t} \tag{2}$$

and then quickly transitioned to the idea that a wave (as shown in optics) is best represented by the form:

$$Z = ae^{i\phi} \tag{3}$$

We note that the phase, ϕ, must vary as little as possible, according to Fermat's principle. We take the assumption: why not treat it (the phase, ϕ) as we do "S" for the action integral? This gives way to the wave-function, as in parallel to eqn. 3:

$$\psi = ae^{iS/\hbar} \tag{4}$$

where S and $\hbar$ have the same units to maintain the power as dimensionless. It should be noted that eqn. 4 is not unlike the relationship between energy and frequency given by E=hν.

We continue on with introducing the Hamilton-Jacobi equation:

$$\mathcal{H} + \frac{\partial S}{\partial t} = 0 \tag{5}$$

where $\partial S/\partial t$ can be shown to reduce to -E and is noted as a conservation law. Taking this to then become:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U = 0 \tag{6}$$

We then take eqn. 4 to be related to $\partial S/\partial x$ by:

$$\frac{\partial S}{\partial x} = -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial x} \tag{7}$$

and from here the rest of the derivation is given in the original paper, but we note ψ, ψ^*, $\partial\psi/\partial x$, and $\partial\psi^*/\partial x$ to be the generalized coordinates required for the Euler-Lagrange equations. The result of all of this is, as anticipated, the time-independent Schrödinger equation, as given by:

$$(U - E)\psi - \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = 0 \tag{8}$$

This amazing result is simply achieved by taking the phase of a wave in place of the action, and proceeding as such. However, this is not the full story.

Feynman generalizes this idea by deciding that the phase <u>is</u> the action, but before preceding to this, we discussed more of the history of Feynman's encounter with Professor Herbert Jehle, as well as reviewing Huygens' Principle. We take G(x,y) from Huygens' formulation to be the kernel, and see that all wave-fronts that affect another point in space to be given by:

$$\psi(x_2, t) = \sum_{x_i} g(x,t)\psi(x_1, t) \tag{9}$$

which becomes:

$$\psi(x_2, t) = \int G(x_1, x_2, t)\psi(x_1, t) \tag{10}$$

and we now jump with joy, because we love mathematical niceties, when we notice G(x_1,x_2,t) to be Green's function (i.e., kernel - or, if one is discussing particle physics, it is taken to be the fermion or boson propagator).

Now, Feynman accounts that Dirac stated G(x_1,x_2,t) to be "analogous" to $e^{iS/\hbar}$, but this would not do (such ambiguity), so Feynman took them to be equal. He also applied the ideas behind local vs. global minimization (as discussed above) so that he could use a wide range of approximation techniques. One such approximation being:

$$S = \int \mathcal{L}\, dt \approx \mathcal{L}_{avg.}\epsilon \tag{11}$$

which we reasoned by thinking about the Mean-Value Theorem presented in an introductory calculus course, just in terms of the integration bounds instead of the derivatives of the studied function.

The rest of the derivation is given in the paper, but the conclusion, that setting Green's function equal to the exponential term related to an action, S, does not work out the way we expected it after making the necessary Taylor approximations (spoiler alert...oops, too late). However, Feynman discovered that the terms are proportional to one another, so instead of setting them equal he placed a constant of proportionality in front of the exponential term, as seen by:

$$G(x_1, x_2, t) = Ae^{iS/\hbar} \tag{12}$$

where:

$$A = \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \tag{13}$$

and in the end, as shown in the paper, this assumption does hold, and the outcome is the full form of the Schrödinger equation, as given in eqn. 2 - again, it is strongly recommended that a serious student go through the details of this derivation.

...And Bob's your uncle! We have exactly what we set out to achieve.

3 Conclusion

We see that this incredible result is due to the following: ϕ is S, i.e., the phase <u>is</u> the action; local minimization holds when part of a global minimization; Green's function, from Huygens' formulation, is utilized when incorporating the wave mechanics into the wave-function; approximating the action to be the product of the average of the Lagrangian and a variable ϵ, which leaves us to utilize later in our Taylor approximations; and lastly, proportionality between Green's function and the exponential term that describes wave mechanics. This is quite the conclusion for such simple tools at ones disposal - it should be studied closely by the student.

The meeting ended with Lisong utilizing the path integral to generalize Feynman's result, a technique often used in quantum field theory (QFT) to take local quantization and generalize it to a global case. This result is given by:

$$Z[\varphi] = \int D\,\varphi\, e^{i \int d^4x \left(\mathcal{L} - J(u)\right)} \tag{14}$$

where we have made use of natural units.