Topics in Physics: Problem Set #3 (TA version)

Topics: circular motion, angular momentum, gyroscopes

General TA instructions

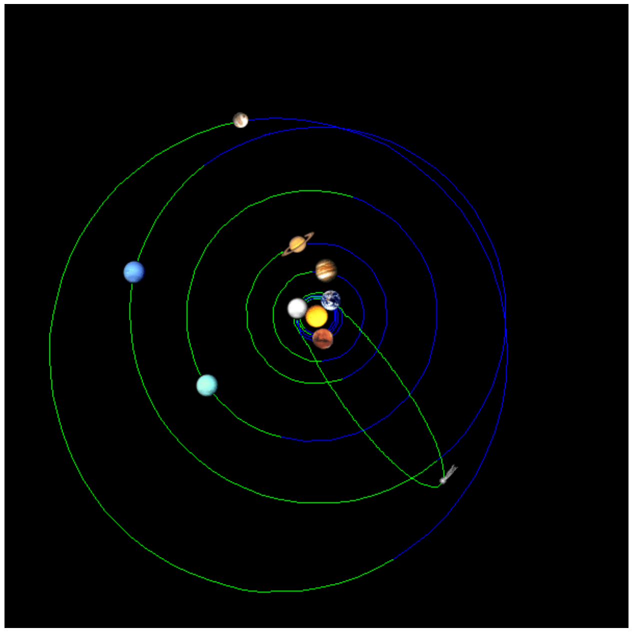
- Give students the estimated time for each section, and plan to spend about 50% of the estimated time going through answers.
- Try not to go over time for the earlier sections, since many of the more interesting activities are at the end of the assignment.
- Ask for student volunteers to explain their answers and discuss discrepancies.
- Be stingy with hints, but if a student seems legitimately stuck or won't finish the section in time, try to walk them through how to solve the problem without giving them the answer. You could also have them discuss with any students who have finished the section.

Practice Problems (approx. 45 min)

You should try to do these problems individually. None of them should take very long to solve; if you get stuck, ask a TA for help!

1. At the equator, the centripetal acceleration from Earth's rotation partially cancels out gravity. How strong is the centripetal acceleration relative to gravity?
   (a) $a_c = \frac{v^2}{r_{\text{earth}}}$, and $v = 2\pi r_{\text{earth}}/1\text{day}$. $a_c = 0.033\text{m/s}^2 = 0.0034g$
2. We are also subject to centripetal acceleration from the rotation of earth around the sun. Would this centripetal acceleration cancel or add to Earth's gravity? (Hint: the answer depends on the time of day.) How big is this acceleration relative to Earth's gravitational acceleration?
   (a) It adds to gravity during the day and subtracts at night. $r = 1$ astronomical unit, $v \sim 10^5\text{km/hr}$, so $a_c = 0.0059\text{m/s}^2 = 0.0006g$.
3. An astronaut uses a can of compressed air to spin himself. Starting with $\omega = 0$, he wants to accelerate and then decelerate so that he stops spinning just as he finishes 1 full revolution. If his maximum angular acceleration is 0.1rad/s^2, how quickly can he complete this maneuver?
   (a) The maneuver takes twice the amount of time it takes to rotate 1/2 revolution accelerating constantly. Using the second angular kinematic, we get $\pi\text{rad} = \frac{1}{2}\alpha t^2 = \frac{1}{2}\left(0.1\text{rad/s}^2\right)t^2$, so $t = 7.9$ seconds. So the full maneuver takes about 16 seconds.

4. Halley's comet follows a highly eccentric elliptical orbit around the sun, shown below, with a period of about 76 years. Its aphelion (farthest orbital radius) is about 5.3 billion km, where it orbits at a snail's pace[1] of 910m/s. The comet's perihelion (closest point to the sun) is 88 million km, where it whips past the sun at blazingly fast speeds. The comet has a mass of 2.2×10^{14}kg.



   (a) Calculate the comet's angular momentum at its aphelion.

      i. $\vec{L} = \vec{r} \times m\vec{v}$, but since $\vec{r}$ and $\vec{v}$ are orthogonal at perihelion and aphelion and we only want $\left\|\vec{L}\right\|$, then $L = rmv = 5.3 \times 10^9\text{km} \cdot 2.2 \times 10^{14}\text{kg} \cdot .910\text{km/s} = 1.061 \times 10^{30}\text{J} \cdot \text{s}$

   (b) How fast is the comet traveling when it is at its perihelion?

      i. Angular momentum is conserved, so at perihelion, $v = \frac{L}{rm} = \frac{1.061 \times 10^{30}\text{J}\cdot\text{s}}{88 \times 10^6\text{km} \cdot 2.2 \times 10^{14}\text{kg}} = 54.8\text{km/s}$.

5. A soccer ball with radius 20cm is rolling at 5 rotations per second and stops after 20m. What is its angular acceleration? How long does it take to stop?

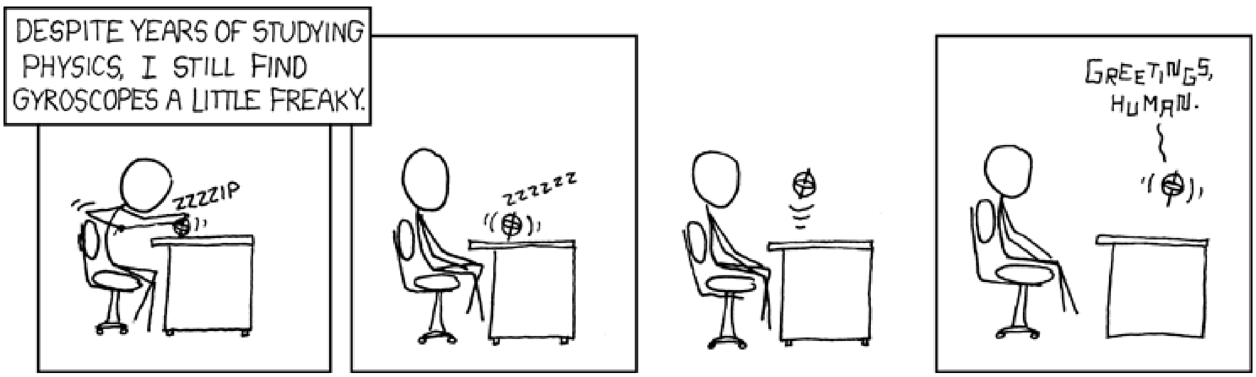
   (a) The circumference of the ball is $2\pi \times 20$cm, so it rolls about 16 times in 20m. That means $\theta = 16 \cdot 2\pi$. We can use $\omega_f^2 = \omega_i^2 + 2\alpha\theta$ to solve for the angular acceleration, which is $\alpha = 4.9\text{rad/s}^2$. Then we can use the $\omega_f = \omega_i + \alpha t$ to find the time, which is $t = 6.4$ seconds.

[1] We're of course talking about space snails, which still travel pretty fast by human standards.

Experiment: fun with gyroscopes (approx. 60-120 min)

TA instructions:

- This is a tricky derivation and might be the hardest problem the students will do in this course. However, I've tried to break it down into small steps and provided enough hints that I think the students should be able to manage this.
- Feel free to be a little bit more liberal with hints on this section if students seem frustrated or are falling behind on time. (Although I think by leaving out the challenge problems and not having a ton of practice problems I've left enough time for this experiment.)
- The derivation builds on itself, so try to check in occasionally with students to make sure they're getting the right answers as they go so they don't have to redo anything.



https://xkcd.com/332/

In this exercise, we'll analyze the physics behind gyroscopes to try to get an estimate of how fast you can spin the flywheel. Gyroscopes are a hard subject to understand (this is probably college level material if you had to do this from scratch!) but we'll walk through the analysis step by step. Take your time with each question and try to think deeply about what is going on.

Note: this experiment follows some derivations which are easy to stumble across if you search for gyroscope precession, so please don't search for anything related to gyroscope precession until you have finished this lab. (General searches for angular kinematics, vectors, etc. are fine.) If you get stuck or are confused, ask a TA!

Consider your gyroscope in the configuration shown below. The gyroscope has m, the moment of inertia of the flywheel about its axle is I, and the center of mass of the flywheel is offset from the pylon by a vector $\vec{r}$, which makes an angle ϕ relative to vertical. Let S denote the contact point between the pylon and the pivot.

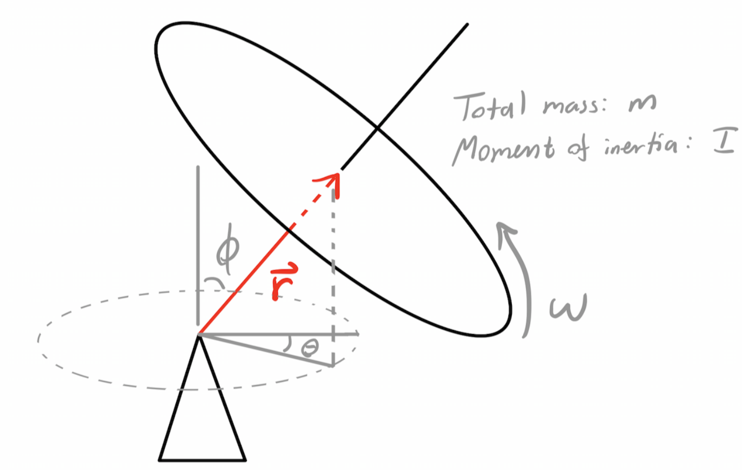


Figure 1: The configuration of your gyroscope as described above.

1. The gyroscope is affected by gravity (believe it or not), and it is held up by the pylon, so there will be a torque resulting from the gravitational force.
   (a) What is the magnitude of the torque about S resulting from the force of gravity?
      i. The torque has a magnitude of $rmg\sin\phi$
   (b) What direction does the torque point when the gyroscope is in the configuration shown in Figure 1?
      i. Directed into the page
   (c) As we learned in class, $\vec{\tau} = \frac{d\vec{L}}{dt}$. Write an expression for the amount $d\vec{L}$ that the angular momentum will change over a time dt.
      i. Just multiply $\vec{\tau}$ by dt: $d\vec{L} = \vec{\tau} \cdot dt = rmg\sin\phi \cdot dt$, directed into the page.
2. Assume the gyroscope is spinning with a constant angular velocity with magnitude ω and angular momentum $\vec{L}$.
   (a) What is the magnitude of $\vec{L}$ in terms of the parameters listed above?
      i. $\left\|\vec{L}\right\| = I\omega$
   (b) What direction does $\vec{L}$ point in?
      i. $\vec{L}$ points in the same direction as $\vec{r}$
   (c) What are the magnitudes of the vertical and horizontal components of $\vec{L}$? (Hint: think about limiting behavior of ϕ!)
      i. Horizontal is $I\omega\sin\phi$ and vertical is $I\omega\cos\phi$
3. The diagram shown in Figure 1 describes the gyroscope angle in spherical coordinates: ϕ denotes the *polar angle*, which is analogous to latitude (except measured from the poles, not the equator), and θ denotes the *azimuthal angle*, which is analogous to longitude.[2] In the first problem, you should have gotten an answer for $d\vec{L}$ which is orthogonal to $\vec{L}$, as shown in Figure 2.

[2]Annoyingly, it is just as common to use θ as the polar angle and ϕ for the azimuthal angle. This conflict has raged between scientists for years and covered the lands in mathematical bloodshed.

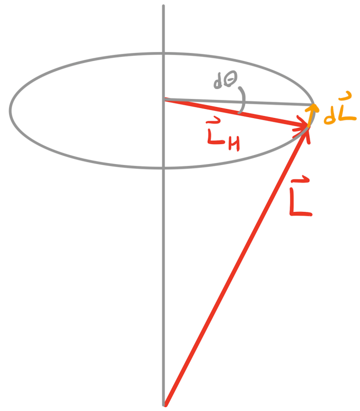


Figure 2: Illustration depicting the relation between $d\theta$ and dL.

(a) Referring to Figure 2, if the angular momentum $\vec{L}$ changes by some amount $d\vec{L}$ over a time dt, what is the resulting change in the angle $d\theta$? (Hint 1: think inverse trig functions!) (Hint 2: remember that θ is defined relative to the horizontal, so you may need to use the decomposed vectors you found in the previous problem in your expression.)

   i. $d\theta = \arctan\left(\frac{dL}{L_H}\right)$

(b) For small values of x, $\arctan(x) \approx x$. Since the change in angular momentum $d\vec{L}$ can be made arbitrarily small by taking $dt \to 0$,[3] we can replace our expression for $d\theta$ with $\arctan(x) \to x$. Using this, write an expression that determines the angle $d\theta$ (from the previous part) that the gyroscope precesses through in a time dt in terms of r, m, g, ϕ, I, ω and dt. (Hint: don't think too hard on this: at this point, aside from replacing $\arctan(x) \to x$, all you are doing is plugging in expressions you have found from the previous problems.)

   i. $d\theta = \frac{dL}{L_H} = \frac{rmg \sin\phi \cdot dt}{I\omega \sin\phi} = \frac{rmg}{I\omega} dt.$

4. You now have an expression which relates $d\theta$ to dt! As we learned in class, the angular frequency (the magnitude of angular velocity) is $\omega' = \frac{d\theta}{dt}$. (I'm using ω' because we're referring to a separate quantity here than ω: ω' measures the frequency of precession, while ω measures the angular frequency of the flywheel.) To find period of the precession, we need to use $T = \frac{2\pi}{\omega'}$.

   (a) What is the period of the gyroscope precession?

      i. The precession frequency is $\omega' = \frac{d\theta}{dt} = \frac{rmg}{I\omega}$, so the precession period is $T = 2\pi\frac{I\omega}{rmg}$.

5. You're almost there! Finally, since we don't know the mass of the flywheel itself, we would like to get an expression for T which is independent of m. We're going to make the (maybe slightly bad[4]) approximation that the flywheel comprises all of the mass of the gyroscope and that it is distributed in a ring with a radius r_{ring}.

   (a) What is the moment of inertia I of the gyroscope?

      i. $I = mr_{\text{ring}}^2$.

   (b) Substitute this into your expression for the precession period T and solve for the flywheel rotation rate ω. Check that your units work out.

      i. $T = 2\pi\frac{\cancel{m}r_{\text{ring}}^2\omega}{r\cancel{m}g}$, so $\omega = \frac{rg}{2\pi r_{\text{ring}}^2} \cdot T$. This has units of $\frac{\text{m}\cdot\frac{\text{m}}{\text{s}^2}}{\text{m}^2} \cdot \text{s}$, which is time^{-1}, which are the expected units for ω.

[3] Remember, this is what the definition of a derivative is!

[4] When I wrote this lab last night, I hadn't actually held the model of gyroscope I'll have given to you tomorrow, only seen pictures of it. The flywheel looks to comprise most of the mass of the gyroscope, but it's hard to tell.

Congratulations! You've made your way through some very sophisticated physics and you now have an expression for ω that is solely in terms of things that you already know or that you can easily measure. Now let's see how well experiment agrees with theory.

1. Using a ruler, measure the values of r (the offset from the pivot) and r_{ring} (the radius of the flywheel).
2. Spin your gyroscope and let it precess. Use a stopwatch to get an estimate of its precession period.
3. How many rpm are you able to spin your flywheel at?

Setting up for next class (10 minutes + download time)

Tomorrow's lecture will deal with computational physics, which is an exciting topic! Please <u>bring your laptop and its charger to class</u> tomorrow. So we don't waste time in the morning session, please set up a Python installation before then.

- Go to `https://www.anaconda.com/distribution` and download the Python 3.7 version for your OS (which should be automatically selected).
- Run the graphical installer and follow all the instructions.
- If prompted to check a box which includes something along the lines of "Add Python to PATH variable", check the box.
- When you are done, restart your computer, then open a terminal (Mac/Linux) or command prompt (Windows) and run the following commands to make sure things work:
  - `python`: will open a python interpreter. Press control+D to exit it.
  - `jupyter lab`: will open the Jupyterlab environment. Press control+C to exit.
- If you have any issues, get a TA to help you.