Topics in Physics: Problem Set #9 (TA version)

Topics: black hole thermodynamics

Reduced-length problem set

This problem set is designed to take only a portion of the afternoon session to complete. Plan to spend the remaining time in the session working on your final projects.

General TA instructions

- Give students the estimated time for each section, and plan to spend about 50% of the estimated time going through answers.
- Try not to go over time for the earlier sections, since many of the more interesting activities are at the end of the assignment.
- Ask for student volunteers to explain their answers and discuss discrepancies.
- Be stingy with hints, but if a student seems legitimately stuck or won't finish the section in time, try to walk them through how to solve the problem without giving them the answer. You could also have them discuss with any students who have finished the section.

Challenge Problems (approx. 60 min)

You may work in small groups to solve these problems, but each student should submit and understand their own answer. These problems are challenging but not impossible to solve. If you get stuck, ask another student or a TA how to approach the problem, and if you are helping another student, try to explain so they understand how to solve the problem (don't just give them the answer). Show all your work and walk the reader through the solution; you may get feedback on both the approach and the clarity of your solutions.

1. Approximately what is the information density (in bits per page) of typical single-sided printed text on paper?

   (a) Each page contains around 500 words, or 3000 characters, according to Google. Typical characters are encoded in ASCII, which is 256 possible characters, which means each character represents $\log_2(256) = 8$ bits of information. Then the information density is:

   $$\rho_{\text{information}} \approx \frac{8\text{bits}}{1\text{character}} \cdot \frac{3000\text{characters}}{1\text{page}} = 24000\frac{\text{bits}}{\text{page}}.$$

2. Bekenstein bound states that the maximum entropy contained in any region of space $S \leq \frac{2\pi k_B R E}{\hbar c}$. In the case of a black hole, this inequality is saturated, so $S_{BH} = \frac{2\pi k_B R E}{\hbar c}$, where R is the Schwarzschild

radius ($R = \frac{2GM}{c^2}$) and E is the total mass-energy of the black hole. The information contained in a black hole is $I_{BH} = \frac{1}{k_B \log 2} S_{BH}$, which represents the number of bits contained in the quantum states contained inside the event horizon. Show that this can be written as:

$$I_{BH} = \frac{4\pi G M^2}{\hbar c \log 2}.$$

(a) $I_{BH} = \frac{1}{k_B \log 2} \cdot \frac{2\pi k_B R E}{\hbar c} = \frac{2\pi R E}{\hbar c \log 2}$. Using $R = \frac{2GM}{c^2}$ and $E = Mc^2$, we have that $I_{BH} = \frac{2\pi}{\hbar c \log 2} \cdot \frac{2GM}{\cancel{c^2}} \cdot M\cancel{c^2} = \frac{4\pi G M^2}{\hbar c \log 2}$.

3. How many pages of paper would it take to print out all of the information contained inside a black hole with the mass of the sun?

   (a) Plugging in `4pi*G*(solarmass)^2/(hbar*c*log(2))` into WolframAlpha, there are about 1.5×10^{77} bits of information contained in the black hole. Then the number of pages required is about $1.5 \times 10^{77}\text{bits}/24000\frac{\text{bits}}{\text{page}} = 6.3 \times 10^{72}$ pages.

4. What would the mass of that many pages be? How does this compare to the mass of the known universe?

   (a) One ream of paper is 500 sheets which weighs about 2kg. Then $6.3 \times 10^{72}\text{pages} \cdot \frac{2\text{kg}}{500\text{pages}} \approx 2.5 \times 10^{70}\text{kg}$. Google points to a Cornell page which says the mass of the universe is around 3×10^{52}kg, so this is about 18 orders of magnitude more mass than what is contained in the universe.

5. If you took that much paper and made a black hole out of it, how many sheets of paper would you need to print out to contain all of the information in the new black hole?

   (a) WolframAlpha: `4pi*G*(2.5*10^70 kg)^2/(hbar*c*log(2))/24000` gives about 1×10^{153} pages of paper.