On the subject of circles

- work and energy
- circular motion
- angular kinematics + rolling
- angular momentum
- gyroscopy
- gravitation and orbits

Work + energy

(~30 min)

What is energy?

↳ good question

most basic concepts are hardest to explain

↳ "what is a number?"

Textbook definition: energy is a quantity that must be transferred to an object to perform work on or to heat the object

Energy is conserved but can change forms

Forms of energy	equation
kinetic	$\frac{1}{2}mv^2$
potential	
gravitational	mgh
elastic	$\frac{1}{2}kx^2$ (spring)
electrostatic	$\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$
magnetic	
chemical	
nuclear	
radiant	
thermal	

Energy conservation is due to <u>Noether's theorem</u>:

For each continuous symmetry a system has, there is a corresponding quantity which is conserved over time.

<u>Symmetry</u>

a physical or mathematical quantity of a system which is unchanged under some transformations

<u>Examples</u>

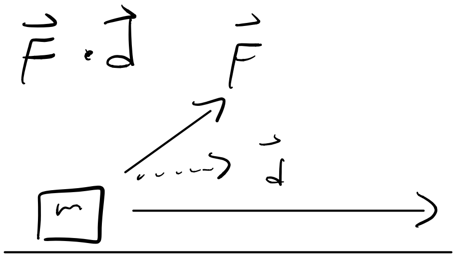
Time translation → energy
Space translation → linear momentum
Spatial rotation → angular momentum

Okay, so... what is work?

→ in mechanics, work is force integrated over distance

→ other definitions for other fields like thermodynamics, but they converge to same defn. for macroscopic systems

$W = \vec{F} \cdot \vec{d}$



$W = \vec{F} \cdot \vec{d}$

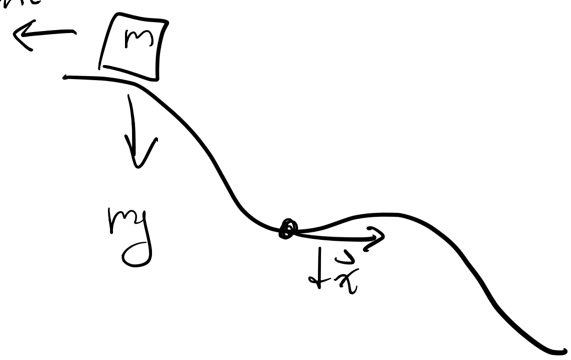
$= \|F\| \, \|d\| \cos\theta$

$= (F\cos\theta)\, d$

is picking up a brick work? yes
is carrying a brick work? no

$$W = \int \vec{F} \cdot d\vec{x} = \int \vec{F} \cdot \vec{v} \; dt$$

$\vec{F}_{fric}$ (always opposite to $\vec{x}$)



$W = \int \vec{F}_{fric} \cdot d\vec{x}$

parametrize $\vec{x}(t)$

$d\vec{x} = \frac{d\vec{x}}{dt} \, dt = \vec{v} \, dt$

$W = \int \vec{F}_{fric}(t) \cdot \vec{v}(t) \; dt$

without slipping $\vec{F}_{fric} \cdot \vec{V} = -F_{fric} v$

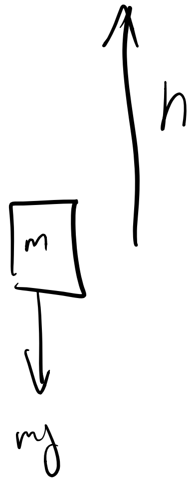
$$W = \int -F(t)\, v(t) \; dt$$

$= -$ (some value)

energy is lost

work can be negative

Ex: potential energy



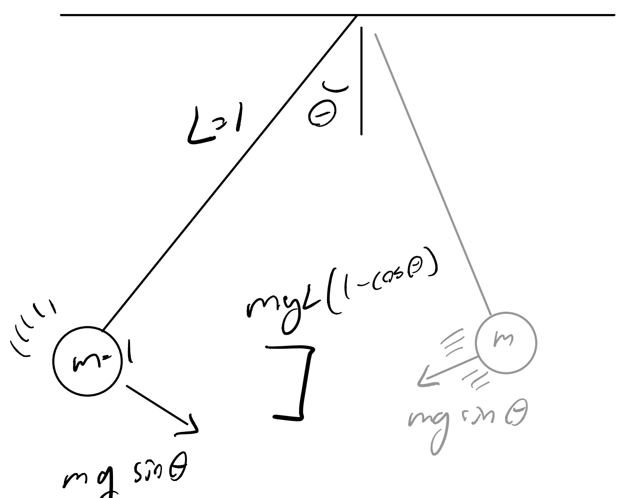
$U = mgh$

$$W = \int \vec{F} \cdot d\vec{x}$$

$$= F \cdot \Delta x$$

$$= mg \cdot h$$

Ex Pendulum energy



Start at $\theta = 30^\circ$ at rest

$KE = 0$

$PE = mgL(1-\cos\theta)$

$= 1 \cdot 10 \cdot 1\left(1 - \frac{\sqrt{3}}{2}\right)$

$= 10\left(1 - \frac{\sqrt{3}}{2}\right)$ J

at $\theta = 0^\circ$ how fast is going?

Kinematics:

$$v = \int a(t)\, dt$$

$$= \int g \sin(\theta(t))\, dt$$

$$\theta(t) = \cos\left(\frac{1}{T} \cdot t\right) = \cos\left(\frac{1}{2\pi\sqrt{\frac{L}{g}}}\, t\right)$$

... complicated!

Energy

$$PE \rightarrow KE = mgL\left(1 - \frac{\sqrt{3}}{2}\right)$$

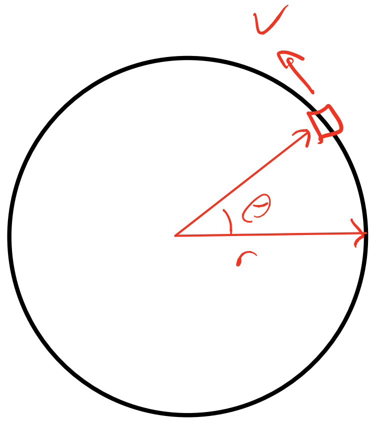
$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gL\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$v = \sqrt{2gL\left(1 - \frac{\sqrt{3}}{2}\right)}$$

$$= \sqrt{2 \cdot 10 \cdot 1\left(1 - \frac{\sqrt{3}}{2}\right)} \approx 1.63\ \text{m/s}$$

Circular motion (20 min)

Consider car on a track:



linear speed v

time to do one circle: period

$$T = \frac{2\pi r}{v}$$

if $|v|$ is constant, what is $\theta(t)$?

$$\theta(t) = 2\pi \cdot \frac{\text{dist}}{\text{circumference}}$$

$$= 2\pi \cdot \frac{vt}{2\pi r} = \frac{vt}{r}$$

Angular kinematic equations: similar to linear kinematics!

Linear	units	Angular	units
x: position	m	θ: angle	rad (unitless)
v: velocity	$m\,s^{-1}$	ω: angular velocity	s^{-1}
a: acceleration	$m\,s^{-2}$	α: angular acceleration	s^{-2}
t: time	s	t: time	s

keep this up on board & add eqns as we go

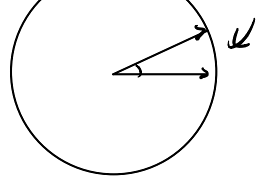
$\theta(t) = \frac{vt}{r}$ is angle, what is angular velocity $\omega(t)$?

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{d}{dt}\left(\frac{v}{r} \cdot t\right) = \frac{v}{r}$$

$$\downarrow$$

$$v = \omega r$$

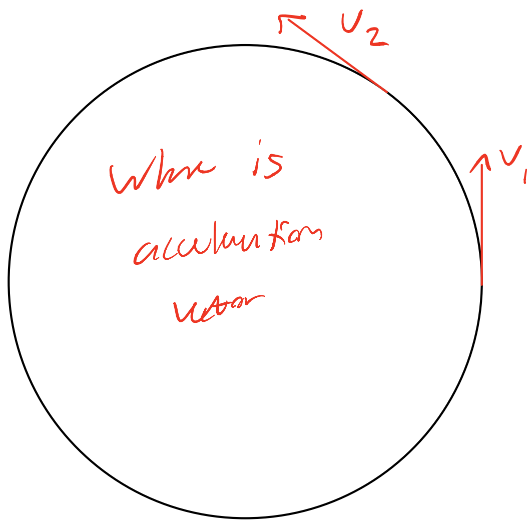
$r\,d\theta = dx$

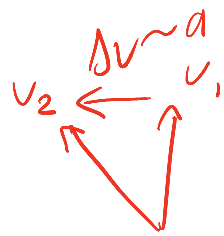


What is angular acceleration α?

$$\alpha(t) = \frac{d}{dt}\omega(t) = \frac{d}{dt}\left(\frac{v}{r}\right) = 0$$

Is the car accelerating? yes (velocity vector changes)

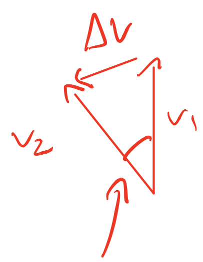




centripetal acceleration / force

derive magnitude:

$$\frac{\Delta v}{v} = \Delta\Theta = \omega\Delta t = \frac{v}{r}\Delta t$$



$\Delta\Theta = \omega\Delta t$

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

$$\frac{dv}{dt} = \frac{v^2}{r}$$

$$a_c = \frac{v^2}{r}$$

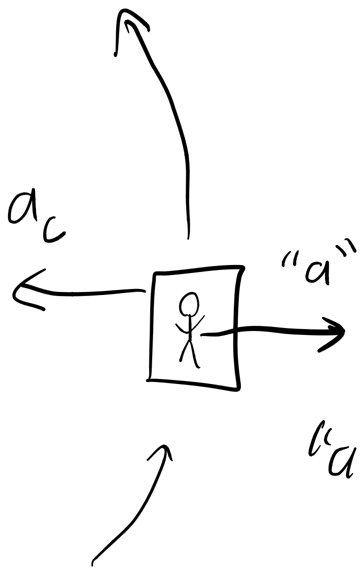
$v = \omega r$

$$a_c = \frac{\omega^2 r^2}{r} = \omega^2 r$$

Centrifugal force

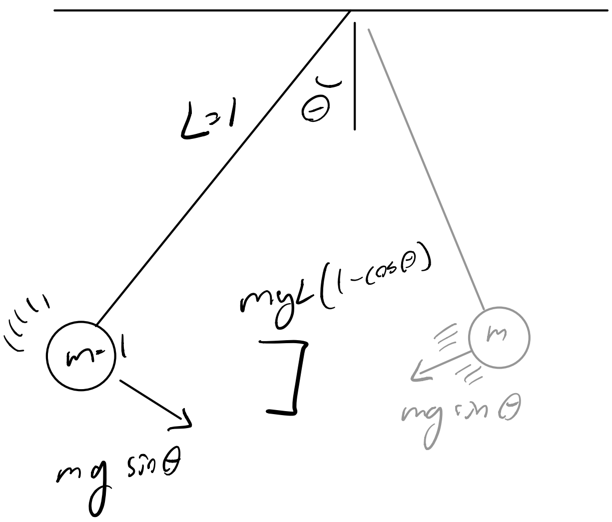
If you are in the car on the track, which way do you feel acceleration?

non-inertial reference frame!



"a" is a "fictitious force"

Pendulum example continued:



Start at $\theta = 30^\circ$ at rest

$KE = 0$

$$PE = mgL(1-\cos\theta) = 1\cdot 10\cdot 1\left(1-\frac{\sqrt{3}}{2}\right) = 10\left(1-\frac{\sqrt{3}}{2}\right)\ J$$

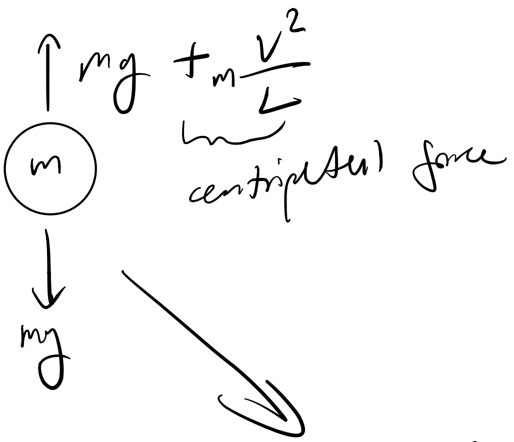
Energy

$$PE \to KE = mgL\left(1-\frac{\sqrt{3}}{2}\right)$$

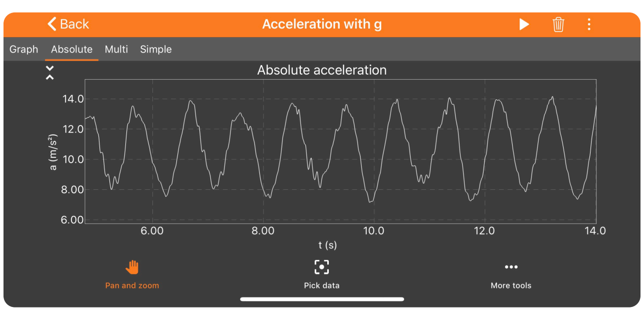
$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gL\left(1-\frac{\sqrt{3}}{2}\right)$$

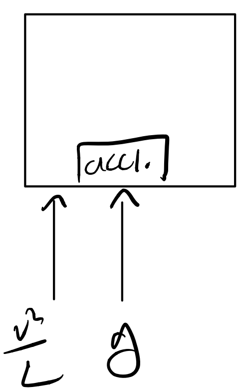
$$v = \sqrt{2gL\left(1-\frac{\sqrt{3}}{2}\right)} = \sqrt{2\cdot 10\cdot 1\left(1-\frac{\sqrt{3}}{2}\right)} \approx 1.63\ m/s$$

What is acceleration at $\theta = 0^\circ$?



in phone's reference frame:





"feels" total acceleration of $g + \frac{v^2}{L}$

$$= 9.8 + \frac{1.63^2}{1^2} = 12.45\ m/s^2$$

Angular dynamics (20 min)

Kinematic equations

Linear	units	Angular	units
x: position	m	θ: angle	rad (unitless)
v: velocity	$m\,s^{-1}$	ω: angular velocity	s^{-1}
a: acceleration	$m\,s^{-2}$	α: angular acceleration	s^{-2}
t: time	s	t: time	s

exactly analogous to linear eqns:

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v = at + v_0 \qquad \omega = \omega_0 + \alpha t$$

$$v_f^2 = v_i^2 + 2a\Delta x \qquad \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

Moment of inertia + torque

energy: $\frac{1}{2}mv^2 \longleftrightarrow \frac{1}{2}(?)\,\omega^2$

rotational energy is just energy! same units!

$$\frac{1}{2}mv^2 = \frac{\text{mass length}^2}{\text{time}^2}\,; \quad \frac{1}{2}(?)\,\omega^2 = (?)\cdot\frac{1}{\text{time}^2} \quad \frac{\text{mass length}^2}{\text{time}^2}$$

↳ (?) has units of mass · length^2

(?) is called moment of inertia ("angular mass")

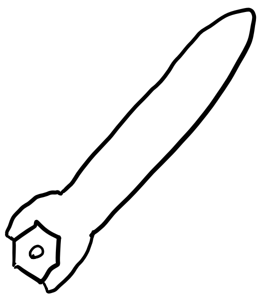
↳ denoted by I

↳ will talk more about this later today

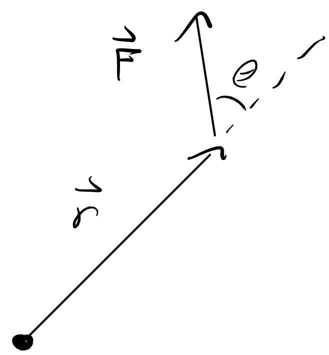
moment of inertia is quantity determining how much torque is needed to get some specified angular acceleration

Torque is angular analogue of force

suppose you want to tighten a bolt. what do you do to maximize effect?



- long lever arm (r)
- big force (F)
- apply force $\perp$ to lever (θ)



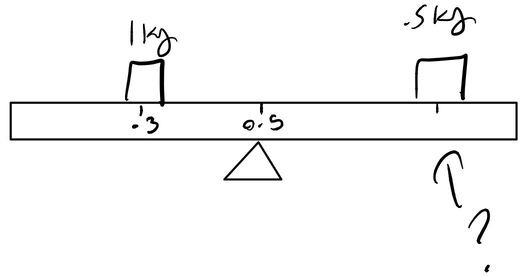
mag of torque is $\|\vec{r}\| \, \|\vec{F}\| \sin\theta$

↳ what's this look like?

Cross product!

$\vec{\tau} = \vec{r} \times \vec{F}$ ← torque is a vector quantity, just like force

Statics example 1



$\tau_1 + \tau_2 = 0$

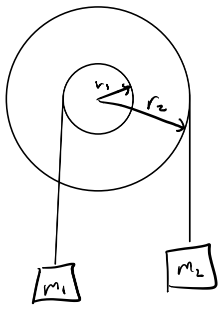
$\tau_1 = r_1 \times F_1 = .2m \cdot 1N$

$\tau_2 = r_2 \times F_2 = ?m \cdot .5N$

$? = .4m$

put at 0.9 m mark

Statics example 2



what condition for equilibrium?

$m_1 r_1 = m_2 r_2$

Newton's laws of (rotational) motion

	linear	angular
1.	objects move w/ const. velocity unless acted by a force	objects move w/ const angular velocity unless acted by a torque
2.	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$
3.	For every force there is equal + opposite force	For every torque there is equal + opposite torque

Final few angular analogues:

work	$\vec{F} \cdot \vec{d}$	$\vec{\tau} \cdot \vec{\theta}$
KE	$\frac{1}{2} m v^2$	$\frac{1}{2} I \omega^2$
power	$\vec{F} \cdot \vec{v}$	$\vec{\tau} \cdot \vec{\omega}$
momentum	$p = m\vec{v}$	$L = I\vec{\omega}$

we'll pick up here after break

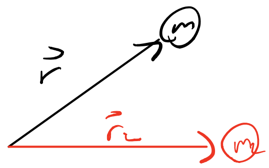
Break + AMA (10min) → at 80-90min

Angular momentum

Moment of inertia, cont. (15 min)

I = mass length^2

consider point mass at $\vec{r}$:

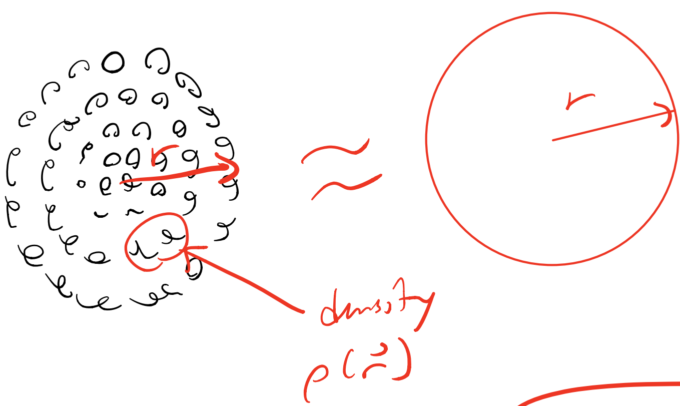


$$I = mr^2$$

$$I = m_1 r_1^2 + m_2 r_2^2$$

if we add another mass, I increases by that amt.

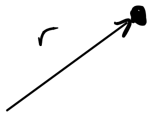
What if we have a bunch of masses?



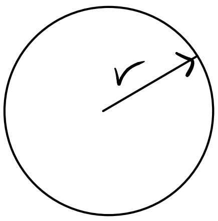
$$I = \sum_i m_i r_i^2 \longrightarrow \boxed{I = \int_{\vec{r} \in V} \rho(\vec{r}) |\vec{r}|^2 \, d\vec{r}}$$

Moment of inertia of an object depends on shape & mass distribution of the object!

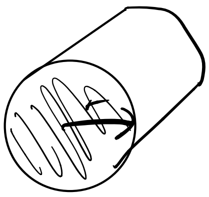
Common moments of inertia:



point mass $I = mr^2$ why?



ring $I = mr^2$



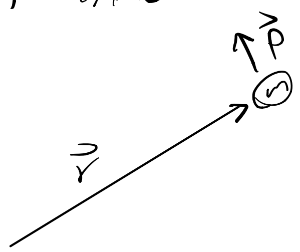
cylinder $I = \frac{1}{2} mr^2$

sphere $I = \frac{2}{5} mr^2$

all these are about symmetry axis,
I is different for a different axis!

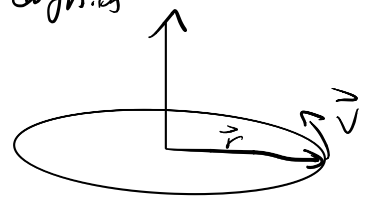
Angular momentum (15-20 min)

just like $\vec{\tau} = \vec{r} \times \vec{F}$, angular momentum of particle is



$$\vec{L} = \vec{r} \times \vec{p}$$

To relate $\vec{L}$ to ω, need vector defn. of $\rightarrow \vec{\omega}$



$\vec{L} = \vec{r} \times \vec{p}$

$= \vec{r} \times m\vec{v}$

$\vec{v} = \vec{\omega} \times \vec{r}$

$\vec{L} = m(\vec{r} \times \vec{\omega} \times \vec{r})$: what direction?

↳ do RHR, direction of $\vec{\omega}$

$\vec{L} = mr^2 \vec{\omega}$

↓ for many m

$$\boxed{\vec{L} = I\vec{\omega}}$$

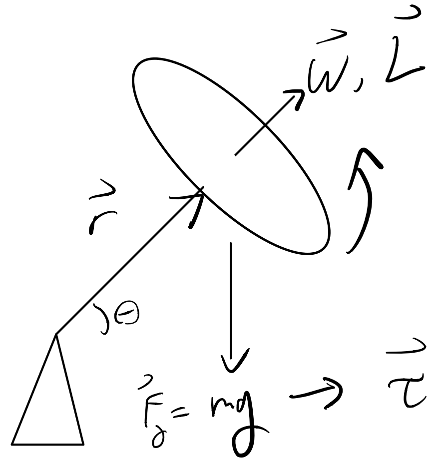
angular momentum is conserved!

↳ spinning chair demo?

Gyroscopes

have students open & play with gyroscopes for 5 min

why does the gyroscope precess?



$\vec{F}_g = mg \rightarrow \vec{\tau} = \vec{r} \times \vec{F}_g =$

$\vec{\tau} = \frac{d\vec{L}}{dt}$