Algebraic Geometry Codes

Background on Coding Theory

Background on Coding Theory

sender

receiver

m

m' = m + e

where e is some low Hamming weight noise term

Background on Coding Theory

sender

receiver

m

m' = m + e

where e is some low Hamming weight noise term

we don't want noise

Background on Coding Theory

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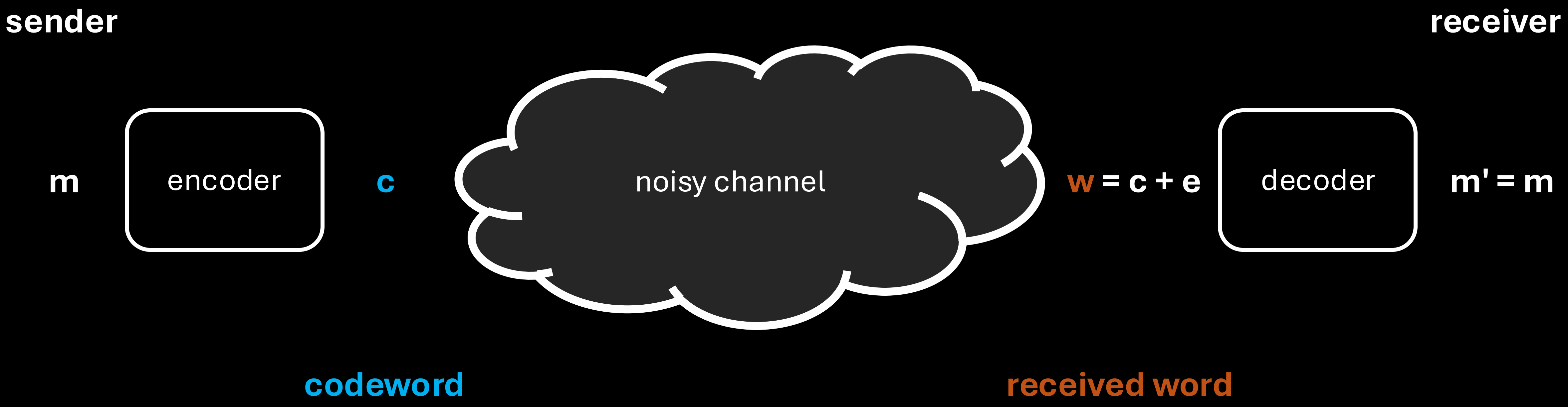
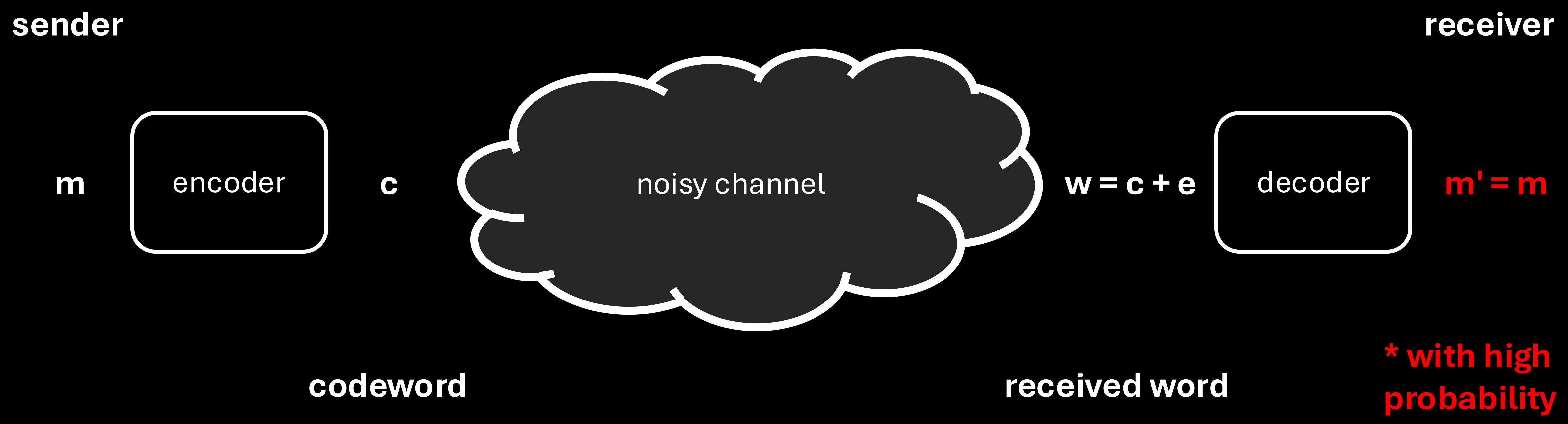
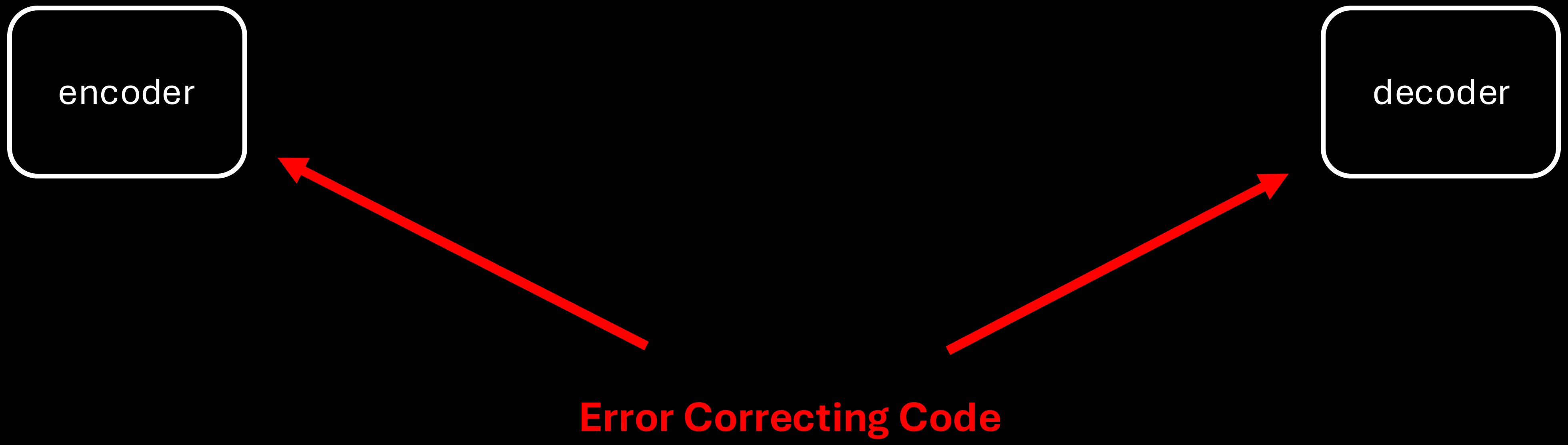


Background on Coding Theory



Background on Coding Theory



Linear Codes

Linear Codes

m

$\mathbb{F}_q^k$

message vector

Linear Codes

m

G

$\mathbb{F}_q^k$

$\mathbb{F}_q^{k \times n}$

message vector

generator matrix

Linear Codes

$$m \quad G \quad = \quad c$$

$\mathbb{F}_q^{k}$	$\mathbb{F}_q^{k \times n}$	$\mathbb{F}_q^{n}$
message vector	**generator matrix**	**codeword**

Linear Codes

the possible codewords form a subspace of $\mathbb{F}_q^n$

c_0

c_1

c_2

c_3

c_4

c_5

c_6

c_7

c_8

Linear Codes

2 important parameters

$R = k/n$

information rate

$D = d/n$

minimum distance

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Linear Codes

2 important parameters

$R = k/n$

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Singleton bound

$d \leq n - k + 1$

Reed-Solomon Codes

Reed-Solomon Codes

m

$m(x)$

$\mathbb{F}_q[x]$

Reed-Solomon Codes

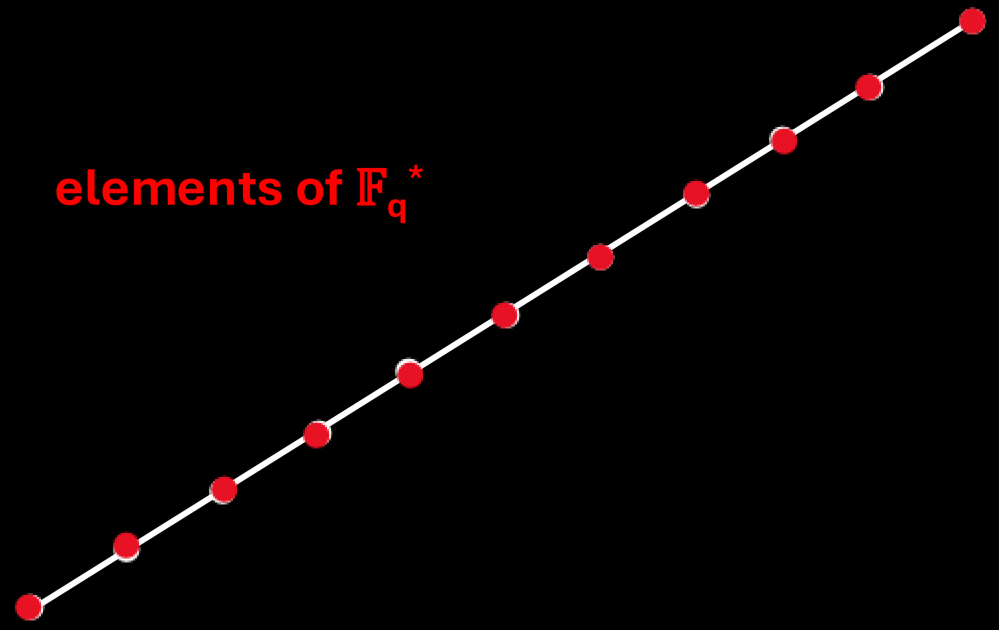
m

$\downarrow$

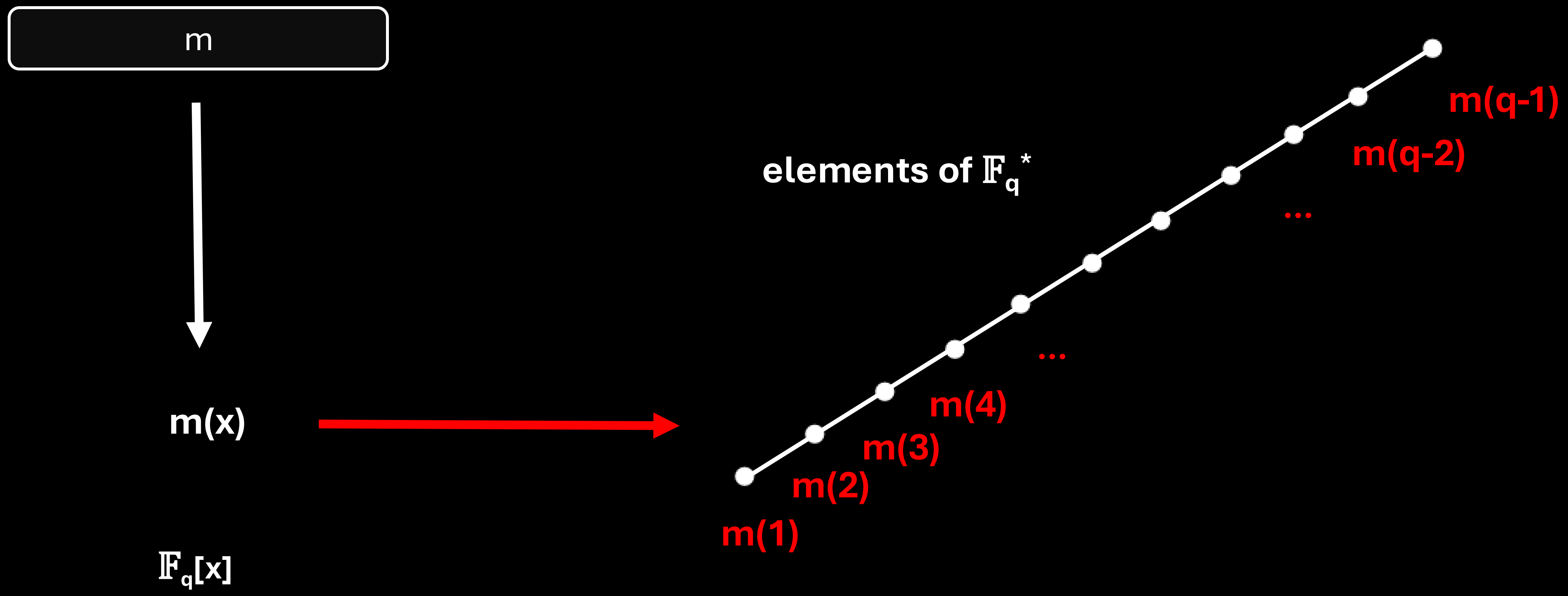
m(x)

$\mathbb{F}_q[x]$

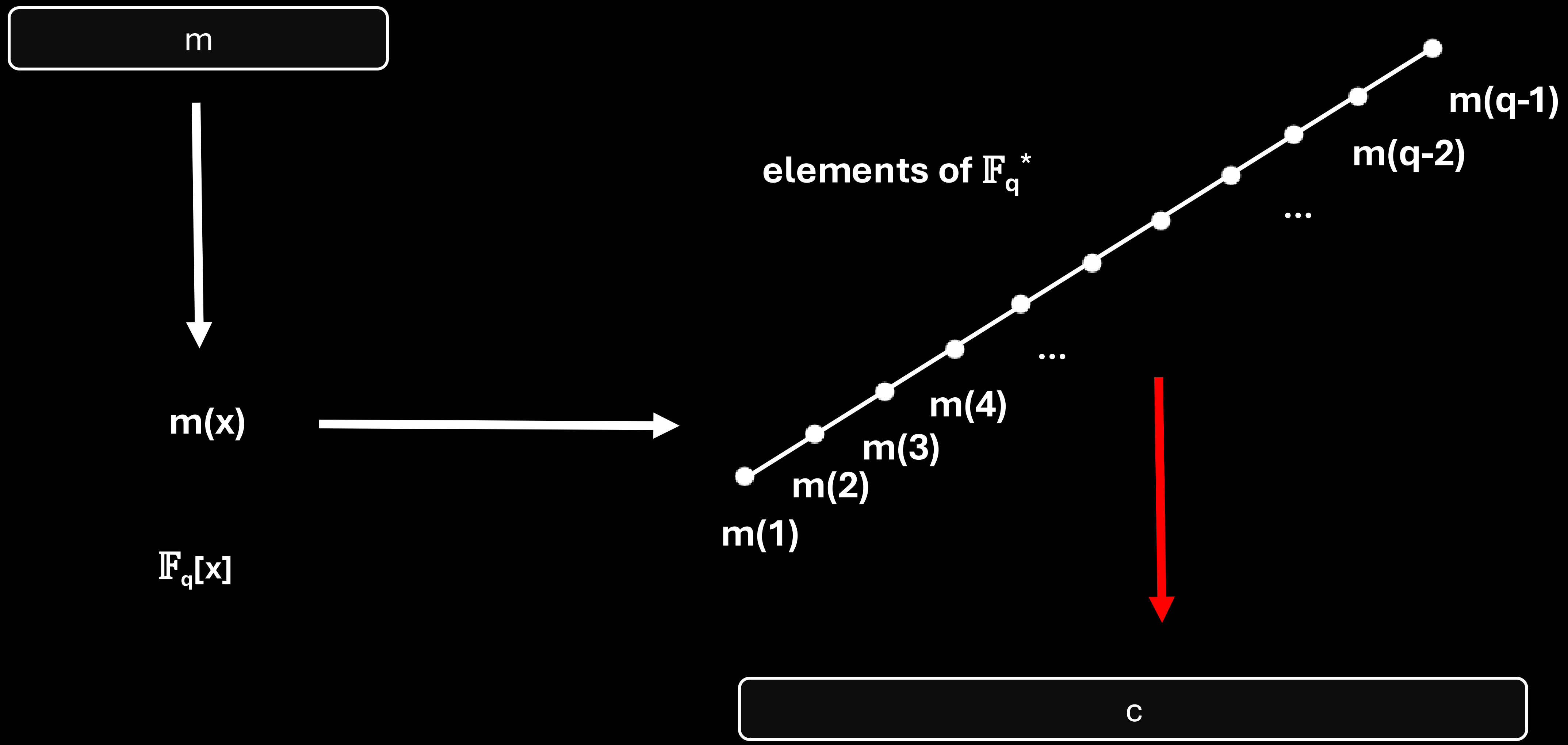
elements of $\mathbb{F}_q^*$



Reed-Solomon Codes

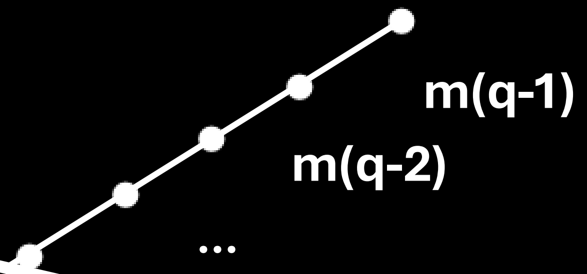


Reed-Solomon Codes



Reed-Solomon Codes

nts of $\mathbb{F}_q^*$



encoding = polynomial evaluation

$\mathbb{F}_q[x]$

c

Reed-Solomon Codes

pros:

MDS code $d = n - k + 1$

cons:

Reed-Solomon Codes

pros:

MDS code $d = n - k + 1$

linear code, easy en/decode

$$\begin{matrix} 1 & g & g^2 & g^3 & g^4 & \dots \\ 1 & g^2 & g^4 & g^6 & g^8 & \dots \\ 1 & g^3 & g^6 & g^9 & g^{12} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{matrix}$$

Vandermonde matrix

cons:

Reed-Solomon Codes

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MDS code $d = n - k + 1$

linear code, easy en/decode

$$
\begin{matrix}
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$$

Vandermonde matrix

cons:

n (#evaluation points) < q (#field elements)

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no asymptotically good family of codes as $n \to \infty$

Algebraic Geometry Codes

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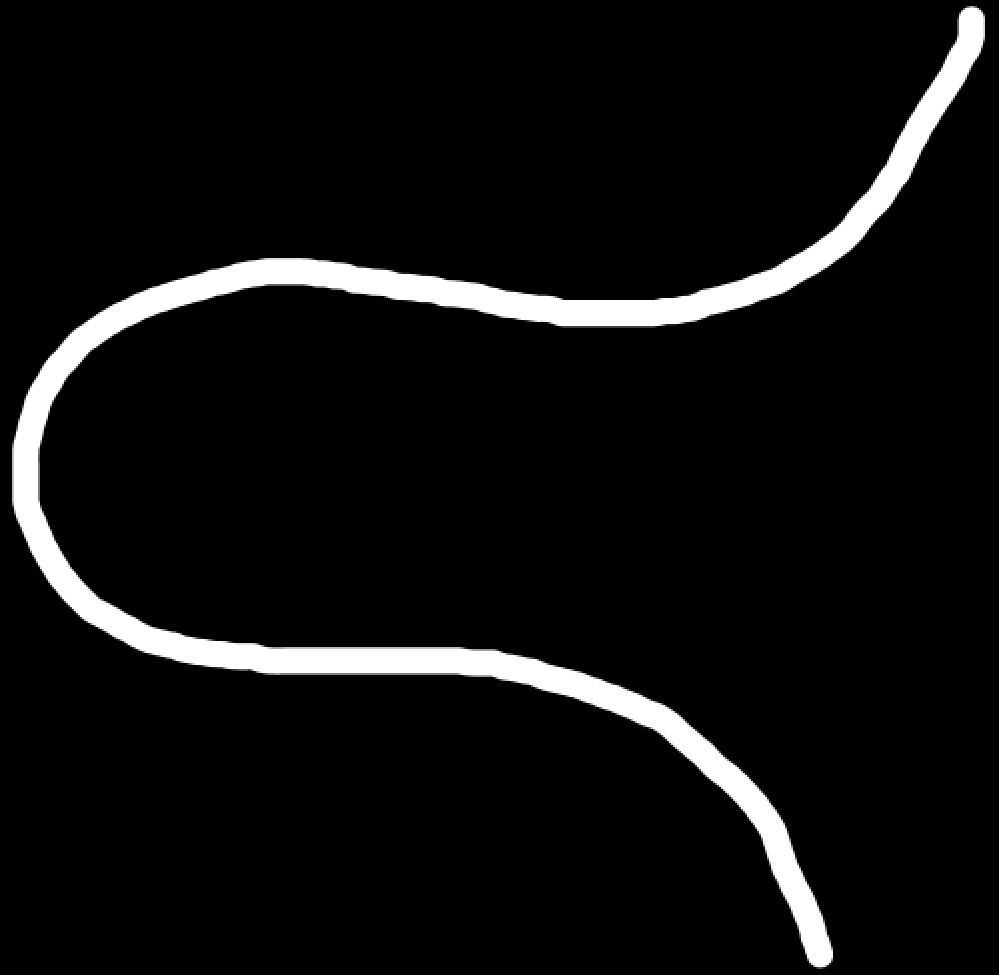
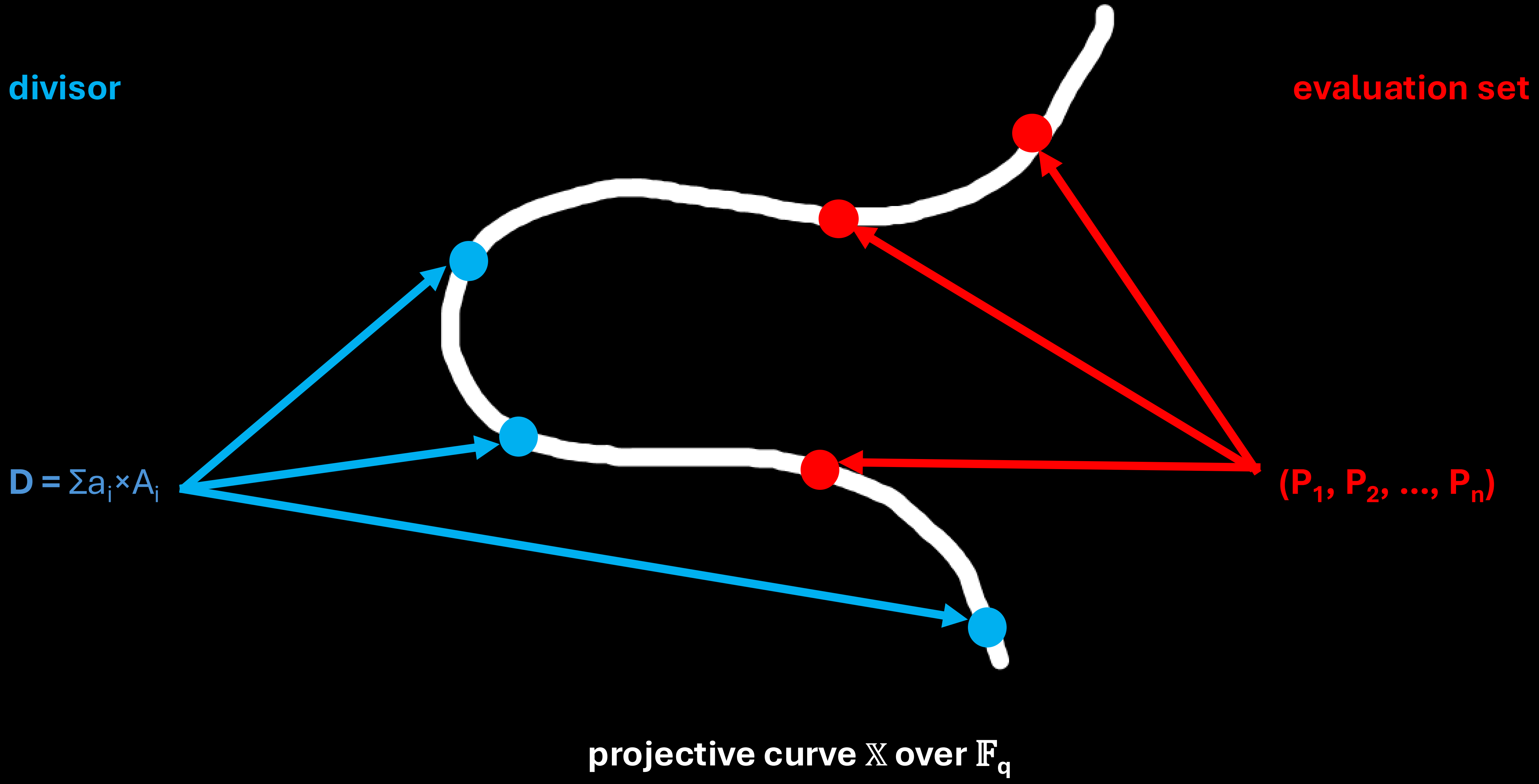


projective curve $\mathbb{X}$ over $\mathbb{F}_q$

Algebraic Geometry Codes



Algebraic Geometry Codes

Reed-Solomon **AG**

ambient function field

$\mathbb{F}_q[x] \longrightarrow F(\mathbb{X}) \approx \mathbb{F}_q[x,y]/(\mathbb{X})$

Algebraic Geometry Codes

Reed-Solomon **AG**

$\mathbb{F}_q[x] \xrightarrow{\text{ambient function field}} F(\mathbb{X}) \approx \mathbb{F}_q[x,y]/(\mathbb{X})$

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Algebraic Geometry Codes

Reed-Solomon		**AG**
$\mathbb{F}_q[x]$	$\xrightarrow{\text{ambient function field}}$	$F(\mathbb{X}) \approx \mathbb{F}_q[x,y]/(\mathbb{X})$
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elements of $\mathbb{F}_q$	$\xrightarrow{\text{evaluation points}}$	$(P_1, P_2, \ldots, P_n)$

Algebraic Geometry Codes

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MDS property	distance/dimension guarantee $\longrightarrow$	Riemann-(Roch) theorem
		$k = \dim(D) = \deg(D) + 1 - g(\mathbb{X})$
$d = n - k + 1$		$d = n - \deg(D)$

Code-based Public-Key Encryption

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hard problem: decoding a linear code, where G is random

Code-based Public-Key Encryption

hard problem: decoding a linear code, where G is random

idea: create a G' that looks random, but we can decode

Code-based Public-Key Encryption

keygen:

1. **generate randomly a structured G that we can decode**
2. **publish the masked G' := SGP**
3. **hope G' looks random to others**

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only known secure instantiation uses AG codes, binary Goppa codes specifically