Undergrad Complexity

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Contents

1 Primitive Recursive Functions

1.1 Various functions and operators

Definition. The **constant function** C_n^k is defined for all $n, k \in \mathbb{N}$ as $C_n^k(x_1, \dots, x_k) = n$.

Definition. The **successor function** S is defined for all $x \in N$ as $S(x) = x + 1$.

Definition. The **projection function** P_i^k is defined for all $i, k \in \mathbb{N}$, $1 \leq i \leq k$ as $P_i^k(x_1, \dots, x_k) = x_i$.

Definition. Given an m-ary function h and m k-ary functions $g_1, \dots, g_m$, the **composition operation** $\circ$ is defined:

$$f = h \circ (g_1, \dots, g_m)$$
$$f(x_1, \dots, x_k) = h(g_1(x_1, \dots, x_k), \dots, g_m(x_1, \dots, x_k))$$

Definition. Given a k-ary function g and a $(k+2)$-ary function h, the **primitive recursion operator** ρ is defined:

$$f = \rho(g, h)$$
$$f(0, x_1, \dots, x_k) = g(x_1, \dots, x_k)$$
$$f(S(y), x_1, \dots, x_k) = h(y, f(y, x_1, \dots, x_k), x_1, \dots, x_k)$$

1.2 Definition and examples

Definition. The constant function, successor function and projection function are **primitive recursive functions**, as well as any finite number of composition or primitive recursive operations on those functions.

Example. $Add : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive, as it can be defined by: $Add = \rho(P_1^1, S \circ P_2^3)$.

$$Add(0, x) = P_1^1(x)$$
$$Add(S(y), x) = S \circ P_2^3(y, f(y, x), x)$$

Let's run $Add(2,3)$:

$$\begin{aligned}
Add(2,3) &= \rho(P_1^1, S \circ P_2^3)(S(1),3) \\
&= (S \circ P_2^3)(1, Add(1,3), 3) \\
&= S(Add(1,3)) \\
&= S(\rho(P_1^1, S \circ P_2^3)(S(0),3)) \\
&= S((S \circ P_2^3)(0, Add(0,3), 3)) \\
&= S(S(Add(0,3))) \\
&= S(S(P_1^1(3))) \\
&= S(S(3)) \\
&= S(4) \\
&= 5
\end{aligned}$$

Example. $Mult : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive, as it can be defined by: $Mult = \rho(C_0^1, Add \circ (P_2^3, P_3^3))$.

$$\begin{aligned}
Mult(0,x) &= C_0^1 \\
Mult(S(y),x) &= Add \circ (P_2^3, P_3^3)(y, Mult(y,x), x)
\end{aligned}$$

1.3 Computability

It's pretty clear that all primitive recursive functions are computable, but are there computable functions that aren't primitive recursive?

Theorem 1. *There exists a computable function which is not primitive recursive.*

Proof. Roughly: enumerate primitive recursive functions, diagonalization argument. □