Undergrad Complexity: Problem Set 1

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Problem 1. A variadic function $f : \mathbb{N}^* \to \mathbb{N}$ is called a **coding function** if there are "inverse" functions $g : \mathbb{N} \to \mathbb{N}$ and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that:

$$g(f(a_1, \ldots, a_n)) = n$$
$$h(f(a_1, \ldots, a_n), i) = a_i, i \leq i \leq n$$

for all sequences $a_1, \ldots, a_n$. Thus g determines the length of a sequence and h decodes it back to its elements. Moreover, h and g are supposed to be easily computable but let's ignore that for the time being. Now consider the pairing function π defined by:

$$\pi(x, y) = \binom{x + y + 1}{2} + x + 1$$

and define f as follows:

$$f(nil) = 0$$
$$f(a) = \pi(0, a)$$
$$f(a_1, \ldots, a_n) = \pi(f(a_2, \ldots, a_n), a_1)$$

1. Show that π is injective.
2. Show that f is a coding function (make sure to explain what the appropriate decoding functions g and h are).
3. What whould happen if we replaced $pi(x, y)$ by $pi(x, y) - 1$? How could you fix the issue?

Answer (1.1)**.** First, note that:

$$\binom{x + y + 1}{2} = \frac{(x + y)(x + y + 1)}{2} \tag{1}$$
$$= 1 + 2 + \ldots + (x + y) \tag{2}$$

Take $(x_1, y_1), (x_2, y_2) \in \mathbb{N} \times \mathbb{N}$ such that $\pi(x_1, y_1) = \pi(x_2, y_2)$. Assume to the contrary that $x_1 + y_1 \neq x_2 + y_2$, and without loss of generality, suppose that $x_1 + y_1 < x_2 + y_2$.

$$\binom{x_1 + y_1 + 1}{2} + x_1 + 1 = \binom{x_2 + y_2 + 1}{2} + x_2 + 1$$

By (1) and (2):

$$\begin{aligned}
1 + 2 + \ldots + (x_1 + y_1) + x_1 + 1 &= 1 + 2 + \ldots + (x_2 + y_2) + x_2 + 1 \qquad (3)\\
x_1 - x_2 &= (1 + 2 + \ldots (x_2 + y_2)) - (1 + 2 + \ldots + (x_1 + y_1))\\
&\geq x_2 + y_2\\
x_1 &\geq 2x_2 + y_2\\
x_1 + y_1 &\geq 2x_2 + y_2 + y_1\\
&\geq x_2 + y_2,
\end{aligned}$$

a contradiction. Thus $x_1 + y_1 = x_2 + y_2$. Then, by (3), $x_1 = x_2$. So $y_1 = y_2$ and π is injective.

Answer (1.2)**.** C