# Complexity Project: The Oslo Model

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**Abstract:** This project imitate the behaviour of one of Self-Organised Critical Model: Oslo Model. And focus on the properties of the system: height of the system, average avalanche size of the system and $k^{th}$ moment of avalanche size. These properties are all depends on the system size. After scaling and data collapsing, all data can be fitted by one curve.

**Word count:** 2350 words in report

# 1 Introduction

Oslo model is one of the simplest models displaying self-organised criticality. It was first came up by Christensen in 1996 [1]. The Oslo model was developed by adapting 1D BTW model which describes sand-pile[2]. To understand the Oslo model, the rice-pile system can be considered as a good example. When we add rice to a rice-pile, the slope of the pile increases and avalanches will be triggered when the slope exceeds the threshold (criticality). After topping, the slope decreases, meanwhile the size of system increases. The property of adjusting the slope on itself is called self-organised criticality. The most difference between BTW model and Oslo model is: the threshold of BTW is fixed while the threshold of Oslo is flexible. We will investigate the properties of Oslo model in 1D with finite system size.

# 2 The Oslo model

## 2.1 Algorithm

Consider a 1D system with size $L$, so $i = 1, 2, 3...L$ are sites of the system. The number of grains on each site $i$ is noted as $h_i$, and the slop at site $i$, is defined as $z_i = h_i - h_{i+1}$. In order to calculate the slop of the last site, we set $h_{L+1} = 0$. The Oslo model system begins with empty state, i.e. $h_i = 0$, for$i = 1, 2, 3...L$, then add grains one-by-one at $i = 1$. The system will topple, for it cannot support a infinite slop, so when the slop exceed the threshold value, the grains will slid down to the next site. The threshold slop is set to be $z_i^{th} = \{1, 2\}$ randomly (since $h_i$ is integer, the slop and threshold slop are integers). Once the grains topple at site $i$, the grain number and slop of the next site increase, i.e. $h_{i+1} \to h_{i+1} + 1$, $z_{i+1} \to z_{i+1} + 1$. Then the threshold slop of site $i$ will reset, choosing a value randomly from $z_i^{th} = \{1, 2\}$. If $i = L$, the change is different, so we have to discuss them separately. The Oslo model can be defined by the following algorithm:[2]

1. *Initialisation.* Prepare the system in he empty configuration with $z_i = 0$ for all $i$ and choose random initial threshold slops for all site $i$,

$$z_i^{th} = \begin{cases} 1 & \text{with probability } p \\ 2 & \text{with probability } 1 - p \end{cases} \tag{1}$$

   in this report, $p = 1/2$

2. *Drive.* Add a grain at the left-most site $i = 1$:

$$\begin{aligned} z_1 &\to z_1 + 1 \\ h_1 &\to h_1 + 1 \end{aligned} \tag{2}$$

3. *Relaxation.* If $z_i > z_i^{th}$, relax site i.
   For $i = 1$:

$$\begin{aligned} z_1 \to z_1 - 2, z_2 \to z_2 + 1 \\ h_1 \to h_1 - 1, h_2 \to h_2 + 1 \end{aligned} \tag{3a}$$

For $i = 2, 3...L-1$:

$$\begin{aligned} z_i &\to z_1 - 2, z_{i\pm1} \to z_{i\pm1} + 1 \\ h_i &\to h_i - 1, h_{i+1} \to h_{i+1} + 1 \end{aligned} \tag{3b}$$

For $i = L$:

$$\begin{aligned} z_L &\to z_L - 1 \\ z_{L-1} &\to z_{L-1} + 1 \\ h_L &\to h_L - 1 \end{aligned} \tag{3c}$$

Choose a new threshold slop as in Eq.(1) for the relaxed site, and repeat the step 3 until all slops are smaller or equal to the corresponding threshold slop i.e.$z_i^{th} \geq z_i$ for all $i$.

4. *Iteration.* Return to step 2.

## 2.2 Implementation of Oslo model-Task 1

To test the model we built up, we need to check if the value of the height and slop of the system are reasonable. For size $L = 16, 32$, drive 100,000 times, and measure the height of the first site ($h_1$). The values of height are 26.496 and 53.732 respectively, and the corresponding average slops are 1.656 and 1.679. These two values are coincide with each other and reasonable. The threshold is set to be 1 or 2 equally and the average should be 1.5 at the beginning, but in the end, number of 2s is more than 1s, for the threshold of 1 is more likely to induce avalanche than the threshold of 2. And after each topple, the threshold will reset to 1 or 2 with 50-50 chance. As a result, more 2s left and the average slop is greater than 1.5.

# 3 Height of piles

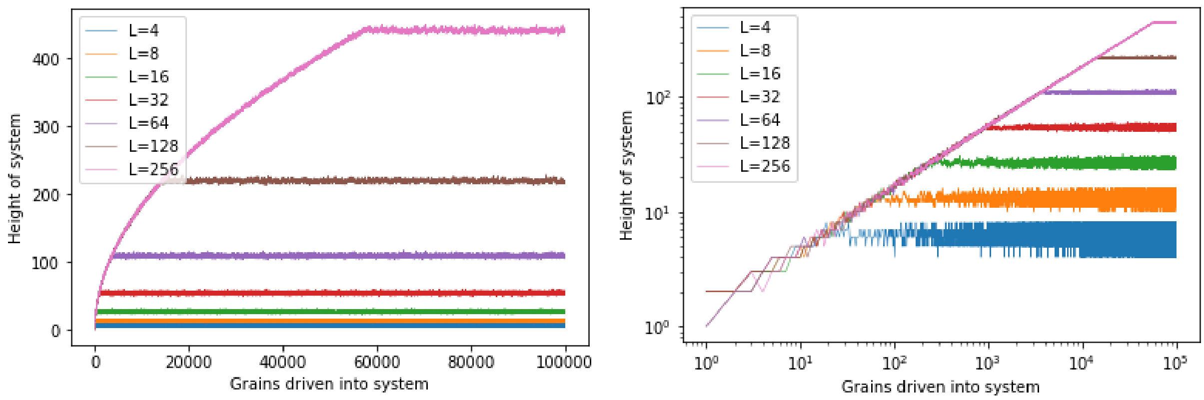
To observable the model better, we manipulate the model on different size of systems, $L = 4, 8, 16, 32, 64, 128, 256$. And the total grains added into the system is 100,000.

## 3.1 Height of piles-Task 2a

The height of the system $h(t; L)$ is defined as the height of the first site after the system reaches stable state, where $t$ is the number of grains added into the system and $L$ is the size of system. The height can be calculated by the following equation:[3]

$$h(t; L) = \sum_{i=1}^{L} z_i(t). \tag{4}$$

Starting from an empty system, the height of the system increases by the power law of the number of grains added in, shown in Fig. 1b, the height is linear to the number of grains in log-scale. The configuration in the increment period is called transit configuration, for it shows up only one time; When the number of grains comes to a critical point, the




(a) (b)

Figure 1: Height of system $h(t; L)$. (a) is in real scale, (b) is in log scale.

height of system will fluctuate around a certain value. The configuration in the fluctuation period is called recurrent configuration, for it shows up repeatedly.

When the system comes to the steady state, the grains added into the system may cause grains dropping out from the system to keep the number of grains be finite. Every topple will reset the slope threshold, which changes the configuration of the system, so the height of system fluctuate around a value.

## 3.2 Average cross-over time $\langle t_c(L)\rangle$-Tasks 2b

In a system of size L, the cross-over time $t_c$ is defined as the number of grains inside the system before an added grain leads to a grain leave the system for the first time,[3]

$$t_c(L) = \sum_{i=1}^{L} z_i \cdot i, \tag{5}$$

where $z_i$ is the slop of the system when the next added grain will cause a grain leaving the system for the first time. Use the equation above to calculate the cross-over time and repeat the process for 15 times so that the average value can be get. The results of calculation is shown in the Fig.2. It is clear that in log-scale, average cross-over time is proportional to size of system $L$ with slope $1.981 \pm 0.024$, which mean in the real scale, $\langle t_c\rangle \propto L^{1.981}$.

## 3.3 Assumption of the system-Task 2c

When the system is steady, we could make an assumption that the shape of the pile is a triangle whose base is $L$ and height is $\langle z\rangle \cdot L$, where $\langle z\rangle$ is a constant represents the average slope of state system, which means height $h \propto L$. In this assumption, the cross-over time should be the area enclosed by the triangle,

$$\langle t_c\rangle = \frac{1}{2} L \cdot \langle z\rangle \cdot L = \frac{\langle z\rangle}{2} L^2. \tag{6}$$

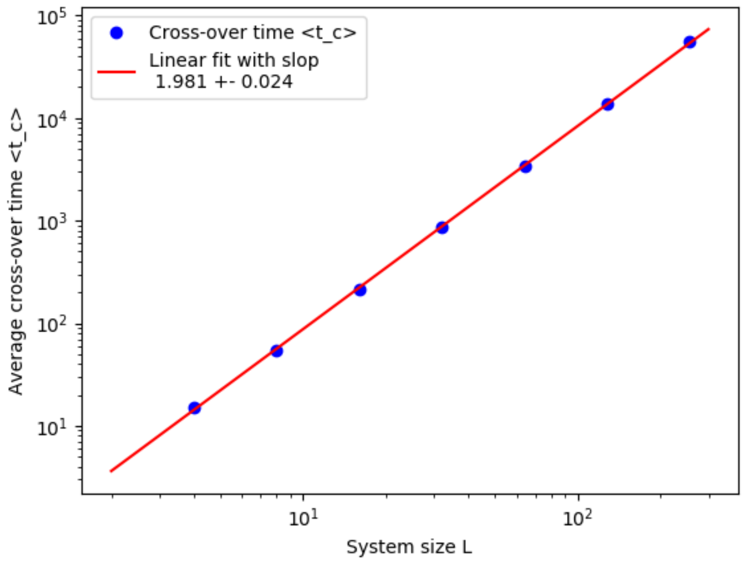



Figure 2: Average cross over time $\langle t_c \rangle$ is linear to system size $L$, with slop $1.982 \pm 0.024$.

Comparing Eq.(6) ($\langle t_c \rangle \propto L^2$) and the result in **section 3.2** ($\langle t_c \rangle \propto L^{1.981}$), we could say the assumption is reasonable.

## 3.4 Data collapse-Task 2d

From previous sections, we get $h \propto L$ and $t_c \propto L^2$. In order to scale it, we can write the scaling function:

$$\frac{\tilde{h}(t;L)}{L} = \mathcal{F}(\frac{t}{L^2}). \tag{7}$$

Let's take $\tilde{h}(t;L)/L$ be the $y$ value and $t/L^2$ be the $x$ value, we will get Fig.3. For convenience, we take $t/L^2$ as $x$. When $x \ll 1$ the scaled height increase linearly in log-scale with slope $0.51021 \pm 0.00002$; When $x \gg 1$, the scaled height becomes a constant, in another word:

$$\mathcal{F}(x) = \begin{cases} x^{0.51021} & x \ll 1 \\ x^0 & x \gg 1 \end{cases}$$

It is easy to understand that we take the cross-over time $t_c$ as one on x-axis and take the height t cross-over time as one on y-axis. When the number of grains smaller than $t_c$, the height has to increase (transit). When the number of grains go over $t_c$, grains might leave the system, and the system become steady state (recurrent). As a result, the height will fluctuate around its maximum value.

Eq.(6) is derived from our assumption, but it is not completely accurate, let's focus on its definition. Because of Eq.(5), we can get the average by

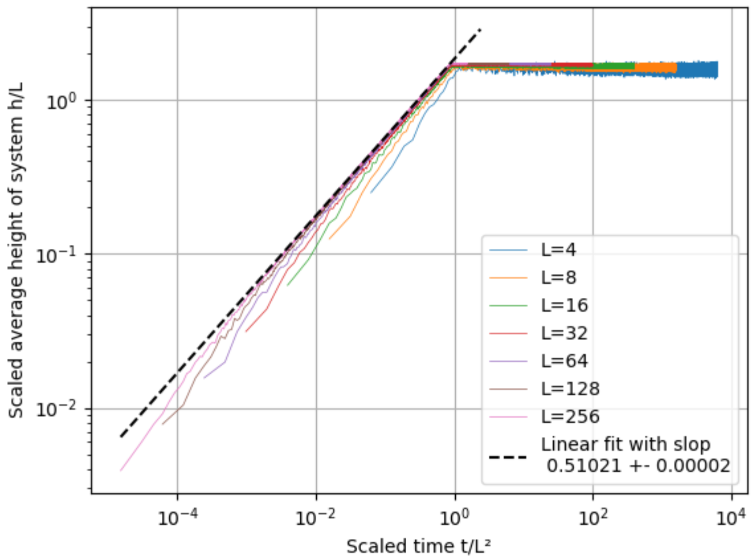



Figure 3: In log scale, the scaled height is proportional to scaled time, when scaled time smaller than one with slope $0.51021 \pm 0.00002$; When the scaled time greater than one, the scaled height fluctuate around one.

$$\langle t_c \rangle = \langle z_i \rangle \cdot \sum_{i=1}^{L} i = \langle z_i \rangle \frac{L(L+1)}{2} = \frac{\langle z \rangle}{2} L^2 \big(1 + \frac{1}{L}\big). \tag{8}$$

From this equation, we know that only when $L \gg 1$, the average of cross-over time $\langle t_c \rangle$ converges to $\frac{\langle z \rangle}{2} L^2$, which is the one we derived from our assumption.

### 3.5 Correction to scaling-Task 2e

To calculate the average recurrent height of system, we will use the equation: [3]

$$\langle h(t;L) \rangle_t = \lim_{T \to \infty} \frac{1}{T} \sum_{t=t_c+1}^{t_c+T} h(t;L),$$

where $t_c$ is the cross-over time. The calculation only take $t > t_c$ into consideration (recurrent configuration). Assume the average of height need to correct and the form of corrections is $\langle h(t;L) \rangle_t = a_0 L(1 - a_1 L^{-\omega_1})$, where $\omega_1 > 0$ and $a_0, a_1$ are constants. There are so many unknown constants, we cannot figure them out in one step. But we know the correction part $(a_1 L^{-\omega_1})$ is very small, and if we ignore it, it won't affect the results much. Based on this, we can calculate a rough $a_0$ value by curve fitting the average height $\langle h \rangle$ and the system size $L$ linearly (Fig.4a): $a_0 = 1.722 \pm 0.002$.

This tell us the accurate value $a_0$ is around 1.722. Rearrange the correction form and take logarithm on both sides, we will get

$$
\begin{aligned}
\langle h \rangle &= a_0 L(1 - a_1 L^{-\omega_1}) \\
\log(a_0 - \frac{\langle h \rangle}{L}) &= \log a_0 + \log a_1 - \omega_1 \log L.
\end{aligned} \tag{9}
$$

To get a precise value $a_0$, choose 150 values between 1.720 and 1.740, substitute them into Eq.(9) and use the function **stats.linregress** to find which $a_0$ fits best. Plot the adjusted $R^2$ respect to the values of $a_0$ and find which $a_0$ corresponds to the maximum $R^2$. Because the large the value $R^2$ is, the better the curve fits (When $R^2 = 1$ means the curve fitting is perfect).

In Fig.4b, we get the optimised value $a_0 = 1.734 \pm 0.001$. Then substitute the value back into Eq.(9), we get the Fig.4c. The slope is $-\omega_1$ and the intercept is $\log a_0 + \log a_1$, so $\omega_1 = 0.579 \pm 0.020$ and $a_1 = 0.211 \pm 0.012$.

### 3.6 Standard deviation of the height-Task 2f

The standard deviation is given by [3]

$$
\begin{aligned}
\sigma_h(L) &= \sqrt{\langle h^2(t;L) \rangle_t - \langle h(t;L) \rangle_t^2} \\
&= \sqrt{\lim_{T \to \infty} \frac{1}{T} \sum_{t=t_c+1}^{t_c+T} h^2(t;L) - \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=t_c+1}^{t_c+T} h(t;L) \right]^2}.
\end{aligned} \tag{10}
$$

Plot the $\sigma_h$ points on a log-scale, we find it is linear to the size of system $L$. Then we find the slope of the curve is $0.235 \pm 0.005$, as shown in Fig.5. As a result, on a real-scale, $\sigma_h \propto L^{0.235}$.

From Eq.(4), we can get

$$
z(t;L) = \frac{1}{L} \sum_{i=i}^{L} z_i = \frac{1}{L} h(t;L) \tag{11}
$$

Clearly, the average of slop over times at recurrent is

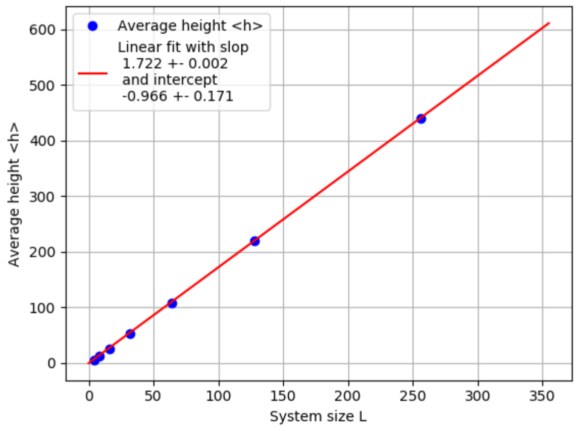
$$
\langle z \rangle = \frac{1}{L} \langle h \rangle = a_0(1 - a_1 L^{-\omega_1}) \tag{12}
$$

$$
\langle z^2 \rangle = \frac{1}{L^2} \langle h^2 \rangle. \tag{13}
$$

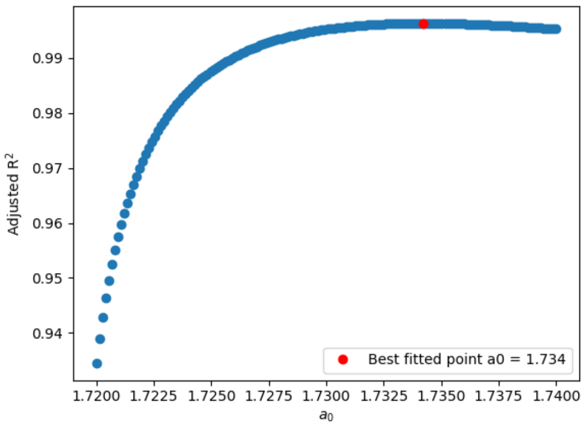
Then, the standard deviation of slope is

$$
\begin{aligned}
\sigma_z &= \sqrt{\langle z^2 \rangle - \langle z \rangle^2} = \sqrt{\frac{1}{L^2} \langle h^2 \rangle - \left[ \frac{1}{L} \langle h \rangle \right]^2} \\
&= \frac{1}{L} \sigma_h \\
&\propto L^{0.235-1} = L^{-0.765}.
\end{aligned} \tag{14}
$$

When $L \to \infty$, the average of slope converges to $a_0$, because the term $a_1 L^{-\omega_1} \to 0$ (Eq.(12)); The standard deviation of average slope $\sigma_z \propto L^{-0.765} \to 0$.




(a)

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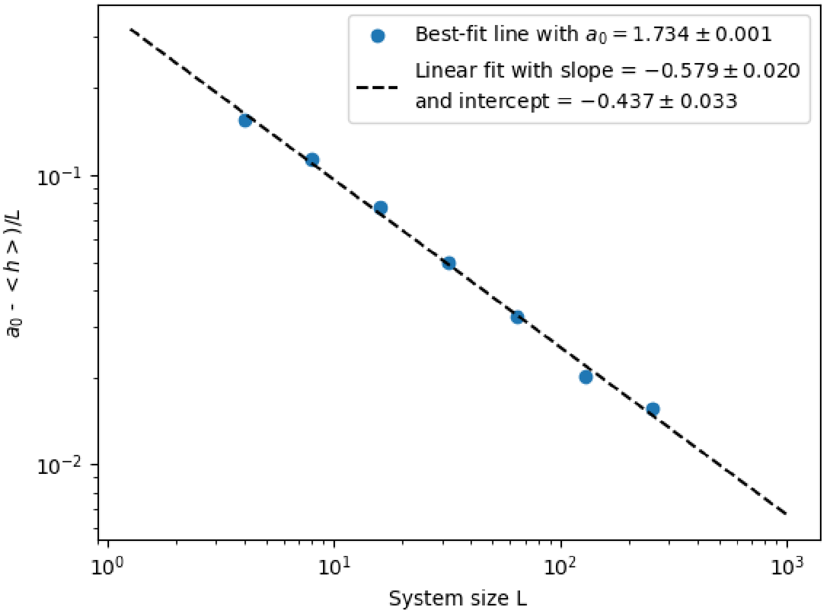


(c)

Figure 4: (a) Average of height $\langle h \rangle$ is proportional to the size of system $L$, with slope $1.722 \pm 0.002$, and the intercept point is $-0.966 \pm 0.171$.
(b)The best fitted curve is when $a_0 = 1.734 \pm 0.001$.
(c)Curve fit Eq.(9),with fixed $a_0 = 1.734$, the slope equals $-0.579 \pm 0.020$ and the intercept is $-0.437 \pm 0.033$.

### 3.7 The height probability-Task 2g

The height probability is defined by

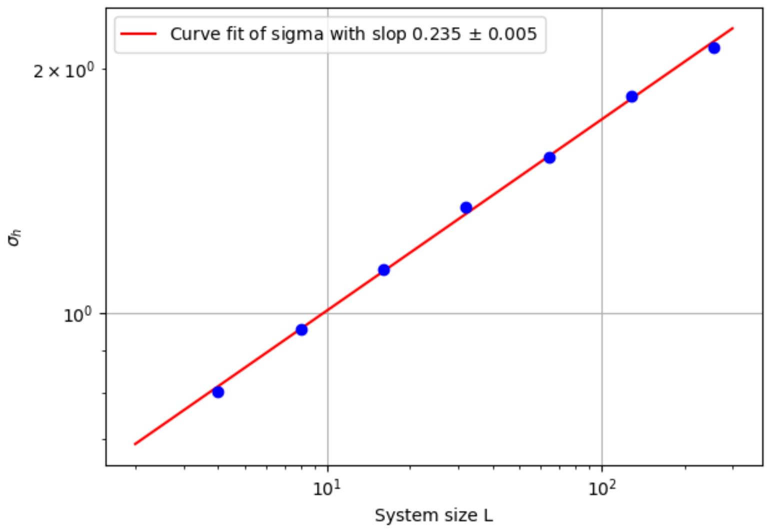


Figure 5: Standard deviation of height $\sigma_h$ is proportional to the size of system $L$ in log-scale, with slope $0.235 \pm 0.005$.

$$P(h;L) = \frac{\text{No. of observed recurrent configurations with height h in pile of size L}}{\text{Total no. observed recurrent configurations.}}$$

The height probability must be normalised, i.e. $\sum_{h=0}^{\infty} P(h;L) = 1$. Using the definition of height (Eq.4), and assume that $z_i$ are independent, identically distributed random variables with finite variance. In central limit theorem, when independent random variables are added, their properly normalised sum tends toward a normal distribution. The variable $z_i$ induces the height $h$ has the same property as $z_i$, so thee distribution of of height probability $P(h;L)$ is expected to be Gaussian when $L \gg 1$. The measured height probability $P(h;L)$ is shown in Fig.6, and the shape of each curve is approximately Gaussian.

The Gaussian distribution has the expression:

$$\begin{aligned} P(h;L) &= \frac{1}{\sigma_h\sqrt{2\pi}} \exp\Big[-\frac{1}{2}\Big(\frac{h-\langle h\rangle}{\sigma_h}\Big)^2\Big] \\ \Rightarrow P(h;L)\sigma_h &= \frac{1}{\sqrt{2\pi}} \exp\Big[-\frac{1}{2}\Big(\frac{h-\langle h\rangle}{\sigma_h}\Big)^2\Big]. \end{aligned} \tag{15}$$

In order to collapse the data, we take $(h - \langle h\rangle)/\sigma_h$ as new variable and replace $P(h;L)$ by $P(h;L)\sigma_h$. Then, we get the collapsed figure $P(h;L)\sigma_h \propto \mathcal{F}((h-\langle h\rangle)/\sigma_h)$ in Fig.7.

# 4 Probability of avalanche

This section will focus on the avalanche-size probability and associated moments. We only consider the avalanches after the system has reached the steady state, i.e. only take the values when $t > t_c$, where $t_c$ is cross-over time.

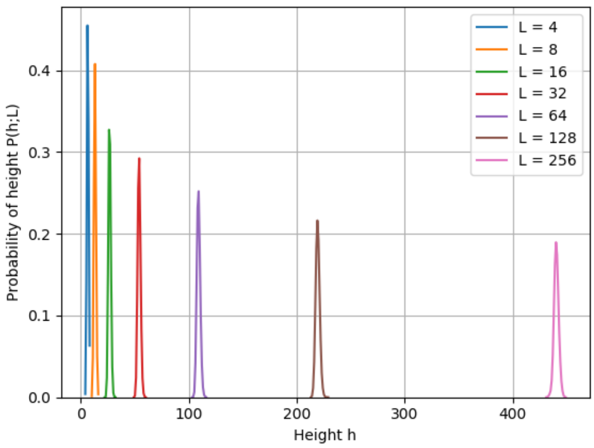


Figure 6: Graph of height probability $P(h; L)$ against height $h$ for a range of system size. Each curve has the shape of Gaussian.

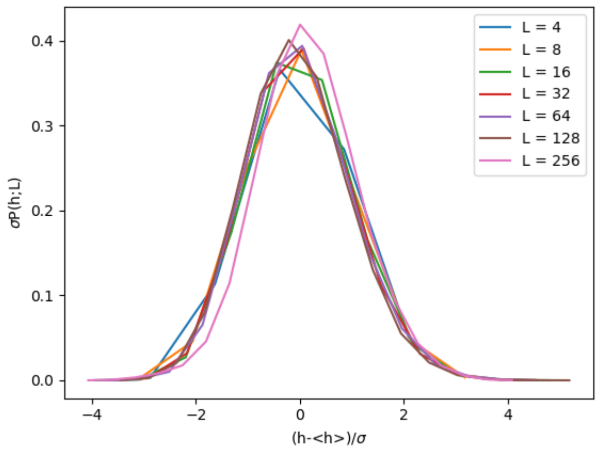


Figure 7: The collapsed figure for various system size of $P(h; L)$

The avalanche-size probability is defined by

$$P_N(s; L) = \frac{\text{No. of avalanches of size s in a system of size L}}{\text{Total no. avalanches N}}$$

The avalanche-size probability must be normalised, i.e. $\sum_{h=0}^{\infty} P_N(h; L) = 1$.

## 4.1 Log-binned data-Task 3a

The log-binned avalanche-size probabilities $\tilde{P}_N(s; L)$ was plotted against avalanche size $s$ in various system size $L$ as shown in Fig.8.

There is a cut-off avalanche size in each system size $L$ as shown in figure. Before the cut-off point, the probability decays linearly to the avalanche size in log-scale, which means the probability $\tilde{P}_N(s; L)$ decays as a power law of avalanche size $s$ in real-scale. The scale factor is the slope in Fig.8, which is $-1.523 \pm 0.012$. After the cut-off point, the probability drops rapidly. The cut-off point is proportional to the size of system. This is because the avalanche size is limited by the finite system size $L$, the maximum of avalanche increases with system size, the larger the system size is, the lager the maximum avalanche size is.

## 4.2 Data collapse-Task 3b

The finite-size scaling ansatz of avalanche-size probability is

$$\tilde{P}_N(s; L) \propto s^{-\tau} \mathcal{G}(s/L^D), \tag{16}$$

for $L \gg 1$ , $s \gg 1$. From last section, the constant $\tau$ has been calculated already, which is the slope we fitted in Fig.8, so $\tau = 1.523 \pm 0.012$. Now, we can scale the y-axis first,

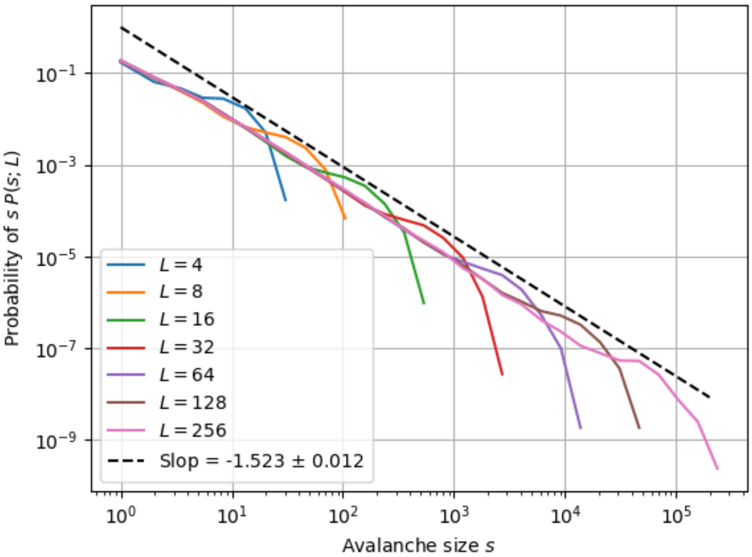


Figure 8: Graph of log-binned avalanche-size probabilities $\tilde{P}_N(s;L)$ was plotted against avalanche size $s$. In log-scale, the probability decays linearly to the avalanche size, with slope -1.523.

then change the $\tilde{P}_N(s;L)$ to $\tilde{P}_N(s;L)\cdot s^{-\tau}$, we will get Fig.9.

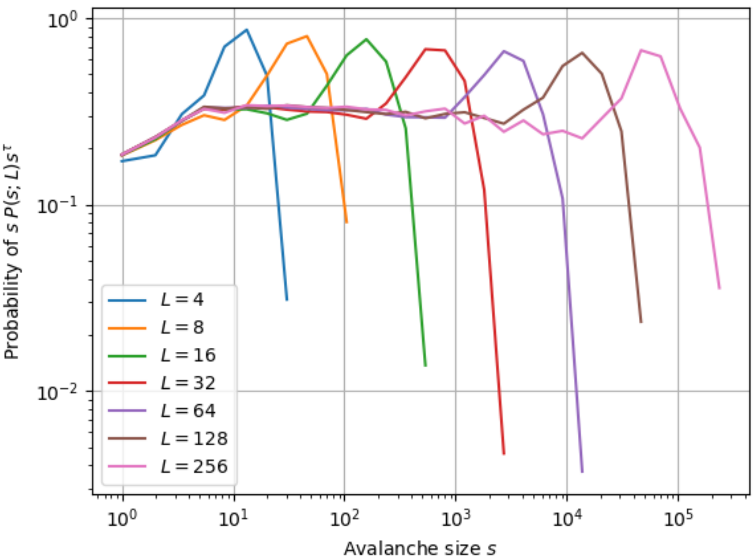


Figure 9: Graph of scaled avalanche-size probabilities $\tilde{P}_N(s;L)\cdot s^{\tau}$ was plotted against avalanche size $s$. The maximum y-value corresponds the cut-off avalanche size.

Record the cut-off avalanche sizes (corresponds to the maximum values on y-axis) for each system size $L$, then find the relation between the cut-off avalanche sizes $s_{cut}$ and

system size $L$. From Eq.(16), the term $\mathcal{G}(s/L^D)$ tells us the avalanche size have a power law of system size, i.e. $s \propto L^D$. So, take logarithm on both avalanche size $s_{cut}$ and system size $L$. Then use linear relation to fit $\log s_{cut}$ and $\log L$, the slope we get is the scale value: $D = 1.995 \pm 0.064$.
Now, using the ansatz in Eq.16, scale both x and y axis by substitute $\tau = 1.523 \pm 0.012$ and $D = 1.995 \pm 0.064$ in, and the data collapsed figure is shown in Fig.10.

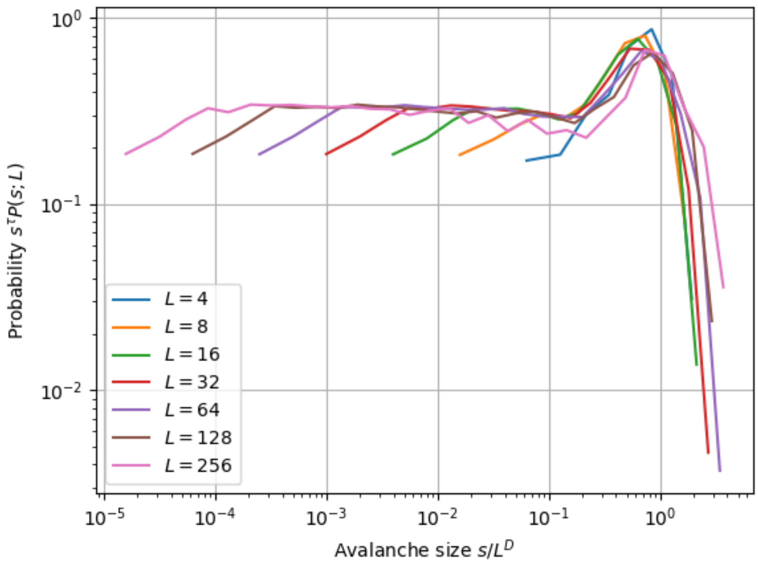


Figure 10: Graph showing data collapse of scaled avalanche-size probabilities $\tilde{P}_N(s; L) \cdot s^\tau$ against scaled avalanche size $s/L^D$, where $\tau = 1.523 \pm 0.012$ and $D = 1.995 \pm 0.064$.

## 4.3 $k^{th}$ moment of avalanche size-Task 3c

The $k^{th}$ moment of avalanche size can be measured by [3]

$$\langle s^k \rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=t_c+1}^{t_c+T} s_t^k, \tag{17}$$

where $t_c$ is the cross-over time, $s_t$ is the avalanche size at time $t$. Avalanche size of zero, $s_t = 0$ is included for normalisation. By using Eq.(17), we could measure the $k^{th}$ moment of avalanche size, which is shown in Fig.11. Obviously, the $k^{th}$ moment of avalanche size is scaled with power law of system size $L$.

Assuming the finite-size scaling (FSS) ansatz (Eq.16) is valid for all avalanche sizes $s$ and approximating the sum with an integral, we find the $k^{th}$ moment by Eq.(18). When $L \gg 1$, the integration will integrate from 0 to $\infty$. As long as the power term of the system size $L$ is greater than zero, i.e. $1 + k - \tau > 0$, the integration will become a constant. Eq.(18) tells us the $k^{th}$ moment of avalanche size has a power law of system size, i.e. $L$, $\langle s^k \rangle \propto L^{D(1+k-\tau)}$. This derivation is coincide with the solution obtained from Fig.11.

To get the coefficients of $D$ and $\tau$, take logarithm on both size: $\log\langle s^k\rangle \propto D(1+k-\tau)\log L$. $\log L$ is the new $x$ value, and $\log\langle s^k\rangle$ is the new $y$ value, then fit them linearly, we will get a slope which is $D(1+k-\tau) = 2.135 \pm 0.018$. Now, we could draw a curve that $D(1+k-\tau)$ against $k$, which is shown Fig.12. The straight line with intersect with $y$-axis at point $b = -1.174$. Substitute the point (0,1.174) back into the straight line, we could find out the value of $\tau$:

$$b = D(1+k-\tau)$$
$$\Rightarrow \tau = 1 - b/D = 1.550 \pm 0.061.$$

$$\begin{aligned}
\langle s^k\rangle &= \sum_{s=1}^{\infty} s^k P(s;L) \\
&= \sum_{s=1}^{\infty} s^{k-\tau}\mathcal{G}(s/L^D) \\
&\propto \int_1^{\infty} s^{k-\tau}\mathcal{G}(s/L^D)ds \\
&= \int_{1/L^D}^{\infty} (L^D u)^{k-\tau}\mathcal{G}(u)L^D ds \text{ change variable} u = s/L^D \Leftrightarrow s = L^D u \\
&= L^{D(1+k-\tau)}\int_{1/L^D}^{\infty} u^{k-\tau}\mathcal{G}(u)du
\end{aligned} \tag{18}$$

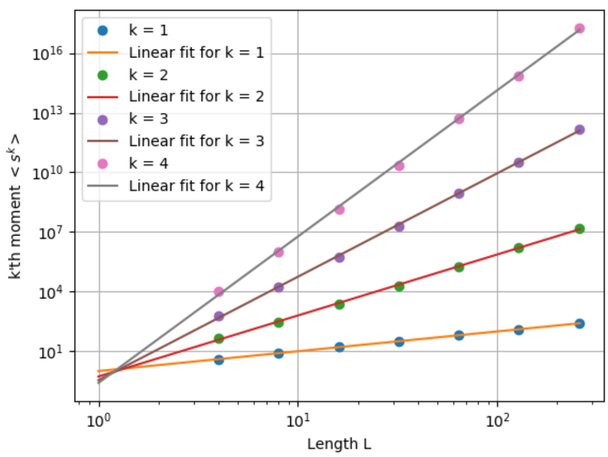


Figure 11: The $k^{th}$ moment $\langle s^k\rangle$,for k = 1,2,3,4 for a various system size. The slope fo each straight line is estimate of $D(1+k-\tau)$.

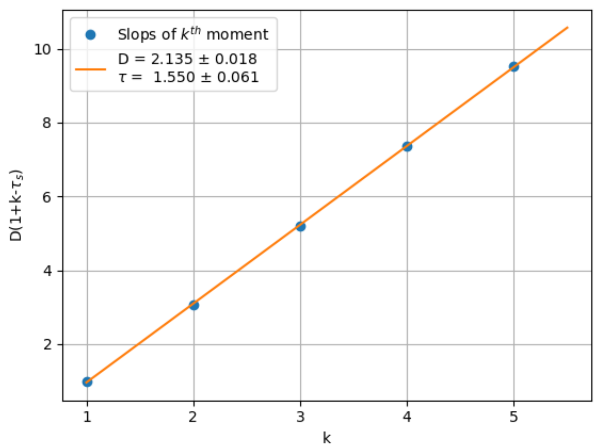


Figure 12: The estimated exponent $D(1+k-\tau)$ vs the moment k. The slop of the straight $D = 2.135 \pm 0.018$, the interecting point on y-axis $b = -1.174 \pm 0.044$, which implies $\tau = 1.550 \pm 0.061$

When the Oslo model become steady state, one added grain should induce one grain leaving the system averagely. In this system, the grain is added at the first site ($i = 1$), and leave the system at the final site ($i = L$). This implies $\langle s\rangle \propto L$. Insert $k = 1$ into

Eq.(18), we will get $\langle s \rangle \propto L^{D(2-\tau)}$. Substitute the values of $D$ and $\tau$ into this equation: $D(2-\tau) = 0.961 \approx 1$, which agree with theoretical expectation.

If we focus on the Fig.11, we will see the intersection point of the straight lines is not on the original point, which means the correction is needed to correct the scaling. The most general correction form is

$$s_c(L) = bL^D(1 + b^1 L^{-\omega_1} + b_2 L^{-\omega_2} + \cdots), \tag{19}$$

where $\omega_i$ are positive values, $D$ is the value we calculated previously $D = 2.135$. But we usually ignore the third term in the bracket, and only $bL^D(1 + b^1 L^{-\omega_1})$ left.

# 5 Conclusion

This report investigates the Oslo model properties: height of system and avalanche size.

The height of the system is proportional to system size, the expression of average of height with expression is: $\langle h \rangle = a_0 L(1 - a_1 L^{-\omega_1})$, where $a_0 = 1.734 \pm 0.001$, $a_1 = 0.211 \pm 0.0012$, $\omega_1 = 0.579 \pm 0.020$. The height probability distribution is Gaussian.

From derivation, the relation between $kth$ moment avalanche size and system $L$ is obtained: $\langle s^k \rangle \propto L^{D(1+k-\tau)}$. By using the method of curving fitting, these coefficient can be calculated: $D = 2.135 \pm 0.018$, $\tau = 1.550 \pm 0.061$. The results from the model is very close to theoretical expectation: $D(2-\tau) = 0.961 \approx 1$, where 0.961 is the value get from model, and 1 is the expected value.

All of the derivations and calculations are based on the assumption that $L \gg 1$, which means the larger the system size $L$ is used, the more precise results could be obtained. If a large system size is applied, for example $L = 1024$, the value $(D(2-\tau))$ obtained from the model should be larger than 0.961, and should be closer to 1.

# References


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