3.5 Cubic spline interpolation — Practice Exams (In-depth)

\documentclass{article}

\usepackage{amsmath}

\begin{document}

\title{Practice Exam: Cubic Spline Interpolation}

\author{}

\date{}

\maketitle

\section{Cubic Spline Interpolation}

\subsection{Introduction}

When interpolating using higher degree polynomials, oscillations can occur. Cubic spline interpolation divides the interval into subintervals and uses piecewise polynomial approximation to address this issue.

\subsection{Cubic Spline Interpolation}

On each subinterval $[x_j, x_{j+1}]$, the interpolating polynomial has the form:

$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

We need to determine the constants a_0, a_1, a_2, a_3 subject to certain conditions.

\subsection{Conditions for Cubic Spline Interpolation}

Given a function f defined on $[a, b]$ and nodes $a = x_0 < x_1 < \ldots < x_n = b$, a cubic spline interpolant S for f satisfies the following conditions:

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\subsection{Boundary Conditions}

\begin{itemize}

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\item Natural or free boundary conditions: <span class="katex"><span c

\item Clamped boundary conditions: <span class="katex"><span class="ka
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\end{itemize}

\subsection{Theorems}

\subsubsection{Theorem 4}

If f is defined at $a = x_0 < x_1 < \ldots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$.

\subsubsection{Theorem 5}

If f is defined at $a = x_0 < x_1 < \ldots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$.

\subsubsection{Theorem 6}

Let $f \in C^4[a, b]$ with $\max_{a \leq x \leq b} |f^{(4)}(x)| = M$. If S is the unique clamped cubic spline interpolant to f with respect to $a = x_0 < x_1 < \ldots < x_n = b$, then for all $x \in [a, b]$, we have:

$$|f(x) - S(x)| \leq \frac{5M}{384} \max_{0 \leq j \leq n-1} (x_{j+1} - x_j)^4$$

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\section{Examples}

\subsection{Example 14}

Construct a natural cubic polynomial that interpolates the data $f(0) = 0, f(1) = 1, f(2) = 2$.

\textbf{Solution:}

Given the data points, we can construct the natural cubic spline interpolant by solving for the coefficients of the cubic polynomials on each subinterval.

Let $S_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ on $[0, 1]$ and $S_1(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ on $[1, 2]$.

Applying the interpolation conditions:

\begin{align*}

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S_0(0) &= a_0 = 0 \\

S_0(1) &= a_0 + a_1 + a_2 + a_3 = 1 \\

S_1(1) &= b_0 + b_1 + b_2 + b_3 = 1 \\

S_1(2) &= b_0 + 2b_1 + 4b_2 + 8b_3 = 2
```

\end{align*}

This system of equations can be solved to find the coefficients $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$.

\subsection{Example 15}

Given a clamped cubic spline $S(x)$ on $[1, 3]$ defined by:

$$S(x) = \begin{cases} 2(x-1) + (x-1)^2 - (x-1)^3, & 1 \le x \le 2 \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 2 \le x \le 3 \end{cases}$$

and $f'(1) = f'(3)$, find a, b, c, d.

\textbf{Solution:}

Given that $f'(1) = f'(3)$, we have:

\begin{align*}

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S_0'(2) &= S_1'(2) \\

2 - 2(2) + 3(2)^2 &= b + 2c(2) + 3d(2)^2
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\end{align*}

This equation can be solved to find the values of a, b, c, d.

\subsection{Example 16}

For a natural cubic spline $S(x)$ given by:

$$S(x) = \begin{cases} 1 + 2x - x^3, & 0 \le x \le 2 \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 2 \le x \le 4 \end{cases}$$

find a, b, c, d.

\textbf{Solution:}

To find a, b, c, d, we apply the interpolation conditions and continuity requirements at the point $x = 2$.

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