Lecture 7: Recurrences

COSC242: Algorithms and Data Structures

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Mergesort again

The time complexity function for mergesort is:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2T(n/2) + n \end{aligned}$$

We call these *recurrence equations* because the function name T recurs on the righthand side of the equation.

It would be easier to compare this function to our landmark functions if we could find a simple non-recurrent formula defining the function T.

Solving is not always possible, but for mergesort we can.

Some technicalities

We're going to keep things simple.

If we were in a maths class, the main equation above would be rendered more precisely as:

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

and we would have to deal with the fact that $2(\lfloor n/2 \rfloor) \leq n$ rather than being equal to n.

By assuming that n is a power of 2 we can avoid these complications.

Solving Recurrences - iteration

The iteration method has two steps:

1. Guess the form of the solution using substitution and iteration
2. Prove the guess is true using induction

A simpler problem

Let's start with a simpler function:

$$f(1) = 2$$
$$f(n) = f(n-1) + 3$$

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We want to eliminate the f on the RHS of the second equation:

$$\begin{aligned} f(n) &= f(n-2) + 3 + 3 \\ &= f(n-3) + 3 + 3 + 3 \\ &\dots \\ ? &= f(n-k) + 3k \end{aligned}$$

Eliminating f on the right

How do we get rid of the f on the right?

We know that $f(1) = 2$, so ideally we want $f(1)$ on the right. In other words, we want to find a k such that $n - k = 1$:

$$f(n)? = f(n-k) + 3k$$

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$$\begin{aligned} f(n) ? &= f(n-k) + 3k \\ ? &= f(n-(n-1)) + 3(n-1) \end{aligned}$$

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\end{aligned}
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$$\begin{aligned} f(n) ?&= f(n-k) + 3k \\ ?&= f(n-(n-1)) + 3(n-1) \\ ?&= f(1) + 3(n-1) \\ ?&= 2 + 3(n-1) \end{aligned}$$

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Proving it

At this point we have an hypothesis which we need to prove:

Prove that the function f:

$$f(1) = 2$$
$$f(n) = f(n-1) + 3$$

and the function g:

$$g(n) = 3n - 1 \tag{1}$$

are the same function.

Use induction

Base case: $f(1) = 2 = g(1)$

Inductive step: assume it's true for $n = k$. That is assume that $f(k) = g(k)$. In other words assume that $f(k) = 3k - 1$.

Show that $f(k+1) = g(k+1)$.

$$LHS = f(k+1)$$

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Show that $f(k+1) = g(k+1)$.

$$\begin{aligned} LHS &= f(k+1) \\ &= f(k) + 3 \end{aligned}$$

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Show that $f(k+1) = g(k+1)$.

$$\begin{aligned} LHS &= f(k+1) \\ &= f(k) + 3 \\ &= 3k - 1 + 3 \\ &= 3(k+1) - 1 \end{aligned}$$

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$$\begin{aligned} LHS &= f(k+1) \\ &= f(k) + 3 \\ &= 3k - 1 + 3 \\ &= 3(k+1) - 1 \\ &= g(k+1) \end{aligned}$$

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Show that $f(k+1) = g(k+1)$.

$$\begin{aligned} LHS &= f(k+1) \\ &= f(k) + 3 \\ &= 3k - 1 + 3 \\ &= 3(k+1) - 1 \\ &= g(k+1) \\ &= RHS \end{aligned}$$

Solve the Mergesort recurrence

$$T(n) = 2T(n/2) + n$$

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$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2\left(2T(n/4) + n/2\right) + n \end{aligned}$$

Solve the Mergesort recurrence

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\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2\left(2T(n/4) + n/2\right) + n \\
&= 4T(n/4) + 2n
\end{aligned}
$$

Solve the Mergesort recurrence

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2\left(2T(n/4) + n/2\right) + n \\ &= 4T(n/4) + 2n \\ &= 4\left(2T(n/8) + n/4\right) + 2n \end{aligned}$$

Solve the Mergesort recurrence

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T(n) &= 2T(n/2) + n \\
&= 2\left(2T(n/4) + n/2\right) + n \\
&= 4T(n/4) + 2n \\
&= 4\left(2T(n/8) + n/4\right) + 2n \\
&= 8T(n/8) + 3n
\end{aligned}$$

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T(n) &= 2T(n/2) + n \\
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&= 4T(n/4) + 2n \\
&= 4\left(2T(n/8) + n/4\right) + 2n \\
&= 8T(n/8) + 3n \\
&= 8\left(2T(n/16) + n\right) + 3n
\end{aligned}
$$

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\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2\left(2T(n/4) + n/2\right) + n \\
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&= 4\left(2T(n/8) + n/4\right) + 2n \\
&= 8T(n/8) + 3n \\
&= 8\left(2T(n/16) + n\right) + 3n \\
&= 16T(n/16) + 4n
\end{aligned}
$$

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&= 4\left(2T(n/8) + n/4\right) + 2n \\
&= 8T(n/8) + 3n \\
&= 8\left(2T(n/16) + n\right) + 3n \\
&= 16T(n/16) + 4n
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We want to get rid of the T on the right. We know that $T(1) = 1$, so what do we need to divide n by to get 1?

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We want to get rid of the T on the right. We know that $T(1) = 1$, so what do we need to divide n by to get 1?

And what do we notice about the other constants (16 and 4 in the last equation)?

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T(n) &= 2T(n/2) + n \\
&= 2\left(2T(n/4) + n/2\right) + n \\
&= 4T(n/4) + 2n \\
&= 4\left(2T(n/8) + n/4\right) + 2n \\
&= 8T(n/8) + 3n \\
&= 8\left(2T(n/16) + n\right) + 3n \\
&= 16T(n/16) + 4n \\
? &= kT(n/k) + n\log k
\end{aligned}
$$

We want to get rid of the T on the right. We know that $T(1) = 1$, so what do we need to divide n by to get 1?

And what do we notice about the other constants (16 and 4 in the last equation)?

Hypothesise a non-recurrent equation

We can now hypothesise that:

$$T(1) = 1$$
$$T(n) = 2T(n/2) + n$$

and

$$g(n) = n \log n + n$$

are the same function for $n >= 1$ and n is a power of two.

Note: we're going to ignore the in between steps when n is not a power of two. This means that our next element is not $k + 1$ it is $2k$, because $n = \{1, 2, 4, 8, \ldots, k, 2k, 4k, \ldots\}$.

Use induction

Base case: $T(1) = 1 = g(1)$

Inductive step: Assume that $T(k) = g(k)$. That is assume that $T(k) = k \log k + k$.

Show that $T(2k) = g(2k)$. That is, show that $T(2k) = (2k)\log(2k) + 2k$.

$$LHS = T(2k) =$$

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$$LHS = T(2k) = 2T(k) + 2k$$

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Show that $T(2k) = g(2k)$. That is, show that $T(2k) = (2k)\log(2k) + 2k$.

$$\begin{aligned} LHS = T(2k) &= 2T(k) + 2k \\ &= 2(k \log k + k) + 2k \end{aligned}$$

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\begin{aligned}
LHS = T(2k) &= 2T(k) + 2k \\
&= 2(k \log k + k) + 2k \\
&= 2k(\log k + 1) + 2k
\end{aligned}
$$

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$$\begin{aligned} LHS = T(2k) &= 2T(k) + 2k \\ &= 2(k \log k + k) + 2k \\ &= 2k(\log k + 1) + 2k \\ &= 2k(\log k + \log 2) + 2k \end{aligned}$$

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LHS = T(2k) &= 2T(k) + 2k \\
&= 2(k \log k + k) + 2k \\
&= 2k(\log k + 1) + 2k \\
&= 2k(\log k + \log 2) + 2k \\
&= 2k \log(2k) + 2k \\
&= RHS
\end{aligned}$$

Relevant parts of the textbook

Chapter 4 looks at recurrences in some detail and is worth the read.

The substitution method is dealt with in Section 4.1, although we apply this method iteratively to develop the method further.

The recursion tree method is dealt with in Section 4.2.

Section 4.3 and 4.4 look at the Master Theorem method which provides exact answers for many cases. But it is rather technical, not very interesting, and not needed for most cases.