Lecture 3: Big-O and Big-Θ

COSC242: Algorithms and Data Structures

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Landmark functions

We saw that the amount of work done by Insertion Sort, in the worst case, is roughly indicated by
$f(n) = 1 + 2 + 3 + \ldots + (n-1) = n(n-1)/2 = (n^2 - n)/2$

We'd like to tie this in to some special **landmark functions,** which are given by assigning to input n the outputs:

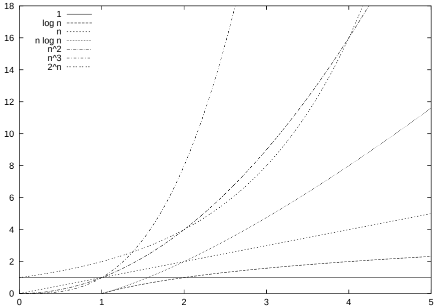
$$
\begin{array}{lll}
f(n) = 1 & f(n) = \log n & f(n) = n \\
f(n) = n \log n & f(n) = n^2 & f(n) = n^3 \\
\quad \ldots & & \\
f(n) = 2^n & f(n) = n! &
\end{array}
$$

Given an algorithm, we estimate how much work it would have to do in the worst case (call this estimate the time-complexity function of the algorithm) and then we identify which landmark it should be grouped with.

Rates of Growth

The landmark functions have different rates of growth.

By the *rate of growth* of f we mean how fast its output $f(n)$ increases in size as the input n gets bigger. Intuitively, a function with a slow rate of growth **scales up more efficiently** than a function with a high rate of growth.



Digression …

Are plots enough?

Θ and big-O

Suppose an algorithm's time-complexity function is f. Which landmark function g should f be grouped with?

We will write $f = \Theta(g)$ to say that f is grouped with g. But to understand Θ notation we first define "big-O" notation.

Intuitively, "$f = O(g)$" means f is no worse than g, i.e. f **scales up at least as well as** g, and maybe better. How can we make this precise?

It's tricky. Some values of f may make f look worse than g even though $f = O(g)$.

For instance, we'll show that $6n = O(n^2)$. But if $n = 1$ then $6n = 6$ whereas $n^2 = 1$ so for this value of n, the function $f(n) = 6n$ is worse than the landmark n^2.

Since we're interested in how f scales up, we ignore small values of n and look at the picture when n gets big.

Formal definition of big-O

Definition of "big-O":
$f = O(g)$ if and only if there are positive integers c and n_0 such that
$f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

In other words, to show that any function $f = O(g)$, we have to find two positive whole numbers, one called c and the other called n_0, so that any output $f(n)$ is no bigger than the output $g(n)$ multiplied by the constant c (at least, any output produced by inputs from the starting point n_0 onwards).

Note that c and n_0 are **positive**, so $c \geq 1$ and $n_0 \geq 1$. Our definition does not allow $c = 0$ nor does it allow $n_0 = 0$.

Example (in class): Let's show that $6n = O(n^2)$.

Important note: it is not necessary to find the smallest c and n_0. Any values that work will do.

big-O proof for Insertion Sort

Consider insertion sort again, which has time complexity $f(n) = n(n-1)/2$. We now show that $n(n-1)/2 = O(n^2)$.

Proof: It is possible to find c and n_0 such that

$$n(n-1)/2 \leq c \cdot n^2 \text{ for all } n \geq n_0.$$

Step 1: choose c, n_0: let $c = 1$ and $n_0 = 1$.

Step 2: show that $n(n-1)/2 \leq 1 \cdot n^2$ for $n \geq 1$. If $n \geq 1$ then $n \neq 0$ and so

$$n(n-1)/2 = (n^2 - n)/2 = n^2/2 - n/2 \leq 1 \cdot n^2 = n^2.$$

Note that since $n(n-1)/2$ is actually $< n^2$, it is also $\leq n^2$.

Θ

It's nice to be able to say that something is no worse than something else, but what about saying it is no better either?

Simple. If $f = O(g)$ says that f is no worse than g then it is also saying that g is no better than f.

So if we want to say that f is no better than g, we may write $g = O(f)$.

Suppose it is true that $f = O(g)$ and also that $g = O(f)$.

This says that f is no worse than g, and f is no better than g. In other words, f scales up about as well as g, so in terms of efficiency, f is **equivalent** to g.

When $f = O(g)$ and $g = O(f)$, then we may write $f = \Theta(g)$. In English, it says "f is equivalent to g in efficiency."

In class example

To prove that $f = \Theta(g)$, the important thing is to remember that there are **two** things to show. We must prove that $f = O(g)$, and we must prove that that $g = O(f)$.

For example, we'll see that $f = \Theta(g)$ where $f(n) = 5n$ and $g(n) = n$. This makes sense: two straight lines scale up about equally well. More formally: linear functions are equivalent in efficiency.

Θ proof for Insertion Sort: $\frac{n(n-1)}{2} = \Theta(n^2)$

We've already shown that $\frac{n(n-1)}{2} = O(n^2)$, so now just show that $n^2 = O(\frac{n(n-1)}{2})$.

If we choose $c = 3$ and $n_0 = 3$, then

$$n^2 \leq c \cdot \frac{n(n-1)}{2} \quad \text{for all} \quad n \geq n_0$$

because:

$$\frac{3n(n-1)}{2} = \frac{3}{2}n^2 - \frac{3}{2}n = n^2 + \frac{n^2}{2} - \frac{3n}{2} = n^2 + \frac{(n^2 - 3n)}{2} \geq n^2$$

since $n \geq 3$ so that $\frac{(n \cdot n - 3 \cdot n)}{2} \geq 0$.

And now, since we have $\frac{n(n-1)}{2} = O(n^2)$ and $n^2 = O(\frac{n(n-1)}{2})$, we have:

$$\frac{n(n-1)}{2} = \Theta(n^2).$$