Lecture 5: Induction Examples

COSC242: Algorithms and Data Structures

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The structure of induction proofs

All induction proofs follow the same recipe:

Base Case: show the statement is true for the first element of the set

Inductive Step: assume the statement is true for the k^{th} element, and show it must therefore be true for the $(k+1)^{th}$ element of the set.

For an induction proof to work, we need some way of getting from one element in the set to one or more "next" elements in the set.

Usually our elements are generated from some set of integers. E.g. the set of numbers where $f(n) = n(n-1)/2$ or where $f(n) = n^3 + 2n$, where $n > 0$ and an integer. But not always.

Recursively defined sets

Technically, we can use induction proofs on any set that is recursively defined.

For example, we can define the natural numbers, $\mathbb{N}$ as:

1. Base case: $0 \in \mathbb{N}$
2. Inductive step: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

Or prime numbers, $\mathbb{P}$ as:

1. Base case: $2 \in \mathbb{P}$
2. Inductive step: $p \in \mathbb{P}$ if and only if $p \neq cn$ for some some constant c and $n \in \mathbb{P}$ and $n < p$.

Binary trees are a particularly relevant non-number computer science example, but we will get to those in a later lecture.

Prove that $n^2 = O(2^n)$

This proof involves both a big-O proof and induction.

First state the definition of big-O:

Show that there exists some c, n_0 such that $n^2 \leq c.2^n$ for all $n \geq n_0$.

then choose appropriate values of c, n_0

Choose $c = 1, n_0 = 4$.

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Prove that $n^2 = O(2^n)$

Show that $n^2 \leq 2^n$ for all $n \geq 4$.

now use induction (check, do we have a recursively defined set?):

Base case:

Inductive step:

Prove that $2^n = O(n!)$

First state the definition of big-O:

Show that there exists some c, n_0 such that $2^n \leq n!$ for all $n \geq n_0$.

then choose appropriate values of c, n_0

Choose $c = 1, n_0 = 4$:

Base case:

Inductive step:

Relevant parts of the textbook

There isn't very much in the textbook on proof techniques - it is assumed that you have a basic knowledge of these. But, there is a very small amount on induction in the textbook. In the second edition, this is in appendix A2 on page 1062, but it is not very illuminating.

Big-O and Big-Θ are discussed in Chapter 3.