HW 2

3.12

CDF

1@

$$F(t) = \begin{cases} 0 & t<1 \\ 1/4 & 1 \le t < 3 \quad 1 \\ 1/2 & 3 \le t < 5 \quad 3 \\ 3/4 & 5 \le t < 7 \quad 5 \\ 1 & t \ge 7 \quad 7 \end{cases}$$

cumulative

CDF

a) $P(T=5)$

$$F(t) = P(T=5) = P(T \le 5) - P(T<5)$$

$$= 3/4 - 1/2$$

$$\boxed{= 1/4}$$

b) $P(T>3)$

PMF: 1, 3, 5, 7

$$F(t) = \begin{cases} \cancel{0} & t = \cancel{0} \\ 1/4 & t=1 \\ 1/2 - 1/4 & t=3 \\ 3/4 - 1/4 - (1/2 - 1/4) & t=5 \\ 1 - 1/4 - (1/2 - 1/4) - [3/4 - 1/4 - (1/2 - 1/4)] & t=7 \end{cases}$$

$$P(T>3) = P(T=5) + P(T=7)$$
$$= 1/4 + 1/4$$
$$\boxed{= 1/2}$$

c) $P(1.4 < T < 6) = P(T=3) + P(T=5)$
$$= 1/4 + 1/4$$
$$\boxed{= 1/2}$$

d) $P(T \leq 5 \overset{\text{given}}{|} T \geq 2) = \dfrac{P(A \overset{\text{and}}{\cap} B)}{P(B)}$

$$= \frac{P(T=3) + P(T=5)}{P(T=3) + P(T=5) + P(T=7)}$$

$$= \frac{1/4 + 1/4}{1/4 + 1/4 + 1/4}$$

$$= \frac{1/2}{3/4}$$

$$\boxed{= 2/3}$$

3.14 CDF:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-8x} & x \geq 0 \end{cases}$$

a) CDF $T < 12\text{min}$ $P(T < 12/6 \text{ hours})$

$$P(X < 0.2) = 1 - e^{-8(.2)}$$
$$1 - e^{-1.6}$$
$$1 - 0.2019$$
$$\boxed{= .798}$$

b) pdf

$$f(x) = \frac{d\,[F(x)]}{dx} = 8e^{-8x}$$

$$P(X < 0.2) = \int_0^{0.2} 8e^{-8x}\,dx$$

$$= \frac{-\cancel{8}e^{-8x}}{\cancel{8}} \Big|_0^{0.2}$$

$$= -e^{-8(.2)} - \left(-e^{-8(0)}\right)$$

$$= 1 - e^{-1.6}$$

$$\boxed{= 0.798}$$

3.30 $f(x) = \begin{cases} K(3-x^2) & -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$

a) $\int_{-\infty}^{\infty} K(3-x^2) = 1$

$\int_{-1}^{1} K(3-x^2)$

$3Kx - \frac{Kx^3}{3} \Big|_{-1}^{1}$

$3K - K/3 - \left(-3K + \frac{K}{3}\right)$

$6K - 2K/3 = 1$

$K(6 - 2/3) = 1$

$(16/3)K = 1$

$\boxed{K = 3/16}$

b) $P(X < 1/2) = \int_{-\infty}^{1/2} 3/16 (3 - x^2)$

$$= \int_{-1}^{1/2} 3/16 (3 - x^2)$$

$$3/16 \left(3x - \frac{x^3}{3}\right) \Big|_{-1}^{1/2}$$

$$3/16 \left(3/2 - \frac{1/8}{3}\right) - \left[3/16 (-3 + 1/3)\right]$$

$$3/16 \left[\frac{3}{2} - \frac{1}{24} + \frac{8}{3}\right]$$

$$\boxed{= 0.7734}$$

c) $P(-.8 \le X \le .8)$

$$\int_{-.8}^{.8} 3/16(3-x^2)\,dx$$

$$3/16\left(3x - \frac{x^3}{3}\right)\Big|_{-.8}^{.8}$$

$$3/16\left(\left(3(.8) - \frac{.8^3}{3}\right) - \left[-3(.8) + \frac{.8^3}{3}\right]\right)$$

$= .8359$

$1 - .835 \boxed{= .1641}$

3.37 a) $F(x,y) = cxy \quad x = 1,2,3 \quad y = 1,2,3$

$$\sum_{x=1}^{3} \sum_{y=1}^{3} cxy = 1$$

$$\sum_{x=1}^{3} cx(1+2+3) = 1$$

$$= \sum_{x=1}^{3} 6cx = 1$$

$$6c(1+2+3) = 1$$

$$36c = 1$$

$$\boxed{c = 1/36}$$

b) $F(x,y) = c|x-y|$ for $x = -2, 0, 2$; $y = 2, 3$

$$\sum_{x=-2}^{2} \sum_{y=2}^{3} c|x-y|$$

x =	-2	0	2
y = -2 c	0	2c	4c
y = 3c	5c	3c	1c

$$2c + 4c + 5c + 3c + 1c$$

$$15c = 1$$

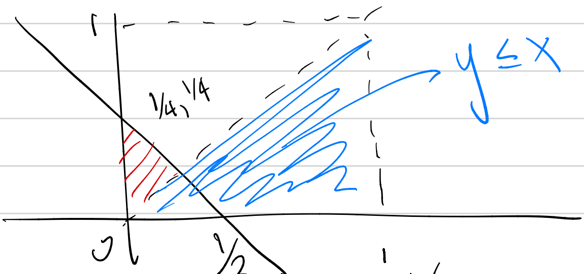
$$\boxed{c = 1/15}$$

3.45 joint density:

y can not be ≤ x

$$f(x,y) = \begin{cases} 1/y & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(X+Y > 1/2) = 1 - P(X+Y \le 1/2)$$



Ind.

$$x+y = 1/2 \longrightarrow y = 1/2 - x$$

$$\int_0^{1/4}\int_0^{1/2-x} 1/y \, dy \, dx$$

$$= \int_0^{1/4} \ln(y) \Big|_x^{1/2-x} dx$$

$$= \int_0^{1/4} \ln(1/2-x) - \ln(x) \, dx$$

$u \quad dv$

$du \quad v$

$$-\left[(1/2-x)\ln(1/2-x) - \cancel{(1/2-x)}\right] - \left(x\ln(x) - \cancel{x}\right) \Big|_0^{1/4}$$

$$= 1/2\ln(2) \qquad \frac{\cancel{1/4\ln(1/4)} + \cancel{1/2} - \cancel{1/4\ln(1/4)} + (1/2\ln 1/2) - \cancel{1/2}}{\cancel{1/4\ln(1/4)}}$$

$\ln(1) = 0$

$$\boxed{1 - 1/2\ln(2)}$$

Marginal pdfs $g(x)$ + $h(y)$

$$g(x) = \int_x^1 F(x,y)\,dy$$

$$= \int_x^1 \frac{1}{y}\,dy$$

$$= \ln(y)\Big|_x^1$$

$$-\ln(x)$$

$$\boxed{g x = \begin{cases} -\ln(x) & 0<x<1 \\ 0 & \text{o.w.} \end{cases}}$$

$$h(y) = \int_0^y F(x,y)\,dx$$

$$= \int_0^y 1/y\,dx$$

$$= \frac{1}{y}\int_0^y dx$$

$$= \frac{1}{y}(x)\Big|_0^y$$

$$= 1 - 0$$

$$= 1$$

$$\boxed{h(y) = \begin{cases} 1 & 0<y<1 \\ \cancel{0} & \text{o.w.} \end{cases}}$$

conditional pdf $f(x|y)$

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{1/y}{1} = 1/y$$

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{1/y}{-\ln(x)}$$

$g(x) = -\ln(x)$; $h(y) = 1$

$g(x) \cdot h(y) \neq f(x, y)$

$-\ln(x) \cdot 1 \neq 1/y$

$\therefore$ x + y are not independent b/c $g(x) \cdot h(y) \neq f(x, y)$. There is no x varible to even separate in $f(x,y)$, but bounds are not separate.

3.61 $f(x,y) = \begin{cases} 24xy & 0 \le x, y \le 1 \quad \underbrace{x+y \le 1}_{\text{not separate}} \\ 0 & \text{o.w.} \end{cases}$

a) $P(X > 1/2)$

$$\int_{1/2}^{1} \int_{0}^{1-x} f(x, y)\, dy\, dx$$

$$\int_{1/2}^{1}\int_{0}^{1-x} 24xy \, dy \, dx$$

$$\int_{1/2}^{1} \frac{24xy^2}{2}\Big|_{0}^{1-x} dx$$

$$12x(1-x)^2 - \cancel{0}$$

$$12x(1-2x+x^2)$$

$$\int_{1/2}^{1} 12x - 24x^2 + 12x^3 \, dx$$

$$6x^2 - 8x^3 + 3x^4 \Big|_{1/2}^{1}$$

$$= 5/16 = \boxed{.3125}$$

b) $hy = \int_0^{1-y} 24xy \, dx$

$24y \int_0^{1-y} x \, dx$

$24y\left(\frac{x^2}{2}\right)\Big|_0^{1-y}$

$12y(1-y)^2$

$$\boxed{hy = \begin{cases} 12y(1-y)^2 & 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}}$$

c) $P(X < 1/4 \mid y = 3/4)$

$$\int_0^{1/8} \frac{f(x,y)}{hy} = \int_0^{1/4} \frac{\overset{2}{\cancel{24}}x\cancel{y}}{\cancel{12}\cancel{y}(1-y)^2} \, dx$$

$$\int_0^{1/8} \frac{2x}{\underbrace{(1-3/4)^2}_{1/4^2}} = \int_0^{1/8} 2x \cdot 16$$

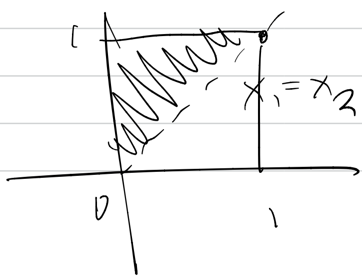
$$= \int_0^{1/8} 32x$$

$$\cancel{\frac{32x^2}{2}}\,16 \Big|_0^{1/8}$$

$$\frac{16}{64} = 1/4 = \boxed{.25}$$

3.75 $f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{o.w.} \end{cases}$

a) $g(x_1) = \int_{x_1}^{1} 2 \, dx_2$



$$2x_2 \Big|_{x_1}^{1}$$

$$= 2 - 2x_1 \quad \boxed{f(x_1) = \begin{cases} 2(1-x_1) & 0<x_1<1 \\ 0 & \text{o.w.} \end{cases}}$$

b) $h(x_2) = \int_0^{x_2} 2\, dx_1$

$$2x_1 \Big|_0^{x_2}$$

$$= 2x_2 \quad \boxed{f(x_2) = \begin{cases} 2x_2 & 0<x_2<1 \\ 0 & \text{o.w.} \end{cases}}$$

c) $P(x_1 < 0.2 \cap x_2 > 0.5)$

$$2\int_0^{0.2}\int_{0.5}^{1} dx_2\, dx_1$$

$$\int_0^{0.2} 2x_2 \Big|_{0.5}^{1} dx_1$$

$$= 2 - 1$$

$$= \int_0^{0.2} 1 \quad dx_1$$

$$x_1 \Big|_0^{0.2}$$

$$\boxed{= 0.2}$$

d.) $f_{x_1|x_2}(x_1|x_2)$

$$= \frac{f(x,y)}{f(x_2)}$$

$$= \frac{\cancel{2}}{\cancel{2}x_2} = 1/x_2$$

$$\boxed{f_{x_1/x_2}(x_1|x_2) = 1/x_2 \, , \quad 0<x_1<x_2}$$

3.74 $\quad f(x_1,x_2) = \begin{cases} 6x_2 & 0<x_2<x_1<1 \\ 0 & \text{o.w.} \end{cases}$

a) $f(x_1) = \int_0^{x_1} 6x_2 \, dx_2$

$$\frac{6x_2^2}{2}\Bigg|_0^{x_1}$$

$$= 3x_1^2$$

$$\boxed{f(x_1) = \begin{cases} 3x_1^2 & 0 < x_1 < 1 \\ 0 & \text{o.w.} \end{cases}}$$

check:

$$\int_0^1 f_{x_1}(x_1)\,dx_1 = \frac{\cancel{3}x^3}{\cancel{3}}\Bigg|_0^1$$

$$= 1 \checkmark$$

b) $P(X_2 < 0.5 \mid X_1 = 0.7)$

$$\int_0^{0.5} \frac{6x_2}{3x_1^2}\,dx_2$$

$$= \int_0^{0.5} \frac{2x_2}{x_1^2}\,dx_2$$

$$\frac{2x_2}{(0.7)^2} = \frac{x_2}{0.245}$$

$$\left.\frac{x_2^2}{2(0.245)}\right|_0^{0.5}$$

$$\boxed{= 0.5102}$$