HW 3

1. WMMY 4.14

$$f(x) = \begin{cases} \frac{2(x+2)}{5}, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x \frac{2(x+2)}{5} dx$$

$$= \frac{2}{5}\int_0^1 x^2 + 2x \, dx$$

$$= \frac{2}{5}\left[\frac{x^3}{3} + x^2\right]_0^1$$

$$= \frac{2}{5}\left(\frac{4}{3}\right) = \boxed{\frac{8}{15}}$$

2. WMMY 4.38

$$\sigma^2 = E[(x-\mu)^2] = E(x^2) - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 \frac{2(x+2)}{5} dx - \left(\frac{8}{15}\right)^2$$

$$= \frac{2}{5}\int_0^1 x^3 + 2x^2 dx - \left(\frac{8}{15}\right)^2$$

$$= \frac{2}{5}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_0^1 - \left(\frac{8}{15}\right)^2$$

$$= \frac{2}{5}\left(\frac{1}{4} + \frac{2}{3}\right) - \left(\frac{8}{15}\right)^2$$

$$= \boxed{\frac{37}{450}}$$

3. WMMY 4.40

$g(X) = 3X^2 + 4$; $\mu_{g(X)} = \frac{8}{15}$

$E(x^4) = \int_0^1 x^4 \frac{(2x+4)}{5} dx$

$= \frac{2}{5} \int_0^1 x^5 + 2x^4 \, dx$

$= \frac{2}{5} \left[\frac{x^6}{6} + \frac{2x^5}{5} \right]_0^1$

$= \frac{2}{5}\left(\frac{1}{6} + \frac{2}{5}\right) = \frac{17}{75}$

$\sigma^2(x^2) = E(x^4) - (E(x^2))^2$

$= \frac{17}{75} - \left(\frac{11}{30}\right)^2 = 0.0923$

$\sigma^2_{g(x)} = \sigma^2_{3x^2+4} = (3)^2 \sigma^2(x^2)$

$= 9 \cdot 0.0923$

$= 0.83$

4. WMMY 4.62

$\sigma_X^2 = 5; \quad \sigma_Y^2 = 3; \quad Z = -2X + 4Y - 3$

by theorem 4.9, $\sigma^2_{aX+bY+c} = a\sigma_X^2 + b^2\sigma_Y^2 + \underbrace{2ab\sigma_{XY}}$

$=0$ because $\sigma_{XY} = 0$ because X and Y are independent

$\therefore \sigma_Z^2 = a\sigma_X^2 + b\sigma_Y^2$

$= (-2)^2(5) + (4)^2(3)^2 = \boxed{32}$

5. WMMY 4.63

$\sigma_Z = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}^2$

$= 32 + (2)(-2)(4)(1)$

$= \boxed{16}$

Correlation coefficient: $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

$= \frac{1}{\sqrt{(5)(3)}}$

$= \boxed{\frac{1}{\sqrt{15}}}$

6. WMMY 4.83

From 3.41 (in-class): $f(x,y) = 24xy\, I(0 \le x \le 1)\, I(0 \le y \le 1)\, I(x+y \le 1)$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$E(XY) = \int_0^1 \int_0^{1-x} 24x^2y^2 \, dy \, dx$$

$$= 24 \int_0^1 x^2 \left[\tfrac{1}{3}y^3\right]_0^{1-x} dx$$

$$= 24 \int_0^1 \tfrac{1}{3}x^2 - x^3 + x^4 - \tfrac{x^5}{3} \, dx$$

$$= 24 \left[\tfrac{1}{9}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{18}x^6\right]_0^1$$

$$= \tfrac{2}{15}$$

$$E(X) = \int_0^1 \int_0^{1-x} 24x^2 y \, dy dx$$

$$= 24 \int_0^1 x^2 \left[\tfrac{1}{2}y^2\right]_0^1 dx$$

$$= 24 \int_0^1 x^2\left(\tfrac{1}{2} - x + \tfrac{1}{2}x^2\right) dx$$

$$= 24 \int_0^1 \tfrac{1}{2}x^2 - x^3 + \tfrac{1}{2}x^4 \, dx$$

$$= 24 \left[\tfrac{1}{6}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{10}x^5\right]_0^1$$

$$= \boxed{\tfrac{2}{5}}$$

cont.

$E(Y) = E(X)$ (same bounds, same steps on integral but with y instead of x)
$= 2/5$

$\therefore \sigma_{XY} = E(XY) - E(X)E(Y)$

$= \frac{2}{15} - (\frac{2}{5})(\frac{2}{5}) = \boxed{-\frac{2}{75}}$

correlation: $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

$\sigma_X^2 = E(X^2) - \mu_X^2$

$= \int_0^1 \int_0^{1-x} 24x^3 y \, dy \, dx$

$= 24 \int_0^1 x^3 [\frac{1}{2}y^2]_0^{1-x} dx$

$= 24 \int_0^1 \frac{1}{2}x^3 - x^4 - \frac{1}{2}x^5 \, dx$

$= 24 [\frac{1}{8}x^4 - \frac{1}{5}x^5 - \frac{1}{12}x^6]_0^1$

$= 1/5$ also, $\sigma_y^2 = \sigma_x^2$ so $\sigma_y^2 = \frac{1}{5}$ for similar reasons to above

finally, $\rho_{XY} = \frac{(-\frac{2}{75})}{\sqrt{(\frac{1}{5})(\frac{1}{5})}} = \boxed{-\frac{2}{15}}$

7. WMMY 4.84

The expected weight for the sum of creams and toffees is simply $E(XY)$

$$E(XY) = \int_0^1 \int_0^{1-x} 24x^2y^2 \, dy \, dx$$

$$= 24 \int_0^1 \left[x^2 \left[\tfrac{1}{3}y^3 \right]_0^{1-x} \right] dx$$

$$= 24 \int_0^1 \tfrac{1}{3}x^2 - x^3 + x^4 - \frac{x^5}{3} \, dx$$

$$= 24 \left[\tfrac{1}{9}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{18}x^6 \right]_0^1$$

$$= \boxed{2/15}$$