pg. 1 STA3032 HW 2 Benjamin Ruddy

1. WMMY 3.12

$\swarrow$ CDF

$$F(t) = \begin{cases} 0, & t < 1 \\ 1/4, & 1 \le t < 3 \\ 1/2, & 3 \le t < 5 \\ 3/4, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

a) $P(T=5) = f(5)$
$= F(5) - F(4)$
$= 3/4 - 1/2 = \boxed{1/4}$

b) $P(T>3) = 1 - P(T \le 3)$
$= 1 - F(3)$
$= \boxed{1/2}$

c) $P(1.4 < T < 6) = P(T<6) - P(T<1.4)$
$= P(T \le 6) - P(T \le 1.4)$ by CDF intervals
$= F(6) - F(1.4)$
$= \boxed{1/2}$

d) $P(T \le 5 \mid T \ge 2) = \dfrac{P(2 \le T \le 5)}{P(T \ge 2)}$

$= \dfrac{F(5) - P(T<2)}{1 - P(T<2)}$

$= \dfrac{F(5) - F(2)}{1 - F(2)}$ b/c $P(T<2) = P(T \le 2)$

$= \dfrac{1/2}{3/4}$

$= 4/6 = \boxed{2/3}$

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2. WMMY 3.14

a) $P(X < 12) = F(12)$

$= 1 - e^{-8\left(\frac{12}{60}\right)}$

$= \boxed{0.798}$

b) $f(x) = \frac{dF(x)}{dx}$

$= \frac{d}{dx} \cdot \left[(1 - e^{-8x}) I(x \geq 0) \right]$

$= 8e^{-8x} \, I(x \geq 0)$

c) $P(X < 12) = \int_0^{12} 8e^{-8t} \, dt$

$\left[-e^{-8t} \right]_0^{\frac{12}{60}}$

$= -e^{(-8)\left(\frac{12}{60}\right)} + 1 = \boxed{0.798}$

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3. WMMY 3.30

a) 1: $f(x) \geq 0 \ \forall x \in \mathbb{R}$ iff. $k \geq 0$

2: $\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} k(3-x^2)\, I(-1 \leq x \leq 1)\, dx = 1$

$$= \int_{-1}^{1} k(3-x^2)\, dx = 1$$

$$= \int_{-1}^{1} 3k - kx^2\, dx = 1$$

$$\left[3kx - \tfrac{1}{3}kx^3\right]_{-1}^{1} = 1$$

$$\frac{16}{3}k = 1 \quad \therefore \boxed{k = 3/16}$$

3: $P(a \leq x \leq b)\ \int_a^b f(x)dx$ ✓

b) $P(X < 1/2) = \int_{-\infty}^{1/2} \frac{3}{16}(3-x^2)\, I(-1 \leq x \leq 1)$

$$= \int_{-1}^{1/2} \frac{9}{16} - \frac{3}{16}x^2\, dx$$

$$= \left[9/16\, x - \frac{1}{16}x^3\right]_{-1}^{1/2} = \boxed{\frac{99}{128}}$$

c) $P(X > 0.8) = 1 - P(X \leq 0.8)$

$$= 1 - \int_{-1}^{0.8} \frac{3}{16}(3-x^2)\, dx$$

$$= 1 - \left[9/16\, x - 1/16\, x^3\right]_{-1}^{0.8} = \boxed{0.082}$$

4. WMMY 3.37

a) $f(x,y) = cxy$, for $x = 1,2,3$; $y = 1,2,3$;

1: $f(x,y) \geq 0 \ \forall (x,y) \Rightarrow c \geq 0$

2: $\sum_x \sum_y f(x,y) = 1$

$$\sum_x \sum_y cxy = 1$$
$$c \sum_x x \sum_y y = 1$$
$$36c = 1$$
$$\boxed{c = 1/36}$$

3: $P(X=x, Y=y) = f(x,y)$ ✓

b) $f(x,y) = c|x-y|$, for $x = -2,0,2$; $y = -2,3$

1: $f(x,y) \geq 0 \ \forall (x,y) \Rightarrow c \geq 0$

2: $\sum_x \sum_y f(x,y) = 1$

$$c \sum_x \sum_y |x-y| = 1$$

$$c \sum_y |-2-y| + |0-y| + |2-y|$$
$$c\big[|-2-(-2)| + |-2-3| + |0-(-2)| + |0-3| + |2-(-2)| + |2-3|\big] = 1$$
$$15c = 1$$
$$c = \boxed{1/15}$$

3: $P(X=x, Y=y) = f(x,y)$ ✓

$x < 1$
$y < 1$
$x < y$

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2. $f(x,y) \begin{cases} \frac{1}{y}, & 0 < x < y < 1 \\ 0, & o/w \end{cases}$

y from $(y = x)$ to $y = \frac{1}{2} - x$

$P(X+Y > \frac{1}{2}) = 1 - P(X+Y \leq \frac{1}{2})$ → x from 0 to $(x = \frac{1}{2} - x)$ ⇓ $(x = \frac{1}{4})$

$$= 1 - \int_0^{1/4} \int_x^{(\frac{1}{2}-x)} \frac{1}{y} \, dy dx$$

$$= 1 - \int_0^{1/4} [\ln y]_x^{\frac{1}{2}-x} dx$$

$$= 1 - \int_0^{1/4} \ln(\tfrac{1}{2}-x) - \ln x$$

$$= 1 - \left[-\left((\tfrac{1}{2}-x)\ln(\tfrac{1}{2}-x) - (\tfrac{1}{2}-x)\right) - (x\ln(x) - x)\right]_0^{1/4}$$

$$= 1 - \left[-(\tfrac{1}{2}-x)\ln(\tfrac{1}{2}-x) + \tfrac{1}{2} - x\ln x\right]_0^{1/4}$$

$$= \boxed{0.654}$$

<u>Marginal Pdfs.</u>

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^{1/2 - x} \frac{1}{y} \, dy$$

$$= [\ln(y)]_x^{1/2-x}$$

$$= \boxed{\ln(\tfrac{1}{2}-x) - \ln(x)}$$

$$h(x) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_y^{1/2-y} \frac{1}{y} \, dx$$

$$= \left[\frac{x}{y}\right]_y^{\frac{1}{2}-y}$$

$$= \frac{(\frac{1}{2}-y)}{y} - \frac{y}{y}$$

$$= \boxed{\frac{1}{2y} - 2}$$

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$$f(x|y) = \frac{f(x,y)}{h(y)} = \boxed{\frac{1}{\frac{1}{2}-2y}}$$

$$f(y|x) = \frac{f(x,y)}{g(x)} = \boxed{\frac{1}{y(\ln(\frac{1}{2}-x) - \ln(x))}}$$

independent?

$$f(x,y) = h(x)\,g(x)$$

$$\neq (\tfrac{1}{2y}-2)(\ln(\tfrac{1}{2}-x) - \ln(x)) \qquad X$$

$\boxed{\text{NO}}$

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6. WMMY 3.61

$$f(x,y) = \begin{cases} 24xy, & 0 \le x, y \le 1, \; x+y \le 1 \\ 0, & o/w \end{cases}$$

a) $P(X > 1/2) = P(1/2 < X \le 1)$

$$= \int_{1/2}^{1} \int_{0}^{1-x} f(x,y)\, dy\, dx$$

$$= \iint 24xy \, dy\, dx$$

$$= \int [12xy^2]_0^{1-x} dx$$

$$= \int_{1/2}^{1} 12x - 24x^2 + 12x^3 \, dx$$

$$= \left[6x^2 - 8x^3 + 3x^4\right]_{1/2}^{1}$$

$$= 1 - \tfrac{11}{16} = \boxed{5/16}$$

b) $h(y) = \int f(x,y)\, dx$

$$= \int_0^{1-y} 24xy\, dy\, dx$$

$$= [12x^2y]_0^{1-y} = \boxed{12y(1-y)^2 \, I(0 \le y < 1)}$$

c) $P(X < 1/8 \mid Y = 3/4) = \int_a^b f(x|y)$

$$= \int_a^b \frac{f(x,y)}{h(y)} dx$$

$$= \int_0^{1/8} \frac{24x(3/4)}{12(3/4)(1-(3/4))^2} dx = \int_0^{1/8} 32x\, dx$$

$$= [16x^2]_0^{1/8} = \boxed{1/4}$$

7. WMMY 3.75

$$f(x_1, x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1 \\ 0, & o/w \end{cases}$$

a) $g(x_1) = \int_{x_1}^{1} f(x_1, x_2)\, dx_2$

$= \int_{x_1}^{1} 2\, dx_2 = \boxed{2(1-x_1)\, I(0 \le x_1 \le 1)}$

b) $h(x_2) = \int_{0}^{x_2} 2\, dx_1$

$= 2(x_2 - 0) = \boxed{2x_2\, I(0 \le x_1 < 1)}$

c) $P(X_1 < 0.2,\ X_2 > 0.5) = P(0 < x_1 < 0.2,\ 0.5 < x_2 < 1)$

$= \int_{0}^{0.2} \int_{0.5}^{1} f(x_1, x_2)\, dx_2\, dx_2$

$= \int_{0}^{0.2} \int_{0.5}^{1} 2\, dx_2\, dx_1$

$= \int_{0}^{0.2} 2(0.5)\, dx$

$= \int_{0}^{0.2} dx_1 = x_1 \Big|_{0}^{0.2} = \boxed{0.2}$

d) $f(x_1 \mid x_2) = \frac{2}{2x_2} I(0 \le x_1 < x_2)$

$= \boxed{1/x_2\, I(0 < x_1 < x_2)}$

8. WMMY 3.76

$$f(x_1, x_2) = \begin{cases} 6x_2, & 0 < x_2 < x_1 < 1 \\ 0, & o/w \end{cases}$$

a) $f_{X_1}(x_1) = \int_0^{x_1} 6x_2 \, dx_2$

$$= \boxed{3x_1^2 \; I(0 < x_1 < 1)}$$

1. $f(x_1) \geq 0 \;\; \forall x_1 \in R$ ✓
2. $\int_{-\infty}^{\infty} 3x_1^2 \, dx_1 = [x^3]_0^1 = 1$ ✓
3. $P(X_1 = x_1, Y_1 = y) = \int_a^b f(x_1, x_2)$ ✓

b) $P(X_2 < 0.5 \mid X_1 = 0.7) = \frac{f(x_1, x_2)}{f_{x_2}(x_2)}$

$$= \frac{6x_2}{3(0.7)^2} = \boxed{\frac{200x_2}{49} \; I(0 < x_2 < 0.7)}$$