Chapter 4: Mathematical Expectation

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Mean of a Random Variable

Definition (Mean/Expected Value)

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x xf(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

if X is continuous.

Mean of a Random Variable

Example (Example 4.2 of WMMY)

A salesperson for a medical device company has two appointments on a given day. At the first appointment, he believes that he has a 70% chance to make the deal, from which he can earn \$1000 commission if successful. On the other hand, he thinks he only has a 40% chance to make the deal at the second appointment, from which, if successful, he can make \$1500. What is his expected commission based on his own probability belief? Assume that the appointment results are independent of each other.

Example (Example 4.3 of WMMY)

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Mean of a Random Variable

Theorem (Theorem 4.1 of WMMY)

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

if X is continuous.

Mean of a Random Variable

Example (Example 4.5 of WMMY)

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Mean of a Random Variable

Definition (Mean/Expected Value: Two Random Variables)

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dxdy$$

if X and Y are continuous.

Mean of a Random Variable

Example (Example 4.7 of WMMY)

Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Variance and Covariance of Random Variables

Definition (Variance and Standard Deviation)

Let X be a random variable with probability distribution $f(x)$ and mean μ. The **variance** of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

if X is discrete, and

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

if X is continuous.
The positive square root of the variance, σ, is called the **standard deviation** of X.

Theorem (Theorem 4.2 of WMMY)

The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

Variance and Covariance of Random Variables

Example (Example 4.9 of WMMY)

Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Calculate σ^2.

Example (Example 4.10 of WMMY)

The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X.

Variance and Covariance of Random Variables

Theorem (Theorem 4.3 of WMMY)

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma^2_{g(X)} = E[\{g(X) - \mu_{g(X)}\}^2] = \sum_x \{g(x) - \mu_{g(X)}\}^2 f(x)$$

if X is discrete, and

$$\sigma^2_{g(X)} = E[\{g(X) - \mu_{g(X)}\}^2] = \int_{-\infty}^{\infty} \{g(x) - \mu_{g(X)}\}^2 f(x) dx$$

if X is continuous.

Example (Example 4.12 of WMMY)

Let X be a random variable having the density function given in Example 4.5. Find the variance of the random variable $g(X) = 4X + 3$.

Variance and Covariance of Random Variables

Definition (Covariance)

Let X and Y be random variables with joint probability distribution $f(x, y)$. The **covariance** of X and Y is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) f(x, y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x, y) dxdy$$

if X and Y are continuous.

Theorem (Theorem 4.4 of WMMY)

The covariance of two random variables X and Y with means μ_X and μ_Y, respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Variance and Covariance of Random Variables

Example (Example 4.14 of WMMY)

The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x, y) = \begin{cases} 8xy, & 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

Variance and Covariance of Random Variables

The covariance σ_{XY} is not scale-free and it does not indicate anything regarding the strength of the relationship.

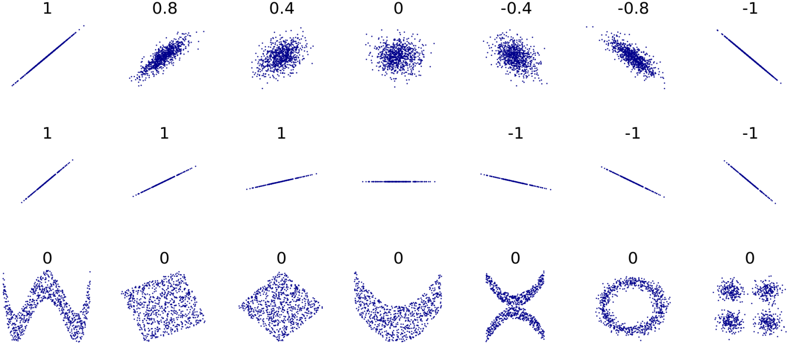
Definition (Correlation Coefficient)

Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively. The **correlation coefficient** of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

- The correlation coefficient ρ_{XY} is free of the units of X and Y.
- $-1 \leq \rho_{XY} \leq 1$ (proof is an application of the Cauchy-Schwarz inequality).
- If $Y = a + bX$, an exact linear relationship, $\rho_{XY} = \text{sign}(b) \cdot 1$.

Variance and Covariance of Random Variables



Example (Example 4.16 of WMMY)

Find the correlation coefficient of X and Y in Example 4.14.

See also Examples 4.6, 4.13, and 4.15 of WMMY for the discrete case.

Means and Variances of Linear Combinations of Random Variables

Theorem (Theorem 4.5 of WMMY)

If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

Example (Example 4.18 of WMMY)

Applying Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$, rework Example 4.5.

Means and Variances of Linear Combinations of Random Variables

Theorem (Theorem 4.6 of WMMY)

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Example (Example 4.20 of WMMY)

The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$, where X has the density function

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of the weekly demand for the drink.

Means and Variances of Linear Combinations of Random Variables

Theorem (Theorem 4.7 of WMMY)

The expected value of the sum or difference of two or more functions of the random variables X and Y is the sum or difference of the expected values of the functions. That is,

$$E[g(X, Y) \pm h(X, Y)] = E[g(X, Y)] \pm E[h(X, Y)].$$

Means and Variances of Linear Combinations of Random Variables

Theorem (Theorem 4.8 of WMMY)

Let X and Y be two **independent** *random variables. Then*

$$E(XY) = E(X)E(Y).$$

Theorem (Corollary 4.5 of WMMY)

Let X and Y be two **independent** *random variables. Then*

$$\sigma_{XY} = 0.$$

Means and Variances of Linear Combinations of Random Variables

Example (Example 4.21 of WMMY)

Suppose X and Y are independent random variables with the joint pdf

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that

$$E(XY) = E(X)E(Y).$$

Means and Variances of Linear Combinations of Random Variables

Theorem (Theorem 4.9 of WMMY)

If X and Y are random variables with joint probability distribution $f(x, y)$ and a, b, and c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

Theorem (Corollary 4.9 of WMMY)

If X and Y are **independent** *random variables, then*

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

Theorem (Corollary 4.11 of WMMY)

If $X_1, X_2, \ldots, X_n$ are **independent** *random variables, then*

$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}.$$

Means and Variances of Linear Combinations of Random Variables

Example (Example 4.22 of WMMY)

If X and Y are random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.

Example (Example 4.23 of WMMY)

Suppose X and Y are independent random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 3$. Find the variance of the random variable $Z = 3X - 2Y + 5$.