Chapter 2: Probability

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Sample Space

Definition (Sample Space)

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Example

- Tossing a coin once: $S = \{H, T\}$.
- Tossing a coin twice: $S = \{HH, HT, TH, TT\}$.
- Tossing a coin three times:
  $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
- Rolling a die once: $S = \{1, 2, 3, 4, 5, 6\}$.
- Counting goals scored in a certain season of the Premier League:
  $S = \{0, 1, 2, \ldots\}$.
- Computing the GPA of a UF student: $S = \{x | 0 \leq x \leq 4.0\}$.
- Recording one's commuting time: $S = \{x | x \geq 0\}$.

Events

Definition (Event)

An **event** is a subset of a sample space.

Example

- The event of observing two heads from tossing a coin twice: $A = \{HH\} \subset S = \{HH, HT, TH, TT\}$.
- The event of observing an even number from rolling a die once: $A = \{2, 4, 6\} \subset S = \{1, 2, 3, 4, 5, 6\}$.
- The event that a UF student graduates with a GPA of at least 3.7: $A = \{x | x \geq 3.7\} \subset S = \{x | 0 \leq x \leq 4.0\}$.

Events

Definition (Complement)

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A^c.

Definition (Intersection)

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Definition (Mutually Exclusive)

Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

Definition (Union)

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Events

Example (Rolling a Die)

Consider rolling a die once.

- Let $A = \{2, 4, 6\}$, the event of observing an even number. Then $A^c = \{1, 3, 5\}$, the event of observing an odd number.
- Let $B = \{4, 5, 6\}$, the event of observing a number greater than 3. Then $A \cap B = \{4, 6\}$ and $A \cup B = \{2, 4, 5, 6\}$.
- Let $C = \{1, 2, 3\}$, the event of observing a number less than or equal to 3. Then $B \cap C = \{4, 5, 6\} \cap \{1, 2, 3\} = \phi$ and B and C are mutually exclusive.

Some Results

1. $A \cap \phi = \phi$.
2. $A \cup \phi = A$.
3. $A \cap A^c = \phi$.
4. $A \cup A^c = S$.
5. $S^c = \phi$.
6. $\phi^c = S$.
7. $(A^c)^c = A$.
8. $(A \cap B)^c = A^c \cup B^c$.
9. $(A \cup B)^c = A^c \cap B^c$.

Read Section 2.3 of WMMY by yourself.

Probability of and Event

Definition (Probability)

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \; P(\phi) = 0, \text{ and } P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots,$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Example (Example 2.24 of WMMY)

A coin is tossed twice. What is the probability that at least 1 head occurs?

Example (Examples 2.25 and 2.26 of WMMY)

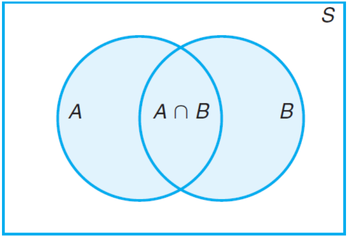
A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.
Let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

Additive Rules

Theorem (Theorem 2.7 of WMMY)

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Theorem (Corollary 2.1 of WMMY)

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Additive Rules

In general,

Theorem (Corollary 2.2 of WMMY)

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Definition (Partition)

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called a **partition** of S if $A_1, A_2, \ldots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \cdots \cup A_n = S$.

Theorem (Corollary 2.3 of WMMY)

If $A_1, A_2, \ldots, A_n$ are a partition of sample space S, then

$$1 = P(S) = P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Additive Rules

Example (Example 2.29 of WMMY)

After being interviewed at two companies, John assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Additive Rules

Theorem (Theorem 2.9 of WMMY)

If A and A^c are complementary events, then

$$P(A) + P(A^c) = 1.$$

Example (Example 2.32 of WMMY)

If the probabilities that an automobile mechanic will service $3, 4, 5, 6, 7$, or 8 or more cars on any given workday are, respectively, $0.12, 0.19, 0.28, 0.24, 0.10$, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Conditional Probability

Definition (Conditional Probability)

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0.$$

Example (Example 2.34 of WMMY)

The probability that a flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane arrives on time,

a. given that it departed on time;
b. given that it did not depart on time.

Independence and the Product Rule

Definition (Independence)

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \text{ or } P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Theorem (Theorem 2.10 of WMMY)

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \textit{ provided } P(A) > 0.$$

Example (Example 2.37 of WMMY)

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Independence and the Product Rule

Theorem (Theorem 2.11 of WMMY)

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Example (Example 2.38 of WMMY)

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Independence and the Product Rule

Theorem (Theorem 2.12 of WMMY)

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

Example (Example 2.40 of WMMY)

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Independence and the Product Rule

Definition (Mutual Independence)

A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

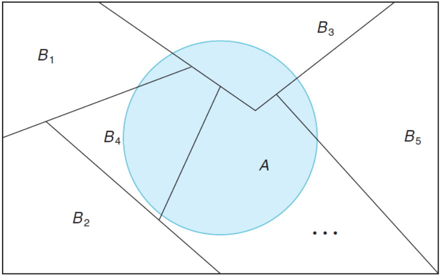
$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

Bayes' Rule

Theorem (Theorem 2.13 of WMMY: Law of Total Probability)

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$



Bayes' Rule

Example (Example 2.41 of WMMY)

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Bayes' Rule

Theorem (Theorem 2.14 of WMMY: Bayes' Rule)

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

for $r = 1, 2, \ldots, k$.

Example (Example 2.42 of WMMY)

With reference to Example 2.41, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?