Correlation length of finite order for the 4-dimensional continuous-time weakly self-avoiding walk and the $|\varphi|^4$ model

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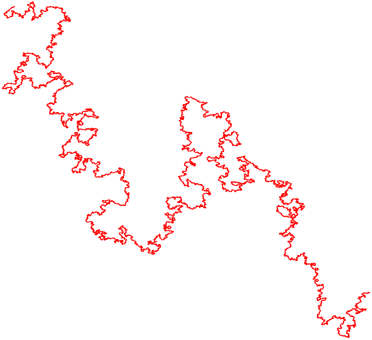

Figure: SAW with 1 million steps (Tom Kennedy)

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- Conclusion: $c_n \approx \mu^n$

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- Mean-field behaviour
- Heuristic reasoning:
  - BM paths have Hausdorff dimension 2 (for $d \geq 2$)
  - 2-dimensional objects generically do not intersect in $d \geq 4$

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- Ratio test + $c_n \geq 0 \Rightarrow$ dominant singularity at $z_c = \mu^{-1}$

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- Prediction: $\chi(z) \sim C(z - z_c)^{-\gamma}$
- (Hara-Slade 92) This is true with $\gamma = 1$ in $d > 4$ (mean-field behaviour)

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- Prediction:

$$G_{z_c}(x) \sim C|x|^{-(d-2+\eta)}$$

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- So $G_{z_c}^{\mathrm{SRW}}(x) \sim C|x|^{-(d-2)}$ and $\eta^{\mathrm{SRW}} = 0$

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- Prediction (mean-field behaviour)

$$G_{z_c}(x) \sim C|x|^{-(d-2+\eta)}$$

with

$$\eta = \begin{cases} \frac{5}{24} & d = 2 \\ 0.03\ldots & d = 3 \\ 0 & d = 4 \\ 0 & d > 4 \quad \text{(Hara-Slade 92)} \end{cases}$$

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- Confirmed by a rigorous renormalisation group method for SAW on a hierarchical lattice (Brydges-Evans-Imbrie 92, Brydges-Imbrie 02)

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- $c_{g,t}(x) = \mathbb{E}[e^{-gI(t)} \mathbb{1}_{\{X(t)=x\}}], \quad g > 0$
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- Two-point function and susceptibility

$$G_{g,\nu}(x) = \int_0^\infty c_{g,t}(x) e^{-\nu t} \mathrm{d}t$$

$$\chi(g,\nu) = \int_0^\infty c_{g,t} e^{-\nu t} \mathrm{d}t$$

RG results

There exists $\nu_c = \nu_c(g) \leq 0$ such that

$$G_{g,\nu}(x) \text{ and } \chi(g,\nu) \begin{cases} < \infty, & \nu > \nu_c \\ = \infty, & \nu < \nu_c \end{cases}$$

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Theorem (Bauerschmidt-Brydges-Slade 14)

For $d = 4$ and $g > 0$ sufficiently small,

$$\chi(g,\nu) \sim C(\nu-\nu_c)^{-1}(\log(\nu-\nu_c)^{-1})^{1/4}$$
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Remark

Analogous results proved for the $|\varphi|^4$ model ("soft" Ising/O(n) model)

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- Correlation length of order p

$$\xi_p(g,\nu) = \left(\frac{1}{\chi(g,\nu)} \sum_{x\in\mathbb{Z}^d} |x|^p G_{g,\nu}(x) \right)^{1/p}$$

expected to scale like ξ near ν_c

Conjectures and rigorous results

- It is expected that

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- (Chen-Sakai 11) proved mean-field behaviour of a related quantity above the upper critical dimension for SAW with long-range steps

The free correlation length of order 2

- Fourier transform

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$$\sum_{x\in\mathbb{Z}^d} |x|^2 G_{0,\nu}(x) = -\Delta_{\mathbb{R}^d} \left(4\sum_{i=1}^{d} \sin^2(k_i/2) + \nu\right)^{-1}\Bigg|_{k=0} = \frac{2d}{\nu^2}$$

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- Since $\chi(0,\nu) \sim \nu^{-1}$,

$$\xi_2(0,\nu) = \left(\frac{1}{\chi(0,\nu)} \sum_{x \in \mathbb{Z}^d} |x|^2 G_{0,\nu}(x) \right)^{1/2} \sim \left(\frac{2d}{\nu} \right)^{1/2}.$$

Main result

Theorem (Bauerschmidt-Slade-Tomberg-W, in progress)

For $d = 4$ and $p \geq 1$, if $g > 0$ is small (depending on p),

$$\xi_p(g, \nu) \sim C(\nu - \nu_c)^{-1/2}(\log(\nu - \nu_c)^{-1})^{1/8}.$$

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Results extends to the $|\varphi|^4$ model

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Open problem

Open problem: replace ξ_p by ξ

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- Torus $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$

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- Probability measure

$$\mathrm{d}\,\mathbb{P}_{g,\nu,N}(\varphi) = \frac{1}{Z_{g,\nu,N}} e^{-U_{g,\nu,N}(\varphi)} \mathrm{d}\varphi$$

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- Interaction polynomial

$$U_{g,\nu,N}(\varphi) = \sum_{x\in\Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{2} \varphi_x (-\Delta \varphi)_x \right)$$

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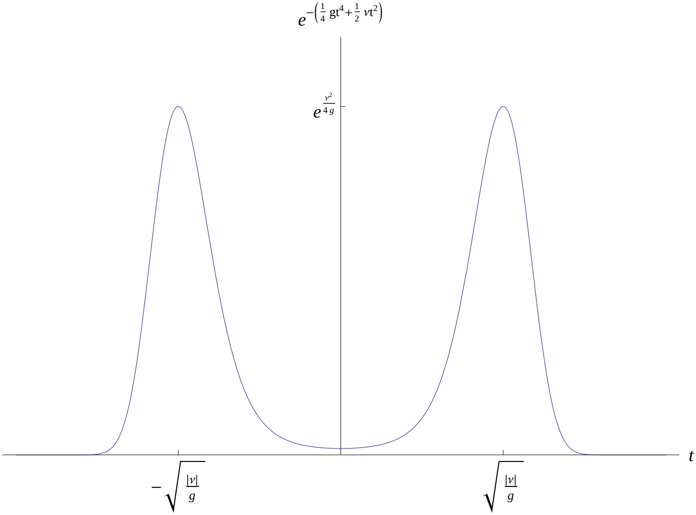
The $|\varphi|^4$ spin system

With $\nu = -g < 0$, get the Ising model as $g \to \infty$



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- Summation by parts [[check this]]

$$\sum_{x\in\Lambda} \varphi_x (-\Delta\varphi)_x = \sum_{x\in\Lambda} \sum_{y\sim x} (\varphi_y - \varphi_x)^2 \tag{1}$$

The two-point function and susceptibility

- Two-point function

$$G_{g,\nu,N}(x) = \frac{1}{n}\langle \varphi_0 \cdot \varphi_x \rangle_{g,\nu,N}$$
$$G_{g,\nu}(x) = \lim_{N\to\infty} G_{g,\nu,N}(x)$$

- Susceptibility

$$\chi_N(g,\nu) = \sum_{x\in\Lambda} G_{g,\nu,N}(x)$$
$$\chi(g,\nu) = \lim_{N\to\infty} \chi_N(g,\nu)$$

The free case

Set $g = 0$ and $\nu \geq 0$

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- Positive-definite quadratic form

$$U_{0,\nu,N}(\varphi) = \frac{1}{2}\sum_{x\in\Lambda}\left(\nu|\varphi_x|^2 + \varphi_x(-\Delta\varphi)_x\right) = \frac{1}{2}\varphi(-\Delta+\nu)\varphi$$

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with covariance $C = (-\Delta + \nu)^{-1}$

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- Thus,

$$G_{0,\nu=0}(x) = (-\Delta)^{-1}_{0x} \sim C|x|^{-(d-2)}, \quad |x| \to \infty$$
$$\chi(0,\nu) \sim C' \nu^{-1}, \quad \nu \downarrow 0$$

Correlation length

Correlation length of order p

$$\xi_p(g,\nu) = \left(\frac{1}{\chi(g,\nu)} \sum_{x \in \mathbb{Z}^d} |x|^p G_{g,\nu}(x) \right)^{1/p}$$

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For $d = 4$ and $p \geq 1$, if $g > 0$ is small (depending on p),

$$\xi_p(g,\nu) \sim C(\nu - \nu_c)^{-1/2} (\log(\nu - \nu_c)^{-1})^{\frac{1}{2}\frac{n+2}{n+8}}.$$

Correlation length

Correlation length of order p

$$\xi_p(g,\nu) = \left(\frac{1}{\chi(g,\nu)} \sum_{x \in \mathbb{Z}^d} |x|^p G_{g,\nu}(x) \right)^{1/p}$$

Theorem (Bauerschmidt-Slade-Tomberg-W, in progress)

For $d = 4$ and $p \geq 1$, if $g > 0$ is small (depending on p),

$$\xi_p(g,\nu) \sim C(\nu - \nu_c)^{-1/2} (\log(\nu - \nu_c)^{-1})^{\frac{1}{2}\frac{n+2}{n+8}}.$$

Remark

$n = 0$ corresponds to WSAW with

$$\frac{1}{2}\frac{n+2}{n+8} = \frac{1}{8}$$

Thank you