Equilibrium stat mech.

Ref. S. Friedli + Y. Velenik
Statistical mechanics of lattice systems

Physical system.
- State space (Ω, λ)
- Hamiltonian $H : \Omega \to \mathbb{R}$ (usu. smooth/bdd below)

Ex. (n particles).

$\Omega = (D \times \mathbb{R}^3)^n$, $D \subseteq \mathbb{R}^3$

$$H(q, p) = \underbrace{\frac{1}{2m} |p|^2}_{\text{kin.}} + \underbrace{U(q)}_{\text{pot.}}$$

$$\begin{cases} \dfrac{dq}{dt} = \nabla_p H \\ -\dfrac{dp}{dt} = \nabla_q H \end{cases}$$

(Stat mech: n large)

Closed system: Constant energy

- Sys. evolves on $\{H = E\} =: S_E$
  energy shell

- Microcanonical ensemble: prob. meas. μ on S_E

$$\int f(x)\mu(dx) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{S_{[E,E+\delta]}} f(x)\,dx.$$

Note. μ singular wrt λ ~~...~~ (hard to work w/)

System in thermal equilibrium: Constant temp.

or $\int H\,d\mu = E$ const.

Ex. (Ω finite) (57)

Employ Jaynes' principle of max entropy

i.e. find μ maximizing $-\sum_{\omega\in\Omega} \mu(\omega)\log\mu(\omega) =: S(\mu)$

subject to $\sum_{\omega} H(\omega)\mu(\omega) = E$

and $\sum_{\omega} \mu(\omega) = 1.$

Define Lagrangian

$$\mathcal{L}(\mu, \lambda_1, \lambda_2) = \mathcal{S}(\mu) - \lambda_1 \left[\int H \, d\mu - E \right] - \lambda_2 \left[\int d\mu - 1 \right]$$

and solve $\nabla \mathcal{L} = 0$.

So $$\frac{\partial \mathcal{L}}{\partial \mu(\omega)} = -\log \mu(\omega) - 1 - \lambda_1 H(\omega) - \lambda_2 = 0$$

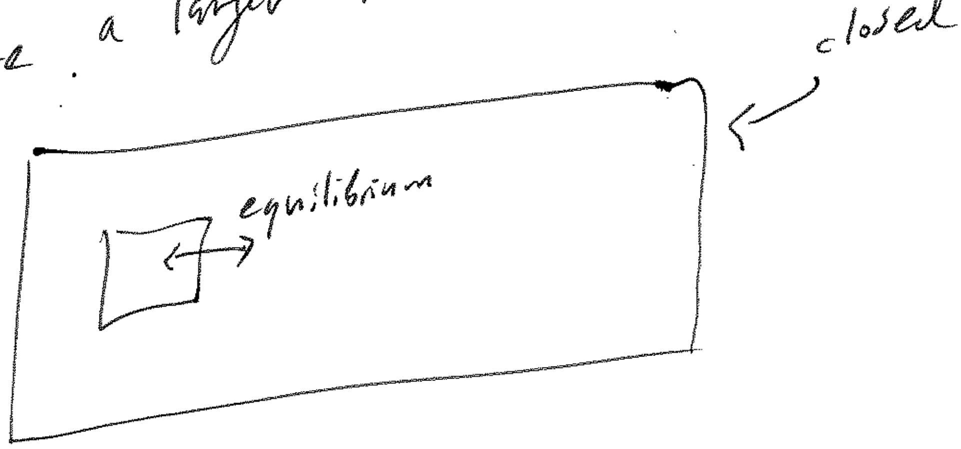
$$\Rightarrow \mu(\omega) \propto e^{-\lambda_1 H(\omega)}.$$

Canonical measure: $\frac{1}{Z} e^{-\beta H(\omega)}$ ($\beta \to$ inverse temp.)

($\beta \to \infty \Rightarrow$ concentration on low energy states)

Other justifications:

- Marginalize a larger microcanonical meas.



- Variational principle (Dobrushin 68, Lanford–Ruelle 69):

$$\cancel{} \; -\frac{1}{\beta}\log Z_\beta = \cancel{} \int H \, d\mu - \beta^{-1} \cancel{} \mathcal{S}(\mu)$$

i.e. free energy = internal energy − temp × entropy

and the RHS is <u>minimized</u> by such measures.

[This holds in <u>great</u> generality, see e.g. Georgii 88]

- Hammersley–Clifford thm.

$$\left.\begin{array}{l} \Omega = \{ f : \text{graph} \longrightarrow \text{countable} \} \\ H = \sum \text{contributions from complete subgraphs} \end{array}\right\} \text{Gibbs random field}$$

<u>Thm.</u> Gibbs random fields $\Longleftrightarrow$ Markov random fields

(see Grimmett 2010)

Relation to quantum theory.

$$H(x,p) = \frac{1}{2m}|p|^2 + U(x)$$

quantize: $x_j \rightsquigarrow \hat{x}_j$, $p_j \rightsquigarrow -i\hbar \frac{\partial}{\partial x_j}$

$$\hat{H} = H(\hat{x},\hat{p}) = -\frac{\hbar^2}{2m} + U(\hat{x})$$

$\psi \in L^2(\mathbb{R}^{3n})$ evolves according to Schrödinger:

$$i\hbar \frac{d\psi}{dt} = \hat{H}\psi$$

Assume soln. operator has kernel K_t:

$$e^{-it\hat{H}/\hbar} f = \int K_t(\cdot, y) f(y)\,dy$$

Feynman integral formulation:

$$K_t(a,b) = \int_{W_t(a,b)} e^{(i/\hbar)\int_0^t L(x(s),\dot{x}(s))} \mathcal{D}x$$

$$W_t(a,b) = \{\text{paths } [0,t] \to \mathbb{R}^{3n} \text{ from } a \to b\}$$

$$L(x,\dot{x}) = \frac{1}{2}m|\dot{x}|^2 - U(x).$$

Wick rotation: $t \longmapsto \psi(-it)$ has soln. kernel

$$K_{-it}(a,b) = \int_{W_{-it}(a,b)} e^{(i/\hbar)\int_0^{-it} L(x(s),\dot{x}(s))\,ds}\,\mathcal{D}x$$

$\tilde{x}(t) = x(-it)$

Change vars $s = -iu$:

$$\frac{i}{\hbar}\int_0^{-it} L(x(s), \dot{x}(s))\,ds = \frac{1}{\hbar}\int_0^t L(\tilde{x}(u), i\dot{\tilde{x}}(u))\,du$$

$$= -\frac{1}{\hbar}\int_0^t H(\tilde{x}(u), m\dot{\tilde{x}}(u))\,du.$$

Use $W_{it} \simeq W_t$ to get

$$K_{-it}(a,b) = \int_{W_t(a,b)} \underbrace{e^{-(1/\hbar)\int_0^t H(x(u), m\dot{x}(u))\,du}\,\mathcal{D}x}_{\text{Gibbs-type meas. on paths}}$$

<u>Feynman–Kac</u>:

$\tilde{\psi}(t) = \psi(-it)$ should solve

$$\frac{d\tilde{\psi}}{dt} = -\hat{H}\tilde{\psi} \qquad (\hbar = 1, m = 1) \qquad (*)$$

- $U=0 \Rightarrow \hat{H} = -\frac{1}{2}\Delta$

  $\Rightarrow (*)$ is heat eqn., $\int_0^t H$ is pos-def quadratic

  and K_{-it} is a "Gaussian" integral

Rigorously, $\tilde{\Psi}(t,x) = \mathbb{E}\left(\tilde{\Psi}(0,B_t) \mid B_0 = x\right)$

$\hookrightarrow$ Brownian motion

(under right conditions)

- $U \neq 0 \Rightarrow$ Feynman-Kac formula

$$\tilde{\Psi}(t,x) = \mathbb{E}\left(\tilde{\Psi}(0,B_t)\, e^{\int_0^t U(B_s)\,ds} \mid B_0 = x\right)$$

cf. K_{-it}

QFT.
Replace paths (fcns. on $\mathbb{R}$) by fields (fcns. on $\mathbb{R}^d$)

Ising model.

Graphs.

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

[undirected / $\mathcal{V}$ countable / no self-loops $\{x\} \in \mathcal{E}$]

Weights $J \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$, $J_{xy} \geq 0$ w/ equality iff $x \sim y$

$D \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ diag. w/ $D_{xx} = d_x = \sum_{y \sim x} J_{xy}$

Laplacian $-\Delta = D - J$

Spin systems.

$d\lambda^0$ meas. on $S \subset \mathbb{R}^n$

$d\lambda = \prod_{x \in \Lambda} d\lambda^0$ on $\Omega = S^{\mathcal{V}}$

$\varphi \in \Omega$ a <u>field</u> or <u>spin configuration</u>

w/ spins in S

$H : \Omega \to \mathbb{R}$ st. $\int e^{-H} d\lambda < \infty$.

A spin system w/ Hamiltonian H at inv. temp. $\beta > 0$

is $d\mu_\beta(\varphi) = \frac{1}{Z_\beta} e^{-\beta H(\varphi)} d\lambda(\varphi)$.

It is ferromagnetic if

$H(\varphi) = -\varphi \cdot M \varphi$ w/ $M_{xy} \geq 0$ $\forall x, y$

$$\overset{\text{(note)}}{=} \sum_{i=1}^{n} \sum_{x,y \in V} \underbrace{\left(-\varphi_x^i M_{xy} \varphi_y^i\right)}_{\text{min @ } \varphi_x^i = \varphi_y^i}$$

Ex. ($O(n)$/n-vector model).

$H(\varphi) = -\frac{1}{2} \varphi \cdot J \varphi$ for $\varphi \in (S^{n-1})^V$

$$\begin{cases} n = 1 \Rightarrow \text{Ising} \\ n = 2 \Rightarrow XY \\ n = 3 \Rightarrow \text{classical Heisenberg} \end{cases}$$

Phase transitions:

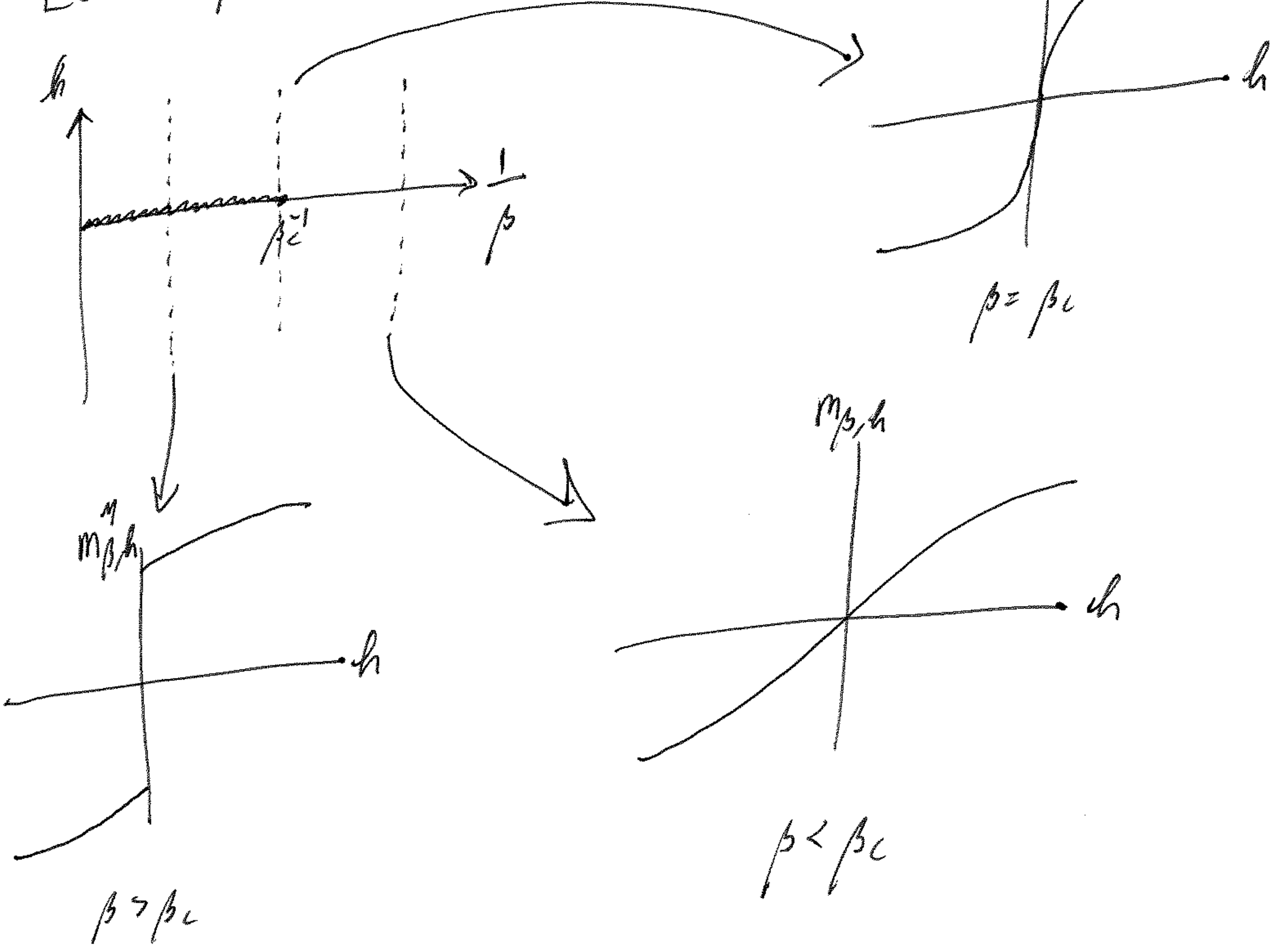
Fix $\mathcal{G} = \Lambda \subset \mathbb{Z}^d$ a box

Consider ~~$O(n)$~~ Ising model w/ bdry condition η outside Λ + ext field $h \in \mathbb{R}$

$$\Longrightarrow d\mu_{\beta,h}^{\eta,\Lambda}$$

Suppose $d\mu_{\beta,h}^{\eta} = \lim_{\Lambda \uparrow \mathbb{Z}^d} d\mu_{\beta,h}^{\eta,\Lambda}$ exists (in some sense)

Let $m_{\beta,h}^{\eta} = \langle \sigma_0 \rangle_{\beta,h}^{\eta}$ be the *magnetization*.



<u>Case $\beta > \beta_c$</u>: • depends on η

• reflects existence of multiple (distinct) Gibbs states for $h = 0$

i.e. $\mu^+_{\beta,0} \neq \mu^-_{\beta,0}$

$$\mu^{\eta}_{\beta,0} \text{ w/ } \eta \equiv 1$$

This is a <u>first-order</u> or <u>discts.</u> phase transition / <u>spontaneous symmetry breaking</u>.

($\mu^{\pm}_{\beta,0}$ not invariant under $\omega \mapsto -\omega$)

<u>To come</u>.

• <u>Next</u>: investigate <u>far from critical</u> regimes ($\beta \gg \beta_c$, $\beta \ll \beta_c$, $|h| \gg 0$) using <u>expansion methods</u>

$\Rightarrow$ cluster expansion

• <u>Other possible topics</u>.

- Other methods to study <u>non-critical</u> regime ($(\beta, h) \neq (\beta_c, 0)$)
- Models w/ continuous symmetry ($n > 1$)
- Renormalisation group for $\beta = \beta_c$ (critical regime)

Gibbs measures for the Ising model.

- Focus on classical nn Ising on $\mathbb{Z}^d$ (avoid conditioning on meas. 0 events)
  Follow Georgii - Häggström - Maes '99
- Generalize to compact spin space + long-range interactions in Georgii '11 or Friedli-Velenik '17
- For unbdd. spins see Lebowitz - Presutti '76

Notation.

$\partial\Lambda$ = ext bdry of $\Lambda \subset \mathbb{Z}^d$

$\Omega_\Lambda = \{\pm 1\}^\Lambda$

restrict $\omega \in \Omega$ to $\omega_\Lambda \in \Omega_\Lambda$

abuse notation: $\omega_\Lambda \simeq \{\tilde\omega : \tilde\omega_\Lambda = \omega_\Lambda\}$

Def. Hamiltonian $H^{\eta,\Lambda}_{\beta,h} : \Omega \to \mathbb{R}$ at inv temp $\beta > 0$, ext field $h \in \mathbb{R}$, bdry condition $\eta \in \Omega$ off of $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$:

$$H^{\eta,\Lambda}_{\beta,h}(\omega) = -\beta\left[\sum_{\substack{x\sim y\\ x,y\in\Lambda}} \omega_x\omega_y + h\sum_{x\in\Lambda}\omega_x + \sum_{\substack{x\sim y\\ x\in\Lambda\\ y\in\partial\Lambda}} \omega_x\eta_y\right]$$

<u>Gibbs specification</u> $\mu^{\eta,\Lambda}_{\beta,h}(\omega) = \dfrac{1}{Z^{\eta,\Lambda}_{\beta,h}} e^{-H^{\eta,\Lambda}_{\beta,h}(\omega)}$,

Prob. meas. μ on Ω is a <u>Gibbs meas.</u> if the DLR eqns.

$$\mu(\omega_\Lambda \mid \eta_{\partial\Lambda}) = \mu^{\eta,\Lambda}(\omega), \qquad \forall\, \omega, \eta, \Lambda$$

are satisfied.

Let $\mu^{\pm,\Lambda} = \mu^{\eta,\Lambda}$ w/ $\eta \equiv 1$.

<u>Thm</u>. The weak lims. $\mu^{\pm} := \lim_{\Lambda\uparrow\mathbb{Z}^d} \mu^{\pm,\Lambda}$ exist and are Gibbs measures.

<u>Note</u>. $\lim_{\Lambda\uparrow\mathbb{Z}^d}(\ldots)$ means $\lim_{\Lambda_n\uparrow\mathbb{Z}^d}(\ldots)$ $\forall\, \Lambda_n\uparrow\mathbb{Z}^d$.

If all these lims exist, they are the same.

<u>Markov prop</u> – $\Lambda_1 \subset \Lambda_2 \Rightarrow \mu^{\xi,\Lambda_1}(\omega) \cancel{= \mu^{\Lambda_2}(\omega \mid \xi_{\Lambda_2\setminus\Lambda_1})}$.

$$= \mu^{\tilde{\xi},\Lambda_2}(\omega \mid \xi_{\Lambda_2\setminus\Lambda_1})$$

Pf. (idea)
RHS has form $\mu^{\eta,\Lambda}_{\beta,h}$ so just check that params η, Λ, β, h same on LHS and RHS. $\square$

Next. Reduce pf. of thm.

Lem. Suffices to show μ^+ exists.

Pf. $\mu^+(\omega_\Lambda \mid \xi_{\partial\Lambda}) = \lim_{\Delta \uparrow \mathbb{Z}^d} \mu^{+,\Delta}(\omega_\Lambda \mid \xi_{\partial\Lambda})$

~~(Markov)~~ $\cancel{= \mu^{\xi,\Lambda}(\omega_\Lambda)}$

$\overset{\text{(Markov)}}{=} \lim_{\Delta} \mu^{\xi,\Lambda}(\omega_\Lambda) = \mu^{\xi,\Lambda}(\omega_\Lambda)$. $\square$

$f : \Omega \to \mathbb{R}$ is local if it only depends on spins in a finite set Λ. Thus, it can be identified w/ an element of $\mathbb{R}^{\Omega_\Lambda}$.

Lem. Any bdd. cts. fn. on Ω (prod. topology) can be unif. approximated (ie. in the sup norm) by local fns. Thus, it suffices to check convergence of $\mu^{+,\Lambda}(f)$ for local fns. f.

Pf. (idea).

Given bdd. cts f and arbitrary $\tilde{\omega} \in \Omega$,

let $g_\Lambda(\omega) = f(\omega_\Lambda \tilde{\omega}_{\Lambda^c})$. So g_Λ local.

Tychonoff $\Rightarrow \Omega$ compact $\Rightarrow f - g_\Lambda$ unif cts

$\Rightarrow \|f - g_\Lambda\|_\infty \to 0$ as $\Lambda \uparrow \mathbb{Z}^d$. ∎

For $A \subset \Lambda$ let $\sigma_A : \Omega_\Lambda \to \mathbb{R}$ be

$$\sigma_A(\omega) = \prod_{x \in A} \omega_x .$$

Lem. $\{\sigma_A : A \subset \Lambda\}$ is an ON basis for $\mathbb{R}^{\Omega_\Lambda}$ under $\langle f, g \rangle = 2^{-|\Lambda|} \sum_{\tilde{\omega} \in \Omega_\Lambda} f(\tilde{\omega}) g(\tilde{\omega})$.

Pf. $|2^\Lambda| = 2^{|\Lambda|} = |\Omega_\Lambda|$ so just check ON

~~$A \Delta A \Rightarrow \omega_x^2$~~

$\omega_x^2 = 1 \Rightarrow \langle \sigma_A, \sigma_A \rangle = 1.$

For $A \neq B$:

$$2^{|\Lambda|}\langle \sigma_A, \sigma_B \rangle = \sum_{\tilde\omega} \prod_{x \in A \Delta B} \tilde\omega_x = 0$$

Claim. $\forall$ finite Λ and $S \subset \Lambda$,

$$\sum_{\tilde\omega \in \Omega_\Lambda} \prod_{x \in S} \tilde\omega_x = 0.$$

Follows by induction: for $|S| = n+1$,

$$\sum_{\tilde\omega \in \Omega_\Lambda} \prod_{x \in S} \tilde\omega_x = \sum_{\tilde\omega \in \Omega_\Lambda} \tilde\omega_y \prod_{\substack{x \neq y \\ x \in S}} \tilde\omega_x$$

$$= (\dots) - (\dots) = 0. \quad \blacksquare$$

Let $\mathbb{1}_{\delta A}$ be the indicator fcn. for $\delta A = \{\omega_x = 1 \ \forall x \in A\}$.

Lem. $\{\mathbb{1}_{\delta A} : A \subset \Lambda\}$ is a basis for $\mathbb{R}^{\Omega_\Lambda}$.

Pf. Suffices to show spanning.

Write ~~$\sum_{A \subset \Lambda} f_A$~~ $f \in \mathbb{R}^{\Omega_\Lambda}$ as

$$f = \sum_{A \subset \Lambda} f_A \sigma_A, \quad f_A = \langle f, \sigma_A \rangle$$

Expand σ_A in terms of indicators using

$$\sigma_x = 2\mathbb{1}_{\delta^x} - 1$$

and binomial thm.

$$\prod_{x \in X} (a_x + b_x) = \sum_{Y \subset X} \left(\prod_{x \in X \setminus Y} a_x\right)\left(\prod_{y \in Y} b_y\right).$$

Get $\sigma_A = \prod_{x \in A} (2\mathbb{1}_{\delta^x} - 1) = \sum_{Y \subset A} (-1)^{|A \setminus Y|} 2^{|Y|} \mathbb{1}_{\delta^Y}$.

Thus, $f = \sum_{Y \subset \Lambda} \tilde{f}_Y \mathbb{1}_{\delta^Y}$

$\left(\text{w/ } \tilde{f}_Y = \sum_{A \supset Y} f_A (-1)^{|A \setminus Y|} 2^{|Y|}\right)$. $\square$

<u>Conclusion</u>. Suffices to show: $\forall A \subset \mathbb{Z}^d$ finite,

$$\mu^{+,\Lambda}(\delta^A) = \mu^{+,\Lambda}(\mathbb{1}_{\delta^A}) \quad \text{converges.}$$

<u>Plan</u>. $\mu^{+,\Lambda}(\delta^A)$ bdd, so monotonicity suffices.

Intuitively, $\mu^{+,\Lambda}(\delta^A)$ decreases as Λ increases.

Def. (Stochastic dominance).

$\mu_1 \le \mu_2$ if $\mu_1(f) \le \mu_2(f)$ ~~[illegible]~~

$\forall$ bdd. cts. $\underbrace{\text{inc.}}_{\text{wrt natural partial order}}$ f.

Thus, we will show that $\Lambda_1 \subset \Lambda_2 \Rightarrow \mu^{+,\Lambda_2} \le \mu^{+,\Lambda_1}$.

Since $\mathbb{1}_{\xi^A}$ is bdd. $\underbrace{\text{cts.}}_{\text{wrt prod. top.}}$ inc., get convergence.

Accomplish this using:

Thm. (Strassen). TFAE:

(1) $\mu_1 \le \mu_2$

(2) There is a coupling μ of (μ_1, μ_2) s.t. $(X_1, X_2) \sim \mu \Rightarrow \mathbb{P}(X_1 \le X_2) = 1$. } monotone coupling

Note. We only use (2) $\Rightarrow$ (1), which is easy:

f inc. + $X_1 \le X_2 \Rightarrow f(X_1) \le f(X_2) \Rightarrow \mu_1(f) = \mathbb{E} f(X_1) \le \mathbb{E} f(X_2) = \mu_2(f)$.

[Converse harder but not needed].

Correlation inequalities.

Def. μ prob meas on Ω_Λ,

G_μ graph w/ $V = \{\omega \in \Omega_\Lambda : \mu(\omega) > 0\}$

and $\omega \sim \omega'$ if $\omega_x = \omega'_x$ $\forall$ but one $x \in \Lambda$.

μ *irreducible* if G_μ connected.

Thm. (Holley inequality).
μ_1, μ_2 prob meas on Ω_Λ. Assume μ_2 irred. and $\mu_2(\delta^\Lambda) > 0$. Suppose $\forall x \in \Lambda$ and $\xi \leq \eta$ w/ $\mu_1(\xi_{\Lambda \setminus x}) > 0$ and $\mu_2(\eta_{\Lambda \setminus x}) > 0$ that

$$\mu_1(\delta^x \mid \xi_{\Lambda \setminus x}) \leq \mu_2(\delta^x \mid \eta_{\Lambda \setminus x}).$$

Then $\mu_1 \leq \mu_2$.

Holley's inequality and $\mu^{\pm}$.

Recall. Our plan is as follows:

(A) Construct monotone coupling of $(\mu^{+,\Lambda_2}, \mu^{+,\Lambda_1})$ for $\Lambda_1 \subset \Lambda_2$

(Strassen) $\Rightarrow$ (B) $\mu^{+,\Lambda_2} \leq \mu^{+,\Lambda_1}$

$\Rightarrow$ (C) $\left(\mu^{+,\Lambda}(\delta^A)\right)_{\Lambda \uparrow \mathbb{Z}^d}$ decreasing $\forall A$ finite

$\Rightarrow$ (D) This sequence converges

$\Rightarrow$ (E) $\mu^{+,\Lambda}(f)$ converges $\forall$ local f

$\Rightarrow$ (F) $\mu^{+,\Lambda}(f)$ converges $\forall$ bdd cts f

$\Rightarrow$ (G) $\mu^{+} = \lim_{\Lambda} \mu^{+,\Lambda}$ exists + is a Gibbs meas.

We prove (A) in greater generality as Holley's inequality.

Correlation inequalities.

Def. μ prob. meas. on Ω_Λ. G_μ graph w/ $V = \Omega_\Lambda$ and $V = \{\omega \in \Omega_\Lambda : \mu(\omega) > 0\}$ and $\omega \sim \omega'$ if $\omega_x = \omega'_x$ $\forall$ but one x. μ <u>irreducible</u> if G_μ connected.

Thm. (Holley ineq.) μ_1, μ_2 prob. meas. on Ω_Λ. Assume μ_2 irred. and $\mu_2(\delta^\Lambda) > 0$. Suppose $\forall x \in \Lambda$ and $\xi \leq \eta$ w/ $\mu_1(\xi_{\Lambda \setminus x}) > 0$, $\mu_2(\eta_{\Lambda \setminus x}) > 0$ that

$$\mu_1(\delta^x \mid \xi_{\Lambda \setminus x}) \leq \mu_2(\delta^x \mid \eta_{\Lambda \setminus x}).$$

Then $\mu_1 \leq \mu_2$.

<u>Gibbs sampler</u>. Markov chain on Ω_Λ w/ stationary μ.

If $\sigma^i = \omega$, then:

(1) pick $x \in \Lambda$ unif.

(2) re-sample ω_x by $\mu(\cdot \mid \omega_{\Lambda \setminus x})$

Output is $\sigma^{i+1} = \omega'$.

μ irred $\Rightarrow$ σ irred. MC.

(Clearly $\sum_{\omega_{\Lambda\setminus x}} \mu(\omega_x^i | \omega_{\Lambda\setminus x})\mu(\omega_{\Lambda\setminus x}) = \mu(\omega_x^i)$) .

(9)

Pf. (Holley).

Couple Gibbs samplers σ, τ for μ_1, μ_2, resp.

U_k iid on $[0,1]$ unif.

If $(\sigma^i, \tau^i) = (\xi, \eta)$, define $(\sigma^{i+1}, \tau^{i+1})$ by:

(1) $x \in \Lambda$ unif.

(2) $\sigma_x^{i+1} = \begin{cases} 1, & \mu_1(\delta^x | \xi_{\Lambda\setminus x}) \geq U_i \\ -1, & o/w \end{cases}$

$\tau_x^{i+1} = \begin{cases} 1, & \mu_2(\delta^x | \eta_{\Lambda\setminus x}) \geq U_i \\ -1, & o/w \end{cases}$.

Then $\mathbb{P}(\sigma_x^{i+1} = 1 | \sigma^i = \xi) = \mathbb{P}(\mu_1(\delta^x | \xi_{\Lambda\setminus x}) \geq U_i)$

$= \mu_1(\delta^x | \xi_{\Lambda\setminus x})$

So marginals are Gibbs samplers.

Moreover: wp1 $\exists i$ st. $\sigma^i \leq \tau^i$

(worst case $\exists i$ st $\tau^i \equiv 1$ by irred.)

Then $\sigma^j \leq \tau^j$ $\forall j \geq i$ (by defn. of coupling)

Clearly (σ, τ) aperiodic.

Thus, (σ, τ) has lim. meas. μ

[Supp. on $\{\xi \le \eta\}$ since (σ,τ) not irred.]. ∎

μ is a monotone coupling of (μ_1, μ_2).

~~Thm. (FKG).~~

Def. μ monotone if $\forall x \in \Lambda$ $\forall \xi \le \eta$ w/ $\mu(\xi_{\Lambda \setminus x}), \mu(\eta_{\Lambda \setminus x}) > 0$,

$$\mu(\delta^x \mid \xi_{\Lambda \setminus x}) \le \mu(\delta^x \mid \eta_{\Lambda \setminus x}).$$

Ex. $\mu^{+,\Lambda}$ monotone: computation

Thm. (FKG). $\mu(\omega) > 0$ $\forall \omega \in \Omega_\Lambda$.

μ monotone $\Rightarrow$ μ has positive correlations, i.e.

$$\mu(fg) \ge \mu(f)\mu(g) \quad \forall f, g \text{ inc. bdd. } \Omega_\Lambda \to \mathbb{R}$$

Pf. Assume wlog $g > 0$ (o/w replace by $g - \min(g) + 1$)

Then $\frac{1}{Z} g \, d\mu$, is a prob. meas w/ $Z = \mu_1(g)$.

Apply Holley w/ $d\mu_1 = d\mu$, $d\mu_2 = \frac{1}{Z} g \, d\mu_1$

$\Rightarrow \mu_1 \le \mu_2 \Rightarrow \mu_1(fg) = Z\mu_2(f) \ge Z\mu_1(f) = \mu_1(f)\mu_1(g)$.

First verify hypothesis:

$$\frac{\mu_2(\delta^x|\xi_{\Lambda\setminus x})}{\mu_2(-\delta^x|\xi_{\Lambda\setminus x})} = \frac{\mu_1(\delta^x\xi_{\Lambda\setminus x})\,g(\delta^x\xi_{\Lambda\setminus x})}{\mu_1((-\delta^x)\xi_{\Lambda\setminus x})\,g((-\delta^x)\xi_{\Lambda\setminus x})}$$

$$\geq \frac{\mu_1(\delta^x\xi_{\Lambda\setminus x})}{\mu_1((-\delta^x)\xi_{\Lambda\setminus x})} \qquad (g \text{ inc.})$$

$$= \frac{\mu_1(\delta^x|\xi_{\Lambda\setminus x})}{\mu_1(-\delta^x|\xi_{\Lambda\setminus x})}$$

$\overset{(\text{solve})}{\Longrightarrow}$ $\mu_2(\delta^x|\xi_{\Lambda\setminus x}) \geq \mu_1(\delta^x|\xi_{\Lambda\setminus x})$ $\square$

$\Rightarrow$ $\mu_1(\delta^x|\xi_{\Lambda\setminus x}) \leq \mu_2(\delta^x|\xi_{\Lambda\setminus x}) \leq \mu_2(\delta^x|\eta_{\Lambda\setminus x})$.

(monotonicity)

Pf. (existence of $\mu^\pm$). For bdd. inc. f, $\Lambda_1 \subset \Lambda_2$

$$\mu^{+,\Lambda_1}(f) = \mu^{+,\Lambda_2}(f\,|\,\delta^{\Lambda_2\setminus\Lambda_1}) \qquad [\text{Markov}]$$

$$= \frac{\mu^{+,\Lambda_2}(f\,\mathbb{1}_{\delta^{\Lambda_2\setminus\Lambda_1}})}{\mu^{+,\Lambda_2}(\mathbb{1}_{\delta^{\Lambda_2\setminus\Lambda_1}})}$$

$$\geq \mu^{+,\Lambda_2}(f) \qquad [\text{FKG}]. \quad \blacksquare$$

Other properties.

- $\mu^{\pm}$ are invariant under any $\mathbb{Z}^d$-automorphism T:

  $\forall$ cylinder set C,

$$\mu^{+}(T^{-1}C) = \lim_{\Lambda} \mu^{+,\Lambda}(T^{-1}C) = \lim_{\Lambda} \mu^{+,T\Lambda}(C) = \mu^{+}(C).$$

- If $\mu^{\omega} := \lim_{\Lambda} \mu^{\omega,\Lambda}$ exist for $\omega = \xi, \eta$, then

$$\xi \leq \eta \Rightarrow \mu^{\xi} \leq \mu^{\eta}.$$

Pf. Computation shows

$$\mu^{\eta,\Lambda}(f) = \frac{\mu^{\xi,\Lambda}(f\, I^{\Lambda}_{\xi,\eta})}{\mu^{\xi,\Lambda}(I^{\Lambda}_{\xi,\eta})} \quad \text{for } I^{\Lambda}_{\xi,\eta} \text{ bdd cts inc.}$$

Thus, for f bdd cts inc,

$$\mu^{\eta,\Lambda}(f) \geq \frac{\mu^{\xi,\Lambda}(f)\,\cancel{\mu^{\xi,\Lambda}(I^{\Lambda}_{\xi,\eta})}}{\cancel{\mu^{\xi,\Lambda}(I^{\Lambda}_{\xi,\eta})}}.$$

Then take $\Lambda \to \mathbb{Z}^d$.

- Follows that $\mu^{-} \leq \mu \leq \mu^{+}$ $\forall$ Gibbs μ.

- If $\mu^+ = \lambda\mu_1 + (1-\lambda)\mu_2$, $\lambda \in (0,1)$ for Gibbs μ_1, μ_2, then $\mu_1 = \mu_2 = \mu^+$ ($\mu^\pm$ are <u>extremal</u>).

<u>Pf</u>. $\mu_1 \neq \mu^+ \Rightarrow \mu_1 < \mu^+ \Rightarrow \mu^+ = \lambda\mu_1 + (1-\lambda)\mu_2 < \mu^+ \Rightarrow \Leftarrow$. ■

<u>Note</u>. (1) Aizenman (80), Higuchi (81):
$\mu^\pm$ are the <u>only</u> extremal states <u>if $d=2$</u>.

(2) Dobrushin (72):
This is false <u>if $d>2$</u>.

- TFAE:
  (a) There is a unique inf-vol state
  (b) $\mu^+ = \mu^-$
  (c) $\mu^+(\sigma^x) = \mu^-(\sigma^x)$

<u>Pf</u>. [Show (c) $\Rightarrow$ (b) $\Rightarrow$ (a).]

(c) $\Rightarrow$ (c) $\forall x$ by translation-invariance.

(c)$\Rightarrow$(b) $\{$ Strassen $\Rightarrow \exists$ monotone coupling μ of (μ^-, μ^+).
Let $(\sigma^1, \sigma^2) \sim \mu$. So $\sigma^1_x = 1 \Rightarrow \sigma^2_x = 1$ and $\sigma^2_x = -1 \Rightarrow \sigma^1_x = -1$.
Thus, $\mathbb{P}(\sigma^1_x = \sigma^2_x) = \mathbb{P}(\sigma^1_x = 1) + \mathbb{P}(\sigma^2_x = -1) \overset{(c)}{=} \mathbb{P}(\sigma^2_x = 1) + \mathbb{P}(\sigma^2_x = -1) = 1$ $\forall x \Rightarrow$ (b).

(b) $\Rightarrow$ (a) by $\mu^- \le \mu \le \mu^+$. ■

Pressure + Magnetization

Recall.

- $\mu^{\pm}_{\beta,h} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \mu^{\pm,\Lambda}_{\beta,h}$ exist $\forall \beta>0,\ h\in\mathbb{R}$
- Phase transition occurs $\iff \langle\sigma_0\rangle^+ \neq \langle\sigma_0\rangle^-$

How do we relate this to spontaneous magnetization?

Pressure.

Let $\psi^{\eta,\Lambda}(\beta,h) = \frac{1}{|\Lambda|}\log Z^{\eta,\Lambda}_{\beta,h}$

Hölder $\Rightarrow Z^{\eta,\Lambda}_{\alpha\beta_1+(1-\alpha)\beta_2,\ \alpha h_1+(1-\alpha)h_2} \leq \left(Z^{\eta,\Lambda}_{\beta_1,h_1}\right)^{\alpha}\left(Z^{\eta,\Lambda}_{\beta_2,h_2}\right)^{1-\alpha}$

$\Rightarrow \psi^{\eta,\Lambda}$ convex

Thm. The pressure $\psi = \lim_{\Lambda\uparrow\mathbb{Z}^d} \psi^{\eta,\Lambda}$ exists, ~~and~~ is indep. of η, convex, and even in h.

Here, $\Lambda\Uparrow\mathbb{Z}^d$ is van Hove convergence, ie. $\Lambda_n\uparrow\mathbb{Z}^d$ w/ $|\partial\Lambda_n|/|\Lambda_n| \to 0$ ($\partial\Lambda$ = interior bdry for today).

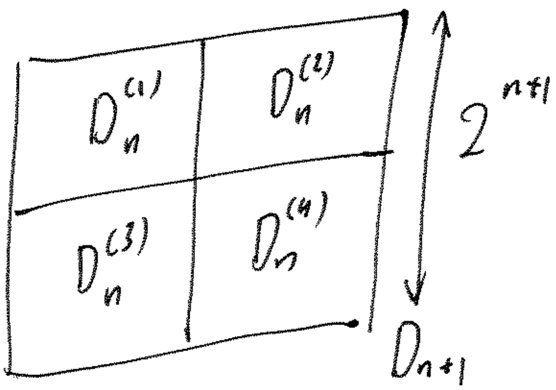
Pf. (A) Free bdry conditions

i.e. $H^{\phi,\Lambda} = -\beta \left[\sum_{x\sim y} \omega_x \omega_y + h \sum_x \omega_x \right]$.

(1) Convergence along exponential boxes.

Let $D_n = \{1, \dots, 2^n\}^d$

Then $D_{n+1} = \bigcup_{i=1}^{2^d} D_n^{(i)}$ w/ $D_n^{(i)}$ translates of D_n



$$H^{\phi, D_{n+1}} = \sum_{i=1}^{2^d} H^{\phi, D_n^{(i)}} + R_n, \quad |R_n| \le O\left((2^{n+1})^{d-1}\right)$$

$$\Rightarrow Z^{\phi, D_{n+1}} \le e^{O((2^{n+1})^{d-1})} \underbrace{\sum_{\omega \in \Omega_{D_{n+1}}} \prod_{i=1}^{2^d} e^{-H^{\phi, D_n^{(i)}}(\omega)}}_{\prod_{i=1}^{2^d} \sum_{\omega^{(i)} \in \Omega_{D_n^{(i)}}} e^{-H^{\phi, D_n^{(i)}}(\omega^{(i)})}} = \left(Z^{\phi, D_n}\right)^{2^d}$$

$$\Rightarrow \psi^{\phi, D_{n+1}} \le O\left(2^{-(n+1)}\right) + \psi^{\phi, D_n}$$

Sim., $\psi^{\phi, D_n} \leq O(2^{-(n+1)}) + \psi^{\phi, D_{n+1}}$

i.e. $|\psi^{\phi, D_n} - \psi^{\phi, D_{n+1}}| \leq O(2^{-(n+1)})$.

Iterated Δ inequality $\Rightarrow$ ψ^{ϕ, D_n} Cauchy.

Write $\psi = \lim_n \psi^{\phi, D_n}$.

(2) Convergence on arbitrary $\Lambda_n \uparrow \mathbb{Z}^d$.

Cover $\mathbb{Z}^d$ by translates of D_k

Let $[\Lambda_n] = \bigcup_{j=1}^{N(n)} D_k^{(j)}$ w/ $\{D_k^{(j)}\}$ a min. covering of Λ_n by translates of D_k

Write

$$|\psi^{\phi,\Lambda_n} - \psi| \leq \underbrace{|\psi^{\phi,\Lambda_n} - \psi^{\phi,[\Lambda_n]}|}_{a} + \underbrace{|\psi^{\phi,[\Lambda_n]} - \psi^{\phi,D_k}|}_{b} + \underbrace{|\psi^{\phi,D_k} - \psi|}_{c}$$

- Then $c \xrightarrow{k \to \infty} 0$ by (1).
- Bound on b sim. to (1):

$$H^{\phi,[\Lambda_n]} \leq \sum_{j=1}^{N(n)} H^{\phi, D_k^{(j)}} + \underbrace{O\left(\overset{|[\Lambda_n]|/|D_k| = |[\Lambda_n]|/(2^k)^d}{N(n)} (2^k)^{d-1}\right)}_{O(|[\Lambda_n]| \, 2^{-k})}$$

$$\Rightarrow Z^{\Phi,[\Lambda_n]} \leq \left(Z^{\Phi,D_k}\right)^{N(n)} e^{\mathcal{O}(|[\Lambda_n]|2^{-k})}$$

$$\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \underbrace{\frac{N(n)}{|[\Lambda_n]|}}_{\to \frac{1}{|D_k|}} \log \cancel{Z}\, Z^{\Phi,D_k} + \mathcal{O}(2^{-k})$$

$$= \psi^{\Phi,D_k} + \mathcal{O}(2^{-k}) \xrightarrow[\text{by } \cancel{\mathcal{O}}(1)]{k\to\infty} \psi$$

(Sim. lower bd.)

- For ⓐ let $\Delta_n = [\Lambda_n] \setminus \Lambda_n$

$$|H^{\Phi,\Lambda_n} - H^{\Phi,[\Lambda_n]}| \leq \mathcal{O}(|\Delta_n|)$$

$$\Rightarrow Z^{\Phi,[\Lambda_n]} \leq \underbrace{\sum_{\omega\in\Omega_{\Lambda_n}} e^{-H^{\Phi,\Lambda_n}(\omega)}}_{Z^{\Phi,\Lambda_n}} \underbrace{\sum_{\tilde\omega\in\Omega_{\Delta_n}} e^{\mathcal{O}(|\Delta_n|)}}_{2^{|\Delta_n|}e^{\mathcal{O}(|\Delta_n|)}} = e^{\mathcal{O}(|\Delta_n|)}$$

$$\cancel{\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \psi^{\Phi,\Lambda_n} + \mathcal{O}\left(\frac{|\Delta_n|}{|[\Lambda_n]|}\right)}$$

$$\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \underbrace{\frac{|\Lambda_n|}{|[\Lambda_n]|}}_{\leq 1} \cdot \psi^{\Phi,\Lambda_n} + \underbrace{\mathcal{O}\left(\frac{|\Delta_n|}{|[\Lambda_n]|}\right)}_{\mathcal{O}\left(\frac{|\partial\Lambda_n||D_k|}{|[\Lambda_n]|}\right) \leq \mathcal{O}\left(\frac{|\partial\Lambda_n||D_k|}{|\Lambda_n|}\right) \xrightarrow{n\to\infty} 0}$$

(Sim. ~~opp~~ opposite bd).

(B) Independence of bdry condition.

Use $|H^{\eta,\Lambda} - H^{\phi,\Lambda}| \le O(|\partial\Lambda|)$

w/ van Hove.

(C) Convexity/evenness.

Preserved under limit. ■

Magnetization.

$m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x$, $\quad m^{\eta,\Lambda} = \langle m_\Lambda\rangle^{\eta,\Lambda}$

Note. Cumulant-gen. fcn. of $\sum_{x\in\Lambda}\sigma_x$ is

$$\log\langle e^{t\sum\sigma_x}\rangle^{\eta,\Lambda}_{\beta,h} = |\Lambda|\left(\psi^{\eta,\Lambda}(\beta, h+t) - \psi^{\eta,\Lambda}(\beta,h)\right).$$

In particular, $m^{\eta,\Lambda} = \frac{\partial\psi^{\eta,\Lambda}}{\partial h}$.

Does this hold in thermodynamic limit?

Let $\mathcal{B}_\beta = \{h : \psi(\beta, \cdot)$ not differentiable$\}$

$$= \left\{h : \frac{\partial \psi}{\partial h^-} \neq \frac{\partial \psi}{\partial h^+}\right\} \text{ by convexity.}$$

<u>Cor</u>. For $h \notin \mathcal{B}_\beta$, $m^\eta = \lim_{\Lambda \uparrow \mathbb{Z}^d} m^{\eta,\Lambda}$ exists, is indep. of η, and equals $\frac{\partial \psi}{\partial h}$. Moreover, it is cts and nondecreasing in h off $\mathcal{B}_\beta$ and discts. on $\mathcal{B}_\beta$, and $\lim_{h' \uparrow\downarrow h} m = \frac{\partial \psi}{\partial h^\pm}$ ~~so~~ the <u>spontaneous magnetization</u> $m^*(\beta) = \lim_{h \downarrow 0} m(\beta, h)$ exists.

<u>Pf</u>. ~~D~~ (A) <u>Existence/indep. of η</u>.

Interchange lims. (justified by convexity).

(B) <u>Cts/monotone</u>.

By convexity of ψ

(C) <u>Left/right lims</u>.

(A) + direct computation. ∎

Prop. $m^\pm = \lim_{\Lambda \nearrow \mathbb{Z}^d} m^{\pm,\Lambda}$ exist and equal $\langle \sigma_0 \rangle^\pm$.

Pf. $\langle \sigma_0 \rangle^+ = \langle m_{\Lambda_n} \rangle^+$ (trans-inv)

$$\leq \langle m_{\Lambda_n} \rangle^{+,\Lambda_n} \quad (m_{\Lambda_n} \text{ inc fn})$$

$$\Rightarrow \langle \sigma_0 \rangle^+ \leq \liminf_n \underbrace{\langle m_{\Lambda_n} \rangle^{+,\Lambda_n}}_{m^{+,\Lambda_n}}$$

Conversely, fix $k \geq 1$, $x \in \Lambda_n$.

$$x + B(k) \subset \Lambda_n \Rightarrow \langle \sigma_x \rangle^{+,\Lambda_n} \leq \langle \sigma_x \rangle^{+,x+B(k)} = \langle \sigma_0 \rangle^{+,B(k)}.$$

$B(k) = \{-k, \dots, k\}^d$

Thus, $$\langle m_{\Lambda_n} \rangle^{+,\Lambda_n} = \frac{1}{|\Lambda_n|} \sum_{\substack{x \in \Lambda_n \\ x+B(k) \subset \Lambda_n}} \langle \sigma_x \rangle^{+,\Lambda_n} + \frac{1}{|\Lambda_n|} \sum_{x+B(k) \not\subset \Lambda_n} \overbrace{\langle \sigma_x \rangle^{+,\Lambda_n}}^{\leq 1}$$

$$\leq \langle \sigma_0 \rangle^{+,B(k)} + \frac{1}{|\Lambda_n|} \underbrace{\left(|B(k)| |\partial \Lambda| \right)}_{\text{upper bd on \# ways } x+B(k) \text{ can intersect } \partial\Lambda}$$

$$\Rightarrow \limsup_n \langle m_{\Lambda_n} \rangle^{+,\Lambda_n} \leq \langle \sigma_0 \rangle^+ \text{ by above.} \quad \square$$

~~Thm~~ **Lem.** (1) $\langle\sigma_0\rangle^+_{\beta,h}$ nondec/right-cts in h

(2) $h\geq 0 \Rightarrow \langle\sigma_0\rangle^+_{\beta,h}$ nondec. in β

Pf. ~~(1) FKG ⇒~~

(1) $\frac{\partial}{\partial h}\langle\sigma_0\rangle^{+,\Lambda} = \sum_{x\in\Lambda}\underbrace{\left(\langle\sigma_0\sigma_x\rangle^{+,\Lambda} - \langle\sigma_0\rangle^{+,\Lambda}\langle\sigma_x\rangle^{+,\Lambda}\right)}_{\geq 0 \text{ by FKG}}$ by quotient rule

(Take $\Lambda\uparrow\mathbb{Z}^d$)

$\Rightarrow$ nondecreasing.

Right-cts by interchanging lims

(2) Sim. but use GKS ineq.: $\langle\sigma_0\sigma_x\sigma_y\rangle^{+,\Lambda} \geq \langle\sigma_0\rangle^{+,\Lambda}\langle\sigma_x\sigma_y\rangle^{+,\Lambda}$. $\square$

Note. Take $h=0$. Then $\langle\sigma_0\rangle^+_{\beta,0} = -\langle\sigma_0\rangle^-_{\beta,0}$.

~~Thus, uniqueness $\Leftrightarrow \langle\sigma_0\rangle^+_{\beta,0}=0$~~ Thus, uniqueness $\Leftrightarrow \langle\sigma_0\rangle^+_{\beta,0}=0$.

But $m^*(\beta) = m(\beta,0^+) = m^+(\beta,0^+)$ (since $m=m^+$ at all but countably many pts)

$= \langle\sigma_0\rangle^+$

So uniqueness $\Leftrightarrow m^*=0$.

Moreover, $m^* = \langle\sigma_0\rangle^+$ is monotone in β, so let

$$\beta_c = \inf(\beta : m^*>0) = \sup(\beta : m^*=0).$$

<u>Thm</u>. $\frac{\partial \psi}{\partial h^\pm} = m^\pm$.

<u>Pf</u>. $\frac{\partial \psi}{\partial h^+} = \lim_{h' \downarrow h} \frac{\partial \psi}{\partial h}(\beta, h')$ ($\mathcal{B}_\beta^{h'}$ countable)

$= \lim_{h' \downarrow h} m(\beta, h') = \lim_{h' \downarrow h} m^+(\beta, h') = m^+(\beta, h)$. ∎

right-cts.

Thus, uniqueness $\iff m^+ = m^- \iff \frac{\partial \psi}{\partial h}$ exists.

<u>Note</u>. Non-uniqueness is a <u>first-order</u> phase transition.

More generally, we have a <u>n^{th}-order</u> PT if

$\frac{\partial \psi}{\partial h^k}$ exists iff $k < n$.

Usually only distinguish btwn <u>cts</u> $(n > 1)$ and <u>discts</u> $(n = 1)$ PTs.

High-temperature regime

Recall.

- Pressure $\psi(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z^{\eta,\Lambda}_{\beta,h}$ is indep of η and convex.
- Magnetization $m^{\pm}(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \left\langle \sum_{x\in\Lambda} \sigma_x \right\rangle^{\pm,\Lambda}_{\beta,h}$

$$= \langle \sigma_0 \rangle^{\pm}_{\beta,h} = \frac{\partial \psi}{\partial h^{\pm}} (\beta,h)$$

- Spontaneous magnetization $m^*(\beta) = \lim_{h \downarrow 0} m^+(\beta,h)$
- Critical point $\beta_c = \inf(\beta : m^*(\beta) > 0)$.
- A first-order/discts PT (ie $\mu^+ \neq \mu^-$) occurs iff $\frac{\partial \psi}{\partial h^-} \neq \frac{\partial \psi}{\partial h^+}$ (equiv. to $m^* > 0$ when $h=0$).

Today. $\beta_c > 0$ ie there is a single-phase region $\{(\beta,0) : \beta < \beta_c\}$

High-temp expansion:

ie expand about $\beta = 0$

$$e^{\pm\beta} = \cosh\beta \pm \sinh\beta \iff e^{\beta\omega_x\omega_y} = \cosh\beta + \omega_x\omega_y \sinh\beta$$
$$= \cosh\beta\,(1 + \omega_x\omega_y \tanh\beta).$$

Let $\Omega^{+,\Lambda} = \{\omega \in \Omega \ : \ \omega_x = 1 \ \forall x \notin \Lambda\}$

Fix $h = 0$, $\omega \in \Omega^{+,\Lambda}$, write

$$H_\beta^\Lambda(\omega) = H_{\beta,0}^{+,\Lambda}(\omega) = -\beta\Big[\sum_{\substack{x\sim y\\ x,y\in\Lambda}} \omega_x\omega_y + \sum_{\substack{x\sim y\\ x\in\Lambda\\ y\notin\Lambda}} \omega_x \underset{+1}{\overset{\shortparallel}{\omega_y}}\Big]$$

$$= -\beta \sum_{x\sim y} \omega_x\omega_y .$$

Then $e^{-H_\beta^\Lambda(\omega)} = \prod_{x\sim y} e^{\beta\omega_x\omega_y} = (\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|} \prod_{x\sim y}(1+\omega_x\omega_y\tanh\beta)$

where $\mathcal{E}(\bar\Lambda) = \{\text{edges in } \bar\Lambda\}$ and $\bar\Lambda = \Lambda \cup \partial^{ext}\Lambda$.

Expand product using binomial theorem:

$|S| < \infty$, $f, g : S \to \mathbb{R}$

$$\prod_{s \in S} \big(f(s) + g(s) \big) = \sum_{A \subset S} \Big(\prod_{a \in A} f(a) \Big) \Big(\prod_{b \notin A} g(b) \Big).$$

Thus,

$$e^{-H_\beta^\Lambda(\omega)} = (\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|} \sum_{E \subset \mathcal{E}(\bar\Lambda)} \underbrace{\prod_{xy \in E} \omega_x \omega_y \tanh\beta}_{(A)}.$$

With

$$(A) = (\tanh\beta)^{|E|} \prod_{x \in \Lambda} \omega_x^{I(x,E)}$$

where $I(x,E) = \#\{ y \in \mathbb{Z}^d : xy \in E \}$, get

$$Z_{\beta,0}^{+,\Lambda} = (\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|} \sum_{E \subset \mathcal{E}(\bar\Lambda)} (\tanh\beta)^{|E|} \underbrace{\sum_{\omega \in \Omega^{+,\Lambda}} \prod_{x \in \Lambda} \omega_x^{I(x,E)}}_{(B)}$$

But $(B) = \prod_{x\in\Lambda} \sum_{\omega_x=\pm1} \omega_x^{I(x,E)} = \prod_{x\in\Lambda} 2\mathbb{1}(I(x,E) \text{ even})$

$$\Rightarrow Z^{+,\Lambda}_{\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\Lambda)|} \sum_{E\subset\mathcal{E}(\Lambda)_e} (\tanh\beta)^{|E|}$$

where $\mathcal{E}(\Lambda)_e = \{E\subset\mathcal{E}(\Lambda) : I(x,E) \text{ even } \forall x\in\Lambda\}$.

(High-temp representation van der Waerden 1941)

Sim.,

$$\sum_{\omega\in\Omega^{+,\Lambda}} \omega_0 e^{-H^\Lambda_\beta(\omega)}$$

$$= (\cosh\beta)^{|\mathcal{E}(\Lambda)|} \sum_{E\subset\mathcal{E}(\Lambda)} (\tanh\beta)^{|E|} \underbrace{\sum_{\omega\in\Omega^{+,\Lambda}} \omega_0 \prod_{x\in\Lambda} \omega_x^{I(x,E)}}_{(C)}$$

$$\text{w/ } (C) = \underbrace{\left(\sum_{\omega_0=\pm1} \omega_0^{1+I(0,E)}\right)}_{2\mathbb{1}(I(0,E)\text{ odd})} \left(\prod_{x\neq 0} \sum_{\omega_x=\pm1} \omega_x^{I(x,E)}\right)$$

Thus,

$$\langle \sigma_0 \rangle^{+,\Lambda}_{\beta,0} = \frac{\displaystyle\sum_{E\subset \mathcal{E}(\bar\Lambda)_0} (\tanh\beta)^{|E|}}{\displaystyle\sum_{E\subset\mathcal{E}(\bar\Lambda)_e} (\tanh\beta)^{|E|}}$$

w/ $\mathcal{E}(\bar\Lambda)_0 = \{E\subset\mathcal{E}(\bar\Lambda) : I(x,E) \text{ even } \forall x\in\Lambda\setminus\{0\},\ I(0,E) \text{ odd}\}$.

Decompose $E\subset\mathcal{E}(\bar\Lambda)_0$ as $E = E_0\cup E'$

w/ E_0 the max. conn. component of 0

and $E'\in\mathcal{E}(\bar\Lambda)_e$ in the "complement" E^* of E_0.

Then

$$\langle\sigma_0\rangle = \sum_{0\in E_0\in\mathcal{E}(\bar\Lambda)_0 \text{ conn.}} (\tanh\beta)^{|E_0|} \underbrace{\frac{\displaystyle\sum_{\substack{E'\in\mathcal{E}(\bar\Lambda)_e \\ E'\subset E_0^*}} (\tanh\beta)^{|E'|}}{\displaystyle\sum_{E\in\mathcal{E}(\bar\Lambda)_e}(\tanh\beta)^{|E|}}}_{\le 1}$$

Bounds: On any connected graph, from any vertex, is a path crossing each edge exactly twice (induction)

Thus, $\#\left\{\begin{matrix}\text{summands in } \langle\sigma_0\rangle \\ \text{w/ } \ell \text{ edges}\end{matrix}\right\} \le \#\left\{\begin{matrix}\text{paths from } 0 \text{ in } \mathbb{Z}^d \\ \text{of length } 2\ell\end{matrix}\right\} = (2d)^{2\ell} = (4d^2)^\ell$

Since $\sum_x I(x, E_0) = 2|E_0|$ even + $I(0,E_0)$ odd,

$\exists y \neq 0$ w/ $I(y, E_0)$ odd.

But $I(y, E_0)$ even $\forall y \in \Lambda \setminus \{0\} \Rightarrow y \notin \Lambda$.

That is, $0 \xleftrightarrow{E_0} y$.

Setting $\Lambda = B(n) = \{-n, \dots, n\}^d$, get $|E_0| \ge n$.

Therefore,

$$\langle \sigma_0 \rangle \le \sum_{\ell \ge n} \underbrace{(\tanh \beta)^\ell}_{\le \beta^\ell} (4d^2)^\ell \le \sum_{\ell \ge n} (4d^2 \beta)^\ell .$$

$\beta < \frac{1}{4d^2} \Rightarrow$ series converges $\Rightarrow$ tail $\xrightarrow{n \to \infty} 0$ i.e. $\langle \sigma_0 \rangle^+ = 0$

So $\beta_c \ge \frac{1}{4d^2} > 0$.

1D case.

If $d=1$, $\Lambda = B(n)$,

$$\mathcal{E}(\bar{\Lambda})_e = \{\emptyset, \mathcal{E}(\bar{\Lambda})\}$$

$$\mathcal{E}(\bar{\Lambda})_o = \{\text{line graphs on } \{-n-1, \dots, 0\} \text{ and } \{0, \dots, n+1\}\}$$

Thus,

$$\langle \sigma_0 \rangle = \frac{\sum_{E \in \mathcal{E}(\bar{\Lambda})_o} (\tanh \beta)^{|E|}}{\sum_{E \in \mathcal{E}(\bar{\Lambda})_e} (\tanh \beta)^{|E|}}$$

$$= \frac{2(\tanh\beta)^{n+1}}{1+(\tanh\beta)^{2(n+1)}} \xrightarrow{n\to\infty} 0 \quad \forall \beta < \infty.$$

$\Rightarrow \beta_c = \infty$ if $d=1$. (also possible by exact soln. via transfer matrix ⇐ Lenz)

Next time. $d > 1 \Rightarrow \beta_c < \infty$.

[In fact, $d=1 \Rightarrow \psi = \log\left[e^{\beta}\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}\right]$]

Low temperature and cluster expansion.

Recall.

- High-temp expansion

$$Z^{+,\Lambda}_{\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|} \sum_{E\in\mathcal{E}(\bar\Lambda)_e} (\tanh\beta)^{|E|}$$

w/ $\bar\Lambda = \Lambda \cup \partial^{ext}\Lambda$

$$\mathcal{E}(\bar\Lambda)_e = \{E \subset \mathcal{E}(\bar\Lambda) : I(x,E) \text{ even } \forall x\in\Lambda\}$$

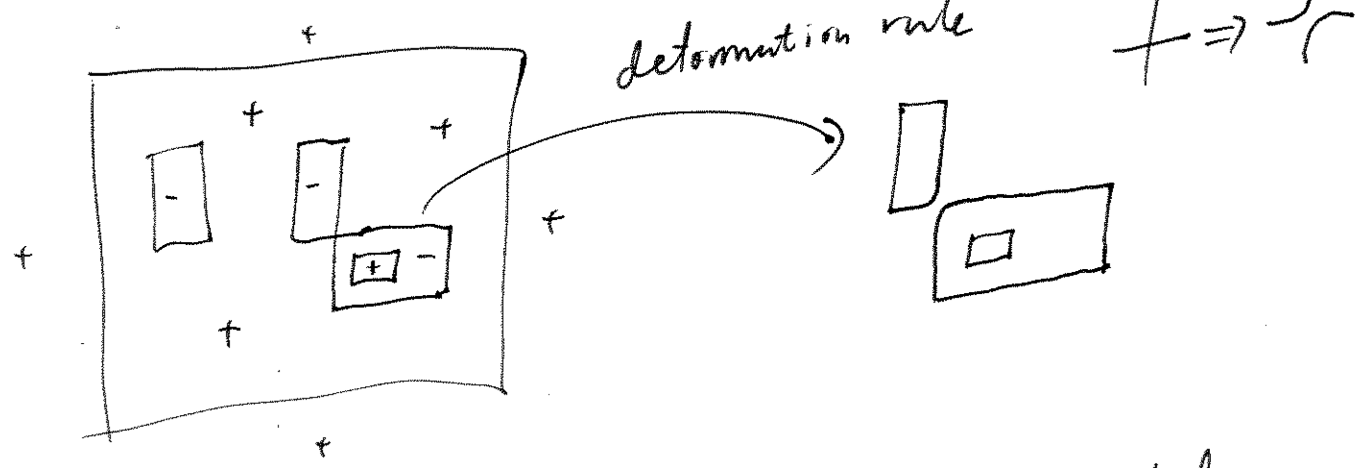
- $\beta_c > 0$ (uniqueness at high temp)
- $d=1 \Rightarrow \beta_c = \infty$ (no phase transition)

Low-temperature expansion: $(d=2)$

$$H^{\Lambda}_{\beta,0}(\omega) = -\beta \sum_{xy\in\mathcal{E}(\bar\Lambda)} \cancel{\sigma_x\sigma_y}\, \omega_x\omega_y = -\beta|\mathcal{E}(\bar\Lambda)| + \sum_{xy\in\mathcal{E}(\bar\Lambda)} \beta \underbrace{\overbrace{(1-\omega_x\omega_y)}^{\in\{0,2\}}}_{2\mathbb{1}(\omega_x\neq\omega_y)}$$

$$= -\beta|\mathcal{E}(\bar\Lambda)| + 2\beta\,\#\{xy : \omega_x\neq\omega_y\}, \qquad \omega\in\Omega^+_\Lambda$$

Associate to $\omega\in\Omega^+_\Lambda$ the corresponding set $\Gamma(\omega) = \{\gamma_1,\dots,\gamma_n\}$ of contours in the dual lattice $\mathbb{Z}^*_2$



Then $\omega_x \neq \omega_y$ iff x and y are separated by a dual edge $xy_\perp$ in γ_* for some $\gamma_* \in \Gamma(\omega)$.

That is,

$$\#\{xy : \omega_x \neq \omega_y\} = \sum_{\gamma \in \Gamma(\omega)} |\gamma|$$

where $|\gamma| = \#\{\text{dual edges in } \gamma\}$

Thus, $H(\omega) = -\beta|\mathcal{E}(\bar{\Lambda})| + 2\beta \sum_{\gamma \in \Gamma(\omega)} |\gamma|$

$$\Rightarrow Z^{+,\Lambda}_{\beta,0} = e^{\beta|\mathcal{E}(\bar{\Lambda})|} \sum_{\omega \in \Omega^+_\Lambda} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}$$

<u>Plan</u>. Show $\mu^{+,B(n)}_{\beta,0}(\sigma_0 = -1) \leq \delta(\beta) \downarrow 0$ as $\beta \to \infty$

Then $\langle \sigma_0 \rangle^{+,\cancel{B(n)}}_{\beta,0} = \mu^+_{\beta,0}(\sigma_0 = 1) - \mu^+_{\beta,0}(\sigma_0 = -1)$

$\quad\;\; \| $

$m^*(\beta)$ $\qquad = 1 - 2\mu^+_{\beta,0}(\sigma_0 = -1) \geq 1 - 2\delta(\beta) > 0$ for β large.

As w/ $Z^{+,\Lambda}_{\beta,0}$, can show

$$\mu^{+,\Lambda}_{\beta,0}(\omega) = \frac{1}{Z^{+,\Lambda}_{\beta,0}} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}. \quad (*)$$

Now set $\Lambda = B(n)$.

If $\omega_0 = -1$, $\exists \gamma_* \in \Gamma(\omega) : \mathrm{Int}(\gamma_*) \ni 0$.

Thus, $\mu^{+,B(n)}_{\beta,0}(\sigma_0 = -1) \leq \sum_{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0} \mu^{+,B(n)}_{\beta,0}(\Gamma \ni \gamma_*)$

__Lem.__ For any contour γ_*,

$$\mu^{+,B(n)}(\Gamma \ni \gamma_*) \leq e^{-2\beta|\gamma_*|}.$$

__Pf.__ By $(*)$,

$$\mu^{+,B(n)}(\Gamma \ni \gamma_*) = \sum_{\omega : \Gamma(\omega) \ni \gamma_*} \mu^{+,B(n)}(\omega)$$

$$= e^{-2\beta|\gamma_*|} \frac{\sum_{\Gamma(\omega) \ni \gamma_*} \prod_{\gamma \in \Gamma \setminus \{\gamma_*\}} e^{-2\beta|\gamma|}}{\sum_{\omega} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}}$$

↳ Show ≤ 1.

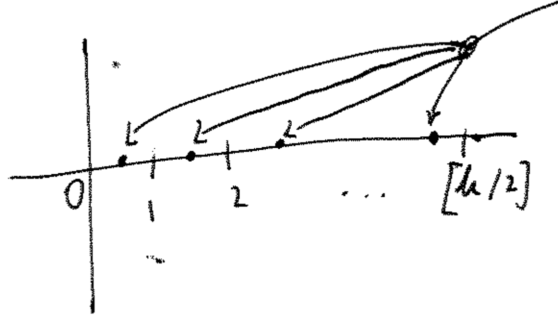
Let $\mathcal{C}(\gamma_*) = \{$ configs. $\hat{\omega}'$ obtainable by "removing" γ_* from some $\omega\}$

("remove" γ_* = flip spins in $\mathrm{Int}(\gamma_*)$)

Then
$$\sum_{\omega:\Gamma(\omega)\ni\gamma_*} \prod_{\gamma\in\Gamma(\omega)\setminus\{\gamma_*\}} e^{-2\beta|\gamma|} = \sum_{\omega'\in\mathcal{C}(\gamma_*)} \prod_{\gamma'\in\Gamma(\omega')} e^{-2\beta|\gamma'|}.$$

Thus, ratio ≤ 1. $\square$

Follows that
$$\mu^{+,B(n)}(\sigma_0=-1) \leq \sum_{\gamma_*:\mathrm{Int}(\gamma_*)\ni 0} e^{-2\beta|\gamma_*|}$$
$$= \sum_{k\geq 4} \sum_{\substack{\mathrm{Int}(\gamma_*)\ni 0\\ |\gamma_*|=k}} e^{-2\beta k}$$
$$= \sum_{k\geq 4} e^{-2\beta k}\,\#\{\gamma_* : \mathrm{Int}(\gamma_*)\ni 0,\ |\gamma_*|=k\}.$$



Contour about 0 of length k must cross one

Since # contours of length k from some vertex is ~~3k-1~~ $4 \times 3^{k-1}$, get

$$\mu^{+,B(n)}(\sigma_0 = -1) \leq \sum_{k \geq 4} e^{-2\beta k} \times \frac{k}{2} \times 4 \times 3^{k-1}$$

$$= \frac{2}{3} \sum_{k \geq 4} k\, 3^k e^{-2\beta k} = \delta(\beta)$$

and $\delta(\beta) \downarrow 0$ iff $\delta(\beta) < \infty$ iff $3e^{-2\beta} < 1$

i.e. $\beta > \frac{1}{2} \log 3$

Note. Exact soln. $\Rightarrow \beta_c(2) = \frac{1}{2} \operatorname{arcsinh}(1) \approx 0.441$

(cf. $\frac{1}{2} \log 3 \approx$ ~~0.5493~~ 0.549)

Cluster expansion:

Γ finite set, elements $\gamma \in \Gamma$ called polymers

weight $w : \Gamma \to \mathbb{R}$ (or $\mathbb{C}$)
(or activity)

interaction $\delta : \Gamma \times \Gamma \to [-1, 1]$ symm.

s.t. $\delta(\gamma, \gamma) = 0$.

Def. (Polymer model).

Assign to $\Gamma' \subset \Gamma$ the probability

$$\frac{1}{\Xi}\Big(\prod_{\gamma \in \Gamma'} w(\gamma')\Big)\Big(\prod_{\{\gamma,\gamma'\}\subset\Gamma'} \delta(\gamma,\gamma')\Big)$$

where the polymer partition fcn is

$$\Xi = \sum_{\Gamma' \subset \Gamma} (\dots)(\dots)$$

$$= 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_n \in \Gamma} \Big(\prod_{i=1}^n w(\gamma_i)\Big)\Big(\prod_{i<j} \delta(\gamma_i,\gamma_j)\Big).$$

Ex. (1) High-temp.

$$Z = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\Lambda)|} \underbrace{\sum_{E \in \mathcal{E}(\Lambda)_e} (\tanh\beta)^{|E|}}_{\Xi}$$

$$\Xi = 1 + \sum_n \frac{1}{n!} \sum_{E_1} \cdots \sum_{E_n} \left(\prod w(E_i)\right)\left(\prod \delta(E_i, E_j)\right)$$

w/ $w(E_i) = (\tanh\beta)^{|E_i|}\, \mathbb{1}(E_i \in \mathcal{E}(\Lambda)_e \text{ connected})$

(so $E_1 \cup \dots \cup E_n$ is decomp into connected components)

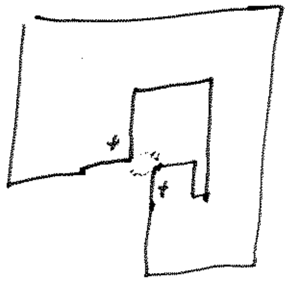
and $\delta(E_i, E_j) = \mathbb{1}(E_i, E_j \text{ share no vertices})$.

(so this decomposition is maximal)

(2). Low-temp.

Λ "reasonable" eg. $B(n)$

avoid eg.



Then a collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of Peierls contours can arise from a config ω iff $\gamma_i \cap \gamma_j = \emptyset$ $\forall i \neq j$.

Let Γ_Λ = {all possible ~~[illegible]~~ contours}.

$$Z^{+,\Lambda} = e^{\beta|\mathcal{E}(\Lambda)|}\,\Xi$$

$$\Xi = 1 + \sum_n \frac{1}{n!} \sum_{\gamma_1 \in \Gamma_\Lambda} \cdots \sum_{\gamma_n \in \Gamma_\Lambda} \left(\prod_i w(\gamma_i)\right)\left(\prod_{i<j} \delta(\gamma_i, \gamma_j)\right)$$

w/ $w(\gamma) = e^{-2\beta|\gamma|}$

$$\delta(\gamma_i, \gamma_j) = \mathbb{1}(\gamma_i \cap \gamma_j = \emptyset)$$

<u>Next time</u>:

$$\log \Xi = \sum_{m \geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi_m(\gamma_1, \ldots, \gamma_m) \prod_{i=1}^{m} w(\gamma_i)$$

w/ $\varphi_m(\gamma_1, \ldots, \gamma_m) = \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn.}}} \prod_{ij \in G} (\delta(\gamma_i, \gamma_j) - 1)$

and G_m = complete graph on m vertices

(The φ_m are called <u>Ursell fcns</u>)

Cluster expansion

Recall.

Polymer partition fcn

$$\Xi = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_n \in \Gamma} \left(\prod_{i=1}^{n} w(\gamma_i) \right) \left(\prod_{i<j} \delta(\gamma_i, \gamma_j) \right)$$

where $|\Gamma| < \infty$, $w : \Gamma \to \mathbb{R}$, $\delta : \Gamma \times \Gamma \to [-1, 1]$ symm

and $\delta(\gamma, \gamma) = 0 \quad \forall \gamma \in \Gamma$.

Today. Compute $\log \Xi$.

Let G_n be the complete graph on $V_n = \{1, \dots, n\}$

w/ edges E_n.

Write $\prod_{i=1}^{n} = \prod_{i \in V_n}$ and $\prod_{i<j} = \prod_{ij \in E_n}$

Let $\zeta(\gamma, \gamma') = \delta(\gamma, \gamma') - 1$

Binomial thm $\Rightarrow \prod_{ij \in E_n} \delta(\gamma_i, \gamma_j) = \sum_{E \subset E_n} \prod_{ij \in E} \zeta(\gamma_i, \gamma_j)$

Thus, $\Xi = 1 + \sum_{n} \frac{1}{n!} \sum_{G \subset G_n} \underbrace{\sum_{\gamma_1} \cdots \sum_{\gamma_n} \left(\prod_{i \in V_n} w(\gamma_i) \right) \left(\prod_{ij \in E} \zeta(\gamma_i, \gamma_j) \right)}_{Q[G],\ G = (V_n, E)}$

Note. $Q[G] = Q[G']$ if $G \simeq G'$

and if G has max conn components $G = (G_1', \dots, G_k')$

(write $G \doteq (G_1', \dots, G_k')$), then $Q[G] = \prod_{r=1}^{k} Q[G_r']$.

Thus,

$$\frac{1}{n!} \sum_{G \subset G_n} Q[G] = \cancel{\tfrac{1}{n!}} \sum_{k=1}^{n} \frac{1}{k!} \sum_{m_1 + \dots + m_k = n} \frac{\cancel{n!}}{m_1! \dots m_k!} \sum_{\substack{G_1' \subset G_{m_1} \\ \text{conn}}} \cdots \sum_{\substack{G_k' \subset G_{m_k} \\ \text{conn}}} \prod_{r=1}^{k} Q[G_r']$$

$$= \sum_{k \le n} \frac{1}{k!} \sum_{m_1 + \dots + m_k = n} \cancel{\sum_{\substack{G_1' \subset G_{m_1} \\ \text{conn}}} \cdots \sum} \prod_{r=1}^{k} \left(\frac{1}{m_r!} \sum_{\substack{G_r' \subset G_{m_r} \\ \text{conn}}} Q[G_r'] \right)$$

Formally,

$$\sum_{n} \sum_{k \le n} \sum_{m_1 + \dots + m_k = n} = \sum_{k} \sum_{n \ge k} \sum_{m_1 + \dots + m_k = n} = \sum_{k} \sum_{m_1, \dots, m_k}$$

Thus,

$$\Xi = 1 + \sum_{k} \frac{1}{k!} \sum_{m_1, \dots, m_k} \prod_{r=1}^{k} \left(\frac{1}{m_r!} \sum_{\substack{G_r' \subset G_{m_r} \\ \text{conn}}} Q[G_r'] \right).$$

$$= 1 + \sum_{k} \frac{1}{k!} \left(\sum_{m} \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} Q[G] \right)^{k}$$

$$= \exp \left(\sum_{m} \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} Q[G] \right).$$

Lastly, ~~we~~ define the Ursell fns

$$\varphi_m(r_1,\dots,r_m) = \frac{1}{m!}\sum_{\substack{G\subset G_m\\ \text{conn}}} \prod_{ij\in G} \zeta(r_i,r_j)$$

so that

$$\frac{1}{m!}\sum_{\substack{G\subset G_m\\ \text{conn}}} \omega[G] = \sum_{r_1}\dots\sum_{r_m} \varphi_m(r_1,\dots,r_m)\prod_{i\in V_m} w(r_i)$$

i.e. $$\Xi = \exp\Big(\sum_m \sum_{r_1}\dots\sum_{r_m}\varphi_m(r_1,\dots,r_m)\prod_{i\in V_m} w(r_i)\Big).$$

Convergence.

Thm. (Ueltschi '04).

If $\exists\, a: \Gamma\to(0,\infty)$ st. $\forall r_* \in\Gamma$,

$$\sum_r |w(r)|e^{a(r)}|\zeta(r,r_*)| \le a(r_*),$$

then $\forall r_1\in\Gamma$,

$$1+\sum_{k\ge 2} k\sum_{r_2}\dots\sum_{r_k}|\varphi(r_1,\dots,r_k)|\prod_{j=2}^{k}|w(r_j)| \le e^{a(r_1)}$$

and so the cluster expansion converges.

Pf. Show

$$1+\sum_{k=2}^{N} k \sum_{r_2} \cdots \sum_{r_k} |\mathcal{C}(r_1,\dots,r_k)| \prod_{j=2}^{k} |w(r_j)| \leq e^{a(r_1)} . \quad (*)$$

Case $N=2$.

$G \subset G_2$ conn $\Rightarrow G = \{\{1,2\}\}$

$$\Rightarrow \mathcal{C}_2(r_1,r_2) = \frac{1}{2}\zeta(r_1,r_2)$$

$$\Rightarrow 1+2\sum_{r_2} |\mathcal{C}(r_1,r_2)|\,|w(r_2)|$$

$$= 1+\sum_{r_2} |\zeta(r_1,r_2)|\,|w(r_2)|$$

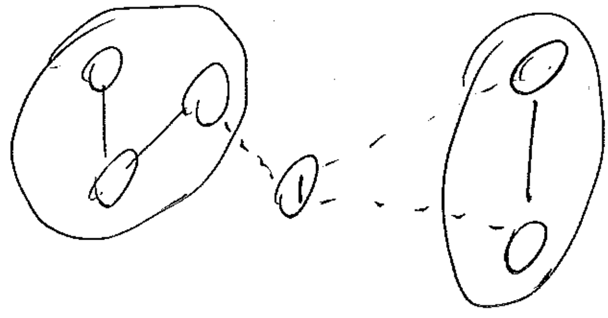
$$\leq 1+\sum_{r_2} |\zeta(r_1,r_2)|\, e^{a(r_2)}|w(r_2)|$$

$$\leq 1+a(r_1) \leq e^{a(r_1)} .$$

Inductive step ($N \Rightarrow N+1$)

Consider $k \leq N+1$ and $G \subset G_k$ conn

Delete 1 + assoc. edges from $G \Rightarrow G' = (G_1',\dots,G_l')$, $l \leq k-1$



$$\varphi_k(r_1,\dots,r_k)$$

$$= \frac{1}{k!}\sum_{l=1}^{k} \frac{1}{l!} \sum_{V_1',\dots,V_l'} \Phi_l \, \Psi_l$$

$$\Phi_l = \sum_{\substack{G_1' \text{ conn} \\ V(G_1') = V_1'}} \cdots \sum_{\substack{G_l' \text{ conn} \\ V(G_l') = V_l'}} \prod_{m=1}^{l} \prod_{ij \in G_m'} \zeta(r_i, r_j)$$

$$\left[= \prod_{m=1}^{l} \sum_{\substack{G_m' \text{ conn} \\ V(G_m') = V_m'}} \prod_{ij \in G_m'} \zeta(r_i, r_j) \right] = \prod_{m=1}^{l} |V_m'|! \, \varphi\big((r_j)_{j \in V_m'}\big)$$

$$\Psi_l = \sum_{\emptyset \neq K_1 \subset V_1'} \cdots \sum_{\emptyset \neq K_l \subset V_l'} \prod_{m=1}^{l} \prod_{j \in K_m} \zeta(r_1, r_j)$$

$$\left[= \prod_{m=1}^{l} \sum_{\emptyset \neq K_m \subset V_m'} \prod_{j \in K_m} \zeta(r_1, r_j) . \right]$$

Bounds.

By induction,

$$|1+\alpha_k| \le 1 \ \forall k \Rightarrow \left|\prod_{k=1}^{n}(1+\alpha_k) - 1\right| \le \sum_{k=1}^{n} |\alpha_k|$$

In particular, by binomial thm.,

$$|\Psi_l| \le \prod_{m=1}^{l} \left|\prod_{j\in V_m'} (1+\zeta(r_1, r_j)) - 1\right|$$

$$\le \prod_{m=1}^{l} \sum_{j\in V_m'} |\zeta(r_1, r_j)|$$

Thus,

$$\sum_{k=2}^{N+1} k \sum_{r_2} \cdots \sum_{r_k} |\varphi_k(r_1, \dots, r_k)| \prod_{j=2}^{k} |w(r_j)|$$

$$\le \sum_{k=2}^{N+1} k \sum_{r_2} \cdots \sum_{r_k} \frac{1}{k!} \sum_{l=1}^{k} \frac{1}{l!} \sum_{V_1', \dots, V_l'} \left(\prod_{m=1}^{l} \left(|V_m'|! \, |\varphi((r_j)_{j\in V_m'})| \sum_{j\in V_m'} |\zeta(r_1, r_j)| \right) \right)$$

$$\times \prod_{j=2}^{k} |w(r_j)|$$

$$= \sum_{k=2}^{N+1} \frac{1}{\cancel{(k-1)!}} \sum_{l=1}^{k} \frac{1}{l!} \sum_{m_1+\dots+m_l=k-1} \frac{\cancel{(k-1)!}}{\cancel{m_1! \dots m_k!}}$$

$$\times \prod_{j=1}^{l} \left(\cancel{m_j!} \sum_{r_1} \dots \sum_{r_{m_j}} |\varphi(r_1,\dots,r_{m_j})| \sum_{i=1}^{m_j} |\zeta(r_1, r_i)| \prod_{i=1}^{m_j} |w(r_i)| \right)$$

$$\le \sum_{l\ge 1} \frac{1}{l!} \prod_{j=1}^{l} \sum_{m_j=1}^{N} \left(\dots \right) \qquad (A)$$

Claim. Inductive hyp $\Rightarrow$ $\forall r_A$,

$$\sum_{k=1}^{N} \sum_{r_1} \dots \sum_{r_k} |\varphi(r_1,\dots,r_k)| \sum_{i=1}^{k} |\zeta(r_i, r_A)| \prod_{i=1}^{k} |w(r_i)| \le a(r_A).$$

Follows that ~~the statement~~ (A)

$$\le \sum_{l} \frac{1}{l!} (a(r_1))^l = e^{a(r_1)} - 1. \quad \blacksquare$$

Pf. (Claim).

Inductive hyp states that

$$1+\sum_{k=2}^{N} k \sum_{\gamma_2}\cdots\sum_{\gamma_k} |\varphi(\gamma_1,\dots,\gamma_k)| \prod_{j=2}^{k} |w(\gamma_j)| \le e^{a(\gamma_1)} .$$

Multiply by $|\zeta(\gamma_*,\gamma_1)||w(\gamma_1)|$ and sum over γ_1:

$$\sum_{k=1}^{N} k \sum_{\gamma_1}\cdots\sum_{\gamma_k} |\zeta(\gamma_*,\gamma_1)||\varphi(\gamma_1,\dots,\gamma_k)| \prod_{j=1}^{k} |w(\gamma_j)|$$

$$\le \sum_{\gamma_1} |w(\gamma_1)|\, e^{a(\gamma_1)} |\zeta(\gamma_*,\gamma_1)|$$

$$\le a(\gamma_*) \quad \text{[by hyp]}. \blacksquare$$

Applications of cluster expansion.

Recall.

If $\Xi = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \left(\prod_{i=1}^{n} w(\gamma_i) \right) \left(\prod_{i<j} \delta(\gamma_i, \gamma_j) \right)$

then $\log \Xi = \sum_{m\geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi_m(\gamma_1, \dots, \gamma_m) \prod_{i \in V_m} w(\gamma_i)$

with $\varphi_m(\gamma_1, \dots, \gamma_m) = \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} \prod_{ij \in G} \underbrace{\zeta(\gamma_i, \gamma_j)}_{\delta(\gamma_i, \gamma_j) - 1}$

where $G_n = (V_n, E_n)$ is the complete graph.

This series converges if $\exists\, a : \Gamma \to \cancel{\mathbb{R}} \; (0, \infty)$ st $\forall \gamma_*$,

$$\sum_{\gamma} |w(\gamma)| e^{a(\gamma)} |\zeta(\gamma, \gamma_*)| \leq a(\gamma_*).$$

Today. Large $h > 0$ expansion w/

$$H = -\beta \sum_{xy} \sigma_x \sigma_y - h \sum_x \sigma_x$$

Write

$$H = -\beta|\mathcal{E}(\pi)| - h|\Lambda| - \beta \sum_{xy} \underbrace{(\sigma_x\sigma_y - 1)}_{-2\mathbb{1}(xy\in\partial\Lambda^-)} - h \sum_x \underbrace{(\sigma_x - 1)}_{-2\mathbb{1}(x\in\Lambda^-)}$$

where $\Lambda^- = \{x \in \Lambda : \sigma_x = -1\}$ and ∂ = ext edge bdry

So $H = -\beta|\mathcal{E}(\pi)| - h|\Lambda| + 2\beta|\partial\Lambda^-| + 2h|\Lambda^-|$

Then $Z = e^{\beta|\mathcal{E}(\pi)| + h|\Lambda|}\,\Xi$

w/ $\Xi = 1 + \sum_{\emptyset \neq \Lambda^- \subset \Lambda} e^{-2\beta|\partial\Lambda^-| - 2h|\Lambda^-|}$

$$= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\emptyset \neq S_1 \subset \Lambda \\ \text{conn}}} \cdots \sum_{\substack{\emptyset \neq S_n \subset \Lambda \\ \text{conn}}} \left(\prod_{i=1}^n w_h(S_i)\right)\left(\prod_{i<j} \delta(S_i, S_j)\right)$$

$$w_h(S_i) = e^{-2\beta|\partial S_i| - 2h|S_i|}$$

$$\delta(S_i, S_j) = \mathbb{1}\left(d_1(S_i, S_j) > 1\right)$$

Thus, $\zeta(S, S_*) = \delta(S, S_*) - 1 \neq 0$ iff $S \cap [S_*] \neq \emptyset$,

where $[S_*] = \{x : d(x, S_*) \leq 1\}$.

③

So

$$\sum_{\substack{S \subset \Lambda \\ (\neq \emptyset, \text{ connected})}} |w_h(S)| e^{a(S)} |\zeta(S, S_*)| \qquad (\text{NTS } \le a(S_*))$$

$$\le \sum_{j \in [S_*]} \sum_{S \ni j} (\dots)$$

$$\le |[S_*]| \max_{j \in [S_*]} \sum_{S \ni j} |w_h(S)| e^{a(S)}$$

(trans-inv)

$$\le |[S_*]| \sum_{S \ni 0} (\dots)$$

$$\le |[S_*]| \sum_{S \ni 0} e^{-2h|S| + a(S)} \qquad \left(\text{ignore surface term}\right)$$

Set $a(S) := |[S]| \le 2d|S|$ and take h large to get:

$$\le |[S_*]| \sum_{h \ge 1} e^{-2(h-2d)h} \underbrace{\# \{S \ni 0 : |S| = h\}}_{\le (2d)^{2h}}$$

(use: every conn graph has a path from every vertex crossing each edge 2x)

$$\le |[S_*]| = a(S_*) \quad \text{for } h \text{ large} .$$

Note. The bd. $\le (2d)^{2h}$ is over all $S \ni 0$, $|S| = 2h$, not just $S \subset \Lambda$.

Thermodynamic limit:

Call $X = (S_1, \ldots, S_m)$ a cluster if $\varphi_m(S_1, \ldots, S_m) \neq 0$,

i.e. if $\zeta(S_i, S_j) \neq 0$ [equiv. $S_i \cap [S_j] \neq \phi$] $\forall i, j$

We have

$$\log \Xi = \sum_m \sum_{S_1 \subset \Lambda} \cdots \sum_{S_m \subset \Lambda} \varphi_m(S_1, \ldots, S_m) \prod_{i=1}^m w_\Lambda(S_i)$$

$$= \sum_{\substack{\text{clusters } X \\ \bar{X} \subset \Lambda}} \Psi(X)$$

where $\bar{X} = S_1 \cup \ldots \cup S_m$ (support)

and $\Psi(X) = \left(\prod_{S \subset \Lambda} \frac{1}{n_X(S)!} \right) \left(\sum_{\substack{G \subset G_m \\ \text{conn}}} \prod_{ij \in G} \zeta(S_i, S_j) \right) \prod_{i=1}^m w(S_i)$

$n_X(S) = \#$ times S appears in X.

(Note. Combinatorial factors comes from $\frac{1}{m!}$ in φ_m
and $\frac{m!}{\prod_S n_X(S)!}$ ways for X to appear)

Now $\sum_{\bar{X} \subset \Lambda} \Psi(X) = \sum_{x \in \Lambda} \sum_{x \in \bar{X} \subset \Lambda} \frac{1}{|\bar{X}|} \Psi(X)$

$$= \sum_{x \in \Lambda} \Big[\overbrace{\sum_{\bar{X} \ni x} \frac{1}{|\bar{X}|} \Psi(X)}^{A} - \overbrace{\sum_{x \in \bar{X} \not\subset \Lambda} \frac{1}{|\bar{X}|} \Psi(X)}^{B} \Big].$$

trans-inv $\Rightarrow$ (A) indep. of X

$$\Rightarrow \sum_{X\in\Lambda} (A) = c|\Lambda|$$

Also, $\left|\sum_{X\in\Lambda} (B)\right| \leq |\partial\Lambda| \max_{j\in\partial\Lambda} \sum_{\bar{X}\ni j} |\Phi(X)|$

$$\leq |\partial\Lambda| \sum_{m\geq 1} n \sum_{S_1\ni x} \sum_{S_2} \cdots \sum_{S_m} |\varphi_m(\ldots)| \prod |w_\alpha(S_k)|$$

$$\sum_{m\geq 1} + \sum_{m\geq 1} \leq e^{|[S_1]|} \quad (\text{calculus})$$

$$\leq |\partial\Lambda| \underbrace{\sum_{x\in S_1} |w_\alpha(S_1)|\, e^{|[S_1]|}}_{\leq 1} \leq |\partial\Lambda|$$

Therefore, $Z = e^{\beta|\mathcal{E}(\Lambda)| + h|\Lambda|}\,\Xi$

$$\Rightarrow \frac{1}{|\Lambda|}\log Z = \beta\frac{|\mathcal{E}(\Lambda)|}{|\Lambda|} + h + (A) + O\left(\frac{|\partial\Lambda|}{|\Lambda|}\right),$$

$(\Lambda = B(n) \to \mathbb{Z}^d)$

$$\Rightarrow \psi(h) = \beta d + h + \sum_{\bar{X}\ni 0} \Phi(X).$$

Note. $\dfrac{-H_{B(n)}(\eta\equiv 1)}{|B(n)|} \longrightarrow \beta d + h.$

In fact, dependence on h is through

$$w_h(S) = e^{-2\beta|\partial S| - 2h|S|}.$$

Thus,

$$\psi(h) = \beta d + h + \sum_{k\geq 1} a_k z^k, \qquad z = e^{-h}$$

and the coeffs. a_k can be computed systematically.

In particular, $h > 0$ large $\Rightarrow$ no phase transition.

<u>Note</u>. Lee-Yang $\Rightarrow$ no PT for $h \neq 0$.

Virial and Mayer expansions

Recall.

Ising model pressure is

$$\psi(h) = \beta d + h + \sum_{k\geq 1} a_k z^k, \quad z = e^{-h}, \quad h \text{ large}$$

The a_k can be computed from

$$\psi(h) = \beta d + h + \sum_{X \ni 0} \frac{1}{|X|} \Phi(X)$$

w/ $\Phi(X) = \left(\prod_{S \subset \Lambda} \frac{1}{\#X(S)!}\right)\left(\sum_{\substack{G \subset G_m \\ \text{conn}}} \prod_{ij \in G} \zeta(S_i, S_j)\right) \prod_{i=1}^{m} w_h(S_i)$

for $X = (S_1, \dots, S_m)$ a cluster, i.e. $\zeta(S_i, S_j) \neq 0$.

In particular, since dependence on h is only through

$$w_h(S) = e^{-2\beta|\partial S| - 2h|S|},$$

$$\cancel{a_1 = w_h(\{0\})}$$

$$a_1 = e^{2h|\{0\}|} w_h(\{0\}) = e^{-2\beta|\partial\{0\}|} = e^{-4d\beta} \neq 0.$$

Nearest-neighbour lattice gas.

$\eta \in \{0,1\}^\Lambda$, $H_\Lambda = -\sum_{x\sim y} \eta_x \eta_y$
(repulsion) , (attraction)

Grand canonical ensemble

$$\nu_{\Lambda;\beta,\mu}(\eta) = \frac{1}{\Theta_{\Lambda;\beta,\mu}} \exp\left(-\beta\left(H_\Lambda(\eta) - \mu N_\Lambda(\eta)\right)\right)$$

w/ $N_\Lambda(\eta) = \sum_{x\in\Lambda} \eta_x$ the # of particles

and $\mu \in \mathbb{R}$ the <u>chemical potential</u>.

Mapping to Ising model.

$\eta_x \longmapsto \omega_x = 2\eta_x - 1 \in \{\pm 1\}$

Let $\Lambda = B(n)$ so $\forall x \notin \partial\Lambda$, $\sum_{y\sim x} 1 = 2d$

and $|\partial\Lambda| = o(|\Lambda|)$ as $n \to \infty$.

Get $\beta(H(\eta) - \mu N(\eta)) = -\frac{\beta}{4}\sum_{x\sim y}\omega_x\omega_y - \frac{\beta}{2}(d+\mu)\sum_x \omega_x$

$-\beta\left(\frac{\mu}{2} + \frac{d}{4}\right)|\Lambda| + o(|\Lambda|)$

so $\Theta_{B(n);\beta,\mu} = e^{\beta(\frac{\mu}{2} + \frac{d}{4})|B(n)|} Z_{B(n);\beta',h'}$

w/ $\beta' = \frac{1}{4}\beta$, $h' = \frac{\beta}{2}(d+\mu)$.

Define the pressure

$$\cancel{p} \; p_\beta(\mu) := \lim_{n\to\infty} \underbrace{\frac{1}{\beta|\Lambda|} \log \Theta_{\Lambda;\beta,\mu}}_{p_{\Lambda;\beta}(\mu)}$$

$$= \frac{\mu}{2} + \frac{d}{4} + \frac{1}{\beta}\psi_{\beta'}(h')$$

Also, μ large $\Rightarrow h'$ large $\Rightarrow$ no PT $\Rightarrow p_\beta$ analytic

Density.

Note that $\frac{\partial p_\Lambda}{\partial \mu} = \frac{1}{\beta|\Lambda|} \cdot \underbrace{\frac{1}{\Theta_\Lambda} \sum_n \beta N(n) e^{-\beta(H_\Lambda(n) - N_\Lambda(n))}}_{\beta\langle N\rangle}$

$$= \left\langle \frac{N}{|\Lambda|} \right\rangle =: \rho_{\Lambda;\beta}(\mu) \text{ (density)}$$

So $\rho_\beta(\mu) = \frac{\partial p_\beta(\mu)}{\partial \mu} = \frac{1}{2} + \frac{1}{\beta} \cdot \frac{\partial \psi}{\partial h'} \cdot \underbrace{\frac{\partial h'}{\partial \mu}}_{\beta/2}$

$$= \frac{1}{2}\left(1 + m_{\beta'}(h')\right)$$

Also, $\rho(\mu) \in (0,1)$, since $m(h') \in (-1,1)$, and ρ is $\underbrace{\text{ct. / inc.}}$ ~~away from for μ large~~ away from CPs,

and $\lim_{\mu \to \pm\infty} \rho(\mu) = \cancel{0}\ 1$ or 0.

(exercise. differentiate ρ)

Thus, $\rho(\mu) = \rho_*$ has soln. $\mu(\rho_*)$ for $\rho_* \in (0,1)$

Let $\tilde{p}(\rho) = p(\mu(\rho))$

Mayer expansion:

Consider $-\mu$ large ($-h'$ large)

Have $\psi_{\beta'}(h') = \psi_{\beta'}(-h')$

$$= \beta' d - h' + \sum_{k \geq 1} a_k (z')^k, \quad z' = e^{2h'}$$

Thus, $\beta p(\mu) = \frac{\beta\mu}{2} + \frac{\beta d}{4} + \psi_{\beta'}(h')$

$$= \left(\frac{\beta\mu}{2} + \frac{\beta d}{4}\right) + \left(\frac{\beta d}{4} - \frac{\beta d}{2} - \frac{\beta\mu}{2}\right) + \psi_{\beta'}(h')$$

$$= \psi_{\beta'}(h') = \sum_{k\geq 1} a_k (z')^k ,$$

which is the <u>Mayer series</u>.

Differentiating yields

$$\rho(\mu) = \frac{\partial p}{\partial \mu} = \frac{1}{\beta} \sum k\, a_k (z')^k \underbrace{\frac{dz'}{dh'}}_{2z'} \cdot \underbrace{\frac{\partial h'}{\partial \mu}}_{\beta/2}$$

$$= \sum \underbrace{k\, a_k}_{\tilde{a}_k} (z')^k =: \phi(z') .$$

<u>Virial expansion</u>:

Invert ϕ : $\frac{d\phi}{dz'}(0) = \tilde{a}_1 = a_1 \neq 0$

$\Rightarrow$ exists local $\phi^{-1}(\rho) = z'$ st $\phi(z') = \rho$,

~~Note~~

Thus, $\beta \tilde{p}(\rho) = \sum a_k \left(\phi^{-1}(\rho)\right)^k$

$$= \sum a_k \left(\sum_m c_m \rho^m\right)$$

$$= \sum_n b_n \rho^n$$

In particular, $b_1 = a_1 c_1$

Since, $c_1 = (\phi^{-1})'(0) = \dfrac{1}{\phi'(\phi^{-1}(0))} = \dfrac{1}{\phi'(0)} = \dfrac{1}{a_1}$,

get $b_1 = 1$

Thus, $\tilde{p}(\rho) = \rho\beta^{-1} + \beta^{-1} \sum_{n \geq 2} b_n \rho^n$. (virial expansion)

The leading part is the ideal gas law

pressure = density × R × temp

w/ ideal gas const $R = 1$.

Random walk representation.

Consider $|\varphi|^4$ model:

- Hamiltonian $H(\varphi) = -\sum_{x,y} J_{xy}\, \varphi_x \cdot \varphi_y$, $\varphi_x \in \mathbb{R}^n$

- Single-spin meas. $g(t) = e^{-\frac{1}{4}|t|^4 - \frac{1}{2}|t|^2}$, $t \in \mathbb{R}^n$

View as perturbed Gaussian measure.
Expected to exhibit same (mean-field) behaviour as Gaussian near β_c, if $d \geq 5$.

For instance, susceptibility
$$\chi(\beta) = \lim_N \sum_{x \in \Lambda_N} \langle \varphi_0 ; \varphi_x \rangle_{\Lambda_N}$$
should blow up as $\chi(\beta) \sim C(\beta - \beta_c)^{-1}$

Easy to see in Gaussian case:
$$\langle \varphi_0 ; \varphi_x \rangle = (-\Delta + m^2)^{-1} = (2d(1-P) + m^2)^{-1}$$
$$= (2d + m^2)^{-1} \left(1 - \frac{2d}{2d+m^2} P\right)^{-1}$$
$$= (2d+m^2)^{-1} \sum_n \left(\frac{2d}{2d+m^2}\right)^n P^n$$

$$\Rightarrow \chi = (2d+m^2)^{-1} \sum_n \left(\frac{2d}{2d+m^2}\right)^n = \frac{1}{m^2}.$$

This computation is based on SRW and we proceed similarly for φ^4, etc.

~~Def~~
Random walk.

$J \in \mathbb{R}^{V\times V}$, $J_{xy} = J_{yx}$, $J_{xx} = 0$, summable on rows

$X = (X_t)_{t\geq 0}$ cts-time SRW gen by J,

i.e. Markov process w/ generator ~~Q_{xx}~~

$$Q_{xy} = \left(\sum_z J_{xz}\right)\delta_{xy} - J_{xy} \qquad \text{(Laplacian)}$$

Thus, $\frac{d}{dt}\Big|_{t=0} \mathbb{E}_x(f(X_t)) = -Qf(x)$.

Define local time $L_x^T = \int_0^T \mathbb{1}(X_t = x)\,dt$.

Feynman-Kac. $b, f \in \mathbb{R}^V$, $T > 0$:

$$\left(e^{-T(Q+b)}f\right)_x = \mathbb{E}_x\left(e^{-(b, L^T)} f_{X_T}\right) = \mathbb{E}_x\left(e^{-\int_0^T b_{X_t}dt} f_{X_T}\right).$$

Pf. $(b, L^T) = \sum_x b_x \int_0^T \mathbb{1}(X_t = x)\,dt = \int_0^T b_{X_t} \underbrace{\sum_x \mathbb{1}(X_t = x)}_{1}\,dt$

Let $(P_t f)_x$ = RHS.

Markov property $\Rightarrow$ (P_t) a semigroup.

So suffices to show it has generator $-(Q+b)$. ③

i.e. that $\frac{1}{T}((P_T - P_0)f)_x \xrightarrow{T \downarrow 0} [-(Q+b)f]_x$.

Do so by writing

$$[(P_T - P_0)f]_x = \int_0^T \frac{d}{dt}(P_t f)_x \, dt .$$

and computing. ∎

Cor. $(Q+b)^{-1}_{xy} = \int_0^\infty \mathbb{E}_x\left(e^{-(b,L^T)} \mathbb{1}(X_T = y)\right) dT$.

Pf. Take $f = \delta_y$ and integrate over $T \in (0,\infty)$. ∎

Gaussian field.

Let $D \in \mathbb{R}^{V \times V}$ be diag., $\mathrm{Re}\, D_{xx} > \sum_y J_{xy}$

so that $A = D - J$ has $\mathrm{Re}(A) > 0$.

(A = massive Laplacian)

Let $(\varphi_x)_{x \in V}$ be the n-(independent)-component

Gaussian field w/ cov. $C = A^{-1}$

Let $\tau_x = \frac{1}{2} |\varphi_x|^2$.

Thm. (BFS)

For $x, y \in V$, nice $g : \mathbb{R}_+^V \to \mathbb{R}$,

$$\tilde{\mathbb{E}}\left(g(\tau)\varphi_x^1 \varphi_y^1\right) = \int_0^\infty \tilde{\mathbb{E}}_x\left(g(L^T+\tau)\, e^{-(\overbrace{A-Q}^{\text{diag.}},\, L^T)} \mathbb{1}(X_T = y)\right) dT$$

Pf. $g(t) = e^{(b,t)}$, small $b : V \to \mathbb{R}$

$$\Rightarrow \int_0^\infty \tilde{\mathbb{E}}_x\Big(\underbrace{g(L^T+\tau)}_{e^{-(b,L^T)} e^{-\frac{1}{2}(\varphi, b\varphi)}} e^{-(A-Q,\, L^T)} \mathbb{1}(X_T = y)\Big) dT$$

$$= \tilde{\mathbb{E}}\left(e^{-\frac{1}{2}(\varphi, b\varphi)}\right) \int_0^\infty \mathbb{E}_x\left(e^{-(b+A-Q,\, L^T)} \mathbb{1}(X_T = y)\right) dT$$

$$= \frac{N(b)}{N(0)} \underbrace{\left(\cancel{Q} + b + A - \cancel{Q}\right)^{-1}_{xy}}_{\text{integrated FK}}$$

$$N(b) = \int e^{-\frac{1}{2}(\varphi, (A+b)\varphi)} d\varphi$$

$$= \tilde{\mathbb{E}}\left(e^{-\frac{1}{2}(\varphi, b\varphi)} \varphi_x^1 \varphi_y^1\right) = \tilde{\mathbb{E}}\left(g(\tau)\varphi_x^1 \varphi_y^1\right). \quad \blacksquare$$

Take $A \downarrow Q$ (when possible) to get

$$\tilde{\mathbb{E}}\left(g(\tau)\varphi'_x\varphi'_y\right) = \int_0^\infty \tilde{\mathbb{E}}_x\left(g(L^T+\tau)\mathbb{1}(X_T=y)\right)dT$$

eg. $g(t) = e^{-(t,b)} \Rightarrow$

$$(A+b)^{-1}_{xy} = \frac{1}{N(b)}\int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)}\varphi_x\varphi_y\,d\varphi = \frac{N(0)}{N(b)}\tilde{\mathbb{E}}\left(e^{-(\tau,b)}\varphi_x\varphi_y\right)$$

$$= \frac{N(0)}{N(b)}\int_0^\infty \tilde{\mathbb{E}}_x\left(e^{-(L^T+\tau,b)}\mathbb{1}(X_T=y)\right)dT$$

In particular, $b=0 \Rightarrow C_{xy} = \mathbb{E}_x(L^T_y)$.

Now for $g : \mathbb{R}^V_+ \to \mathbb{R}$ nice, $t \in \mathbb{R}^V_+$, let

$$\langle F\rangle_t = \frac{1}{Z(t)}\mathbb{E}(F(\varphi)g(\tau+t)),\quad Z(t) = \mathbb{E}(g(\tau+t)),\quad \langle\cdot\rangle = \langle\cdot\rangle_0$$

Lem. (GIP).

$$\tilde{\mathbb{E}}(\varphi'_x F(\varphi)) = \sum_y C_{xy}\tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\right) = \sum_y\int_0^\infty\tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\mathbb{1}(X_T=y)\right)dT$$

Pf. By IBP,

$$\sum_y C_{xy}\tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\right) = \sum_y C_{xy}\frac{1}{N(0)}\int\frac{\partial F}{\partial\varphi'_y}e^{-\frac{1}{2}(\varphi,A\varphi)}d\varphi$$

$$= -\sum_y C_{xy}\frac{1}{N(0)}\int F(\varphi)\sum_z A_{xz}\varphi'_z e^{-\frac{1}{2}(\varphi,A\varphi)}d\varphi$$

(5)

$$= \frac{1}{N(0)} \int F(\varphi) e^{-\frac{1}{2}(\varphi, A\varphi)} \underbrace{\sum_z \delta_{xz} \varphi'_z}_{\varphi'_x} \, d\varphi$$

$$= \tilde{\mathbb{E}}(\varphi'_x F(\varphi)).$$

For second equality use $C_{xy} = \mathbb{E}_x(L^T_y) = \int_0^\infty \mathbb{E}_x(\mathbb{1}(X_T = y))\, dT$.

<u>Lem.</u> (BFS-IBP).

$$\langle \varphi'_x F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT$$

w/ $Z(t) = \dfrac{z(t)}{z(0)}$.

<u>Pf.</u> $g(t) = e^{-(b,t)}$

$$z(0) \langle \varphi'_x F(\varphi) \rangle = \tilde{\mathbb{E}} \Big(\varphi'_x F(\varphi) \underbrace{e^{-\frac{1}{2}(\varphi, b\varphi)}}_{g(\tau)} \Big)$$

$$= \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi, (A+b)\varphi)} \varphi'_x F(\varphi)\, d\varphi$$

$$= \frac{1}{N(0)} \sum_y (A+b)^{-1}_{xy} \int e^{-\frac{1}{2}(\varphi, (A+b)\varphi)} \frac{\partial F}{\partial \varphi'_y}\, d\varphi$$

(GIBP)

$$= \frac{1}{N(0)} \int F(\varphi) e^{-\frac{1}{2}(\varphi, A\varphi)} \underbrace{\sum_z \delta_{xz} \varphi_z^1}_{\varphi_x^1} \, d\varphi$$

$$= \tilde{\mathbb{E}}\left(\varphi_x^1 F(\varphi)\right).$$

For second equality use $C_{xy} = \int_0^\infty \mathbb{E}_x\left(\mathbb{1}(X_T = y)\right) dT$. ∎

<u>Lem.</u> (BFS–IBP).

$$\langle \varphi_x^1 F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x\left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_{L^T} \mathbb{1}(X_T = y)\right) dT$$

$$\text{w/ } Z(t) = \frac{Z(t)}{Z(0)}.$$

<u>Cor.</u> (Gauss. UB).

For $|\varphi|^4$, $n = 1, 2$, and related models,

$$\langle \varphi_x^1 F(\varphi) \rangle = \sum_y \langle \varphi_x^1 \varphi_y^1 \rangle \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle.$$

Proof of Lem. + Cor. next time.

Ex. ($n=1$ for simplicity).

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2p}} \rangle = \sum_{y=x_2,\dots,x_{2p}} \langle \varphi_{x_1} \varphi_y \rangle \underbrace{\left\langle \frac{\partial}{\partial \varphi_y} \varphi_{x_2} \cdots \varphi_{x_{2p}} \right\rangle}_{\leq \cdots}$$

$$\vdots$$

$$\leq \sum_{\pi} \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \cdots \langle \varphi_{x_{\pi(2p-1)}} \varphi_{x_{\pi(2p)}} \rangle$$

w/ π running over perfect matchings of $\{1,\dots,2p\}$.

In particular, we have the Lebowitz inequality:

$$u_4(x_1,x_2,x_3,x_4) := \langle \varphi_{x_1} \varphi_{x_2} \varphi_{x_3} \varphi_{x_4} \rangle - \sum_{\pi} \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \cdots \langle \varphi_{x_{\pi(3)}} \varphi_{x_{\pi(4)}} \rangle$$

$$\leq 0.$$

As an application:

$$\beta \chi'(\beta) = C\chi^2(\beta) + \frac{1}{2} \sum_{x,y,z} J_{xy} u_4(0,x,y,z) \leq C\chi^2(\beta)$$

$$\Rightarrow \chi(\beta) \geq \frac{1}{C(\beta_c - \beta)} \quad \text{i.e.} \quad \boxed{\gamma \geq 1}.$$

Final remarks/critical phenomena.

Recall.
$\varphi \in (\mathbb{R}^n)^V$, $J_{xy} = J_{yx}$, $J_{xx} = 0$

$Q_{xy} = (\sum_z J_{xz})\delta_{xy} - J_{xy}$ generates $\mathbb{E}_x$ (Laplacian)

$\tilde{\mathbb{E}}$ Gauss w/ cov. $C = A^{-1}$, $A = D - J$ (Laplacian + pos. potent.)

$$\langle F \rangle_\ell = \frac{1}{Z(\ell)} \tilde{\mathbb{E}}\big(F(\varphi) g(\tau + \ell)\big),$$

$$g : \mathbb{R}_+^V \to \mathbb{R}, \quad \ell \in \mathbb{R}_+^V, \quad \tau_x = \frac{1}{2}|\varphi_x|^2$$

$\langle \cdot \rangle = \langle \cdot \rangle_0$, usu. set $\ell = L^T$ local time

Prop. (BFS-IBP).

$$\langle \varphi_x^i F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x\left(\mathcal{Z}(L^T) \left\langle \frac{\partial F}{\partial \varphi_y^i} \right\rangle_{L^T} \mathbb{1}(X_T = y)\right) dT,$$

$$\mathcal{Z}(\ell) = \frac{Z(\ell)}{Z(0)}.$$

Pf. $g(\cdot) = e^{-(b,\cdot)}$, $N(b) := \int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)} d\varphi$

$$Z(0)\langle \varphi_x^i F(\varphi) \rangle = \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)} \varphi_x^i F(\varphi)\, d\varphi$$

$$= \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)} \sum_y (A+b)^{-1}_{xy} \frac{\partial F}{\partial \varphi_y^i}\, d\varphi \quad [\text{GIP}]$$

$$= \sum_y \int_0^\infty \tilde{\mathbb{E}}_x \left(e^{-(\tau + L^T, b)} \frac{\partial F}{\partial \varphi_y'} \mathbb{1}(X_T = y) \right) dT \qquad [\text{IFK}]$$

$$= \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi_y'} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT. \qquad \blacksquare$$

<u>Cor.</u> (GUB). If $\frac{\partial}{\partial \ell_x} \left\langle \frac{\partial F}{\partial \varphi_y'} \right\rangle_\ell \leq 0 \quad \forall x, y$, then

$$\langle \varphi_x' F(\varphi) \rangle \leq \sum_y \langle \varphi_x' \varphi_y' \rangle \left\langle \frac{\partial F}{\partial \varphi_y'} \right\rangle.$$

<u>Ex.</u> $g(\ell) = e^{-\sum_x V_x(\ell_x)}$

$$\Rightarrow \frac{\partial}{\partial \ell_x} \langle F \rangle_\ell = - \langle F ; V_x'(\tau_x + \ell_x) \rangle$$

So Griffiths ineq. $\langle \varphi^A ; \varphi^B \rangle \geq 0$, $A, B \subset V$ ~~$[\ldots]$~~

[holds for $n=1$/flip-inv./ferro.]

implies RHS ≤ 0 eg. for $F(\varphi) = \varphi^A$ and $V_x(\ell_x) = g t_x^2 + \nu t_x$, $g \geq 0$.

<u>Pf.</u> $\langle \varphi_x' F(\varphi) \rangle$

$$= \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi_y'} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT \qquad [\text{BFS-IBP}]$$

$$\leq \quad \cdots\cdots \langle \cdot \rangle_0 \cdots\cdots$$

$$= \sum_y \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle \frac{1}{Z(\tau)} \underbrace{\int_0^\infty \tilde{E}_x \left(g(L^T + \tau) \mathbb{1}(X_T = y) \right) dT}_{\tilde{\mathbb{E}}(g(\tau)\varphi'_x \varphi'_y) \text{ by BFS}}$$

$$= \sum_y \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle \langle \varphi'_x \varphi'_y \rangle .$$

Application:

GHS $\Rightarrow$ Lebowitz ineq.:

$$u_4(x_1, \dots, x_4) := \langle \varphi_{x_1} \dots \varphi_{x_4} \rangle - \sum_\pi \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \langle \varphi_{x_{\pi(3)}} \varphi_{x_{\pi(4)}} \rangle$$

$$\leq 0$$

Susceptibility $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \varphi_0 \varphi_x \rangle$

$$\Rightarrow \chi'(\beta) = \sum_z \langle \varphi_0 \varphi_z ; -H(\varphi) \rangle$$

$$= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(\langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle - \langle \varphi_0 \varphi_z \rangle \langle \varphi_x \varphi_y \rangle \right)$$

$$= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(u_4(0,x,y,z) + \langle \varphi_0 \varphi_x \rangle \langle \varphi_y \varphi_z \rangle + \langle \varphi_0 \varphi_y \rangle \langle \varphi_x \varphi_z \rangle \right)$$

$$= \frac{1}{2} \sum_{x,y,z} J_{xy} u_4(0,x,y,z) + \frac{1}{2} \underbrace{\sum_x \langle \varphi_0 \varphi_x \rangle}_{\chi} \underbrace{\sum_y J_{xy}}_{|J|} \underbrace{\sum_z \langle \varphi_y \varphi_z \rangle}_{\chi}$$

$$+ \frac{1}{2} \underbrace{\sum_y \langle \varphi_0 \varphi_y \rangle}_{\chi} \underbrace{\sum_z \langle \varphi_x \varphi_z \rangle}_{\chi} \underbrace{\sum_x J_{xy}}_{|J|}$$

(assume trans-inv)

$$= \frac{1}{2} \sum_{x,y,z} \underbrace{J_{xy}}_{\geq 0} \underbrace{u_4(0,x,y,z)}_{\leq 0} + |J| \chi^2(\beta)$$

$$\leq |J| \chi^2$$

Let $f(\varepsilon) = [\chi(\beta_c - \varepsilon)]^{-1}$

w/ $\beta_c := \sup(\beta : \chi(\beta) < \infty)$ $[\varepsilon < 0 \Rightarrow f(\varepsilon) = 0]$

Then $f'(\varepsilon) = \dfrac{\chi'(\beta_c - \varepsilon)}{\chi(\beta_c - \varepsilon)} \leq |J|$

$\Rightarrow f(\varepsilon) = \int_0^\varepsilon f'(t)\,dt \leq |J| \varepsilon$

i.e. $\chi(\beta_c - \varepsilon) \geq \frac{1}{|J|} \varepsilon^{-1}$

So if $\chi(\beta_c - \varepsilon) \sim C \varepsilon^{-\gamma}$,

then $\boxed{\gamma \geq 1}$.

These [RW] methods, together w/ IR bds, can be used to show the opposite ineq.

Can also show triviality in $d > 4$, i.e. continuum limits [$J(a)$, $g(a)$, $\nu(a)$ depend on lattice scaling a] must be Gaussian

Note. These results w/ $n=1$ hold by Griffiths and extend to $n=2$ by Ginibre [proved using polar coords] not known to hold for $n > 2$.

Critical behaviour.

For wide classes of models, expect

$$\langle \varphi_0 ; \varphi_x \rangle \sim C_1 \, |x|^{-(d-2+\eta)}, \quad |x| \to \infty$$

$$\chi(\beta) \sim C_2 \, (\beta_c - \beta)^{-\gamma}, \quad \beta \uparrow \beta_c$$

$$\xi(\beta) \sim C_3 \, (\beta_c - \beta)^{-\nu}, \quad \beta \uparrow \beta_c$$

where $\xi = \limsup_{k \to \infty} \dfrac{-k}{\log \langle \varphi_0 ; \varphi_{ke} \rangle}$ so that

$\langle \varphi_0 ; \varphi_x \rangle \approx C \, f(x) \, e^{-|x|/\xi}$ w/ $f(x)$ sub-exp.

(Other relations expected as well)

Exponents $\eta, \gamma, \nu, \dots$ should be universal, i.e. depend only on range and symmetry (d, n).

d	2	3	≥ 4
η	1/4	0.0362...	0
γ	7/4	1.2370...	1
ν	1	0.6299...	1/2

$(n=1)$

Log corrections predicted if $d=4$, (for γ, ν), eg.

$$\chi(\beta) \sim C_2 (\beta_c - \beta)^{-1} \left(-\log(\beta_c - \beta)\right)^{\frac{n+2}{n+8}}$$

[Shown in BBS 14 for small coupling φ^4 and SAW/$n=0$]

Also extensions for $n=0$ SAW predicted.

$d>4$. • Aizenman 82, Fröhlich 82, $n=1,2$

~~Slade Hara-Slade~~

• Brydges-Spencer 85, Hara-Slade 92, $n=0$

$d=4$. • BBS 14 RG, all n

$d=2$. • Onsager $\Rightarrow$ Ising

• Ising interfaces $\Rightarrow SLE_3$ (Chelkak et al 14)

• SAW $\rightarrow$ SLE8/3 if limit exists + is conformally invariant (Lawler-Schramm-Werner 04)

• BKT transition $n=2$ predicted

$d=3$. • magnetization $\rightarrow 0$ at β_c ($n=1$) (Aizenman et al 15)

[analogous to percolation problem]

Equilibrium stat mech.

Ref. S. Friedli + Y. Velenik
Statistical mechanics of lattice systems

Physical system.
- State space (Ω, λ)
- Hamiltonian $H : \Omega \to \mathbb{R}$ (usu. smooth / bdd below)

Ex. (n particles).

$$\Omega = (D \times \mathbb{R}^3)^n, \quad D \subseteq \mathbb{R}^3$$

$$H(q,p) = \underbrace{\frac{1}{2m}|p|^2}_{\text{kin.}} + \underbrace{U(q)}_{\text{pot.}}$$

$$\begin{cases} \dfrac{dq}{dt} = \nabla_p H \\ -\dfrac{dp}{dt} = \nabla_q H \end{cases}$$

(Stat mech: n large)

Closed system: Constant energy

- Sys. evolves on $\underset{\text{energy shell}}{\underset{\wedge}{\{H=E\}}} =: S_E$

- Microcanonical ensemble: prob. meas. μ on S_E

$$\int f(x)\mu(dx) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{S_{[E,E+\delta]}} f(x)\,dx.$$

Note. μ singular wrt λ ~~...~~ (hard to work w/)

System in thermal equilibrium: Constant temp.

or $\int H\,d\mu = E$ const.

Ex. (Ω finite) (57)

Employ Jaynes' principle of max entropy

i.e. find μ maximizing $-\sum_{\omega\in\Omega} \mu(\omega)\log\mu(\omega) =: \mathcal{S}(\mu)$

subject to $\sum_{\omega} H(\omega)\mu(\omega) = E$

and $\sum_{\omega} \mu(\omega) = 1.$

Define Lagrangian

$$\mathcal{L}(\mu, \lambda_1, \lambda_2) = \mathcal{S}(\mu) - \lambda_1 \left[\int H d\mu - E\right] - \lambda_2 \left[\int d\mu - 1\right]$$

and solve $\nabla \mathcal{L} = 0$.

So $$\frac{\partial \mathcal{L}}{\partial \mu(\omega)} = -\log \mu(\omega) - 1 - \lambda_1 H(\omega) - \lambda_2 = 0$$

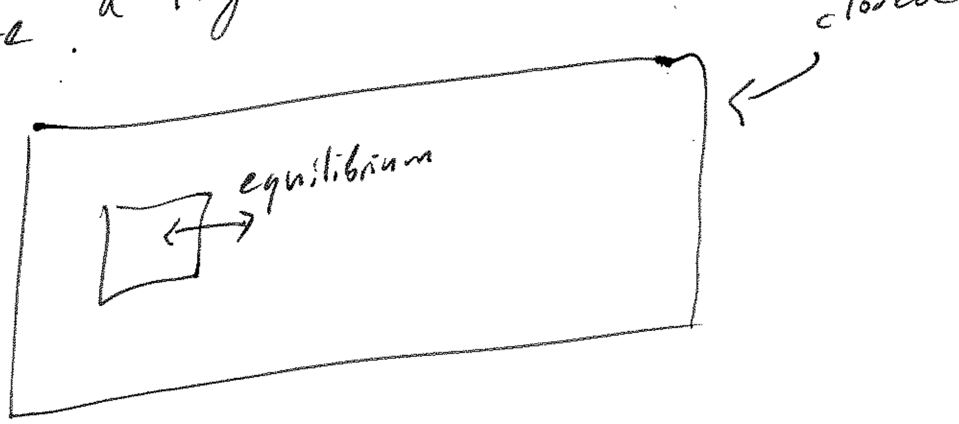
$$\Rightarrow \mu(\omega) \propto e^{-\lambda_1 H(\omega)}.$$

Canonical measure: $\frac{1}{Z} e^{-\beta H(\omega)}$ (β → inverse temp.)

($\beta \to \infty \Rightarrow$ concentration on low energy states)

Other justifications:

- Marginalize a larger microcanonical meas.



- Variational principle (Dobrushin 68, Lanford–Ruelle 69):

$$-\frac{1}{\beta}\log Z_\beta = \int H\,d\mu - \beta^{-1} \mathcal{S}(\mu)$$

i.e. free energy = internal energy − temp × entropy

and the RHS is <u>minimized</u> by such measures.

[This holds in <u>great</u> generality, see eg. Georgii 88]

- Hammersley–Clifford thm.

$\Omega = \{ f : \text{graph} \to \text{countable} \}$

$H = \sum$ contributions from complete subgraphs

} Gibbs random field

<u>Thm.</u> Gibbs random fields $\Longleftrightarrow$ Markov random fields

(see Grimmett 2010)

Relation to quantum theory.

$$H(x,p) = \frac{1}{2m}|p|^2 + U(x)$$

quantize: $x_j \rightsquigarrow \hat{x}_j$

$p_j \rightsquigarrow -i\hbar \frac{\partial}{\partial x_j}$

$$\hat{H} = H(\hat{x},\hat{p}) = -\frac{\hbar^2}{2m} + U(\hat{x})$$

$\psi \in L^2(\mathbb{R}^{3n})$ evolves according to Schrödinger:

$$i\hbar \frac{d\psi}{dt} = \hat{H}\psi$$

Assume soln. operator has kernel K_t:

$$e^{-it\hat{H}/\hbar} f = \int K_t(\cdot, y) f(y)\, dy$$

Feynman integral formulation:

$$K_t(a,b) = \int_{W_t(a,b)} e^{(i/\hbar)\int_0^t L(x(s),\dot{x}(s))}\, \mathcal{D}x$$

$$W_t(a,b) = \{\text{paths } [0,t] \longrightarrow \mathbb{R}^{3n} \text{ from } a \to b\}$$

$$L(x,\dot{x}) = \frac{1}{2}m|\dot{x}|^2 - U(x).$$

Wick rotation: $t \longmapsto \Psi(-it)$ has soln. kernel

$$K_{-it}(a,b) = \int_{W_{-it}(a,b)} e^{(i/\hbar)\int_0^{-it} L(x(s),\dot{x}(s))\,ds}\,\mathcal{D}x$$

Change vars $s = -iu$:

$\tilde{x}(t) = x(-it)$

$$\frac{i}{\hbar}\int_0^{-it} L(x(s),\dot{x}(s))\,ds = \frac{1}{\hbar}\int_0^t L(\tilde{x}(u), i\dot{\tilde{x}}(u))\,du$$

$$= -\frac{1}{\hbar}\int_0^t H(\tilde{x}(u), m\dot{\tilde{x}}(u))\,du.$$

Use $W_{it} \simeq W_t$ to get

$$K_{-it}(a,b) = \int_{W_t(a,b)} \underbrace{e^{-(1/\hbar)\int_0^t H(x(u), m\dot{x}(u))\,du}\,\mathcal{D}x}_{\text{Gibbs-type meas. on paths}}$$

Feynman-Kac.

$\tilde{\Psi}(t) = \Psi(-it)$ should solve

$$\frac{d\tilde{\Psi}}{dt} = -\hat{H}\tilde{\Psi} \quad (\hbar = 1, m = 1) \qquad (*)$$

- $U = 0 \Rightarrow \hat{H} = -\frac{1}{2}\Delta$

$\Rightarrow$ (*) is heat eqn., $\int_0^t H$ is pos-det quadratic

and K_{-it} is a "Gaussian" integral

Rigorously, $\tilde{\Psi}(t,x) = \mathbb{E}\left(\tilde{\Psi}(0, B_t) \mid B_0 = x\right)$

$\hookrightarrow$ Brownian motion

(under right conditions)

- $U \neq 0 \Rightarrow$ Feynman-Kac formula

$$\tilde{\Psi}(t,x) = \mathbb{E}\left(\tilde{\Psi}(0,B_t)\, e^{\int_0^t U(B_s)\,ds} \mid B_0 = x\right)$$

cf. K_{-it}

QFT.

Replace paths (fcns. on $\mathbb{R}$) by fields (fcns. on $\mathbb{R}^d$)

Ising model.

Graphs.

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

[undirected/ $\mathcal{V}$ countable/ no self-loops $\{x\} \in \mathcal{E}$]

Weights $J \in \mathbb{R}^{V \times V}$, $J_{xy} \geq 0$ w/ equality iff $x \sim y$

$D \in \mathbb{R}^{V \times V}$ diag. w/ $D_{xx} = d_x = \sum_{y \sim x} J_{xy}$

Laplacian $-\Delta = D - J$

Spin systems.

$d\lambda^0$ meas. on $S \subset \mathbb{R}^n$

$d\lambda = \prod_{x \in \Lambda} d\lambda^0$ on $\Omega = S^V$

$\varphi \in \Omega$ a field or spin configuration

w/ spins in S

$H : \Omega \to \mathbb{R}$ st. $\int e^{-H} d\lambda < \infty$.

A spin system w/ Hamiltonian H at inv. temp. $\beta > 0$ ⑨

is $d\mu_\beta(\varphi) = \frac{1}{Z_\beta} e^{-\beta H(\varphi)} d\lambda(\varphi)$.

It is ferromagnetic if

$H(\varphi) = -\varphi \cdot M \varphi$ w/ $M_{xy} \geq 0$ $\forall x, y$

$$\overset{(\text{note})}{=} \sum_{i=1}^{n} \sum_{x,y \in V} \underbrace{\left(-\varphi_x^i M_{xy} \varphi_y^i \right)}_{\min \text{ @ } \varphi_x^i = \varphi_y^i}$$

Ex. ($O(n)$/n-vector model).

$H(\varphi) = -\frac{1}{2} \varphi \cdot J \varphi$ for $\varphi \in (S^{n-1})^V$

$$\begin{cases} n = 1 \Rightarrow \text{Ising} \\ n = 2 \Rightarrow XY \\ n = 3 \Rightarrow \text{classical Heisenberg} \end{cases}$$

Phase transitions:

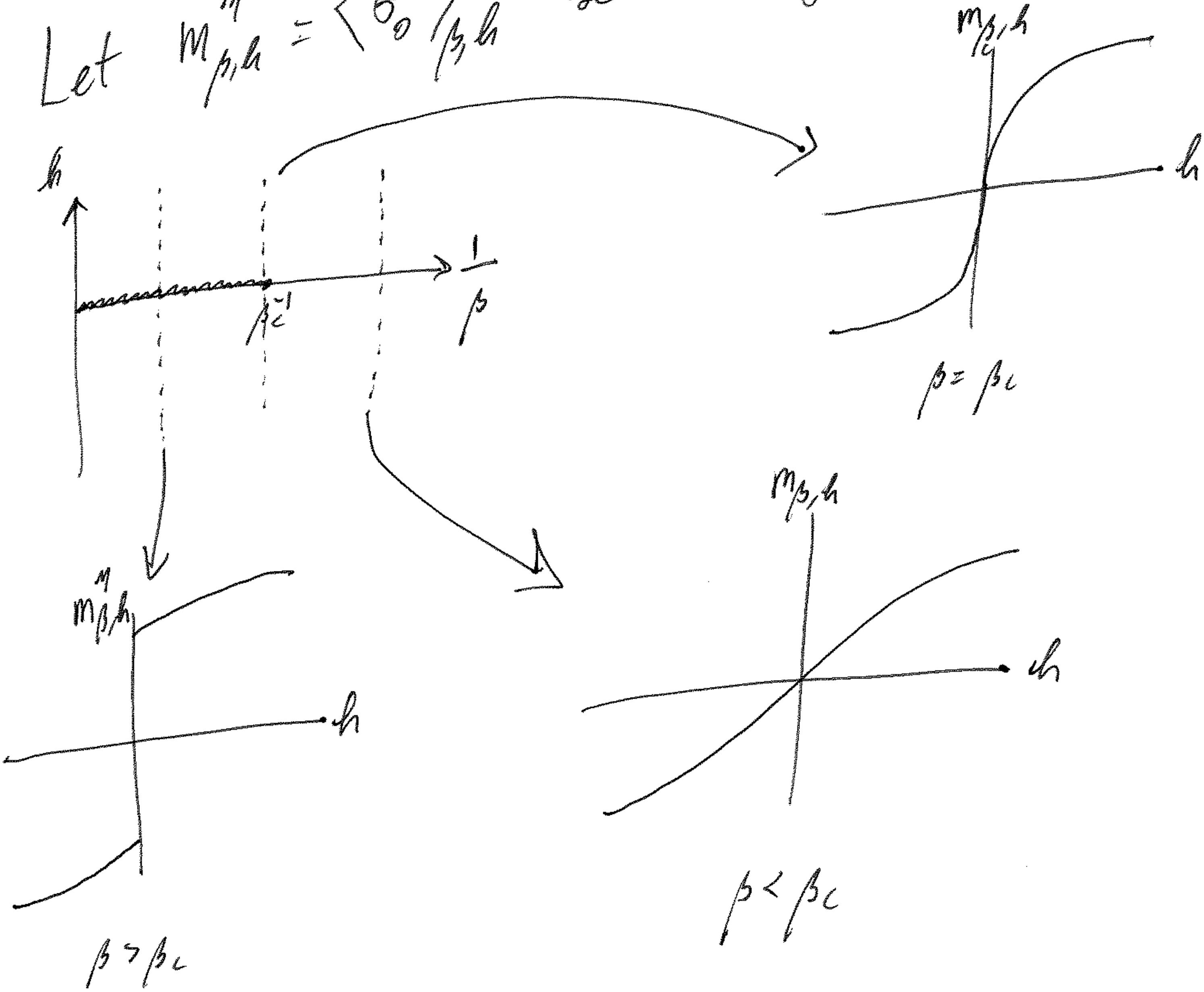
Fix $\mathcal{B} = \Lambda \subset \mathbb{Z}^d$ a box

Consider ~~$O(n)$~~ Ising model w/ bdry condition η outside Λ + ext field $h \in \mathbb{R}$

$$\Rightarrow d\mu^{\eta,\Lambda}_{\beta,h}$$

Suppose $d\mu^{\eta}_{\beta,h} = \lim_{\Lambda \uparrow \mathbb{Z}^d} d\mu^{\eta,\Lambda}_{\beta,h}$ exists (in some sense)

Let $m^{\eta}_{\beta,h} = \langle \sigma_0 \rangle^{\eta}_{\beta,h}$ be the magnetization.



Case $\beta > \beta_c$: • depends on η

• reflects existence of multiple (distinct) Gibbs states for $h = 0$

i.e. $\mu^{+}_{\beta,0} \neq \mu^{-}_{\beta,0}$

$$\mu^{+}_{\beta,0} \overset{..}{=} \mu^{\eta}_{\beta,0} \text{ w/ } \eta \equiv 1$$

This is a first-order or discts. phase transition / spontaneous symmetry breaking.

($\mu^{\pm}_{\beta,0}$ not invariant under $\omega \mapsto -\omega$)

To come.

• Next: investigate far from critical regimes ($\beta \gg \beta_c$, $\beta \ll \beta_c$, $|h| \gg 0$) using expansion methods

$\Longrightarrow$ cluster expansion

• Other possible topics.

- Other methods to study non-critical regime ($(\beta, h) \neq (\beta_c, 0)$)
- Models w/ continuous symmetry ($n > 1$)
- Renormalisation group for $\beta = \beta_c$ (critical regime)

Gibbs measures for the Ising model.

- Focus on classical nn Ising on $\mathbb{Z}^d$ (avoid conditioning on meas. 0 events)
  Follow Georgii-Häggström-Maes '99
- Generalize to compact spin space + long-range interactions in Georgii '11 or Friedli-Velenik '17
- For unbdd. spins see Lebowitz-Presutti '76

Notation.

$\partial\Lambda$ = ext bdry of $\Lambda \subset \mathbb{Z}^d$

$\Omega_\Lambda = \{\pm 1\}^\Lambda$

restrict $\omega \in \Omega$ to $\omega_\Lambda \in \Omega_\Lambda$

abuse notation: $\omega_\Lambda \simeq \{\tilde{\omega} : \tilde{\omega}_\Lambda = \omega_\Lambda\}$

Def. Hamiltonian $H^{\eta,\Lambda}_{\beta,h} : \Omega \to \mathbb{R}$ at inv temp $\beta > 0$, ext field $h \in \mathbb{R}$, bdry condition $\eta \in \Omega$ off of $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$:

$$H^{\eta,\Lambda}_{\beta,h}(\omega) = -\beta\Big[\sum_{\substack{x\sim y\\ x,y\in\Lambda}} \omega_x\omega_y + h\sum_{x\in\Lambda}\omega_x + \sum_{\substack{x\sim y\\ x\in\Lambda\\ y\in\partial\Lambda}} \omega_x\eta_y\Big]$$

<u>Gibbs specification</u> $\mu^{\eta,\Lambda}_{\beta,h}(\omega) = \frac{1}{Z^{\eta,\Lambda}_{\beta,h}} e^{-H^{\eta,\Lambda}_{\beta,h}(\omega)}$,

Prob. meas. μ on Ω is a <u>Gibbs meas.</u> if the DLR eqns.

$$\mu(\omega_\Lambda \mid \eta_{\partial\Lambda}) = \mu^{\eta,\Lambda}(\omega), \qquad \forall\, \omega, \eta, \Lambda$$

are satisfied.

Let $\mu^{\pm,\Lambda} = \mu^{\eta,\Lambda}$ w/ $\eta \equiv 1$.

<u>Thm.</u> The weak lims. $\mu^{\pm} := \lim_{\Lambda\uparrow\mathbb{Z}^d} \mu^{\pm,\Lambda}$ exist and are Gibbs measures.

<u>Note.</u> $\lim_{\Lambda\uparrow\mathbb{Z}^d}(\ldots)$ means $\lim_{\Lambda_n\uparrow\mathbb{Z}^d}(\ldots)$ $\forall\, \Lambda_n \uparrow \mathbb{Z}^d$.

If all these lims exist, they are the same.

<u>Markov prop.</u> $\Lambda_1 \subset \Lambda_2 \Rightarrow \mu^{\xi,\Lambda_1}(\omega) = \cancel{\mu^{\Lambda_2}(\omega \mid \xi_{\Lambda_2\setminus\Lambda_1})}$.

$$= \mu^{\tilde{\xi},\Lambda_2}(\omega \mid \xi_{\Lambda_2\setminus\Lambda_1})$$

Pf. (idea)
RHS has form $\mu^{\eta,\Lambda}_{\beta,h}$ so just check that params η,Λ,β,h same on LHS and RHS. $\square$

Next. Reduce pf. of thm.

Lem. Suffices to show μ^+ exists.

Pf. $\mu^+(\omega_\Lambda \mid \xi_{\partial\Lambda}) = \lim_{\Delta \uparrow \mathbb{Z}^d} \mu^{+,\Delta}(\omega_\Lambda \mid \xi_{\partial\Lambda})$

$\overset{(\text{Markov})}{=} \lim_{\Delta} \mu^{\xi,\Lambda}(\omega_\Lambda) = \mu^{\xi,\Lambda}(\omega_\Lambda)$. $\square$

$f : \Omega \to \mathbb{R}$ is local if it only depends on spins in a finite set Λ. Thus, it can be identified w/ an element of $\mathbb{R}^{\Omega_\Lambda}$.

Lem. Any bdd. cts. fn. on Ω (prod. topology) can be unif. approximated (ie. in the sup norm) by local fns. Thus, it suffices to check convergence of $\mu^{+,\Lambda}(f)$ for local fns. f.

Pf. (idea).

Given bdd. cts f and arbitrary $\tilde{\omega} \in \Omega$,

let $g_\Lambda(\omega) = f(\omega_\Lambda \tilde{\omega}_{\Lambda^c})$. So g_Λ local.

Tychonoff $\Rightarrow$ Ω compact $\Rightarrow$ $f - g_\Lambda$ unif cts

$\Rightarrow \|f - g_\Lambda\|_\infty \to 0$ as $\Lambda \uparrow \mathbb{Z}^d$. ∎

For $A \subset \Lambda$ let $\sigma_A : \Omega_\Lambda \to \mathbb{R}$ be

$$\sigma_A(\omega) = \prod_{x \in A} \omega_x .$$

Lem. $\{\sigma_A : A \subset \Lambda\}$ is an ON basis for $\mathbb{R}^{\Omega_\Lambda}$ under $\langle f, g \rangle = 2^{-|\Lambda|} \sum_{\tilde{\omega} \in \Omega_\Lambda} f(\tilde{\omega}) g(\tilde{\omega})$.

Pf. $|2^\Lambda| = 2^{|\Lambda|} = |\Omega_\Lambda|$ so just check ON

~~$A \Delta A \Rightarrow \omega_x^2$~~

$\omega_x^2 = 1 \Rightarrow \langle \sigma_A, \sigma_A \rangle = 1.$

For $A \neq B$:

$$2^{|\Lambda|}\langle \sigma_A, \sigma_B \rangle = \sum_{\tilde{\omega}} \prod_{x \in A \Delta B} \tilde{\omega}_x = 0$$

Claim. $\forall$ finite Λ and $S \subset \Lambda$,

$$\sum_{\tilde{\omega} \in \Omega_\Lambda} \prod_{x \in S} \tilde{\omega}_x = 0.$$

Follows by induction: for $|S| = n+1$,

$$\sum_{\tilde{\omega} \in \Omega_\Lambda} \prod_{x \in S} \tilde{\omega}_x = \sum_{\tilde{\omega} \in \Omega_\Lambda} \tilde{\omega}_y \prod_{\substack{x \neq y \\ x \in S}} \tilde{\omega}_x$$

$$= (\dots) - (\dots) = 0. \quad \blacksquare$$

Let $\mathbb{1}_{\delta A}$ be the indicator fcn. for $\delta A = \{\omega_x = 1 \ \forall x \in A\}$.

Lem. $\{\mathbb{1}_{\delta A} : A \subset \Lambda\}$ is a basis for $\mathbb{R}^{\Omega_\Lambda}$.

Pf. Suffices to show spanning.

Write $\cancel{\sum_{A \subset \Lambda} f_A}$ $f \in \mathbb{R}^{\Omega_\Lambda}$ as

$$f = \sum_{A \subset \Lambda} f_A \sigma_A, \quad f_A = \langle f, \sigma_A \rangle$$

Expand σ_A in terms of indicators using

$$\sigma_x = 2\mathbb{1}_{\delta^x} - 1$$

and binomial thm.

$$\prod_{x \in X} (a_x + b_x) = \sum_{Y \subset X} \left(\prod_{x \in X \setminus Y} a_x\right)\left(\prod_{y \in Y} b_y\right).$$

Get $\sigma_A = \prod_{x \in A} (2\mathbb{1}_{\delta^x} - 1) = \sum_{Y \subset A} (-1)^{|A \setminus Y|} 2^{|Y|} \mathbb{1}_{\delta^Y}$.

Thus, $f = \sum_{Y \subset \Lambda} \tilde{f}_Y \mathbb{1}_{\delta^Y}$

$\left(w/ \ \tilde{f}_Y = \sum_{A \supset Y} f_A (-1)^{|A \setminus Y|} 2^{|Y|} \right)$. $\square$

<u>Conclusion</u>. Suffices to show: $\forall A \subset \mathbb{Z}^d$ finite,

$\mu^{+,\Lambda}(\delta^A) = \mu^{+,\Lambda}(\mathbb{1}_{\delta^A})$ converges.

<u>Plan</u>. $\mu^{+,\Lambda}(\delta^A)$ bdd, so monotonicity suffices.

Intuitively, $\mu^{+,\Lambda}(\delta^A)$ decreases as Λ increases.

Def. (Stochastic dominance).

$\mu_1 \leq \mu_2$ if $\mu_1(f) \leq \mu_2(f)$ ~~[illegible]~~

$\forall$ bdd. cts. $\underbrace{\text{inc.}}_{\text{wrt natural partial order}}$ f.

Thus, we will show that $\Lambda_1 \subset \Lambda_2 \Rightarrow \mu^{+,\Lambda_2} \leq \mu^{+,\Lambda_1}$.

Since $\mathbb{1}_{\delta A}$ is bdd. $\underbrace{\text{cts.}}_{\text{wrt prod. top.}}$ inc., get convergence.

Accomplish this using:

Thm. (Strassen). TFAE:

(1) $\mu_1 \leq \mu_2$

(2) There is a coupling μ of (μ_1, μ_2) s.t. $(X_1, X_2) \sim \mu \Rightarrow \mathbb{P}(X_1 \leq X_2) = 1$. } monotone coupling

Note. We only use $(2) \Rightarrow (1)$, which is easy:

f inc. + $X_1 \leq X_2 \Rightarrow f(X_1) \leq f(X_2) \Rightarrow \mu_1(f) = \mathbb{E} f(X_1) \leq \mathbb{E} f(X_2) = \mu_2(f)$.

[Converse harder but not needed].

Correlation inequalities.

Def. μ prob meas on Ω_Λ.

G_μ graph w/ $V = \{\omega \in \Omega_\Lambda : \mu(\omega) > 0\}$

and $\omega \sim \omega'$ if $\omega_x = \omega'_x$ $\forall$ but one $x \in \Lambda$.

μ irreducible if G_μ connected.

Thm. (Holley inequality). μ_1, μ_2 prob meas on Ω_Λ. Assume μ_2 irred. and $\mu_2(\delta^\Lambda) > 0$. Suppose $\forall x \in \Lambda$ and $\xi \leq \eta$ w/ $\mu_1(\xi_{\Lambda \setminus x}) > 0$ and $\mu_2(\eta_{\Lambda \setminus x}) > 0$ that

$$\mu_1(\delta^x \mid \xi_{\Lambda \setminus x}) \leq \mu_2(\delta^x \mid \eta_{\Lambda \setminus x}).$$

Then $\mu_1 \leq \mu_2$.

Holley's inequality and $\mu^{\pm}$.

Recall. Our plan is as follows:

(A) Construct monotone coupling of $(\mu^{+,\Lambda_2}, \mu^{+,\Lambda_1})$ for $\Lambda_1 \subset \Lambda_2$

(Strassen) $\Rightarrow$ (B) $\mu^{+,\Lambda_2} \leq \mu^{+,\Lambda_1}$

$\Rightarrow$ (C) $\left(\mu^{+,\Lambda}(\delta^A)\right)_{\Lambda \uparrow \mathbb{Z}^d}$ decreasing $\forall A$ finite

$\Rightarrow$ (D) This sequence converges

$\Rightarrow$ (E) $\mu^{+,\Lambda}(f)$ converges $\forall$ local f

$\Rightarrow$ (F) $\mu^{+,\Lambda}(f)$ converges $\forall$ bdd cts f

$\Rightarrow$ (G) $\mu^{+} = \lim_{\Lambda} \mu^{+,\Lambda}$ exists + is a Gibbs meas.

We prove (A) in greater generality as <u>Holley's inequality</u>.

Correlation inequalities.

Def. μ prob. meas. on Ω_Λ. G_μ graph w/ ~~$V = \Omega_\Lambda$ and~~ $V = \{\omega \in \Omega_\Lambda : \mu(\omega) > 0\}$ and $\omega \sim \omega'$ if $\omega_x = \omega'_x$ $\forall$ but one x. μ <u>irreducible</u> if G_μ connected.

Thm. (Holley ineq.) μ_1, μ_2 prob. meas. on Ω_Λ. Assume μ_2 irred. and $\mu_2(\delta^\Lambda) > 0$. Suppose $\forall x \in \Lambda$ and $\xi \le \eta$ w/ $\mu_1(\xi_{\Lambda \setminus x}) > 0$, $\mu_2(\eta_{\Lambda \setminus x}) > 0$ that

$$\mu_1(\delta^x \mid \xi_{\Lambda\setminus x}) \le \mu_2(\delta^x \mid \eta_{\Lambda \setminus x}).$$

Then $\mu_1 \le \mu_2$.

<u>Gibbs sampler</u>. Markov chain on Ω_Λ w/ stationary μ.
If $\sigma^i = \omega$, then:

(1) pick $x \in \Lambda$ unif.

(2) re-sample ω_x by $\mu(\cdot \mid \omega_{\Lambda\setminus x})$

Output is $\sigma^{i+1} = \omega'$.

μ irred $\Rightarrow$ σ irred. MC.

(Clearly $\sum_{\omega_{\Lambda\setminus x}} \mu(\omega_x^i | \omega_{\Lambda\setminus x}) \mu(\omega_{\Lambda\setminus x}) = \mu(\omega_x^i)$).

Pf. (Holley).

Couple Gibbs samplers σ, τ for μ_1, μ_2, resp.

U_k iid on $[0,1]$ unif.

If $(\sigma^i, \tau^i) = (\xi, \eta)$, define $(\sigma^{i+1}, \tau^{i+1})$ by:

(1) $x \in \Lambda$ unif.

(2) $\sigma_x^{i+1} = \begin{cases} 1, & \mu_1(\delta^x | \xi_{\Lambda\setminus x}) \geq U_i \\ -1, & o/w \end{cases}$

$\tau_x^{i+1} = \begin{cases} 1, & \mu_2(\delta^x | \eta_{\Lambda\setminus x}) \geq U_i \\ -1, & o/w \end{cases}$.

Then $\mathbb{P}(\sigma_x^{i+1} = 1 | \sigma^i = \xi) = \mathbb{P}(\mu_1(\delta^x | \xi_{\Lambda\setminus x}) \geq U_i)$

$= \mu_1(\delta^x | \xi_{\Lambda\setminus x})$

So marginals are Gibbs samplers.

Moreover: wp1 $\exists i$ st. $\sigma^i \leq \tau^i$

(worst case $\exists i$ st $\tau^i \equiv 1$ by irred.)

Then $\sigma^j \leq \tau^j$ $\forall j \geq i$ (by defn. of coupling)

Clearly (σ,τ) aperiodic.

Thus, (σ,τ) has lim. meas. μ

[Supp. on $\{\xi \le \eta\}$ since (σ,τ) not irred.].

μ is a monotone coupling of (μ_1,μ_2). ■

~~Thm. (FKG)~~

Def. μ monotone if $\forall x\in\Lambda$ $\forall \xi\le\eta$ w/ $\mu(\xi_{\Lambda\setminus x}),\mu(\eta_{\Lambda\setminus x})>0$,

$$\mu(\delta^x|\xi_{\Lambda\setminus x})\le\mu(\delta^x|\eta_{\Lambda\setminus x}).$$

Ex. $\mu^{+,\Lambda}$ monotone: computation

Thm. (FKG). $\mu(\omega)>0$ $\forall\omega\in\Omega_\Lambda$.

μ monotone $\Rightarrow$ μ has positive correlations, i.e.

$$\mu(fg)\ge\mu(f)\mu(g)\quad\forall f,g \text{ inc. bdd. } \Omega_\Lambda\to\mathbb{R}$$

Pf. Assume wlog $g>0$ (o/w replace by $g-\min(g)+1$)

Then $\frac{1}{Z}g\,d\mu_1$ is a prob. meas w/ $Z=\mu_1(g)$.

Apply Holley w/ $d\mu_1=d\mu$, $d\mu_2=\frac{1}{Z}g\,d\mu_1$

$$\Rightarrow\mu_1\le\mu_2\Rightarrow\mu_1(fg)=Z\mu_2(f)\ge Z\mu_1(f)=\mu_1(f)\mu_1(g).$$

First verify hypothesis:

$$\frac{\mu_2(\delta^x \mid \xi_{\Lambda\setminus x})}{\mu_2(-\delta^x \mid \xi_{\Lambda\setminus x})} = \frac{\mu_1(\delta^x \xi_{\Lambda\setminus x})\, g(\delta^x \xi_{\Lambda\setminus x})}{\mu_1((-\delta^x)\xi_{\Lambda\setminus x})\, g((-\delta^x)\xi_{\Lambda\setminus x})}$$

$$\geq \frac{\mu_1(\delta^x \xi_{\Lambda\setminus x})}{\mu_1((-\delta^x)\xi_{\Lambda\setminus x})} \qquad (g \text{ inc.})$$

$$= \frac{\mu_1(\delta^x \mid \xi_{\Lambda\setminus x})}{\mu_1(-\delta^x \mid \xi_{\Lambda\setminus x})}$$

$$\overset{(\text{solve})}{\Longrightarrow} \mu_2(\delta^x \mid \xi_{\Lambda\setminus x}) \geq \mu_1(\delta^x \mid \xi_{\Lambda\setminus x}) \qquad \square$$

$$\Rightarrow \mu_1(\delta^x \mid \xi_{\Lambda\setminus x}) \leq \mu_2(\delta^x \mid \xi_{\Lambda\setminus x}) \leq \mu_2(\delta^x \mid \eta_{\Lambda\setminus x}).$$

(monotonicity)

$\underline{\text{Pf}}$. (existence of $\mu^\pm$). For bdd. inc. f, $\Lambda_1 \subset \Lambda_2$

$$\mu^{+,\Lambda_1}(f) = \mu^{+,\Lambda_2}(f \mid \delta^{\Lambda_2\setminus\Lambda_1}) \qquad [\text{Markov}]$$

$$= \frac{\mu^{+,\Lambda_2}(f\, \mathbb{1}_{\delta^{\Lambda_2\setminus\Lambda_1}})}{\mu^{+,\Lambda_2}(\mathbb{1}_{\delta^{\Lambda_2\setminus\Lambda_1}})}$$

$$\geq \mu^{+,\Lambda_2}(f) \qquad [\text{FKG}]. \qquad \blacksquare$$

Other properties.

- $\mu^{\pm}$ are invariant under any $\mathbb{Z}^d$-automorphism T:

  $\forall$ cylinder set C,

$$\mu^+(T^{-1}C) = \lim_\Lambda \mu^{+,\Lambda}(T^{-1}C) = \lim_\Lambda \mu^{+,T\Lambda}(C) = \mu^+(C).$$

- If $\mu^\omega := \lim_\Lambda \mu^{\omega,\Lambda}$ exist for $\omega = \xi, \eta$, then

  $\xi \leq \eta \Rightarrow \mu^\xi \leq \mu^\eta$.

  Pf. Computation shows

$$\mu^{\eta,\Lambda}(f) = \frac{\mu^{\xi,\Lambda}(f\, I^\Lambda_{\xi,\eta})}{\mu^{\xi,\Lambda}(I^\Lambda_{\xi,\eta})} \quad \text{for } I^\Lambda_{\xi,\eta} \text{ bdd cts inc.}$$

  Thus, for f bdd cts inc,

$$\mu^{\eta,\Lambda}(f) \geq \frac{\mu^{\xi,\Lambda}(f)\,\cancel{\mu^{\xi,\Lambda}(I^\Lambda_{\xi,\eta})}}{\cancel{\mu^{\xi,\Lambda}(I^\Lambda_{\xi,\eta})}}.$$

  Then take $\Lambda \to \mathbb{Z}^d$.

- Follows that $\mu^- \leq \mu \leq \mu^+$ $\forall$ Gibbs μ.

- If $\mu^+ = \lambda\mu_1 + (1-\lambda)\mu_2$, $\lambda \in (0,1)$ for Gibbs μ_1, μ_2,
then $\mu_1 = \mu_2 = \mu^+$ ($\mu^\pm$ are <u>extremal</u>).

<u>Pf</u>. $\mu_1 \neq \mu^+ \Rightarrow \mu_1 < \mu^+ \Rightarrow \mu^+ = \lambda\mu_1 + (1-\lambda)\mu_2 < \mu^+ \Rightarrow \Leftarrow$. ■

<u>Note</u>. (1) Aizenman (80), Higuchi (81):
$\mu^\pm$ are the <u>only</u> extremal states <u>if $d=2$</u>.

(2) Dobrushin (72):
This is false if $\underline{d > 2}$.

- TFAE:
(a) There is a unique inf-vol state
(b) $\mu^+ = \mu^-$
(c) $\mu^+(\delta^x) = \mu^-(\delta^x)$

<u>Pf</u>. [Show (c) $\Rightarrow$ (b) $\Rightarrow$ (a).]

(c) $\Rightarrow$ (c) $\forall x$ by translation-invariance.

(c)$\Rightarrow$(b): Strassen $\Rightarrow$ $\exists$ monotone coupling μ of (μ^-, μ^+).
Let $(\sigma^1, \sigma^2) \sim \mu$. So $\sigma^1_x = 1 \Rightarrow \sigma^2_x = 1$ and $\sigma^2_x = -1 \Rightarrow \sigma^1_x = -1$.
Thus, $\mathbb{P}(\sigma^1_x = \sigma^2_x) = \mathbb{P}(\sigma^1_x = 1) + \mathbb{P}(\sigma^2_x = -1) \overset{(c)}{=} \mathbb{P}(\sigma^2_x = 1) + \mathbb{P}(\sigma^2_x = -1) = 1$ $\forall x \Rightarrow$ (b).

(b) $\Rightarrow$ (a) by $\mu^- \le \mu \le \mu^+$. ■

Pressure + Magnetization

Recall.

- $\mu^{\pm}_{\beta,h} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu^{\pm,\Lambda}_{\beta,h}$ exist $\forall \beta > 0,\ h \in \mathbb{R}$
- Phase transition occurs $\iff \langle \sigma_0 \rangle^+ \neq \langle \sigma_0 \rangle^-$

How do we relate this to spontaneous magnetization?

Pressure.

Let $\psi^{\eta,\Lambda}(\beta, h) = \frac{1}{|\Lambda|} \log Z^{\eta,\Lambda}_{\beta,h}$

Hölder $\Rightarrow Z^{\eta,\Lambda}_{\alpha\beta_1 + (1-\alpha)\beta_2,\ \alpha h_1 + (1-\alpha) h_2} \leq \left(Z^{\eta,\Lambda}_{\beta_1,h_1}\right)^{\alpha} \left(Z^{\eta,\Lambda}_{\beta_2,h_2}\right)^{1-\alpha}$

$\Rightarrow \psi^{\eta,\Lambda}$ convex

Thm. The pressure $\psi = \lim_{\Lambda \uparrow \mathbb{Z}^d} \psi^{\eta,\Lambda}$ exists, ~~and~~ is indep. of η, convex, and even in h.

Here, $\Lambda \Uparrow \mathbb{Z}^d$ is van Hove convergence, ie. $\Lambda_n \uparrow \mathbb{Z}^d$ w/ $|\partial \Lambda_n| / |\Lambda_n| \to 0$ ($\partial\Lambda$ = interior bdry for today).

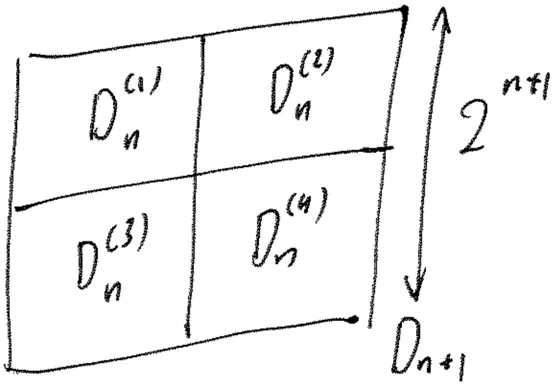
Pf. (A) Free bdry conditions

i.e. $H^{\phi,\Lambda} = -\beta\left[\sum_{x\sim y}\omega_x\omega_y + h\sum_x \omega_x\right]$.

(1) Convergence along exponential boxes.

Let $D_n = \{1,\dots,2^n\}^d$

Then $D_{n+1} = \bigcup_{i=1}^{2^d} D_n^{(i)}$ w/ $D_n^{(i)}$ translates of D_n



$$H^{\phi,D_{n+1}} = \sum_{i=1}^{2^d} H^{\phi,D_n^{(i)}} + R_n, \quad |R_n| \le O\left((2^{n+1})^{d-1}\right)$$

$$\Rightarrow Z^{\phi,D_{n+1}} \le e^{O((2^{n+1})^{d-1})} \underbrace{\sum_{\omega\in\Omega_{D_{n+1}}} \prod_{i=1}^{2^d} e^{-H^{\phi,D_n^{(i)}}(\omega)}}_{\prod_{i=1}^{2^d}\sum_{\omega^{(i)}\in\Omega_{D_n^{(i)}}} e^{-H^{\phi,D_n^{(i)}}(\omega^{(i)})}} = \left(Z^{\phi,D_n}\right)^{2^d}$$

$$\Rightarrow \psi^{\phi,D_{n+1}} \le O\left(2^{-(n+1)}\right) + \psi^{\phi,D_n}$$

Sim., $\psi^{\phi, D_n} \leq O(2^{-(n+1)}) + \psi^{\phi, D_{n+1}}$

i.e. $|\psi^{\phi, D_n} - \psi^{\phi, D_{n+1}}| \leq O(2^{-(n+1)})$.

Iterated Δ inequality $\Rightarrow$ ψ^{ϕ, D_n} Cauchy.

Write $\psi = \lim_n \psi^{\phi, D_n}$.

(2) <u>Convergence on arbitrary $\Lambda_n \uparrow \mathbb{Z}^d$.</u>

Cover $\mathbb{Z}^d$ by translates of D_k

Let $[\Lambda_n] = \bigcup_{j=1}^{N(n)} D_k^{(j)}$ w/ $\{D_k^{(j)}\}$ a min. covering of Λ_n by translates of D_k

Write

$$|\psi^{\phi, \Lambda_n} - \psi| \leq \underbrace{|\psi^{\phi, \Lambda_n} - \psi^{\phi, [\Lambda_n]}|}_{a} + \underbrace{|\psi^{\phi, [\Lambda_n]} - \psi^{\phi, D_k}|}_{b} + \underbrace{|\psi^{\phi, D_k} - \psi|}_{c}$$

- Then $c \xrightarrow{k \to \infty} 0$ by (1).
- Bound on b sim. to (1):

$$H^{\phi, [\Lambda_n]} \leq \sum_{j=1}^{N(n)} H^{\phi, D_k^{(j)}} + \underbrace{O(N(n)(2^k)^{d-1})}_{O(|[\Lambda_n]| 2^{-k})}$$

$|[\Lambda_n]| / |D_k| = |[\Lambda_n]| / (2^k)^d$

$$\Rightarrow Z^{\Phi,[\Lambda_n]} \leq \left(Z^{\Phi,D_k}\right)^{N(n)} e^{O(|[\Lambda_n]|2^{-k})}$$

$$\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \underbrace{\frac{N(n)}{|[\Lambda_n]|}}_{\to \frac{1}{|D_k|}} \log Z^{\Phi,D_k} + O(2^{-k})$$

$$= \psi^{\Phi,D_k} + O(2^{-k}) \xrightarrow[\text{by (1)}]{k\to\infty} \psi$$

(Sim. lower bd.)

- For (a) let $\Delta_n = [\Lambda_n] \setminus \Lambda_n$

$$|H^{\Phi,\Lambda_n} - H^{\Phi,[\Lambda_n]}| \leq O(|\Delta_n|)$$

$$\Rightarrow Z^{\Phi,[\Lambda_n]} \leq \underbrace{\sum_{\omega\in\Omega_{\Lambda_n}} e^{-H^{\Phi,\Lambda_n}(\omega)}}_{Z^{\Phi,\Lambda_n}} \underbrace{\sum_{\tilde\omega\in\Omega_{\Delta_n}} e^{O(|\Delta_n|)}}_{2^{|\Delta_n|}e^{O(|\Delta_n|)}} = e^{O(|\Delta_n|)}$$

$$\cancel{\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \psi^{\Phi,\Lambda_n} + O\left(\frac{|\Delta_n|}{|[\Lambda_n]|}\right)}$$

$$\Rightarrow \psi^{\Phi,[\Lambda_n]} \leq \underbrace{\frac{|\Lambda_n|}{|[\Lambda_n]|}}_{\leq 1} \cdot \psi^{\Phi,\Lambda_n} + \underbrace{O\left(\frac{|\Delta_n|}{|[\Lambda_n]|}\right)}_{O\left(\frac{|\partial\Lambda_n||D_k|}{|[\Lambda_n]|}\right) \leq O\left(\frac{|\partial\Lambda_n||D_k|}{|\Lambda_n|}\right) \xrightarrow{n\to\infty} 0}$$

(Sim. opposite bd).

(B) Independence of bdry condition.

Use $|H^{\eta,\Lambda} - H^{\emptyset,\Lambda}| \leq O(|\partial\Lambda|)$

w/ van Hove.

(C) Convexity / evenness.

Preserved under limit.

Magnetization.

$$m_\Lambda = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x, \qquad m^{\eta,\Lambda} = \langle m_\Lambda\rangle^{\eta,\Lambda}$$

Note. Cumulant-gen. fcn. of $\sum_{x\in\Lambda}\sigma_x$ is

$$\log\langle e^{t\sum\sigma_x}\rangle^{\eta,\Lambda}_{\beta,h} = |\Lambda|\left(\psi^{\eta,\Lambda}(\beta,h+t)-\psi^{\eta,\Lambda}(\beta,h)\right).$$

In particular, $m^{\eta,\Lambda} = \dfrac{\partial\psi^{\eta,\Lambda}}{\partial h}$.

Does this hold in thermodynamic limit?

Let $\mathcal{B}_\beta = \{h : \psi(\beta, \cdot) \text{ not differentiable}\}$

$= \{h : \frac{\partial \psi}{\partial h^-} \neq \frac{\partial \psi}{\partial h^+}\}$ by convexity.

Cor. For $h \notin \mathcal{B}_\beta$, $m^\eta = \lim_{\Lambda \uparrow \mathbb{Z}^d} m^{\eta,\Lambda}$ exists, is indep. of η, and equals $\frac{\partial \psi}{\partial h}$. Moreover, it is cts and nondecreasing in h off $\mathcal{B}_\beta$ and discts. on $\mathcal{B}_\beta$, and the spontaneous magnetization $m^*(\beta) = \lim_{h \downarrow 0} m(\beta, h)$ exists.

$\lim_{h' \downarrow \uparrow h} m = \frac{\partial \psi}{\partial h^\pm}$

Pf. (A) Existence/indep. of η.

Interchange lims. (justified by convexity).

(B) Cts/monotone.

By convexity of ψ

(C) Left/right lims.

(A) + direct computation. ∎

Prop. $m^{\pm} = \lim_{\Lambda \nearrow \mathbb{Z}^d} m^{\pm,\Lambda}$ exist and equal $\langle \sigma_0 \rangle^{\pm}$.

Pf. $\langle \sigma_0 \rangle^{+} = \langle m_{\Lambda_n} \rangle^{+}$ (trans-inv)

$\leq \langle m_{\Lambda_n} \rangle^{+,\Lambda_n}$ (m_{Λ_n} inc fn)

$\Rightarrow \langle \sigma_0 \rangle^{+} \leq \liminf_n \underbrace{\langle m_{\Lambda_n} \rangle^{+,\Lambda_n}}_{m^{+,\Lambda_n}}$

Conversely, fix $k \geq 1$, $x \in \Lambda_n$.

$x + \underset{\substack{\| \\ \{-k,\dots,k\}^d}}{B(k)} \subset \Lambda_n \Rightarrow \langle \sigma_x \rangle^{+,\Lambda_n} \leq \langle \sigma_x \rangle^{+,x+B(k)} = \langle \sigma_0 \rangle^{+,B(k)}$.

Thus, $\langle m_{\Lambda_n} \rangle^{+,\Lambda_n} = \frac{1}{|\Lambda_n|} \sum_{\substack{x \in \Lambda_n \\ x+B(k) \subset \Lambda_n}} \langle \sigma_x \rangle^{+,\Lambda_n} + \frac{1}{|\Lambda_n|} \sum_{x+B(k) \not\subset \Lambda_n} \overbrace{\langle \sigma_x \rangle^{+,\Lambda_n}}^{\leq 1}$

$\leq \langle \sigma_0 \rangle^{+,B(k)} + \frac{1}{|\Lambda_n|} \underbrace{\left(|B(k)| \, |\partial \Lambda| \right)}_{\text{upper bd on \# ways } x+B(k) \text{ can intersect } \partial\Lambda}$

$\Rightarrow \limsup_n \langle m_{\Lambda_n} \rangle^{+,\Lambda_n} \leq \langle \sigma_0 \rangle^{+}$ by vHove. $\square$

Lem. (1) $\langle\sigma_0\rangle^+_{\beta,h}$ nondec/right-cts in h

(2) $h\geq 0 \Rightarrow \langle\sigma_0\rangle^+_{\beta,h}$ nondec. in β

Pf. ~~FKG~~

(1) $\frac{\partial}{\partial h}\langle\sigma_0\rangle^{+,\Lambda} = \sum_{x\in\Lambda}\underbrace{\left(\langle\sigma_0\sigma_x\rangle^{+,\Lambda} - \langle\sigma_0\rangle^{+,\Lambda}\langle\sigma_x\rangle^{+,\Lambda}\right)}_{\geq 0 \text{ by FKG}}$ by quotient rule

(Take $\Lambda\uparrow\mathbb{Z}^d$)

$\Rightarrow$ nondecreasing.

Right-cts by interchanging lims

(2) Sim. but use GKS ineq.: $\langle\sigma_0\sigma_x\sigma_y\rangle^{+,\Lambda} \geq \langle\sigma_0\rangle^{+,\Lambda}\langle\sigma_x\sigma_y\rangle^{+,\Lambda}$. $\square$

Note. Take $h=0$. Then $\langle\sigma_0\rangle^+_{\beta,0} = -\langle\sigma_0\rangle^-_{\beta,0}$.

~~Thus, uniqueness $\Leftrightarrow \langle\sigma_0\rangle^+_{\beta,0}=0$~~ Thus, uniqueness $\Leftrightarrow \langle\sigma_0\rangle^+_{\beta,0}=0$.

But $m^*(\beta) = m(\beta,0^+) = m^+(\beta,0^+)$ (since $m=m^+$ at all but countably many pts)

$=\langle\sigma_0\rangle^+$

So uniqueness $\Leftrightarrow$ $m^*=0$.

Moreover, $m^* = \langle\sigma_0\rangle^+$ is monotone in β, so let

$$\beta_c = \inf(\beta : m^*>0) = \sup(\beta : m^*=0).$$

Thm. $\dfrac{\partial \psi}{\partial h^{\pm}} = m^{\pm}$.

Pf. $\dfrac{\partial \psi}{\partial h^{+}} = \lim\limits_{h' \downarrow h} \dfrac{\partial \psi}{\partial h}(\beta, h')$ ($\mathcal{B}_{\beta}^{h'}$ countable)

$= \lim\limits_{h' \downarrow h} m(\beta, h') = \lim\limits_{h' \downarrow h} m^{+}(\beta, h') = m^{+}(\beta, h)$. ∎

right-cts.

Thus, uniqueness $\iff m^{+} = m^{-} \iff \dfrac{\partial \psi}{\partial h}$ exists.

Note. Non-uniqueness is a first-order phase transition.

More generally, we have a n^{th}-order PT if

$\dfrac{\partial \psi}{\partial h^{k}}$ exists iff $k < n$.

Usually only distinguish btwn cts $(n > 1)$ and

discts $(n = 1)$ PTs.

High-temperature regime

Recall.

- **Pressure** $\psi(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z^{\eta,\Lambda}_{\beta,h}$ is indep of η and convex.
- **Magnetization** $m^{\pm}(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \left\langle \sum_{x\in\Lambda} \sigma_x \right\rangle^{\pm,\Lambda}_{\beta,h}$

$$= \langle \sigma_0 \rangle^{\pm}_{\beta,h} = \frac{\partial \psi}{\partial h^{\pm}}(\beta,h)$$

- **Spontaneous magnetization** $m^*(\beta) = \lim_{h\downarrow 0} m^+(\beta,h)$
- **Critical point** $\beta_c = \inf(\beta : m^*(\beta) > 0)$.
- A **first-order/discts** PT (ie $\mu^+ \neq \mu^-$) occurs iff $\frac{\partial \psi}{\partial h^-} \neq \frac{\partial \psi}{\partial h^+}$ (equiv. to $m^* > 0$ when $h=0$).

Today. $\beta_c > 0$ ie there is a single-phase region $\{(\beta,0) : \beta < \beta_c\}$

High-temp expansion:

ie expand about $\beta = 0$

$$e^{\pm\beta} = \cosh\beta \pm \sinh\beta \iff e^{\beta\omega_x\omega_y} = \cosh\beta + \omega_x\omega_y \sinh\beta$$

$$= \cosh\beta\,(1 + \omega_x\omega_y \tanh\beta).$$

Let $\Omega^{+,\Lambda} = \{\omega \in \Omega : \omega_x = 1 \ \forall x \notin \Lambda\}$

Fix $h = 0$, $\omega \in \Omega^{+,\Lambda}$, write

$$H_\beta^\Lambda(\omega) = H_{\beta,0}^{+,\Lambda}(\omega) = -\beta\Big[\sum_{\substack{x\sim y\\ x,y\in\Lambda}} \omega_x\omega_y + \sum_{\substack{x\sim y\\ x\in\Lambda\\ y\notin\Lambda}} \omega_x \overset{+1}{\overset{\parallel}{\omega_y}}\Big]$$

$$= -\beta\sum_{x\sim y} \omega_x\omega_y .$$

Then $$e^{-H_\beta^\Lambda(\omega)} = \prod_{x\sim y} e^{\beta\omega_x\omega_y} = (\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|}\prod_{x\sim y}(1+\omega_x\omega_y\tanh\beta)$$

where $\mathcal{E}(\bar\Lambda) = \{\text{edges in } \bar\Lambda\}$ and $\bar\Lambda = \Lambda \cup \partial^{ext}\Lambda$.

Expand product using <u>binomial theorem</u>:

$|S| < \infty$, $f, g : S \to \mathbb{R}$

$$\prod_{s\in S}\big(f(s)+g(s)\big) = \sum_{A\subset S}\Big(\prod_{a\in A} f(a)\Big)\Big(\prod_{b\notin A} g(b)\Big).$$

Thus,

$$e^{-H_\beta^\Lambda(\omega)} = (\cosh\beta)^{|\mathcal{E}(\Lambda)|}\sum_{E\subset\mathcal{E}(\Lambda)}\underbrace{\prod_{xy\in E}\omega_x\omega_y\tanh\beta}_{(A)}.$$

With

$$(A) = (\tanh\beta)^{|E|}\prod_{x\in\Lambda}\omega_x^{I(x,E)}$$

where $I(x,E) = \#\{y\in\mathbb{Z}^d : xy\in E\}$, get

$$Z_{\beta,0}^{+,\Lambda} = (\cosh\beta)^{|\mathcal{E}(\Lambda)|}\sum_{E\subset\mathcal{E}(\Lambda)}(\tanh\beta)^{|E|}\underbrace{\sum_{\omega\in\Omega^{+,\Lambda}}\prod_{x\in\Lambda}\omega_x^{I(x,E)}}_{(B)}$$

But $(B) = \prod_{x\in\Lambda} \sum_{\omega_x=\pm 1} \omega_x^{I(x,E)} = \prod_{x\in\Lambda} 2\mathbb{1}(I(x,E) \text{ even})$

$$\Rightarrow Z^{+,\Lambda}_{\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\Lambda)|} \sum_{E\subset\mathcal{E}(\Lambda)_e} (\tanh\beta)^{|E|}$$

where $\mathcal{E}(\Lambda)_e = \{E\subset\mathcal{E}(\Lambda) : I(x,E) \text{ even } \forall x\in\Lambda\}$.

(High-temp representation van der Waerden 1941)

Sim.,

$$\sum_{\omega\in\Omega^{+,\Lambda}} \omega_0 e^{-H^\Lambda_\beta(\omega)}$$

$$= (\cosh\beta)^{|\mathcal{E}(\Lambda)|} \sum_{E\subset\mathcal{E}(\Lambda)} (\tanh\beta)^{|E|} \underbrace{\sum_{\omega\in\Omega^{+,\Lambda}} \omega_0 \prod_{x\in\Lambda} \omega_x^{I(x,E)}}_{(C)}$$

$$\text{w/ } (C) = \underbrace{\left(\sum_{\omega_0=\pm 1} \omega_0^{1+I(0,E)}\right)}_{2\mathbb{1}(I(0,E)\text{ odd})} \left(\prod_{x\neq 0} \sum_{\omega_x=\pm 1} \omega_x^{I(x,E)}\right)$$

Thus,

$$\langle \sigma_0 \rangle_{\beta,0}^{+,\Lambda} = \frac{\sum_{E \subset \mathcal{E}(\bar\Lambda)_0} (\tanh\beta)^{|E|}}{\sum_{E \subset \mathcal{E}(\bar\Lambda)_e} (\tanh\beta)^{|E|}}$$

w/ $\mathcal{E}(\bar\Lambda)_0 = \{ E \subset \mathcal{E}(\bar\Lambda) : I(x,E) \text{ even } \forall x \in \Lambda \setminus \{0\},\ I(0,E) \text{ odd} \}$.

Decompose $E \subset \mathcal{E}(\bar\Lambda)_0$ as $E = E_0 \cup E'$

w/ E_0 the max. conn. component of 0

and $E' \in \mathcal{E}(\bar\Lambda)_e$ in the "complement" E^* of E_0.

Then

$$\langle \sigma_0 \rangle = \sum_{0 \in E_0 \in \mathcal{E}(\bar\Lambda)_0 \text{ conn.}} (\tanh\beta)^{|E_0|} \underbrace{\frac{\sum_{\substack{E' \in \mathcal{E}(\bar\Lambda)_e \\ E' \subset E_0^*}} (\tanh\beta)^{|E'|}}{\sum_{E \in \mathcal{E}(\bar\Lambda)_e} (\tanh\beta)^{|E|}}}_{\le 1}$$

Bounds. On any connected graph, from any vertex, is a path crossing each edge exactly twice (induction)

Thus, $\#\left\{\begin{matrix}\text{summands in } \langle\sigma_0\rangle \\ \text{w/ } \ell \text{ edges}\end{matrix}\right\} \leq \#\left\{\begin{matrix}\text{paths from } 0 \text{ in } \mathbb{Z}^d \\ \text{of length } 2\ell\end{matrix}\right\} = (2d)^{2\ell} = (4d^2)^\ell$

Since $\sum_x I(x, E_0) = 2|E_0|$ even + $I(0,E_0)$ odd,

$\exists y \neq 0$ w/ $I(y, E_0)$ odd.

But $I(y, E_0)$ even $\forall y \in \Lambda \setminus \{0\}$ $\Rightarrow y \notin \Lambda$.

That is, $0 \overset{E_0}{\longleftrightarrow} y$.

Setting $\Lambda = B(n) = \{-n, \ldots, n\}^d$, get $|E_0| \geq n$.

Therefore,

$$\langle\sigma_0\rangle \leq \sum_{\ell \geq n} \underbrace{(\tanh\beta)^\ell}_{\leq \beta^\ell} (4d^2)^\ell \leq \sum_{\ell \geq n} (4d^2\beta)^\ell .$$

$\beta < \frac{1}{4d^2} \Rightarrow$ series converges $\Rightarrow$ tail $\xrightarrow{n\to\infty} 0$ i.e. $\langle\sigma_0\rangle^+ = 0$

So $\beta_c \geq \frac{1}{4d^2} > 0$.

1D case.

If $d=1$, $\Lambda = B(n)$,

$$\mathcal{E}(\bar{\Lambda})_e = \{\emptyset, \mathcal{E}(\bar{\Lambda})\}$$

$$\mathcal{E}(\bar{\Lambda})_o = \{\text{line graphs on } \{-n-1, \dots, 0\} \text{ and } \{0, \dots, n+1\}\}$$

Thus,

$$\langle \sigma_0 \rangle = \frac{\sum_{E \in \mathcal{E}(\bar{\Lambda})_o} (\tanh\beta)^{|E|}}{\sum_{E \in \mathcal{E}(\bar{\Lambda})_e} (\tanh\beta)^{|E|}}$$

$$= \frac{2(\tanh\beta)^{n+1}}{1+(\tanh\beta)^{2(n+1)}} \xrightarrow{n\to\infty} 0 \quad \forall \beta < \infty.$$

$\Rightarrow \beta_c = \infty$ if $d=1$. (also possible by exact soln. via transfer matrix $\Leftarrow$ Lenz)

Next time: $d > 1 \Rightarrow \beta_c < \infty$.

[In fact, $d=1 \Rightarrow \psi = \log\left[e^{\beta}\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}\right]$]

Low temperature and cluster expansion.

Recall.

- High-temp expansion

$$Z^{+,\Lambda}_{\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\bar\Lambda)|}\sum_{E\in\mathcal{E}(\bar\Lambda)_e}(\tanh\beta)^{|E|}$$

w/ $\bar\Lambda = \Lambda \cup \partial^{ext}\Lambda$

$$\mathcal{E}(\bar\Lambda)_e = \{E\subset\mathcal{E}(\bar\Lambda) : I(x,E) \text{ even } \forall x\in\Lambda\}$$

- $\beta_c > 0$ (uniqueness at high temp)
- $d=1 \Rightarrow \beta_c = \infty$ (no phase transition)

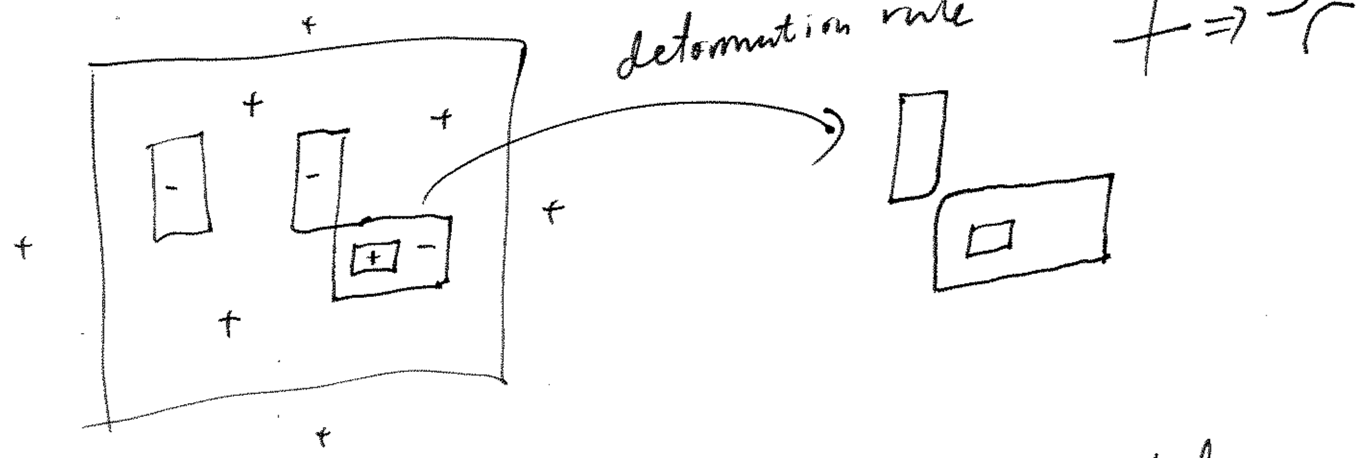
Low-temperature expansion: $(d=2)$

$$H^{\Lambda}_{\beta,0}(\omega) = -\beta\sum_{xy\in\mathcal{E}(\bar\Lambda)} \omega_x\omega_y = -\beta|\mathcal{E}(\bar\Lambda)| + \sum_{xy\in\mathcal{E}(\bar\Lambda)}\beta\underbrace{\overbrace{(1-\omega_x\omega_y)}^{\in\{0,2\}}}_{2\mathbb{1}(\omega_x\neq\omega_y)}$$

$$= -\beta|\mathcal{E}(\bar\Lambda)| + 2\beta\,\#\{xy : \omega_x\neq\omega_y\}, \qquad \omega\in\Omega^+_\Lambda$$

Associate to $\omega\in\Omega^+_\Lambda$ the corresponding set

$\Gamma(\omega) = \{\gamma_1,\dots,\gamma_n\}$ of <u>contours</u> in the dual lattice $\mathbb{Z}^*_2$



Then $\omega_x \neq \omega_y$ iff x and y are separated by a dual edge $xy_\perp$ in γ_* for some $\gamma_* \in \Gamma(\omega)$.

That is,

$$\#\{xy : \omega_x \neq \omega_y\} = \sum_{\gamma \in \Gamma(\omega)} |\gamma|$$

where $|\gamma| = \#\{\text{dual edges in } \gamma\}$

Thus, $H(\omega) = -\beta|\mathcal{E}(\bar{\Lambda})| + 2\beta \sum_{\gamma \in \Gamma(\omega)} |\gamma|$

$$\Rightarrow Z^{+,\Lambda}_{\beta,0} = e^{\beta|\mathcal{E}(\bar{\Lambda})|} \sum_{\omega \in \Omega^+_\Lambda} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}$$

Plan. Show $\mu^{+,B(n)}_{\beta,0}(\sigma_0 = -1) \le \delta(\beta) \downarrow 0$ as $\beta \to \infty$

Then $\underset{m^*(\beta)}{\overset{\|}{\langle \sigma_0 \rangle^{+,\cancel{B(n)}}_{\beta,0}}} = \mu^+_{\beta,0}(\sigma_0 = 1) - \mu^+_{\beta,0}(\sigma_0 = -1)$

$= 1 - 2\mu^+_{\beta,0}(\sigma_0 = -1) \ge 1 - 2\delta(\beta) > 0$ for β large.

As w/ $Z^{+,\Lambda}_{\beta,0}$, can show

$$\mu^{+,\Lambda}_{\beta,0}(\omega) = \frac{1}{Z^{+,\Lambda}_{\beta,0}} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|} . \quad (*)$$

Now set $\Lambda = B(n)$.

If $\omega_0 = -1$, $\exists\, \gamma_* \in \Gamma(\omega) : \mathrm{Int}(\gamma_*) \ni 0$.

Thus, $\mu^{+,B(n)}_{\beta,0}(\sigma_0 = -1) \leq \sum_{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0} \mu^{+,B(n)}_{\beta,0}(\Gamma \ni \gamma_*)$

<u>Lem</u>. For any contour γ_*,

$$\mu^{+,B(n)}(\Gamma \ni \gamma_*) \leq e^{-2\beta|\gamma_*|} .$$

<u>Pf</u>. By $(*)$,

$$\mu^{+,B(n)}(\Gamma \ni \gamma_*) = \sum_{\omega : \Gamma(\omega) \ni \gamma_*} \mu^{+,B(n)}(\omega)$$

$$= e^{-2\beta|\gamma_*|} \frac{\sum_{\Gamma(\omega) \ni \gamma_*} \prod_{\gamma \in \Gamma \setminus \{\gamma_*\}} e^{-2\beta|\gamma|}}{\sum_{\omega} \prod_{\gamma \in \Gamma} e^{-2\beta|\gamma|}}$$

↳ Show ≤ 1.

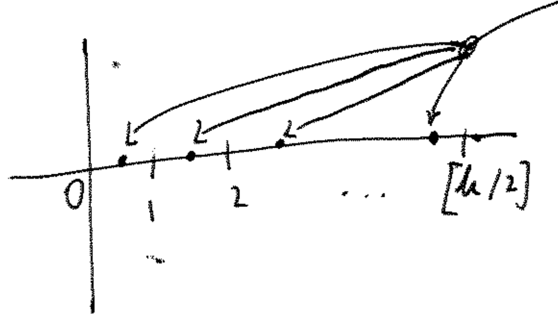
Let $\mathcal{C}(\gamma_*) = \{$ configs. $\hat{\omega}'$ obtainable by "removing" γ_* from some $\omega\}$

("remove" γ_* = flip spins in $\mathrm{Int}(\gamma_*)$)

Then
$$\sum_{\omega:\Gamma(\omega)\ni\gamma_*} \prod_{\gamma\in\Gamma(\omega)\setminus\{\gamma_*\}} e^{-2\beta|\gamma|} = \sum_{\omega'\in\mathcal{C}(\gamma_*)} \prod_{\gamma'\in\Gamma(\omega')} e^{-2\beta|\gamma'|}.$$

Thus, ratio ≤ 1. $\square$

Follows that
$$\mu^{+,B(n)}(\sigma_0 = -1) \le \sum_{\gamma_*:\mathrm{Int}(\gamma_*)\ni 0} e^{-2\beta|\gamma_*|}$$
$$= \sum_{k\ge 4} \sum_{\substack{\mathrm{Int}(\gamma_*)\ni 0\\ |\gamma_*|=k}} e^{-2\beta k}$$
$$= \sum_{k\ge 4} e^{-2\beta k}\,\#\{\gamma_* : \mathrm{Int}(\gamma_*)\ni 0,\ |\gamma_*|=k\}.$$



Contour about 0 of length k must cross one

Since # contours of length k from some vertex is ~~3×4~~ $4 \times 3^{k-1}$, get

$$\mu^{+,B(n)}(\sigma_0 = -1) \le \sum_{k \ge 4} e^{-2\beta k} \times \frac{k}{2} \times 4 \times 3^{k-1}$$

$$= \frac{2}{3} \sum_{k \ge 4} k\, 3^k e^{-2\beta k} = \delta(\beta)$$

and $\delta(\beta) \downarrow 0$ iff $\delta(\beta) < \infty$ iff $3e^{-2\beta} < 1$

ie. $\beta > \frac{1}{2} \log 3$

Note. Exact soln. $\Rightarrow \beta_c(2) = \frac{1}{2} \operatorname{arcsinh}(1) \approx 0.441$

(cf. $\frac{1}{2} \log 3 \approx$ ~~...~~ 0.549)

Cluster expansion:

Γ finite set, elements $\gamma \in \Gamma$ called polymers

weight (or activity) $w : \Gamma \to \mathbb{R}$ (or $\mathbb{C}$)

interaction $\delta : \Gamma \times \Gamma \to [-1, 1]$ symm.

st. $\delta(\gamma, \gamma) = 0$.

Def. (Polymer model).

Assign to $\Gamma' \subset \Gamma$ the probability

$$\frac{1}{\Xi}\left(\prod_{\gamma\in\Gamma'} w(\gamma')\right)\left(\prod_{\{\gamma,\gamma'\}\subset\Gamma'} \delta(\gamma,\gamma')\right)$$

where the polymer partition fun is

$$\Xi = \sum_{\Gamma'\subset\Gamma} (\ldots)(\ldots)$$

$$= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1\in\Gamma} \cdots \sum_{\gamma_n\in\Gamma} \left(\prod_{i=1}^{n} w(\gamma_i)\right)\left(\prod_{i<j} \delta(\gamma_i,\gamma_j)\right).$$

Ex. (1) High-temp.

$$Z = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{E}(\Lambda)|} \underbrace{\sum_{E\in\mathcal{E}(\Lambda)_e} (\tanh\beta)^{|E|}}_{\Xi}$$

$$\Xi = 1 + \sum_n \frac{1}{n!} \sum_{E_1} \cdots \sum_{E_n} \left(\prod w(E_i)\right)\left(\prod \delta(E_i, E_j)\right)$$

w/ $w(E_i) = (\tanh\beta)^{|E_i|} \mathbb{1}(E_i \in \mathcal{E}(\Lambda)_e \text{ connected})$

(so $E_1 \cup \ldots \cup E_n$ is decomp into connected components)

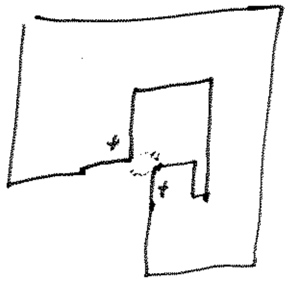
and $\delta(E_i, E_j) = \mathbb{1}(E_i, E_j \text{ share no vertices})$

(so this decomposition is maximal).

(2). Low-temp.

Λ "reasonable" eg. $B(n)$

avoid eg.



Then a collection $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of Peierls contours can arise from a config ω iff $\gamma_i \cap \gamma_j = \emptyset \; \forall i \neq j$.

Let $\Gamma_\Lambda = \{$all possible contours$\}$.

$$Z^{+,\Lambda} = e^{\beta|E(\Lambda)|}\, \Xi$$

$$\Xi = 1 + \sum_n \frac{1}{n!} \sum_{\gamma_1 \in \Gamma_\Lambda} \cdots \sum_{\gamma_n \in \Gamma_\Lambda} \left(\prod_i w(\gamma_i)\right)\left(\prod_{i<j} \delta(\gamma_i, \gamma_j)\right)$$

w/ $w(\gamma) = e^{-2\beta|\gamma|}$

$$\delta(\gamma_i, \gamma_j) = \mathbb{1}(\gamma_i \cap \gamma_j = \varphi)$$

<u>Next time.</u>

$$\log \Xi = \sum_{m \geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi_m(\gamma_1, \dots, \gamma_m) \prod_{i=1}^{m} w(\gamma_i)$$

$$\text{w/ } \varphi_m(\gamma_1, \dots, \gamma_m) = \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn.}}} \prod_{ij \in G} (\delta(\gamma_i, \gamma_j) - 1)$$

and G_m = complete graph on m vertices

(The φ_m are called <u>Ursell fcns</u>)

Cluster expansion

Recall.

Polymer partition fcn

$$\Xi = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1\in\Gamma} \cdots \sum_{\gamma_n\in\Gamma} \left(\prod_{i=1}^{n} w(\gamma_i)\right)\left(\prod_{i<j} \delta(\gamma_i,\gamma_j)\right)$$

where $|\Gamma|<\infty$, $w:\Gamma\to\mathbb{R}$, $\delta:\Gamma\times\Gamma\to[-1,1]$ symm

and $\delta(\gamma,\gamma)=0 \quad \forall \gamma\in\Gamma$.

Today. Compute $\log \Xi$.

Let G_n be the complete graph on $V_n=\{1,\dots,n\}$

w/ edges E_n.

Write $\prod_{i=1}^{n} = \prod_{i\in V_n}$ and $\prod_{i<j} = \prod_{ij\in E_n}$

Let $\zeta(\gamma,\gamma') = \delta(\gamma,\gamma')-1$

Binomial thm $\Rightarrow \prod_{ij\in E_n} \delta(\gamma_i,\gamma_j) = \sum_{E\subset E_n} \prod_{ij\in E} \zeta(\gamma_i,\gamma_j)$

Thus,
$$\Xi = 1 + \sum_{n} \frac{1}{n!} \sum_{G\subset G_n} \underbrace{\sum_{\gamma_1} \cdots \sum_{\gamma_n} \left(\prod_{i\in V_n} w(\gamma_i)\right)\left(\prod_{ij\in E} \zeta(\gamma_i,\gamma_j)\right)}_{Q[G],\ G=(V_n,E)}$$

Note. $Q[G] = Q[G']$ if $G \simeq G'$

and if G has max conn components $G = (G_1', \dots, G_k')$

(write $G = (G_1', \dots, G_k')$), then $Q[G] = \prod_{r=1}^{k} Q[G_r']$.

Thus,

$$\frac{1}{n!} \sum_{G \subset G_n} Q[G] = \sum_{k=1}^{n} \frac{1}{k!} \sum_{m_1 + \dots + m_k = n} \cancel{\frac{n!}{m_1! \dots m_k!}} \sum_{\substack{G_1' \subset G_{m_1} \\ \text{conn}}} \dots \sum_{\substack{G_k' \subset G_{m_k} \\ \text{conn}}} \prod_{r=1}^{k} Q[G_r']$$

$$= \sum_{k \le n} \frac{1}{k!} \sum_{m_1 + \dots + m_k = n} \cancel{\sum_{\substack{G_1' \subset G_{m_1} \\ \text{conn}}} \dots \sum} \prod_{r=1}^{k} \left(\frac{1}{m_r!} \sum_{\substack{G_r' \subset G_{m_r} \\ \text{conn}}} Q[G_r'] \right)$$

Formally,

$$\sum_{n} \sum_{k \le n} \sum_{m_1 + \dots + m_k = n} = \sum_{k} \sum_{n \ge k} \sum_{m_1 + \dots + m_k = n} = \sum_{k} \sum_{m_1, \dots, m_k}$$

Thus,

$$\Xi = 1 + \sum_{k} \frac{1}{k!} \sum_{m_1, \dots, m_k} \prod_{r=1}^{k} \left(\frac{1}{m_r!} \sum_{\substack{G_r' \subset G_{m_r} \\ \text{conn}}} Q[G_r'] \right).$$

$$= 1 + \sum_{k} \frac{1}{k!} \left(\sum_{m} \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} Q[G] \right)^{k}$$

$$= \exp\left(\sum_{m} \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} Q[G] \right).$$

Lastly, ~~we~~ define the Ursell fns

$$\varphi_m(\gamma_1,\dots,\gamma_m) = \frac{1}{m!}\sum_{\substack{G\subset G_m\\ \text{conn}}}\prod_{ij\in G}\zeta(\gamma_i,\gamma_j)$$

so that

$$\frac{1}{m!}\sum_{\substack{G\subset G_m\\ \text{conn}}}\mathcal{Z}[G] = \sum_{\gamma_1}\dots\sum_{\gamma_m}\varphi_m(\gamma_1,\dots,\gamma_m)\prod_{i\in V_m}w(\gamma_i)$$

i.e. $\Xi = \exp\left(\sum_m\sum_{\gamma_1}\dots\sum_{\gamma_m}\varphi_m(\gamma_1,\dots,\gamma_m)\prod_{i\in V_m}w(\gamma_i)\right).$

Convergence

Thm. (Ueltschi '04).

If $\exists\, a:\Gamma\to(0,\infty)$ s.t. $\forall \gamma_*\in\Gamma$,

$$\sum_{\gamma}|w(\gamma)|e^{a(\gamma)}|\zeta(\gamma,\gamma_*)| \le a(\gamma_*),$$

then $\forall \gamma_1\in\Gamma$,

$$1+\sum_{k\ge 2}k\sum_{\gamma_2}\dots\sum_{\gamma_k}|\varphi(\gamma_1,\dots,\gamma_k)|\prod_{j=2}^{k}|w(\gamma_j)| \le e^{a(\gamma_1)}$$

and so the cluster expansion converges.

Pf. Show

$$1+\sum_{k=2}^{N} k \sum_{r_2} \cdots \sum_{r_k} |\varphi(r_1,\dots,r_k)| \prod_{j=2}^{k} |w(r_j)| \le e^{a(r_1)} . \quad (*)$$

Case $N=2$.

$G \subset G_2$ conn $\Rightarrow G = \{\{1,2\}\}$

$$\Rightarrow \varphi_2(r_1,r_2) = \frac{1}{2}\zeta(r_1,r_2)$$

$$\Rightarrow 1+2\sum_{r_2} |\varphi(r_1,r_2)|\,|w(r_2)|$$

$$= 1+\sum_{r_2} |\zeta(r_1,r_2)|\,|w(r_2)|$$

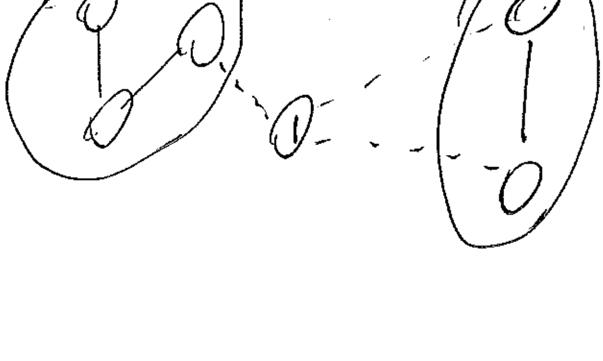
$$\le 1+\sum_{r_2} |\zeta(r_1,r_2)|\, e^{a(r_2)}|w(r_2)|$$

$$\le 1+a(r_1) \le e^{a(r_1)} .$$

Inductive step $(N \Rightarrow N+1)$

Consider $k \le N+1$ and $G \subset G_k$ conn

Delete 1 + assoc. edges from G $\Rightarrow G' = (G'_1,\dots,G'_l)$, $l \le k-1$



$$\varphi_k(r_1, \dots, r_k)$$

$$= \frac{1}{k!} \sum_{l=1}^{k} \frac{1}{l!} \sum_{V_1', \dots, V_l'} \Phi_l \Psi_l$$

$$\Phi_l = \sum_{\substack{G_1' \text{ conn} \\ V(G_1') = V_1'}} \cdots \sum_{\substack{G_l' \text{ conn} \\ V(G_l') = V_l'}} \prod_{m=1}^{l} \prod_{ij \in G_m'} \zeta(r_i, r_j)$$

$$\left[= \prod_{m=1}^{l} \sum_{\substack{G_m' \text{ conn} \\ V(G_m') = V_m'}} \prod_{ij \in G_m'} \zeta(r_i, r_j) \right] = \prod_{m=1}^{l} |V_m'|! \, \varphi((r_j)_{j \in V_m'})$$

$$\Psi_l = \sum_{\phi \neq K_1 \subset V_1'} \cdots \sum_{\phi \neq K_l \subset V_l'} \prod_{m=1}^{l} \prod_{j \in K_m} \zeta(r_1, r_j)$$

$$\left[= \prod_{m=1}^{l} \sum_{\phi \neq K_m \subset V_m'} \prod_{j \in K_m} \zeta(r_1, r_j). \right]$$

Bounds.

By induction,

$$|1+\alpha_k| \le 1 \ \forall k \Rightarrow \left|\prod_{k=1}^{n}(1+\alpha_k) - 1\right| \le \sum_{k=1}^{n} |\alpha_k|$$

In particular, by binomial thm.,

$$|\Psi_l| = \prod_{m=1}^{l} \left|\prod_{j\in V_m'} (1+\zeta(r_1,r_j)) - 1\right|$$

$$\le \prod_{m=1}^{l} \sum_{j\in V_m'} |\zeta(r_1,r_j)|$$

Thus,

$$\sum_{k=2}^{N+1} k \sum_{r_2} \cdots \sum_{r_k} |\varphi_k(r_1,\dots,r_k)| \prod_{j=2}^{k} |w(r_j)|$$

$$\le \sum_{k=2}^{N+1} k \sum_{r_2} \cdots \sum_{r_k} \frac{1}{k!} \sum_{l=1}^{k} \frac{1}{l!} \sum_{V_1',\dots,V_l'} \prod_{m=1}^{l} \left(|V_m'|!\,|\varphi((r_j)_{j\in V_m'})| \sum_{j\in V_m'} |\zeta(r_1,r_j)| \right)$$

$$\times \prod_{j=2}^{k} |w(r_j)|$$

$$= \sum_{k=2}^{N+1} \frac{1}{\cancel{(k-1)!}} \sum_{l=1}^{k} \frac{1}{l!} \sum_{m_1+\dots+m_l=k-1} \cancel{\frac{(k-1)!}{m_1! \cdots m_k!}}$$

$$\times \prod_{j=1}^{l} \left(\cancel{m_j!} \sum_{x_1} \cdots \sum_{x_{m_j}} |\varphi(x_1,\dots,x_{m_j})| \sum_{i=1}^{m_j} |\zeta(x_1,x_i)| \prod_{i=1}^{m_j} |w(x_i)| \right)$$

(A)

$$\leq \sum_{l \geq 1} \frac{1}{l!} \prod_{j=1}^{l} \sum_{m_j=1}^{N} \left(\dots \right)$$

Claim. Inductive hyp $\Rightarrow$ $\forall x_*$,

$$\sum_{k=1}^{N} \sum_{x_1} \cdots \sum_{x_k} |\varphi(x_1,\dots,x_k)| \sum_{i=1}^{k} |\zeta(x_i,x_*)| \prod_{i=1}^{k} |w(x_i)| \leq a(x_*).$$

Follows that (A)

$$\leq \sum_{l} \frac{1}{l!} (a(x_1))^l = e^{a(x_1)} - 1. \quad \blacksquare$$

Pf. (Claim).

Inductive hyp states that

$$1+\sum_{k=2}^{N} k \sum_{r_2} \cdots \sum_{r_k} |\varphi(r_1,\dots,r_k)| \prod_{j=2}^{k} |w(r_j)| \le e^{a(r_1)} .$$

Multiply by $|\zeta(r_A, r_1)||w(r_1)|$ and sum over r_1:

$$\sum_{k=1}^{N} k \sum_{r_1} \cdots \sum_{r_k} |\zeta(r_A, r_1)||\varphi(r_1,\dots,r_k)| \prod_{j=1}^{k} |w(r_j)|$$

$$\le \sum_{r_1} |w(r_1)| e^{a(r_1)} |\zeta(r_A, r_1)|$$

$$\le a(r_A) \quad \text{[by hyp]}. \quad \blacksquare$$

Applications of cluster expansion.

<u>Recall.</u>

If $\Xi = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \left(\prod_{i=1}^{n} w(\gamma_i)\right)\left(\prod_{i<j} \delta(\gamma_i,\gamma_j)\right)$

then $\log \Xi = \sum_{m\geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi_m(\gamma_1,\dots,\gamma_m) \prod_{i\in V_m} w(\gamma_i)$

with $\varphi_m(\gamma_1,\dots,\gamma_m) = \frac{1}{m!} \sum_{\substack{G \subset G_m \\ \text{conn}}} \prod_{ij\in G} \underbrace{\zeta(\gamma_i,\gamma_j)}_{\delta(\gamma_i,\gamma_j)-1}$

where $G_n = (V_n, E_n)$ is the complete graph.

This series converges if $\exists\, a : \Gamma \to (0,\infty)$ st $\forall \gamma_*$,

$$\sum_{\gamma} |w(\gamma)| e^{a(\gamma)} |\zeta(\gamma,\gamma_*)| \leq a(\gamma_*).$$

<u>Today.</u> Large $h>0$ expansion w/

$$H = -\beta \sum_{xy} \sigma_x\sigma_y - h\sum_x \sigma_x$$

Write

$$H = -\beta|\mathcal{E}(\Lambda)| - h|\Lambda| - \beta \sum_{xy} \underbrace{(\sigma_x \sigma_y - 1)}_{-2\mathbb{1}(xy \in \partial\Lambda^-)} - h \sum_x \underbrace{(\sigma_x - 1)}_{-2\mathbb{1}(x \in \Lambda^-)}$$

where $\Lambda^- = \{x \in \Lambda : \sigma_x = -1\}$ and ∂ = ext edge bdry

So $H = -\beta|\mathcal{E}(\Lambda)| - h|\Lambda| + 2\beta|\partial\Lambda^-| + 2h|\Lambda^-|$

Then $Z = e^{\beta|\mathcal{E}(\Lambda)| + h|\Lambda|}\,\Xi$

w/ $\Xi = 1 + \sum_{\emptyset \neq \Lambda^- \subset \Lambda} e^{-2\beta|\partial\Lambda^-| - 2h|\Lambda^-|}$

$$= 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\emptyset \neq S_1 \subset \Lambda \\ \text{conn}}} \cdots \sum_{\substack{\emptyset \neq S_n \subset \Lambda \\ \text{conn}}} \left(\prod_{i=1}^n w_h(S_i)\right)\left(\prod_{i<j} \delta(S_i, S_j)\right)$$

$$w_h(S_i) = e^{-2\beta|\partial S_i| - 2h|S_i|}$$

$$\delta(S_i, S_j) = \mathbb{1}\left(d_1(S_i, S_j) > 1\right)$$

~~Note~~ Thus, $\zeta(S, S_*) = \delta(S, S_*) - 1 \neq 0$ iff $S \cap [S_*] \neq \emptyset$,

where $[S_*] = \{x : d(x, S_*) \leq 1\}$.

③

So

$$\sum_{\substack{S \subset \Lambda \\ (\neq \emptyset,\ \text{connected})}} |w_h(S)| e^{a(S)} |\zeta(S, S_*)| \qquad (\text{NTS } \le a(S_*))$$

$$\le \sum_{j \in [S_*]} \sum_{S \ni j} (\dots)$$

$$\le |[S_*]| \max_{j \in [S_*]} \sum_{S \ni j} |w_h(S)| e^{a(S)}$$

$$\le |[S_*]| \sum_{S \ni 0} (\dots) \qquad (\text{trans-inv})$$

$$\le |[S_*]| \sum_{S \ni 0} e^{-2h|S| + a(S)} \qquad \left(\begin{array}{c}\text{ignore surface}\\ \text{term}\end{array}\right)$$

Set $a(S) := |[S]| \le 2d|S|$ and take h large to get:

$$\le |[S_*]| \sum_{k \ge 1} e^{-2(h-2d)k} \underbrace{\# \{S \ni 0 : |S| = k\}}_{\le (2d)^{2k}}$$

(use: every conn graph has a path from every vertex crossing each edge $2\times$)

$$\le |[S_*]| = a(S_*) \quad \text{for } h \text{ large}.$$

Note. The bd. $\le (2d)^{2k}$ is over all $S \ni 0$, $|S| = 2k$, not just $S \subset \Lambda$.

Thermodynamic limit:

Call $X = (S_1, \ldots, S_m)$ a cluster if $\varphi_m(S_1, \ldots, S_m) \neq 0$,

i.e. if $\zeta(S_i, S_j) \neq 0$ [equiv. $S_1 \cap [S_2] \neq \phi$] $\forall i, j$

We have

$$\log \Xi = \sum_m \sum_{S_1 \subset \Lambda} \cdots \sum_{S_m \subset \Lambda} \varphi_m(S_1, \ldots, S_m) \prod_{i=1}^{m} w_\Lambda(S_i)$$

$$= \sum_{\substack{\text{clusters } X \\ \bar{X} \subset \Lambda}} \Psi(X)$$

where $\bar{X} = S_1 \cup \ldots \cup S_m$ (support)

and $\Psi(X) = \left(\prod_{S \subset \Lambda} \frac{1}{n_X(S)!} \right) \left(\sum_{\substack{G \subset G_m \\ \text{conn}}} \prod_{ij \in G} \zeta(S_i, S_j) \right) \prod_{i=1}^{m} w(S_i)$

$n_X(S) = \#$ times S appears in X.

(Note. Combinatorial factors comes from $\frac{1}{m!}$ in φ_m
and $\frac{m!}{\prod_S n_X(S)!}$ ways for X to appear)

Now $$\sum_{\bar{X} \subset \Lambda} \Psi(X) = \sum_{x \in \Lambda} \sum_{x \in \bar{X} \subset \Lambda} \frac{1}{|\bar{X}|} \Psi(X)$$

$$= \sum_{x \in \Lambda} \Big[\underbrace{\sum_{\bar{X} \ni x} \frac{1}{|\bar{X}|} \Psi(X)}_{A} - \underbrace{\sum_{x \in \bar{X} \not\subset \Lambda} \frac{1}{|\bar{X}|} \Psi(X)}_{B} \Big].$$

trans-inv $\Rightarrow$ (A) indep. of X

$$\Rightarrow \sum_{X \in \Lambda} (A) = C|\Lambda|$$

Also, $\left|\sum_{X \in \Lambda} (B)\right| \le |\partial\Lambda| \max_{j \in \partial\Lambda} \sum_{X \ni j} |\Phi(X)|$

$$\le |\partial\Lambda| \sum_{m \ge 1} n \sum_{S_1 \ni x} \sum_{S_2} \cdots \sum_{S_m} |\varphi_m(\ldots)| \prod |w_\alpha(S_k)|$$

$$\sum_{m \ge 1} + \sum_{m \ge 1} \le e^{|[S_1]|} \quad \text{(Ursell's bdd)}$$

$$\le |\partial\Lambda| \underbrace{\sum_{X \in S_1} |w_\alpha(S_1)|\, e^{|[S_1]|}}_{\le 1} \le |\partial\Lambda|$$

Therefore, $Z = e^{\beta|\mathcal{E}(\Lambda)| + h|\Lambda|}\,\Xi$

$$\Rightarrow \frac{1}{|\Lambda|} \log Z = \beta \frac{|\mathcal{E}(\Lambda)|}{|\Lambda|} + h + (A) + \cancel{O(|\partial\Lambda|)}\ O\left(\frac{|\partial\Lambda|}{|\Lambda|}\right)$$

$(\Lambda = B(n) \to \mathbb{Z}^d)$

$$\Rightarrow \psi(h) = \beta d + h + \sum_{X \ni 0} \Phi(X).$$

Note. $\dfrac{-H_{B(n)}(\eta \equiv 1)}{|B(n)|} \longrightarrow \beta d + h.$

In fact, dependence on h is through

$$w_h(S) = e^{-2\beta|\partial S| - 2h|S|}.$$

Thus,

$$\psi(h) = \beta d + h + \sum_{k \geq 1} a_k z^k, \qquad z = e^{-h}$$

and the coeffs. a_k can be computed systematically.

In particular, $h > 0$ large $\Rightarrow$ no phase transition.

<u>Note</u>. Lee-Yang $\Rightarrow$ no PT for $h \neq 0$.

Virial and Mayer expansions

Recall.

Ising model pressure is

$$\psi(h) = \beta d + h + \sum_{k\geq 1} a_k z^k, \quad z = e^{-h}, \quad h \text{ large}$$

The a_k can be computed from

$$\psi(h) = \beta d + h + \sum_{X \ni 0} \frac{1}{|X|} \Psi(X)$$

w/ $\Psi(X) = \left(\prod_{S\subset\Lambda} \frac{1}{N_X(S)!}\right)\left(\sum_{\substack{G\subset G_m\\ \text{conn}}} \prod_{ij\in G} \zeta(S_i, S_j)\right)\prod_{i=1}^{m} w_h(S_i)$

for $X = (S_1, \dots, S_m)$ a cluster, i.e. $\zeta(S_i, S_j) \neq 0$.

In particular, since dependence on h is only through

$$w_h(S) = e^{-2\beta|\partial S| - 2h|S|},$$

~~$a_1 = w_h(\dots)$~~

$$a_1 = e^{2h|\{0\}|} w_h(\{0\}) = e^{-2\beta|\partial\{0\}|} = e^{-4d\beta} \neq 0.$$

Nearest-neighbour lattice gas.

$\eta \in \{0,1\}^\Lambda$, $H_\Lambda = -\sum_{x\sim y} \eta_x \eta_y$
(repulsion) , (attraction)

Grand canonical ensemble

$$\nu_{\Lambda;\beta,\mu}(\eta) = \frac{1}{\Theta_{\Lambda;\beta,\mu}} \exp\left(-\beta\left(H_\Lambda(\eta) - \mu N_\Lambda(\eta)\right)\right)$$

w/ $N_\Lambda(\eta) = \sum_{x\in\Lambda} \eta_x$ the # of particles

and $\mu \in \mathbb{R}$ the chemical potential.

Mapping to Ising model.

$\eta_x \longmapsto \omega_x = 2\eta_x - 1 \in \{\pm 1\}$

Let $\Lambda = B(n)$ so $\forall x \notin \partial\Lambda$, $\sum_{y\sim x} 1 = 2d$

and $|\partial\Lambda| = o(|\Lambda|)$ as $n \to \infty$.

Get $\beta\left(H(\eta) - \mu N(\eta)\right) = -\frac{\beta}{4}\sum_{x\sim y} \omega_x\omega_y - \frac{\beta}{2}(d+\mu)\sum_x \omega_x$

$$-\beta\left(\frac{\mu}{2} + \frac{d}{4}\right)|\Lambda| + o(|\Lambda|)$$

so $\Theta_{B(n);\beta,\mu} = e^{\beta(\frac{\mu}{2}+\frac{d}{4})|B(n)|} Z_{B(n);\beta',h'}$

w/ $\beta' = \frac{1}{4}\beta$, $h' = \frac{\beta}{2}(d+\mu)$.

Define the pressure

$$\underbrace{~~P_\beta(\mu) := \lim_{n\to\infty} \frac{1}{\beta|\Lambda|} \log \Theta_{\Lambda;\beta,\mu}}_{p_{\Lambda;\beta}(\mu)}$$

$$= \frac{\mu}{2} + \frac{d}{4} + \frac{1}{\beta}\psi_{\beta'}(h')$$

Also, μ large $\Rightarrow h'$ large $\Rightarrow$ no PT $\Rightarrow P_\beta$ analytic

Density.

Note that $\frac{\partial p_\Lambda}{\partial\mu} = \frac{1}{\beta|\Lambda|}\cdot\frac{1}{\Theta_\Lambda}\underbrace{\sum_\eta \beta N(\eta) e^{-\beta(H_\Lambda(\eta)-N_\Lambda(\eta))}}_{\beta\langle N\rangle}$

$$= \left\langle \frac{N}{|\Lambda|}\right\rangle =: \rho_{\Lambda;\beta}(\mu)\ (\text{density})$$

So $\rho_\beta(\mu) = \frac{\partial p_\beta(\mu)}{\partial \mu} = \frac{1}{2} + \frac{1}{\beta} \cdot \frac{\partial \psi}{\partial h'} \cdot \underbrace{\frac{\partial h'}{\partial \mu}}_{\beta/2}$

$$= \frac{1}{2}\left(1 + m_{\beta'}(h')\right)$$

Also, $\rho(\mu) \in (0,1)$, since $m(h') \in (-1,1)$, and ρ is $\underbrace{\text{cts./inc.}}$ ~~away from for μ large~~ away from CPs,

and $\lim_{\mu \to \pm\infty} \rho(\mu) = 1$ or 0.

(exercise. differentiate ρ)

Thus, $\rho(\mu) = \rho_*$ has soln. $\mu(\rho_*)$ for $\rho_* \in (0,1)$

Let $\tilde{p}(\rho) = p(\mu(\rho))$

Mayer expansion:

Consider $-\mu$ large ($-h'$ large)

Have $\psi_{\beta'}(h') = \psi_{\beta'}(-h')$

$$= \beta' d - h' + \sum_{k \geq 1} a_k (z')^k, \quad z' = e^{2h'}$$

Thus, $\beta p(\mu) = \frac{\beta\mu}{2} + \frac{\beta d}{4} + \Psi_{\beta'}(h')$

$$= \left(\frac{\beta\mu}{2} + \frac{\beta d}{4}\right) + \left(\frac{\beta d}{4} - \frac{\beta d}{2} - \frac{\beta\mu}{2}\right) + \Psi_{\beta'}(h')$$

$$= \Psi_{\beta'}(h') = \sum_{k\geq 1} a_k (z')^k ,$$

which is the <u>Mayer series</u>.

Differentiating yields

$$\rho(\mu) = \frac{\partial p}{\partial \mu} = \frac{1}{\beta} \sum k\, a_k (z')^k \underbrace{\frac{dz'}{dh'}}_{2z'} \cdot \underbrace{\frac{\partial h'}{\partial \mu}}_{\beta/2}$$

$$= \sum \underbrace{k\, a_k}_{\tilde{a}_k} (z')^k =: \phi(z') .$$

<u>Virial expansion</u>:

Invert ϕ: $\frac{d\phi}{dz'}(0) = \tilde{a}_1 = a_1 \neq 0$

$\Rightarrow$ exists local $\phi^{-1}(\rho) = z'$ st $\phi(z') = \rho$,

~~Note~~

Thus, $\beta \tilde{p}(\rho) = \sum a_k \left(\phi^{-1}(\rho)\right)^k$

$$= \sum a_k \left(\sum_m c_m \rho^m \right)$$

$$= \sum_n b_n \rho^n$$

In particular, $b_1 = a_1 c_1$

Since, $c_1 = (\phi^{-1})'(0) = \dfrac{1}{\phi'(\phi^{-1}(0))} = \dfrac{1}{\phi'(0)} = \dfrac{1}{a_1}$,

get $b_1 = 1$

Thus, $\tilde{p}(\rho) = \rho\beta^{-1} + \beta^{-1} \sum_{n \geq 2} b_n \rho^n$. (virial expansion)

The leading part is the ideal gas law

pressure = density × R × temp

w/ ideal gas const $R = 1$.

Random walk representation.

Consider $|\varphi|^4$ model:

- Hamiltonian $H(\varphi) = -\sum_{x,y} J_{xy}\, \varphi_x \cdot \varphi_y$, $\varphi_x \in \mathbb{R}^n$

- Single-spin meas. $g(t) = e^{-\frac{1}{4}|t|^4 - \frac{1}{2}|t|^2}$, $t \in \mathbb{R}^n$

View as perturbed Gaussian measure.
Expected to exhibit same (mean-field) behaviour as Gaussian near β_c, if $d \geq 5$.

For instance, susceptibility

$$\chi(\beta) = \lim_N \sum_{x \in \Lambda_N} \langle \varphi_0 ; \varphi_x \rangle_{\Lambda_N}$$

should blow up as $\chi(\beta) \sim C(\beta - \beta_c)^{-1}$

Easy to see in Gaussian case:

$$\langle \varphi_0 ; \varphi_x \rangle = (-\Delta + m^2)^{-1} = (2d(1-P) + m^2)^{-1}$$

$$= (2d + m^2)^{-1} \left(1 - \frac{2d}{2d+m^2} P\right)^{-1}$$

$$= (2d+m^2)^{-1} \sum_n \left(\frac{2d}{2d+m^2}\right)^n P^n$$

$$\Rightarrow \chi = (2d+m^2)^{-1} \sum_n \left(\frac{2d}{2d+m^2}\right)^n = \frac{1}{m^2}.$$

This computation is based on SRW and we proceed similarly for φ^4, etc.

~~Def~~

Random walk.

$J \in \mathbb{R}^{V \times V}$, $J_{xy} = J_{yx}$, $J_{xx} = 0$, summable in rows

$X = (X_t)_{t \geq 0}$ cts-time SRW gen by J,

i.e. Markov process w/ generator ~~Q_{xx}~~

$$Q_{xy} = \left(\sum_z J_{xz}\right)\delta_{xy} - J_{xy} \quad \text{(Laplacian)}$$

Thus, $\frac{d}{dt}\Big|_{t=0} \mathbb{E}_x\left(f(X_t)\right) = -Qf(x)$.

Define local time $L_x^T = \int_0^T \mathbb{1}(X_t = x)\, dt$.

Feynman-Kac. $b, f \in \mathbb{R}^V$, $T > 0$:

$$\left(e^{-T(Q+b)} f\right)_x = \mathbb{E}_x\left(e^{-(b, L^T)} f_{X_T}\right) = \mathbb{E}_x\left(e^{-\int_0^T b_{X_t} dt} f_{X_T}\right).$$

Pf. $(b, L^T) = \sum_x b_x \int_0^T \mathbb{1}(X_t = x)\, dt = \int_0^T b_{X_t} \underbrace{\sum_x \mathbb{1}(X_t = x)}_{1}\, dt$

Let $(P_t f)_x = $ RHS.

Markov property $\Rightarrow (P_t)$ a semigroup.

So suffices to show it has generator $-(Q+b)$. ③

i.e. that $\frac{1}{T}((P_T - P_0)f)_x \xrightarrow{T\downarrow 0} [-(Q+b)f]_x$.

Do so by writing

$$[(P_T - P_0)f]_x = \int_0^T \frac{d}{dt}(P_t f)_x \, dt.$$

and computing. ■

Cor. $(Q+b)^{-1}_{xy} = \int_0^\infty E_x\left(e^{-(b,L^T)} \mathbb{1}(X_T = y)\right) dT$.

Pf. Take $f = \delta_y$ and integrate over $T \in (0,\infty)$. ■

Gaussian field.

Let $D \in \mathbb{R}^{V\times V}$ be diag., $\operatorname{Re} D_{xx} > \sum_y J_{xy}$

So that $A = D - J$ has $\operatorname{Re}(A) > 0$.

(A = massive Laplacian)

Let $(\varphi_x)_{x\in V}$ be the n-(independent)-component

Gaussian field w/ cov. $C = A^{-1}$

Let $\tau_x = \frac{1}{2}|\varphi_x|^2$.

Thm. (BFS)

For $x, y \in V$, nice $g : \mathbb{R}_+^V \to \mathbb{R}$,

$$\tilde{\mathbb{E}}\left(g(\tau)\varphi_x^1\varphi_y^1\right) = \int_0^\infty \tilde{\mathbb{E}}_x\left(g(L^T+\tau)e^{-(\underset{\text{diag.}}{\underbrace{A-Q}}, L^T)}\mathbb{1}(X_T = y)\right)dT$$

Pf. $g(t) = e^{(b,t)}$, small $b : V \to \mathbb{R}$

$$\Rightarrow \int_0^\infty \tilde{\mathbb{E}}_x\Big(\underbrace{g(L^T+\tau)}_{e^{-(b,L^T)}e^{-\frac{1}{2}(\varphi, b\varphi)}}e^{-(A-Q, L^T)}\mathbb{1}(X_T = y)\Big)dT$$

$$= \tilde{\mathbb{E}}\left(e^{-\frac{1}{2}(\varphi, b\varphi)}\right)\int_0^\infty \mathbb{E}_x\left(e^{-(b+A-Q, L^T)}\mathbb{1}(X_T = y)\right)dT$$

$$= \underbrace{\frac{N(b)}{N(0)}}_{N(b) = \int e^{-\frac{1}{2}(\varphi, (A+b)\varphi)}d\varphi}\underbrace{\left(\cancel{Q} + b + A - \cancel{Q}\right)^{-1}_{xy}}_{\text{integrated FK}}$$

$$= \tilde{\mathbb{E}}\left(e^{-\frac{1}{2}(\varphi, b\varphi)}\varphi_x^1\varphi_y^1\right) = \tilde{\mathbb{E}}\left(g(\tau)\varphi_x^1\varphi_y^1\right). \quad \blacksquare$$

Take $A \downarrow Q$ (when possible) to get

$$\tilde{\mathbb{E}}\left(g(\tau)\varphi'_x\varphi'_y\right) = \int_0^\infty \tilde{\mathbb{E}}_x\left(g(L^T+\tau)\mathbb{1}(X_T=y)\right)dT$$

e.g. $g(t) = e^{-(t,b)} \Rightarrow$

$$(A+b)^{-1}_{xy} = \frac{1}{N(b)}\int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)}\varphi_x\varphi_y\,d\varphi = \frac{N(0)}{N(b)}\tilde{\mathbb{E}}\left(e^{-(\tau,b)}\varphi_x\varphi_y\right)$$

$$= \frac{N(0)}{N(b)}\int_0^\infty \tilde{\mathbb{E}}_x\left(e^{-(L^T+\tau,b)}\mathbb{1}(X_T=y)\right)dT$$

In particular, $b=0 \Rightarrow C_{xy} = \mathbb{E}_x(L_y^T)$.

Now for $g: \mathbb{R}_+^V \to \mathbb{R}$ nice, $t \in \mathbb{R}_+^V$, let

$$\langle F\rangle_t = \frac{1}{Z(t)}\mathbb{E}(F(\varphi)g(\tau+t)),\quad Z(t) = \mathbb{E}(g(\tau+t)),\quad \langle\cdot\rangle = \langle\cdot\rangle_0.$$

Lem. (GIP).

$$\tilde{\mathbb{E}}(\varphi'_x F(\varphi)) = \sum_y C_{xy}\tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\right) = \sum_y\int_0^\infty \tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\mathbb{1}(X_T=y)\right)dT$$

Pf. By IBP,

$$\sum_y C_{xy}\tilde{\mathbb{E}}\left(\frac{\partial F}{\partial\varphi'_y}\right) = \sum_y C_{xy}\frac{1}{N(0)}\int\frac{\partial F}{\partial\varphi'_y}e^{-\frac{1}{2}(\varphi,A\varphi)}d\varphi$$

$$= -\sum_y C_{xy}\frac{1}{N(0)}\int F(\varphi)\sum_z A_{xz}\varphi'_z e^{-\frac{1}{2}(\varphi,A\varphi)}d\varphi$$

(5)

$$= \frac{1}{N(0)} \int F(\varphi) e^{-\frac{1}{2}(\varphi, A\varphi)} \underbrace{\sum_z \delta_{xz} \varphi'_z}_{\varphi'_x} \, d\varphi$$

$$= \tilde{\mathbb{E}}(\varphi'_x F(\varphi)).$$

For second equality use $C_{xy} = \mathbb{E}_x(L^T_y) = \int_0^\infty \mathbb{E}_x(\mathbb{1}(X_T = y))\, dT$.

<u>Lem.</u> (BFS-IBP).

$$\langle \varphi'_x F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x\left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT$$

w/ $Z(t) = \dfrac{z(t)}{z(0)}$.

<u>Pf.</u> $g(t) = e^{-(\delta, t)}$

$$z(0) \langle \varphi'_x F(\varphi) \rangle = \tilde{\mathbb{E}}\Big(\varphi'_x F(\varphi)\, \underbrace{e^{-\frac{1}{2}(\varphi, \delta\varphi)}}_{g(\tau)}\Big)$$

$$= \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi, (A+\delta)\varphi)} \varphi'_x F(\varphi)\, d\varphi$$

$$= \frac{1}{N(0)} \sum_y (A+\delta)^{-1}_{xy} \int e^{-\frac{1}{2}(\varphi, (A+\delta)\varphi)} \frac{\partial F}{\partial \varphi'_y}\, d\varphi$$

(GIBP)

$$= \frac{1}{N(0)} \int F(\varphi) e^{-\frac{1}{2}(\varphi, A\varphi)} \underbrace{\sum_z \delta_{xz} \varphi'_z}_{\varphi'_x} \, d\varphi$$

$$= \tilde{\mathbb{E}}(\varphi'_x F(\varphi)).$$

For second equality use $C_{xy} = \int_0^\infty \mathbb{E}_x(\mathbb{1}(X_T = y))\, dT$. ∎

<u>Lem.</u> (BFS-IBP).

$$\langle \varphi'_x F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT$$

$$\text{w/ } Z(t) = \frac{Z(t)}{Z(0)}.$$

<u>Cor.</u> (Gauss. UB).

For $|\varphi|^4$, $n = 1, 2$, and related models,

$$\langle \varphi'_x F(\varphi) \rangle = \sum_y \langle \varphi'_x \varphi'_y \rangle \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle.$$

Proof of Lem. + Cor. next time.

Ex. ($n=1$ for simplicity).

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2p}} \rangle = \sum_{y = x_2, \dots, x_{2p}} \langle \varphi_{x_1} \varphi_y \rangle \underbrace{\left\langle \frac{\partial}{\partial \varphi_y} \varphi_{x_2} \cdots \varphi_{x_{2p}} \right\rangle}_{\leq \cdots}$$

$$\vdots$$

$$\leq \sum_{\pi} \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \cdots \langle \varphi_{x_{\pi(2p-1)}} \varphi_{x_{\pi(2p)}} \rangle$$

w/ π running over perfect matchings of $\{1, \dots, 2p\}$.

In particular, we have the Lebowitz inequality:

$$u_4(x_1, x_2, x_3, x_4) := \langle \varphi_{x_1} \varphi_{x_2} \varphi_{x_3} \varphi_{x_4} \rangle - \sum_{\pi} \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \langle \varphi_{x_{\pi(3)}} \varphi_{x_{\pi(4)}} \rangle$$

$$\leq 0.$$

As an application:

$$\beta \chi'(\beta) = C \chi^2(\beta) + \frac{1}{2} \sum_{x, y, z} J_{xy} u_4(0, x, y, z) \leq C \chi^2(\beta)$$

$$\Rightarrow \chi(\beta) \geq \frac{1}{C(\beta_c - \beta)} \quad \text{i.e. } \boxed{\gamma \geq 1}.$$

Final remarks/critical phenomena.

Recall.
$\varphi \in (\mathbb{R}^n)^V$, $J_{xy} = J_{yx}$, $J_{xx} = 0$

$Q_{xy} = (\sum_z J_{xz})\delta_{xy} - J_{xy}$ generates E_x (Laplacian)

$\tilde{E}$ Gauss w/ cov. $C = A^{-1}$, $A = D - J$ (Laplacian + pos. potent.)

$$\langle F \rangle_\ell = \frac{1}{Z(\ell)} \tilde{E}\big(F(\varphi) g(\tau + \ell)\big),$$

$$g : \mathbb{R}_+^V \to \mathbb{R}, \quad \ell \in \mathbb{R}_+^V, \quad \tau_x = \frac{1}{2}|\varphi_x|^2$$

usu. set $\ell = L^T$ local time

$\langle \cdot \rangle = \langle \cdot \rangle_0$,

Prop. (BFS-IBP).

$$\langle \varphi_x^1 F(\varphi) \rangle = \sum_y \int_0^\infty E_x\Big(\mathcal{Z}(L^T) \Big\langle \frac{\partial F}{\partial \varphi_y^1} \Big\rangle_{L^T} \mathbb{1}(X_T = y)\Big)\, dT,$$

$$\mathcal{Z}(\ell) = \frac{Z(\ell)}{Z(0)}.$$

Pf. $g(\cdot) = e^{-(b,\cdot)}$, $\quad N(b) := \int e^{-\frac{1}{2}(\varphi, (A+b)\varphi)} d\varphi$

$$Z(0) \langle \varphi_x^1 F(\varphi) \rangle = \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)} \varphi_x^1 F(\varphi)\, d\varphi$$

$$= \frac{1}{N(0)} \int e^{-\frac{1}{2}(\varphi,(A+b)\varphi)} \sum_y (A+b)^{-1}_{xy} \frac{\partial F}{\partial \varphi_y^1}\, d\varphi \quad [\text{GIP}]$$

$$= \sum_y \int_0^\infty \tilde{\mathbb{E}}_x \left(e^{-(\tau + L^T, b)} \frac{\partial F}{\partial \varphi'_y} \mathbb{1}(X_T = y) \right) dT \qquad [\text{IFK}]$$

$$= \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT. \qquad \blacksquare$$

<u>Cor</u>. (GUB). If $\frac{\partial}{\partial l_x} \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_l \leq 0 \quad \forall x, y$, then

$$\langle \varphi'_x F(\varphi) \rangle \leq \sum_y \langle \varphi'_x \varphi'_y \rangle \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle.$$

<u>Ex</u>. $g(l) = e^{-\sum_x v_x(l_x)}$

$$\Rightarrow \frac{\partial}{\partial l_x} \langle F \rangle_l = - \langle F ; v'_x(\tau_x + l_x) \rangle$$

So Griffiths ineq. $\langle \varphi^A ; \varphi^B \rangle \geq 0$, $A, B \subset V$

[holds for $n=1$/flip-inv./ferro.]

implies RHS ≤ 0 eg. for $F(\varphi) = \varphi^A$ and $v_x(l_x) = g t_x^2 + \nu t_x$, $g \geq 0$.

<u>Pf</u>. $\langle \varphi'_x F(\varphi) \rangle$

$$= \sum_y \int_0^\infty \mathbb{E}_x \left(Z(L^T) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L^T} \mathbb{1}(X_T = y) \right) dT \qquad [\text{BFS-IBP}]$$

$$\leq \quad \cdots\cdots \langle \cdot \rangle_0 \cdots\cdots$$

$$= \sum_y \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle \frac{1}{Z(\emptyset)} \underbrace{\int_0^\infty \tilde{\mathbb{E}}_x \left(g(L^T + \tau) \mathbb{1}(X_T = y) \right) dT}_{\tilde{\mathbb{E}}(g(\tau)\varphi'_x \varphi'_y) \text{ by BFS}}$$

$$= \sum_y \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle \langle \varphi'_x \varphi'_y \rangle . \quad \blacksquare$$

Application:

GHS $\Rightarrow$ Lebowitz ineq.:

$$u_4(x_1, \dots, x_4) := \langle \varphi_{x_1} \dots \varphi_{x_4} \rangle - \sum_\pi \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle \langle \varphi_{x_{\pi(3)}} \varphi_{x_{\pi(4)}} \rangle$$

$$\leq 0$$

Susceptibility $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \varphi_0 \varphi_x \rangle$

$$\Rightarrow \chi'(\beta) = \sum_z \langle \varphi_0 \varphi_z ; -H(\varphi) \rangle$$

$$= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(\langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle - \langle \varphi_0 \varphi_z \rangle \langle \varphi_x \varphi_y \rangle \right)$$

$$= \frac{1}{2} \sum_{x,y,z} J_{xy} \left(u_4(0,x,y,z) + \langle \varphi_0 \varphi_x \rangle \langle \varphi_y \varphi_z \rangle + \langle \varphi_0 \varphi_y \rangle \langle \varphi_x \varphi_z \rangle \right)$$

$$= \frac{1}{2}\sum_{x,y,z} J_{xy}\, u_4(0,x,y,z) + \frac{1}{2}\underbrace{\sum_x \langle \varphi_0 \varphi_x\rangle}_{\chi}\underbrace{\sum_y J_{xy}}_{|J|}\underbrace{\sum_z \langle \varphi_y \varphi_z\rangle}_{\chi}$$

$$+ \frac{1}{2}\underbrace{\sum_y \langle \varphi_0 \varphi_y\rangle}_{\chi}\underbrace{\sum_z \langle \varphi_x \varphi_z\rangle}_{\chi}\underbrace{\sum_x J_{xy}}_{|J|}$$

(assume trans-inv)

$$= \frac{1}{2}\sum_{x,y,z} \underbrace{J_{xy}}_{\geq 0}\, \underbrace{u_4(0,x,y,z)}_{\leq 0} + |J|\,\chi^2(\beta)$$

$$\leq |J|\,\chi^2$$

Let $f(\varepsilon) = [\chi(\beta_c - \varepsilon)]^{-1}$

w/ $\beta_c := \sup(\beta : \chi(\beta) < \infty)$ $[\varepsilon < 0 \Rightarrow f(\varepsilon) = 0]$

Then $f'(\varepsilon) = \dfrac{\chi'(\beta_c - \varepsilon)}{\chi(\beta_c - \varepsilon)} \leq |J|$

$\Rightarrow f(\varepsilon) = \int_0^\varepsilon f'(t)\,dt \leq |J|\,\varepsilon$

i.e. $\chi(\beta_c - \varepsilon) \geq \frac{1}{|J|} \varepsilon^{-1}$

So if $\chi(\beta_c - \varepsilon) \sim C \varepsilon^{-\gamma}$,

then $\boxed{\gamma \geq 1}$.

These [RW] methods, together w/ IR bds, can be used to show the opposite ineq.

Can also show triviality in $d > 4$, i.e. continuum limits [$J(a), g(a), \nu(a)$ depend on lattice scaling a] must be Gaussian

Note. These results w/ $n=1$ hold by Griffiths and extend to $n=2$ by Ginibre [proved using polar coords] not known to hold for $n > 2$.

Critical behaviour.

For wide classes of models, expect

$$\langle \varphi_0 ; \varphi_x \rangle \sim C_1 \, |x|^{-(d-2+\eta)}, \quad |x| \to \infty$$

$$\chi(\beta) \sim C_2 \, (\beta_c - \beta)^{-\gamma}, \quad \beta \uparrow \beta_c$$

$$\xi(\beta) \sim C_3 \, (\beta_c - \beta)^{-\nu}, \quad \beta \uparrow \beta_c$$

where $\xi = \limsup_{k \to \infty} \dfrac{-k}{\log \langle \varphi_0 ; \varphi_{ke} \rangle}$ so that

$\langle \varphi_0 ; \varphi_x \rangle \approx C f(x) \, e^{-|x|/\xi}$ w/ $f(x)$ sub-exp.

(Other relations expected as well)

Exponents $\eta, \gamma, \nu, \ldots$ should be universal, ie.

depend only on range and symmetry (d, n).

d	2	3	≥ 4
η	$1/4$	$0.0362\ldots$	0
γ	$7/4$	$1.2370\ldots$	1
ν	1	$0.6299\ldots$	$1/2$

$(n = 1)$

Log corrections predicted if $d=4$,
(for γ, ν), eg.

$$\chi(\beta) \sim C_2 (\beta_c - \beta)^{-1} \left(-\log(\beta_c-\beta)\right)^{\frac{n+2}{n+8}}$$

[Shown in BBS 14 for small coupling φ^4 and SAW/$n=0$]

Also extensions for $n=0$ SAW predicted.

<u>$d>4$</u>. • Aizenman 82, Fröhlich 82, $n=1,2$

~~Slade Hara-Slade~~

• Brydges-Spencer 85, Hara-Slade 92, $n=0$

<u>$d=4$</u>. • BBS 14 RG, all n

<u>$d=2$</u>. • Onsager ⇒ Ising

• Ising interfaces ⇒ SLE_3 (Chelkak et al 14)

• SAW ⟶ $SLE_{8/3}$ if limit exists + is conformally invariant
(Lawler-Schramm-Werner 04)

• BKT transition $n=2$ predicted

<u>$d=3$</u>. • magnetization ⟶ 0 at β_c ($n=1$)
(Aizenman et al 15)

[analogous to percolation problem]