Finite-order correlation length of the $|\varphi|^4$ spin model in four dimensions

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Joint work with R. Bauerschmidt[2], G. Slade[1], and A. Tomberg

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Background

Definition of the model

Let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ be discrete tori ($L \gg 1$ fixed and $N \to \infty$)

For a *spin field* $\varphi : \Lambda \to \mathbb{R}^n$, let $(\Delta\varphi)_x = \sum_{y \sim x} (\varphi_y - \varphi_x)$

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The $|\varphi|^4$ **spin model** is defined by the Hamiltonian

$$U_{g,\nu,1} = \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{2} \varphi_x \cdot (-\Delta\varphi)_x \right)$$

That is, we study the measure on $(\mathbb{R}^n)^\Lambda$ with expectation

$$\langle F \rangle_{g,\nu,1}^{(N)} = \frac{1}{Z_{g,\nu,1}} \int F(\varphi) e^{-U_{g,\nu,1}(\varphi)} \, d\varphi$$

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For $g = 0$ and *mass* $\nu > 0$, this is the discrete massive GFF on Λ:

$$U_{0,\nu,1}(\varphi) = \frac{1}{2} \varphi \cdot (-\Delta + \nu) \varphi$$

Definition of the model

Take $\nu < 0 < g$.

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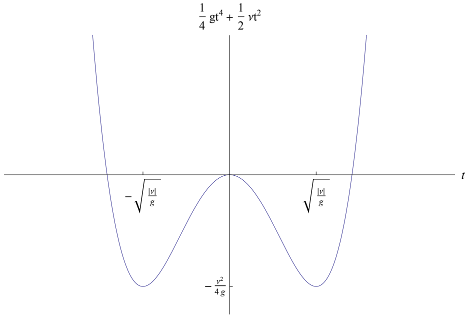

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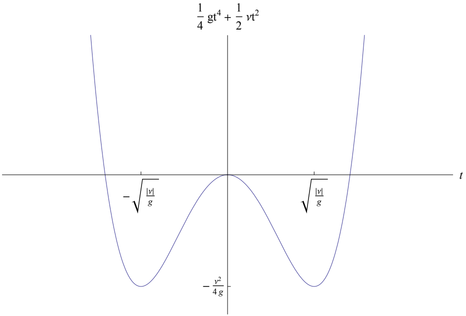

Spins are encouraged to align:

$$\sum_{x\in\Lambda} \varphi_x \cdot (-\Delta\varphi)_x = \frac{1}{2} \sum_{x\in\Lambda} \sum_{y\sim x} |\varphi_y - \varphi_x|^2$$

Relation to self-avoiding walk

Formal $n \to 0$ limit

- De Gennes ('72)

Supersymmetric integral representation

- McKane ('80)
- Parisi and Sourlas ('80)

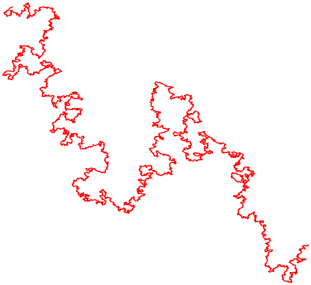

Figure: SAW with 1 million steps (Tom Kennedy)

The critical two-point function

Define the **two-point function**

$$G_{x,N}(g,\nu;n) = \frac{1}{n}\langle \varphi_0 \cdot \varphi_x \rangle_{g,\nu,1}^{(N)}$$

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Predicted behaviour ($d > 2$ or $d = 2$ and $n = 0, 1$):

$$G_x(g,\nu_c;n) \sim C|x|^{-(d-2+\eta)}, \quad |x| \to \infty$$

whereas $G_x(g,\nu;n) = O(e^{-m|x|})$ for $\nu > \nu_c$

Note: $\nu_c = \nu_c(g;n) < 0$

Critial exponent η

Predicted values:

$$\eta = \eta(d;n) = \begin{cases} \frac{5+n}{24}, & d = 2, n = 0, 1 \\ 0.03\ldots, & d = 3 \\ 0, & d \geq 4 \end{cases}$$

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Mean-field behaviour: same as if $g = 0$:

$$G_x(0, \nu_c = 0; n) = (-\Delta)^{-1}_{0x} \sim C|x|^{-(d-2+0)}$$

The near-critical two-point function

The **correlation length** is (one over) the rate of exponential decay:

$$\xi(g,\nu;n) = \limsup_{k\to\infty} \frac{-k}{\log G_{ke}(g,\nu;n)}, \quad e = (1,0,\ldots,0)$$

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Mean-field exponent $\tilde{\nu} = \frac{1}{2}$ expected for $d \geq 4$

$$G_x(0,\nu;n) = (-\Delta+\nu)^{-1} \sim C|x|^{-\alpha}e^{-|x|/\nu^{-1/2}}$$

The finite-order correlation length

The **correlation length of order** p is defined by

$$\xi_p(g,\nu;n) = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^p G_x(g,\nu;n)}{\chi(g,\nu;n)} \right)^{1/p},$$

Here, the **susceptibility** χ is defined by

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Hence,

$$\xi_p^p(g,\nu;n) \approx \frac{\xi(g,\nu;n)^{-(\alpha-p)}}{\xi(g,\nu;n)^{-\alpha}} = \xi^p(g,\nu;n)$$

The case $n = 0$

For the weakly self-avoiding walk (WSAW)

$$\xi_p^p(g,\nu;0) = \frac{\int_0^\infty \langle |X(T)|^p \rangle c_T e^{-\nu T}\, dT}{\int_0^\infty c_T e^{-\nu T}\, dT},$$

where $\langle \cdot \rangle$ is the expectation on WSAW of length T and c_T is the partition function for such walks

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It is expected that

$$\langle |X(T)|^p \rangle^{1/p} \sim C T^{\tilde{\nu}}$$

$$\Updownarrow$$

$$\xi_p(g,\nu;0) \sim C'(\nu - \nu_c)^{-\tilde{\nu}}$$

Previous results

Block-spin RG ($n = 1$, $d = 4$, small g)

- Hara and Tasaki ('87):

$$\xi(g, \nu_c + \varepsilon; 1) \asymp \varepsilon^{-\frac{1}{2}} (\log \varepsilon^{-1})^{\frac{1}{6}}$$

($a \asymp b$ means $C^{-1} \leq a/b \leq C$ for some $C > 0$)

Lace expansion (SAW, $d > 4$)

- Hara and Slade ('91):

$$\xi, \xi_2 \sim C\varepsilon^{-\frac{1}{2}}$$

RG (hierarchical $n = 0$, $d = 4$, small g)

- Brydges and Imbrie ('03):

$$\langle |X(T)|^2 \rangle^{\frac{1}{2}} \sim CT^{\frac{1}{2}} (\log T)^{\frac{1}{8}}$$

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Theorem 1 (Bauerschmidt, Slade, Tomberg, W. (2016))

Let $d = 4$, $n \geq 0$, and $p > 0$. For L large and $g > 0$ small (both depending on p and n),

$$\xi_p(g, \nu_c + \varepsilon; n) \sim C_{g,n,p} \varepsilon^{-\frac{1}{2}} (\log \varepsilon^{-1})^{\frac{1}{2} \frac{n+2}{n+8}}$$

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The constant is given by

$$C_{g,n,p} = \left(\frac{g}{4\pi^2}\right)^{\frac{1}{2} \frac{n+2}{n+8}} \left(\int_{\mathbb{R}^4} |x|^p (-\Delta_{\mathbb{R}^d} + 1)^{-1}_{0x} \, dx \right)^{\frac{1}{p}} (1 + O(g))$$

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For simplicity: we will prove $\asymp$ rather than $\sim$

Key lemma: approximation of $G_x(g,\nu;n)$

Proposition 1 (BBS (2015), ST (2016))

Let $g > 0$ be sufficiently small and let $\nu = \nu_c + \varepsilon$. For any small $\varepsilon > 0$,

$$\frac{G_x(g,\nu;n)}{\chi(g,\nu;n)} = (1+\tilde{R}_x(m^2))\frac{G_x(0,m^2;n)}{\chi(0,m^2;n)} + (m/|x|)^2 R_x(m^2),$$

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Sufficient for studying the critical two-point function

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Proof of main result: main term

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Compute the p-th moment of both sides:

$$\sum_{x\in\mathbb{Z}^d} |x|^p \frac{G_x(0,m^2;n)}{\chi(0,m^2;n)} = \xi_p^p(0,m^2;n)$$

$$\sim Cm^{-p}$$

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It remains to show that

$$m^2 \sum_{x\in\mathbb{Z}^d} |x|^{p-2} R_x(m^2) = O(m^{-p})$$

(Can be improved to $o(m^{-p})$ to get $\sim$)

Proof of main result: remainder

Since

$$R_x(m^2) \le O(1) \begin{cases} 1, & |x| \le m^{-1} \\ (m|x|)^{-2s}, & |x| > m^{-1}, \end{cases}$$

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Note: the decay of R_x is insufficient for computing ξ

The renormalisation group

Preparation: Gaussian approximation

Set $n = 1$ ($\varphi_x \in \mathbb{R}$) and drop it from the notation

Generalize the Hamiltonian $U_{g,\nu,1}$:

$$U_{g,\nu,z}(\varphi) = \sum_{x \in \Lambda} \left(\frac{1}{4} g \varphi_x^4 + \frac{1}{2} \nu \varphi_x^2 + \frac{1}{2} z \varphi_x (-\Delta \varphi)_x \right)$$

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Split the Hamiltonian:

$$U_{g,\nu,1}(\varphi) = U_{0,\nu,1}(\varphi) + U_{g,0,0}(\varphi)$$

Not positive-definite: $\nu < 0$

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Corresponds to Gaussian part with covariance

$$(1+z_0)(-\Delta + m^2)^{-1}$$

Preparation: Gaussian approximation

Splitting yields

$$G_{x,N}(g,\nu) = \frac{\int e^{-U_{g,\nu,1}(\varphi)}\varphi_0\varphi_x \, d\varphi}{\int e^{-U_{g,\nu,1}(\varphi)} \, d\varphi} = (1+z_0)\frac{\mathbb{E}_C\left(e^{-U_0(\Lambda)}\varphi_0\varphi_x\right)}{\mathbb{E}_C\left(e^{-U_0(\Lambda)}\right)},$$

where $\mathbb{E}_C$ is Gaussian expectation with

$$C = (-\Delta + m^2)^{-1}$$

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Put $\varphi_0\varphi_x$ in the exponent:

$$G_{x,N}(g,\nu) = (1+z_0)\frac{\partial^2}{\partial\sigma_0\partial\sigma_x}\log\mathbb{E}_C\Big(e^{-\overbrace{(U_0(\Lambda) - \sigma_0\varphi_0 - \sigma_x\varphi_x)}^{V_0(\Lambda)}}\Big)\Big|_{\sigma=0}$$

Progressive integration

Problem: Evaluate $\mathbb{E}_C e^{-V_0(\Lambda)}$

If $C = C' + \tilde{C}$ and (correspondingly) $\varphi \stackrel{\mathcal{D}}{=} \varphi' + \zeta$:

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Recall: we need to integrate $Z_0 = e^{-V_0(\Lambda)}$

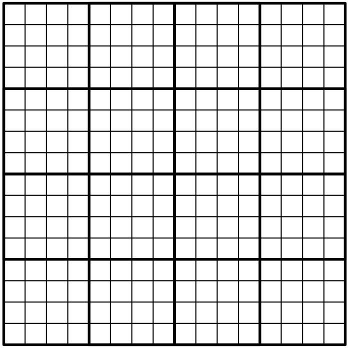
Renormalisation group map: integrate out small fluctuations

$$Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j$$

Covariance decomposition

The choice of decomposition $C = C_1 + \cdots + C_N$ is fundamental

Block spin example: $C_{j;xy}$ constant on "blocks" of side $O(L^j)$



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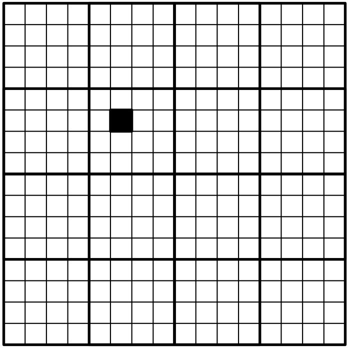

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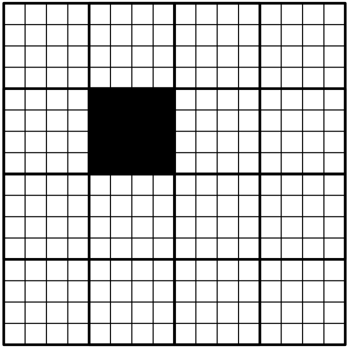

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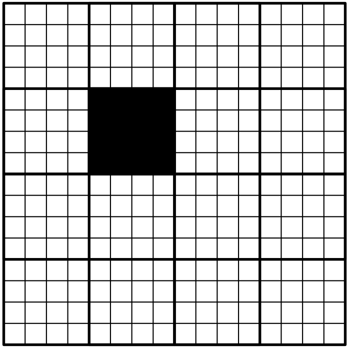

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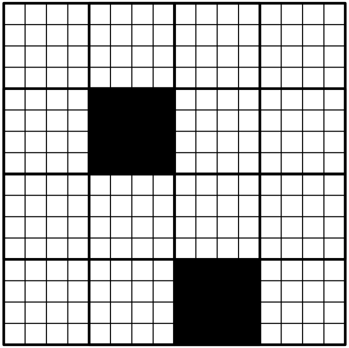

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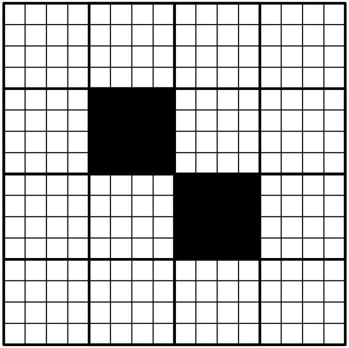
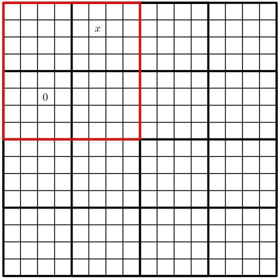


Covariance decomposition

Coalescence scale: $j_x = \min(j : C_{j;0x} \neq 0) \sim \log_L(2|x|)$



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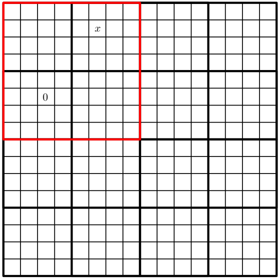

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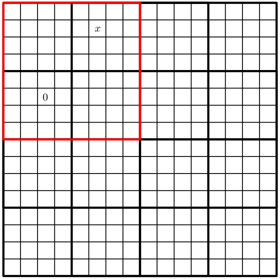

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$$\leq O(L^{-j_x(d-2)}) = O(|x|^{-(d-2)})$$

Covariance decomposition

Mass scale: $j_m \sim \log_L m^{-1}$

Proof of key lemma: take advantage of m^2-dependent bounds

$$|C_{j;xy}| \leq O(L^{-j(d-2)-2s(j-j_m)_+}), \quad \forall s$$

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Appears in remainder term of key lemma

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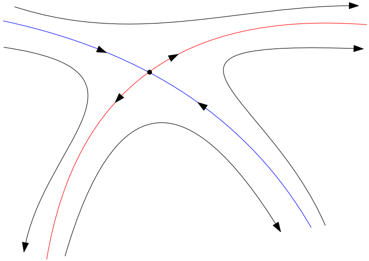
Ken Wilson ('74):

- Scale-invariant theories $\leftrightarrow$ fixed points
- Universality classes $\leftrightarrow$ stable set of fixed point
- Critical exponents $\leftrightarrow$ dynamics near a fixed point



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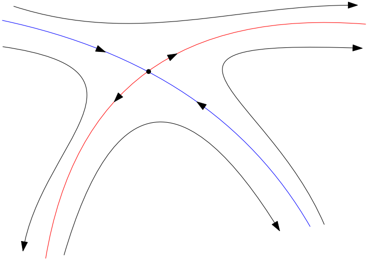

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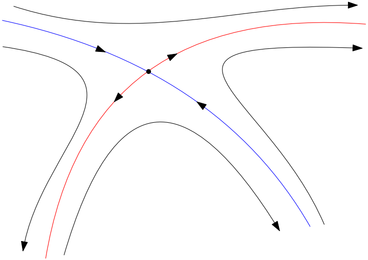

Approximation of the two-point function

Try to maintain the form $Z_j \approx e^{-V_j(\Lambda)}$ with

$$V_j(\Lambda) = \sum_{y\in\Lambda}\left(\frac{1}{4}g_j\varphi_y^4 + \frac{1}{2}\nu_j\varphi_y^2 + \frac{1}{2}z_j\varphi_y(-\Delta\varphi)_y\right) + u_j|\Lambda| - \lambda_{0,j}\varphi_0\sigma_0 - \lambda_{x,j}\varphi_x\sigma_x - \frac{1}{2}(q_{0,j}+q_{x,j})\sigma_0\sigma_x$$

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$$\log \mathbb{E}_C Z_0 = \log Z_N(\varphi = 0) \approx -V_N(\Lambda; \varphi = 0)$$

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Thus,

$$G_{x,N}(g,\nu) = (1 + z_0) \frac{\partial^2}{\partial \sigma_0 \partial \sigma_x} \log \mathbb{E}_C Z_0 \Big|_{\sigma=0} \approx \frac{1 + z_0}{2}(q_{0,N} + q_{x,N})$$

Flow of coupling constants

$V_j \mapsto V_{j+1}$ determined by solving an equation of the form

$$\mathbb{E}_{C_{j+1}} e^{-V_j}(1 + O(V_j^2)) = e^{-V_{j+1}}(1 + O(V_{j+1}^2)) + O(V_{j+1}^3)$$

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Combine this with

$$G_x(g, \nu) = \frac{1 + z_0}{2} \lim_{N \to \infty} (q_{0,N} + q_{x,N})$$

to get the key lemma (without remainder)

A hint of log corrections

The coefficient β_j satisfies

$$\beta_j = O(L^{-j(d-4)-2s(j-j_m)_+})$$

If $d = 4$,

$$\beta_j \approx \begin{cases} c, & j \leq j_m \\ 0, & j > j_m \end{cases}$$

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Origin of log:

$$g_\infty \sim g_{j_m} \sim \frac{1}{cj_m} \sim \frac{1}{c \log_L m^{-1}}$$

Relevant, marginal, and irrelevant fields

Dimensional analysis

$$|B||\varphi_x^p| \approx L^{jd}|C_j|^{p/2} = O\left(L^{-j\left(\frac{d-2}{2}p-d\right)}\right)$$

suggests

$$\varphi_x^p \text{ is } \begin{cases} \text{relevant}, & p < \frac{2d}{d-2} = 4 \\ \text{marginal}, & p = \frac{2d}{d-2} = 4 \\ \text{irrelevant}, & p > \frac{2d}{d-2} = 4 \end{cases}$$

RG coordinates

Parameterize Z_j by (expanding, contracting) coordinates (I_j, K_j)

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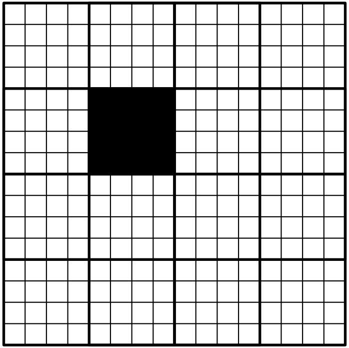
$$Z_j = \sum_{X \in \mathcal{P}_j} I_j(\Lambda \setminus X) K_j(X) = I_j(\Lambda) + \cdots$$



Block at scale j

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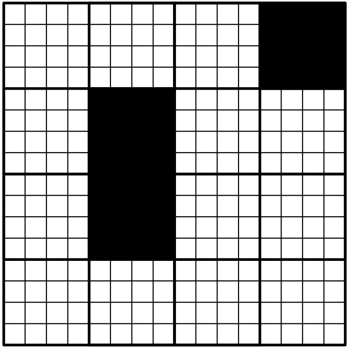

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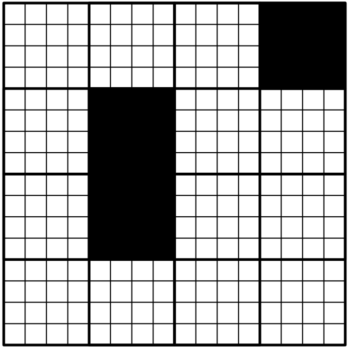

Leading-order term: $X = \varnothing$

Controlling the remainder

Full RG map $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$ must maintain the form

$$Z_j = (I_j \circ K_j)(\Lambda) := \sum_{Y \in \mathcal{P}_j} I_j(\Lambda \setminus X) K_j(X)$$

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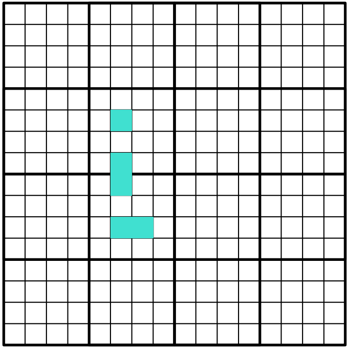

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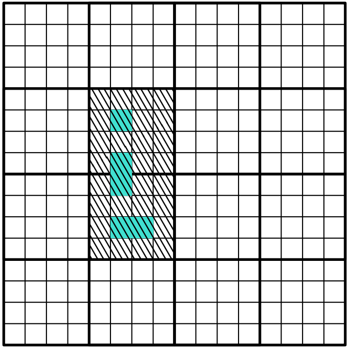

Controlling the remainder

To first order in K_j,

$$
\begin{aligned}
K_{j+1}(X) &= \sum_{\bar{Y}=X} I_{j+1}(X \setminus Y)\mathbb{E}_{C_{j+1}}(K_j \circ \delta I_{j+1})(Y) \\
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Problem

- K_j tracks irrelevant terms, e.g. φ_x^6
- Expectation of irrelevant terms gives rise to relevant/marginal terms:

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\mathbb{E}_{C_{j+1}} \varphi_x^6 &= \mathbb{E}_{C_{j+1}} (\varphi_x + \zeta_x)^6 \\
&= \varphi_x^6 + \binom{6}{4} \varphi_x^4 C_{j+1;00} + \cdots
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$$

Controlling the remainder

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Solution

- Extract marginal/relevant part prior to integration
- Estimate extracted terms $\rightarrow$ structural stability

Thank you!