Two-point function of $O(n)$ models below the critical dimension

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Background

The $|\varphi|^4$ model

Setup:

- Torus $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$
- Spins $\varphi_x \in \mathbb{R}^n$ for $x \in \Lambda_N$
- Symmetric matrix $M \in \mathbb{R}^{\Lambda \times \Lambda}$

Hamiltonian with interaction M:

$$V_{g,\nu}(\varphi) = \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{2} \varphi_x \cdot (M\varphi)_x \right)$$

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Gaussian case: $g = 0 < \nu$ and M positive-definite

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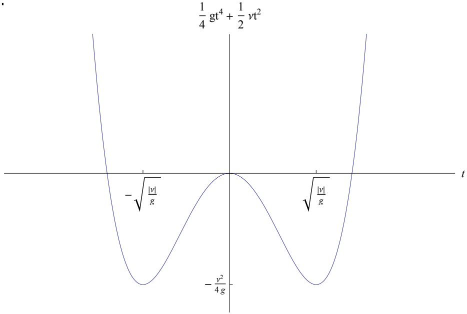
Case $\nu < 0 < g$:



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Examples:

- Nearest-neighbour: $M = -\Delta_\Lambda = -\sum_{|e|=1} \nabla^e$
- Long-range: $M = (-\Delta_\Lambda)^{\alpha/2}$ with $\alpha \in (0, 2)$

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Two-point function:

$$G_{a,b}(g,\nu;n) = \frac{1}{n} \lim_{N \to \infty} \langle \varphi_a \cdot \varphi_b \rangle_{g,\nu,N} = \lim_{N \to \infty} \langle \varphi_a^1 \varphi_b^1 \rangle_{g,\nu,N}$$

Relation to (weakly) self-avoiding walk

Formal $n \to 0$ limit

- De Gennes ('72)

Supersymmetric integral representation

- McKane ('80)
- Parisi-Sourlas ('80)

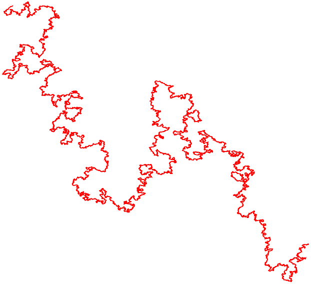

Figure: SAW with 1 million steps (Tom Kennedy)

Mean-field behaviour: nearest-neighbour case

Prediction: there is $\nu_c = \nu_c(g; n)$ such that

$$G_{a,b}(g, \nu_c; n) \sim C|a - b|^{-(d-2+\eta)}$$

Mean-field (Gaussian) behaviour:

$$\eta = 0 \text{ if } d \geq d_c = 4$$

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Rigorous results (mostly for small $g > 0$)

- $d > 4$
  - $n = 1, 2$ (continuum limit): Aizenman '82, Fröhlich '82
  - $n = 1$: Sakai '15
  - $n = 1, 2$: Brydges-Helmuth-Holmes
- $d = 4$
  - $n = 1$: Gawedzki-Kupiainen '85, Feldman-Magnen-Rivasseau-Sénéor '87
  - $n \geq 0$: Bauerschmidt-Brydges-Slade '14

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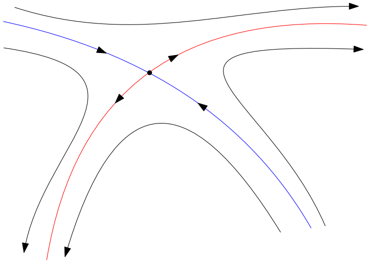
Anomalous ($\eta \neq 0$) behaviour predicted in $d \leq 3$

RG idea (Ken Wilson)

With respect to a "renormalisation group map" on a space of models:

- Fixed points $\leftrightarrow$ scaling limits
- Invariant manifolds $\leftrightarrow$ universality classes
- Dynamics near fixed point $\leftrightarrow$ critical behaviour



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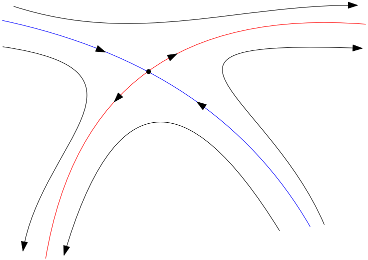

Bifurcation at $d = 4$:

- If $d > 4$: Gaussian fixed point is stable
- If $d < 4$: Wilson-Fisher fixed point is stable
- Wilson-Fisher '72: analysis of WF fixed point in $d = 4 - \epsilon$

Long-range models

The fractional Laplacian

For $x, y \in \mathbb{Z}^d$,

$$(-\Delta) \qquad = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left[4 \sum_{j=1}^{d} \sin^2(k_j/2) \right] \quad e^{ik\cdot(x-y)} \, dk$$

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Long-range interaction if $\alpha < 2$:

$$-(-\Delta)^{\alpha/2}_{x,y} \asymp |x-y|^{-(d+\alpha)}$$

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On the torus Λ, let

$$(-\Delta_\Lambda)^{\alpha/2}_{x,y} = \sum_{z\in\mathbb{Z}^d} (-\Delta_\Lambda)^{\alpha/2}_{x,y+zL^N}$$

The bubble diagram B

$-(-\Delta)^{\alpha/2}$ generator of a Markov process X

Expected number of intersections of X with an independent copy Y:

$$B = \sum_{x \in \mathbb{Z}^d} [(-\Delta)^{-\alpha/2}_{0,x}]^2$$

Use $(-\Delta)^{-\alpha/2}_{0,x} \asymp |x|^{-(d-\alpha)}$ to conclude

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Mean-field behaviour above d_c

- Aizenman-Fernandez ('88)
- Heydenreich-van der Hofstad-Sakai ('08)
- Heydenreich ('11)
- Chen-Sakai ('11, '15)

Long-range models

With $\alpha = \frac{1}{2}(d+\epsilon)$ and $d = 1, 2, 3$,

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Rigorous RG

- Control of RG trajectory: Brydges-Dimock-Hurd '98, Brydges-Mitter-Scoppola '03, Abdesselam '07, Mitter-Scoppola '08
- Slade '16 $\rightarrow$ susceptibility exponent

$$\gamma = 1 + \frac{n+2}{n+8}\frac{\epsilon}{\alpha} + O(\epsilon^2)$$

Main result

For long-range models, we expect

$$G_{a,b}(g,\nu_c;n) \sim C|a-b|^{-(d-\alpha+\eta)}$$

Fisher-Ma-Nickel ('72) predicted that

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Theorem 1 (Lohmann-Slade-W)

Let $d = 1,2,3$, $n \geq 0$, L large, $\epsilon > 0$ small. Then for $g \asymp \epsilon$,

$$G_{a,b}(g,\nu_c;n) = (1+O(\epsilon))((-\Delta)^{\alpha/2})^{-1}_{a,b} \asymp \frac{1}{|a-b|^{d-\alpha}}$$

Renormalisation group

Gaussian approximation

Write

$$V_{g,\nu} = \text{quadratic part} + \text{quartic remainder}$$

With $g_0 = g$ and $\nu_0 = \nu - m^2$,

$$\begin{aligned} V_{g,\nu}(\varphi) &= V_{0,m^2}(\varphi) + V_{g_0,\nu_0}(\varphi) \\ &= \frac{1}{2}\varphi \cdot \underbrace{\left((-\Delta_\Lambda)^{\alpha/2} + m^2\right)}_{C^{-1}} \varphi + V_{g_0,\nu_0}(\varphi) \end{aligned}$$

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Then

$$G_{a,b,N}(g,\nu;n) = \frac{\mathbb{E}_C\big(\varphi_a^1 \varphi_b^1 e^{-V_{g_0,\nu_0}(\varphi)}\big)}{\mathbb{E}_C\big(e^{-V_{g_0,\nu_0}(\varphi)}\big)}$$

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Critical: choice of $m^2 = m^2(g,\nu)$

Generating function

Put $\varphi_a^1 \varphi_b^1$ into the exponent:

$$
\begin{aligned}
G_{a,b,N}(g,\nu;n) &= \frac{\mathbb{E}_C\big(\varphi_a^1 \varphi_b^1 e^{-V_{g_0,\nu_0}(\varphi)}\big)}{\mathbb{E}_C\big(e^{-V_{g_0,\nu_0}(\varphi)}\big)} \\
&= \frac{\partial^2}{\partial\sigma_a \partial\sigma_b}\Big|_0 \log \mathbb{E}_C\, e^{-U_0(\Lambda)}
\end{aligned}
$$

with

$$
U_0(\Lambda) = V_{g_0,\nu_0}(\varphi) - \sigma_a \varphi_a^1 - \sigma_b \varphi_b^1
$$

Progressive integration

Evaluate $\mathbb{E}_C$ using finite-range decomposition (Bauerschmidt '13)

$$C = C_1 + \cdots + C_N$$

Define convolution (conditional expectation)

$$[\mathbb{E}_C\, \theta F](\varphi) = \mathbb{E}_C\, F(\varphi + \zeta)$$

Then

$$\mathbb{E}_C\, \theta F(\varphi) = \mathbb{E}_{C_N}\, \theta \ldots \mathbb{E}_{C_1}\, \theta F(\varphi)$$

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Define the RG map

$$F \mapsto \mathbb{E}_{C_{j+1}}\, \theta F$$

This induces the sequence

$$Z_{j+1} = \mathbb{E}_{C_{j+1}}\, \theta Z_j, \quad Z_0 = e^{-U_0(\Lambda)}$$

Perturbative flow

Suppose $Z_j \approx e^{-U_j(\Lambda)}$ with

$$U_j(\Lambda) = \sum_{x\in\Lambda}\Big(\frac{1}{4}g_j|\varphi_x|^4 + \frac{1}{2}\nu_j|\varphi_x|^2 + u_j\Big)$$

$$- \lambda_{a,j}\sigma_a\varphi_a^1 - \lambda_{b,j}\sigma_b\varphi_b^1 - \frac{1}{2}(q_{a,j} + q_{b,j})\sigma_a\sigma_b$$

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Then

$$G_{a,b,N}(g, \nu; n) = \frac{\partial^2}{\partial \sigma_a \partial \sigma_b}\Big|_0 \overbrace{\log \mathbb{E}_C \, e^{-U_0(\Lambda)}}^{\approx - U_N(\Lambda)}$$

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Control of remainder: Brydges-Slade ('15)

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Second-order part of RG map: $U_j \mapsto U_{j+1}$ such that

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This gives

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Two-point function

Cluster expansion $\Rightarrow \lambda_{x,j} = 1 + O(\epsilon)$ for $x = a, b$

Solve the recursion:

$$\begin{aligned} q_{j+1} &= q_j + \lambda_{a,j}\lambda_{b,j} C_{j+1;a,b} \\ &= q_j + (1 + O(\epsilon)) C_{j+1;ab} \\ &= (1 + O(\epsilon)) \sum_{i=1}^{j+1} C_{i;a,b} \end{aligned}$$

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Take $N \to \infty$ and $\nu \downarrow \nu_c \Leftrightarrow m^2 \downarrow 0$:

$$\begin{aligned} G_{a,b}(g, \nu_c; n) &\approx \frac{1}{2} \lim_{m^2 \downarrow 0} \lim_{N \to \infty} (q_{a,N} + q_{b,N}) \\ &= (1 + O(\epsilon)) \sum_{i=0}^{\infty} C_{i;a,b} \\ &= (1 + O(\epsilon)) ((-\Delta)^{\alpha/2})^{-1}_{a,b} \end{aligned}$$

Thank you