HirzebruchPolyptych_LatticePoints-Copy1

February 2, 2022

[1]:
```
from sympy import *
from sympy.vector import Vector
from sympy.vector import CoordSys3D
import IPython.display as disp

N = CoordSys3D('N')

t, k = symbols( 't k' )
n, m, d = symbols('n m d', positive=True, integer=True)
# init_printing(use_unicode=True)
init_printing(use_latex='mathjax')

# Basis for the edge/weight vectors for the points

v1 = N.i
v2 = N.j

# Define the vector which is not parallel to any edge vector, which will tend␣
 ↪to zero:

Phi = t*(v1 + 3*v2)

# Set the fixed points of the action; P denotes those that belong
# to the core, and Q those that come from the cut extended core:

def P12(n,m):
    return Vector.zero

def P23(n,m):
    return (m+n)*v1

def P34(n,m):
    return m*v1 + n*v2

def P14(n,m):
    return n*v2
```

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def P13(n,m):
    return (m+n)*v2
```

[2]:
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def Q12_1(n,m,d):
    return (-m-2*d)*v2

def Q12_2(n,m,d):
    return (-m-2*d)*v1

def Q23_2(n,m,d):
    return (2*m+n+2*d)*v1

def Q23_3(n,m,d):
    return (2*m+n+2*d)*v1 + (-m-2*d)*v2

def Q34_4(n,m,d):
    return (2*m + 2*d)*v1 + n*v2

def Q13_1(n,m,d):
    return (m+n+d)*v2

def Q13_3(n,m,d):
    return -d*v1 + (m+n+d)*v2

def Q14_4(n,m,d):
    return (-m-2*d)*v1 + n*v2

# Define the term which is summed over each fixed point,
# representing the character for the representation

def P(P, edge1, edge2, edge3, edge4):
    return exp( Phi.dot(P) ) / ( (1 - exp( Phi.dot(edge1) ) ) * ( 1 - exp( Phi.
 ↪dot(edge2) ) ) * ( 1 - exp( Phi.dot(edge3) ) ) * ( 1 - exp( Phi.dot(edge4) )␣
 ↪) )


def Q(P, edge1, edge2):
    return exp( Phi.dot(P) ) / ( (1 - exp( Phi.dot(edge1) ) ) * ( 1 - exp( Phi.
 ↪dot(edge2) ) ) )

def Exp(p, q):
    return exp( 2*pi*I*Rational(p,q) )

def OrbiFactor(p, q, edge):
    return ( 1 - ( Exp(p,q) * exp( Rational(p,q) * Phi.dot(edge) ) ) )

def OrbiSum(Q, p, q, edge1, edge2, edge3, edge4):
```

```
    return ( Rational(1,q) * exp( Phi.dot(Q) ) ) / ( (1 - ( Exp(p,q) * exp(␣
↪Rational(1,q) * Phi.dot(edge1) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q)␣
↪* Phi.dot(edge2) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q) * Phi.
↪dot(edge3) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q) * Phi.dot(edge4) ) )␣
↪) )

def OrbiHalfSum(Q, p, q, edge1, edge2, edge3, edge4):
    return ( Rational(1,q) * exp( Phi.dot(Q) ) ) / ( (1 - ( Exp(p,q) * exp(␣
↪Rational(1,q) * Phi.dot(edge1) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q)␣
↪* Phi.dot(edge2) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q) * Phi.
↪dot(edge3) ) ) ) * (1 - ( Exp(p,q) * exp( Rational(1,q) * Phi.dot(edge4) ) )␣
↪) )
```

[3]:
```
def TermP12(n,m):
    return P(P12(n,m), v1, v2, -v1, -v2)

def TermP23(n,m):
    return P(P23(n,m), v1, -v1 + v2, -v1, v1 - v2)

def TermP34(n,m):
    return P(P34(n,m), v1, -v1 + v2, -v1, v1 - v2)

def TermP14(n,m):
    return P(P14(n,m), v1, v2, -v1, -v2)

def TermP13(n,m):
    return P(P13(n,m), v1 - v2, v2, -v1 + v2, -v2)

def InteriorSum(n,d):
    return TermP12(n,m) + TermP23(n,m) + TermP34(n,m) + TermP14(n,m) +␣
↪TermP13(n,m)

InteriorSum(n,d)
```

[3]:

$$\frac{e^{t(m+n)}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} + \frac{e^{mt+3nt}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})} + \frac{e^{3nt}}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} + \frac{1}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})} + \frac{e^{3t(m+n)}}{(1-e^{-3t})(1-e^{-2t})(1-e^{2t})(1-e^{3t})}$$

[7]:
```
def TermQ12_1(n,m,d):
    return P(Q12_1(n,m,d), v2, v2, v1, -v1 + v2)

def TermQ12_2(n,m,d):
    return P(Q12_2(n,m,d), v1, v1, v2, v1 - v2)
```

```python
def TermQ23_2(n,m,d):
    return P(Q23_2(n,m,d), -v1, -v1, -v2, -v1 + v2)

def TermQ23_3(n,m,d):
    return P(Q23_3(n,m,d), -v1 + v2, -v1 + v2, -v1, v2)

def TermQ34_4(n,m,d):
    return P(Q34_4(n,m,d), -v1, -v1, -2*v1 + v2, v1 - v2)

# Q13_1 is an orbifold point of order 2:

def TermQ13_1(n,m,d):
    return OrbiSum(Q13_1(n,m,d), 0, 2, 2*v1 - v2, -v2, -v2, -2*v1) + OrbiSum(Q13_1(n,m,d), 1, 2, 2*v1 - v2, -v2, -v2, -2*v1)

# Q13_3 is an orbifold point of order 2:

def TermQ13_3(n,m,d):
    return OrbiSum(Q13_3(n,m,d), 0, 2, 2*v1, v1 - v2, v1 - v2, -v1 - v2) + OrbiSum(Q13_3(n,m,d), 1, 2, 2*v1, v1 - v2, v1 - v2, -v1 - v2)

def TermQ14_4(n,m,d):
    return P(Q14_4(n,m,d), v1, v1, v1 + v2, -v2)
```

[8]:
```python
def InteriorSum(n,m):
    return TermP12(n,d) + TermP23(n,m) + TermP34(n,m) + TermP13(n,m) + TermP14(n,m)

def ExteriorSum(n,m,d):
    return TermQ12_1(n,m,d) + TermQ12_2(n,m,d) + TermQ23_2(n,m,d) + TermQ23_3(n,m,d) + TermQ34_4(n,m,d) + TermQ13_1(n,m,d) + TermQ13_3(n,m,d) + TermQ14_4(n,m,d)

ExteriorSum(n,m,d)
```

[8]:

$$\frac{e^{3t(-2d-m)}}{(1-e^{t})(1-e^{2t})(1-e^{3t})^{2}} + \frac{e^{3t(-2d-m)+t(2d+2m+n)}}{(1-e^{-t})(1-e^{2t})^{2}(1-e^{3t})} + \frac{e^{3t(d+m+n)}}{2\left(1+e^{-\frac{3t}{2}}\right)^{2}(1+e^{-t})\left(1+e^{-\frac{t}{2}}\right)} + \frac{e^{3t(d+m+n)}}{2\left(1-e^{-\frac{3t}{2}}\right)^{2}(1-e^{-t})\left(1-e^{-\frac{t}{2}}\right)} + \frac{e^{-dt+3t(d+m+n)}}{2(1+e^{-2t})(1+e^{-t})^{2}(e^{t}+1)} + \frac{e^{t(-2d-m)}}{(1-e^{-2t})(1-e^{t})^{2}(1-e^{3t})} + \frac{e^{-dt+3t(d+m+n)}}{2(1-e^{-2t})(1-e^{-t})^{2}(1-e^{t})} + \frac{e^{3nt+t(2d+2m)}}{(1-e^{-2t})(1-e^{-t})^{2}(1-e^{t})} + \frac{e^{3nt+t(-2d-m)}}{(1-e^{-3t})(1-e^{t})^{2}(1-e^{4t})} + \frac{e^{t(2d+2m+n)}}{(1-e^{-3t})(1-e^{-t})^{2}(1-e^{2t})}$$

[9]:
```
def TotalSum(n,m,d):
    return InteriorSum(n,m) + ExteriorSum(n,m,d)

TotalSum(n,m,d)
```

[9]:

$$\frac{e^{3t(-2d-m)}}{(1-e^{t})(1-e^{2t})(1-e^{3t})^{2}}+\frac{e^{3t(-2d-m)+t(2d+2m+n)}}{(1-e^{-t})(1-e^{2t})^{2}(1-e^{3t})}+\frac{e^{3t(d+m+n)}}{2\left(1+e^{-\frac{3t}{2}}\right)^{2}(1+e^{-t})\left(1+e^{-\frac{t}{2}}\right)}+\frac{e^{3t(d+m+n)}}{2\left(1-e^{-\frac{3t}{2}}\right)^{2}(1-e^{-t})\left(1-e^{-\frac{t}{2}}\right)}+\frac{e^{-dt+3t(d+m+n)}}{2(1+e^{-2t})(1+e^{-t})^{2}(e^{t}+1)}+\frac{e^{t(-2d-m)}}{(1-e^{-2t})(1-e^{t})^{2}(1-e^{3t})}+\frac{e^{t(m+n)}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})}+\frac{e^{mt+3nt}}{(1-e^{-2t})(1-e^{-t})(1-e^{t})(1-e^{2t})}+\frac{e^{-dt+3t(d+m+n)}}{2(1-e^{-2t})(1-e^{-t})^{2}(1-e^{t})}+\frac{e^{3nt+t(2d+2m)}}{(1-e^{-2t})(1-e^{-t})^{2}(1-e^{t})}+\frac{e^{3nt+t(-2d-m)}}{(1-e^{-3t})(1-e^{t})^{2}(1-e^{4t})}+\frac{e^{3nt}}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})}+\frac{1}{(1-e^{-3t})(1-e^{-t})(1-e^{t})(1-e^{3t})}+\frac{e^{t(2d+2m+n)}}{(1-e^{-3t})(1-e^{-t})^{2}(1-e^{2t})}+\frac{e^{3t(m+n)}}{(1-e^{-3t})(1-e^{-2t})(1-e^{2t})(1-e^{3t})}$$

[14]:
```
factor(limit(TotalSum(n,m,d), t, 0))
```

[14]:

$$\frac{544d^4+1536d^3m+768d^3n+1632d^3+1536d^2m^2+1536d^2mn+3456d^2m+192d^2n^2+1728d^2n+1856d^2+640dm}{}$$

[31]:
```
def Euler(m,n,d):
    return factor((544*d**4 + 1536*d**3*m + 768*d**3*n + 1632*d**3 +
 ↪1536*d**2*m**2 + 1536*d**2*m*n + 3456*d**2*m + 192*d**2*n**2 + 1728*d**2*n +
 ↪1856*d**2 + 640*d*m**3 + 960*d*m**2*n + 2304*d*m**2 + 192*d*m*n**2 +
 ↪2304*d*m*n + 2624*d*m + 288*d*n**2 + 1248*d*n + 948*d + 96*m**4 + 192*m**3*n
 ↪+ 480*m**3 + 48*m**2*n**2 + 720*m**2*n + 864*m**2 + 144*m*n**2 + 816*m*n +
 ↪672*m + 96*n**2 + 288*n + 195)/192)

def Diff(m,n,d):
    return simplify(Euler(m,n,d) - Euler(m,n,d-1))

Diff(m,n,d)
```

[31]:

$$\frac{34d^3}{3}+24d^2m+12d^2n+\frac{17d^2}{2}+16dm^2+16dmn+12dm+2dn^2+6dn+\frac{31d}{6}+\frac{10m^3}{3}+5m^2n+4m^2+mn^2+4mn+\frac{11m}{3}+\frac{n^2}{2}+\frac{3n}{2}+\frac{15}{16}$$

[34]:
```
Diff(m,n,0)
```

[34]:

$$\frac{10m^3}{3}+5m^2n+4m^2+mn^2+4mn+\frac{11m}{3}+\frac{n^2}{2}+\frac{3n}{2}+\frac{15}{16}$$